Vladimir Nazaikinskii,

Bert-Wolfgang Schulze, and Boris Sternin

QUANTIZATION METHODS

in

DIFFERENTIAL EQUATIONS

Doctor Vladimir Nazaikinskii
Moscow State University
E-mail: nazaikinskii@mtu-net.ru
nazaikinskii@math.uni-potsdam.de

Professor Bert-Wolfgang Schulze
Potsdam University
E-mail: schulze@math.uni-potsdam.de

Professor Boris Sternin
Moscow State University
E-mail: sternine@mtu-net.ru
sternin@math.uni-potsdam.de

Chapter 2

Exactly Soluble Commutation Relations (The Simplest Class of Classical Mechanics)

Contents

In § 1.5 of the preceding chapter, we introduced the notions of left and right ordered representations of a system of relations. Namely, recall that for a given system Σ of relations,

$$\Sigma : \varphi_\alpha \big(\overset{k_1}{A}_{j_1}, \ldots, \overset{k_m}{A}_{j_m} \big) = 0, \quad \alpha = 1, \ldots, N,$$

operators $l_1, \ldots, l_n$ (respectively, $r_1, \ldots, r_n$) acting in the space on n-ary symbols are called operators of left (respectively, right) ordered

representation if

$$(l_j f)(A) = A_j f(A),$$

$$(r_j f)(A) = f(A) A_j$$

for any n-ary symbol $f(x_1, \ldots, x_n) \in \mathcal{F}_n$ whenever $A = (\overset{1}{A_1}, \ldots, \overset{n}{A_n})$ is an n-tuple of $\mathcal{F}$-generators satisfying Σ. If a relation system Σ admits a left (or right) regular representation by $\mathcal{F}$-generators in $\mathcal{F}_n$, then the set

$$\Lambda = \{ f(A) \,|\, f \in \mathcal{F}_n \}$$

is an algebra, and the composition law is given by

$$f(A)g(A) = (f \underset{l}{*} g)(A) = (f \underset{r}{*} g)(A),$$

where

$$f \underset{l}{*} g = f(l)g, \quad f \underset{r}{*} g = g(\overset{\leftarrow}{r})f,$$

$$l = (\overset{1}{l_1}, \ldots, \overset{n}{l_n}), \quad \overset{\leftarrow}{r} = (\overset{n}{r_1}, \ldots, \overset{1}{r_n}).$$

(In most cases,

$$f \underset{l}{*} g = f \underset{r}{*} g \overset{\text{def}}{\equiv} f * g;$$

in particular, this is always the case if the Jacobi condition holds.)

Thus, the existence of an ordered representation permits one to move the arguments of f (or g) in the product

$$f(A)g(A) \equiv [\![f(\overset{1}{A_1}, \ldots, \overset{n}{A_n})]\!] \, [\![g(\overset{1}{A_1}, \ldots, \overset{n}{A_n})]\!]$$

to their respective places so as to obtain a single expression of the form $h(\overset{1}{A_1}, \ldots, \overset{n}{A_n})$. For that reason, relation systems admitting an ordered representation (left, or, equivalently, right) will be called *systems of commutation relations*, or, for short, simply *commutation relations*.

Given a system of relations (probably arising from some specific problem in quantum mechanics, differential equations, etc.), we face the question as to whether this is a system of commutation relations

and, if this is the case, how to compute the left (or right) ordered representation. These questions are by no means easy, and often the answer cannot be given just by analyzing the form of the relations (see the very first example in § 1 below). However, there exist wide classes of relations systems for which the (positive) answer is known, and these classes, along with certain examples that do not fit into these classes but still can be treated individually, are the subject of this chapter. In this chapter, we deal with "exactly soluble" commutation relations. The description of the class of exactly soluble relations is fuzzy in that it is based on human ability rather than on purely mathematical facts. By terming some relations "exactly soluble," we only mean that a proof is known of the fact that these relations are commutation relations and a method is available for writing out the ordered representation.

The present chapter is naturally divided into three sections. Section 1 comprises a series of simple examples in which the ordered representation is computed. Each example usually starts from some specific problem (often simplified and purely illustrative) in which the need for computing the ordered representation of some specific relation system arises. However, in treating the problem we do not go far beyond the theme of this chapter, that is, we stop once the representation is computed and, possibly, the Jacobi condition is verified. At most, we give a few hints as to what might follow next. However, we return to the consideration of some of these problems (in a setting that is closer to real-life issues) in subsequent chapters.

In § 2 we deal with a series of increasingly complicated classes of *Lie commutation relations* (that is, relations covered by the theory of Lie algebras, Lie groups, and their representations.) Although the task of noncommutative analysis is not the same as that of Lie theory, the interaction between the two areas proves to be interesting and fruitful.

In § 3 we consider certain classes of *non-Lie commutation relations* (often referred to as "*nonlinear*," which is technically incorrect, since Lie relations also involve nonlinear terms) and establish a technique (actually, imitating the one for the Lie case) for constructing ordered representations.

In closing, we point out that in this chapter we have tried to make it clear that noncommutative analysis, like Earth in an ancient model of the universe, rests on three whales:

- ordered operators notation, including *Feynman indices* and *autonomous brackets*;

- the *commutation rule*

$$AB = CA \Rightarrow Af(B) = f(C)A$$

  (and its many-operator version);

- the quantization *uniqueness theorem*

$$A = C \Rightarrow f(A) = f(C).$$

Essentially, most of the complicated computations sometimes occurring in noncommutative analysis are none other than the reduction of a fact to be proved to the simplest commutation rule, often with the help of the uniqueness theorem. The standard trick in such a reduction is to use block matrix operators (we have already seen numerous examples of that in Chapter 1) and symbols involving difference derivatives.

Having this in mind may well help the reader realize that noncommutative analysis in no more intricate than the well-known differential and integral calculus (which have their own whales like the Leibniz rule, Newton–Leibniz formula, etc.).

2.1   Some examples

We start by giving a fairly simple example that shows that the property of being a system of commutation relations is rather subtle and may even depend on the Feynman ordering of the operators in the tuple as well as on the choice of the class of admissible symbols.

Example 1 ([3]) We consider the system consisting of the single relation

$$AB = B^2 A \tag{2.1}$$

for two operators A and B. The symbol class $\mathcal{F}$ is assumed to be the algebra of polynomials. Then the following assertion holds:

For tuples of the form $(\overset{1}{A}, \overset{2}{B})$, relation (2.1) is a system of commutation relations. For tuples of the form $(\overset{2}{A}, \overset{1}{B})$, relation (2.1) is not a system of commutation relations.

