Degenerated operator equations of higher order

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1 Introduction

The main object of the present paper is the differential–operator equation

$$\mathcal{L}u \equiv (-1)^m D_t^m(t^\alpha D_t^m u) + AD_t^{m-1}(t^{\alpha-1}D_t^m u) + t^{\alpha-2m} P u = f(t), \tag{1.1}$$

where m is a natural number, $t \in (0,b)$, $b < \infty$, $\alpha \geq 0$, $\alpha \neq 1, 3, ..., 2m-1$, $D_t \equiv d/dt$, $f \in L_{2,-\alpha}((0,b),\mathcal{H}) \equiv H$, A and P are operators acting in some Hilbert space $\mathcal{H}$, commuting with D_t and possessing a complete system of eigenfunctions $\{\varphi_k\}_{k=1}^\infty$ that forms a Riesz basis in $\mathcal{H}$.

We are interested in the character of boundary conditions for $t = 0, b$, which guarantee the existence and uniqueness of solution of (1.1) for every $f \in H$. These conditions depend on α and on properties of the operators A and P.

In the case when $m = 1$, A is the operation of multiplication by a constant, $P \equiv D_x^2$ is an operator on a closed interval and $\alpha \geq 1$, the dependence on the sign of A of the character of the conditions with respect to t was first observed by Keldiš [6]. Later, the corresponding effect was studied by Višik [13]. For $0 \leq \alpha \leq 2m$ this problem has been considered in [2], [10], [11]. For the case $\alpha > 2m$ the factors $t^{\alpha-1}$ and $t^{\alpha-2m}$ in (1.1) are essential, and instead of the usual $L_2(0,b)$ we consider the weighted space $L_{2,-\alpha}(0,b)$.

Our approach is close to that of Dezin [2] and is based on the case A and P are the operators of multiplication by numbers a and p. We describe the spectrum of one–dimensional operator and prove embedding theorems for weighted Sobolev spaces.

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2 The one–dimensional case

2.1 The space W_α^m

For simplicity, in this section we assume the function $u(t)$ to be real–valued. Let $\dot{C}^m$ be the set of m times continuously differentiable functions on $[0, b]$ satisfying the conditions

$$u^{(k)}(t)\Big|_{t=0} = u^{(k)}(t)\Big|_{t=b} = 0, \;\; k = 0, 1, ..., m-1. \tag{2.1}$$

Denote by $\dot{W}^m$ the completion of $\dot{C}^m$ in the norm generated by the inner product

$$\{u, v\} = (u^{(m)}, v^{(m)}),$$

where $(\cdot\,,\cdot)$ stands for the inner product in $L_2(0, b)$. Moreover, let W_α^m denote the completion of $\dot{W}^m$ in the norm

$$|u, W_\alpha^m|^2 = \int_0^b t^\alpha\, |u^{(m)}|^2\, dt. \tag{2.2}$$

Define the inner product in W_α^m by

$$\{u, v\}_\alpha = (t^\alpha\, u^{(m)}, v^{(m)}).$$

It is clear that the embedding $\dot{W}^m \subset W_\alpha^m$ is proper and that the space $\dot{W}^m$ coincides with the class of $(m-2)$ times continuously differentiable functions $u(t)$ for which $u^{(m-1)}(t)$ is absolutely continuous and (2.1) is fulfilled. It is easy to check that the norms of $\dot{W}^m$ and W_α^m are equivalent on $[\eta, b]$, $\eta > 0$. Hence it is enough to study the properties of functions from W_α^m for t in a neighbourhood of 0. For the proof of the following proposition we refer to [11].

Proposition 2.1 *For every* $u \in W_\alpha^m$

$$|u^{(k)}(t)|^2 \le C_k\, t^{2m-2k-1-\alpha}\, |u, W_\alpha^m|^2, \tag{2.3}$$

where

$$\alpha \ne 2n+1, \;\; n = 0, 1, ..., m-1, \;\; k = 0, 1, ..., m-1.$$

For $\alpha = 2n+1$, $n = 0, 1, ..., m-1$ *in* (2.3) *the factor* $t^{2m-2k-1-\alpha}$ *should be replaced by* $t^{2m-2k-2n-2}|\ln t|$, $k = 0, 1, ..., m-1$.

The inequality (2.3) implies that for $\alpha < 1$ (weak degeneracy) the conditions (2.1) are "retained", while for $\alpha \ge 1$ (strong degeneracy) not all boundary conditions are "retained". For instance, for $1 \le \alpha < 3$, $u^{(m-1)}(t)|_{t=0}$ can become infinite, while for $\alpha \ge 2m-1$ all $u^{(k)}(t)|_{t=0}$ may be infinite for $k = 0, 1, ..., m-1$.

