On the Inverse of Parabolic Systems of Partial Differential Equations of General Form in an Infinite Space–Time Cylinder[1]

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Abstract

We consider general parabolic systems of equations on the infinite time interval in case of the underlying spatial configuration is a closed manifold. The solvability of equations is studied both with respect to time and spatial variables in exponentially weighted anisotropic Sobolev spaces, and existence and maximal regularity statements for parabolic equations are proved. Moreover, we analyze the long-time behaviour of solutions in terms of complete asymptotic expansions.

These results are deduced from a pseudodifferential calculus that we construct explicitly. This algebra of operators is specifically designed to contain both the classical systems of parabolic equations of general form and their inverses, parabolicity being reflected purely on symbolic level. To this end, we assign $t = \infty$ the meaning of an anisotropic conical point, and prove that this interpretation is consistent with the natural setting in the analysis of parabolic PDE. Hence, major parts of this work consist of the construction of an appropriate anisotropic cone calculus of so-called Volterra operators.

In particular, which is the most important aspect, we obtain the complete characterization of the microlocal and the global kernel structure of the inverse of parabolic systems in an infinite space–time cylinder. Moreover, we obtain perturbation results for parabolic equations from the investigation of the ideal structure of the calculus.

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Contents

Introduction

Parabolic partial differential equations arise in the modelling of time-dependent phenomena, e. g., in the description of diffusion processes, such as heat diffusion, as well as in probability, where we often find probability densities of stochastic processes as solutions of associated parabolic equations or systems, as it is the case for (certain) Markov chains. Moreover, there are deep connections between the analysis of the heat equation associated with geometric operators and (spectral) geometry.
In general, external influences, such as exterior sources, and interactions with the geometry of the underlying spatial configuration, lead to non-autonomous equations or systems, i. e., the coefficients may depend on time, and we have to solve them with inhomogeneous data. To give an example, consider the heat diffusion flow in a body, represented by a closed manifold X. Then, as the diffusion u interacts with the geometry of X, perturbed by an exterior source f, we find it as a solution of the heat equation $\big(\partial_t - \Delta_{g_X(t)}\big)u = f$, where $g_X(t)$ is a family of Riemannian metrics, depending on time.

The analysis of parabolic partial differential equations is concerned, in particular, with the following questions:

- Existence and uniqueness of solutions in appropriate function spaces.
- Local properties of solutions, such as regularity and local bounds for the derivatives, on a finite time interval.
- Global properties, e. g., global bounds and/or integrability conditions, as well as stabilization of the solution and its derivatives, especially asymptotics, on the infinite time interval.

In this work, we consider general parabolic systems of equations on the infinite time interval in case of the underlying spatial configuration is a closed manifold. The solvability of equations is studied both with respect to time and spatial variables in exponentially weighted anisotropic Sobolev-Slobodeckij spaces (which will be called just Sobolev spaces in the sequel). In particular, this leads to fine analysis of regularity, which results into existence and maximal regularity statements for parabolic equations. Moreover, we analyze the long-time behaviour of solutions in terms of complete asymptotic expansions.

These results are deduced from the concept of regularity of a pseudodifferential calculus that we construct explicitly. This algebra of operators is specifically designed to contain both the classical systems of parabolic equations of general form and their inverses, parabolicity being reflected purely on symbolic level. To this end, we assign $t = \infty$ the meaning of an anisotropic conical point, and prove that this interpretation is consistent with the natural setting in the analysis of parabolic PDE (see [34]). Hence, major parts of this work consist of the construction of an appropriate anisotropic cone calculus of so-called Volterra operators.

In particular, which is the most important aspect of this work, we obtain the complete characterization of the microlocal and the global kernel structure of the inverse of parabolic systems on the infinite time interval. Moreover, we obtain perturbation results for parabolic equations from the investigation of the ideal structure of the calculus.

Let us enter this subject in more detail with some historical and methodical remarks first.

Elements of the classical theory

In parabolic partial differential equations there is a canonical splitting of variables into the (preferred) time and the spatial variables, and the analysis requires the anisotropic treatment of these. The most direct approach is based upon the anisotropic treatment of space and time on side of the covariables. Parabolicity in this framework is regarded as some "strong" anisotropic ellipticity with the time covariable polynomially rescaled in order to compensate the different orders between spatial and time derivatives. This point of view goes back essentially to the classical works of Petrovskij [48]. A deep connection between parabolicity and anisotropic parameter-dependent ellipticity is given via regarding the time covariable as a parameter for the operators acting in space. This was observed and systematically exploited by Agranovich and Vishik [2] in their work about boundary value problems for parabolic partial differential equations of general form.
The solvability of equations is established both with respect to time and spatial variables in anisotropic Sobolev spaces. This framework is particularly well-suited for the analysis of regularity of distribution solutions.
As general references in this context we want to mention the works of Ejdel'man [14], Ejdel'man and Zhitarashu [15], Friedman [16], Ladyzhenskaya, Solonnikov, and Uraltseva [38], Lions and Magenes [40], and Solonnikov [66], [67], [68].

The concept of anisotropic parameter-dependent ellipticity also plays an important role in spectral theory and the analysis of resolvents (see, e. g., Shubin [65]).

This can be regarded as a link between the microlocal approach, i. e., anisotropic treatment of space and time on side of the covariables as just discussed, and the semigroup-theoretical approach to parabolic equations, which is, roughly speaking, based upon anisotropic treatment on side of the variables (see, e. g., Amann [3], Lunardi [41], Pazy [47], and Tanabe [69]). The solution there is given in terms of an evolution operator, and can be seen in the context of singular (Volterra) integral operators, where the fundamental solution plays the role of an operator-valued convolution kernel. Hence, starting from a partial differential equation, the solution operator is rather implicit due to the emphasis of the kernel level, and the microlocal character is not reflected. We shall not pursue this discussion further.

The global long-time behaviour of solutions is an important feature in the study of equations posed on the infinite time interval. The analysis is essentially devoted to establish bounds at infinity, and it is most natural to ask for solutions satisfying exponential estimates (see, e. g., Agranovich and Vishik [2]); of course, such can be expected only under suitable assumptions on the coefficients of the equation as well as on the inhomogeneous data. More refined control of the global behaviour of solutions is reflected by asymptotic stabilization, or even asymptotic expansions. Asymptotic analysis of partial differential equations is for itself a field of independent interest in mathematics with a long tradition. Concerning long-time behaviour and exponentially stable solutions we just want to mention the works of Agmon and Nirenberg [1], Maz'ya and Plamenevskij [42], and Pazy [46], [47]. However, a sufficiently complete analysis including perturbation theory of the long-time asymptotical behaviour for solutions to parabolic equations seems not to be available in the literature yet, even under rather strong assumptions on the coefficients of the equations and the inhomogeneous data.

Pseudodifferential analysis of parabolic equations

The basic idea in pseudodifferential analysis in general is to embed differential operators, which are "typical" for a certain problem setting, into an algebra of operators with symbolic structure, and to study, e. g., the solvability and regularity of equations therein. The symbolic structure plays the dominating role in all investigations, and conditions and manipulations on the microlocal side are reflected on the operator level – the quantized objects – usually up to a small ideal of residual elements that qualitatively can be neglected in the considerations in question.
This concept is particularly well-established in the theory of elliptic equations, where ellipticity is determined by the invertibility of the symbolic components, and the existence of parametrices within the calculus is proved. In particular, Fredholm solvability for elliptic equations is achieved in natural scales of Sobolev spaces, with the parametrix being a Fredholm inverse. The analysis of the operators in the algebra, applied to the parametrix and the remainders, provides detailed elliptic regularity results, including the asymptotic behaviour of solutions

near the singular sets in the theory of degenerate elliptic equations. Moreover, this naturally implies perturbation results for elliptic equations.
See, e. g., Boutet de Monvel [5], Grubb [25], Rempel and Schulze [53] for boundary value problems with the transmission property, and Schulze [59], [60], [61] for the more general case of pseudodifferential theory of degenerate elliptic operators, where the degeneracy reflects the presence of geometric singularities on the underlying manifold in a typical way.

In contrast to elliptic theory, we classically expect unique solvability for parabolic equations. However, it is still mostly desirable to have the achievements of elliptic theory at hand also in the framework of parabolicity, i. e., to take advantage in the study of equations from a specifically designed calculus of pseudodifferential operators. Hence, the program to be carried out is the following:

- Completion of the most natural systems of non-autonomous parabolic equations of general form to an enveloping algebra of pseudodifferential operators.
- Characterization of parabolicity purely on symbolic level by means of the invertibility of the symbolic components.
- Proof of the equivalence of symbolic and operational invertibility, i. e., parabolic operators are invertible, and the inverses belong to the calculus.
- Representation of the algebra as bounded operators acting in the natural scale of anisotropic Sobolev spaces (with an exponential weight at infinity).
- Extension of the concept of regularity for the calculus in the sense, that the analysis of both smoothness (via the smoothness-parameters of the Sobolev spaces) and asymptotics (via subspaces that carry the asymptotic information) of solutions is a consequence of the general mapping properties of the operators in these spaces.

As a consequence, the microlocal character of the solution operator and its global kernel structure are clarified, and an extensive study of regularity and global behaviour of solutions, as well as perturbation theory, is available purely in algebraic terms on side of the algebras of symbols and operators, as well as their ideal structure.

A first step towards this program was done in the works of Piriou [49], [50]; however, his approach was not really widely applied. Let us shortly summarize the important contents:
He introduced the class of anisotropic scalar pseudodifferential operators with the Volterra property in order to investigate parabolic pseudodifferential equations on a finite time interval, where the underlying spatial configuration is a closed manifold. The significant feature of these operators is that they are built upon

anisotropic symbols, the anisotropy referring to time and space, that extend holomorphically in the time covariable to the lower complex half-plane, including the symbol estimates. It is proved that this class remains preserved under compositions. Parabolicity is defined by the invertibility of the anisotropic homogeneous principal symbol, extended to the half-plane, and a parametrix construction is carried out within the algebra of operators with the Volterra property. Due to the Paley-Wiener theorem, the Schwartz kernel of a Volterra pseudodifferential operator is supported below the diagonal with respect to the time variable; in fact, this is the justification for this notion. As a consequence, the operators are "one-sided local" with respect to time, i. e., the support of a distribution is preserved by the action of the operator from the positive side. Hence, a Neumann series argument is applicable to the kernels of the remainders of the parametrix construction, which insures the invertibility of parabolic operators within the calculus, as they are considered on the finite time interval only.

Stimulated by Piriou's results, Rempel and Schulze [53] initiated similar investigations for parabolic boundary value problems, and subsequently first steps were done by Buchholz [6], and Buchholz and Schulze [7], [8], to approach the case of the underlying spatial manifold having geometric singularities. However, these studies were restricted to problems on the finite time interval, while in the present work we fully carry out the above program in case of operators on the infinite time interval, and thus the analysis of the relevant effects near $t = \infty$ is included (see also [34]). As turns out, the non-compactness with respect to time is responsible for the presence of an additional operator-valued symbol in the regulation of parabolicity, and the before-mentioned concepts for the calculus on a finite time interval have to be extended considerably.

A basic observation to achieve the desired results is, that it is possible to interpret $t = \infty$ as a conical point of the infinite space-time configuration, and this interpretation is consistent with the natural setting in parabolic problems. Let us illustrate this a bit more for differential operators:

Consider an anisotropic differential operator on $\mathbb{R}\times X$

$$\left.\begin{aligned} &A = \sum_{j=0}^{\mu/\ell} a_j(t)\partial_t^j, \\ &a_j \in C^\infty(\mathbb{R}, \mathrm{Diff}^{\,\mu-\ell j}(X; E, F)) \end{aligned}\right\} \tag{1}$$

of general form, where $\mu \in \ell\mathbb{N}_0$ is the anisotropic order of A, and $\mathrm{Diff}^{\,\mu-\ell j}(X; E, F)$ denotes the space of differential operators of order $\mu-\ell j$ acting in (smooth) sections of the vector bundles E and F on X. The anisotropy $\ell \in \mathbb{N}$ refers to the different treatment of space and time for the operator A; for the heat operator, e. g., we have $\ell = 2$ and $\mu = 2$.

Writing A in the coordinates $r = e^{-t}$ leads to

$$\left.\begin{aligned} \tilde{A} &= \sum_{j=0}^{\mu/\ell} \tilde{a}_j(r)(-r\partial_r)^j, \\ \tilde{a}_j(r) &= a_j(-\log(r)), \end{aligned}\right\} \tag{2}$$

and the effects near $t = \infty$ are now located at $r = 0$. We assume that the coefficients $\tilde{a}_j(r)$ extend as smooth functions up to the origin $r = 0$ – in the original coordinates this corresponds to exponential stabilization as $t \to \infty$ – hence, the operator $\tilde{A}$ can be regarded as an anisotropic totally characteristic operator. Notice that operators A with coefficients not depending on time for $t \gg 0$ sufficiently large belong to our setting (see also Agranovich and Vishik [2]).

In singular analysis, operators of the form (2) are widely investigated in the framework of elliptic theory, for this is precisely the form of the typical differential operators near a conical singularity. The natural function spaces are Mellin Sobolev spaces, and the anisotropic variants of these are exactly the exponentially weighted Sobolev spaces on the cylinder $\mathbb{R}\times X$, written in the new coordinates on $\mathbb{R}_+ \times X$.

Hence, for the study of parabolicity of the operator (1), we consider it from the very beginning as given in the form (2). Our construction of the enveloping pseudodifferential algebra then relies on techniques which originate from elements of the cone calculus introduced by Schulze (in an anisotropic setting, see [13], [59], [60], or [61]), and, which is the crucial step, on establishing the analogue of Piriou's operators with the Volterra property in this framework.

The *Volterra cone calculus* is given in Chapter 6: Section 6.3 deals with the algebraic properties, and in Section 6.4 we discuss parabolicity and the invertibility of the operators within the algebra. As a by-product, we furthermore obtain a calculus for anisotropic elliptic totally characteristic operators, and a Fredholm theory for these in anisotropic weighted Sobolev spaces.

The concept of regularity of this calculus covers the control of conormal asymptotics, i. e., asymptotic expansions for functions $\tilde{u}$ of the form

$$\tilde{u}(r,x) \sim \sum_j \sum_{k=0}^{m_j} c_{j,k}(x) \log^k(r) r^{-p_j}, \quad r \to 0, \tag{3}$$

where the $p_j \in \mathbb{C}$ are complex numbers, only finitely many located in every strip parallel to the imaginary axis over a compact real interval, and the $c_{j,k}$ are smooth sections in a vector bundle over X. Equivalently, in the original space-time coordinates, these take the form

$$u(t,x) \sim \sum_j \sum_{k=0}^{m_j} c_{j,k}(x) t^k e^{p_j t}, \quad t \to \infty, \tag{4}$$

of exponential long-time asymptotics as desired.

Organization of the text and further comments

In Chapter 1 we give an account on the notations, and shortly summarize some preliminary material, e. g., about the Mellin transform, that we freely use throughout this work.
Due to the role played by meromorphic operator functions in the symbolic calculus of the final algebra as to control the asymptotic behaviour of solutions (see, in particular, Sections 5.1, 5.2, and 6.2), we decided to supply Section 1.2, where we recall the classical theorem on the inversion of finitely meromorphic Fredholm families (see Gohberg and Sigal [19], Gramsch [20], and Gramsch and Kaballo [22]). This theorem is used in the construction of parametrices of parabolic operators, more precisely in symbolic inversion.
Moreover, we recall in Section 1.3 in some detail elements of the theory of Volterra integral operators with operator-valued L^2-kernel functions. Provided that the kernel is continuous and fulfills suitable weighted estimates we give a proof that the associated operator is quasinilpotent, i. e., its spectrum consists of zero only. This observation is crucial for the analysis of remainders of the parametrix construction to parabolic pseudodifferential operators, and leads to the invertibility of these within the calculus.
We conclude the chapter with some notes on abstract kernels. The mapping properties of an operator within a scale of suitable function spaces is closely related to the behaviour of its Schwartz kernel; in applications, the residual elements of a pseudodifferential calculus are usually characterized by such mapping properties. In order to be able to apply the inversion result for Volterra integral operators to these operators, we have to conclude that the Schwartz kernels satisfy certain weighted estimates. From the abstract point of view, the relationship between mapping properties and kernels is given by means of tensor product representations.

Chapter 2 is devoted to recall some basic elements of pseudodifferential calculus with operator-valued symbols. In general, a global calculus of pseudodifferential operators is built upon underlying structures of operator-valued symbols, e. g., passing to local coordinates in the interior of a manifold gives rise to matrix-valued symbols. Moreover, in parameter-dependent calculi we often find residual elements characterized as operator-valued symbols within a suitable scale of Sobolev spaces, while the parameter-dependent calculus itself embeds into a space of operator-valued symbols. Consequently, we find operator-valued symbols both as a sub- and superstructure, which enables us to trace back many global constructions to the abstract calculus of pseudodifferential operators with operator-valued symbols. Though some technical properties of our Volterra cone calculus could be deduced in a more direct way, we prefer to make as much use of the abstract setting given in Chapter 2 as possible. On the one hand, this shows that the more complicated constructions later are in fact based upon some few analytic principles that we formulate explicitly, and on the other hand, it demonstrates that our methods should be extendable to apply to more complicated situations, e. g., parabolic equations with geometric singularities on the spatial configuration.

Intuitively, the abstract calculus considered in Sections 2.1 – 2.4 should be thought of as operators acting in spatial direction with the time covariable unaffected as a parameter, while in the remaining sections of Chapter 2 we have the converse situation, i. e., operators acting in time with the spatial covariables unaffected. For our purposes, the calculi of Volterra symbols and operators are of course crucial, and many of the basic general constructions in pseudodifferential calculus have to be considerably modified to apply to this framework.
A rigorous treatment of the abstract general calculus of Volterra pseudodifferential operators with "twisted" operator-valued symbols is given in Krainer [32], [33]; see also Buchholz and Schulze [8]. In [32] and [33] the reader will also find those details of proofs that were skipped in the present exposition. Material on the general calculus of pseudodifferential operators with "twisted" operator-valued symbols as introduced by Schulze can be found, e. g., in [59], [60], [61].

In Chapter 3 we recall the calculus of anisotropic parameter-dependent pseudodifferential operators acting in sections of vector bundles on a closed manifold. The interpretation is that the parameter should be regarded as the time covariable. Moreover, we study the subcalculus of parameter-dependent Volterra operators, where the parameter-space is a complex half-plane, and the operator families depend holomorphically on the parameter. We define the corresponding notions of parameter-dependent ellipticity and parabolicity for such operators, and carry out the parametrix construction within the (Volterra) calculus.
The definitions and arguments are traced back to the considerations from Sections 2.1 – 2.4; in a local chart, we find operators that are built upon matrix-valued (Volterra) symbols, while the global smoothing remainders are precisely the regularizing operator-valued (Volterra) symbols in the standard Sobolev spaces of distributional sections in the bundles. Using elementary norm estimates of the operators in terms of the parameter, we conclude that the calculus itself embeds into a suitable space of operator-valued (Volterra) symbols in the Sobolev spaces. This observation, in particular, enables us to add some necessary supplements, such as kernel cut-off, simply via restriction from the abstract setting.

Chapter 4 is devoted to state the basic definitions and properties of the weighted anisotropic Sobolev spaces on the transformed space-time configuration in that form as they are needed in this exposition – the formulation in global terms via parameter-dependent reductions of orders admits, e. g., to deal in Chapter 5 in an efficient way with the continuity of pseudodifferential operators that are built upon symbols which themselves are parameter-dependent operators on the manifold. The elementary analysis of anisotropic Sobolev spaces is widely available in the literature, cf. Agranovich and Vishik [2], Grubb and Solonnikov [27], Lions and Magenes [40]). Material about (isotropic) Mellin Sobolev spaces can be found in the monographs of Schulze, concerning Mellin Sobolev spaces with discrete conormal asymptotics we refer, in particular, to [59]. Finally, we want to point out that though we employ the notion of "cone Sobolev spaces" due to superficial similarities with the corresponding spaces in the analysis on manifolds with conical singularities, the spaces in our framework are essentially different from these, see Section 4.3.

In Section 5.1 and 5.2 we introduce certain spaces of meromorphic functions taking values in the pseudodifferential operators on the manifold, which will later serve as meromorphic operator-valued (Volterra) Mellin symbols in the final algebra near the origin $r = 0$. We define the notions of ellipticity and parabolicity and prove inversion results under these conditions. In this context, we decided to supply Section 1.2.
Section 5.3 and 5.4 are concerned with pseudodifferential calculi where the symbols are built upon parameter-dependent (Volterra) operators on the manifold – these operators now act in function spaces on the full (transformed) space-time configuration. The pseudodifferential properties, such as composition, are consequences of the results in the general abstract setting. We define the notion of parabolicity for Volterra operators and establish the existence of Volterra parametrices. In fact, the arguments rely on the results for the parameter-dependent calculus of Chapter 3. In addition, we handle ellipticity for general anisotropic symbols.

The (holomorphic) Mellin calculus from Section 5.3 is of major importance, for these operators contribute to the final algebra near the origin $r = 0$, which corresponds to $t \to \infty$ in the original coordinates; the calculus of Section 5.4 will be employed away from $r = 0$, i. e., near $t = -\infty$. Isotropic meromorphic Mellin symbols and Mellin pseudodifferential calculus play an important role in the elliptic theory on manifolds with singularities, see [13], [59], [60], [61]; via specializing in Section 5.1 to $\ell = 1$, e. g., we find the symbol spaces which are considered in the cone calculus. For the treatment of parabolicity, however, we have to impose additional structures, and a much more refined analysis is required. In isotropic elliptic theory, a global Fourier calculus and an ellipticity criterion for operators considered in Section 5.4 were obtained by Seiler [64].

In Chapter 6 we establish the Volterra cone calculus. The definition of the calculus and its symbolic structure, as well as the analysis of the algebraic properties, is given in Section 6.3. Near the origin $r = 0$ we employ Volterra Mellin operators with meromorphic symbols, and away from $r = 0$ we find the operators from Section 5.4. In addition, there arise Volterra Green operators (Section 6.1) as residual elements. Section 6.4 is devoted to study parabolicity, and to establish the invertibility of parabolic operators within the calculus. In Section 6.2 we introduce an auxiliary algebra which allows us to present the analysis of regularity with asymptotics (the conormal effects) in a transparent form.

Parabolicity is determined by three symbols: The interior symbol (in the classical case this is just the anisotropic homogeneous principal symbol), the conormal symbol that controls the effects at $r = 0$, and the exit symbol which reflects the behaviour as $r \to \infty$ – all symbols extend in a canonical way holomorphically in the (transformed) time covariable to a half-plane. Provided that all symbols are invertible, the invertibility of the operator within the algebra follows; in particular, the operator is invertible in the Sobolev spaces on $\mathbb{R}_+ \times X$.

In parabolic partial differential equations there is usually an initial time $t_0 \in \mathbb{R}$, and the problem is posed on the time interval $[t_0, \infty)$, which corresponds in our setting to an interval $(0, r_0]$ with $r_0 \in \mathbb{R}_+$ – there are no effects as $r \to \infty$. Indeed, if we are just interested in the invertibility of an operator on an interval of this form, we can drop the parabolicity assumption for the exit symbol and still find the inverse operator in the calculus, but now restricted to subspaces of the Sobolev spaces which consist of all distributions with support in $(0, r_0] \times X$ (see Section 6.4).

We conclude the chapter with a proof of the existence of parabolic reductions of orders in our calculus; in particular, there are parabolic operators for any given order. This result, e. g., simplifies the analysis of parabolic boundary value problems, for they then are reduced to the case of the interior parabolic operator as well as all boundary conditions having the same unified pseudodifferential order (see [34]).

Finally, in Chapter 7, we give some more remarks about how the classical theory of parabolic partial differential equations fits into the framework of our Volterra cone calculus. To this end, we discuss the classical notion of parabolicity, as well as the results about solvability and regularity, for a generalized heat operator, and draw the connection to the functional analytic structure of our calculus for this example. In particular, the chapter may be thought of as an additional guide to the previous chapters along the lines of a particularly simple example.

Concluding remarks and future prospects

The achievements of pseudodifferential theory affected the analysis of parabolic equations in various other directions, in particular, what the study of equations of pseudodifferential character is concerned; see, e. g., Grubb [23], [24], Grubb and Solonnikov [27], Iwasaki [28], and Purmonen [52].

The present work, however, aims at another direction. Though the equations under consideration were modelled over a closed manifold, we have proved that, due to the additional non-compactness in time, we ended up with a theory for degenerate operators. In fact, this instance should be thought of as a general rule (see also [34]). Thus, in view of the insights from singular analysis that are nowadays available, our results should also be regarded as a step towards singular (or degenerate) parabolic problems, and the more advanced analysis of higher singularities in the future will rely upon them.

Indeed, many interesting and challenging problems from theory and applications belong to the singular problems:

- The non-compatible case of parabolic initial-boundary value problems is of high relevance and natural in models of applications. This is to a large extent not yet treated in the literature in a sufficiently general form.
- Parabolic mixed boundary problems, e. g., of Dirichlet/Neumann type (i. e., like in Zaremba's problem), see, for instance, Chan Žui Cho and Eskin [9].
- Parabolicity for degenerate cases, e. g., for boundary conditions of type of the oblique derivative problem; see Paneah [44], Popivanov and Palagachev [51].
- Parabolicity for geometric singularities of the spatial configuration, e. g., stratified spaces, where the singularities induce a hierarchy of (operator-valued) symbolic levels, see Schulze [62]. Necessary results in this direction can be found in Krainer [32], [33].
- Long-time asymptotics for singular spatial configurations, characterization of adequate asymptotic terms; see, e. g., Krainer and Schulze [35].

Chapter 1

Preliminary material

1.1 Basic notation and general conventions

Sets of real and complex numbers

- We denote:

$\mathbb{C}$	the complex numbers,
$\mathbb{R}$	the reals,
$\mathbb{R}_+, \mathbb{R}_-$	the positive (negative) reals,
$\overline{\mathbb{R}}_+, \overline{\mathbb{R}}_-$	the non-negative (non-positive) reals,
$\mathbb{Z}$	the integers,
$\mathbb{N}$	the positive integers,
$\mathbb{N}_0$	the non-negative integers.

- Let $\mathbb{C}^N$ and $\mathbb{R}^N$ denote the complex N-space, respectively the Euclidean N-space, in the variables $(z_1, \dots, z_N) \in \mathbb{C}^N$ or $(x_1, \dots, x_N) \in \mathbb{R}^N$, respectively. In general, we allow N to be zero, and in this case these spaces degenerate to the set containing a single point only.

- The upper half-plane in $\mathbb{C}$ will be denoted as

$$\mathbb{H} := \{z \in \mathbb{C};\ \mathrm{Im}(z) \geq 0\}.$$

  Moreover, for $\beta \in \mathbb{R}$ let

$$\Gamma_\beta := \{z \in \mathbb{C};\ \mathrm{Re}(z) = \beta\},$$
$$\mathbb{H}_\beta := \{z \in \mathbb{C};\ \mathrm{Re}(z) \geq \beta\}.$$

  We refer to Γ_β also as a *weight line*. With the splitting $z = \beta + i\tau$ into real and imaginary part we shall identify Γ_β with $\mathbb{R}$ via $\Gamma_\beta \ni z = \beta + i\tau \leftrightarrow \tau \in \mathbb{R}$.

Analogously, we have an identification of $\mathbb{H}_\beta$ with the right half-plane $\mathbb{H}_0$ via $\mathbb{H}_\beta \ni z = \beta + \zeta \leftrightarrow \zeta \in \mathbb{H}_0$, i. e. $\mathbb{H}_\beta$ originates from $\mathbb{H}_0$ via translation, and we shall also employ the identification of $\mathbb{H}_\beta$ with the upper half-plane $\mathbb{H}$ via $\mathbb{H} \ni \zeta \leftrightarrow \beta - i\zeta \in \mathbb{H}_\beta$.

- The Euclidean norm of $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ is denoted as $|x| = \Big(\sum\limits_{j=1}^{N} x_j^2\Big)^{\frac{1}{2}}$. Moreover, let $\langle x \rangle = \big(1 + |x|^2\big)^{\frac{1}{2}}$ be the standard regularized distance in $\mathbb{R}^N$. The inner product in $\mathbb{R}^N$ is denoted as $\langle x, \xi \rangle = x\xi = \sum\limits_{j=1}^{N} x_j \xi_j$.

Multi-index notation

We employ the standard multi-index notation.

For multi-indices $\alpha = (\alpha_1, \dots, \alpha_N), \beta = (\beta_1, \dots, \beta_N) \in \mathbb{N}_0^N$ we denote

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_N}{\beta_N} \qquad \alpha! = \prod_{j=1}^{N} \alpha_j! \qquad |\alpha| = \sum_{j=1}^{N} \alpha_j.$$

We write $\alpha \leq \beta$ if the inequality holds componentwise. Moreover, (normalized) partial derivatives with respect to the variables $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ are written as

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N}} \qquad D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha.$$

In case a function $f(x, \lambda)$ depends on the group of complex variables $\lambda \in \mathbb{C}^M$ we also use the notations

$$\partial_\lambda^\beta f = \frac{\partial^{|\beta|}}{\partial_{\lambda_1}^{\beta_1} \dots \partial_{\lambda_M}^{\beta_M}} f \qquad D_\lambda^\beta f = (-i)^{|\beta|} \partial_\lambda^\beta f,$$

$$\partial_{\overline{\lambda}}^\beta f = \frac{\partial^{|\beta|}}{\partial_{\overline{\lambda}_1}^{\beta_1} \dots \partial_{\overline{\lambda}_M}^{\beta_M}} f \qquad D_{\overline{\lambda}}^\beta f = (-i)^{|\beta|} \partial_{\overline{\lambda}}^\beta f.$$

For $z = (z_1, \dots, z_N) \in \mathbb{C}^N$ and $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ we write $z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_N^{\alpha_N}$.

Functional analysis and basic function spaces

Unless stated explicitly otherwise, the spaces in this work are always assumed to be complex. For topological vector spaces E and F we denote the space of continuous linear operators $E \longrightarrow F$ as $\mathcal{L}(E, F)$. Moreover, the topological dual of E is denoted as E'. We write $E \otimes F$ for the algebraical tensor product of E and F. The projective and injective topology on $E \otimes F$ is indicated by the subscripts $E \otimes_\pi F$ and $E \otimes_\varepsilon F$, respectively, while $E \widehat{\otimes}_\pi F$ and $E \widehat{\otimes}_\varepsilon F$ denote the corresponding completions. We employ the notation $\langle \cdot, \cdot \rangle_{E,F}$, or just $\langle \cdot, \cdot \rangle$, when we deal with a duality $E \times F \longrightarrow \mathbb{C}$. The inner product in a Hilbert space E is also denoted as $\langle \cdot, \cdot \rangle_E$, or simply as $\langle \cdot, \cdot \rangle$.