Indeed,

$$B[\![f(\overset{1}{A}, \overset{2}{B})]\!] = \overset{2}{B} \, f(\overset{1}{A}, \overset{2}{B})$$

and

$$A[\![f(\overset{1}{A}, \overset{2}{B})]\!] = \overset{3}{A} \, f(\overset{1}{A}, \overset{2}{B}) = \overset{1}{A} \, f(\overset{1}{A}, (\overset{2}{B})^2)$$

by the commutation theorem. Consequently, the left ordered representation operators for the ordering $(\overset{1}{A}, \overset{2}{B})$ exist and have the form (where the arguments of a symbol f are denoted by $(x, y) : f = f(x, y)$)

$$l_A f(x, y) = x f(x, y^2), \quad l_B f(x, y) = y f(x, y). \tag{2.2}$$

We note that the operators (2.2) satisfy relation (2.1):

$$l_A l_B = (l_B)^2 l_A.$$

Thus, the Jacobi condition is satisfied, and the algebra $\mathcal{A}$ with two generators A, B and relation (2.1) is isomorphic as a linear space to the space of polynomials $f(x, y)$ in two variables. The composition law is given by

$$f * g = f(\overset{1}{l_A}, \overset{2}{l_B}) g$$

or, more specifically, if

$$f = \sum a_{jk} x^j y^k, \quad g = \sum b_{jk} x^j y^k,$$

then

$$f * g = \sum_{j,k,l,m} a_{jk} b_{lm} x^{j+l} y^{k+2^l m}.$$

Now we are in a position to prove that there is no left ordered representation for the ordering $(\overset{2}{A}, \overset{1}{B})$. Otherwise, we would have, for some symbol f

$$BA = f(\overset{2}{A}, \overset{1}{B}) = \sum a_{jk} A^j B^k = \sum a_{jk} B^{2^j k} A^j$$

(we have used the multiplication law for the ordering $(\overset{1}{A}, \overset{2}{B})$). Since l_A and l_B satisfy the Jacobi condition, it follows that

$$xy = \sum a_{jk} x^j y^{2^j k},$$

which is impossible (the right-hand side does not contain odd powers of y).

However, if, instead of polynomials, we admit symbols involving arbitrary fractional powers of y, then the ordered representation for the ordering $(\overset{2}{A}, \overset{1}{B})$ exists. In this case, it is more convenient to compute the right ordered representation first. We have (the tilde is used to distinguish from the old ordering)

$$\widetilde{r}_B f(x, y) = y f(x, y), \quad \widetilde{r}_A f(x, y) = x f(x, \sqrt{y}).$$

One obviously has[1]

$$\begin{aligned}
\widetilde{r}_B \widetilde{r}_A f(x, y) &= yx f(x, \sqrt{y}), \\
\widetilde{r}_A \widetilde{r}_B^2 f(x, y) &= \widetilde{r}_A(y^2 f(x, y)) = xy f(x, \sqrt{y}),
\end{aligned}$$

and so the Jacobi condition is satisfied:

$$\widetilde{r}_B \widetilde{r}_A f(x, y) = \widetilde{r}_A \widetilde{r}_B^2$$

(note the reverse order of operators).

Now, by the general formula,

$$\begin{aligned}
\widetilde{l}_A f(x, y) &= f(\overset{1}{\widetilde{r}}_A, \overset{2}{\widetilde{r}}_B) x = x f(x, y) \\
\widetilde{l}_B f(x, y) &= f(\overset{1}{\widetilde{r}}_A, \overset{2}{\widetilde{r}}_B) y
\end{aligned}$$

(unfortunately, no simple explicit formula is available for the operator $\widetilde{l}_B$).

The preceding example was only intended to illustrate the complicated nature of the property of being commutation relations. We proceed to examples that are more important practically.

---

[1]Needless to say, we assume that the variable y ranges in $\mathbf{R}_+$.

Example 2 *Coordinate and momentum operators.* We consider functions of the operators

$$q_1, \ldots, q_n, \quad \hat{p}_1 = -ih\frac{\partial}{\partial q_1}, \ldots, \quad \hat{p}_n = -ih\frac{\partial}{\partial q_n} \qquad (2.3)$$

of coordinates and momenta in quantum mechanics. We shall use the conventional ordering

$$(\overset{2}{q}_1, \ldots, \overset{2}{q}_n, \overset{1}{\hat{p}}_1, \ldots, \overset{1}{\hat{p}}_n).$$

An important example of such functions is the Schrödinger operator

$$\hat{H} = H\left(\overset{2}{q}_1, \ldots, \overset{2}{q}_n, -ih\,\frac{\overset{1}{\partial}}{\partial q_1}, \ldots, -ih\,\frac{\overset{1}{\partial}}{\partial q_n}\right) \equiv H\left(\overset{2}{q}, -ih\,\frac{\overset{1}{\partial}}{\partial q}\right), \quad (2.4)$$

where $H(q, p)$ is the classical Hamiltonian function, for example,

$$H(q, p) = \frac{p^2}{2m} + V(q).$$

(One possible definition of the operator $\hat{H}$ for fairly general Hamiltonian functions (symbols) $H \in S^\infty(\mathbf{R}^{2n})$ is given by

$$[\hat{H}u](q) = \left(\frac{i}{2\pi h}\right)^n \int e^{\frac{i}{h}pq} H(q, p)\tilde{u}(p)\, dp, \qquad (2.5)$$

where $\tilde{u}(p)$ is the quantum Fourier transform of u.) We wish to learn how to compute products of the form $\hat{H}_1\hat{H}_2$. This may be helpful, for example, in solving the Cauchy problem for the operator $\hat{H}_1$. One seeks the solution of the problem

$$-ih\frac{\partial u}{\partial t} + \hat{H}_1 u = 0, \quad u\big|_{t=0} = u_0(q) \qquad (2.6)$$

in the form

$$u = \hat{H}_2(t)u_0.$$

Then for the symbol $H_2(t) \equiv H_2(q, p, t)$ of the resolving operator of problem (2.6) we obtain the problem

$$-ih\frac{\partial H_2}{\partial t} + H_1 * H_2 = 0, \quad H_2\big|_{t=0} = 1. \qquad (2.7)$$

So we are really interested in computing the twisted product $*$. To simplify the notation, we consider the case $n = 1$. The operators q and $\hat{p}$ satisfy the Heisenberg commutation relation

$$[\hat{p}, q] = -ih, \tag{2.8}$$

which will be used as a basis for computing the regular representation. We have

$$q[\![f(\overset{2}{q}, \overset{1}{\hat{p}})]\!] = qf(\overset{2}{q}, \overset{1}{\hat{p}}),$$

and so

$$l_q f(q, p) = qf(q, p), \quad l_q = q. \tag{2.9}$$

On the other hand, by Theorem 5 of Chapter 1,

$$\left[\hat{p}, [\![f(\overset{2}{q}, \overset{1}{\hat{p}})]\!]\right] = [\overset{3}{\hat{p}}, q]\frac{\delta f}{\delta q}(\overset{2}{q}, \overset{4}{q}; \overset{1}{\hat{p}}) = -ih\frac{\partial f}{\partial q}(\overset{2}{q}, \overset{1}{\hat{p}}) \tag{2.10}$$

(the commutator is a scalar, and we can move the Feynman indices over $\overset{2}{q}$ and $\overset{4}{q}$ together), whence we readily obtain

$$l_p = p - ih\partial/\partial q. \tag{2.11}$$

Note that the Jacobi condition is obviously satisfied,

$$[l_p, l_q] = -ih.$$

The composition law for h^{-1}-pseudodifferential (i.e. Hamiltonian) operators reads

$$H_1 * H_2 = H_1\left(\overset{2}{q}, p - ih\,\frac{\overset{1}{\partial}}{\partial q}\right)(H_2(q, p)). \tag{2.12}$$

Example 3 *Coordinate and momenta operators revisited. The Wigner equation.*

Let ψ, the "wave function," be a solution of the Schrödinger equation

$$-ih\frac{\partial \psi}{\partial t} + \hat{H}\psi = 0, \tag{2.13}$$

where $\hat{H}$ is the Schrödinger operator (2.4). The corresponding *density matrix* (Blokhintsev matrix) $\hat{\rho}$, that is, the rank one projection on ψ given by the formulas

$$\hat{\rho}u = \psi(\psi, u)$$

(where $(\cdot, \cdot)$ is the L_2 inner product), satisfies the (reversed) Heisenberg equation

$$-ih\frac{\partial\hat{\rho}}{\partial t} + [\hat{H}, \hat{\rho}] = 0. \qquad (2.14)$$