Proposition 2.2 *For every* $\alpha \ge 0$, $\alpha \ne 1, 3, ..., 2m-1$, *we have the embedding*

$$W_\alpha^m \subset L_{2,\alpha-2m}. \tag{2.4}$$

Proof. Since $\dot{C}^m$ is dense in W_α^m, it is enough to prove (2.4) in the case where $u \in \dot{C}^m$. First, for $m = 1$ have

$$\begin{aligned}\left|\int_0^b t^{\alpha-2}\, u^2(t)\, dt\right|^2 &= \left|\frac{1}{\alpha-1}\int_0^b u^2(t)\, dt^{\alpha-1}\right|^2 \\ &= \left|\frac{2}{\alpha-1}\int_0^b t^{\alpha/2-1}\, u'(t)\, t^{\alpha/2}\, u(t)\, dt\right|^2 \\ &\leq \frac{4}{(\alpha-1)^2}\int_0^b t^\alpha\, (u'(t))^2\, dt \int_0^b t^{\alpha-2}\, u^2(t)\, dt.\end{aligned}$$

This implies

$$\int_0^b t^{\alpha-2}\, u^2(t)\, dt \leq \frac{4}{(\alpha-1)^2}\int_0^b t^\alpha\, (u'(t))^2\, dt.$$

Then, repeating this procedure m times we obtain

$$\int_0^b t^{\alpha-2m}\, u^2(t)\, dt \leq \frac{4^m}{(\alpha-1)^2\,(\alpha-3)^2\cdots(\alpha-(2m-1))^2}\int_0^b t^\alpha\, (u^{(m)})^2\, dt. \tag{2.5}$$

This completes the proof of Proposition 2.2.

Remark 2.3 *The embedding* (2.4) *breaks down for* $\alpha = 1, 3, ..., 2m-1$.

We prove this assertion for $\alpha = 1$ and $m = 1$. Consider the function $u(t) = |\ln t|^\beta$, $t \in (0, a)$, $a < \min(1, b)$. In $L_{2,-1}(0, a)$ its norm is finite for $\beta < -1/2$, while in $W_1^1(0, a)$ it is finite for $\beta < 1/2$. Therefore, for $-1/2 \leq \beta < 1/2$ the embedding (2.4) breaks down.

Remark 2.4 *The embedding* (2.4) *is not compact.*

For simplicity we verify this assertion for $m = 1$ and $\alpha = 2$. Consider the bounded in $W_2^1(0, a)$ sequence of functions $u_n(t) = n^{-1/2}\, t^{-1/2}\, |\ln t|^{-1/2-1/2n}$, where $a < \min(1, b)$. It is easy to check that there is no subsequence of $u_n(t)$ convergent in the metric of $L_2(0, a)$ (see [3]).

Remark 2.5 *For* $\alpha \geq 0$, $\alpha \neq 1, 3, ..., 2m-1$ *in the space* W_α^m *we can define the norm*

$$\|u\|_\alpha^2 = \int_0^b \left|\left(t^{\alpha/2}\, u\right)^{(m)}\right|^2 dt \tag{2.6}$$

that is equivalent to (2.2).

The proof of equivalence of the norms (2.2) and (2.6) follows the same lines as those of Proposition 2.2 and the inequality (2.5) (see [4]). Therefore, we can write

$$W_\alpha^m = t^{-\alpha/2}\, \dot{W}^m.$$

This means that $u \in \dot{W}^m$ implies $v = t^{-\alpha/2} u \in W_\alpha^m$. Observe that for $\alpha = 1, 3, ..., 2m-1$ the norms (2.2) and (2.6) are not equivalent. Denote by $\tilde{W_\alpha^m}$ the completion of $\dot{W}^m$ by the norm (2.6). The inequality (2.3) implies the embedding

$$\tilde{W_\alpha^m} \subset W_\alpha^m.$$

With the help of Mellin transform (see [9])

$$\mathcal{M}u(z) = \int_0^\infty t^{z-1} u(t)\, dt\,,$$

a norm equivalent to (2.2) can be introduced.

Let $v_0, v_1 \in C^\infty[0,b]$, $v_1 = 1 - v_0$ be cut–off functions with $v_0(x) = 0\,(v_0(x) = 1)$ in some neighbourhood of $x = b\,(x = 0)$.