Moreover, we have the following spaces of E-valued functions on M (where M and E are appropriate):

$L^p(M, E)$	measurable functions u with $\int\limits_M \|u(x)\|_E^p \, dx < \infty$ (with respect to Lebesgue measure, $1 \leq p < \infty$),
$C(M, E)$	continuous functions,
$\mathcal{A}(M, E)$	analytic functions,
$C^k(M, E)$	k-times continuously differentiable functions,
$C^\infty(M, E)$	smooth functions,
$C_0^\infty(M, E)$	smooth functions with compact support,
$C_b^\infty(M, E)$	smooth functions with bounded derivatives,
$\mathcal{S}(M, E)$	rapidly decreasing functions,
$\mathcal{D}'(M, E) = \mathcal{L}(C_0^\infty(M), E)$	distributions,
$\mathcal{E}'(M, E) = \mathcal{L}(C^\infty(M), E)$	distributions with compact support,
$\mathcal{S}'(M, E) = \mathcal{L}(\mathcal{S}(M), E)$	tempered distributions.

If $E = \mathbb{C}$ we drop it from the notation.

The following spaces of smooth, bounded functions naturally occur in Mellin pseudodifferential calculus:

Let E be a Fréchet space. Define

$$C_B^\infty((\mathbb{R}_+)^q, E) := \{u \in C^\infty((\mathbb{R}_+)^q, E);\ \big((-r\partial_r)^k u\big)((\mathbb{R}_+)^q) \subseteq E \text{ is bounded for all } k \in \mathbb{N}_0^q\},$$

endowed with the Fréchet topology of uniform convergence of $(-r\partial_r)^k u$ on $(\mathbb{R}_+)^q$ for every $k \in \mathbb{N}_0^q$. Here we use the notation $(-r\partial_r)^k := (-r_1\partial_{r_1})^{k_1} \ldots (-r_q\partial_{r_q})^{k_q}$ for $r = (r_1, \ldots, r_q) \in (\mathbb{R}_+)^q$ and $k = (k_1, \ldots, k_q) \in \mathbb{N}_0^q$.

Moreover, let $C_B^\infty((\overline{\mathbb{R}}_+)^q, E) := C_B^\infty((\mathbb{R}_+)^q, E) \cap C_b^\infty((\mathbb{R}_+)^q, E)$ be the subspace of all functions that extend smoothly to $(\overline{\mathbb{R}}_+)^q$.

Hilbert triples and (formal) adjoint operators

A triple $\{E_0, E, E_1\}$ of Hilbert spaces E_0, E, E_1 is called a *Hilbert triple*, if the following conditions are fulfilled:

a) There exists a Hausdorff topological vector space X such that E_0, E and E_1 are embedded in X.

b) $E_0 \cap E \cap E_1$ is dense in E_0, E and E_1.

c) The inner product on E induces a non-degenerate sesquilinear pairing $\langle \cdot, \cdot \rangle : E_0 \times E_1 \to \mathbb{C}$, that provides antilinear isomorphisms $E_0' \cong E_1$ and $E_1' \cong E_0$.

Let $\{E_0, E, E_1\}$ and $\{\tilde{E}_0, \tilde{E}, \tilde{E}_1\}$ be Hilbert triples. Then, for each $A \in \mathcal{L}(E_0, \tilde{E}_0)$, there is a unique operator $A^* \in \mathcal{L}(\tilde{E}_1, E_1)$ such that $\langle Ae_0, \tilde{e}_1 \rangle_{\tilde{E}} = \langle e_0, A^*\tilde{e}_1 \rangle_E$ for all $e_0 \in E_0$ and $\tilde{e}_1 \in \tilde{E}_1$. A^* is called the *(formal) adjoint operator* of A.

The mapping $A \mapsto A^*$ provides an antilinear isomorphism $\mathcal{L}(E_0, \tilde{E}_0) \to \mathcal{L}(\tilde{E}_1, E_1)$.

Tempered distributions and the Fourier transform

Let E be a Hilbert space. Partial derivatives of a distribution $u \in \mathcal{S}'(\mathbb{R}^n, E)$ are defined as $\langle \partial_x^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial_x^\alpha \varphi \rangle$, while multiplication with a function ψ of tempered growth is given as $\langle \psi u, \varphi \rangle = \langle u, \psi \varphi \rangle$. A distribution $u \in \mathcal{S}'(\mathbb{R}^n, E)$ is called regular, if u is a Bochner measurable function, and there exists $N \in \mathbb{N}_0$ with $\int_{\mathbb{R}^n} \langle x \rangle^{-N} \|u(x)\|_E \, dx < \infty$. Note that we identify regular distributions with their densities. In this sense we in particular have $L^p(\mathbb{R}^n, E) \hookrightarrow \mathcal{S}'(\mathbb{R}^n, E)$.

We employ the normalized Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$, i. e.

$$(\mathcal{F}u)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} u(x) \, dx,$$

$$(\mathcal{F}^{-1}u)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} u(\xi) \, d\xi,$$

for $u \in \mathcal{S}(\mathbb{R}^n)$. For Fréchet spaces E the Fourier transform extends to an isomorphism $\mathcal{S}(\mathbb{R}^n, E) \longrightarrow \mathcal{S}(\mathbb{R}^n, E)$ via $\mathcal{F} = \mathcal{F} \widehat{\otimes}_\pi \mathrm{id}_E$, noting that $\mathcal{S}(\mathbb{R}^n, E) \cong \mathcal{S}(\mathbb{R}^n) \widehat{\otimes}_\pi E$. If E is a Hilbert space we have $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n, E) \longrightarrow \mathcal{S}'(\mathbb{R}^n, E)$ via $\langle \mathcal{F}u, \varphi \rangle = \langle u, \mathcal{F}\varphi \rangle$.

Elementary symbol spaces

For a Fréchet space E we denote the space of symbols of order $\mu \in \mathbb{R}$ with values in E as $S^\mu(\mathbb{R}^n, E)$, i. e. a function $a \in C^\infty(\mathbb{R}^n, E)$ belongs to $S^\mu(\mathbb{R}^n, E)$ if and only if $\partial_\xi^\alpha a(\xi) = O(\langle \xi \rangle^{\mu - |\alpha|})$ as $|\xi| \to \infty$, for all $\alpha \in \mathbb{N}_0^n$.

Similarly, the space $S^{\mu,\varrho}(\mathbb{R}^n \times \mathbb{R}^n, E)$ of symbols of order (μ, ϱ) is the space of all $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, E)$ such that $\partial_x^\alpha \partial_\xi^\beta a(x, \xi) = O(\langle x \rangle^{\varrho - |\alpha|} \langle \xi \rangle^{\mu - |\beta|})$ as $|x, \xi| \to \infty$, for all $\alpha, \beta \in \mathbb{N}_0^n$.

These spaces are Fréchet spaces in a canonical way. If any one of the orders equals $-\infty$ in the notations involved, we mean the corresponding intersection over all spaces with finite orders.

Preliminaries on function spaces and the Mellin transform

Let E be a Fréchet space.

- For $\gamma \in \mathbb{R}$ let

  $$\mathcal{T}_\gamma(\mathbb{R}_+, E) := \{u \in C^\infty(\mathbb{R}_+, E);\ \big(r^{\frac{1}{2}-\gamma}\langle\log(r)\rangle^m(-r\partial_r)^k u\big)(\mathbb{R}_+) \subseteq E \text{ is bounded for all } k, m \in \mathbb{N}_0\}.$$

  This space is endowed with the Fréchet topology of uniform convergence of $r^{\frac{1}{2}-\gamma}\langle\log(r)\rangle^m(-r\partial_r)^k u$ on $\mathbb{R}_+$ for every $k, m \in \mathbb{N}_0$.

  Note that for every $\delta \in \mathbb{R}$ the operator of multiplication with the function r^δ induces a topological isomorphism $r^\delta : \mathcal{T}_\gamma(\mathbb{R}_+, E) \longrightarrow \mathcal{T}_{\gamma+\delta}(\mathbb{R}_+, E)$.

- For $\gamma \in \mathbb{R}$ define the operator

  $$S_\gamma : u(r) \longmapsto e^{(\gamma-\frac{1}{2})t}u(e^{-t}) \tag{1.1.i}$$

  and its inverse

  $$S_\gamma^{-1} : u(t) \longmapsto r^{\gamma-\frac{1}{2}}u(-\log r). \tag{1.1.ii}$$

  The operator (1.1.i) is well-defined as a topological isomorphism

  $$S_\gamma : \mathcal{D}'(\mathbb{R}_+) \to \mathcal{D}'(\mathbb{R})$$

  and restricts to topological isomorphisms on various subspaces, e. g.

  $$S_\gamma : \begin{cases} C_0^\infty(\mathbb{R}_+) \longrightarrow C_0^\infty(\mathbb{R}) \\ \mathcal{T}_\gamma(\mathbb{R}_+) \longrightarrow \mathcal{S}(\mathbb{R}). \end{cases}$$

  This shows, in particular, that $\mathcal{T}_\gamma(\mathbb{R}_+)$ is a nuclear Fréchet space with $C_0^\infty(\mathbb{R}_+)$ as a dense subspace, and we have a canonical isomorphism $\mathcal{T}_\gamma(\mathbb{R}_+, E) \cong \mathcal{T}_\gamma(\mathbb{R}_+)\widehat{\otimes}_\pi E$.

- The (weighted) Mellin transform (defined on $C_0^\infty(\mathbb{R}_+)$)

  $$(\mathcal{M}_\gamma u)(z) := \int_{\mathbb{R}_+} r^z u(r)\,\frac{dr}{r}$$

  for $z \in \Gamma_{\frac{1}{2}-\gamma}$ with its inverse

  $$(\mathcal{M}_\gamma^{-1} u)(r) := \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{2}-\gamma}} r^{-z}u(z)\,dz$$

extends via $\mathcal{M}_\gamma = \mathcal{M}_\gamma \widehat{\otimes}_\pi \mathrm{id}_E$ to a topological isomorphism

$$\mathcal{M}_\gamma : \mathcal{T}_\gamma(\mathbb{R}_+, E) \longrightarrow \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma}, E).$$

For $u \in \mathcal{T}_\gamma(\mathbb{R}_+, E)$ we have

$$\mathcal{M}_\gamma((-r\partial_r)u)(z) = z\mathcal{M}_\gamma(u)(z), \qquad \mathcal{M}_\gamma((\log r)u)(z) = D_\tau \mathcal{M}_\gamma(u)(z),$$
$$\mathcal{M}_{\gamma+\delta}(r^\delta u)(z) = \mathcal{M}_\gamma(u)(z+\delta). \tag{1.1.iii}$$

- For $u \in C_0^\infty(\mathbb{R}_+, E)$ the Mellin transform $\mathcal{M}u$ extends to an entire function such that $\mathcal{M}u|_{\Gamma_{\frac{1}{2}-\gamma}} = \mathcal{M}_\gamma u$, and the mapping

$$\mathbb{R} \ni \gamma \mapsto \mathcal{M}_\gamma u \in \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma}, E)$$

is a C^∞-function taking values in the rapidly decreasing functions.

A Paley–Wiener type theorem

Let E be a Fréchet space.

- For $t_0 \in \mathbb{R}$ let

$$\mathcal{S}_0((t_0,\infty), E) := \{u \in \mathcal{S}(\mathbb{R}, E);\ \mathrm{supp}(u) \subseteq [t_0, \infty)\},$$
$$\mathcal{S}_0((-\infty,t_0), E) := \{u \in \mathcal{S}(\mathbb{R}, E);\ \mathrm{supp}(u) \subseteq (-\infty, t_0]\},$$

as well as

$$\mathcal{A}(\mathbb{H}_{(-)}, E; t_0) := \{f \in C^\infty(\mathbb{H}_{(-)}, E) \cap \mathcal{A}(\mathring{\mathbb{H}}_{(-)}, E);$$
$$[\mathbb{H}_{(-)} \ni z \mapsto e^{it_0 z} f(z) \in E] \in \mathcal{S}(\mathbb{H}_{(-)}, E)\}$$

with either the upper half-plane $\mathbb{H}$ or the lower half-plane $\mathbb{H}_- := \{z \in \mathbb{C};\ \mathrm{Im}(z) \le 0\}$ in $\mathbb{C}$ involved; these spaces are Fréchet with the projective topology with respect to the mappings

$$\mathcal{A}(\mathbb{H}_{(-)}, E; t_0) \ni f \longmapsto e^{it_0 z} f(z) \in \mathcal{S}(\mathbb{H}_{(-)}, E).$$

Then the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}, E) \xrightarrow{\cong} \mathcal{S}(\mathbb{R}, E)$ restricts to topological isomorphisms

$$\mathcal{F} : \begin{cases} \mathcal{S}_0((-\infty, t_0), E) \xrightarrow{\cong} \mathcal{A}(\mathbb{H}, E; t_0) \\ \mathcal{S}_0((t_0, \infty), E) \xrightarrow{\cong} \mathcal{A}(\mathbb{H}_-, E; t_0) \end{cases}$$

for every $t_0 \in \mathbb{R}$.

- For $r_0 \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}$ let
$$\mathcal{T}_{\gamma,0}((0,r_0),E) := \{u \in \mathcal{T}_\gamma(\mathbb{R}_+,E);\ \mathrm{supp}(u) \subseteq (0,r_0]\},$$
and let
$$\mathcal{A}(\mathbb{H}_{\frac{1}{2}-\gamma},E;r_0) := \{f \in C^\infty(\mathbb{H}_{\frac{1}{2}-\gamma},E) \cap \mathcal{A}(\mathring{\mathbb{H}}_{\frac{1}{2}-\gamma},E);\\ [\mathbb{H}_{\frac{1}{2}-\gamma} \ni z \mapsto r_0^{-z} f(z) \in E] \in \mathcal{S}(\mathbb{H}_{\frac{1}{2}-\gamma},E)\}.$$
The latter is a Fréchet space with the projective topology with respect to the mapping
$$\mathcal{A}(\mathbb{H}_{\frac{1}{2}-\gamma},E;r_0) \ni f \longmapsto r_0^{-z} f(z) \in \mathcal{S}(\mathbb{H}_{\frac{1}{2}-\gamma},E).$$
Then the weighted Mellin transform $\mathcal{M}_\gamma : \mathcal{T}_\gamma(\mathbb{R}_+,E) \xrightarrow{\cong} \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma},E)$ restricts to a topological isomorphism
$$\mathcal{M}_\gamma : \mathcal{T}_{\gamma,0}((0,r_0),E) \xrightarrow{\cong} \mathcal{A}(\mathbb{H}_{\frac{1}{2}-\gamma},E;r_0)$$
for every $r_0 \in \mathbb{R}_+$ and every $\gamma \in \mathbb{R}$.

The Mellin transform in distributions

Let E be a Hilbert space.

- For $\gamma \in \mathbb{R}$ the space $\mathcal{T}'_\gamma(\mathbb{R}_+,E)$ consists of all continuous linear functionals $\mathcal{T}_{-\gamma}(\mathbb{R}_+) \to E$. Consequently, we have $\mathcal{T}'_\gamma(\mathbb{R}_+,E) \subseteq \mathcal{D}'(\mathbb{R}_+,E)$ in view of the density of $C_0^\infty(\mathbb{R}_+)$ in $\mathcal{T}_{-\gamma}(\mathbb{R}_+)$.
- A distribution $u \in \mathcal{T}'_\gamma(\mathbb{R}_+,E)$ is called *regular* if
$$\langle u,\varphi\rangle = \int_0^\infty \tilde{u}(r)\varphi(r)\,dr, \qquad \varphi \in \mathcal{T}_{-\gamma}(\mathbb{R}_+),$$
for some Bochner measurable function $\tilde{u}$ such that $r^{-(\frac{1}{2}+\gamma)}\langle \log r\rangle^{-N}\tilde{u}(r) \in L^1(\mathbb{R}_+,E)$ for some $N \in \mathbb{N}_0$. In particular, we have $\mathcal{T}_\gamma(\mathbb{R}_+,E) \subseteq \mathcal{T}'_\gamma(\mathbb{R}_+,E)$, and more generally even $L^{2,\gamma}(\mathbb{R}_+,E) := r^\gamma L^2(\mathbb{R}_+,E) \subseteq \mathcal{T}'_\gamma(\mathbb{R}_+,E)$ as regular distributions.
- For every $\delta \in \mathbb{R}$ the product with functions $\psi \in C^\infty(\mathbb{R}_+)$ such that $r^{-\delta}(-r\partial_r)^\nu\psi(r)$ is majorized by some power of $\langle \log r\rangle$, uniformly on $\mathbb{R}_+$ for every $\nu \in \mathbb{N}_0$, provides an operator $\mathcal{T}'_\gamma(\mathbb{R}_+,E) \to \mathcal{T}'_{\gamma+\delta}(\mathbb{R}_+,E)$. Recall that $\langle \psi u,\varphi\rangle = \langle u,\psi\varphi\rangle$ for $\varphi \in \mathcal{T}_{-(\gamma+\delta)}(\mathbb{R}_+)$.

- The totally characteristic derivative $(-r\partial_r) : \mathcal{D}'(\mathbb{R}_+, E) \longrightarrow \mathcal{D}'(\mathbb{R}_+, E)$ restricts to $\mathcal{T}'_\gamma(\mathbb{R}_+, E)$, i. e. $(-r\partial_r)(\mathcal{T}'_\gamma(\mathbb{R}_+, E)) \subseteq \mathcal{T}'_\gamma(\mathbb{R}_+, E)$.

- The isomorphism $S_\gamma : \mathcal{D}'(\mathbb{R}_+, E) \longrightarrow \mathcal{D}'(\mathbb{R}_+, E)$ from (1.1.i) restricts to an isomorphism $S_\gamma : \mathcal{T}'_\gamma(\mathbb{R}_+, E) \longrightarrow \mathcal{S}'(\mathbb{R}, E)$. Note that we may write $\langle S_\gamma u, \varphi\rangle = \langle u, S_{-\gamma}^{-1}\varphi\rangle$ for $\varphi \in \mathcal{S}(\mathbb{R})$.

- The weighted Mellin transform $\mathcal{M}_\gamma$ extends to $\mathcal{T}'_\gamma(\mathbb{R}_+, E)$ by means of the identity

$$\big(\mathcal{M}_\gamma u\big)\big(\frac{1}{2} - \gamma + i\tau\big) = \big(\sqrt{2\pi}\mathcal{F}S_\gamma u\big)(\tau), \tag{1.1.iv}$$

which provides an isomorphism $\mathcal{M}_\gamma : \mathcal{T}'_\gamma(\mathbb{R}_+, E) \longrightarrow \mathcal{S}'(\Gamma_{\frac{1}{2}-\gamma}, E)$. It restricts to an isomorphism

$$\mathcal{M}_\gamma : L^{2,\gamma}(\mathbb{R}_+, E) \longrightarrow L^2(\Gamma_{\frac{1}{2}-\gamma}, E), \tag{1.1.v}$$

and we have *Parseval's identity*

$$\langle u, v\rangle_{L^{2,\gamma}(\mathbb{R}_+,E)} = \frac{1}{2\pi}\langle \mathcal{M}_\gamma u, \mathcal{M}_\gamma v\rangle_{L^2(\Gamma_{\frac{1}{2}-\gamma},E)}. \tag{1.1.vi}$$

The relations in (1.1.iii) hold in the distributional sense.

Global analysis

- In this work, we consider C^∞-manifolds X. TX denotes the tangent bundle over X, while T^*X is the cotangent bundle. Let $\mathrm{Vect}(X)$ be the set of complex vector bundles over X. The pull-back of a bundle F with respect to a smooth mapping g is denoted as g^*F. This is mainly employed with the projection $\pi : T^*X \setminus 0 \longrightarrow X$, where 0 is the zero section in T^*X. Let $\mathrm{Hom}(E, F)$ be the bundle of homomorphisms acting in the fibres of the bundles E and F. E^* denotes the dual bundle to E, and for vector bundles $E \in \mathrm{Vect}(X)$ and $F \in \mathrm{Vect}(Y)$ the external tensor product is denoted as $E \boxtimes F \in \mathrm{Vect}(X \times Y)$.

- For a vector bundle E let $C^\infty(X, E)$ denote the space of smooth sections in E, and $C_0^\infty(X, E)$ is the space of smooth sections with compact support. Let $\mathcal{D}'(X, E)$ denote the distributional sections in the bundle E. With the density bundle Γ^1 this space is globally given as $\mathcal{D}'(X, E) = C_0^\infty(X, \Gamma^1 \otimes E)'$. Any choice of a smooth positive section in the density bundle provides an isomorphism $\Gamma^1 \cong X \times \mathbb{C}$, and consequently $\mathcal{D}'(X, E) \cong C_0^\infty(X, E)'$. Note that a Riemannian metric gives rise to a canonical positive section.

- We will be concerned mainly with closed manifolds X, i. e. X is compact and $\partial X = \emptyset$. Then we have invariantly the space $L^2(X, E)$ of measurable absolutely square integrable sections. Any choice of a Riemannian metric and a Hermitean inner product on E induces a canonical scalar product on $L^2(X, E)$.

  More generally, the Sobolev spaces $H^s(X, E)$ of distributional sections of smoothness $s \in \mathbb{R}$ are well-defined, where in particular $H^0(X, E) = L^2(X, E)$. These are hilbertizable spaces, but the choice of an inner product is non-canonical. $H^s(X, E)$ and $H^{-s}(X, E)$ are dual to each other via the sesquilinear pairing induced by the $L^2(X, E)$-inner product. By the Sobolev embedding theorem we have $C^\infty(X, E) \cong \text{proj-lim}_{s\to\infty} H^s(X, E)$, and hence we also have an identification $\mathcal{D}'(X, E) \cong \text{ind-lim}_{s\to-\infty} H^s(X, E)$.

- Let X be closed, and let $A : C^\infty(X, E) \longrightarrow \mathcal{D}'(X, F)$ be continuous. Then the Schwartz kernel of A belongs to $C^\infty(X \times X, F \boxtimes E^*)$ if and only if A extends to a continuous operator $A : H^s(X, E) \longrightarrow H^t(X, F)$ for all $s, t \in \mathbb{R}$.

1.2 Finitely meromorphic Fredholm families in Ψ-algebras

1.2.1 Remark. Ψ- and Ψ*-algebras were introduced by Gramsch [21]. These are certain topological Fréchet-algebras which share many important properties of Banach- and C^*-algebras. We include in this section some results about the inversion of meromorphic Fredholm families taking values in Ψ-algebras which are needed in this exposition (see also Gohberg and Sigal [19], Gramsch [20], and Gramsch and Kaballo [22]).

1.2.2 Definition. Let Ψ be a subalgebra of the unital Banach-algebra $\mathcal{B}$. Then Ψ is called a Ψ-algebra in $\mathcal{B}$ if

i) Ψ is a locally convex Fréchet space with respect to the topology $\tau(\Psi)$ which is finer than the induced topology of $\mathcal{B}$.

ii) $1_\Psi = 1_\mathcal{B}$.

iii) Ψ is “spectrally invariant” in $\mathcal{B}$, i. e., for the groups of invertible elements we have $\mathcal{B}^{-1} \cap \Psi = \Psi^{-1}$.

If $\mathcal{B}$ is a C^*-algebra and Ψ a symmetric Ψ-algebra in $\mathcal{B}$, then Ψ is called a Ψ*-algebra in $\mathcal{B}$.

1.2.3 Remark. By the left-regular representation of $\mathcal{B}$ in $\mathcal{L}(\mathcal{B})$ we may assume $\mathcal{B} = \mathcal{L}(X)$ with a Banach space X. In the case of Ψ^*-algebras we may assume $\mathcal{B} = \mathcal{L}(H)$ with a Hilbert space H due to the Gelfand-Neumark-Segal theorem.

The multiplication in a Ψ-algebra is jointly continuous with respect to $\tau(\Psi)$ which follows from the closed graph theorem. Note that Ψ^{-1} is open and consequently the inversion $\cdot^{-1} : \Psi^{-1} \to \Psi^{-1}$ is continuous, since $(\Psi, \tau(\Psi))$ is Fréchet. Moreover, also the $*$-operation is continuous in a Ψ^*-algebra by the closed graph theorem. Note furthermore that a Ψ-algebra Ψ is invariant with respect to the (one-dimensional) holomorphic functional calculus.

1.2.4 Definition. Let X be a Banach space and Ψ a Ψ-algebra in $\mathcal{L}(X)$. Let $\Omega \subseteq \mathbb{C}$ be an open set and $D \subseteq \Omega$ a discrete subset. A function $T \in \mathcal{A}(\Omega \setminus D, \Psi)$ is called a *finitely meromorphic Fredholm family* in Ω if

i) T takes values in the Fredholm operators $\Phi(X)$.

ii) For $p \in D$ there exists a neighbourhood $U(p) \subseteq \Omega$ such that T can be written

$$T(z) = \sum_{k=-N}^{-1} F_k (z-p)^k + T_0(z), \qquad z \in U(p) \setminus \{p\},$$

with finite-dimensional operators $F_k \in \mathcal{F}(X)$ and a holomorphic function $T_0 \in \mathcal{A}(U(p), \mathcal{L}(X))$ such that $T_0(U(p)) \subseteq \Phi(X)$.

1.2.5 Remark. Note that we allow $D = \emptyset$, i. e., the case of holomorphic Fredholm families is contained in this definition. By Cauchy's integral formulas for the Laurent coefficients we have

$$F_k = \frac{1}{2\pi i} \int_{\partial U_\delta(p)} \frac{T(\zeta)}{(\zeta - p)^{k+1}}\, d\zeta \qquad \text{for } k = -N, \dots, -1.$$

It follows $F_k \in \Psi$, and consequently $T_0 \in \mathcal{A}(U(p) \setminus \{p\}, \Psi)$. But from Cauchy's integral formula

$$T_0(z) = \frac{1}{2\pi i} \int_{\partial U_\delta(p)} \frac{T_0(\zeta)}{\zeta - z}\, d\zeta, \qquad z \in U_\delta(p),$$

we obtain $T_0 \in \mathcal{A}(U(p), \Psi)$.

Finally, we want to point out that $T_0(p)$ is required to be Fredholm which is necessary for the validity of the theorem about the inversion of meromorphic Fredholm families.

1.2.6 Theorem. *Let Ψ be a Ψ-algebra in $\mathcal{L}(X)$ with a Banach space X. Let $\Omega \subseteq \mathbb{C}$ be a connected domain and $T \in \mathcal{A}(\Omega \setminus D, \Psi)$ a finitely meromorphic Fredholm family in Ω. Let $z^* \in \Omega \setminus D$ such that $T(z^*)$ is invertible in $\mathcal{L}(X)$.*

Then there exists a discrete set $\tilde{D} \subseteq \Omega$, $D \subseteq \tilde{D}$, such that $T(z)$ is invertible in $\mathcal{L}(X)$ for $z \in \Omega \setminus \tilde{D}$. Moreover, we have $T^{-1} \in \mathcal{A}(\Omega \setminus \tilde{D}, \Psi)$ and T^{-1} extends to a finitely meromorphic Fredholm family in the sense of Definition 1.2.4.

For the proof of this theorem we need some preparations. First recall the following theorem on inversion of holomorphic Fredholm families.

1.2.7 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a connected domain and $T \in \mathcal{A}(\Omega, \mathcal{L}(X))$ taking values in $\Phi(X)$. Let $z^* \in \Omega$ such that $T(z^*)$ is invertible in $\mathcal{L}(X)$.*

Then there exists a discrete set $D \subseteq \Omega$ such that $T(z)$ is invertible in $\mathcal{L}(X)$ for $z \in \Omega \setminus D$. Moreover, we have $T^{-1} \in \mathcal{A}(\Omega \setminus D, \mathcal{L}(X))$ and for $p \in D$ we can write in a neighbourhood $U(p)$:

$$T(z)^{-1} = \sum_{k=-N}^{-1} F_k (z-p)^k + T_0(z), \qquad z \in U(p) \setminus \{p\},$$

with finite-dimensional operators $F_k \in \mathcal{F}(X)$ and $T_0 \in \mathcal{A}(U(p), \mathcal{L}(X))$, $T_0(U(p)) \subseteq \Phi(X)$.

Proof. Let $z_0 \in \Omega$ such that $T(z_0)$ is not invertible in $\mathcal{L}(X)$. Since Ω is connected and $T(z^*) \in \mathcal{L}(X)^{-1}$ we conclude $T(\Omega) \subseteq \Phi_0(X)$ where $\Phi_0(X)$ denotes the Fredholm operators of index zero. Recall that the index is locally constant on $\Phi(X)$. Thus we have $0 < \dim N(T(z_0)) = \operatorname{codim} R(T(z_0))$. By Kato's lemma the range of Fredholm operators is closed and thus we can find a direct decomposition $X = N(T(z_0)) \oplus_{\text{top}} X_1 = X_2 \oplus_{\text{top}} R(T(z_0))$. By choosing an isomorphism $N(T(z_0)) \cong X_2$ we find a finite-dimensional operator $F \in \mathcal{F}(X)$ such that $T(z_0) - F \in \mathcal{L}(X)^{-1}$ and consequently $T(z) - F \in \mathcal{L}(X)^{-1}$ for $|z - z_0| < \varepsilon$ with $\varepsilon > 0$ sufficiently small. With the projections $P = P^2 : X \to X_2$, $N(P) = R(T(z_0))$, and $Q = I - P$ we may write for $|z - z_0| < \varepsilon$:

$$\begin{aligned}
T(z) &= (I + PF(z))(T(z) - F) \quad \text{with} \\
F(z) &:= F(T(z) - F)^{-1} \quad \text{and thus} \\
T(z) &= (I + PF(z))(T(z) - F) \\
&= \underbrace{(I + PF(z)Q)}_{=:\,C(z)}(I + PF(z)P)\underbrace{(T(z) - F)}_{=:\,B(z)}.
\end{aligned}$$

Note that $B(z)$ as well as $C(z)$ are invertible for $|z - z_0| < \varepsilon$; we can write as a triangular matrix

$$C(z) = \begin{pmatrix} Q & 0 \\ PF(z)Q & P \end{pmatrix} \quad \text{with “invertible diagonal”.}$$

Note also that $B(z)$, $C(z)$ and $F(z)$ are holomorphic for $|z-z_0|<\varepsilon$. Thus we have for $|z-z_0|<\varepsilon$

$$T(z) = C(z)\cdot\begin{pmatrix} Q & 0 \\ 0 & P(I+F(z))P \end{pmatrix}\cdot B(z).$$

This implies: $T(\tilde z)$ is invertible for $|\tilde z - z_0| < \varepsilon$ if and only if $P(I+F(\tilde z))P$ is invertible in $\mathcal{L}(X_2)$. Then we have with the inverse $M(\tilde z) = [P(I+F(\tilde z))P]^{-1} \in \mathcal{L}(X_2)$:

$$T(\tilde z)^{-1} = B(\tilde z)^{-1}\cdot\begin{pmatrix} Q & 0 \\ 0 & PM(\tilde z)P \end{pmatrix}\cdot C(\tilde z)^{-1}. \tag{1}$$

In the case $\dim X < \infty$ the assertion of the theorem is obvious due to Cramer's rule for the inversion of a matrix and due to the scalar analysis of meromorphic functions in connected domains (applied to the determinant of component-functions of a matrix-valued function).