Let us represent $\hat{\rho}$ as a function of the coordinate and momenta operators (2.3): $\hat{\rho} = \rho(\overset{2}{\hat{q}}, \overset{1}{\hat{p}})$. Using the general formula (2.5) for functions of the operators (2.3) is not difficult to find that[2]

$$\rho(q, p) = \left(\frac{1}{2\pi h}\right)^n \psi(q)\widetilde{\psi}(p)e^{\frac{ipx}{h}}. \qquad (2.15)$$

Let us find the equation satisfied by the function (2.15), which is known as the *density function*. In principle, this equation could be derived directly from Eq. (2.13) with the use of properties of the quantum Fourier transform. However, we prefer to use (2.14) and the composition formula, thus obtaining

$$-ih\frac{\partial\rho}{\partial t} + H * \rho - \rho * H = 0. \qquad (2.16)$$

The straightforward application of the expression (2.12) for the twisted product is not advisable in the last term on the left-hand side in (2.16), because in this case we would obtain the operator $\rho\left(\overset{2}{q}, p - ih\ \overset{1}{\frac{\partial}{\partial q}}\right)$ occurring in our equation, which is intended to be an equation for ρ. However the remedy is clear. In the last term on the left-hand side, we must use the right ordered representation operators:

$$\rho * H = H(\overset{1}{\hat{r}}_q, \overset{2}{\hat{r}}_p)\rho. \qquad (2.17)$$

Let us compute these operators.

---

[2]We do not explicitly write out the argument t.

We have

$$r_p f(q,p) = f\left(\overset{2}{q}, p - ih\,\overset{1}{\frac{\partial}{\partial q}}\right)p = pf(q,p),$$

that is, $r_p = p$, and

$$
\begin{aligned}
r_q f(q,p) &= f\left(\overset{2}{q}, p - ih\,\overset{1}{\frac{\partial}{\partial q}}\right)q \\[2mm]
&= f(q,p)q + \sum_{j=1}^{\infty} \frac{f_p^{(j)}(q,p)}{j!}\left(-ih\frac{\partial}{\partial q}\right)^j (q) \\[2mm]
&= f(q,p)q - ih\frac{\partial f}{\partial p}(q,p).
\end{aligned}
$$

(The trick with the Taylor series expansion on the right-hand side can easily be justified, say, by using the Taylor formula with remainder in Hadamard's form.) Thus, $r_q = q - i\frac{\partial}{\partial p}$, and substituting all this into (2.16), we finally obtain the following equation for the density function:

$$
\begin{aligned}
-ih\frac{\partial\rho(q,p,t)}{\partial t} + H\left(\overset{2}{q}, p - ih\,\overset{1}{\frac{\partial}{\partial q}}\right)\rho(q,p,t) \\[3mm]
-H\left(q - ih\,\overset{1}{\frac{\partial}{\partial p}}, \overset{2}{p}\right)\rho(q,p,t) = 0.
\end{aligned}
\qquad (2.18)
$$

This is the Wigner equation, well known in quantum mechanics. From this equation, it is easy to obtain the asymptotics of the density function as $h \to 0$ in the weak topology, that is, in the sense of distributions. Indeed, suppose that $\rho(q,p,t)$ can be expanded in a regular asymptotic series in powers of h:

$$\rho(q,p,t) \cong \rho_0(q,p,t) - ih\rho_1(q,p,t) - (ih)^2\rho_2(q,p,t)\ldots, \qquad (2.19)$$

where the coefficients ρ_j are independent of h.

The expansion (2.19) means that

$$< \rho, f > \cong \sum_{j=0}^{\infty} (-ih)^j < \rho_j, f >, \qquad (2.20)$$

for any Hamiltonian function $f(q,p)$ independent of h, where

$$< \rho, f > = \int \rho(q,p) f(q,p) \, dq \, dp \qquad (2.21)$$

is the value of the distribution ρ on a test function ρ. The quantity (2.20) is very important in quantum mechanics. This is just the mean value of the dynamic variable f in the state described by ρ:

$$< \rho, f > \equiv \mathrm{tr} \, (\hat{\rho} \hat{f}) \equiv (\hat{f} \psi, \psi). \qquad (2.22)$$

Then we can expand the operators occurring in (2.18) in asymptotic series in powers of $(-ih)$ (in fact, this is just the Taylor expansion) and obtain the following system of equations for the coefficients of the expansion (2.19):

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial H}{\partial p} \frac{\partial \rho_0}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial \rho_0}{\partial p} \equiv \frac{\partial \rho_0}{\partial t} + \{H, \rho_0\} = 0, \qquad (2.23)$$

where $\{\,\cdot\,,\,\cdot\,\}$ is the Poisson bracket,

$$\frac{\partial \rho_1}{\partial t} + \{H, \rho_1\} = -\frac{\partial^2 H}{\partial p^2} \frac{\partial^2 \rho_0}{\partial q^2} + -\frac{\partial^2 H}{\partial q^2} \frac{\partial^2 \rho_0}{\partial p^2}, \qquad (2.24)$$

etc. (We do not write out the subsequent equations.) Equation (2.23) is just the well-known *Liouville equation* for ρ_0. It has the solution

$$\rho_0(t) = (g_H^{-t})^* \rho_0(0), \qquad (2.25)$$

where g_H^t is the Hamiltonian flow corresponding to the Hamilton function H.

It remains to note that the expansion (2.19) holds indeed for semiclassical wave functions ψ. For example, if

$$\psi(q) = e^{\frac{i}{\hbar} S(q)} a(q),$$

then

$$\rho(q,p) = a(q) \delta \left(p - \frac{\partial S}{\partial q} \right)$$

is a measure concentrated on the Lagrangian manifold $p = \frac{\partial S}{\partial q}$.

Remark 1 In the preceding examples, the operators or ordered representation can be shown to be S^∞-generators in S^∞. It follows that the corresponding twisted product is well defined on the entire symbol space $S^\infty(\mathbf{R}^{2n})$.

Example 4 *Creation–annihilation operators.* In various settings of the second quantization method, the Hamiltonian of a system of particles, as well as other dynamic variables, is represented via so-called *creation–annihilation operators* a_j, a_j^+ (where the index j numbers possible states of particles), which satisfy the commutation relations

$$
\begin{aligned}
[a_j, a_k^+] &\equiv a_j a_k^+ - a_k^+ a_j = \delta_{jk}, \\
[a_j, a_k] &= [a_j^+, a_k^+] = 0
\end{aligned}
\tag{2.26}
$$

if the particles in questions are bosons and the anticommutation relations

$$
\begin{aligned}
[a_j, a_k^+]_+ &\equiv a_j a_k^+ + a_k^+ a_j = \delta_{jk}, \\
[a_j, a_k]_+ &= [a_j^+, a_k^+]_+ = 0
\end{aligned}
\tag{2.27}
$$

if the particles are fermions.