Remark 2.6 *The norm given by the formula*

$$\begin{aligned} ||u||_{m,\alpha}^2 &= \int_{\Gamma_{1/2+\alpha/2-m}} \left(1 + |z|^2\right)^m |(\mathcal{M} v_0 u)(z)|^2\, |dz| \\ &+ \int_{\Gamma_{1/2-m}} \left(1 + |z|^2\right)^m |(\mathcal{M}\mathcal{R} v_1 u)(z)|^2\, |dz|, \end{aligned}$$

where $\Gamma_\beta = \{z \in \mathbb{C} : \operatorname{Re} z = \beta\}$, $\mathcal{R}v(x) = v(b-x)$ *and* $\alpha < 1$, *is equivalent to* (2.2).

For the proof we refer to [4].

Denote by $L_{2,\alpha}$ the weight space

$$L_{2,\alpha} = \{u(t) :\ |u, L_{2,\alpha}|^2 = \int_0^b t^\alpha\, |u(t)|^2\, dt < \infty\}$$

.

Proposition 2.7 *For every* $\alpha \geq 0$ *the embedding*

$$W_\alpha^m \subset L_{2,\alpha} \tag{2.7}$$

is compact.

Proof. First consider the case when $\alpha \neq 1, 3, ..., 2m-1$. It follows from (2.3) that

$$|u(t)|^2 \leq C_k\, t^{2m-1-\alpha}\, |u, W_\alpha^m|^2.$$

This implies

$$|u, L_{2,\alpha}| \leq c_1\, |u, W_\alpha^m|. \tag{2.8}$$

To prove that the embedding (2.7) is compact we use (2.3) for $k = 1$ and write

$$
\begin{aligned}
|u(t+h) - u(t), L_{2,\alpha}|^2 &= \int_0^b t^\alpha \, |u(t+h) - u(t)|^2 \, dt \\
&= \int_0^b t^\alpha \left| \int_t^{t+h} u'(\tau) \, d\tau \right|^2 \, dt \\
&\leq c_1 \, |u, W_\alpha^m|^2 \int_0^b t^\alpha \left| \int_t^{t+h} \tau^{(2m-3-\alpha)/2} \, d\tau \right|^2 \, dt \\
&= c_2 \, |u, W_\alpha^m|^2 \, |h|^2 \int_0^b t^\alpha \, \xi^{2m-3-\alpha} \, dt \\
&\leq c \, |h|^2 \, |u, W_\alpha^m|^2,
\end{aligned}
$$

where $\xi \in [t, t+h]$. Therefore,

$$|u(t+h) - u(t), L_{2,\alpha}| \leq c \, |h| \, |u, W_\alpha^m|. \tag{2.9}$$

The result now follows from compactness criterion in $L_{2,\alpha}$ (see [2], [10]). For $\alpha = 1, 3, ..., 2m-1$ the proof is similar.

Djarov in [3] has proved, that the embedding

$$W_\alpha^m \subset L_{2,\beta},$$

is compact for every $\beta > \alpha - 2$ and $m = 1$. For us it is sufficient the case $\beta = \alpha$.

2.2 Self–adjoint Equation

We consider the one–dimensional version of equation (1.1) for $A = 0$:

$$B\,u \equiv (-1)^m \, D_t^m (t^\alpha \, D_t^m \, u) + p \, t^{\alpha - 2m} \, u = f(t), \tag{2.10}$$

where $\alpha \geq 0$, $\alpha \neq 1, 3, ..., 2m-1$, $f(t) \in L_{2,-\alpha}$, and p is a constant.

Definition 2.8 *A function $u \in W_\alpha^m$ is called a generalized solution of equation* (2.10), *if for every $v \in W_\alpha^m$*

$$\{u, v\}_\alpha + p(t^{\alpha - 2m} \, u, v) = (f, v). \tag{2.11}$$

We set

$$d(m, \alpha) = 4^{-m} \, (\alpha - 1)^2 \, (\alpha - 3)^2 \cdots (\alpha - (2m-1))^2. \tag{2.12}$$

Theorem 2.9 *Assume that $p + d(m, \alpha) > 0$, $\alpha \geq 0$ and $\alpha \neq 1, 3, ..., 2m-1$. Then equation* (2.10) *has a unique generalized solution for every $f \in L_{2,-\alpha}$.*