Thus it remains to prove the existence of $\tilde z \in \Omega$, $|\tilde z - z_0| < \varepsilon$, such that $T(\tilde z)$ is invertible in $\mathcal{L}(X)$. Employing the finite-dimensional result with the function $P(I+F(z))P$ and inverse $M(z)$ we then see that $T(z)$ is invertible for $0<|z-z_0|<\delta<\varepsilon$ and by (1) we have that T^{-1} is meromorphic in z_0 and the Laurent-coefficients of the principal part are finite-dimensional operators. Let $D := \{z_0 \in \Omega;\ T(z_0) \text{ is not invertible}\}$. D is a closed subset in Ω. We will show $\overset{\circ}{D} = \emptyset$, i. e., $D = \partial D$. Assume that there exists a point $z_1 \in \overset{\circ}{D}$. Since Ω is connected, we may choose a path $\gamma : [0,1] \to \Omega$, $\gamma(0) = z_1$, $\gamma(1) = z^*$. Let

$$s := \sup\{t>0;\ T(\gamma(\tau)) \text{ is not invertible in } \mathcal{L}(X) \text{ for } \tau \in [0,t)\}.$$

By assumption we have $0<s<1$ and $\gamma(s) \in \partial D \subsetneq \Omega$. The first part of the proof implies the existence of $0<\delta$ such that $T(z)$ in invertible in $\mathcal{L}(X)$ for $0<|z-\gamma(s)|<\delta$. This leads to a contradiction. Thus we have $D = \partial D$ and by the first part of the proof D is consequently discrete in Ω. This proves that $T^{-1} \in \mathcal{A}(\Omega\setminus D, \mathcal{L}(X))$, and for $p \in D$ we have in a neighbourhood U(p):

$$T(z)^{-1} = \sum_{k=-N}^{-1} F_k (z-p)^k + T_0(z), \quad z \in U(p)\setminus\{p\},$$

with finite-dimensional operators $F_k \in \mathcal{F}(X)$ and $T_0 \in \mathcal{A}(U(p), \mathcal{L}(X))$, $T_0(U(p)\setminus\{p\}) \subseteq \Phi(X)$. It remains to prove $T_0(p) \in \Phi(X)$. We can write for $z \in U(p)\setminus\{p\}$:

$$\begin{aligned} I = T(z)T(z)^{-1} &= T(z)\cdot\Big[\sum_{k=-N}^{-1} F_k(z-p)^k\Big] + T(z)T_0(z) \\ &= T(z)^{-1}T(z) = \Big[\sum_{k=-N}^{-1} F_k(z-p)^k\Big]\cdot T(z) + T_0(z)T(z). \end{aligned}$$

Since the functions $T_0(z)T(z)$ and $T(z)T_0(z)$ extend holomorphically into p the functions $T(z) \cdot \left[\sum_{k=-N}^{-1} F_k(z-p)^k\right]$ and $\left[\sum_{k=-N}^{-1} F_k(z-p)^k\right] \cdot T(z)$ necessarily extend also holomorphically into p. But the latter functions take values in the finite-dimensional operators on $U(p) \setminus \{p\}$, and thus their values in p are compact operators. Hence $T(p)$ inverts $T_0(p)$ modulo compact operators which shows $T_0(p) \in \Phi(X)$. □

1.2.8 Lemma. *Let X be a vector space and $E_1, \ldots, E_N \subseteq X$ be subspaces of finite codimension. Then $E_1 \cap \ldots \cap E_N \subseteq X$ is of finite codimension in X.*

Proof. Consider the mapping $J : X \to \bigoplus_{j=1}^{N} X/E_j$ given by the canonical quotient mappings. J is linear, and we have $N(J) = \bigcap_{j=1}^{N} E_j$. Consequently

$$X/\bigcap_{j=1}^{N} E_j \cong R(J) \subseteq \bigoplus_{j=1}^{N} X/E_j,$$

where $\dim \bigoplus_{j=1}^{N} X/E_j < \infty$ by assumption. This implies $\dim X/\bigcap_{j=1}^{N} E_j < \infty$ which shows $\operatorname{codim} \bigcap_{j=1}^{N} E_j < \infty$. □

1.2.9 Lemma. *Let X be a Banach space and $\Omega \subseteq \mathbb{C}$ be a connected open neighbourhood of $0 \in \mathbb{C}$. Let $A_{-1}, \ldots, A_{-N} \in \mathcal{F}(X)$ be finite-dimensional operators. Let $H \in \mathcal{A}(\Omega, \mathcal{L}(X))$ such that $H(z)u = 0$, $z \in \Omega$, for $u \in K_0 \subseteq X$, where K_0 is a closed subspace of X of finite codimension. Consider the function*

$$F(z) := I + H(z) + \sum_{k=-N}^{-1} A_k z^k, \quad z \in \Omega \setminus \{0\}.$$

Assume that there exists a $z^* \in \Omega \setminus \{0\}$ such that $F(z^*)$ is invertible in $\mathcal{L}(X)$. Then there exists a $\delta > 0$ such that $F(z)$ is invertible for $0 < |z| < \delta$. Moreover, we can write for $0 < |z| < \delta$

$$F(z)^{-1} = \sum_{k=-M}^{-1} F_k z^k + F_0(z)$$

with finite-dimensional operators $F_k \in \mathcal{F}(X)$ and $F_0 \in \mathcal{A}(U_\delta(0), \mathcal{L}(X))$. Furthermore we have $(I + H)(\Omega) \subseteq \Phi(X)$ and $F_0(U_\delta(0)) \subseteq \Phi(X)$.

Proof. We will first prove that $(I+H)(\Omega) \subseteq \Phi_0(X)$, more precisely $H(z) \in \mathcal{F}(X)$ for $z \in \Omega$. Since $K_0 \subseteq N(H(z))$ we have that the canonical mapping $X/K_0 \longrightarrow X/N(H(z))$ is onto. But since $\dim X/K_0 < \infty$ we conclude $\dim X/N(H(z)) < \infty$, i. e., $H(z) \in \mathcal{F}(X)$. Let $K_1 := \bigcap_{k=-N}^{-1} N(A_k)$ and $K := K_0 \cap K_1$. Then K is a closed subspace of X and by Lemma 1.2.8 we have $\operatorname{codim} K < \infty$. Let $L \subseteq X$ be a finite-dimensional subspace such that $X = K \oplus_{\text{top}} L$ and $P = P^2 \in \mathcal{L}(X)$, $R(P) = L$, $N(P) = K$. Consequently we may write for $z \in \Omega \setminus \{0\}$, $Q = I - P$:

$$F(z) = \begin{pmatrix} Q & QF(z)P \\ 0 & PF(z)P \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & PF(z)P \end{pmatrix} \cdot \begin{pmatrix} Q & QF(z)P \\ 0 & P \end{pmatrix}.$$

Since $C(z) := \begin{pmatrix} Q & QF(z)P \\ 0 & P \end{pmatrix}$ is invertible for all $z \in \Omega \setminus \{0\}$ we see that $F(\tilde{z})$ is invertible in $\mathcal{L}(X)$ if and only if $PF(\tilde{z})P$ is invertible in $\mathcal{L}(L)$ for $\tilde{z} \in \Omega \setminus \{0\}$ where $\dim L < \infty$. With the inverse $M(\tilde{z}) = (PF(\tilde{z})P)^{-1} \in \mathcal{L}(L)$ we then may write

$$F(\tilde{z})^{-1} = C(\tilde{z})^{-1} \cdot \begin{pmatrix} Q & 0 \\ 0 & PM(\tilde{z})P \end{pmatrix} = \begin{pmatrix} Q & -QF(\tilde{z})P \\ 0 & P \end{pmatrix} \cdot \begin{pmatrix} Q & 0 \\ 0 & PM(\tilde{z})P \end{pmatrix}. \quad (1)$$

For $z \in \Omega \setminus \{0\}$ we have $C(z)^{-1} = I - QH(z)P - \sum_{k=-N}^{-1} QA_kPz^k$, i. e., $C(z)^{-1}$ is meromorphic in 0 and the Laurent coefficients of the principal part are finite-dimensional operators. Since $\dim L < \infty$ the function $PF(z)P$ can be regarded as a holomorphic matrix-valued function on $z \in \Omega \setminus \{0\}$ which is meromorphic in 0. The determinant of the component functions is consequently a holomorphic scalar function which is meromorphic in 0. Since $F(z^*)$ is invertible in $\mathcal{L}(X)$, we have that $PF(z^*)P$ is invertible in $\mathcal{L}(L)$, i. e., the determinant of the component functions is a meromorphic scalar function in Ω which is not identically zero. From Cramer's rule for the inversion of a matrix we now get that the function $M(z) = (PF(z)P)^{-1}$ is a meromorphic $\mathcal{L}(L)$-valued function. Note that Ω is assumed to be connected and thus the scalar meromorphic functions in Ω form a field. In particular, we see that there exists a $\delta > 0$ such that $\begin{pmatrix} Q & 0 \\ 0 & PF(z)P \end{pmatrix}$ is invertible in $\mathcal{L}(X)$ for $0 < |z| < \delta$ (and consequently also $F(z)$), and we may write for $0 < |z| < \delta$

$$\begin{pmatrix} Q & 0 \\ 0 & PF(z)P \end{pmatrix}^{-1} = \begin{pmatrix} Q & 0 \\ 0 & PM(z)P \end{pmatrix} = Z_0(z) + \sum_{k=-\tilde{M}}^{-1} Z_k z^k$$

with finite-dimensional operators $Z_k \in \mathcal{F}(X)$ and $Z_0 \in \mathcal{A}(U_\delta(0), \mathcal{L}(X))$. From the identity (1) we now get that $F(z)^{-1}$, $z \in U_\delta(0) \setminus \{0\}$, is a product of two meromorphic functions (meromorphic in 0), whose Laurent coefficients of the principal parts are finite-dimensional operators. This proves that for $0 < |z| < \delta$ we may write

$F(z)^{-1} = \sum_{k=-M}^{-1} F_k z^k + F_0(z)$ with finite-dimensional operators $F_k \in \mathcal{F}(X)$ and $F_0 \in \mathcal{A}(U_\delta(0), \mathcal{L}(X))$, $F_0(U_\delta(0) \setminus \{0\}) \subseteq \Phi(X)$. It remains to prove $F_0(0) \in \Phi(X)$. Let $[\cdot] : \mathcal{L}(X) \longrightarrow \mathcal{L}(X)/\mathcal{K}(X)$ be the canonical quotient mapping, where $\mathcal{K}(X)$ denotes the ideal of compact operators. For $0 < |z| < \delta$ we have

$$\begin{aligned}[I] &= [F(z)F(z)^{-1}] = [F(z)][F(z)^{-1}] = [I + H(z)][F_0(z)] \\ &= [F(z)^{-1}F(z)] = [F(z)^{-1}][F(z)] = [F_0(z)][I + H(z)],\end{aligned}$$

and consequently $[I] = [I + H(0)][F_0(0)] = [F_0(0)][I + H(0)]$, since the functions $I + H$ and F_0 are holomorphic in 0. This shows that $I + H(0)$ inverts $F_0(0)$ modulo compact operators, i. e., $F_0(0) \in \Phi(X)$. □

Proof of Theorem 1.2.6.

We will first prove the assertion in the case $\Psi = \mathcal{L}(X)$.

Since D is discrete in Ω we have that $\Omega \setminus D$ is a connected domain. Hence we may apply Theorem 1.2.7 on the inversion of holomorphic Fredholm families to the function $T \in \mathcal{A}(\Omega \setminus D, \mathcal{L}(X))$. It follows the existence of a discrete set $D' \subseteq (\Omega \setminus D)$ such that $T(z)$ is invertible in $\mathcal{L}(X)$ for $z \in (\Omega \setminus D) \setminus D'$ and the inverse T^{-1} is a meromorphic Fredholm family in $(\Omega \setminus D) \setminus D'$ in the sense of Definition 1.2.4. It remains to prove that no point $p \in D$ is an accumulation point of D' and that T^{-1} extends meromorphically into $p \in D$, i. e., there exists a neighbourhood $U(p)$ of p such that we can write for $z \in U(p) \setminus \{p\}$: $T^{-1}(z) = \sum_{k=-M}^{-1} \tilde{F}_k (z-p)^k + \tilde{F}_0(z)$ with $\tilde{F}_k \in \mathcal{F}(X)$ and $\tilde{F}_0 \in \mathcal{A}(U(p), \mathcal{L}(X))$, $\tilde{F}_0(U(p)) \subseteq \Phi(X)$. Let $p \in D$. By assumption we find $\varepsilon > 0$ such that for $0 < |z-p| < \varepsilon$ we have $T(z) = \sum_{k=-N}^{-1} F_k(z-p)^k + T_0(z)$ with $F_k \in \mathcal{F}(X)$ and $T_0 \in \mathcal{A}(U_\varepsilon(p), \mathcal{L}(X))$, $T_0(U_\varepsilon(p)) \subseteq \Phi(X)$. Since D' is discrete in $\Omega \setminus D$ there exists a $\hat{z} \in U_\varepsilon(p) \setminus \{p\}$ such that $T(\hat{z})$ is invertible in $\mathcal{L}(X)$. Consequently $T_0(U_\varepsilon(p)) \subseteq \Phi_0(X)$ where $\Phi_0(X)$ denotes the subset of all Fredholm operators with index equal to zero. Recall that the index is locally constant on $\Phi(X)$ and that $U_\varepsilon(p)$ is connected. Let $F_0 \in \mathcal{F}(X)$ be a finite-dimensional operator such that $T_0(p) - F_0$ is invertible in $\mathcal{L}(X)$. Thus $T_0(z) - F_0$ is invertible in $\mathcal{L}(X)$ for all z in a small neighbourhood of p. Without loss of generality we may assume that $T_0(z) - F_0$ is invertible for $z \in U_\varepsilon(p)$. For $0 < |z - p| < \varepsilon$ we can write

$$\begin{aligned}(T_0(z) - F_0)^{-1}T(z) &= (T_0(z) - F_0)^{-1}\Big[\sum_{k=-N}^{-1} F_k(z-p)^k\Big] + (T_0(z) - F_0)^{-1}T_0(z) \\ &= I + (T_0(z) - F_0)^{-1}\Big[\sum_{k=-N}^{0} F_k(z-p)^k\Big] =: F(z).\end{aligned}$$

We have $(T_0(z) - F_0)^{-1}\big[\sum_{k=-N}^{0} F_k(z-p)^k\big] = H(z) + \sum_{k=-N}^{-1} A_k(z-p)^k$ with finite-dimensional operators $A_k \in \mathcal{F}(X)$ and $H \in \mathcal{A}(U_\varepsilon(p), \mathcal{L}(X))$. Set $K_0 := \big(\bigcap_{k=-N}^{0} N(F_k)\big) \cap \big(\bigcap_{k=-N}^{-1} N(A_k)\big)$. Then K_0 is a closed subspace of X and by Lemma 1.2.8 we have $\operatorname{codim} K_0 < \infty$. This implies $H(z)u = 0$ for $u \in K_0$, $z \in U_\varepsilon(p) \setminus \{p\}$, but from Cauchy's integral formula

$$H(p) = \frac{1}{2\pi i} \int_{\partial U_{\frac{\varepsilon}{2}}(p)} \frac{H(\zeta)}{\zeta - p}\, d\zeta$$

we also obtain $H(p)u = 0$ for $u \in K_0$. Thus we may apply Lemma 1.2.9 to the function $F(z)$ in $U_\varepsilon(p) \setminus \{p\}$ which shows the existence of $0 < \delta < \varepsilon$, such that $F(z) = (T_0(z) - F_0)^{-1} T(z)$ is invertible for $0 < |z-p| < \delta$. Consequently $T(z)$ is invertible in $\mathcal{L}(X)$ for $0 < |z-p| < \delta$ which implies that p is no accumulation point of D', i. e., $\tilde{D} := D \cup D'$ is discrete in Ω. Moreover, we can write for $0 < |z-p| < \delta$:

$$F(z)^{-1} = T(z)^{-1} \cdot (T_0(z) - F_0) = \sum_{k=-M}^{-1} \tilde{A}_k (z-p)^k + \tilde{A}_0(z)$$

with finite-dimensional operators $\tilde{A}_k \in \mathcal{F}(X)$ and $\tilde{A}_0 \in \mathcal{A}(U_\delta(p), \mathcal{L}(X))$, and hence

$$\begin{aligned} T(z)^{-1} &= \Big[\sum_{k=-M}^{-1} \tilde{A}_k (z-p)^k\Big] \cdot (T_0(z) - F_0)^{-1} + \tilde{A}_0(z) \cdot (T_0(z) - F_0)^{-1} \\ &= \sum_{k=-M}^{-1} \tilde{F}_k (z-p)^k + \tilde{F}_0(z) \end{aligned}$$

with $\tilde{F}_k \in \mathcal{F}(X)$ and $\tilde{F}_0 \in \mathcal{A}(U_\delta(p), \mathcal{L}(X))$. Since $\tilde{F}_0(U_\delta(p) \setminus \{p\}) \subseteq \Phi(X)$ it remains to prove that $\tilde{F}_0(p) \in \Phi(X)$.

Let $[\cdot] : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$ be the canonical quotient mapping, where $\mathcal{K}(X)$ denotes the ideal of compact operators. For $0 < |z-p| < \delta$ we have

$$\begin{aligned} [I] &= [T(z) \cdot T(z)^{-1}] = [T(z)][T(z)^{-1}] = [T_0(z)][\tilde{F}_0(z)] \\ &= [T(z)^{-1} \cdot T(z)] = [T(z)^{-1}][T(z)] = [\tilde{F}_0(z)][T_0(z)] \end{aligned}$$

and consequently $[I] = [T_0(p)][\tilde{F}_0(p)] = [\tilde{F}_0(p)][T_0(p)]$ since the functions $\tilde{F}_0$ and T_0 are holomorphic in p. This shows $\tilde{F}_0(p) \in \Phi(X)$ and finishes the proof of Theorem 1.2.6 in the case $\Psi = \mathcal{L}(X)$.

In the general case we may first apply the result for $\mathcal{L}(X)$. It remains to show that $T(z)^{-1} \in \Psi$ for $z \in \Omega \setminus \tilde{D}$ and that $T^{-1} \in \mathcal{A}(\Omega \setminus \tilde{D}, \Psi)$. But by Definition 1.2.2 of Ψ-algebras we have $T(z)^{-1} \in \Psi$, and since the inversion ${}^{-1} : \Psi^{-1} \to \Psi^{-1}$ is continuous we obtain the holomorphy of T^{-1} in $\Omega \setminus \tilde{D}$ from the holomorphy of T (as Ψ - valued functions). This completes the proof of Theorem 1.2.6.

1.2.10 Example. We conclude this section with an example where the validity of Theorem 1.2.6 is violated due to the holomorphic part of a meromorphic function evaluated at a pole not being a Fredholm operator.

Let X be a Banach space and $P = P^2 \in \mathcal{L}(X)$ be a non-trivial finite-dimensional projection. Assume $\dim X = \infty$. Consider the function $T \in \mathcal{A}(\mathbb{C}\setminus\{0\}, \mathcal{L}(X))$ given by $T(z) := zI - \frac{1}{z}P$. We have $T(\mathbb{C}\setminus\{0\}) \subseteq \Phi_0(X)$, and $T(z)$ is invertible in $\mathcal{L}(X)$ for $|z| > \|P\|^{\frac{1}{2}}$. For $|z|$ large we may write

$$\begin{aligned} T(z)^{-1} &= \frac{1}{z}(I - \frac{1}{z^2}P)^{-1} = \frac{1}{z} \cdot \sum_{k=0}^{\infty} \frac{1}{z^{2k}} P^k \\ &= \frac{1}{z} \cdot \Big[I + \Big(\sum_{k=1}^{\infty} \frac{1}{z^{2k}} \Big) P \Big] = (I - P)\frac{1}{z} + \frac{z}{z^2 - 1} P. \end{aligned}$$

Since $T(z)$ is invertible for $z \in \mathbb{C} \setminus \{0, -1, +1\}$ we conclude from uniqueness of analytic continuation that

$$T(z)^{-1} = (I - P)\frac{1}{z} + \frac{z}{z^2 - 1} P \text{ for } z \in \mathbb{C} \setminus \{0, -1, +1\}.$$

Hence T^{-1} is meromorphic in 0 but the residue is $(I - P) \notin \mathcal{F}(X)$.

1.3 Volterra integral operators

1.3.1 Remark. In this section we discuss integral operators with operator-valued kernel functions that are supported on one side of the diagonal. The theory of operators of such kind is classical, and they arise, e. g., in the study of (Volterra) integral equations. From our point of view the main property of these operators is, that under some natural assumptions they turn out to be quasinilpotent, i. e. their spectrum consists of zero only. This observation will be employed later in the analysis of remainders of the parametrix construction to parabolic pseudodifferential operators, and it is crucial for the proof of the invertibility of these operators within the calculus.

1.3.2 Remark. Let E, $\tilde{E}$ and $\hat{E}$ be Hilbert spaces, and let $I \subseteq \mathbb{R}$ be an interval.

a) With a kernel function $k \in L^2(I \times I, \mathcal{L}(E, \tilde{E}))$ we associate an operator $T_k \in \mathcal{L}(L^2(I, E), L^2(I, \tilde{E}))$ via

$$\big(T_k u\big)(t) := \int_I k(t, t')u(t')\, dt'$$

for $u \in L^2(I,E)$. The mapping

$$T : L^2(I\times I, \mathcal{L}(E,\tilde{E})) \ni k \longmapsto T_k \in \mathcal{L}(L^2(I,E), L^2(I,\tilde{E}))$$

is one-to-one and bounded with norm $\|T\| = 1$.

In particular, T is an isomorphism of $L^2(I\times I, \mathcal{L}(E,\tilde{E}))$ onto its range in $\mathcal{L}(L^2(I,E), L^2(I,\tilde{E}))$, and via T we transfer the kernel topology to the operator space. Thus we obtain a Banach subspace of $\mathcal{L}(L^2(I,E), L^2(I,\tilde{E}))$ endowed with a finer topology.

b) Let $k_1 \in L^2(I\times I, \mathcal{L}(\tilde{E},\hat{E}))$ and $k_2 \in L^2(I\times I, \mathcal{L}(E,\tilde{E}))$. Then the composition $T_{k_1} \circ T_{k_2} \in \mathcal{L}(L^2(I,E), L^2(I,\hat{E}))$ equals $T_{k_1\circ k_2}$ with the function

$$\big(k_1 \circ k_2\big)(t,t') := \int_I k_1(t,s) k_2(s,t')\, ds \in L^2(I\times I, \mathcal{L}(E,\hat{E})).$$

The mapping

$$\circ : L^2(I\times I, \mathcal{L}(\tilde{E},\hat{E})) \times L^2(I\times I, \mathcal{L}(E,\tilde{E})) \longrightarrow L^2(I\times I, \mathcal{L}(E,\hat{E}))$$

is bilinear and continuous; more precisely we have $\|k_1 \circ k_2\|_{L^2} \leq \|k_1\|_{L^2} \|k_2\|_{L^2}$.

1.3.3 Definition. Let $k \in L^2(I\times I, \mathcal{L}(E,\tilde{E}))$. The operator T_k is called a *Volterra integral operator* provided that one of the following equivalent conditions is fulfilled:

i) For every $t_0 \in I$ we have $\big(T_k u\big)(t) \equiv 0$ for $t > t_0$ for all $u \in L^2(I,E)$ such that $u(t) \equiv 0$ for $t > t_0$.

ii) For every $u \in L^2(I,E)$ and every $v \in L^2(I,\tilde{E})$ such that $\operatorname{supp}(u) < \operatorname{supp}(v)$ we have $\langle T_k u, v \rangle_{L^2(I,\tilde{E})} = 0$.

iii) $k(t,t') \equiv 0$ for $t > t'$.

A kernel k satisfying iii) is called *Volterra integral kernel.*

1.3.4 Proposition. *a) The space of Volterra integral kernels is a closed subspace of $L^2(I\times I, \mathcal{L}(E,\tilde{E}))$.*

b) Let $k_1 \in L^2(I\times I, \mathcal{L}(\tilde{E},\hat{E}))$ and $k_2 \in L^2(I\times I, \mathcal{L}(E,\tilde{E}))$ be Volterra integral kernels. Then also $k_1 \circ k_2 \in L^2(I\times I, \mathcal{L}(E,\hat{E}))$ is a Volterra integral kernel. If k_1 and k_2 are continuous then $k_1 \circ k_2$ is continuous.

On the level of operators this means that the space of Volterra integral operators is a closed subspace of all integral operators with L^2-kernel functions, and it is closed with respect to taking compositions.

Proof. These assertions are obvious. For the continuity of $k_1 \circ k_2$ in b) let us note the following:

To every point $(t_0, t_0') \in I \times I$ there exists a neighbourhood $U(t_0, t_0') \subseteq I \times I$ and a compact subinterval $J \subseteq I$ such that

$$\big(k_1 \circ k_2\big)(t, t') = \int_J k_1(t, s) k_2(s, t')\, ds$$

for $(t, t') \in U(t_0, t_0')$. Thus the continuity follows from the continuity of k_1 and k_2 and Lebesgue's dominated convergence theorem. □

1.3.5 Lemma. *Let $k \in L^2(I \times I, \mathcal{L}(E))$ be a continuous Volterra integral kernel. Moreover, let $g, h \in C(I)$ be everywhere positive functions, and assume that*

$$C := \sup\{g(t)h(t')\|k(t,t')\|_{\mathcal{L}(E)};\ (t,t') \in I \times I\} < \infty.$$

For short we write

$$k_{(N)} := \underbrace{k \circ \ldots \circ k}_{N} \in L^2(I \times I, \mathcal{L}(E))$$

for $N \in \mathbb{N}$. Then $k_{(N)}$ is a continuous Volterra integral kernel, and for $t' \geq t$ we have the pointwise estimate

$$g(t)h(t')\|k_{(N)}(t,t')\|_{\mathcal{L}(E)} \leq C^N \frac{1}{(N-1)!} \Big(\int_t^{t'} \frac{1}{g(s)h(s)}\, ds \Big)^{N-1}.$$

Proof. $k_{(N)}$ is a continuous Volterra integral kernel by Proposition 1.3.4. It remains to prove the pointwise estimate. Let $F \in C^1(I)$ such that $F' = \frac{1}{gh}$. We proceed by induction: For $N = 1$ the estimate is true by assumption. Now assume it holds for some $N \in \mathbb{N}$. Then we have for $t' \geq t$:

$$\begin{aligned}
g(t)h(t')\|k_{(N+1)}(t,t')\|_{\mathcal{L}(E)} &= g(t)h(t')\|\big(k \circ k_{(N)}\big)(t,t')\|_{\mathcal{L}(E)} \\
&= g(t)h(t') \Big\| \int_t^{t'} k(t,s) k_{(N)}(s,t')\, ds \Big\|_{\mathcal{L}(E)} \\
&= \Big\| \int_t^{t'} \frac{1}{g(s)h(s)} \big(g(t)h(s)k(t,s)\big)\big(g(s)h(t')k_{(N)}(s,t')\big)\, ds \Big\|_{\mathcal{L}(E)} \\
&\leq C^{N+1} \frac{1}{(N-1)!} \int_t^{t'} F'(s)\big(F(t') - F(s)\big)^{N-1}\, ds \\
&= C^{N+1} \frac{1}{N!} \big(F(t') - F(t)\big)^N.
\end{aligned}$$

This finishes the proof of the lemma. □

1.3.6 Theorem. *Let $g, h \in C(I)$ be everywhere positive functions with*

$$\int_I \frac{1}{g(s)^2}\, ds < \infty \qquad \text{and} \qquad \int_I \frac{1}{h(s)^2}\, ds < \infty.$$

Moreover, let $k \in C(I\times I, \mathcal{L}(E))$ such that $k(t,t') \equiv 0$ for $t > t'$, and

$$\sup\{g(t)h(t')\|k(t,t')\|_{\mathcal{L}(E)};\ (t,t') \in I\times I\} < \infty.$$

Then $k \in L^2(I\times I, \mathcal{L}(E))$ is a continuous Volterra integral kernel, and the Volterra integral operator $T_k \in \mathcal{L}(L^2(I, \mathcal{L}(E)))$ is quasinilpotent. For $0 \neq \lambda \in \mathbb{C}$ we have

$$\big(\lambda Id - T_k\big)^{-1} = \frac{1}{\lambda} Id - T_{k'}$$

with a Volterra integral operator $T_{k'}$.