Customarily, there are infinitely many possible states of particles and accordingly, infinitely many operators a_j and a_j^+. We consider the simplified model in which there are only $n < \infty$ possible states. (And we actually carry out our computations only for $n = 1$.) Let us first consider the case of bosons. We have two operators a and a^+ satisfying the relation

$$
[a, a^+] = 1.
\tag{2.28}
$$

Let us construct the left ordered representation for the orderings $(\overset{1}{a}, \overset{2}{a^+})$ (the operator $H(\overset{1}{a}, \overset{2}{a^+})$ is said to be in the Wick normal form) and $(\overset{2}{a}, \overset{1}{a^+})$ (the anti-Wick normal form). For the first case, we have

$$
a^+[\![f(\overset{1}{a}, \overset{2}{a^+})]\!] = \overset{2}{a^+} f(\overset{1}{a}, \overset{2}{a^+}),
\tag{2.29}
$$

$$
\begin{aligned}
a[\![f(\overset{1}{a}, \overset{2}{a^+})]\!] &= \overset{3}{a}\, f(\overset{1}{a}, \overset{2}{a^+}) \\
&= \overset{3}{a}\, f(\overset{1}{a}, \overset{4}{a^+}) + \overset{3}{a}\left[f(\overset{1}{a}, \overset{2}{a^+}) - f(\overset{1}{a}, \overset{4}{a^+})\right].
\end{aligned}
\tag{2.30}
$$

By transforming the second term, we obtain

$$a[\![f(\overset{1}{a},\overset{2}{a}{}^+)]\!] = \overset{1}{a}\, f(\overset{1}{a},\overset{2}{a}{}^+) + \overset{3}{a}\,(\overset{2}{a}{}^+ - \overset{4}{a}{}^+)\frac{\delta f}{\delta y}(\overset{1}{a},\overset{2}{a}{}^+,\overset{4}{a}{}^+)$$

$$= \overset{1}{a}\, f(\overset{1}{a},\overset{2}{a}{}^+) + \frac{\partial f}{\partial y}(\overset{1}{a},\overset{2}{a}{}^+).$$

Thus we have the following formulas for the left ordered representation operators in this case:

$$l^+ = y, \quad l = x + \frac{\partial}{\partial y}. \tag{2.31}$$

Likewise, for the anti-Wick ordering we have

$$\widetilde{l}{}^+ = y - \frac{\partial}{\partial x}, \quad \widetilde{l} = x. \tag{2.32}$$

We point out that the *operators l and $\widetilde{l}{}^+$ are not S^∞-generators in S^∞*. Indeed, let us, say, consider the operator l. The corresponding one-parameter group

$$U(t) = e^{ilt} \tag{2.33}$$

has the form

$$[U(t)f](x,y) = e^{itx}f(x,y+it) \tag{2.34}$$

and hence is well defined only on analytic symbols. Accordingly, one is forced to deal with classes of symbols represented by entire analytic functions (cf. [4]).

Now let us consider the case of fermions. In the simplest case, we have two operators, a and a^+, satisfying the three anticommutation relations

$$[a,a^+]_+ = 1, \tag{2.35}$$

$$[a,a]_+ = [a^+,a^+]_+ = 0. \tag{2.36}$$

In fact, relation (2.35) alone is sufficient for computing the regular representation operators. Note that (2.35) can be rewritten in the form

$$\overset{2}{a}\,(\overset{1}{a}{}^+ + \overset{3}{a}{}^+) = 1. \tag{2.37}$$

Let us compute the left ordered representation for the Wick ordering $(\overset{1}{a}, \overset{2}{a}{}^+)$. (Since (2.35) is symmetric with respect to the change $a \Rightarrow a^+$, the formulas for the anti-Wick ordering will be similar, with an obvious change in notation.) We have

$$a^+ [\![f(\overset{1}{a}, \overset{2}{a}{}^+)]\!] = \overset{2}{a}{}^+ f(\overset{1}{a}, \overset{2}{a}{}^+),$$

and so

$$l^+ = y; \tag{2.38}$$

now, to compute l, we observe that

$$a [\![f(\overset{1}{a}, \overset{2}{a}{}^+)]\!] = \overset{3}{a}\, f(\overset{1}{a}, \overset{4}{a}{}^+) = \overset{3}{a}\, f(\overset{1}{a}, -\overset{4}{a}{}^+) + \overset{3}{a}\, (f(\overset{1}{a}, \overset{2}{a}{}^+) - f(\overset{1}{a}, -\overset{4}{a}{}^+)).$$

Let us transform the last term, using difference derivatives as usual:

$$
\begin{aligned}
\overset{3}{a}\, (f(\overset{1}{a}, \overset{2}{a}{}^+) - f(\overset{1}{a}, -\overset{4}{a}{}^+)) &= \overset{3}{a}\, (\overset{2}{a}, \overset{4}{a}{}^+) \frac{\delta f}{\delta y}(\overset{1}{a}; \overset{2}{a}{}^+, -\overset{4}{a}{}^+) \\[2mm]
&= \overset{3}{a}\, (\overset{2}{a}{}^+ + \overset{4}{a}{}^+) \frac{\partial f}{\partial y}(\overset{0}{a}; \overset{1}{a}{}^+, -\overset{5}{a}{}^+) \\[2mm]
&= \frac{\delta f}{\delta y}(\overset{1}{a}; \overset{2}{a}{}^+, -\overset{2}{a}{}^+) \tag{2.39}
\end{aligned}
$$

(here we have used (2.37)). Let us introduce the operator κ of inversion with respect to the variable y:

$$\kappa f(x, y) = f(x, -y).$$

The difference derivative in (2.39) can be represented in the form

$$\frac{\delta f}{\delta y} = \frac{1}{2y}(1 - \kappa)f. \tag{2.40}$$

All in all, we obtain

$$l = x + \frac{1}{2y}(1 - \kappa). \tag{2.41}$$

Let us verify the Jacobi condition. We have

$$
\begin{aligned}
[l^+, l]_+ f(x, y) &= l^+ l f + l l^+ f \\
&= y x f + \frac{1}{2}(f - \kappa f) + x y f + \frac{1}{2}(f + \kappa f) \\
&= 2 x y f + f, \tag{2.42} \\
[l^+, l^+]_+ f(x, y) &= y^2 f(x, y), \tag{2.43} \\
[l^-, l^-]_+ f(x, y) &= x^2 f(x, y) + \frac{x}{y}(f - \kappa f) + \frac{1}{4y^2}(1 - \kappa^2) f \\
&= x^2 f + \frac{x}{y}(f - \kappa f). \tag{2.44}
\end{aligned}
$$

Thus, the Jacobi condition is not satisfied! This, of course, does not prevent one from using the ordered representation operators (2.38) and (2.41). However, if one wishes to "force" relations (2.35), (2.36) to satisfy the Jacobi condition, in view of all the nice consequences of that condition, then the idea of using symbols that are ordinary functions must be abandoned. Instead, one can use symbols that are functions of anticommuting variables, i.e., symbols that are elements of Grassmann algebras. We refer the reader to [4], where this topic is covered in deep detail.

Example 5 Our next example deals with finite-difference approximations (difference schemes). A typical finite-difference equation has the form (we consider the one-dimensional case)

$$
\hat{D}u \equiv \sum_\alpha a_\alpha(x) u(x + \alpha h) = f(x), \tag{2.45}
$$

where the summation with respect to α ranges over some set of integers (the stencil) and h is the grid increment. The known right-hand side $f(x)$, as well as the unknown function $u(x)$, are usually treated as grid functions. However, it is more convenient for us to assume that $f(x)$ and $u(x)$ are defined for all $x \in \mathbf{R}$. (To achieve this for the right-hand side $f(x)$, one can, say, use interpolation.)

Equation (2.45) can be represented via functions of x and $-i\frac{\partial}{\partial x}$ in the form

$$H\left(\overset{2}{x}, -i\,\overset{1}{\frac{\partial}{\partial x}}\right)u = f, \tag{2.46}$$

where

$$H(x,p) = \sum a_\alpha(x)e^{i\alpha ph}. \tag{2.47}$$

However, equally useful is the representation, which will be derived below, via difference derivative operators.