Proof. Uniqueness of the generalized solution of (2.10) follows from (2.5) and (2.11) with $f = 0$ and $v = u$. To prove the existence we consider the functional $l_f(v) \equiv (f, v)$, $f \in L_{2,-\alpha}$ over the space W^m_α. Using (2.8) we write

$$\begin{aligned} |l_f(v)|^2 &= \left| \int_0^b f(t)\, \overline{v(t)}\, dt \right|^2 \\ &= \left| \int_0^b t^{-\alpha/2} f(t)\, t^{\alpha/2}\, \overline{v(t)}\, dt \right|^2 \\ &\le \int_0^b t^{-\alpha} |f(t)|^2\, dt \int_0^b t^\alpha |v(t)|^2\, dt \\ &\le c\, |f, L_{2,-\alpha}|^2\, |v, W^m_\alpha|^2. \end{aligned}$$

Therefore, $l_f(v)$ is a linear bounded functional over the space W^m_α. The result now follows from Riesz's lemma on the representation of such functionals (see [2]). Theorem 2.9 is proved.

Remark 2.10 *Observe that the generalized solution* $u(t)$ *of equation* (2.10) *belongs to* $W_2^{2m}(\delta, b-\delta)$ *for every* $\delta > 0$.

Hence in each interval $(\delta, b-\delta)$ the generalized solution $u(t)$ coincides with the usual solution of (2.10).

An element $f \in L_{2,-\alpha}$ can be represented in the form

$$f(t) = t^\alpha f_1(t) \tag{2.13}$$

It is clear that $f_1 \in L_{2,\alpha}$ and

$$|f, L_{2,-\alpha}| = |f_1, L_{2,\alpha}|.$$

Now, using Definition 2.8 we define an operator $\mathbb{B} : W^m_\alpha \subset L_{2,\alpha} \to L_{2,\alpha}$.

Definition 2.11 *We say that a function* $u(t) \in W^m_\alpha$ *belongs to the domain* $D(\mathbb{B})$ *of an operator* $\mathbb{B}$ *if* (2.11) *is fulfilled for some* $f \in L_{2,-\alpha}$. *In this case we will write* $\mathbb{B}u = f_1$, *where the function* f_1 *is specified by* (2.13).

It follows from Definition 2.11 that the operator $\mathbb{B}$ acts by the formula

$$\mathbb{B}u \equiv t^{-\alpha}\{(-1)^m D_t^m (t^\alpha D_t^m u) + p t^{\alpha-2m} u\}$$

and for every $u \in D(\mathbb{B})$, $v \in W^m_\alpha$ and $f \in L_{2,-\alpha}$

$$(\mathbb{B}u, v)_\alpha = (f, v),$$

where $(\cdot\,,\cdot)_\alpha$ stands for the inner product in $L_{2,\alpha}$.

Theorem 2.12 *Under the assumptions of Theorem* 2.12 *the operator* $\mathbb{B}$ *is positive and self-adjoint in* $L_{2,\alpha}$. *Moreover,* $\mathbb{B}^{-1} : L_{2,\alpha} \to L_{2,\alpha}$ *is a compact operator.*

Proof. The symmetry and positivity of $\mathbb{B}$ is an immediate consequence of Definition 2.11. The self–adjointness of the symmetric operator $\mathbb{B}$ follows from the fact that, according to Theorem 2.9, for every $f \in L_{2,-\alpha}$ the equation (1.1) is solvable, that is, for every $f_1 \in L_{2,\alpha}$ of the form (2.13) the equation $\mathbb{B}u = f_1$ is solvable. Using (2.5), (2.11) with $v = u$ and the embedding $L_{2,\alpha-2m} \subset L_{2,\alpha}$ we can write

$$\begin{aligned}(d(m,\alpha)+p)\,c\,|u, L_{2,\alpha}|^2 &\leq (d(m,\alpha)+p)\,|u, L_{2,\alpha-2m}|^2 \\ &\leq |u, W_\alpha^m|^2 + p\,|u, L_{2,\alpha-2m}|^2 \\ &\leq |f, L_{2,-\alpha}|\,|u, L_{2,\alpha}| = |f_1, L_{2,\alpha}|\,|u, L_{2,\alpha}|.\end{aligned}$$

Therefore,

$$|\mathbb{B}^{-1} f_1, L_{2,\alpha}| \leq c_1\,|f_1, L_{2,\alpha}|.$$

This implies that $\mathbb{B}^{-1}$ is bounded. To prove that $\mathbb{B}^{-1}$ is compact it remains to observe that, according to Proposition 2.2, the embedding $D(\mathbb{B}) \subset W_\alpha^m \subset L_{2,\alpha}$ is compact. Theorem 2.12 is proved.