Proof. Clearly, $k \in L^2(I\times I, \mathcal{L}(E))$ is a continuous Volterra integral kernel. Let

$$C := \sup\{g(t)h(t')\|k(t,t')\|_{\mathcal{L}(E)};\ (t,t') \in I\times I\}.$$

Then we have for $t' \geq t$ in the notation from Lemma 1.3.5:

$$\begin{aligned} g(t)h(t')\|k_{(N)}(t,t')\|_{\mathcal{L}(E)} &\leq C^N \frac{1}{(N-1)!}\Big(\int_t^{t'} \frac{1}{g(s)h(s)}\, ds\Big)^{N-1} \\ &\leq C^N \frac{1}{(N-1)!}\Big\|\frac{1}{g}\Big\|_{L^2(I)}^{N-1}\Big\|\frac{1}{h}\Big\|_{L^2(I)}^{N-1}, \end{aligned}$$

and consequently

$$\|k_{(N)}\|_{L^2(I\times I, \mathcal{L}(E))} \leq \frac{\Big(C\big\|\frac{1}{g}\big\|_{L^2(I)}\big\|\frac{1}{h}\big\|_{L^2(I)}\Big)^N}{(N-1)!}.$$

This shows that

$$\|T_k^N\|_{\mathcal{L}(L^2(I,\mathcal{L}(E)))}^{\frac{1}{N}} \leq \|k_{(N)}\|_{L^2(I\times I,\mathcal{L}(E))}^{\frac{1}{N}} \underset{N\to\infty}{\longrightarrow} 0,$$

i. e. T_k is quasinilpotent. Moreover, for $\lambda \neq 0$ the series

$$k' := -\sum_{N=1}^{\infty} \frac{1}{\lambda^{N+1}} k_{(N)} \in L^2(I\times I, \mathcal{L}(E))$$

converges and defines a Volterra integral kernel, and we have

$$\big(\lambda \mathrm{Id} - T_k\big)^{-1} = \frac{1}{\lambda}\mathrm{Id} - T_{k'}.$$

□

Some notes on abstract kernels

1.3.7 Remark. In many situations the residual elements of a pseudodifferential calculus are characterized by their mapping properties in a scale of suitable function spaces. In algebras consisting of Volterra operators we are interested to invert the remainders of the parametrix construction of parabolic elements within the calculus, where Theorem 1.3.6 serves as the key for the proof. In order to be able to apply this result we are in need to obtain information about the Schwartz kernels of the residual elements from their mapping properties.

To this end recall the following facts (cf. e. g. [29]):

a) Let E and F be (Hausdorff) locally convex spaces. Then $E\otimes F$ is realized within $E\varepsilon F := \mathcal{L}_e(E'_c, F)$ via

$$\sum_{i=1}^{n} e_i\otimes f_i \longmapsto \Big(e' \mapsto \sum_{i=1}^{n}\langle e', e\rangle_{E',E}\, f_i\Big).$$

Here the subscript c denotes the topology of uniform convergence on precompact subsets in E, while the subscript e denotes the topology of uniform convergence on equicontinuous subsets in E'. The induced topology of $E\varepsilon F$ on $E\otimes F$ is the ε-topology, i. e. we obtain the injective tensor product $E\otimes_\varepsilon F$ of the spaces E and F.

b) $E\varepsilon F$ is complete if and only if F is complete.

c) If E and F are complete then $E\varepsilon F$ and $F\varepsilon E$ are topologically isomorphic via transposition, i. e.

$$E\varepsilon F \ni G \longmapsto G^t \in F\varepsilon E.$$

Passing to completions shows that this isomorphism induces in any case a topological isomorphism $E\otimes_\varepsilon F \cong F\otimes_\varepsilon E$.

d) If F has the approximation property, which in particular holds for hilbertizable and consequently for nuclear spaces F, then $E\otimes F$ is dense in $E\varepsilon F$.

e) From b) and d) we conclude that $E\widehat{\otimes}_\varepsilon F = E\varepsilon F$ if F is complete and has the approximation property.

f) If E or F is nuclear we have $E\otimes_\pi F = E\otimes_\varepsilon F$. In particular, if F is complete and has the approximation property, we have $E\widehat{\otimes}_\pi F = E\widehat{\otimes}_\varepsilon F = E\varepsilon F$.

g) Let E and F be Fréchet spaces and assume that E is nuclear. Then we have $\mathcal{L}_\beta(E'_\beta, F) \cong E\widehat{\otimes}_\pi F$ in the canonical way. Here the subscript β indicates that the spaces are endowed with the strong topology.

1.3.8 Remark. Let $\{E_0, E, E_1\}$ be a Hilbert triple. Then the inner product in E induces a canonical (antilinear) isomorphism $E_1 \cong E_0'$ via

$$E_1 \ni e_1 \longmapsto \langle \cdot, e_1 \rangle_E \in E_0'.$$

Let $\tilde{E}$ be another Hilbert space. Via this isomorphism we may identify the nuclear operators $E_0 \longrightarrow \tilde{E}$ as

$$\ell^1(E_0, \tilde{E}) = E_0' \widehat{\otimes}_\pi \tilde{E} \cong E_1 \widehat{\otimes}_\pi \tilde{E},$$

i. e. $G \in \ell^1(E_0, \tilde{E})$ if and only if there exist sequences $(\lambda_j) \in \ell^1$ and $(x_j) \subseteq E_1$, $(\tilde{e}_j) \subseteq \tilde{E}$ tending to zero such that

$$G = \sum_{j=1}^{\infty} \lambda_j \langle \cdot, x_j \rangle_E \, \tilde{e}_j.$$

The tensor product representations in Proposition 1.3.9 below are to be understood in this sense.

1.3.9 Proposition. *Let $\{E_0, E, E_1\}$ and $\{\tilde{E}_0, \tilde{E}, \tilde{E}_1\}$ be Hilbert triples, and let F and $\tilde{F}$ be nuclear Fréchet spaces such that $F \hookrightarrow E_1$ and $\tilde{F} \hookrightarrow \tilde{E}_0$. Moreover, let $G \in \mathcal{L}(E_0, \tilde{E}_0)$ be given.*

a) We have $G(E_0) \subseteq \tilde{F}$ if and only if

$$G \in E_1 \widehat{\otimes}_\pi \tilde{F} \hookrightarrow \ell^1(E_0, \tilde{E}_0).$$

b) $G(E_0) \subseteq \tilde{F}$ and $G^*(\tilde{E}_1) \subseteq F$ if and only if

$$G \in \big(E_1 \widehat{\otimes}_\pi \tilde{F}\big) \cap \big(F \widehat{\otimes}_\pi \tilde{E}_0\big) \hookrightarrow \ell^1(E_0, \tilde{E}_0).$$

c) Let $\{\hat{E}_0, \hat{E}, \hat{E}_1\}$ be another Hilbert triple, and assume that $G = AB^*$ with $A \in \mathcal{L}(\hat{E}_0, \tilde{E}_0)$ and $B \in \mathcal{L}(\hat{E}_1, E_1)$ such that $A(\hat{E}_0) \subseteq \tilde{F}$ and $B(\hat{E}_1) \subseteq F$. Then we have

$$G \in F \widehat{\otimes}_\pi \tilde{F} \hookrightarrow \ell^1(E_0, \tilde{E}_0).$$

Proof. For the proof of a) note first that $G(E_0) \subseteq \tilde{F}$ if and only if $G \in \mathcal{L}(E_0, \tilde{F})$ by the closed graph theorem. Clearly, every element in $E_1 \widehat{\otimes}_\pi \tilde{F} \cong E_0' \widehat{\otimes}_\pi \tilde{F}$ induces an operator in $\mathcal{L}(E_0, \tilde{F})$. Now assume that $G \in \mathcal{L}(E_0, \tilde{F})$ is given. Then we have by the nuclearity of $\tilde{F}$ that

$$G^t \in \mathcal{L}((\tilde{F}')_\beta, E_0') = \mathcal{L}((\tilde{F}')_c, E_0') = \tilde{F} \varepsilon E_0',$$

and thus by Remark 1.3.7 (and 1.3.8)

$$G = \big(G^t\big)^t \in E_0' \varepsilon \tilde{F} = E_0' \widehat{\otimes}_\pi \tilde{F} \cong E_1 \widehat{\otimes}_\pi \tilde{F}.$$

Assume that G fulfills the mapping properties in b). From a) we obtain that $G \in E_1 \widehat{\otimes}_\pi \tilde{F}$ and $G^* \in \tilde{E}_0 \widehat{\otimes}_\pi F \cong \tilde{E}_1' \widehat{\otimes}_\pi F$. Since $G = (G^*)^*$ we conclude that $G \in F \widehat{\otimes}_\pi \tilde{E}_0$, i. e.

$$G \in \big(E_1 \widehat{\otimes}_\pi \tilde{F}\big) \cap \big(F \widehat{\otimes}_\pi \tilde{E}_0\big)$$

as desired. The converse assertion in b) is immediate.

For the proof of c) note first that by a) we have $A \in \hat{E}_1 \widehat{\otimes}_\pi \tilde{F}$. Consequently, there are sequences $(\lambda_j) \in \ell^1$ and $(x_j) \subseteq \hat{E}_1$, $(\tilde{f}_j) \subseteq \tilde{F}$ tending to zero such that

$$A(\hat{e}) = \sum_{j=1}^{\infty} \lambda_j \langle \hat{e}, x_j \rangle_{\hat{E}} \, \tilde{f}_j$$

for $\hat{e} \in \hat{E}_0$. Thus we may write for $e \in E_0$:

$$A(B^* e) = \sum_{j=1}^{\infty} \lambda_j \langle B^* e, x_j \rangle_{\hat{E}} \, \tilde{f}_j = \sum_{j=1}^{\infty} \lambda_j \langle e, B(x_j) \rangle_E \, \tilde{f}_j .$$

Since $B \in \mathcal{L}(\hat{E}_1, F)$ we conclude that the sequence $(B(x_j)) \subseteq F$ converges to zero, i. e. $G = AB^* \in F \widehat{\otimes}_\pi \tilde{F}$ as asserted. $\square$

Chapter 2

Abstract Volterra pseudodifferential calculus

2.1 Anisotropic parameter-dependent symbols

2.1.1 Definition. Let $\ell \in \mathbb{N}$ be a given anistropy.

a) For $(\xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^q$ define

$$|\xi, \lambda|_\ell := (|\xi|^{2\ell} + |\lambda|^2)^{\frac{1}{2\ell}},$$
$$\langle \xi, \lambda \rangle_\ell := (1 + |\xi|^{2\ell} + |\lambda|^2)^{\frac{1}{2\ell}},$$

where $|\cdot|$ denotes the Euclidean norm.

b) For a multi-index $\beta = (\alpha, \alpha') \in \mathbb{N}_0^{n+q}$ let

$$|\beta|_\ell := |\alpha| + \ell \cdot |\alpha'|,$$

where $|\cdot|$ denotes the usual length of a multi-index as the sum of its components.

2.1.2 Lemma. *There exists a constant $c > 0$ such that for all $s \in \mathbb{R}$ and $\xi_1, \xi_2 \in \mathbb{R}^n$, $\lambda_1, \lambda_2 \in \mathbb{R}^q$ the following inequality is fulfilled* (Peetre's inequality)*:*

$$\langle \xi_1 + \xi_2, \lambda_1 + \lambda_2 \rangle_\ell^s \leq c^{|s|} \langle \xi_1, \lambda_1 \rangle_\ell^{|s|} \langle \xi_2, \lambda_2 \rangle_\ell^s . \tag{2.1.i}$$

Morover, we can compare the regularized "anisotropic distance" $\langle \cdot, \cdot \rangle_\ell$ with the "isotropic distance", i. e. there exist constants $c_1, c_2 > 0$ such that

$$c_1 \langle \xi, \lambda \rangle_\ell \leq \langle \xi, \lambda \rangle \leq c_2 \langle \xi, \lambda \rangle_\ell^\ell . \tag{2.1.ii}$$

2.1.3 Definition. Let E and $\tilde{E}$ be Hilbert spaces. For $\mu \in \mathbb{R}$ we define

$$S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) := \{a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q, \mathcal{L}(E, \tilde{E}));$$
$$p_k(a) := \sup_{\substack{(\xi,\lambda)\in\mathbb{R}^n\times\mathbb{R}^q \\ |\beta|_\ell \le k}} \|\partial^\beta_{(\xi,\lambda)} a(\xi,\lambda)\| \, \langle \xi,\lambda\rangle_\ell^{-\mu+|\beta|_\ell} < \infty \text{ for all } k \in \mathbb{N}_0 \}.$$

This is a Fréchet space with the topology induced by the seminorm-system $\{p_k;\ k \in \mathbb{N}_0\}$. Define

$$S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) := \bigcap_{\mu\in\mathbb{R}} S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}).$$

By (2.1.ii) this space does not depend on $\ell \in \mathbb{N}$, and we have $S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) = \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^q, \mathcal{L}(E, \tilde{E}))$. Moreover, for $\mu \in \mathbb{R}$ the spaces of x- (resp. x'-) and (x, x')-dependent symbols are defined as

$$S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) := C_b^\infty(\mathbb{R}^n, S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})),$$
$$S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) := C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n, S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})).$$

Analogously, we obtain the spaces of order $-\infty$. If $E = \tilde{E} = \mathbb{C}$ we suppress the Hilbert spaces from the notation.

Let $\{E_j\}_{j\in\mathbb{N}}$ and $\{\tilde{E}_j\}_{j\in\mathbb{N}}$ be scales of Hilbert spaces such that $E_j \hookrightarrow E_{j+1}$ and $\tilde{E}_{j+1} \hookrightarrow \tilde{E}_j$ for $j \in \mathbb{N}$. Define

$$S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \operatorname*{ind-lim}_{j\in\mathbb{N}} E_j, \operatorname*{proj-lim}_{k\in\mathbb{N}} \tilde{E}_k) := \bigcap_{j,k\in\mathbb{N}} S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E_j, \tilde{E}_k)$$

with the natural Fréchet topologies induced. The spaces of order $-\infty$ are defined in an analogous manner, as well as the symbol spaces with dependence on $x, x' \in \mathbb{R}^n$. With this notion the case of single Hilbert spaces E and $\tilde{E}$ corresponds to the constant scales.

2.1.4 Definition. Let E and $\tilde{E}$ be Hilbert spaces. A function $f : (\mathbb{R}^n \times \mathbb{R}^q) \setminus \{0\} \to \mathcal{L}(E, \tilde{E})$ is called *(anisotropic) homogeneous* of degree $\mu \in \mathbb{R}$, if for $(\xi, \lambda) \in (\mathbb{R}^n \times \mathbb{R}^q) \setminus \{0\}$ and $\varrho > 0$

$$f(\varrho\xi, \varrho^\ell\lambda) = \varrho^\mu f(\xi, \lambda). \tag{2.1.iii}$$

A function $f : \mathbb{R}^n \times \mathbb{R}^q \to \mathcal{L}(E, \tilde{E})$ is called *(anisotropic) homogeneous* of degree $\mu \in \mathbb{R}$ for large (ξ, λ), if for $(\xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^q$ with $|(\xi, \lambda)|$ sufficiently large and $\varrho \ge 1$

$$f(\varrho\xi, \varrho^\ell\lambda) = \varrho^\mu f(\xi, \lambda). \tag{2.1.iv}$$

In this work, homogeneity always is meant in this anisotropic sense.

2.1.5 Remark. Let $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q, \mathcal{L}(E, \tilde{E}))$ be homogeneous of degree $\mu \in \mathbb{R}$ for large (ξ, λ). Then $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$.

Asymptotic expansion

2.1.6 Definition. Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces in the sense of Definition 2.1.3. For short, we set

$$\mathcal{E} := \operatorname*{ind\text{-}lim}_{j\in\mathbb{N}} E_j \quad \text{and} \quad \tilde{\mathcal{E}} := \operatorname*{proj\text{-}lim}_{j\in\mathbb{N}} \tilde{E}_j.$$

Let $(\mu_k) \subseteq \mathbb{R}$ be a sequence of reals such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max_{k\in\mathbb{N}} \mu_k$. Moreover, let $a_k \in S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$. A symbol $a \in S^{\overline{\mu};\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$ is called the *asymptotic expansion* of the a_k, if for every $R \in \mathbb{R}$ there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$

$$a - \sum_{j=1}^{k} a_j \in S^{R;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}}).$$

The symbol a is uniquely determined modulo $S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$.

For short we write $a \sim \sum_{j=1}^{\infty} a_j$.

2.1.7 Lemma. *Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces, and $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as in Definition 2.1.6. Let $(\mu_k) \subseteq \mathbb{R}$ such that $\mu_k > \mu_{k+1} \underset{k\to\infty}{\longrightarrow} -\infty$. Furthermore, for each $k \in \mathbb{N}$ let $(A_{k_j})_{j\in\mathbb{N}} \subseteq S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$ be a countable system of bounded sets. Let $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q)$ be a 0-excision function. Then there is a sequence $(c_i) \subseteq \mathbb{R}_+$ with $c_i < c_{i+1} \underset{i\to\infty}{\longrightarrow} \infty$ such that for each $k \in \mathbb{N}$*

$$\sum_{i=k}^{\infty} \sup_{a\in A_{i_j}} p\Big(\chi\Big(\frac{\xi}{d_i}, \frac{\lambda}{d_i^\ell}\Big) a(\xi,\lambda)\Big) < \infty \tag{2.1.v}$$

for all continuous seminorms p on $S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$ and every $j \in \mathbb{N}$, and for all sequences $(d_i) \subseteq \mathbb{R}_+$ with $d_i \geq c_i$.

Proof. The proof of this lemma is a variant of the standard Borel-argument. □

2.1.8 Theorem. *Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces, and $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as in Definition 2.1.6. Let $(\mu_k) \subseteq \mathbb{R}$ such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max_{k\in\mathbb{N}} \mu_k$. Moreover, let $a_k \in S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$. Then there exists $a \in S^{\overline{\mu};\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$ such that $a \sim \sum_{j=1}^{\infty} a_j$, and a is uniquely determined modulo $S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$.*

Proof. Without loss of generality we may assume that $\mu_k > \mu_{k+1} \underset{k\to\infty}{\longrightarrow} -\infty$. For $k, j \in \mathbb{N}$ let

$$A_{k_j} := \{\partial_x^\alpha a_k(x);\ x \in \mathbb{R}^n,\ |\alpha| \le j\}.$$

Then $A_{k_j} \subseteq S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$ is bounded. Let $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q)$ be a 0-excision function. Now apply Lemma 2.1.7. With a suitable sequence $(c_i) \subseteq \mathbb{R}_+$ formula (2.1.v) becomes

$$\sum_{i=k}^\infty \sup\{p(\chi\Big(\frac{\xi}{c_i}, \frac{\lambda}{c_i^\ell}\Big)(\partial_x^\alpha a_i(x))(\xi,\lambda));\ x \in \mathbb{R}^n,\ |\alpha| \le j\} < \infty$$

for all continuous seminorms p on $S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$, which shows that for every $k \in \mathbb{N}$ the sum

$$\sum_{i=k}^\infty \chi\Big(\frac{\cdot}{c_i}, \frac{\cdot}{c_i^\ell}\Big) a_i$$

is unconditionally convergent in $S^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}})$. The assertion of the theorem follows with

$$a := \sum_{i=1}^\infty \chi\Big(\frac{\cdot}{c_i}, \frac{\cdot}{c_i^\ell}\Big) a_i \in S^{\mu_1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \mathcal{E}, \tilde{\mathcal{E}}).$$

□

Classical symbols

2.1.9 Definition. Let E and $\tilde{E}$ be Hilbert spaces. For $\mu \in \mathbb{R}$ define

$$S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) := \Big\{a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E});\ a \sim \sum_{k=0}^\infty \chi a_{(\mu-k)}\Big\},$$

where $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q)$ is a 0-excision function, and $a_{(\mu-k)} \in C^\infty((\mathbb{R}^n \times \mathbb{R}^q) \setminus \{0\}, \mathcal{L}(E, \tilde{E}))$ are (anisotropic) homogeneous functions of degree $\mu - k$, the so called *homogeneous components* of a.

2.1.10 Remark. By 2.1.5 the space $S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ is well-defined.

The homogeneous components of $a \in S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ are uniquely determined by a. They can iteratively be recovered from the relation

$$\frac{1}{\varrho^{\mu-k}}\Big(a(\varrho\xi, \varrho^\ell\lambda) - \sum_{j=0}^{k-1} a_{(\mu-j)}(\varrho\xi, \varrho^\ell\lambda)\Big) \underset{\varrho\to\infty}{\longrightarrow} a_{(\mu-k)}(\xi,\lambda) \tag{2.1.vi}$$

with convergence in $\mathcal{L}(E,\tilde{E})$, which holds locally uniformly for $0 \neq (\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^q$.

Note that $S^{\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ is a Fréchet space with respect to the projective topology of the mappings

$$S^{\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) \ni a \longmapsto \begin{cases} a - \sum\limits_{j=0}^{k-1} \chi a_{(\mu-j)} & \in S^{\mu-k;\ell}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) \\ a_{(\mu-k)} & \in C^\infty((\mathbb{R}^n \times \mathbb{R}^q) \setminus \{0\}, \mathcal{L}(E,\tilde{E})) \end{cases}$$

for $k \in \mathbb{N}_0$.

The spaces of x- (resp. x'-) and (x,x')-dependent classical symbols are defined as

$$S^{\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) := C^\infty_b(\mathbb{R}^n, S^{\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})),$$
$$S^{\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) := C^\infty_b(\mathbb{R}^n \times \mathbb{R}^n, S^{\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})).$$

Clearly, the spaces of classical symbols are closed with respect to taking asymptotic expansions if the sequence $(\mu_k)_{k\in\mathbb{N}_0}$ of orders is given as $\mu_k := \mu - k$ for some $\mu \in \mathbb{R}$.

2.1.11 Remark. The notions of parameter-dependent symbols are analogous if the parameter-space $\mathbb{R}^q$ is replaced by a conical subset $\emptyset \neq \Lambda \subseteq \mathbb{R}^q$, which is the closure of its interior. There only arise notational modifications. In this work, we will mainly make use of parameter-dependent symbols and operators with the parameter running over $\mathbb{R}$ or over a half-plane $\mathbb{H} \subseteq \mathbb{C} \cong \mathbb{R}^2$.

2.2 Anisotropic parameter-dependent operators

2.2.1 Definition. Let E and $\tilde{E}$ be Hilbert spaces, and let $\mu \in \mathbb{R}$. With a double-symbol $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ we associate a family of *pseudodifferential operators* $\mathrm{op}_x(a)(\lambda) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n, E), \mathcal{S}(\mathbb{R}^n, \tilde{E}))$ for $\lambda \in \mathbb{R}^q$ by means of the following oscillatory integral:

$$\begin{aligned}(\mathrm{op}_x(a)(\lambda)u)(x) :&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i(x-x')\xi} a(x,x',\xi,\lambda) u(x')\, dx'\, đ\xi \\ &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{-ix'\xi} a(x,x+x',\xi,\lambda) u(x+x')\, dx'\, đ\xi\end{aligned}$$

where as usually $đ\xi := (2\pi)^{-n} d\xi$.

The space of these operators is denoted by

$$L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E}) := \{\mathrm{op}_x(a)(\lambda);\ a \in S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})\}.$$

In the case of $E = \tilde{E} = \mathbb{C}$ the Hilbert spaces are suppressed from the notation.

Elements of the calculus

2.2.2 Theorem. *Let $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$. Then there exist unique left- and right-symbols $a_L(x,\xi,\lambda)$, $a_R(x',\xi,\lambda) \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ such that $op_x(a)(\lambda) = op_x(a_L)(\lambda) = op_x(a_R)(\lambda)$ as operators on $\mathcal{S}(\mathbb{R}^n, E)$.*

These symbols are given by means of the following oscillatory integrals:

$$a_L(x,\xi,\lambda) = \iint e^{-iy\eta} a(x, y+x, \xi+\eta, \lambda)\, dy\, đ\eta,$$
$$a_R(x',\xi,\lambda) = \iint e^{-iy\eta} a(x'+y, x', \xi-\eta, \lambda)\, dy\, đ\eta.$$

The mappings $a \mapsto a_L$ and $a \mapsto a_R$ are continuous. Moreover, we have the asymptotic expansions

$$a_L(x,\xi,\lambda) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha D_{x'}^\alpha a(x,x',\xi,\lambda)|_{x'=x},$$
$$a_R(x',\xi,\lambda) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} (-1)^{|\alpha|} \partial_\xi^\alpha D_{x}^\alpha a(x,x',\xi,\lambda)|_{x=x'}.$$

If a is classical, so are a_L and a_R, and the mappings $a \mapsto a_L$ and $a \mapsto a_R$ are continuous with respect to the (stronger) topology of classical symbols.

2.2.3 Remark. By Theorem 2.2.2 the mapping op_x provides an isomorphism between the space of x-dependent symbols ("left-symbols") and pseudodifferential operators:

$$S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E}) \xrightarrow[\cong]{\mathrm{op}_x} L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E}).$$

Via op_x we transfer the topology, which makes $L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E})$ a Fréchet space.

Moreover, we have the space of parameter-dependent operators of order $-\infty$ which is independent of $\ell \in \mathbb{N}$:

$$\begin{aligned} L^{-\infty}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E}) &= \bigcap_{\mu\in\mathbb{R}} L^{\mu;\ell}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E}) = \mathcal{S}(\mathbb{R}^q, L^{-\infty}(\mathbb{R}^n; E, \tilde{E})) \\ &= \{\mathrm{op}_x(a)(\lambda);\ a \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})\}. \end{aligned}$$

2.2.4 Theorem. a) *Let E, $\tilde{E}$ and $\hat{E}$ be Hilbert spaces. Let $A(\lambda) = op_x(a)(\lambda) \in L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; \tilde{E}, \hat{E})$ and $B(\lambda) = op_x(b)(\lambda) \in L^{\mu';\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; E, \tilde{E})$ with $a \in S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \hat{E})$ and $b \in S^{\mu';\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$. Then the composition as operators on $\mathcal{S}(\mathbb{R}^n, E)$ belongs to $L^{\mu+\mu';\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; E, \hat{E})$.*

More precisely, we have $A(\lambda)B(\lambda) = C(\lambda) = op_x(a\#b)(\lambda)$ *with the symbol* $a\#b \in S^{\mu+\mu';\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \hat{E})$ *given by the oscillatory integral formula*

$$a\#b(x,\xi,\lambda) = \iint e^{-iy\eta} a(x,\xi+\eta,\lambda) b(x+y,\xi,\lambda)\, dy\, đ\eta. \tag{2.2.i}$$

Moreover, the following asymptotic expansion holds for $a\#b$:

$$a\#b \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} (\partial_\xi^\alpha a)(D_x^\alpha b). \tag{2.2.ii}$$

The mapping $(a,b) \mapsto a\#b$ *is bilinear and continuous. The symbol* $a\#b$ *is called the* Leibniz-product *of* a *and* b.

b) *Let* $\{E_0, E, E_1\}$ *and* $\{\tilde{E}_0, \tilde{E}, \tilde{E}_1\}$ *be Hilbert triples, and* $A(\lambda) = op_x(a)(\lambda) \in L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; E_0, \tilde{E}_0)$ *with the symbol* $a \in S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E_0, \tilde{E}_0)$. *Then the formal adjoint operators belong to* $L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; \tilde{E}_1, E_1)$, *i. e., for* $u \in \mathcal{S}(\mathbb{R}^n, E_0)$ *and* $v \in \mathcal{S}(\mathbb{R}^n, \tilde{E}_1)$ *we have* $\langle A(\lambda)u, v\rangle_{L^2(\mathbb{R}^n,\tilde{E})} = \langle u, A(\lambda)^{(*)}v\rangle_{L^2(\mathbb{R}^n,E)}$ *with* $A(\lambda)^{(*)} = op_x(a^{(*)})(\lambda)$, *where* $a^{(*)} \in S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}_1, E_1)$ *is given by means of the oscillatory integral*

$$a^{(*)}(x,\xi,\lambda) = \iint e^{-iy\eta} a^*(x+y,\xi+\eta,\lambda)\, dy\, đ\eta, \tag{2.2.iii}$$

and the following asymptotic expansion is valid:

$$a^{(*)} \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha a^*. \tag{2.2.iv}$$

The mapping $a \mapsto a^{(*)}$ *is antilinear and continuous. The symbol* $a^{(*)}$ *is called the* adjoint symbol *to* a.

Proof. To prove a), we associate to the operators $B(\lambda)$ the right-symbol $b_R(x',\xi,\lambda)$ according to Theorem 2.2.2. Then the composition $A(\lambda)B(\lambda)$ has the double-symbol $c(x,x',\xi,\lambda) = a(x,\xi,\lambda)b_R(x',\xi,\lambda) \in S^{\mu+\mu';\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \hat{E})$. Employing again 2.2.2, we obtain $a\#b$ as the corresponding left-symbol associated to c. This also implies the continuity of the bilinear mapping $(a,b) \mapsto a\#b$.

Assertion b) follows directly from Theorem 2.2.2, noting that $\big(a(x',\xi,\lambda)\big)^*$ is the right-symbol for $A(\lambda)^{(*)}$. □

2.2.5 Remark. As an immediate consequence of Theorem 2.2.2 we obtain the pseudolocality property of the operators:

Let $A(\lambda) \in L^{\mu;\ell}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$ be given by $A(\lambda) = \mathrm{op}_x(a)(\lambda)$ with a double-symbol $a(x,x',\xi,\lambda) \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;E,\tilde{E})$, such that $a(x,x',\xi,\lambda) \equiv 0$ for $|x-x'| < \varepsilon$ for a sufficiently small $\varepsilon > 0$. Then $A(\lambda) \in L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$.

In particular, if $A(\lambda) \in L^{\mu;\ell}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$, and $\varphi,\psi \in C_b^\infty(\mathbb{R}^n)$ such that $\mathrm{dist}(\mathrm{supp}\varphi,\mathrm{supp}\psi) > 0$, then $\varphi A(\lambda)\psi \in L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$.

2.2.6 Definition. Let $A(\lambda) = \mathrm{op}_x(a)(\lambda) \in L_{cl}^{\mu;\ell}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$, where $a \in S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;E,\tilde{E})$. By Theorem 2.2.2 the symbol a is uniquely determined by $A(\lambda)$, and so are the homogeneous components of a by (2.1.vi). We define $\sigma_\wedge^{\mu;\ell}(A)(x,\xi,\lambda) := a_{(\mu)}(x,\xi,\lambda)$ as the homogeneous component of highest order and call $\sigma_\wedge^{\mu;\ell}(A)$ the parameter-dependent homogeneous principal symbol of $A(\lambda)$ or simply *principal symbol*. The mapping $A(\lambda) \mapsto \sigma_\wedge^{\mu;\ell}(A)$ is continuous.

In case of $E = \mathbb{C}^{N_-}$ and $\tilde{E} = \mathbb{C}^{N_+}$ we write as usual $\sigma_\psi^{\mu;\ell}(A)$ instead of $\sigma_\wedge^{\mu;\ell}(A)$.

2.2.7 Remark. With the notations of Theorem 2.2.4 we obtain for classical operators the following relations for the principal symbols of compositions and adjoints: $\sigma_\wedge^{\mu+\mu';\ell}(AB) = \sigma_\wedge^{\mu;\ell}(A)\sigma_\wedge^{\mu';\ell}(B)$ and $\sigma_\wedge^{\mu;\ell}(A^{(*)}) = \sigma_\wedge^{\mu;\ell}(A)^*$.