Consider the operators $\delta_\pm$ given by

$$
\begin{aligned}
(\delta_+ f)(x) &= \frac{f(x+h) - f(x)}{h}, \\[2mm]
(\delta_- f)(x) &= \frac{f(x) - f(x-h)}{h}
\end{aligned}
\tag{2.48}
$$

(the forward and backward difference derivative). Then

$$
\begin{aligned}
f(x+h) &= (1 + h\delta_+)f, \\
f(x-h) &= (1 - h\delta_-)f
\end{aligned}
\tag{2.49}
$$

and the operator D occurring in (2.45) can be rewritten in the form

$$\hat{D} = \sum_{\alpha \geq 0} a_\alpha(x)(1 + h\delta_+)^\alpha + \sum_{\alpha < 0} a_\alpha(x)(1 + h\delta_-)^{-\alpha}. \tag{2.50}$$

At first glance, the representation (2.50) does not seem to be particularly simple, but it usually becomes so if the original operator $\hat{D}$ stems from a finite-difference representation of some differential operator. For example, the usual approximations to $\frac{\partial}{\partial x}$ are

$$\delta_+, \quad \delta_-, \quad \frac{1}{2}(\delta_+ + \delta_-),$$

and the most widely used approximation to $\frac{\partial^2}{\partial x^2}$ is

$$
\begin{aligned}
\Delta &= \delta_+ \delta_- = \delta_- \delta_+, \\
[\Delta f](x) &= \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.
\end{aligned}
\tag{2.51}
$$

The three operators x, δ_-, and δ_+ satisfy the commutation relations

$$
\begin{aligned}
{[\delta_+, x]} &= 1 + h\delta_+, \\
{[\delta_-, x]} &= 1 - h\delta_-, \\
{[\delta_+, \delta_-]} &= 0.
\end{aligned}
\tag{2.52}
$$

$$\tag{2.53}$$

Let us construct the operators of ordered representation for the triple $(\overset{2}{x}, \overset{1}{\delta}_+, \overset{1}{\delta}_-)$. These representation operators will be denoted by l_x, l_+, l_-.

We have, first of all,

$$
x[\![f(\overset{2}{x}, \overset{1}{\delta}_+, \overset{1}{\delta}_-)]\!] = \overset{2}{x}\, f(\overset{2}{x}, \overset{1}{\delta}_+, \overset{1}{\delta}_-),
$$

and so $l_x = x$. To find l_+, let us rewrite the first relation in (2.52) in the form

$$
\delta_+ x = (x + h)\delta_+ + 1.
\tag{2.54}
$$

Then we readily obtain

$$
\begin{aligned}
\delta_+[\![f(\overset{2}{x}, \overset{1}{\delta}_+, \overset{1}{\delta}_-)]\!] &= \overset{3}{\delta}_+\, f(\overset{2}{x}, \overset{1}{\delta}_+, \overset{1}{\delta}_-) \\
&= \overset{3}{\delta}_+\, f(\overset{4}{x} + h, \overset{1}{\delta}_+, \overset{1}{\delta}_-) \\
&\quad + \overset{3}{\delta}_+\, (\overset{4}{x} + h - \overset{2}{x}) \frac{\delta f}{\delta x}(\overset{2}{x}, \overset{4}{x} + h, \overset{1}{\delta}_+, \overset{1}{\delta}_-) \\
&= \overset{1}{\delta}_+\, f(\overset{2}{x} + h, \overset{1}{\delta}_+, \overset{1}{\delta}_-) + \frac{\delta f}{\delta x}(\overset{2}{x}, \overset{2}{x} + h, \overset{1}{\delta}_+, \overset{1}{\delta}_-).
\end{aligned}
\tag{2.55}
$$

Hence

$$
l_+ = y(1 + h\delta_+) + \delta_+.
\tag{2.56}
$$

Likewise, similar computations give

$$
l_- = y(1 - h\delta_-) + \delta_-.
\tag{2.57}
$$

Let us verify the Jacobi condition, We obviously have

$$
[l_+, l_-] = 0.
$$

Next,

$$
\begin{aligned}
[l_+, l_x] &= [y(1 + h\delta_+) + \delta_+, x] = (1 + yh)[\delta_+, x] \\
&= (1 + yh)(1 + h\delta_+) = 1 + h[y(1 + h\delta_+) + \delta_+] \\
&= 1 + hl_+,
\end{aligned}
$$

and similarly for $[l_-, l_x]$. Hence the Jacobi condition is satisfied.

Example 6 We consider degenerate differential equations of the Fuchsian type:

$$
H\left(\overset{2}{r}, ir\, \frac{\overset{1}{d}}{dr}\right) u = f. \tag{2.58}
$$

The operators occurring in such equations are represented as functions of the ordered pair $(\overset{1}{A}, \overset{2}{B})$, where

$$
A = ir\frac{d}{dr}, \quad B = r. \tag{2.59}
$$

Let us evaluate the left and right ordered representations for this pair. We have the commutation relation

$$
[A, B] = iB, \tag{2.60}
$$

which can also be rewritten in the form

$$
AB = B(A + i) \tag{2.61}
$$

suitable for the application of the simplest commutation rule. Let us first evaluate the *right* ordered representation, We have

$$
[\![f(\overset{1}{A}, \overset{2}{B})]\!]A = f(\overset{1}{A}, \overset{2}{B})\,\overset{1}{A} \tag{2.62}
$$

and

$$
[\![f(\overset{1}{A}, \overset{2}{B})]\!]B = f(\overset{1}{A}, \overset{2}{B})\,\overset{0}{B} = f(\overset{1}{A} + i, \overset{2}{B})\,\overset{2}{B} \tag{2.63}
$$

by the commutation rule with regard for (2.61). Consequently, the ordered representation operators have the form

$$
r_A = x, \quad r_B = yT, \tag{2.64}
$$

where T is the shift operator with unit imaginary step,

$$Tf(x, y) = f(x + i, y).$$

Symbolically, T can be written in the form

$$T = e^{i\frac{\partial}{\partial x}}. \tag{2.65}$$

We obviously have

$$r_B r_A = (r_A + i)r_B, \tag{2.66}$$

and so the Jacobi condition is satisfied. Now let us compute the left ordered representation. To this end, we note that

$$\varphi(T)x = i\varphi'(1) + x\varphi(1). \tag{2.67}$$

Indeed, consider the operator T in the two-dimensional space $\mathcal{H}$ of linear functions of x:

$$\mathcal{H} = \{u(x) \,|\, u(x) = ax + b, \ \ a, c \in \mathbf{C}\}. \tag{2.68}$$

This operator leaves the space $\mathcal{H}$ invariant; it acts on the basis functions as follows:

$$\begin{aligned} Tx &= x + i, \\ T1 &= 1. \end{aligned} \tag{2.69}$$

Now we have, by induction,

$$\begin{aligned} T^k x &= x + k, \\ T^k 1 &= 1 \end{aligned} \tag{2.70}$$

and for an arbitrary polynomial $p(T) = \sum a_k T^k$,

$$\begin{aligned} p(T)1 &= \sum a_k, \\ p(T)x &= \sum a_k(x + ik) = p(1)x + ip'(1). \end{aligned} \tag{2.71}$$

Now we arrive at formula (2.66) for arbitrary symbols by the uniqueness theorem. Having this in mind, we readily obtain

$$l_B = f(\overset{2}{r}_A, \overset{1}{r}_B)y = yf(x,y),$$

$$l_A = f(\overset{2}{r}_A, \overset{1}{r}_B)x = xf(x,y) + iy\frac{\partial f}{\partial y}(x,y), \qquad (2.72)$$

that is,

$$l_A = x + iy\frac{\partial}{\partial y}, \quad l_B = y. \qquad (2.73)$$

It is not at all obvious from the expressions (2.73) that the class of symbols for which functions of the operators $A = ir\frac{\partial}{\partial r}$ and $B = r$ form an algebra cannot be S^∞ and must satisfy the severe restriction of analyticity. However, this becomes pretty obvious once we look at the operators r_A and r_B. The operator r_B involves shift into the complex plane with respect to the variable x, and so reasonable class of symbols must contain only symbols analytic in the variable x.