Applying standard properties of the spectra of self–adjoint compact operators we get the following corollary (see [5]).

Corollary 2.13 *The operator $\mathbb{B}$ has a pure point spectrum, and the system of corresponding eigenfunctions is dense in $L_{2,\alpha}$.*

Observe that if λ is an eigenvalue of $\mathbb{B}$, and $u(t)$ is the corresponding eigenfunction, then according to Definition 2.11,

$$(-1)^m\,D_t^m(t^\alpha\,D_t^m\,u) + p\,t^{\alpha-2m}\,u = \lambda\,t^\alpha\,u.$$

Remark 2.14 *Note that, if $p = 0$, $\alpha = 2m$ and $f \in L_2(0,b)$, then the spectrum of the operator B is pure continuous and coincides with the ray $[d(m,2m);+\infty)$ (when $p \neq 0$, then we can take $\lambda - p$ instead of the spectral parameter λ).*

For the proof we refer to [11] and note that it is in fact a consequence of the embedding (2.4).

Now we consider equation (2.10), as above, with $p = 0$ and $f \in L_{2,2m-\alpha}$

$$Qu \equiv (-1)^m\,D_t^m(t^\alpha\,D_t^m\,u) = f(t),\ f \in L_{2,2m-\alpha}. \tag{2.14}$$

We can define (as for equation (2.10)) generalized solutions for equation (2.14) and prove that for every $f \in L_{2,2m-\alpha}$ a generalized solution exists and is unique. Let $f = t^{\alpha-2m} f_1$. It is clear that $f_1 \in L_{2,\alpha-2m}$. Then we can define an operator $\mathbb{Q} : L_{2,\alpha-2m} \to L_{2,\alpha-2m}$ as in Definition 2.11.

Theorem 2.15 *The operator $\mathbb{Q}$ has a pure continuous spectrum which coincides with the ray $[d(m,\alpha);+\infty)$.*

Theorem 2.15 can be proved similarly to [11] with the help of embedding (2.4).

2.3 Non–selfadjoint Equation

Now we consider the one–dimensional version of equation (1.1)

$$S\,u \equiv (-1)^m\, D_t^m (t^\alpha\, D_t^m\, u) + aD_t^{m-1}(t^{\alpha-1} D_t^m u) + pt^{\alpha-2m}\, u = f(t), \tag{2.15}$$

where $\alpha \geq 0,\ \alpha \neq 1, 3, ..., 2m-1,\ f(t) \in L_{2,-\alpha}$, a and p are constant.

Definition 2.16 *A function $u \in W_\alpha^m$ is called generalized solution of equation* (2.15)*, if for every $v \in W_\alpha^m$*

$$\{u, v\}_\alpha + a(-1)^{m-1}(t^{\alpha-1} D_t^m\, u, D_t^{m-1} u) + p(t^{\alpha-2m}\, u, v) = (f, v). \tag{2.16}$$

Theorem 2.17 *Let the following condition be fulfilled*

$$a(\alpha-1)(-1)^m > 0,\ \gamma = d(m,\alpha) + a/2(\alpha-1)(-1)^m d(m-1,\alpha-2) + p > 0, \tag{2.17}$$

where $d(m,\alpha)$ is defined in (2.12)*. Then equation* (2.15) *has a unique generalized solution for every $f(t) \in L_{2,-\alpha}$.*

Proof. *Uniqueness.* For the proof of uniqueness in equality (2.16) we set $f = 0$ and $u = v$. Let $\alpha > 1$ (in the case $\alpha < 1$ the proof is similar and we use that $\left(t^{\alpha-1}\big(u^{(m-1)}(t)\big)^2\right)|_{t=0} = 0$ [see Proposition 2.1]). Then integrating by parts we get

$$\begin{aligned}\left(t^{\alpha-1}u^{(m)}\,,\,u^{(m-1)}\right) &= \int_0^b t^{\alpha-1}u^{(m)}(t)u^{(m-1)}(t)\,dt \\ &= -\frac{1}{2}\left(t^{\alpha-1}\left(u^{(m-1)}(t)\right)^2\right)|_{t=0} \\ &- \frac{\alpha-1}{2}\int_0^b t^{\alpha-2}\left(u^{(m-1)}(t)\right)^2 dt.\end{aligned}$$

It follows from (2.3) that $\left(t^{\alpha-1}\big(u^{(m-1)}(t)^2\right)|_{t=0}$ is finite. Using (2.5) we can write

$$\int_0^b t^{\alpha-2}\left(u^{(m-1)}(t)\right)^2 dt \geq d(m-1,\alpha-2)\int_0^b t^{\alpha-2m}(u(t))^2\,dt.$$