This follows from the asymptotic expansions for the Leibniz-product and the adjoint symbol in Theorem 2.2.4.

Ellipticity and parametrices

2.2.8 Definition. A symbol $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;E,\tilde{E})$ is called *parameter-dependent elliptic*, if there is a symbol $b \in S^{-\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\tilde{E},E)$ such that

$$a{\cdot}b - 1 \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\tilde{E},\tilde{E}),$$
$$b{\cdot}a - 1 \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;E,E)$$

for some $\varepsilon > 0$.

Let $K \Subset \mathbb{R}^n$ be compact. A symbol $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;E,\tilde{E})$ is called parameter-dependent elliptic on K, if there is a symbol $b \in S^{-\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\tilde{E},E)$ such that $ab-1$ and $ba-1$ coincide with symbols of order $-\varepsilon$ for some $\varepsilon > 0$ in a neighbourhood $U(K)$ of K.

In particular we see, that the condition of parameter-dependent ellipticity is not affected by perturbations of lower-order terms.

An operator $A(\lambda) = \mathrm{op}_x(a)(\lambda) \in L^{\mu;\ell}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$ is called *parameter-dependent elliptic (on K)*, if a is parameter-dependent elliptic (on K).

2.2.9 Remark. The following characterizations of parameter-dependent ellipticity are valid:

a) Let $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$. Then a is parameter-dependent elliptic if and only if for some $R > 0$ there exists $(a(x,\xi,\lambda))^{-1} \in \mathcal{L}(\tilde{E}, E)$ for all $x \in \mathbb{R}^n$, $(\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^q$ with $|\xi,\lambda|_\ell \geq R$, and

$$\sup\{\|(a(x,\xi,\lambda))^{-1}\| \, \langle\xi,\lambda\rangle_\ell^\mu \,;\ x \in \mathbb{R}^n,\ |\xi,\lambda|_\ell \geq R\} < \infty.$$

If $a \in S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$, then a is parameter-dependent elliptic if and only if the homogeneous component $a_{(\mu)}(x,\xi,\lambda) \in \mathcal{L}(E, \tilde{E})$ of highest order is invertible for all $x \in \mathbb{R}^n$ and $0 \neq (\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^q$ and

$$\sup\{\|(a_{(\mu)}(x,\xi,\lambda))^{-1}\|;\ x \in \mathbb{R}^n,\ |\xi,\lambda|_\ell = 1\} < \infty.$$

b) Let $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ and $K \Subset \mathbb{R}^n$ be compact. Then a is parameter-dependent elliptic on K if and only if for some $R > 0$ there exists $(a(x,\xi,\lambda))^{-1} \in \mathcal{L}(\tilde{E}, E)$ for all $x \in K$, $(\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^q$ with $|\xi,\lambda|_\ell \geq R$, and

$$\sup\{\|(a(x,\xi,\lambda))^{-1}\| \, \langle\xi,\lambda\rangle_\ell^\mu \,;\ x \in K,\ |\xi,\lambda|_\ell \geq R\} < \infty.$$

If $a \in S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$, then a is parameter-dependent elliptic on K if and only if $a_{(\mu)}(x,\xi,\lambda) \in \mathcal{L}(E, \tilde{E})$ is invertible for all $x \in K$ and $0 \neq (\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^q$.

c) Let $a \in S_{cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$. Then a is parameter-dependent elliptic if and only if there exists $b \in S_{cl}^{-\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, E)$ such that $ab - 1 \in S_{cl}^{-1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \tilde{E})$ and $ba - 1 \in S_{cl}^{-1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, E)$.

   a is parameter-dependent elliptic on a compact set $K \Subset \mathbb{R}^n$ if and only if there exists $b \in S_{cl}^{-\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, E)$ such that $ab - 1$ and $ba - 1$ coincide with classical symbols of order -1 in a neighbourhood $U(K)$ of K.

Proof. Note first that in view of Definition 2.2.8 the conditions in a) and b) are clearly necessary for parameter-dependent ellipticity. To prove the sufficiency of the conditions in a) let $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q)$ such that $\chi \equiv 0$ for $|\xi,\lambda|_\ell \leq R + 1$ and $\chi \equiv 1$ for $|\xi,\lambda|_\ell \geq R + 2$. For $(x,\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q$ define

$$b(x,\xi,\lambda) := \begin{cases} \chi(\xi,\lambda)(a(x,\xi,\lambda))^{-1} & \text{in the general case} \\ \chi(\xi,\lambda)(a_{(\mu)}(x,\xi,\lambda))^{-1} & \text{in the classical case.} \end{cases}$$

Thus we see that $b \in S_{(cl)}^{-\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, E)$, and moreover $ab - 1 \in S_{(cl)}^{-1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \tilde{E})$ and $ba - 1 \in S_{(cl)}^{-1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, E)$. This proves a) and the first assertion in c).

Now assume that the conditions in b) hold. Note that they not only hold for $x \in K$, but also in a neighbourhood $V(K)$ of K. Let $\hat{\varphi} \in C_0^\infty(V(K))$ with $\hat{\varphi} \equiv 1$

in a neighbourhood $U(K)$. Let $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q)$ such that $\chi \equiv 0$ for $|\xi,\lambda|_\ell \le R+1$ and $\chi \equiv 1$ for $|\xi,\lambda|_\ell \ge R+2$. For $(x,\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q$ define

$$b(x,\xi,\lambda) := \begin{cases} \chi(\xi,\lambda)\hat{\varphi}(x)(a(x,\xi,\lambda))^{-1} & \text{in the general case} \\ \chi(\xi,\lambda)\hat{\varphi}(x)(a_{(\mu)}(x,\xi,\lambda))^{-1} & \text{in the classical case.} \end{cases}$$

We thus see that $b \in S^{-\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, E)$ with $ab - \hat{\varphi}I \in S^{-1;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \tilde{E})$ and $ba - \hat{\varphi}I \in S^{-1;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, E)$ which shows b) and completes the proof of c). □

2.2.10 Theorem. *Let $A(\lambda) \in L^{\mu;\ell}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$. Then the following are equivalent:*

a) $A(\lambda)$ is parameter-dependent elliptic.

b) There exists an operator $P(\lambda) \in L^{-\mu;\ell}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},E)$, such that $A(\lambda)P(\lambda)-1 \in L^{-\varepsilon;\ell}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},\tilde{E})$ and $P(\lambda)A(\lambda)-1 \in L^{-\varepsilon;\ell}(\mathbb{R}^n;\mathbb{R}^q;E,E)$ for some $\varepsilon > 0$.

c) There exists an operator $P(\lambda) \in L^{-\mu;\ell}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},E)$, such that $A(\lambda)P(\lambda)-1 \in L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},\tilde{E})$ and $P(\lambda)A(\lambda)-1 \in L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;E,E)$.

If even $A(\lambda) \in L^{\mu;\ell}_{cl}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$ is parameter-dependent elliptic then every $P(\lambda)$ satisfying c) belongs to $L^{-\mu;\ell}_{cl}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},E)$.

Every $P(\lambda) \in L^{-\mu;\ell}_{(cl)}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},E)$ satisfying c) is called a (parameter-dependent) parametrix *of $A(\lambda)$.*

Proof. Assume that a) holds. Let $A(\lambda) = \mathrm{op}_x(a)(\lambda)$ with $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$. Let $b \in S^{-\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, E)$ satisfying the condition of Definition 2.2.8. Now the asymptotic expansion of the Leibniz-product in Theorem 2.2.4 (2.2.ii) gives that $b\#a - 1 \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, E)$ and $a\#b - 1 \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \tilde{E})$ for some $\varepsilon > 0$ which implies b). If $a \in S^{\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, \tilde{E})$ we choose $b \in S^{-\mu;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, E)$ satisfying condition c) of Remark 2.2.9. We then even obtain $b\#a - 1 \in S^{-1;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E, E)$ and $a\#b - 1 \in S^{-1;\ell}_{cl}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \tilde{E})$.

Now assume that b) is fulfilled. Let $P(\lambda) = \mathrm{op}_x(b)(\lambda)$ and $A(\lambda)P(\lambda) = 1 - \mathrm{op}_x(r)(\lambda)$ with $r \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \tilde{E})$. From Theorem 2.1.8 and Theorem 2.2.4 we see that there is a symbol $c \in S^{-\varepsilon;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; \tilde{E}, \tilde{E})$ such that $c \sim \sum_{j\in\mathbb{N}} \#^{(j)} r$. Now define $P_{(R)}(\lambda) := \mathrm{op}_x(b\#(1+c))(\lambda)$. Then we have $A(\lambda)P_{(R)}(\lambda) - 1 \in L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},\tilde{E})$ as desired. Analogously, we obtain a parametrix $P_{(L)}(\lambda)$ from the left. But both the left- and the right-parametrix differ only by a term in $L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},E)$ which we see from considering the product $P_{(L)}(\lambda)A(\lambda)P_{(R)}(\lambda)$. This implies c). Note that if we had started with the case $\varepsilon = 1$ and $P(\lambda)$ as well as the remainder being classical, we would have obtained also a classical parametrix which proves the second assertion of the theorem (cf. Remark 2.1.10).

c) implies a) follows at once from Theorem 2.2.4. □

2.2.11 Corollary. *Let $A(\lambda) \in L^{\mu;\ell}_{(cl)}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$ and $K \Subset \mathbb{R}^n_x$ be compact. Then $A(\lambda)$ is parameter-dependent elliptic on K if and only if there are $\varphi,\ \psi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi\psi = \varphi$, $\varphi \equiv 1$ on K, and $P(\lambda) \in L^{-\mu;\ell}_{(cl)}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},E)$ such that $\varphi(A(\lambda)P(\lambda) - 1)\psi \in L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;\tilde{E},\tilde{E})$ and $\varphi(P(\lambda)A(\lambda) - 1)\psi \in L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;E,E)$.*

Sobolev spaces and continuity

2.2.12 Definition. Let E be a Hilbert space. For $s \in \mathbb{R}$ the Sobolev space $H^s(\mathbb{R}^n,E)$ is defined to consist of all $u \in \mathcal{S}'(\mathbb{R}^n,E)$ such that $\mathcal{F}u$ is a regular distribution, and

$$\|u\|_{H^s(\mathbb{R}^n,E)} := \Big(\int_{\mathbb{R}^n} \langle\xi\rangle^{2s}\|\mathcal{F}u(\xi)\|_E^2\,d\xi\Big)^{\frac{1}{2}} < \infty.$$

In case of $E = \mathbb{C}$ the space is suppressed from the notation.

2.2.13 Theorem. *Let E and $\tilde{E}$ be Hilbert spaces. Let $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;E,\tilde{E})$ and $s,\nu \in \mathbb{R}$ where $\nu \geq \mu$. Then $op_x(a)(\lambda)$ extends for $\lambda \in \mathbb{R}^q$ by continuity to an operator $op_x(a)(\lambda) \in \mathcal{L}(H^s(\mathbb{R}^n,E),H^{s-\nu}(\mathbb{R}^n,\tilde{E}))$, and we have the following estimate for the norm:*

$$\|op_x(a)(\lambda)\|_{\mathcal{L}(H^s(\mathbb{R}^n,E),H^{s-\nu}(\mathbb{R}^n,\tilde{E}))} \leq \begin{cases} C_{s,\nu}\langle\lambda\rangle^{\frac{\mu}{\ell}} & \nu \geq 0 \\ C_{s,\nu}\langle\lambda\rangle^{\frac{\mu-\nu}{\ell}} & \nu \leq 0, \end{cases} \tag{2.2.v}$$

where $C_{s,\nu} > 0$ is a constant depending on s,ν and a.

More precisely, this induces a continuous embedding

$$L^{\mu;\ell}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E}) \hookrightarrow \begin{cases} S^{\frac{\mu}{\ell}}(\mathbb{R}^q;H^s(\mathbb{R}^n,E),H^{s-\nu}(\mathbb{R}^n,\tilde{E})) & \nu \geq 0 \\ S^{\frac{\mu-\nu}{\ell}}(\mathbb{R}^q;H^s(\mathbb{R}^n,E),H^{s-\nu}(\mathbb{R}^n,\tilde{E})) & \nu \leq 0 \end{cases} \tag{2.2.vi}$$

into the space of operator-valued symbols in the Sobolev spaces.

Coordinate invariance

2.2.14 Definition. Let $U \subseteq \mathbb{R}^n$ be an open set. Then $A(\lambda) \in L^{\mu;\ell}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$ is said to be *compactly supported in* U, if for some $\varphi,\psi \in C_0^\infty(U)$, and some $B(\lambda) \in L^{\mu;\ell}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$, we have $A(\lambda) = \varphi B(\lambda)\psi$.

In other words: $A(\lambda)$ is compactly supported in U if and only if there is a compact set $K \subseteq U \times U$ such that

$$\mathrm{supp} K_{A(\lambda)} \subseteq K \text{ for all } \lambda \in \mathbb{R}^q, \tag{2.2.vii}$$

where $K_{A(\lambda)} \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(E,\tilde{E}))$ denotes the operator-valued Schwartz kernel of the operator $A(\lambda)$.

For each compact set $K \subseteq U \times U$ the space of compactly supported operators $A(\lambda) \in L^{\mu;\ell}_{(cl)}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$ satisfying (2.2.vii) is a closed subspace of $L^{\mu;\ell}_{(cl)}(\mathbb{R}^n;\mathbb{R}^q;E,\tilde{E})$.

Let $L^{\mu;\ell}_{\mathrm{comp}\,(cl)}(U;\mathbb{R}^q;E,\tilde{E})$ denote the space of all (classical) parameter-dependent pseudodifferential operators that are compactly supported in U. We endow this space with the inductive limit topology of the subspaces of operators with Schwartz kernels satisfying (2.2.vii) (taken over all compact sets $K \subseteq U \times U$). Thus it becomes a strict countable inductive limit of Fréchet spaces.

Note that $A(\lambda) = \mathrm{op}_x(a)(\lambda) \in L^{\mu;\ell}_{\mathrm{comp}}(U;\mathbb{R}^q;E,\tilde{E})$ acts as a family of continuous operators $A(\lambda) : C_0^\infty(U,E) \to C_0^\infty(U,\tilde{E})$, and its symbol $a(x,\xi,\lambda)$ is uniquely determined by this action.

2.2.15 Theorem. *Let U, $V \subseteq \mathbb{R}^n$ be open subsets and $\chi : U \to V$ a diffeomorphism. Then the operator pull-back $\chi^* A(\lambda)$ defined as*

$$(\chi^* A(\lambda))u := \chi^*(A(\lambda)(\chi_* u)) \tag{2.2.viii}$$

for $u \in C_0^\infty(U,E)$ and $A(\lambda) \in L^{\mu;\ell}_{comp}(V;\mathbb{R}^q;E,\tilde{E})$, with the pull-back χ^* and push-forward χ_* for C_0^∞-functions, defines a topological isomorphism $\chi^* : L^{\mu;\ell}_{comp\,(cl)}(V;\mathbb{R}^q;E,\tilde{E}) \to L^{\mu;\ell}_{comp\,(cl)}(U;\mathbb{R}^q;E,\tilde{E})$.

Moreover, given $A(\lambda) = op_x(a)(\lambda) \in L^{\mu;\ell}_{comp\,(cl)}(V;\mathbb{R}^q;E,\tilde{E})$, then $\chi^* A(\lambda) = op_x(b)(\lambda)$ with a symbol $b \in S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;E,\tilde{E})$ having the following asymptotic expansion in terms of a and χ:

$$b(x,\xi,\lambda) \sim \sum_{\alpha \in \mathbb{N}_0^n} (\partial_\xi^\alpha a)(\chi(x), [D\chi(x)^{-1}]^t \xi, \lambda)\varphi_\alpha(x,\xi) \tag{2.2.ix}$$

with polynomials $\varphi_\alpha(x,\xi)$ in ξ of degree less or equal to $\frac{|\alpha|}{2}$ and $\varphi_0 \equiv 1$, that are given completely in terms of the diffeomorphism χ.

Note that the symbol a vanishes identically outside a compact set in V which gives a meaning to this asymptotic expansion.

In particular, we obtain $b(x,\xi,\lambda) - a(\chi(x),[D\chi(x)^{-1}]^t\xi,\lambda) \in S^{\mu-1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q; E,\tilde{E})$. This yields in the classical case to the following relation for the principal symbols:

$$\sigma_\wedge^{\mu;\ell}(\chi^*A)(x,\xi,\lambda) = \sigma_\wedge^{\mu;\ell}(A)(\chi(x),[D\chi(x)^{-1}]^t\xi,\lambda). \tag{2.2.x}$$

This also shows, that $\chi^*A(\lambda)$ is parameter-dependent elliptic on a compact set $K \subseteq U$ if and only if $A(\lambda)$ is parameter-dependent elliptic on $\chi(K) \subseteq V$.

2.3 Parameter-dependent Volterra symbols

Let

$$\mathbb{H} := \{z \in \mathbb{C};\ \mathrm{Im}(z) \geq 0\} \subseteq \mathbb{C} \cong \mathbb{R}^2$$

be the upper half-plane in $\mathbb{C}$. The significant property of Volterra operators, resp. symbols with the Volterra property is, that in addition to the symbol estimates we employ the analyticity in the interior of $\mathbb{H}$.

2.3.1 Definition. Let E and $\tilde{E}$ be Hilbert spaces. For $\mu \in \mathbb{R}$ we define

$$S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E,\tilde{E}) := S_{(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E,\tilde{E}) \cap \mathcal{A}(\mathring{\mathbb{H}}, C^\infty(\mathbb{R}^n, \mathcal{L}(E,\tilde{E}))),$$

which is a closed subspace of $S_{(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E,\tilde{E})$. Analogously, we define

$$S_V^{-\infty}(\mathbb{R}^n \times \mathbb{H}; E,\tilde{E}) := \bigcap_{\mu\in\mathbb{R}} S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E,\tilde{E}),$$

as well as the spaces of x- (resp. x'-) and (x,x')-dependent symbols

$$S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E,\tilde{E}) := C_b^\infty(\mathbb{R}^n, S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E,\tilde{E})),$$
$$S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E,\tilde{E}) := C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n, S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E,\tilde{E})).$$

These symbols are called symbols with the *Volterra property* or simply *Volterra symbols* which is indicated by the subscript V.

Of course, this notion also applies to the case of scales of Hilbert spaces involved instead of the single spaces only, and we shall employ the same conventions as in the case without the extra analyticity condition, see Definition 2.1.3.

2.3.2 Proposition. a) *The restriction of the parameter to the real line induces a continuous embedding $S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E,\tilde{E}) \hookrightarrow S_{(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E,\tilde{E})$.*

b) *The homogeneous components of a symbol $a \in S_{V\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E,\tilde{E})$ are analytic in $\mathring{\mathbb{H}}$.*

Kernel cut-off and asymptotic expansion of Volterra symbols

2.3.3 Definition. Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces. For short, we set

$$\mathcal{E} := \operatorname*{ind\text{-}lim}_{j\in\mathbb{N}} E_j \quad \text{and} \quad \tilde{\mathcal{E}} := \operatorname*{proj\text{-}lim}_{j\in\mathbb{N}} \tilde{E}_j.$$

Let $(\mu_k) \subseteq \mathbb{R}$ be a sequence of reals such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max_{k\in\mathbb{N}} \mu_k$. Moreover, let $a_k \in S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$. A symbol $a \in S_V^{\overline{\mu};\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ is called the *asymptotic expansion* of the a_k, if for every $R \in \mathbb{R}$ there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$

$$a - \sum_{j=1}^{k} a_j \in S_V^{R;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}).$$

The symbol a is uniquely determined modulo $S_V^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$.

For short we write $a \underset{V}{\sim} \sum_{j=1}^{\infty} a_j$.

2.3.4 Remark. Note that the notion of asymptotic expansion for Volterra symbols from Definition 2.3.3 is more refined than that of Definition 2.1.6. What makes things more complicated is the extra analyticity condition. In particular, the standard excision function arguments in the proof of the existence of symbols having a prescribed asymptotic expansion, see also Lemma 2.1.7, cannot be applied to obtain corresponding existence results in the Volterra case. The substitute for these are kernel cut-off techniques, see also Proposition 2.3.8.

2.3.5 Definition. Let E and $\tilde{E}$ be Hilbert spaces, and let $\varphi \in C_b^\infty(\mathbb{R})$. On $S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$ define the *kernel cut-off operator* $H(\varphi)$ by means of the following oscillatory integral:

$$\big(H(\varphi)a\big)(\xi,\lambda) := \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-it\tau}\varphi(t)a(\xi,\lambda-\tau)\,dt\,đ\tau \tag{2.3.i}$$

for $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$.

2.3.6 Theorem. *Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces. We again use the abbreviations $\mathcal{E}$ for the inductive limit of the spaces E_j, as well as $\tilde{\mathcal{E}}$ for the projective limit of the spaces $\tilde{E}_j$. Then the mapping*

$$H : \begin{cases} C_b^\infty(\mathbb{R}) \times S_{(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}}) & \longrightarrow S_{(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}}), \\ C_b^\infty(\mathbb{R}) \times S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}) & \longrightarrow S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}) \end{cases}$$

is bilinear and continuous.

The following asymptotic expansion holds for $H(\varphi)a$ in terms of φ and a:

$$H(\varphi)a \underset{(V)}{\sim} \sum_{k=0}^{\infty} \Big(\frac{(-1)^k}{k!} D_t^k \varphi(0) \Big) \cdot \partial_\lambda^k a \tag{2.3.ii}$$

where ∂_λ denotes the complex derivative with respect to $\lambda \in \mathbb{H}$ in case of Volterra symbols.

2.3.7 Corollary. *Let $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi \equiv 1$ near $t = 0$. Then the operator $I - H(\varphi)$ is coninuous in the spaces*

$$S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}}) \to S^{-\infty}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}}),$$
$$S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}) \to S_V^{-\infty}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}).$$

2.3.8 Proposition. *Let $(\mu_k) \subseteq \mathbb{R}$ such that $\mu_k \geq \mu_{k+1} \underset{k\to\infty}{\longrightarrow} -\infty$. Furthermore, for each $k \in \mathbb{N}$ let $(A_{k_j})_{j\in\mathbb{N}} \subseteq S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ be a countable system of bounded sets. Let $\varphi \in C_0^\infty(\mathbb{R})$, and for $c \in [1, \infty)$ let $\varphi_c \in C_0^\infty(\mathbb{R})$ be defined as $\varphi_c(t) := \varphi(ct)$.*

Then there is a sequence $(c_i) \subseteq [1, \infty)$ with $c_i < c_{i+1} \underset{i\to\infty}{\longrightarrow} \infty$ such that for each $k \in \mathbb{N}$

$$\sum_{i=k}^{\infty} \sup_{a\in A_{i_j}} p\big(H(\varphi_{d_i})a\big) < \infty \tag{2.3.iii}$$

for all continuous seminorms p on $S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ and every $j \in \mathbb{N}$, and for all sequences $(d_i) \subseteq \mathbb{R}_+$ with $d_i \geq c_i$.

2.3.9 Theorem. *Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces, and $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as before. Let $(\mu_k) \subseteq \mathbb{R}$ such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max_{k\in\mathbb{N}} \mu_k$. Moreover, let $a_k \in S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$. Then there exists $a \in S_V^{\overline{\mu};\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ such that $a \underset{V}{\sim} \sum_{j=1}^{\infty} a_j$. The asymptotic sum a is uniquely determined modulo $S_V^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$.*

If the sequence $(\mu_k)_{k\in\mathbb{N}_0}$ is given as $\mu_k = \overline{\mu} - k$ and $a_k \in S_{V\,cl}^{\overline{\mu}-k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$, then also $a \in S_{V\,cl}^{\overline{\mu};\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$.

Proof. For the proof we may without loss of generality assume that $\mu_k \geq \mu_{k+1} \underset{k\to\infty}{\longrightarrow} -\infty$. For $k, j \in \mathbb{N}$ let

$$A_{k_j} := \big\{ \partial_x^\alpha a_k(x);\ x \in \mathbb{R}^n,\ |\alpha| \leq j \big\}.$$

Then $A_{k_j} \subseteq S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ is bounded. Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi \equiv 1$ near $t = 0$. Now apply Proposition 2.3.8. With a suitable sequence $(c_i) \subseteq [1, \infty)$ formula (2.3.iii) becomes

$$\sum_{i=k}^{\infty} \sup\{p\big(H(\varphi_{c_i})(\partial_x^\alpha a_i(x))\big);\ x \in \mathbb{R}^n,\ |\alpha| \leq j\} < \infty$$

for all continuous seminorms p on $S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$.

This shows that $\sum\limits_{i=k}^{\infty} H(\varphi_{c_i})a_i$ is unconditionally convergent in $S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ for every $k \in \mathbb{N}$. Now define

$$a := \sum_{i=1}^{\infty} H(\varphi_{c_i})a_i \in S_V^{\mu_1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}).$$

We thus see

$$a - \sum_{i=1}^{k} a_i = \sum_{i=k+1}^{\infty} H(\varphi_{c_i})a_i - \sum_{i=1}^{k}\big(I - H(\varphi_{c_i})\big)a_i$$

where

$$\sum_{i=1}^{k}\big(I - H(\varphi_{c_i})\big)a_i \in S_V^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$$

in view of Corollary 2.3.7. This yields the desired result, since the uniqueness assertion is clear. □

The translation operator in Volterra symbols

2.3.10 Definition. For $z = i\tau \in i\mathbb{R} \subseteq \mathbb{C}$, $\tau \geq 0$, define the *translation operator* $T_{i\tau}$ on $S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})$ via

$$\big(T_{i\tau}a\big)(\xi, \lambda) := a(\xi, \lambda + i\tau).$$

2.3.11 Proposition. *For every $\tau \geq 0$ the translation operator $T_{i\tau}$ acts linear and continuous in the spaces*

$$T_{i\tau} : S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}).$$

Moreover, $T_{i\tau}a$ has the following asymptotic expansion in terms of τ and a:

$$T_{i\tau}a \underset{V}{\sim} \sum_{k=0}^{\infty} \frac{(i\tau)^k}{k!} \cdot \partial_\lambda^k a.$$

In particular, the operator $I - T_{i\tau}$ is continuous in the spaces

$$I - T_{i\tau} : S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S_{V(cl)}^{\mu-\ell;\ell}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}).$$

2.3.12 Notation. For $\mu \in \mathbb{R}$ let $S^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big)$ denote the closed subspace of $C^\infty\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}, \mathcal{L}(E, \tilde{E})\big)$ consisting of all anisotropic homogeneous functions of degree μ. Moreover, let

$$S_V^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big) := S^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big) \\ \cap \mathcal{A}(\mathring{\mathbb{H}}, C^\infty(\mathbb{R}^n, \mathcal{L}(E, \tilde{E}))),$$

which is a closed subspace of $S^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big)$.

2.3.13 Theorem. *For every $\tau > 0$ the mapping $T_{i\tau} : a(\xi, \lambda) \mapsto a(\xi, \lambda + i\tau)$ is continuous in the spaces*

$$T_{i\tau} : S_V^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big) \longrightarrow S_{V\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}).$$

Moreover, for every 0-excision function $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{H})$, the following asymptotic expansion holds for $T_{i\tau}a$:

$$T_{i\tau}a \sim \sum_{k=0}^{\infty} \frac{(i\tau)^k}{k!} \cdot \chi\big(\partial_\lambda^k a\big).$$

This shows, in particular, that for the homogeneous component of order μ we have the identity $\big(T_{i\tau}a\big)_{(\mu)} = a$.

In other words, the restriction of the "principal symbol sequence" (on symbolic level) to Volterra symbols is topologically exact and splits:

$$0 \longrightarrow S_{V\,cl}^{\mu-1;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \longrightarrow S_{V\,cl}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \longrightarrow \\ S_V^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big) \longrightarrow 0.$$

The operator $T_{i\tau}$ provides a splitting of this sequence. Analogous assertions hold in case of scales of Hilbert spaces involved.

2.4 Parameter-dependent Volterra operators

2.4.1 Remark. In this section we first recall the basic elements of pseudodifferential calculus on $\mathbb{R}^n$ built upon parameter-dependent Volterra symbols, which is rather straightforward in view of Sections 2.2 and 2.3.

Secondly, we study *parabolicity* of such Volterra operators which is defined by requiring the parameter-dependent ellipticity of the symbols. The main point here is that we are in need to construct a parametrix which itself has again the Volterra property. The latter cannot be obtained from Theorem 2.2.10 (or its proof) because there are arguments with excision functions involved, which destroy the analyticity in the interior of the half-plane (see also Remark 2.2.9). However, the possibility to carry out asymptotic expansions, see Theorem 2.3.9, which relies on kernel cut-off techniques, and the translation operator in Volterra symbol spaces provide the tools to handle these difficulties.

2.4.2 Definition. Let E and $\tilde{E}$ be Hilbert spaces. For $\mu \in \mathbb{R}$ the space of *Volterra pseudodifferential operators* respectively *operators with the Volterra property* is defined as

$$\begin{aligned} L^{\mu;\ell}_{V(cl)}(\mathbb{R}^n;\mathbb{H};E,\tilde{E}) &:= \{\mathrm{op}_x(a)(\lambda);\ a \in S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};E,\tilde{E})\} \\ &\subseteq L^{\mu;\ell}_{(cl)}(\mathbb{R}^n;\mathbb{H};E,\tilde{E}). \end{aligned}$$

In case of $E = \tilde{E} = \mathbb{C}$ the spaces are suppressed from the notation as usual.

Elements of the calculus

2.4.3 Theorem. *Let $a \in S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};E,\tilde{E})$. Then the unique left- and right-symbol from Theorem 2.2.2 associated to the operator $\mathrm{op}_x(a)(\lambda) \in L^{\mu;\ell}_{V(cl)}(\mathbb{R}^n;\mathbb{H};E,\tilde{E})$ are Volterra symbols, i. e. $a_L(x,\xi,\lambda), a_R(x',\xi,\lambda) \in S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};E,\tilde{E})$.*

Moreover, the asymptotic expansions for a_L and a_R in terms of a are valid in the Volterra sense:

$$\begin{aligned} a_L(x,\xi,\lambda) &\underset{V}{\sim} \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\partial_\xi^\alpha D_{x'}^\alpha a(x,x',\xi,\lambda)|_{x'=x}, \\ a_R(x',\xi,\lambda) &\underset{V}{\sim} \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}(-1)^{|\alpha|}\partial_\xi^\alpha D_x^\alpha a(x,x',\xi,\lambda)|_{x=x'}. \end{aligned}$$

2.4.4 Remark. By Theorem 2.4.3 the mapping op_x restricts to an isomorphism between the space of Volterra left-symbols and Volterra pseudodifferential operators:

$$S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};E,\tilde{E}) \xrightarrow[\cong]{\mathrm{op}_x} L^{\mu;\ell}_{V(cl)}(\mathbb{R}^n;\mathbb{H};E,\tilde{E}).$$

Consequently, $L^{\mu;\ell}_{V(cl)}(\mathbb{R}^n;\mathbb{H};E,\tilde{E})$ is a closed subspace of $L^{\mu;\ell}_{(cl)}(\mathbb{R}^n;\mathbb{H};E,\tilde{E})$.