Example 7 We consider degenerate differential equations with higher order of degeneracy at the point $r = 0$:

$$H\left(r, ir^{k+1}\frac{d}{dr}\right)u = f, \qquad (2.74)$$

where $k \geq 1$ is an integer (for simplicity).

Here the very definition of the operator (2.74) for functions $H(y,x)$ nonanalytic in the second argument is obscure. Indeed, the operator $A = ir^{k+1}\frac{d}{dr}$ even does not generate a one-parameter group! To verify this, consider the equation

$$\frac{du}{dt} = iAu = -r^{k+1}\frac{du}{dr}. \qquad (2.75)$$

The corresponding characteristic equation reads

$$\dot{r} = -r^{k+1}, \qquad (2.76)$$

and the trajectories are readily seen to exit to infinity in finite time. One possible way of improving the situation when dealing with such singular

points is as follows. Note that we are actually interested in the vicinity of the singular point $r = 0$, while the behavior of the coefficients for large $|r|$ seems to be irrelevant to our study. Accordingly, we replace the operator $ir^{k+1}\frac{d}{dr}$ by another operator, which behaves well at infinity and coincides with $ir^{k+1}\frac{d}{dr}$ for small r. Namely, we consider a real-valued function $\varphi(r)$ such that $\varphi(r) \neq 0$ for $r \neq 0$ and

$$\begin{cases} \varphi(r) = r^{k+1} & \text{for } |r| \leq \frac{1}{2}, \\ \varphi(r) = \pm 1 & \text{for } |r| \geq 1. \end{cases} \tag{2.77}$$

We set

$$A = i\varphi(r)\frac{d}{dr}, \quad B = r \tag{2.78}$$

and consider functions of the ordered pair $(\overset{1}{A}, \overset{2}{B})$. The operators A and B satisfy the "nonlinear" commutation relation

$$[A, B] = i\varphi(B), \tag{2.79}$$

and now we shall construct the left ordered representation of this relation. We have

$$B[\![f(\overset{1}{A}, \overset{2}{B})]\!] = \overset{2}{B}\, f(\overset{1}{A}, \overset{2}{B})$$

and

$$\begin{aligned} A[\![f(\overset{1}{A}, \overset{2}{B})]\!] &= \overset{3}{A}\,[\![f(\overset{1}{A}, \overset{2}{B})]\!] \\ &= \overset{1}{A}\, f(\overset{1}{A}, \overset{2}{B}) + [\overset{3}{A}, B]\frac{\delta f}{\delta y}(\overset{1}{A}; \overset{2}{B}, \overset{4}{B}) \\ &= \overset{1}{A}\, f(\overset{1}{A}, \overset{2}{B}) + i\varphi(\overset{2}{B})\frac{\partial f}{\partial y}(\overset{1}{A}, \overset{2}{B}). \end{aligned}$$

Thus, the formula for the left regular representation operators reads

$$l_A = x + i\varphi(y)\frac{\partial}{\partial y}, \quad l_B = y. \tag{2.80}$$

It can be proved that these operators are S^∞-generators in S^∞, and so for higher-order singular points (but not for Fuchsian points) the

algebra of functions of operators "describing" the singularity can be chosen to be the algebra of functions of A and B with S^∞ symbols.

As to our example, it remains to verify that the Jacobi condition holds. This is however easy:

$$[l_A, l_B] = i\varphi'(y) \equiv i\varphi'(l_B).$$

In conclusion, we note that computing the right ordered representation for the relations (2.79) explicitly would be a boring task in view of the nonlinear nature of the commutation relations. Actually, no explicit formula is available for this case.

2.2   Lie commutation relations

Having considered certain examples, we now proceed to some general classes of commutation relations. The first of these classes is the class of so-called *Lie commutation relations*, which naturally enjoys extensive connections with the well-known theory of Lie groups and Lie algebras. Our exposition, however, mostly does not rely upon this theory, and we try to keep it as elementary as possible.

In this section we deal with commutation relations of the form

$$[A_j, A_k] = -i \sum c_{jk}^l A_l, \tag{2.81}$$

where the c_{jk}^l are some constants. The factor $-i$ is introduced in (2.81) for the sake of convenience, so that, for real c_{jk}^l, relation (2.81) could be satisfied by self-adjoint operators A_j, $j = 1, \ldots, n$. Relations (2.81) are referred to as *Lie commutation relations*, because they say that the operators $(A_1, \ldots, A_2)$ form a representation of the Lie algebra with structural constants c_{jk}^l. In view of the skew-symmetry of the commutator, $[A, B] = -[B, A]$, the structural constants change sign under the transposition of subscripts,

$$c_{jk}^l + c_{kj}^l = 0. \tag{2.82}$$

(The violation of this condition would imply that there necessarily exists a nontrivial linear identity satisfied by $A_1, \ldots, A_n$, so that some of

the operators $A_1, \ldots, A_n$ can be expressed via the others and we actually deal with functions of a shorter tuple of operators.) Next, the commutators satisfy the easy-to-verify *Jacobi identity*

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0, \qquad (2.83)$$

and, assuming $A_1, \ldots, A_n$ to be linearly independent, this also implies certain conditions imposed on the structural constants. We do not write out these conditions (the reader way wish to do this himself or herself), but in the following we always assume that this condition is satisfied.

Our task is to compute the ordered representation of the n-tuple $(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)$ (or, more precisely, of relations (2.81) for this particular ordering of operators) and find symbol classes in which these representation operators act.

First, we note that relations (2.81) can be rewritten in "vector form." Namely, let us introduce the column vector

$$\vec{A} = {}^t(A_1, \ldots, A_n) \qquad (2.84)$$

and consider all relations in (2.81) for some given k, while j ranges from 1 to n. These relations can be rewritten as

$$\vec{A} \, A_k = (A_k \otimes I - i\Lambda_k) \, \vec{A}, \qquad (2.85)$$

where I is the $n \times n$ identity matrix and Λ_k is the matrix with entries $(\Lambda_k)_{jl} = c_{jk}^l$, known as the matrix of the *adjoint representation* of the Lie algebra. The obvious advantage of (2.85) is that now we can widely use the standard commutation formula:

$$\vec{A} \, f(A_k) = f(A_k \otimes I - i\Lambda_k) \, \vec{A} \qquad (2.86)$$

for any suitable symbol $f(x)$. We also note that the two terms $A_k \otimes I$ and $-i\Lambda_k$ commute, which will also prove convenient in our subsequent computations. Relation (2.86) can also be rewritten in the form

$$\vec{A} \, f(A_k) = g(A_k) \, \vec{A}, \qquad (2.87)$$

where

$$g(x_k) = e^{-i\Lambda_k \frac{\partial}{\partial x_k}} f(x_k). \qquad (2.88)$$

Let us introduce the *auxiliary* representation operators L_{jk} such that

$$\overset{j+1}{A}_k\, f(\overset{1}{A}_1,\ldots,\overset{j}{A}_j,\overset{j+2}{A}_{j+1},\ldots,\overset{n+1}{A}_n) = (L_{jk} f)(\overset{1}{A}_1,\ldots,\overset{n}{A}_n). \qquad (2.89)$$