From this inequality and (2.5) we get

$$\begin{aligned}0 = \{u\,,\,u\} &+ a(-1)^{m-1}(t^{\alpha-1}u^{(m)}\,,\,u^{(m-1)}) + p(t^{\alpha-2m}u\,,\,u) \\ &\geq \gamma\int_0^b t^{\alpha-2m}(u(t))^2\,dt \\ &+ \frac{1}{2}a(-1)^m\left(t^{\alpha-1}\left(u^{(m-1)}(t)\right)^2\right)|_{t=0}.\end{aligned}$$

Now uniqueness of the generalized solution immediately follows from the condition (2.17). *Existence.* First note that the functional $l_f(v) \equiv (f,v)$ can be represented in the form $(f,v) = \{u^*, v\}$, where $u^* \in W_\alpha^m$ (see the proof of Theorem 2.9). The last two terms in the left-hand side of equality (2.16) also can be regarded as a continuous linear functional relative to u and represented in the form $\{u, \mathcal{K}v\}$, where $\mathcal{K}v \in W_\alpha^m$. Indeed, using inequality (2.5) we can write

$$\begin{aligned}
|a(-1)^{m-1}(t^{\alpha-1}D_t^m u, D_t^{m-1}u) &+ p(t^{\alpha-2m} u, v)| \\
&\le |a(t^{\alpha/2}D_t^m u, t^{\alpha/2-1}D_t^{m-1}u)| + |p(t^{\alpha/2-m} u, t^{\alpha/2-m}v)| \\
&\le c_1|u, W_\alpha^m|\left\{\int_0^b t^{\alpha-2}\left(v^{(m-1)}(t)\right)^2 dt\right\}^{1/2} \\
&+ c_2|u, L_{2,\alpha-2m}||v, L_{2,\alpha-2m}| \\
&\le \frac{2c_1}{|\alpha-1|}|u, W_\alpha^m||v, W_\alpha^m| + c_3|u, W_\alpha^m||v, W_\alpha^m| \\
&= c|u, W_\alpha^m||v, W_\alpha^m|.
\end{aligned}$$

Now from (2.16) we obtain

$$\{u, (I+\mathcal{K})v\} = \{u^*, v\} \tag{2.18}$$

for every $v \in W_\alpha^m$. Note that the image of the operator $I+\mathcal{K}$ is dense in W_α^m. Indeed, if there exists a $u_0 \in W_\alpha^m$ such that

$$\{u_0, (I+\mathcal{K})v\} = 0$$

for every $v \in W_\alpha^m$, we get $u_0 = 0$, since we have already proved uniqueness of the generalized solution for equation (2.15).

Let $0 < \sigma d(m,\alpha) \le \gamma$. Then we can write

$$\begin{aligned}
\{u, (I+\mathcal{K})u\} &\ge \sigma\{u\,,\,u\} + [(1-\sigma)d(m,\alpha) \\
&+ a/2(\alpha-1)(-1)^m d(m-1,\alpha-2) + p]\int_0^b t^{\alpha-2m}u^2(t)\,dt \\
&= \sigma\{u\,,\,u\} + (\gamma - \sigma d(m,\alpha))\int_0^b t^{\alpha-2m}u^2(t)\,dt \\
&\ge \sigma\{u\,,\,u\}.
\end{aligned}$$

Finally, we get

$$\{u, (I+\mathcal{K})u\} \ge \sigma\{u\,,\,u\}. \tag{2.19}$$

From (2.19) it follows that $(I+\mathcal{K})^{-1}$ is defined on W_α^m and is bounded therefore, there exist $I+\mathcal{K}^*$ and $(I+\mathcal{K}^*)^{-1} = ((I+\mathcal{K})^{-1})^*$. Then from (2.18) we obtain

$$u = (I+\mathcal{K}^*)^{-1}u^*.$$

Theorem 2.17 is proved.

Let $f = t^\alpha f_1$. As in the self-adjoint case we can define an operator $\mathbb{S}$, according to Definition 2.11.