The space of parameter-dependent Volterra operators of order $-\infty$ is denoted by

$$\begin{aligned} L_V^{-\infty}(\mathbb{R}^n;\mathbb{H};E,\tilde{E}) &= \bigcap_{\mu\in\mathbb{R}} L_V^{\mu;\ell}(\mathbb{R}^n;\mathbb{H};E,\tilde{E}) \\ &= \{\mathrm{op}_x(a)(\lambda);\ a \in S_V^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};E,\tilde{E})\}. \end{aligned}$$

In view of Proposition 2.3.2 the restriction of the parameter to the real line induces a continuous embedding

$$L^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}; E, \tilde{E}) \hookrightarrow L^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E}).$$

2.4.5 Theorem. *Let E, $\tilde{E}$ and $\hat{E}$ be Hilbert spaces, and $A(\lambda) = op_x(a)(\lambda) \in L^{\mu;\ell}_{V(cl)}(\mathbb{R}^n; \mathbb{H}; \tilde{E}, \hat{E})$ as well as $B(\lambda) = op_x(b)(\lambda) \in L^{\mu';\ell}_{V(cl)}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$ with $a \in S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, \hat{E})$ and $b \in S^{\mu';\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$.*

Then the composition as operators on $\mathcal{S}(\mathbb{R}^n, E)$ belongs to $L^{\mu+\mu';\ell}_{V(cl)}(\mathbb{R}^n; \mathbb{H}; E, \hat{E})$, i. e., the Leibniz-product $a\#b$ of the symbols a and b (cf. Theorem 2.2.4) belongs to $S^{\mu+\mu';\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \hat{E})$.

Moreover, for $a\#b$ the following asymptotic expansion holds:

$$a\#b \underset{V}{\sim} \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} (\partial_\xi^\alpha a)(D_x^\alpha b). \tag{2.4.i}$$

2.4.6 Remark. From Theorem 2.4.3 we conclude the pseudolocality property of the parameter-dependent Volterra calculus (see also Remark 2.2.5):

Let $A(\lambda) \in L^{\mu;\ell}_V(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$ be given by $A(\lambda) = \mathrm{op}_x(a)(\lambda)$ with a double-symbol $a(x, x', \xi, \lambda) \in S^{\mu;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$, such that $a(x, x', \xi, \lambda) \equiv 0$ for $|x - x'| < \varepsilon$ for a sufficiently small $\varepsilon > 0$. Then $A(\lambda) \in L^{-\infty}_V(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$.

In particular, if $\varphi, \psi \in C_b^\infty(\mathbb{R}^n)$ such that $\mathrm{dist}(\mathrm{supp}\varphi, \mathrm{supp}\psi) > 0$, then $\varphi A(\lambda)\psi \in L^{-\infty}_V(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$. The mapping $L^{\mu;\ell}_V(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}) \ni A(\lambda) \longmapsto \varphi A(\lambda)\psi \in L^{-\infty}_V(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$ is continuous.

2.4.7 Theorem. *For every $\mu \in \mathbb{R}$ the principal symbol sequence in Volterra pseudodifferential operators is topologically exact and splits:*

$$\begin{aligned} 0 \longrightarrow L^{\mu-1;\ell}_{V\,cl}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}) &\stackrel{\imath}{\longrightarrow} L^{\mu;\ell}_{V\,cl}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}) \stackrel{\sigma^{\mu;\ell}_\wedge}{\longrightarrow} \\ &C_b^\infty\big(\mathbb{R}^n, S_V^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; E, \tilde{E}\big)\big) \longrightarrow 0. \end{aligned}$$

The translation operator $T_{i\tau}$ for $\tau > 0$ gives rise to a splitting of this sequence. Analogous assertions hold in case of scales of Hilbert spaces involved.

Continuity and coordinate invariance

2.4.8 Theorem. *Let E and $\tilde{E}$ be Hilbert spaces, and let $a \in S^{\mu;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$, as well as $s, \nu \in \mathbb{R}$ with $\nu \geq \mu$. Then $op_x(a)(\lambda)$ extends for $\lambda \in \mathbb{H}$ by* *continuity to an operator* $op_x(a)(\lambda) \in \mathcal{L}(H^s(\mathbb{R}^n, E), H^{s-\nu}(\mathbb{R}^n, \tilde{E}))$*, which induces a continuous embedding*

$$L_V^{\mu;\ell}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E}) \hookrightarrow \begin{cases} S_V^{\frac{\mu}{\ell}}(\mathbb{H}; H^s(\mathbb{R}^n, E), H^{s-\nu}(\mathbb{R}^n, \tilde{E})) & \nu \geq 0 \\ S_V^{\frac{\mu-\nu}{\ell}}(\mathbb{H}; H^s(\mathbb{R}^n, E), H^{s-\nu}(\mathbb{R}^n, \tilde{E})) & \nu \leq 0 \end{cases} \tag{2.4.ii}$$

into the space of operator-valued Volterra symbols in the Sobolev spaces.

2.4.9 Remark. Let $U \subseteq \mathbb{R}^n$ be an open set. Recall from Section 2.2 that an operator $A(\lambda) \in L^{\mu;\ell}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$ is compactly supported in U if and only if there is a compact set $K \subseteq U \times U$ such that

$$\mathrm{supp} K_{A(\lambda)} \subseteq K \text{ for all } \lambda \in \mathbb{H} \tag{2.4.iii}$$

where $K_{A(\lambda)} \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(E, \tilde{E}))$ denotes the operator-valued Schwartz kernel of the operator $A(\lambda)$.

For each compact set $K \subseteq U \times U$ the space of compactly supported Volterra operators $A(\lambda) \in L_{V(cl)}^{\mu;\ell}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$ satisfying (2.4.iii) is a closed subspace of $L_{V(cl)}^{\mu;\ell}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$.

Let $L_{\mathrm{comp}\, V(cl)}^{\mu;\ell}(U; \mathbb{H}; E, \tilde{E})$ denote the space of all (classical) parameter-dependent Volterra pseudodifferential operators that are compactly supported in U. This space is endowed with the inductive limit topology of the subspaces of operators with Schwartz kernels satisfying (2.4.iii) (taken over all compact sets $K \subseteq U \times U$), and thus it is a closed subspace of $L_{\mathrm{comp}\,(cl)}^{\mu;\ell}(U; \mathbb{H}; E, \tilde{E})$.

2.4.10 Theorem. *Let* $U, V \subseteq \mathbb{R}^n$ *be open subsets and* $\chi : U \to V$ *a diffeomorphism. In view of Theorem 2.2.15 the operator pull-back* χ^* *(cf. (2.2.viii)) induces a topological isomorphism*

$$\chi^* : L_{comp\,(cl)}^{\mu;\ell}(V; \mathbb{H}; E, \tilde{E}) \to L_{comp\,(cl)}^{\mu;\ell}(U; \mathbb{H}; E, \tilde{E}).$$

Its restriction to the spaces of compactly supported Volterra pseudodifferential operators acts as a topological isomorphism

$$\chi^* : L_{comp\, V(cl)}^{\mu;\ell}(V; \mathbb{H}; E, \tilde{E}) \to L_{comp\, V(cl)}^{\mu;\ell}(U; \mathbb{H}; E, \tilde{E}).$$

Moreover, given $A(\lambda) = op_x(a)(\lambda) \in L_{comp\, V(cl)}^{\mu;\ell}(V; \mathbb{H}; E, \tilde{E})$*, then* $\chi^* A(\lambda) = op_x(b)(\lambda)$ *with a symbol* $b \in S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ *having the following asymptotic expansion in the Volterra sense in terms of* a *and* χ*:*

$$b(x, \xi, \lambda) \underset{V}{\sim} \sum_{\alpha \in \mathbb{N}_0^n} (\partial_\xi^\alpha a)(\chi(x), [D\chi(x)^{-1}]^t \xi, \lambda) \varphi_\alpha(x, \xi) \tag{2.4.iv}$$

with the universal polynomials $\varphi_\alpha(x,\xi)$ *in* ξ *of degree less or equal to* $\frac{|\alpha|}{2}$ *and* $\varphi_0 \equiv 1$ *depending only on the diffeomorphism* χ *from the asymptotic expansion (2.2.ix) in Theorem 2.2.15.*

In particular, we obtain $b(x,\xi,\lambda) - a(\chi(x),[D\chi(x)^{-1}]^t\xi,\lambda) \in S_V^{\mu-1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$.

Parabolicity for Volterra pseudodifferential operators

2.4.11 Definition. A symbol $a \in S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ is called *parabolic*, if a is parameter-dependent elliptic as an element in $S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$.

Let $K \Subset \mathbb{R}^n$ be compact. A symbol $a \in S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ is called parabolic on K, if a is parameter-dependent elliptic on K as an element in $S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$.

An operator $A(\lambda) = \mathrm{op}_x(a)(\lambda) \in L_V^{\mu;\ell}(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$ is called *parabolic (on K)*, if a is parabolic (on K).

2.4.12 Proposition. *Let* $a \in S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$.

a) a is parabolic if and only if there exists an element $b \in S_{V(cl)}^{-\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, E)$ *such that*

$$a{\cdot}b - 1 \in S_{V(cl)}^{-1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, \tilde{E}),$$
$$b{\cdot}a - 1 \in S_{V(cl)}^{-1;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, E).$$

b) a is parabolic on a compact set $K \Subset \mathbb{R}^n$ *if and only if there exists an element* $b \in S_{V(cl)}^{-\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, E)$ *such that* $a{\cdot}b - 1$ *and* $b{\cdot}a - 1$ *coincide with (classical) Volterra symbols of order* -1 *in a neighbourhood* $U(K)$ *of* K.

Proof. We only have to prove the necessity of the conditions in a) and b), for the sufficiency follows immediately from the definition of parabolicity as parameter-dependent ellipticity (see Definition 2.2.8 and Remark 2.2.9). Assume that $a \in S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ is parabolic. According to Remark 2.2.9, for some sufficiently large $R > 0$ there exists $(a(x,\xi,\lambda))^{-1} \in \mathcal{L}(\tilde{E}, E)$ for all $x \in \mathbb{R}^n$, $(\xi,\lambda) \in \mathbb{R}^n \times \mathbb{H}$ with $|\xi,\lambda|_\ell \geq R$, and

$$\sup\{\|(a(x,\xi,\lambda))^{-1}\| \, \langle \xi,\lambda\rangle_\ell^\mu \, ; \, x \in \mathbb{R}^n, \, |\xi,\lambda|_\ell \geq R\} < \infty.$$

Consequently, if we choose $\tau \in \mathbb{R}_+$ sufficiently large, we conclude that for all $x \in \mathbb{R}^n$ and all $(\xi,\lambda) \in \mathbb{R}^n \times \mathbb{H}$ there exists $\big((T_{i\tau}a)(x,\xi,\lambda)\big)^{-1} \in \mathcal{L}(\tilde{E}, E)$ with

$$\sup\{\|\big((T_{i\tau}a)(x,\xi,\lambda)\big)^{-1}\| \, \langle \xi,\lambda\rangle_\ell^\mu \, ; \, x \in \mathbb{R}^n, \, (\xi,\lambda) \in \mathbb{R}^n \times \mathbb{H}\} < \infty$$

for the symbol $T_{i\tau}a \in S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ (cf. Proposition 2.3.11). Recall that $a - T_{i\tau}a \in S^{\mu-\ell;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$. Consequently we see, using Theorem 2.3.13, that the function

$$b(x,\xi,\lambda) := \begin{cases} \big((T_{i\tau}a)(x,\xi,\lambda)\big)^{-1} & \text{in the general case} \\ T_{i\tau}\big(a_{(\mu)}\big)^{-1}(x,\xi,\lambda) & \text{in the classical case} \end{cases}$$

belongs to $S^{-\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, E)$ and satisfies the asserted condition in a).

Now assume that $a \in S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; E, \tilde{E})$ is parabolic on a compact set $K \Subset \mathbb{R}^n$. Employing again Remark 2.2.9 we see, that there is a neighbourhood $V(K)$ of K and a sufficiently large $R > 0$, such that there exists $(a(x,\xi,\lambda))^{-1} \in \mathcal{L}(\tilde{E}, E)$ for all $x \in V(K)$, $(\xi,\lambda) \in \mathbb{R}^n \times \mathbb{H}$ with $|\xi,\lambda|_\ell \geq R$, and

$$\sup\{\|(a(x,\xi,\lambda))^{-1}\| \langle\xi,\lambda\rangle_\ell^\mu\,;\ x \in V(K),\ |\xi,\lambda|_\ell \geq R\} < \infty.$$

Passing as before to the symbol $T_{i\tau}a$ for sufficiently large $\tau \in \mathbb{R}_+$ we see, that there exists $\big((T_{i\tau}a)(x,\xi,\lambda)\big)^{-1} \in \mathcal{L}(\tilde{E}, E)$ for all $x \in V(K)$ and all $(\xi,\lambda) \in \mathbb{R}^n \times \mathbb{H}$, and

$$\sup\{\|\big((T_{i\tau}a)(x,\xi,\lambda)\big)^{-1}\| \langle\xi,\lambda\rangle_\ell^\mu\,;\ x \in V(K),\ (\xi,\lambda) \in \mathbb{R}^n \times \mathbb{H}\} < \infty.$$

In the classical case we have

$$\sup\{\|T_{i\tau}\big(a_{(\mu)}\big)^{-1}(x,\xi,\lambda)\| \langle\xi,\lambda\rangle_\ell^\mu\,;\ x \in V(K),\ (\xi,\lambda) \in \mathbb{R}^n \times \mathbb{H}\} < \infty.$$

Now choose a function $\hat{\varphi} \in C_0^\infty(V(K))$ such that $\hat{\varphi} \equiv 1$ in a neighbourhood $U(K)$ of K, and define

$$b(x,\xi,\lambda) := \begin{cases} \hat{\varphi}(x)\big((T_{i\tau}a)(x,\xi,\lambda)\big)^{-1} & \text{in the general case} \\ \hat{\varphi}(x)T_{i\tau}\big(a_{(\mu)}\big)^{-1}(x,\xi,\lambda) & \text{in the classical case.} \end{cases}$$

Then b belongs to $S^{-\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \tilde{E}, E)$ and fulfills the asserted condition in b). □

2.4.13 Theorem. *Let $A(\lambda) \in L^{\mu;\ell}_V(\mathbb{R}^n; \mathbb{H}; E, \tilde{E})$. Then the following are equivalent:*

a) *$A(\lambda)$ is parabolic.*

b) *There exists an operator $P(\lambda) \in L^{-\mu;\ell}_V(\mathbb{R}^n; \mathbb{H}; \tilde{E}, E)$, such that $A(\lambda)P(\lambda) - 1 \in L^{-\varepsilon;\ell}_V(\mathbb{R}^n; \mathbb{H}; \tilde{E}, \tilde{E})$ and $P(\lambda)A(\lambda) - 1 \in L^{-\varepsilon;\ell}_V(\mathbb{R}^n; \mathbb{H}; E, E)$ for some $\varepsilon > 0$.*

c) *There exists an operator $P(\lambda) \in L^{-\mu;\ell}_V(\mathbb{R}^n; \mathbb{H}; \tilde{E}, E)$, such that $A(\lambda)P(\lambda) - 1 \in L^{-\infty}_V(\mathbb{R}^n; \mathbb{H}; \tilde{E}, \tilde{E})$ and $P(\lambda)A(\lambda) - 1 \in L^{-\infty}_V(\mathbb{R}^n; \mathbb{H}; E, E)$.*

If $A(\lambda) \in L^{\mu;\ell}_{V\,cl}(\mathbb{R}^n;\mathbb{H};E,\tilde{E})$ is parabolic, then every $P(\lambda)$ satisfying c) belongs to $L^{-\mu;\ell}_{V\,cl}(\mathbb{R}^n;\mathbb{H};\tilde{E},E)$. Every $P(\lambda) \in L^{-\mu;\ell}_{V(cl)}(\mathbb{R}^n;\mathbb{H};\tilde{E},E)$ satisfying c) is called a Volterra parametrix *of $A(\lambda)$.*

Proof. In view of Definition 2.4.11 of parabolicity for Volterra pseudodifferential operators and Theorem 2.2.10 it suffices to show that a) implies b), and b) implies c).

Assume that a) holds. Let $A(\lambda) = \mathrm{op}_x(a)(\lambda)$ with $a \in S^{\mu;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};E,\tilde{E})$. Let $b \in S^{-\mu;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};\tilde{E},E)$ satisfying the condition in a) of Proposition 2.4.12. Now the asymptotic expansion (2.4.i) in the Volterra sense of the Leibniz-product in Theorem 2.4.5 gives that $b\#a - 1 \in S^{-1;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};E,E)$ and $a\#b - 1 \in S^{-1;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};\tilde{E},\tilde{E})$ which implies b).

Now assume that b) is fulfilled. Let $P(\lambda) = \mathrm{op}_x(b)(\lambda)$ and $A(\lambda)P(\lambda) = 1 - \mathrm{op}_x(r)(\lambda)$ with $r \in S^{-\varepsilon;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};\tilde{E},\tilde{E})$. From Theorem 2.3.9 and Theorem 2.4.5 we see that there is a symbol $c \in S^{-\varepsilon;\ell}_V(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};\tilde{E},\tilde{E})$ such that $c \underset{V}{\sim} \sum_{j\in\mathbb{N}} \#^{(j)}r$. Now define $P_{(R)}(\lambda) := \mathrm{op}_x(b\#(1+c))(\lambda)$. Then we have $A(\lambda)P_{(R)}(\lambda) - 1 \in L^{-\infty}_V(\mathbb{R}^n;\mathbb{H};\tilde{E},\tilde{E})$ as desired. Analogously, we obtain a Volterra parametrix $P_{(L)}(\lambda)$ from the left. But both the left- and the right-parametrix differ only by a term in $L^{-\infty}_V(\mathbb{R}^n;\mathbb{H};\tilde{E},E)$ which follows from considering the product $P_{(L)}(\lambda)A(\lambda)P_{(R)}(\lambda)$. This implies c). □

2.4.14 Corollary. *Let $A(\lambda) \in L^{\mu;\ell}_{V(cl)}(\mathbb{R}^n;\mathbb{H};E,\tilde{E})$ and $K \Subset \mathbb{R}^n$ be compact. Then $A(\lambda)$ is parabolic on K if and only if there are $\varphi,\ \psi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi\psi = \varphi$, $\varphi \equiv 1$ on K, and $P(\lambda) \in L^{-\mu;\ell}_{V(cl)}(\mathbb{R}^n;\mathbb{H};\tilde{E},E)$ such that $\varphi(A(\lambda)P(\lambda) - 1)\psi \in L^{-\infty}_V(\mathbb{R}^n;\mathbb{H};\tilde{E},\tilde{E})$ and $\varphi(P(\lambda)A(\lambda) - 1)\psi \in L^{-\infty}_V(\mathbb{R}^n;\mathbb{H};E,E)$.*

2.5 Volterra Mellin calculus

2.5.1 Definition. Let E and $\tilde{E}$ be Hilbert spaces. For $\mu \in \mathbb{R}$ the spaces of (r,r')-resp. r-dependent (classical) parameter-dependent Mellin symbols with respect to the weight $\gamma \in \mathbb{R}$ and parameter-space $\mathbb{R}^n$ are defined as

$$M_\gamma S^{\mu;\ell}_{(cl)}((\mathbb{R}_+)^q \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma};E,\tilde{E}) := C_B^\infty((\mathbb{R}_+)^q, S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma};E,\tilde{E}))$$

for $q = 1,2$.

The spaces of (classical) Volterra Mellin symbols of order μ with respect to the weight $\gamma \in \mathbb{R}$ are defined as

$$M_\gamma S^{\mu;\ell}_{V(cl)}((\mathbb{R}_+)^q \times \mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma};E,\tilde{E}) := C_B^\infty((\mathbb{R}_+)^q, S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma};E,\tilde{E}))$$

for $q = 1, 2$.

Analogously, we obtain the spaces of order $-\infty$ with respect to the weight $\gamma \in \mathbb{R}$. All these spaces carry Fréchet topologies in a canonical way.

With the same conventions as in Definition 2.1.3 we also have the (Volterra) Mellin symbol spaces when we deal with scales of Hilbert spaces instead of single Hilbert spaces only.

The operator of restriction of the half-plane $\mathbb{H}_{\frac{1}{2}-\gamma}$ to the weight line $\Gamma_{\frac{1}{2}-\gamma}$ induces continuous embeddings

$$M_\gamma S^{\mu;\ell}_{V(cl)}\big((\mathbb{R}_+)^q \times \mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; E, \tilde{E}\big) \hookrightarrow M_\gamma S^{\mu;\ell}_{(cl)}\big((\mathbb{R}_+)^q \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E}\big)$$

for $q = 1, 2$, see also Proposition 2.3.2.

2.5.2 Theorem. *Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces, and $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as in Definition 2.1.6. Let $(\mu_k) \subseteq \mathbb{R}$ such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max\limits_{k\in\mathbb{N}_0} \mu_k$. Moreover, let $a_k \in M_\gamma S^{\mu_k;\ell}(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}})$. Then there exists $a \in M_\gamma S^{\overline{\mu};\ell}(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}})$ such that $a \sim \sum\limits_{k=0}^{\infty} a_k$. The asymptotic expansion a is uniquely determined modulo $M_\gamma S^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}})$.*

If $a_k \in M_\gamma S^{\mu_k;\ell}_V(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}})$ are given, then we find $a \in M_\gamma S^{\overline{\mu};\ell}_V(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}})$ such that $a \underset{V}{\sim} \sum\limits_{k=0}^{\infty} a_k$, and a is uniquely determined modulo $M_\gamma S^{-\infty}_V(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}})$.

If the sequence $(\mu_k)_{k\in\mathbb{N}_0}$ is given as $\mu_k = \overline{\mu} - k$ and the a_k are classical (Volterra) Mellin symbols then also a is a classical (Volterra) Mellin symbol of order $\overline{\mu}$.

Proof. This follows in the non-Volterra case as in the proof of Theorem 2.1.8 from Lemma 2.1.7. In the Volterra case we obtain the desired result analogous to the proof of Theorem 2.3.9 from Proposition 2.3.8. □

2.5.3 Definition. Let E and $\tilde{E}$ be Hilbert spaces, and $\mu \in \mathbb{R}$. With a Mellin double-symbol $a \in M_\gamma S^{\mu;\ell}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E})$ we associate a family of *Mellin pseudodifferential operators* $\operatorname{op}_M^\gamma(a)(\xi) \in \mathcal{L}(\mathcal{T}_\gamma(\mathbb{R}_+, E), \mathcal{T}_\gamma(\mathbb{R}_+, \tilde{E}))$ for $\xi \in \mathbb{R}^n$ by means of the following Mellin oscillatory integral:

$$\begin{aligned}(\operatorname{op}_M^\gamma(a)(\xi)u)(r) :&= \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{1}{2}-\gamma}} \int\limits_{\mathbb{R}_+} \Big(\frac{r}{r'}\Big)^{-z} a(r, r', \xi, z) u(r') \,\frac{dr'}{r'}\, dz \\ &= \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}_+} r'^{\frac{1}{2}-\gamma+i\tau} a(r, rr', \xi, \tfrac{1}{2} - \gamma + i\tau) u(rr') \,\frac{dr'}{r'}\, đ\tau.\end{aligned}$$

Taking into account the operator S_γ and its inverse from (1.1.i) and (1.1.ii) we see that we may write

$$\mathrm{op}_M^\gamma(a)(\xi) = S_\gamma^{-1}\mathrm{op}_t(a_\gamma)(\xi)S_\gamma \tag{2.5.i}$$

as operators on $\mathcal{T}_\gamma(\mathbb{R}_+, E)$, where the (Fourier) double-symbol $a_\gamma \in C_b^\infty(\mathbb{R}\times\mathbb{R}, S^{\mu;\ell}(\mathbb{R}^n\times\mathbb{R}; E, \tilde{E}))$ is given as

$$a_\gamma(t,t',\xi,\tau) = a\big(e^{-t}, e^{-t'}, \xi, \frac{1}{2}-\gamma+i\tau\big).$$

From (2.5.i) we thus see that the theory of Mellin pseudodifferential operators can be carried over to some extent from the (standard) setting of operators based on the Fourier transform.

2.5.4 Theorem. *Let $a \in M_\gamma S^{\mu;\ell}_{(cl)}(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n\times\Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E})$. Then there exist unique Mellin left- and right-symbols $a_L(r,\xi,z)$, $a_R(r',\xi,z) \in M_\gamma S^{\mu;\ell}_{(cl)}(\mathbb{R}_+\times\mathbb{R}^n\times\Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E})$ such that $op_M^\gamma(a)(\xi) = op_M^\gamma(a_L)(\xi) = op_M^\gamma(a_R)(\xi)$ as operators on $\mathcal{T}_\gamma(\mathbb{R}_+, E)$.*

These symbols are given by the following Mellin oscillatory integrals:

$$a_L(r,\xi,z) = \int_{\mathbb{R}}\int_{\mathbb{R}_+} s^{i\eta} a(r, sr, \xi, z+i\eta)\,\frac{ds}{s}\,đ\eta,$$

$$a_R(r',\xi,z) = \int_{\mathbb{R}}\int_{\mathbb{R}_+} s^{i\eta} a(sr', r', \xi, z-i\eta)\,\frac{ds}{s}\,đ\eta.$$

The mappings $a \mapsto a_L$ and $a \mapsto a_R$ are continuous. Moreover, we have the asymptotic expansions

$$a_L(r,\xi,z) \sim \sum_{k=0}^\infty \frac{1}{k!} D_\tau^k(-r'\partial_{r'})^k a(r,r',\xi,z)|_{r'=r},$$

$$a_R(r',\xi,z) \sim \sum_{k=0}^\infty \frac{1}{k!} (-1)^k D_\tau^k(-r\partial_r)^k a(r,r',\xi,z)|_{r=r'}.$$

If $a \in M_\gamma S^{\mu;\ell}_{V(cl)}(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{H}_{\frac{1}{2}-\gamma}; E, \tilde{E})$ then also a_L, $a_R \in M_\gamma S^{\mu;\ell}_{V(cl)}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{H}_{\frac{1}{2}-\gamma}; E, \tilde{E})$, and the mappings $a\mapsto a_L$ and $a \mapsto a_R$ are continuous with respect to the topology of the Volterra Mellin symbol spaces. In this case we have the asymptotic expansions

$$a_L(r,\xi,z) \underset{V}{\sim} \sum_{k=0}^\infty \frac{1}{k!}\partial_z^k(-r'\partial_{r'})^k a(r,r',\xi,z)|_{r'=r},$$

$$a_R(r',\xi,z) \underset{V}{\sim} \sum_{k=0}^\infty \frac{1}{k!}(-1)^k\partial_z^k(-r\partial_r)^k a(r,r',\xi,z)|_{r=r'}$$

in the Volterra sense.

2.5.5 Definition. For $\gamma \in \mathbb{R}$ define

$$M_\gamma L^{\mu;\ell}_{(cl)}(\mathbb{R}_+;\mathbb{R}^n;E,\tilde{E}) := \{\mathrm{op}^\gamma_M(a)(\xi);\ a \in M_\gamma S^{\mu;\ell}_{(cl)}(\mathbb{R}_+\times\mathbb{R}^n\times\Gamma_{\frac{1}{2}-\gamma};E,\tilde{E})\},$$
$$M_\gamma L^{\mu;\ell}_{V(cl)}(\mathbb{R}_+;\mathbb{R}^n;E,\tilde{E}) := \{\mathrm{op}^\gamma_M(a)(\xi);\ a \in M_\gamma S^{\mu;\ell}_{V(cl)}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{H}_{\frac{1}{2}-\gamma};E,\tilde{E})\}.$$

In view of Theorem 2.5.4 we conclude that $\mathrm{op}^\gamma_M(\cdot)(\xi)$ provides an isomorphism between these spaces and the corresponding (left-) symbol spaces. Via that isomorphism we carry over the topologies which turns the operator spaces into Fréchet spaces.

2.5.6 Theorem. *Let E, $\tilde{E}$ and $\hat{E}$ be Hilbert spaces. Let $a \in M_\gamma S^{\mu;\ell}_{(cl)}(\mathbb{R}_+\times\mathbb{R}^n\times\Gamma_{\frac{1}{2}-\gamma};\tilde{E},\hat{E})$ and $b \in M_\gamma S^{\mu';\ell}_{(cl)}(\mathbb{R}_+\times\mathbb{R}^n\times\Gamma_{\frac{1}{2}-\gamma};E,\tilde{E})$. Then the composition as operators on $\mathcal{T}_\gamma(\mathbb{R}_+,E)$ may be written as*

$$op^\gamma_M(a)(\xi)\circ op^\gamma_M(b)(\xi) = op^\gamma_M(a\#b)(\xi)$$

with the Leibniz-product $a\#b \in M_\gamma S^{\mu+\mu';\ell}_{(cl)}(\mathbb{R}_+\times\mathbb{R}^n\times\Gamma_{\frac{1}{2}-\gamma};E,\hat{E})$. More precisely, the Leibniz-product is given by the Mellin oscillatory integral formula

$$a\#b(r,\xi,z) = \int_{\mathbb{R}}\int_{\mathbb{R}_+} s^{i\eta}a(r,\xi,z+i\eta)b(rs,\xi,z)\,\frac{ds}{s}\,đ\eta, \tag{2.5.ii}$$

and the following asymptotic expansion holds for $a\#b$:

$$a\#b \sim \sum_{k=0}^{\infty}\frac{1}{k!}(D^k_\tau a)((-r\partial_r)^k b). \tag{2.5.iii}$$

The mapping $(a,b)\mapsto a\#b$ is bilinear and continuous.