In particular, we have

$$L_{nk} = l_k, \quad L_{0k} = r_k \qquad (2.90)$$

(the operators of left and right regular representation, respectively). For given j, by $\vec{L}_j$ we denote the column vector

$$\vec{L}_j = (L_{j1},\ldots,L_{jn}); \qquad (2.91)$$

then

$$\overset{j+1}{\vec{A}}\, f(\overset{1}{A}_1,\ldots,\overset{j}{A}_j,\overset{j+2}{A}_{j+1},\ldots,\overset{n}{A}_n) = (\vec{L}_j\, f)(\overset{1}{A}_1,\ldots,\overset{n}{A}_n). \qquad (2.92)$$

On the other hand, we can rewrite (2.87), (2.88) with additional operator arguments:

$$\overset{j+1}{\vec{A}}\, f(\overset{1}{A}_1,\ldots,\overset{j}{A}_j,\overset{j+2}{A}_{j+1},\ldots,\overset{n}{A}_n) = \overset{j}{\vec{A}}\, g(\overset{1}{A}_1,\ldots,\overset{j-1}{A}_{j-1},\overset{j+1}{A}_j,\ldots,\overset{n}{A}_n),$$
$$(2.93)$$

where

$$g(x_1,\ldots,x_n) = e^{-i\Lambda_j \frac{\partial}{\partial x_j}} f(x_1,\ldots,x_n). \qquad (2.94)$$

Both sides of (2.94) can be reduced to the standard form of functions of $(\overset{1}{A}_1,\ldots,\overset{n}{A}_n)$ with the help of (2.92). Assuming that the Jacobi condition is satisfied, this implies the equality of the corresponding symbols. Thus we obtain

$$\vec{L}_j\, f = \vec{L}_{j-1}\, e^{-i\Lambda_j \frac{\partial}{\partial x_j}} f \qquad (2.95)$$

for any symbol f, or

$$\vec{L}_j = \vec{L}_{j-1}\, e^{-i\Lambda_j \frac{\partial}{\partial x_j}}, \quad j = 1,\ldots,n. \qquad (2.96)$$

This can also be rewritten as follows:

$$\vec{L}_j = \vec{L}_n\, e^{i\Lambda_n \frac{\partial}{\partial x_n}} e^{i\Lambda_{n-1} \frac{\partial}{\partial x_{n-1}}} \ldots e^{i\Lambda_{j+1} \frac{\partial}{\partial x_{j+1}}} \qquad (2.97)$$

(in the equation with $j = n$, the product of exponentials is empty). However, we know certain components of the $\vec{L}_j$ very well. Namely, we have

$$L_{jj} = x_j, \quad j = 1, \dots, n. \tag{2.98}$$

Indeed,

$$\overset{j+1}{A}_j \, f(\overset{1}{A}_1, \dots, \overset{j}{A}_j, \overset{j+2}{A}_{j+1}, \dots, \overset{n}{A}_n) = \overset{j}{A}_j \, f(\overset{1}{A}_1, \dots, \overset{j}{A}_j, \dots, \overset{n}{A}_n), \tag{2.99}$$

since we need not permute A_j with any operators that do not commute with A_j. Thus, we disregard the identity for $j = 0$ in (2.97) and pick up the jth row in each of the remaining identities, $j = 1, \dots, n$:

$$x_j = \sum_{k=1}^{n} l_k \left[e^{i\Lambda_n \frac{\partial}{\partial x_n}} \cdot \dots \cdot e^{i\Lambda_{j+1} \frac{\partial}{\partial x_{j+1}}} \right]_{kj}. \tag{2.100}$$

This is a system of n equations for n unknown operators l_k of the left ordered representation, and if this system is solvable, then the solution provides the desired left ordered representation operators. We readily see that these operators are linear and homogeneous in $(x_1, \dots, x_n)$.

Now let us study system (2.100) more thoroughly. We shall seek the left ordered representation operators in the form

$$l_j = g_j \left(\overset{2}{x}, -i \, \frac{\overset{1}{\partial}}{\partial x} \right), \quad j = 1, \dots, n. \tag{2.101}$$

Then for the symbols $g_j(x, p)$ we obtain the system

$$\sum_{k=1}^{n} [e^{-\Lambda_n p_n} \cdot \dots \cdot e^{-\Lambda_{j+1} p_{j+1}}]_{kj} g_k(x, p) = x_j, \quad j = 1, \dots, n, \tag{2.102}$$

or, in matrix form,

$$A(p)g(x, p) = x,$$

where

$$(A(p))_{jk} = [e^{-{}^t\Lambda_n p_n} \dots e^{-{}^t\Lambda_{j+1} p_{j+1}}]_{jk} \tag{2.103}$$

(here ${}^t\Lambda$ is the transpose of Λ, and the empty product is, by convention, the identity matrix).

Thus the main problem is whether the matrix (2.103) is invertible. Note that this is always the case for small $|p|$, since $A(0)$ is just the identity matrix. Accordingly, the operators of left ordered representation are well defined on the space of functions whose Fourier transform is supported in a sufficiently small neighborhood of the origin. Indeed, if $u(x)$ is such a function, then we can set

$$\vec{l}u = \left(\frac{1}{2\pi}\right)^{n/2} \int e^{ipx} A^{-1}(p) x \widetilde{u}(p)\, dp, \qquad (2.104)$$

which is well defined as long as the support of $\widetilde{u}$ is concentrated near zero. By the Paley–Wiener theorem, this forces $u(x)$ to be an analytic function of certain exponential growth.

However, in some cases we can say more about the solvability of system (2.103). Suppose that all adjoint representation matrices Λ_j, $j = 1, \ldots, n$, are simultaneously upper triangular (or lower triangular). This precisely corresponds to the case in which the corresponding Lie algebra is *soluble* (and the basis operators $A_1, \ldots, A_n$ are chosen in a special way). Then the matrices $e^{-t\Lambda_j p_j}$ are also triangular, and so is any product of these matrices. Let $\lambda_k^{(j)} = c_{jk}^k$ be the diagonal entries of these matrices. The matrix $A(p)$ is triangular with diagonal entries

$$A(p)_{jj} = \exp\left(-\sum_{k=j+1}^{n} p_k \lambda_k^{(k)}\right), \qquad (2.105)$$

whence we see that $A(p)$ is invertible for all p and $(A(p))^{-1}$ has entries that grow at most exponentially in p as $|p| \to \infty$. Accordingly, the left ordered representation operators are defined on functions whose Fourier transform decays faster than some exponential at infinity (the precise order of the exponential depends on the commutation relations).