Definition 2.18 *We say that $u(t) \in D(\mathbb{S})$ if* (2.16) *is fulfilled for some $f \in L_{2,-\alpha}$, and then we will write $\mathbb{S}u = f_1$.*

Proposition 2.19 *Under the assumptions of Theorem* 2.17 *the operator $\mathbb{S}^{-1} : L_{2,\alpha} \to D(\mathbb{S}) \subset L_{2,\alpha}$ is compact.*

Proof. For the proof we first note that

$$|u, L_{2,\alpha}| \leq (d(m,\alpha))^{-1}|f_1, L_{2,\alpha}|. \tag{2.20}$$

Indeed, setting $v = u$ in (2.16) we obtain

$$\begin{aligned} d(m,\alpha)|u, L_{2,\alpha}|^2 &\leq d(m,\alpha)|L_{2,\alpha-2m}|^2 \\ &\leq |(f,u)| \\ &\leq |f, L_{2,-\alpha}||u, L_{2,\alpha}| = |f_1, L_{2,\alpha}||u, L_{2,\alpha}|. \end{aligned}$$

Now to complete the proof of Proposition 2.19 it is enough to apply the compactness of the embedding (2.7). Proposition 2.19 is proved.

For the case $a(\alpha - 1)(-1)^m < 0$ we consider the operator

$$Tv \equiv (-1)^m D_t^m(t^\alpha D_t^m v) + aD_t^m(t^{\alpha-1}D_t^{m-1}v) + pt^{\alpha-2m} v = g(t). \tag{2.21}$$

Definition 2.20 *We say that v is a generalized solution of* (2.21), *if the following equality holds*

$$(Su, v) = (u, g), \tag{2.22}$$

for every $u \in D(S)$.

Let $g = t^\alpha g_1$. Definition 2.20 of a generalized solutions defines an operator $\mathbb{T} : L_{2,\alpha} \to L_{2,\alpha}$ (see Definition 2.18). We can express formula (2.22) in the form $(\mathbb{S}u, v)_\alpha = (u, g)_\alpha$. Since $D(S) = D(\mathbb{S})$ is dense in $L_{2,\alpha}$, we obtain that

$$\mathbb{T} = \mathbb{S}^*$$

in $L_{2,\alpha}$.

Theorem 2.21 *Under the assumptions of Theorem* 2.17, *for every $g \in L_{2,-\alpha}$ a generalized solution of equation* (2.21) *exists and is unique. Moreover, $\mathbb{T}^{-1} : L_{2,\alpha} \to L_{2,\alpha}$ is compact.*

Proof. Solvability of the equation $\mathbb{S}u = f_1$ for any right-hand side implies uniqueness of the solution of (2.21), while existence of the bounded operator $\mathbb{S}^{-1}$ (Proposition 2.19) implies solvability of (2.21) for any $g \in L_{2,-\alpha}$ (see, for example, [1]).

Because of $(\mathbb{S}^*)^{-1} = (\mathbb{S}^{-1})^*$, compactness of the operator $\mathbb{S}^{-1}$ implies compactness of the operator $\mathbb{T}^{-1}$. Theorem 2.21 is proved.

Remark 2.22 *For $\alpha > 1$, and for every generalized solution v of the equation* (2.21), *we have*

$$\left(t^{\alpha-1}(v^{(m-1)}(t))^2\right)|_{t=0} = 0. \tag{2.23}$$

In fact, replacing g by Tv in equality (2.22), integrating by parts the second term and using equality (2.16) we obtain (2.23). Note that for equation (2.15) the left-hand side of (2.23) for $a(-1)^m > 0$ is only bounded.

3 Operator Equation

In this section we consider the operator equation (1.1):

$$\mathcal{L}\,u \equiv (-1)^m\,D_t^m(t^\alpha\,D_t^m\,u) + AD_t^{m-1}(t^{\alpha-1}D_t^m\,u) + t^{\alpha-2m}\,P\,u = f(t),\ f \in H.$$

Recall that if a system $\{\varphi_k\}_{k=1}^\infty$ is a Riesz basis in $\mathcal{H}$, every element $x \in \mathcal{H}$ can be uniquely represented in the form $x = \sum_{k=1}^\infty x_k\varphi_k$, and the inequality

$$c_2\sum_{k=1}^\infty |x_k|^2 \le ||x||^2 \le c_1\sum_{k=1}^\infty |x_k|^2 \tag{3.1}$$

holds, where $||\cdot||$ stands for the norm in $\mathcal{H}$.