If $a \in M_\gamma S^{\mu;\ell}_{V(cl)}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{H}_{\frac{1}{2}-\gamma};\tilde{E},\hat{E})$ and $b \in M_\gamma S^{\mu';\ell}_{V(cl)}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{H}_{\frac{1}{2}-\gamma};E,\tilde{E})$, then also $a\#b \in M_\gamma S^{\mu+\mu';\ell}_{V(cl)}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{H}_{\frac{1}{2}-\gamma};E,\hat{E})$, and the oscillatory integral formula (2.5.ii) is valid for $z \in \mathbb{H}_{\frac{1}{2}-\gamma}$, and the asymptotic expansion (2.5.iii) holds in the Volterra sense, i. e.,

$$a\#b \underset{V}{\sim} \sum_{k=0}^{\infty}\frac{1}{k!}(\partial^k_z a)((-r\partial_r)^k b). \tag{2.5.iv}$$

In this case the mapping $(a,b)\mapsto a\#b$ is bilinear and continuous within the Volterra Mellin symbol spaces.

Proof. The assertion follows from Theorem 2.5.4. Note that $a(r,\xi,z)b_R(r',\xi,z)$ is a double-symbol for the composition, and the Leibniz-product is the associated left-symbol. This also implies the continuity of $(a,b) \mapsto a\#b$.

The oscillatory integral formula (2.5.ii) necessarily holds in the preceding situation, for it holds in the non-Volterra case without parameters, and by uniqueness of analytic continuation the formula is valid within the half-plane $\mathbb{H}_{\frac{1}{2}-\gamma}$.

The asymptotic expansions (2.5.iii) and (2.5.iv) follow from (2.5.ii) via Taylor expansion. □

2.5.7 Remark. The following pseudolocality property holds for Mellin operators: Let $a(r,r',\xi,z) \in M_\gamma S^{\mu;\ell}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E})$, such that $a(r,r',\xi,\lambda) \equiv 0$ for $\left|\frac{r}{r'} - 1\right| < \varepsilon$ for a sufficiently small $\varepsilon > 0$. Then $\mathrm{op}_M^\gamma(a)(\xi) = \mathrm{op}_M^\gamma(c)(\xi)$ with a symbol $c \in M_\gamma S^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E})$. If even $a \in M_\gamma S_V^{\mu;\ell}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; E, \tilde{E})$, then also $c \in M_\gamma S_V^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; E, \tilde{E})$.

In particular, if $a(r,\xi,z) \in M_\gamma S^{\mu;\ell}(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E})$ and $\varphi, \psi \in C_B^\infty(\mathbb{R}_+)$ such that $\mathrm{dist}(\mathrm{supp}\varphi, \mathrm{supp}\psi) > 0$, then $\varphi\, \mathrm{op}_M^\gamma(a)(\xi)\, \psi = \mathrm{op}_M^\gamma(a_{\varphi,\psi})(\xi)$ with a symbol $a_{\varphi,\psi} \in M_\gamma S^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E})$, and the mapping $a \longmapsto a_{\varphi,\psi}$ is continuous. If even $a(r,\xi,z) \in M_\gamma S_V^{\mu;\ell}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; E, \tilde{E})$, then also $a_{\varphi,\psi} \in M_\gamma S_V^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; E, \tilde{E})$. In this case the mapping $a \longmapsto a_{\varphi,\psi}$ is continuous with respect to the Volterra Mellin symbol spaces.

2.5.8 Theorem. *Let $a \in M_\gamma S_V^{\mu;\ell}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; E, \tilde{E})$. Then $op_M^\gamma(a)(\xi)$ restricts for every $r_0 \in \mathbb{R}_+$ to a family of continuous operators*

$$op_M^\gamma(a)(\xi) : \mathcal{T}_{\gamma,0}((0,r_0),E) \longrightarrow \mathcal{T}_{\gamma,0}((0,r_0),\tilde{E}).$$

Proof. Without loss of generality assume $n = 0$. We may write

$$\big(\mathrm{op}_M^\gamma(a)u\big)(r) = \big(\mathcal{M}_{\gamma,z\to r'}^{-1} a(r,z)\mathcal{M}_{\gamma,r'\to z}u\big)|_{r'=r} \tag{1}$$

for $u \in \mathcal{T}_\gamma(\mathbb{R}_+,E)$. Now let $u \in \mathcal{T}_{\gamma,0}((0,r_0),E)$ be given and $r \in \mathbb{R}_+$ fixed. In view of the Paley–Wiener characterizations (see Section 1.1) we have $\mathcal{M}_\gamma u \in \mathcal{A}(\mathbb{H}_{\frac{1}{2}-\gamma}, E; r_0)$. For a is a Volterra symbol by assumption we see that $a(r,z)\big(\mathcal{M}_\gamma u\big)(z)$ may be regarded as an element of $\mathcal{A}(\mathbb{H}_{\frac{1}{2}-\gamma}, \tilde{E}; r_0)$, i. e., a acts as a "multiplier" in the spaces $\mathcal{A}(\mathbb{H}_{\frac{1}{2}-\gamma}, \cdot; r_0)$. Employing again the Paley–Wiener characterizations we now conclude that $\mathcal{M}_{\gamma,z\to r'}^{-1} a(r,z)\mathcal{M}_{\gamma,r'\to z}u \in \mathcal{T}_{\gamma,0}((0,r_0)_{r'}, \tilde{E})$, where the subscript r' indicates that we consider the latter function space in the variable r'. In particular, evaluation at $r' = r$ yields that (1) necessarily vanishes for $r > r_0$ which finishes the proof of the theorem. □

2.5.9 Remark. Theorem 2.5.8 provides the motivation for the name "Volterra" symbols respectively operators:

If we regard the Mellin pseudodifferential operators as

$$\operatorname{op}_M^\gamma(a)(\xi) : C_0^\infty(\mathbb{R}_+) \longrightarrow C^\infty(\mathbb{R}_+, \mathcal{L}(E, \tilde{E})),$$

then we obtain for every $r_0 \in \mathbb{R}_+$ as in Theorem 2.5.8 that $\big(\operatorname{op}_M^\gamma(a)(\xi)u\big)(r) \equiv 0$ for $r > r_0$, if $u \in C_0^\infty(\mathbb{R}_+)$ such that $u \equiv 0$ for $r > r_0$. In other words, the operator-valued Schwartz kernel $K_{\operatorname{op}_M^\gamma(a)(\xi)} \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{L}(E, \tilde{E}))$ satisfies

$$\operatorname{supp} K_{\operatorname{op}_M^\gamma(a)(\xi)} \subseteq \{(r, r') \in \mathbb{R}_+ \times \mathbb{R}_+;\ r \leq r'\}$$

for all $\xi \in \mathbb{R}^n$.

This gives the link to (classical) Volterra integral equations where the kernel is supported on one side of the diagonal only.

Continuity in Mellin Sobolev spaces

2.5.10 Definition. Let E be a Hilbert space. For $s, \gamma \in \mathbb{R}$ define the Mellin Sobolev space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, E)$ to consist of all $u \in \mathcal{T}_\gamma'(\mathbb{R}_+, E)$ such that $\mathcal{M}_\gamma u$ is a regular distribution in $\mathcal{S}'(\Gamma_{\frac{1}{2}-\gamma}, E)$, and

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+,E)} := \Big(\frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{1}{2}-\gamma}} \langle \operatorname{Im}(z)\rangle^{2s} \|\mathcal{M}_\gamma u(z)\|_E^2 \, dz\Big)^{\frac{1}{2}} < \infty.$$

In case of $E = \mathbb{C}$ the space is suppressed from the notation.

The operator S_γ from (1.1.i) provides an isomorphism $S_\gamma : \mathcal{H}^{s,\gamma}(\mathbb{R}_+, E) \longrightarrow H^s(\mathbb{R}, E)$.

For $r_0 \in \mathbb{R}_+$ we define the space $\mathcal{H}_0^{s,\gamma}((0, r_0], E)$ to consist of all $u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+, E)$ such that $\operatorname{supp}(u) \subseteq (0, r_0]$. This is a closed subspace of $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, E)$ and equals the closure of $\mathcal{T}_{\gamma,0}((0, r_0), E)$ in $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, E)$.

2.5.11 Theorem. *Let E and $\tilde{E}$ be Hilbert spaces. Let $a \in M_\gamma S^{\mu;\ell}(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E})$ and $s, \nu \in \mathbb{R}$ where $\nu \geq \frac{\mu}{\ell}$. Then $\operatorname{op}_M^\gamma(a)(\xi)$ extends for $\xi \in \mathbb{R}^n$ by continuity to an operator $\operatorname{op}_M^\gamma(a)(\xi) \in \mathcal{L}(\mathcal{H}^{s,\gamma}(\mathbb{R}_+, E), \mathcal{H}^{s-\nu,\gamma}(\mathbb{R}_+, \tilde{E}))$, which induces a continuous embedding*

$$M_\gamma L^{\mu;\ell}(\mathbb{R}_+; \mathbb{R}^n; E, \tilde{E}) \hookrightarrow \begin{cases} S^\mu(\mathbb{R}^n; \mathcal{H}^{s,\gamma}(\mathbb{R}_+, E), \mathcal{H}^{s-\nu,\gamma}(\mathbb{R}_+, \tilde{E})) & \nu \geq 0 \\ S^{\mu-\nu}(\mathbb{R}^n; \mathcal{H}^{s,\gamma}(\mathbb{R}_+, E), \mathcal{H}^{s-\nu,\gamma}(\mathbb{R}_+, \tilde{E})) & \nu \leq 0 \end{cases} \tag{2.5.v}$$

into the space of operator-valued symbols in the Sobolev spaces.

Moreover, restriction of Volterra pseudodifferential operators to the $\mathcal{H}_0^{s,\gamma}$*-spaces provides continuous mappings*

$$M_\gamma L_V^{\mu;\ell}(\mathbb{R}_+;\mathbb{R}^n;E,\tilde{E}) \longrightarrow \begin{cases} S^\mu(\mathbb{R}^n;\mathcal{H}_0^{s,\gamma}((0,r_0],E),\mathcal{H}_0^{s-\nu,\gamma}((0,r_0],\tilde{E})) & \nu \geq 0 \\ S^{\mu-\nu}(\mathbb{R}^n;\mathcal{H}_0^{s,\gamma}((0,r_0],E),\mathcal{H}_0^{s-\nu,\gamma}((0,r_0],\tilde{E})) & \nu \leq 0 \end{cases}$$

for each $r_0 \in \mathbb{R}_+$*.*

2.5.12 Remark. Employing relation (2.5.i) there is analogously a parameter-dependent pseudodifferential calculus with parameter-space $\mathbb{R}^n$ for operators based on the Fourier transform, where the action in the covariable is carried out in the "Volterra"-covariable, i. e., the covariable which extends holomorphically into an upper or lower half-plane in $\mathbb{C}$.

The analogue of Theorem 2.5.8 is valid within this calculus, which follows in the same way from Paley–Wiener characterizations as in the proof of Theorem 2.5.8, but now with the Fourier transform involved. We will not state the details for they are straightforward in view of the properties of the operator S_γ as well as (2.5.i) (see also Section 2.7).

2.6 Analytic Volterra Mellin calculus

2.6.1 Definition. Let E and $\tilde{E}$ be Hilbert spaces. Moreover, let $z = \beta + i\tau \in \mathbb{C}$ be the splitting of $z \in \mathbb{C}$ in real and imaginary part. For $\mu \in \mathbb{R}$ define the Fréchet spaces

$$S^{\mu;\ell}_{O(cl)}(\mathbb{R}^n \times \mathbb{C}; E, \tilde{E}) := \mathcal{A}(\mathbb{C}, S^\mu(\mathbb{R}^n;E,\tilde{E})) \cap C^\infty(\mathbb{R}_\beta, S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \Gamma_\beta; E, \tilde{E})),$$
$$S^{\mu;\ell}_{V,O(cl)}(\mathbb{R}^n \times \mathbb{C}; E, \tilde{E}) := S^{\mu;\ell}_{O(cl)}(\mathbb{R}^n \times \mathbb{C}; E, \tilde{E}) \cap S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}_0; E, \tilde{E})$$

with the induced topologies. Analogously, we define the corresponding symbol spaces when we deal with scales of Hilbert spaces.

2.6.2 Notation. For an interval $\emptyset \neq I \subseteq \mathbb{R}$ we shall use the notation

$$\Gamma_I := \{z \in \mathbb{C};\ \mathrm{Re}(z) \in I\}$$

for the strip in the complex plane over I.

2.6.3 Proposition. *Let* $\emptyset \neq I \subseteq \mathbb{R}$ *be an open interval and* $\mu \in \mathbb{R}$*. Let*

$$\ell^\infty_{loc}\mathcal{A}^{\mu;\ell}_{(cl)}(E,\tilde{E}) := \{a \in \mathcal{A}(\Gamma_I, S^\mu(\mathbb{R}^n;E,\tilde{E}));\ a|_{\Gamma_\beta} \in S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \Gamma_\beta; E, \tilde{E}) \text{ locally uniformly for } \beta \in I\},$$
$$C^\infty\mathcal{A}^{\mu;\ell}_{(cl)}(E,\tilde{E}) := \{a \in \mathcal{A}(\Gamma_I, S^\mu(\mathbb{R}^n;E,\tilde{E}));\ a \in C^\infty(I_\beta, S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \Gamma_\beta; E, \tilde{E}))\}$$

endowed with their natural Fréchet topologies. Observe that for $I = \mathbb{R}$ we recover $S^{\mu;\ell}_{O(cl)}(\mathbb{R}^n \times \mathbb{C}; E, \tilde{E}) = C^\infty \mathcal{A}^{\mu;\ell}_{(cl)}(E, \tilde{E})$.

a) *The embedding* $\iota : C^\infty \mathcal{A}^{\mu;\ell}_{(cl)}(E, \tilde{E}) \hookrightarrow \ell^\infty_{loc}\mathcal{A}^{\mu;\ell}_{(cl)}(E, \tilde{E})$ *is onto and provides an isomorphism between these spaces.*

b) *The complex derivative is a linear and continuous operator in the spaces* $\partial_z : \ell^\infty_{loc}\mathcal{A}^{\mu;\ell}_{(cl)}(E, \tilde{E}) \to \ell^\infty_{loc}\mathcal{A}^{\mu-\ell;\ell}_{(cl)}(E, \tilde{E})$.

c) *Given* $a \in \ell^\infty_{loc}\mathcal{A}^{\mu;\ell}_{(cl)}(E, \tilde{E})$, *we have the following asymptotic expansion for* $a|_{\Gamma_{\beta_0}}$ *in terms of* $a|_{\Gamma_\beta}$ *for every* $\beta_0, \beta \in I$ *which depends smoothly on* $(\beta_0, \beta) \in I \times I$:

$$a|_{\Gamma_{\beta_0}} \sim \sum_{k=0}^{\infty} \frac{(\beta_0 - \beta)^k}{k!} \big(\partial_z^k a\big)|_{\Gamma_\beta}.$$

d) *For arbitrary* $\beta \in \mathbb{R}$ *we have* $S^{\mu;\ell}_{V,O(cl)}(\mathbb{R}^n \times \mathbb{C}; E, \tilde{E}) \hookrightarrow S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}_\beta; E, \tilde{E})$. *If* $a \in S^{\mu;\ell}_{V,O(cl)}(\mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$, *then we have* $a|_{\mathbb{H}_\beta} \in S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}_\beta; E, \tilde{E})$ *as a smooth function of* $\beta \in \mathbb{R}$, *and the asymptotic expansion*

$$a|_{\mathbb{H}_{\beta_0}} \underset{V}{\sim} \sum_{k=0}^{\infty} \frac{(\beta_0 - \beta)^k}{k!} \big(\partial_z^k a\big)|_{\mathbb{H}_\beta}$$

is valid, which depends smoothly on $(\beta_0, \beta) \in \mathbb{R} \times \mathbb{R}$.

e) *For* $\beta \in I$ *and* $\mu \geq \mu'$ *the identity*

$$\ell^\infty_{loc}\mathcal{A}^{\mu;\ell}(E, \tilde{E}) \cap S^{\mu';\ell}_{(cl)}(\mathbb{R}^n \times \Gamma_\beta; E, \tilde{E}) = \ell^\infty_{loc}\mathcal{A}^{\mu';\ell}_{(cl)}(E, \tilde{E})$$

holds algebraically and topologically.

f) *For* $\beta \in \mathbb{R}$ *and* $\mu \geq \mu'$ *the identity*

$$S^{\mu;\ell}_{V,O}(\mathbb{R}^n \times \mathbb{C}; E, \tilde{E}) \cap S^{\mu';\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}_\beta; E, \tilde{E}) = S^{\mu';\ell}_{V,O(cl)}(\mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$$

holds algebraically and topologically.

From the expansions in c) and d) we see, that in the classical cases the homogeneous principal symbols of the restrictions do not depend on the particular weight line or half-plane.

2.6.4 Proposition. a) *Let* $\emptyset \neq I \subseteq \mathbb{R}$ *be an open interval. Moreover, let* $a \in C^\infty_B(\mathbb{R}_+, \ell^\infty_{loc}\mathcal{A}^{\mu;\ell}(E, \tilde{E}))$ *(cf. Proposition 2.6.3). Then for* $\gamma, \gamma' \in \mathbb{R}$ *such that* $\frac{1}{2} - \gamma, \frac{1}{2} - \gamma' \in I$ *we have* $op^\gamma_M(a)(\xi) = op^{\gamma'}_M(a)(\xi)$ *as operators on* $C^\infty_0(\mathbb{R}_+, E)$.

b) Let $\gamma, \delta \in \mathbb{R}$ *and* $a \in M_\gamma S^{\mu;\ell}(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E})$*. Then we have* $op_M^\gamma(a)(\xi) r^\delta = r^\delta op_M^{\gamma-\delta}(T_{-\delta}a)(\xi)$ *as operators acting in* $\mathcal{T}_{\gamma-\delta}(\mathbb{R}_+, E) \longrightarrow \mathcal{T}_\gamma(\mathbb{R}_+, \tilde{E})$*, where* $T_{-\delta}a \in M_{\gamma-\delta} S^{\mu;\ell}(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma+\delta}; E, \tilde{E})$ *is defined as* $\big(T_{-\delta}a\big)(r, \xi, \frac{1}{2} - \gamma + \delta + i\tau) := a(r, \xi, \frac{1}{2} - \gamma + i\tau)$*.*

Proof. For the proof of a) note that we may write for $u \in C_0^\infty(\mathbb{R}_+, E)$

$$\big(\mathrm{op}_M^\gamma(a)(\xi)u\big)(r) = \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{1}{2}-\gamma}} r^{-\zeta} a(r, \xi, \zeta) \big(\mathcal{M}u\big)(\zeta)\, d\zeta.$$

By Cauchy's theorem we may change the line of integration from $\Gamma_{\frac{1}{2}-\gamma}$ to $\Gamma_{\frac{1}{2}-\gamma'}$ which shows a).

We have to prove the asserted identity in b) only as operators on $C_0^\infty(\mathbb{R}_+, E)$ in view of the density. We may write for $u \in C_0^\infty(\mathbb{R}_+, E)$

$$\begin{aligned}
\big(\mathrm{op}_M^\gamma(a)(\xi)\big(r^\delta u\big)\big)(r) &= \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{1}{2}-\gamma}} r^{-\zeta} a(r, \xi, \zeta) \big(\mathcal{M}u\big)(\zeta + \delta)\, d\zeta \\
&= \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{1}{2}-\gamma+\delta}} r^{-(\zeta-\delta)} \big(T_{-\delta}a\big)(r, \xi, \zeta) \big(\mathcal{M}u\big)(\zeta)\, d\zeta \\
&= r^\delta \big(\mathrm{op}_M^{\gamma-\delta}(T_{-\delta}a)(\xi)u\big)(r).
\end{aligned}$$

□

2.6.5 Definition. Let E and $\tilde{E}$ be Hilbert spaces. For $\mu \in \mathbb{R}$ the spaces of (r, r')-resp. r-dependent (classical) parameter-dependent holomorphic Mellin symbols with parameter-space $\mathbb{R}^n$ are defined as

$$MS_{O(cl)}^{\mu;\ell}\big(\big(\overline{\mathbb{R}}_+\big)^q \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E}\big) := C_B^\infty\big(\big(\overline{\mathbb{R}}_+\big)^q, S_{O(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{C}; E, \tilde{E})\big)$$

for $q = 1, 2$.

The spaces of (classical) holomorphic Volterra Mellin symbols of order μ with parameter-space $\mathbb{R}^n$ are defined as

$$MS_{V,O(cl)}^{\mu;\ell}\big(\big(\overline{\mathbb{R}}_+\big)^q \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E}\big) := C_B^\infty\big(\big(\overline{\mathbb{R}}_+\big)^q, S_{V,O(cl)}^{\mu;\ell}(\mathbb{R}^n \times \mathbb{C}; E, \tilde{E})\big)$$

for $q = 1, 2$.

Analogously, we obtain the spaces of order $-\infty$. All these spaces carry Fréchet topologies in a canonical way.

With the same conventions as in Definition 2.1.3 we also have the (Volterra) Mellin symbol spaces when we deal with scales of Hilbert spaces instead of single Hilbert spaces only.

For every $\gamma \in \mathbb{R}$ the embeddings

$$MS^{\mu;\ell}_{O(cl)}((\overline{\mathbb{R}}_+)^q \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E}) \hookrightarrow M_\gamma S^{\mu;\ell}_{(cl)}((\mathbb{R}_+)^q \times \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E}),$$
$$MS^{\mu;\ell}_{V,O(cl)}((\overline{\mathbb{R}}_+)^q \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E}) \hookrightarrow M_\gamma S^{\mu;\ell}_{V(cl)}((\mathbb{R}_+)^q \times \mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; E, \tilde{E})$$

are well-defined and continuous for $q = 1, 2$.

2.6.6 Definition. Let $(\mu_k) \subseteq \mathbb{R}$ be a sequence of reals such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$, and $\overline{\mu} := \max\limits_{k\in\mathbb{N}} \mu_k$. Moreover, let $a_k \in MS^{\mu_k;\ell}_{(V,)O}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$. A symbol $a \in MS^{\overline{\mu};\ell}_{(V,)O}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$ is called the *asymptotic expansion* of the a_k, if for every $R \in \mathbb{R}$ there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$

$$a - \sum_{j=1}^{k} a_j \in MS^{R;\ell}_{(V,)O}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E}).$$

The symbol a is uniquely determined modulo $MS^{-\infty}_{(V,)O}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$.

We shall again employ the notation $a \underset{(V)}{\sim} \sum\limits_{j=1}^{\infty} a_j$.

Elements of the calculus

2.6.7 Theorem. *Let $\gamma \in \mathbb{R}$ and $a \in MS^{\mu;\ell}_{(V,)O(cl)}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$. Then the Mellin left- and right-symbols $a_L(r,\xi,z)$, $a_R(r',\xi,z)$ associated to the operator $op^\gamma_M(a)(\xi)$ from Theorem 2.5.4 belong to $MS^{\mu;\ell}_{(V,)O(cl)}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$ and do not depend on the particular weight $\gamma \in \mathbb{R}$. The oscillatory integral formulas for a_L and a_R in terms of a from Theorem 2.5.4 hold for $z \in \mathbb{C}$, and the mappings $a \mapsto a_L$ and $a \mapsto a_R$ are continuous.*

Moreover, we have the asymptotic expansions in the sense of Definition 2.6.6:

$$a_L(r,\xi,z) \underset{(V)}{\sim} \sum_{k=0}^{\infty} \frac{1}{k!} \partial_z^k (-r'\partial_{r'})^k a(r,r',\xi,z)|_{r'=r},$$
$$a_R(r',\xi,z) \underset{(V)}{\sim} \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \partial_z^k (-r\partial_r)^k a(r,r',\xi,z)|_{r=r'}.$$

2.6.8 Definition. Define

$$M_O L^{\mu;\ell}_{(cl)}(\mathbb{R}_+;\mathbb{R}^n;E,\tilde{E}) := \{\mathrm{op}^\gamma_M(a)(\xi);\ a \in MS^{\mu;\ell}_{O(cl)}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C};E,\tilde{E})\},$$
$$M_{V,O} L^{\mu;\ell}_{(cl)}(\mathbb{R}_+;\mathbb{R}^n;E,\tilde{E}) := \{\mathrm{op}^\gamma_M(a)(\xi);\ a \in MS^{\mu;\ell}_{V,O(cl)}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C};E,\tilde{E})\}.$$

In view of Theorem 2.6.7 $\mathrm{op}^\gamma_M(\cdot)(\xi)$ provides an isomorphism between the operator spaces and the corresponding (left-) symbol spaces. Via that isomorphism we carry over the topologies which turns the operator spaces into Fréchet spaces.

We do not refer to the particular weight $\gamma \in \mathbb{R}$ which is on the one hand justified by Theorem 2.6.7, and on the other hand by Proposition 2.6.4.

2.6.9 Theorem. *Let E, $\tilde{E}$ and $\hat{E}$ be Hilbert spaces, and let $a \in MS^{\mu;\ell}_{(V,)O(cl)}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C};\tilde{E},\hat{E})$, as well as $b \in MS^{\mu';\ell}_{(V,)O(cl)}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C};E,\tilde{E})$. Then the Leibniz-product $a\#b$ from Theorem 2.5.6 belongs to $MS^{\mu+\mu';\ell}_{(V,)O(cl)}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C};E,\hat{E})$ and is independent of the particular weight $\gamma \in \mathbb{R}$. The oscillatory integral formula (2.5.ii) for $a\#b$ in terms of a and b holds for $z \in \mathbb{C}$, and the mapping $(a,b) \mapsto a\#b$ is bilinear and continuous.*

The following asymptotic expansion holds for $a\#b$ in the sense of Definition 2.6.6:

$$a\#b \underset{(V)}{\sim} \sum_{k=0}^{\infty} \frac{1}{k!}(\partial_z^k a)((-r\partial_r)^k b). \tag{2.6.i}$$

Moreover, we have the following formula for the derivatives of the Leibniz-product:

$$\partial_r^k(a\#b) = \sum_{j=0}^{k} \binom{k}{j} \big(T_{-(k-j)}\partial_r^j a\big)\#\big(\partial_r^{k-j} b\big), \tag{2.6.ii}$$

where T denotes the translation operator for functions in the complex plane.

2.6.10 Definition. Let $\mathrm{op}^\gamma_M(a)(\xi) \in M_O L^{\mu;\ell}(\mathbb{R}_+;\mathbb{R}^n;\tilde{E},\hat{E})$. For $k \in \mathbb{N}_0$ we define the *conormal symbol of order -k* via

$$\sigma_M^{-k}\big(\mathrm{op}^\gamma_M(a)(\xi)\big)(\xi,z) := \frac{1}{k!}\big(\partial_r^k a\big)(0,\xi,z). \tag{2.6.iii}$$

The conormal symbol of order 0 is also called conormal symbol simply.

Let $\mathrm{op}^\gamma_M(b)(\xi) \in M_O L^{\mu';\ell}(\mathbb{R}_+;\mathbb{R}^n;E,\tilde{E})$. Then we obtain from (2.6.ii) the following formula for the conormal symbols of the composition

$$\sigma_M^{-k}\big(\mathrm{op}^\gamma_M(a\#b)(\xi)\big) = \sum_{p+q=k} T_{-q}\sigma_M^{-p}\big(\mathrm{op}^\gamma_M(a)(\xi)\big)\sigma_M^{-q}\big(\mathrm{op}^\gamma_M(b)(\xi)\big), \tag{2.6.iv}$$

where T denotes the translation operator for functions in the complex plane.

2.6.11 Remark. By Theorem 2.6.7 the following pseudolocality property of the calculus is valid:

Let $a(r,r',\xi,z) \in MS^{\mu;\ell}_{(V,)O}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$, such that $a(r,r',\xi,\lambda) \equiv 0$ for $\left|\frac{r}{r'} - 1\right| < \varepsilon$ for a sufficiently small $\varepsilon > 0$. Then $\mathrm{op}^\gamma_M(a)(\xi) = \mathrm{op}^\gamma_M(c)(\xi)$ with a symbol $c \in MS^{-\infty}_{(V,)O}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$.

In particular, if $a(r,\xi,z) \in MS^{\mu;\ell}_{(V,)O}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$, and $\varphi, \psi \in C^\infty_B(\overline{\mathbb{R}}_+)$ such that $\mathrm{dist}(\mathrm{supp}\varphi, \mathrm{supp}\psi) > 0$, then $\varphi\,\mathrm{op}^\gamma_M(a)(\xi)\,\psi = \mathrm{op}^\gamma_M(a_{\varphi,\psi})(\xi)$ with a symbol $a_{\varphi,\psi} \in MS^{-\infty}_{(V,)O}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; E, \tilde{E})$. The mapping $a \longmapsto a_{\varphi,\psi}$ is continuous.

The Mellin kernel cut-off operator and asymptotic expansion

2.6.12 Definition. Let E and $\tilde{E}$ be Hilbert spaces. Define the *Mellin kernel cut-off operator* with respect to the weight $\gamma \in \mathbb{R}$ by means of the Mellin oscillatory integral

$$\big(H_\gamma(\varphi)a\big)(\xi,z) := \int_{\mathbb{R}} \int_0^\infty r^{i\tau} \varphi(r) a(\xi, z - i\tau) \,\frac{dr}{r}\, đ\tau \tag{2.6.v}$$

for $(\xi,z) \in \mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}$ and $\varphi \in C^\infty_B(\mathbb{R}_+)$, $a \in S^{\mu;\ell}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E})$.

Note that we may rewrite (2.6.v) as

$$\big(H_\gamma(\varphi)a\big)(\xi, \frac{1}{2} - \gamma + i\tau) = \big(H\big(S_{\frac{1}{2}}\varphi\big)a_\gamma\big)(\xi,\tau)$$

with the Fourier kernel cut-off operator H as introduced in Definition 2.3.5, the transformation $S_{\frac{1}{2}} : C^\infty_B(\mathbb{R}_+) \longrightarrow C^\infty_b(\mathbb{R})$ from (1.1.i), and $a_\gamma(\xi,\tau) := a(\xi, \frac{1}{2} - \gamma + i\tau)$.

Analogous notions apply in case of scales of Hilbert spaces involved.

2.6.13 Theorem. *Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces as in Definition 2.1.3. We again use the abbreviations*

$$\mathcal{E} := \operatorname*{ind\text{-}lim}_{j\in\mathbb{N}} E_j \quad \textit{and} \quad \tilde{\mathcal{E}} := \operatorname*{proj\text{-}lim}_{j\in\mathbb{N}} \tilde{E}_j.$$

The Mellin kernel cut-off operator with respect to the weight $\gamma \in \mathbb{R}$ acts as a continuous bilinear mapping in the spaces

$$H_\gamma : C^\infty_B(\mathbb{R}_+) \times S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}).$$

It restricts to continuous bilinear mappings in the spaces

$$H_\gamma : \begin{cases} C_B^\infty(\mathbb{R}_+) \times S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}) \\ C_0^\infty(\mathbb{R}_+) \times S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{O(cl)}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}}) \\ C_0^\infty(\mathbb{R}_+) \times S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{V,O(cl)}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}}). \end{cases}$$

The following asymptotic expansion holds for $H_\gamma(\varphi)a \in S^{\mu;\ell}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}})$ in terms of $\varphi \in C_B^\infty(\mathbb{R}_+)$ and $a \in S^{\mu;\ell}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}})$:

$$H_\gamma(\varphi)a \sim \sum_{k=0}^\infty \frac{1}{k!}(r\partial_r)^k \varphi(r)|_{r=1} \cdot D_\tau^k a.$$

In case of Volterra symbols we obtain

$$H_\gamma(\varphi)a \underset{V}{\sim} \sum_{k=0}^\infty \frac{1}{k!}(r\partial_r)^k \varphi(r)|_{r=1} \cdot \partial_z^k a.$$

If $\varphi \in C_0^\infty(\mathbb{R}_+)$ and $a \in S^{\mu;\ell}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}})$ we have for every $\delta \in \mathbb{R}$ the following asymptotic expansion of $H_\gamma(\varphi)a|_{\Gamma_{\frac{1}{2}-\gamma-\delta}} \in S^{\mu;\ell}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma-\delta}; \mathcal{E}, \tilde{\mathcal{E}})$ in terms of φ and a:

$$H_\gamma(\varphi)a|_{\Gamma_{\frac{1}{2}-\gamma-\delta}} \sim \sum_{k=0}^\infty \frac{1}{k!}(r\partial_r)^k r^{-\delta} \varphi(r)|_{r=1} \cdot D_\tau^k a.$$

If $\psi \in C_0^\infty(\mathbb{R}_+)$ such that $\psi \equiv 1$ near $r = 1$, then the operator $I - H_\gamma(\psi)$ is continuous in the spaces

$$I - H_\gamma(\psi) : \begin{cases} S^{\mu;\ell}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S^{-\infty}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}) \\ S_V^{\mu;\ell}(\mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}) \longrightarrow S_V^{-\infty}(\mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}). \end{cases}$$

2.6.14 Theorem. *Let $(\mu_k) \subseteq \mathbb{R}$ such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max_{k\in\mathbb{N}_0} \mu_k$. Moreover, let $a_k \in MS^{\mu_k;\ell}_{(V,)O}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}})$. Then there exists $a \in MS^{\overline{\mu};\ell}_{(V,)O}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}})$ such that $a \underset{(V)}{\sim} \sum_{k=0}^\infty a_k$ in the sense of Definition 2.6.6. The asymptotic expansion a is uniquely determined modulo $MS^{-\infty}_{(V,)O}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}})$.*

If the sequence $(\mu_k)_{k\in\mathbb{N}_0}$ is given as $\mu_k = \overline{\mu} - k$ and the a_k are classical (Volterra) Mellin symbols then also a is a classical (Volterra) Mellin symbol of order $\overline{\mu}$.

Proof. According to Lemma 2.1.7 and Theorem 2.1.8, or Proposition 2.3.8 and Theorem 2.3.9, respectively, we first obtain a symbol

$$\overline{a} \in \begin{cases} C_B^\infty(\overline{\mathbb{R}}_+, S^{\overline{\mu};\ell}(\mathbb{R}^n \times \Gamma_0; \mathcal{E}, \tilde{\mathcal{E}})) \\ C_B^\infty(\overline{\mathbb{R}}_+, S_V^{\overline{\mu};\ell}(\mathbb{R}^n \times \mathbb{H}_0; \mathcal{E}, \tilde{\mathcal{E}})) \end{cases}$$

such that $\overline{a} \sim \sum_{k=0}^{\infty} a_k|_{\Gamma_0}$, respectively $\overline{a} \underset{V}{\sim} \sum_{k=0}^{\infty} a_k|_{\mathbb{H}_0}$. Now define $a := H_{\frac{1}{2}}(\psi)\overline{a}$ with the Mellin kernel cut-off operator $H_{\frac{1}{2}}$, and a function $\psi \in C_0^\infty(\mathbb{R}_+)$ such that $\psi \equiv 1$ near $r = 1$. From Theorem 2.6.13 we now obtain that $a \in MS^{\overline{\mu};\ell}_{(V,)O}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C};\mathcal{E},\tilde{\mathcal{E}})$, and moreover $a|_{\Gamma_0} \sim \overline{a}$ resp. $a|_{\mathbb{H}_0} \underset{V}{\sim} \overline{a}$. From Proposition 2.6.3 we now conclude that indeed $a \underset{(V)}{\sim} \sum_{k=0}^{\infty} a_k$ in the sense of Definition 2.6.6 as asserted. In the classical case we have $\overline{a}$ and consequently also a as classical symbols. □

Degenerate symbols and Mellin quantization

2.6.15 Definition. Let E and $\tilde{E}$ be Hilbert spaces. For $\varphi \in C_0^\infty(\mathbb{R}_+)$ and $a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$ define

$$Q(\varphi,a)(\xi,z) := \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-is\eta}e^{i\eta}s^{-z}\varphi(s)a(\xi,\eta)\,ds\,đ\eta \tag{2.6.vi}$$

for $(\xi,z) \in \mathbb{R}^n \times \mathbb{C}$.

Moreover, for every $\gamma \in \mathbb{R}$ we define for $\psi \in C_0^\infty(\mathbb{R}_+)$ and $a \in S^{\mu;\ell}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E})$

$$\tilde{Q}_\gamma(\psi,a)(\xi,z) := \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{2}-\gamma}}\int_0^\infty s^\zeta e^{i(s-1)z}\psi(s)a(\xi,\zeta)\,\frac{ds}{s}\,d\zeta \tag{2.6.vii}$$

for $(\xi,z) \in \mathbb{R}^n \times \mathbb{C}$.

If $\varphi \equiv 1$ near $r = 1$ respectively $\psi \equiv 1$ near $r = 1$ we simply write $Q(\varphi,a) = Q(a)$ and $\tilde{Q}_\gamma(\psi,a) = \tilde{Q}_\gamma(a)$, respectively. The mapping Q is called *Mellin quantization*, $\tilde{Q}_\gamma$ is called *inverse Mellin quantization with respect to the weight* $\gamma \in \mathbb{R}$.

2.6.16 Theorem. *Let $\{E_j\}$ and $\{\tilde{E}_j\}$ be scales of Hilbert spaces, and $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as before.*

a) The operator Q from (2.6.vi) provides continuous bilinear mappings

$$Q : \begin{cases} C_0^\infty(\mathbb{R}_+)\times S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R};\mathcal{E},\tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{O(cl)}(\mathbb{R}^n\times\mathbb{C};\mathcal{E},\tilde{\mathcal{E}}) \\ C_0^\infty(\mathbb{R}_+)\times S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H};\mathcal{E},\tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{V,O(cl)}(\mathbb{R}^n\times\mathbb{C};\mathcal{E},\tilde{\mathcal{E}}). \end{cases}$$

Moreover, there are universal coefficients $(c_{k,j}(\varphi,\gamma))$ depending neither on a nor on the Hilbert spaces, but only on $\{(\partial_r^\nu\varphi)(1);\ \nu \in \mathbb{N}_0\}$ and $\gamma \in \mathbb{R}$, such *that the following asymptotic expansion holds for $Q(\varphi,a)|_{\Gamma_{\frac{1}{2}-\gamma}}$, respectively $Q(\varphi,a)|_{\mathbb{H}_{\frac{1}{2}-\gamma}}$, in terms of a:*

$$Q(\varphi,a)(\xi,\frac{1}{2}-\gamma+i\tau) \underset{(V)}{\sim} \varphi(1)a(\xi,-\tau)+\sum_{k=1}^{\infty}\sum_{j=0}^{k} c_{k,j}(\varphi,\gamma)(-\tau)^j\big(\partial_\tau^{k+j}a\big)(\xi,-\tau) \tag{2.6.viii}$$

for $\tau\in\mathbb{R}$, respectively $\tau\in\mathbb{H}_-$.

b) The operator $\tilde{Q}_\gamma$ from (2.6.vii) provides continuous bilinear mappings

$$\tilde{Q}_\gamma : \begin{cases} C_0^\infty(\mathbb{R}_+)\times S^{\mu;\ell}_{(cl)}(\mathbb{R}^n\times\Gamma_{\frac{1}{2}-\gamma};\mathcal{E},\tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{iO(cl)}(\mathbb{R}^n\times\mathbb{C};\mathcal{E},\tilde{\mathcal{E}}) \\ C_0^\infty(\mathbb{R}_+)\times S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n\times\mathbb{H}_{\frac{1}{2}-\gamma};\mathcal{E},\tilde{\mathcal{E}}) \longrightarrow S^{\mu;\ell}_{V,iO(cl)}(\mathbb{R}^n\times\mathbb{C};\mathcal{E},\tilde{\mathcal{E}}). \end{cases}$$

The spaces in the image are given by means of the isomorphism

$$S^{\mu;\ell}_{(V,)iO(cl)}(\mathbb{R}^n\times\mathbb{C};\mathcal{E},\tilde{\mathcal{E}}) \ni a(\xi,z)\longmapsto a(\xi,-iz)\in S^{\mu;\ell}_{(V,)O(cl)}(\mathbb{R}^n\times\mathbb{C};\mathcal{E},\tilde{\mathcal{E}}).$$

Moreover, there are universal coefficients $(d_{k,j}(\psi,\gamma))$ depending neither on a nor on the Hilbert spaces, but only on $\{(\partial_r^\nu\psi)(1);\ \nu\in\mathbb{N}_0\}$ and $\gamma\in\mathbb{R}$, such that the following asymptotic expansion holds for $\tilde{Q}_\gamma(\psi,a)|_{\mathbb{R}}$, respectively $\tilde{Q}_\gamma(\psi,a)|_{\mathbb{H}}$, in terms of a:

$$\begin{aligned}\tilde{Q}_\gamma(\psi,a)(\xi,\tau) \underset{(V)}{\sim} {}& \psi(1)a(\xi,\frac{1}{2}-\gamma-i\tau) \\ &+\sum_{k=1}^{\infty}\sum_{j=0}^{k} d_{k,j}(\psi,\gamma)(-i\tau)^j\big(\partial_\tau^{k+j}a\big)(\xi,\frac{1}{2}-\gamma-i\tau)\end{aligned} \tag{2.6.ix}$$

for $\tau\in\mathbb{R}$, respectively $\tau\in\mathbb{H}$.

c) For $\varphi,\psi\in C_0^\infty(\mathbb{R}_+)$ such that $\varphi\equiv 1$ and $\psi\equiv 1$ near $r=1$ we have

$$Q(\tilde{Q}_\gamma(a))-a\in\begin{cases} S^{-\infty}(\mathbb{R}^n\times\Gamma_{\frac{1}{2}-\gamma};\mathcal{E},\tilde{\mathcal{E}}) \\ S^{-\infty}_V(\mathbb{R}^n\times\mathbb{H}_{\frac{1}{2}-\gamma};\mathcal{E},\tilde{\mathcal{E}}), \end{cases}$$

$$\tilde{Q}_\gamma(Q(a))-a\in\begin{cases} S^{-\infty}(\mathbb{R}^n\times\mathbb{R};\mathcal{E},\tilde{\mathcal{E}}) \\ S^{-\infty}_V(\mathbb{R}^n\times\mathbb{H};\mathcal{E},\tilde{\mathcal{E}}). \end{cases}$$

2.6.17 Remark. Let $\varphi,\psi\in C_0^\infty(\mathbb{R}_+)$ such that $\varphi\equiv 1$ and $\psi\equiv 1$ near $r=1$. By Theorem 2.6.16 the mappings $Q: a\mapsto Q(a)$ and $\tilde{Q}_\gamma: a\mapsto\tilde{Q}_\gamma(a)$ provide isomorphisms

$$Q : \begin{cases} S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}})/S^{-\infty}(\mathbb{R}^n \times \mathbb{R}; \mathcal{E}, \tilde{\mathcal{E}}) \\ \qquad \longrightarrow S^{\mu;\ell}_{O(cl)}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}})/S^{-\infty}_{O}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}}) \\ S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}})/S^{-\infty}_{V}(\mathbb{R}^n \times \mathbb{H}; \mathcal{E}, \tilde{\mathcal{E}}) \\ \qquad \longrightarrow S^{\mu;\ell}_{V,O(cl)}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}})/S^{-\infty}_{V,O}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}}), \end{cases}$$

$$\tilde{Q}_\gamma : \begin{cases} S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}})/S^{-\infty}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}) \\ \qquad \longrightarrow S^{\mu;\ell}_{iO(cl)}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}})/S^{-\infty}_{iO}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}}) \\ S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}})/S^{-\infty}_{V}(\mathbb{R}^n \times \mathbb{H}_{\frac{1}{2}-\gamma}; \mathcal{E}, \tilde{\mathcal{E}}) \\ \qquad \longrightarrow S^{\mu;\ell}_{V,iO(cl)}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}})/S^{-\infty}_{V,iO}(\mathbb{R}^n \times \mathbb{C}; \mathcal{E}, \tilde{\mathcal{E}}). \end{cases}$$

On the level of quotient spaces, Q and $\tilde{Q}_\gamma$ are independent of φ and ψ, respectively, and we have $\tilde{Q}_\gamma = Q^{-1}$ (according to part c) of Theorem 2.6.16).

2.6.18 Theorem. *Let E and $\tilde{E}$ be Hilbert spaces, and let $\varphi, \psi \in C_0^\infty(\mathbb{R}_+)$ be fixed such that $\varphi \equiv 1$ and $\psi \equiv 1$ near $r = 1$.*

a) Let $\tilde{a} \in C^\infty(\mathbb{R}_+, S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}; E, \tilde{E}))$, and define $a(r,\xi,\tau) := \tilde{a}(r,\xi,r\tau)$. Then we have for every $\gamma \in \mathbb{R}$

$$op_r(a)(\xi) - op_M^\gamma(Q(\tilde{a}))(\xi) = op_r\big((1-\varphi)(\frac{r'}{r})a\big)(\xi)$$

as operators in $C_0^\infty(\mathbb{R}_+, E) \longrightarrow C^\infty(\mathbb{R}_+, \tilde{E})$.

b) Let $\tilde{a} \in C^\infty(\mathbb{R}_+, S^{\mu;\ell}(\mathbb{R}^n \times \Gamma_{\frac{1}{2}-\gamma}; E, \tilde{E}))$, and define $a(r,\xi,\tau) := \tilde{Q}_\gamma(\tilde{a})(r,\xi,r\tau)$. Then we have

$$op_M^\gamma(\tilde{a})(\xi) - op_r(a)(\xi) = op_M^\gamma((1-\psi)(\frac{r'}{r})\tilde{a})(\xi)$$

as operators in $C_0^\infty(\mathbb{R}_+, E) \longrightarrow C^\infty(\mathbb{R}_+, \tilde{E})$.

2.6.19 Remark. Theorem 2.6.18 gives the explanation for the name "Mellin quantization" for the operator Q, and "inverse Mellin quantization" for $\tilde{Q}$. Together with Theorem 2.6.16 it follows, that modulo "smoothing" (Volterra) operators we obtain isomorphisms between Fourier pseudodifferential operators with degenerate (Volterra) symbols on the half-axis and (Volterra) Mellin pseudodifferential operators.

Note in particular, that if in a) or b) the dependence of the symbol $\tilde{a}$ on $r \in \mathbb{R}_+$ is $C_B^\infty(\overline{\mathbb{R}}_+, \cdot)$, then so is also the dependence of $Q(\tilde{a})$ or $\tilde{Q}(\tilde{a})$, respectively.

2.7 Volterra Fourier operators with global weight conditions

2.7.1 Remark. In this section we briefly recall the elements of a parameter-dependent pseudodifferential calculus based on the Fourier transform, where the action is carried out with respect to the Volterra covariable, and the symbols globally satisfy weighted estimates in the variable. In view of the considerations from Section 2.5, note in particular relation (2.5.i), these are easily obtained together with the general theory of such global operators, see, e. g., Cordes [10], Dorschfeldt, Grieme, and Schulze [11], Parenti [45], Schrohe [55], or Seiler [64].

2.7.2 Definition. Let E and $\tilde{E}$ be Hilbert spaces, and let again $\mathbb{H}$ be the upper half-plane in $\mathbb{C}$, and $\mu, \varrho_1, \varrho_2 \in \mathbb{R}$. The spaces of globally weighted (Volterra) double- resp. left-/right- symbols with parameter-space $\mathbb{R}^n$ are defined as

$$\begin{aligned}
S^{\mu,\varrho_1,\varrho_2;\ell}_{(cl)}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E}) &:= S^{\varrho_1,\varrho_2}(\mathbb{R}\times\mathbb{R},S^{\mu;\ell}_{(cl)}(\mathbb{R}^n\times\mathbb{R};E,\tilde{E})),\\
S^{\mu,\varrho_1;\ell}_{(cl)}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E}) &:= S^{\varrho_1}(\mathbb{R},S^{\mu;\ell}_{(cl)}(\mathbb{R}^n\times\mathbb{R};E,\tilde{E})),\\
S^{\mu,\varrho_1,\varrho_2;\ell}_{V(cl)}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E}) &:= S^{\varrho_1,\varrho_2}(\mathbb{R}\times\mathbb{R},S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n\times\mathbb{H};E,\tilde{E})),\\
S^{\mu,\varrho_1;\ell}_{V(cl)}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E}) &:= S^{\varrho_1}(\mathbb{R},S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n\times\mathbb{H};E,\tilde{E})).
\end{aligned}$$

With a symbol $a \in S^{\mu,\varrho_1,\varrho_2;\ell}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E})$ we associate a family of pseudodifferential operators acting as continuous operators

$$\mathrm{op}_r(a)(\xi) : \mathcal{S}(\mathbb{R},E) \longrightarrow \mathcal{S}(\mathbb{R},\tilde{E})$$

for $\xi \in \mathbb{R}^n$ as in Section 2.2. The corresponding operator spaces are denoted as follows:

$$\begin{aligned}
L^{\mu,\varrho_1;\ell}_{(cl)}(\mathbb{R};\mathbb{R}^n;E,\tilde{E}) &:= \{\mathrm{op}_r(a)(\xi);\ a \in S^{\mu,\varrho_1;\ell}_{(cl)}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E})\},\\
L^{\mu,\varrho_1;\ell}_{V(cl)}(\mathbb{R};\mathbb{R}^n;E,\tilde{E}) &:= \{\mathrm{op}_r(a)(\xi);\ a \in S^{\mu,\varrho_1;\ell}_{V(cl)}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E})\}.
\end{aligned}$$

As before, Theorem 2.7.4 below guarantees that these spaces are well-defined in the sense, that the action applied to left-symbols only already gives the full space of operators, and by means of the uniqueness of the left-symbol for the action of the operator we have the canonical isomorphism between symbols and operators, which induces a Fréchet topology on the operator spaces.

2.7.3 Remark. In the sequel, the asymptotic expansions are to be understood in the following sense:
Let $(\mu_k), (\varrho_k) \subseteq \mathbb{R}$ be sequences such that $\mu_k, \varrho_k \underset{k\to\infty}{\longrightarrow} -\infty$, and $\overline{\mu} := \max\limits_{k\in\mathbb{N}} \mu_k$ as

well as $\overline{\varrho} := \max_{k\in\mathbb{N}} \varrho_k$. Moreover, let

$$a_k \in \begin{cases} S^{\mu_k,\varrho_k;\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E}) \\ S_V^{\mu_k,\varrho_k;\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E}), \end{cases}$$

$$a \in \begin{cases} S^{\overline{\mu},\overline{\varrho};\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E}) \\ S_V^{\overline{\mu},\overline{\varrho};\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E}). \end{cases}$$

We write $a \underset{(V)}{\sim} \sum_{j=1}^{\infty} a_j$ if for every $R\in\mathbb{R}$ there is a $k_0\in\mathbb{N}$ such that for $k>k_0$

$$a - \sum_{j=1}^{k} a_j \in \begin{cases} S^{R,R;\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E}) \\ S_V^{R,R;\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E}). \end{cases}$$

2.7.4 Theorem. *Let $a\in S^{\mu,\varrho_1,\varrho_2;\ell}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E})$. Then there exist unique left- and right-symbols $a_L(r,\xi,\tau)$, $a_R(r',\xi,\tau)\in S^{\mu,\varrho_1+\varrho_2;\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E})$ such that $op_r(a)(\xi)=op_r(a_L)(\xi)=op_r(a_R)(\xi)$ as operators on $\mathcal{S}(\mathbb{R},E)$. These symbols are given by means of oscillatory integral formulas analogous to that of Theorem 2.2.2. The class of Volterra symbols remains preserved, i. e., if $a\in S_V^{\mu,\varrho_1,\varrho_2;\ell}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E})$, then also $a_L,a_R\in S_V^{\mu,\varrho_1+\varrho_2;\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E})$.*

More precisely, the mappings $a\longmapsto a_L,a_R$ are well-defined and continuous in the spaces

$$\left.\begin{matrix} S_{(cl)}^{\mu,\varrho_1,\varrho_2;\ell}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E}) \\ S_{V(cl)}^{\mu,\varrho_1,\varrho_2;\ell}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E}) \end{matrix}\right\} \longrightarrow \begin{cases} S_{(cl)}^{\mu,\varrho_1+\varrho_2;\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E}) \\ S_{V(cl)}^{\mu,\varrho_1+\varrho_2;\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E}). \end{cases}$$

Moreover, we have the asymptotic expansions

$$a_L(r,\xi,\tau) \underset{(V)}{\sim} \sum_{k=0}^{\infty} \frac{1}{k!}\partial_\tau^k D_{r'}^k a(r,r',\xi,\tau)|_{r'=r},$$

$$a_R(r',\xi,\tau) \underset{(V)}{\sim} \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k \partial_\tau^k D_r^k a(r,r',\xi,\tau)|_{r=r'}.$$

2.7.5 Theorem. *Let E, $\tilde{E}$ and $\hat{E}$ be Hilbert spaces. Let*

$$a \in \begin{cases} S_{(cl)}^{\mu,\varrho;\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};\tilde{E},\hat{E}) \\ S_{V(cl)}^{\mu,\varrho;\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};\tilde{E},\hat{E}), \end{cases}$$

$$b \in \begin{cases} S_{(cl)}^{\mu',\varrho';\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R};E,\tilde{E}) \\ S_{V(cl)}^{\mu',\varrho';\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{H};E,\tilde{E}), \end{cases}$$

and $A(\xi) = op_r(a)(\xi)$, $B(\xi) = op_r(b)(\xi)$. Then the composition as operators on $\mathcal{S}(\mathbb{R}, E)$ is given as $A(\xi)B(\xi) = op_r(a\#b)(\xi)$ with the Leibniz-product

$$a\#b \in \begin{cases} S^{\mu+\mu',\varrho+\varrho';\ell}_{(cl)}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}; E, \hat{E}) \\ S^{\mu+\mu',\varrho+\varrho';\ell}_{V(cl)}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{H}; E, \hat{E}). \end{cases}$$

of the symbols a and b. The analogue of the oscillatory integral formula (2.2.i) for the Leibniz-product from Theorem 2.2.4 is valid, and we have the asymptotic expansion

$$a\#b \underset{(V)}{\sim} \sum_{k=0}^{\infty} \frac{1}{k!}(\partial_\tau^k a)(D_r^k b).$$

2.7.6 Remark. From Theorem 2.7.4 we obtain the following pseudolocality property of this calculus:

Let

$$a(r, r', \xi, \tau) \in \begin{cases} S^{\mu,\varrho_1,\varrho_2;\ell}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}; E, \tilde{E}) \\ S^{\mu,\varrho_1,\varrho_2;\ell}_V(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{H}; E, \tilde{E}) \end{cases}$$

such that $a(r, r', \xi, \tau) \equiv 0$ for $|r - r'| < \varepsilon$ for a sufficiently small $\varepsilon > 0$. Then

$$\mathrm{op}_r(a)(\xi) \in \begin{cases} L^{-\infty,-\infty}(\mathbb{R}; \mathbb{R}^n; E, \tilde{E}) \\ L_V^{-\infty,-\infty}(\mathbb{R}; \mathbb{R}^n; E, \tilde{E}). \end{cases}$$

2.7.7 Proposition. *Let $a \in S_V^{\mu,\varrho;\ell}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{H}; E, \tilde{E})$. Then $op_r(a)(\xi)$ restricts for every $r_0 \in \mathbb{R}$ to a family of continuous operators*

$$op_r(a)(\xi) : \mathcal{S}_0((-\infty, r_0), E) \longrightarrow \mathcal{S}_0((-\infty, r_0), \tilde{E}).$$

Proof. This follows analogously to Theorem 2.5.8. □

2.7.8 Notation. Let $[\cdot] : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a smoothed norm function, i. e. $[\cdot] \in C^\infty(\mathbb{R}, \mathbb{R}_+)$, and $[r] \equiv |r|$ for $|r|$ sufficiently large. Note that $[\cdot]^\varrho \in S^\varrho(\mathbb{R})$ for every $\varrho \in \mathbb{R}$.

2.7.9 Theorem. *Let $\{E_0, E, E_1\}$ and $\{\tilde{E}_0, \tilde{E}, \tilde{E}_1\}$ be Hilbert triples, and $A(\xi) = op_r(a)(\xi)$ with $a \in S^{\mu,\varrho;\ell}_{(cl)}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}; E_0, \tilde{E}_0)$. Then the formal adjoint operators $A(\xi)^{(*),\delta}$ with respect to the $[\cdot]^{-\delta}L^2$-inner product belong to $L^{\mu,\varrho;\ell}_{(cl)}(\mathbb{R}; \mathbb{R}^n; \tilde{E}_1, E_1)$. More precisely, for $u \in \mathcal{S}(\mathbb{R}, E_0)$ and $v \in \mathcal{S}(\mathbb{R}, \tilde{E}_1)$ we have*

$$\int_{\mathbb{R}} \langle \big(A(\xi)u\big)(r), v(r)\rangle_{\tilde{E}} [r]^{2\delta}\, dr = \int_{\mathbb{R}} \langle u(r), \big(A(\xi)^{(*),\delta} v\big)(r)\rangle_E [r]^{2\delta}\, dr$$

with $A(\xi)^{(*),\delta} = op_r(a^{(*),\delta})(\xi)$*, where* $a^{(*),\delta} \in S^{\mu,\varrho;\ell}_{(cl)}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}; \tilde{E}_1, E_1)$ *is given as*

$$a^{(*),\delta} = \big([r]^{-2\delta}\big(a(r',\xi,\tau)\big)^*[r']^{2\delta}\big)_L.$$

We have the asymptotic expansion

$$a^{(*),\delta}(r,\xi,\tau) \sim \sum_{k=0}^{\infty} \sum_{p+q=k} \frac{1}{p!q!}\big([r]^{-2\delta}D_r^p[r]^{2\delta}\big)\big(\partial_\tau^k D_r^q(a(r,\xi,\tau))^*\big).$$

2.7.10 Definition. Let E be a Hilbert space. For $s, \delta \in \mathbb{R}$ define

$$H^{s,\delta}(\mathbb{R}, E) := \langle r\rangle^{-\delta} H^s(\mathbb{R}, E)$$

with the Sobolev space $H^s(\mathbb{R}, E)$, see Definition 2.2.12. This space is endowed with the induced scalar product which turns it into a Hilbert space.

Moreover, for $r_0 \in \mathbb{R}$ let $H_0^{s,\delta}((-\infty, r_0], E)$ denote the closed subspace of all $u \in H^{s,\delta}(\mathbb{R}, E)$ such that $\mathrm{supp}(u) \subseteq (-\infty, r_0]$, which equals the closure of $\mathcal{S}_0((-\infty, r_0), E)$ in $H^{s,\delta}(\mathbb{R}, E)$.

2.7.11 Theorem. *Let* E *and* $\tilde{E}$ *be Hilbert spaces. Moreover, let* $a \in S^{\mu,\varrho;\ell}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}; E, \tilde{E})$*, and* $s, \delta, \nu, \delta' \in \mathbb{R}$ *where* $\nu \geq \frac{\mu}{\ell}$ *and* $\delta' \geq \varrho$*.*

Then $op_r(a)(\xi)$ *extends for* $\xi \in \mathbb{R}^n$ *by continuity to an operator* $op_r(a)(\xi) \in \mathcal{L}(H^{s,\delta}(\mathbb{R}, E), H^{s-\nu,\delta-\delta'}(\mathbb{R}, \tilde{E}))$*, which induces a continuous embedding*

$$L^{\mu,\varrho;\ell}(\mathbb{R}; \mathbb{R}^n; E, \tilde{E}) \hookrightarrow \begin{cases} S^{\mu}(\mathbb{R}^n; H^{s,\delta}(\mathbb{R}, E), H^{s-\nu,\delta-\delta'}(\mathbb{R}, \tilde{E})) & \nu \geq 0 \\ S^{\mu-\nu}(\mathbb{R}^n; H^{s,\delta}(\mathbb{R}, E), H^{s-\nu,\delta-\delta'}(\mathbb{R}, \tilde{E})) & \nu \leq 0 \end{cases}$$

into the space of operator-valued symbols in the Sobolev spaces.

Moreover, restriction of Volterra pseudodifferential operators on the $H_0^{s,\delta}$*-spaces provides continuous mappings*

$$L_V^{\mu,\varrho;\ell}(\mathbb{R}; \mathbb{R}^n; E, \tilde{E}) \longrightarrow \begin{cases} S^{\mu}(\mathbb{R}^n; H_0^{s,\delta}((-\infty, r_0], E), H_0^{s-\nu,\delta-\delta'}((-\infty, r_0], \tilde{E})) \\ \qquad \text{if} \quad \nu \geq 0 \\ S^{\mu-\nu}(\mathbb{R}^n; H_0^{s,\delta}((-\infty, r_0], E), H_0^{s-\nu,\delta-\delta'}((-\infty, r_0], \tilde{E})) \\ \qquad \text{if} \quad \nu \leq 0 \end{cases}$$

for each $r_0 \in \mathbb{R}$*.*

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