By assumption, the operators A and P appearing in (1.1) have a complete common system of eigenfunctions $\{\varphi_k\}_{k=1}^\infty$ that forms Riesz basis in $\mathcal{H}$. So we have

$$u(t) = \sum_{k=1}^\infty u_k(t)\,\varphi_k,\ f(t) = \sum_{k=1}^\infty f_k(t)\,\varphi_k,\ A\varphi_k = a_k\varphi_k,\ P\,\varphi_k = p_k\,\varphi_k,\ k \in \mathbb{N}. \tag{3.2}$$

Hence, the operator equation (1.1) can be decomposed into an infinite chain of ordinary differential equations

$$\mathcal{L}_k\,u_k \equiv (-1)^m\,D_t^m(t^\alpha\,D_t^m\,u_k) + a_kD_t^{m-1}(t^{\alpha-1}D_t^m\,u_k) + t^{\alpha-2m}\,p_k\,u_k = f_k(t),\ k \in \mathbb{N}. \tag{3.3}$$

The condition $f \in H$ implies that $f_k \in L_{2,-\alpha}$ for $k \in \mathbb{N}$. For the one–dimensional equation (3.3) we can define the generalized solutions $u_k(t), k \in \mathbb{N}$ (compare with the Definitions 2.8, 2.16 and 2.20).

Definition 3.1 *A function $u \in W_\alpha^m((0,b),\mathcal{H}) \subset L_{2,\alpha}((0,b),\mathcal{H})$ admitting the representation*

$$u(t) = \sum_{k=1}^\infty u_k(t)\varphi_k,$$

where $u_k(t)$ are the generalized solutions of the one–dimensional equation (3.3) *is called a generalized solution of the operator equation* (1.1).

The following theorem is a consequense of the general results of Dezin [1].

Theorem 3.2 *The operator equation* (1.1) *is uniquely solvable if and only if the one–dimensional equation* (3.3) *is uniquely solvable and the inequality*

$$|u_k, L_{2,\alpha}| \leq c|f_k, L_{2,-\alpha}| = c|g_k, L_{2,\alpha}| \tag{3.4}$$

is fulfilled, uniformly with respect to $k \in \mathbb{N}$, *where* $f_k = t^\alpha g_k$.

Theorems 2.9, 2.17 and 2.21 shows us that a sufficient condition for relations (3.4) are either

$$p_k + d(m, \alpha) > \varepsilon > 0,\ k \in \mathbb{N}, \tag{3.5}$$

for all $k \in \mathbb{N}$, with $a_k = 0$, or

$$\gamma_k = d(m, \alpha) + a_k/2(\alpha - 1)(-1)^m d(m-1, \alpha - 2) + p_k > \varepsilon > 0, \tag{3.6}$$

for all $k \in \mathbb{N}$, such that $a_k \neq 0$. Therefore, we can state the following result.

Theorem 3.3 *Let* (3.5) *and* (3.6) *be fulfilled and let* $\alpha \neq 1, 3, ..., 2m-1$. *Then the operator equation* (1.1) *has a unique generalized solution for every* $f \in H$.

Proof. Observe that if u is generalized solution of (1.1), then according to (3.1) and (3.4) we have

$$\begin{aligned} |u, L_{2,\alpha}((0,b),\mathcal{H})|^2 &= \int_0^b t^\alpha \,\|u(t)\|^2\, dt \\ &\leq c_1 \int_0^b t^\alpha \sum_{k=1}^\infty |u_k(t)|^2\, dt \\ &\leq c_2 \sum_{k=1}^\infty |f_k, L_{2,-\alpha}|^2 \\ &\leq c|f, H|^2. \end{aligned} \tag{3.7}$$

Similarly to (2.13) we set $f = t^\alpha g$. It is clear that $g \in L_{2,\alpha}((0,b),\mathcal{H})$ and $|f, H| = |g, L_{2,\alpha}((0,b),\mathcal{H})|$. Inequality (3.7) can be written in the form

$$|u, L_{2,\alpha}((0,b),\mathcal{H})| \leq c\,|g, L_{2,\alpha}((0,b),\mathcal{H})|. \tag{3.8}$$

Analogously to the one–dimensional case the generalized solution of the operator equation (1.1) generates the operator

$$\Lambda : W_\alpha^m((0,b),\mathcal{H}) \subset L_{2,\alpha}((0,b),\mathcal{H}) \to L_{2,\alpha}((0,b),\mathcal{H}).$$

Inequality (3.8) implies that $\Lambda^{-1} : L_{2,\alpha}((0,b),\mathcal{H}) \to W_\alpha^m((0,b),\mathcal{H}) \subset L_{2,\alpha}((0,b),\mathcal{H})$ is a bounded operator. Hence, we have $0 \in \rho(\Lambda)$, where $\rho(\Lambda)$ is the resolvent set of the operator Λ.

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