Next, suppose that the adjoint representation matrices satisfy the following, more restrictive condition: they are strictly triangular, that is (upper- or lower-) triangular with zeros on the main diagonal. This is precisely the case in which the corresponding Lie algebra is *nilpotent* (and the basis operators are chosen in a special way). The matrices Λ_j themselves are nilpotent:

$$(\Lambda_j)^{n+1} = 0 \qquad (2.106)$$

(it may happen that $(\Lambda_j)^s = 0$ for some $s < n + 1$). Accordingly, the exponentials degenerate into polynomials of order $\geq n$; more precisely,

$$e^{-{}^t \Lambda_k p_k} = 1 + D_k(p_k),$$

where $D_k(p_k)$ is a polynomial strictly triangular (and hence nilpotent) matrix. One can readily see that in this case a similar relation holds for $A(p)$:

$$A(p) = 1 + D(p), \qquad (2.107)$$

where $D(p)$ is a strictly triangular polynomial matrix. The inverse of $A(p)$ is hence given by a finite segment of the Neumann series:

$$[A(p)]^{-1} = 1 - D(p) + D(p)^2 - \ldots + (-1)^n (D(p))^n.$$

We see that for the case of nilpotent Lie algebras (and a special choice of the basis) *the ordered representation operators are differential operators with linear coefficients* and hence are well defined on wide symbol classes. This conclusion is valid only for a special basis; in other words, a linear transformation of the original operators $A_1, \ldots, A_n$ might be necessary, so that for the transformed tuple the following condition is satisfied: *The commutator $[A_j, A_k]$ is a linear combination of operators A_l with $l < \min\{j, k\}$.* (For soluble Lie algebras, the condition says that $l \leq \min\{j, k\}$ instead.)

However, there is one more specific case in which the left ordered representation of a tuple $(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)$ forming a representation of a nilpotent Lie algebra can be computed and is given by differential operators even though the adjoint representation matrices corresponding to this basis need not be upper- of lower- triangular. Namely, suppose that the commutation relations have the form[3]

$$[A_j, A_k] = -i A_{l(j,k)}, \quad j > k, \qquad (2.108)$$

where $l : \{1, \ldots, n\} \times \{1, \ldots, n\} \to \{0, \ldots, n\}$ is some mapping and we adopt the convention that $A_0 = 0$. This was the original class of commutation relations considered by Maslov [1]. For these commutation

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[3]One can also introduce a constant factor $\mu(j, k)$ on the right-hand side in (2.108).

relations, one can compute the left ordered representation operators without solving system (2.102), and these operators are differential operators. We proceed as follows. To each A_j, we assign its level $\nu(A_j)$. By definition, $\nu(A_j) = 0$ if A_j commutes with all other operators A_k. Otherwise, $\nu(A_j)$ is the length of the maximal nontrivial commutator that involves A_j. Now let us try to compute the ordered representation operators. In the expression

$$A_j [\![f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)]\!] = \overset{n+1}{A}_j \, f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n),$$

we must move the operator $\overset{n+1}{A}_j$ to its place. To this end, we must permute it with the operators $\overset{j+1}{A}_{j+1}, \ldots, \overset{n}{A}_n$:

$$\overset{n+1}{A}_j \, f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n) = \overset{j}{A}_j \, f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)$$

$$- i \sum_{k=j+1}^{n} \overset{k+1}{A}_{l(j,k)} \, \frac{\delta f}{\delta x_k}(\overset{1}{A}_1, \ldots; \overset{k}{A}_k, \overset{k+2}{A}_k; \overset{k+3}{A}_{k+1}, \ldots, \overset{n+2}{A}_n),$$

where all operators $A_{l(j,k)}$ have a level less than that of A_j. Next, we move the operators $A_{l(j,k)}$ to their respective places. In doing so, we obtain higher order difference derivatives and new commutators (2.108), whose level is again reduced at least by one. In finitely many steps, we arrive at the situation in which all newly obtained commutators have the level 0, that is, commute with all other operators in the tuple $(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)$. These commutators move to their respective places automatically, and the difference derivatives turn into the usual partial derivatives. This provides a differential expression for the operators of left ordered representation. In a similar way, one can obtain the right ordered representation operators. It was also proved in [1] that in this situation the ordered representation operators are S^∞-generators in S^∞.

Now we return to the case of general Lie commutation relations. The subsequent text (up to the end of this section) is not used in the remaining part of the book and is intended for more advanced readers familiar with the theory of Lie groups and their representations.

As we have already mentioned, in the general case the matrix $A(p)$ is not globally invertible, and hence the ordered representation operators

are not defined "globally," i.e., for functions with arbitrary support of the Fourier transform. This is not occasional; the point is that the Fourier transforms of symbols actually live on the corresponding Lie group rather than on the linear space $\mathbf{R}_p^n$. So everything goes fine with our computations as long as the Lie group is nilpotent or soluble and hence isomorphic to $\mathbf{R}_p^n$.

Let us outline the general construction and its relationship with the representation theory of Lie groups.

Let $A_1, \ldots, A_n$ be the generators of a continuous representation $T = G \to \operatorname{End}(V)$ of a Lie group G in a linear space V. (We do not assume that V is a Hilbert or Banach space, since in most applications of noncommutative analysis V is a convergence space of more complicated nature. Since we only outline the subject, we shall not be too pedantic about convergence issues in what follows; instead, we omit them altogether.) Thus,

$$A_j = \left[-i \frac{d}{dt} T(\exp(a_j t)) \right] \Big|_{t=0},$$

where $(a_1, \ldots, a_n)$ is a basis of the Lie algebra $g = T_e G$, $\exp : g \to G$ is the exponential mapping, and $\exp(a_j t)$ is the one-parameter subgroup corresponding to a_j. Let $f(x_1, \ldots, x_n)$ be a given symbol. We intend to define the operator $f(\overset{1}{A_1}, \ldots, \overset{n}{A_n})$. Under suitable analytic conditions, one can set

$$f(\overset{1}{A_1}, \ldots, \overset{n}{A_n}) = \left(\frac{1}{2\pi} \right)^{n/2} \int \widetilde{f}(t_1, \ldots, t_n) e^{iA_n t_n} \ldots e^{iA_1 t_1} dt_1 \ldots dt_n,$$
(2.109)

where $\widetilde{f}(t_1, \ldots, t_n)$ is the Fourier transform of f. Note that

$$e^{iA_j t_j} = T(\exp(a_j t_j))$$

is just the representation of the corresponding one-parameter subgroup. We define an "ordered exponential mapping"

$$\exp_1 : g \to G$$

by setting

$$\exp_1(t) = \exp(a_n t_n) \cdot \ldots \cdot \exp(a_1 t_1). \qquad (2.110)$$

Then we have

$$f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n) = \left(\frac{1}{2\pi}\right)^{n/2} \int \widetilde{f}(t_1, \ldots, t_n) T(\exp_1(t)) dt_1 \ldots dt_n. \quad (2.111)$$

We can interpret (2.111) as an integral over the group G:

$$f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n) = \left(\frac{1}{2\pi}\right)^{n/2} \int_G \breve{f}(g) T(g) \, dg, \quad (2.112)$$

where

$$\breve{f}(\exp_1(t)) = \widetilde{f}(t) \left| \frac{D(\exp_1(t))}{D_t} \right|$$

is the *group Fourier transform* of f and dg is the Haar measure. Thus, $f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)$ is just an element in the representation of the group algebra of G,

$$f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n) = T(\breve{f}), \quad (2.113)$$

where

$$T(\psi) = \left(\frac{1}{2\pi}\right)^{n/2} \int_G \psi(g) T(g) \, dg. \quad (2.114)$$

Now we see that

$$[\![f_1(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)]\!] \, [\![f_2(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)]\!] = T(\breve{f}_1 * \breve{f}_2), \quad (2.115)$$

where

$$\breve{f}_1 * \breve{f}_2 = \int_G \breve{f}_1(gh^{-1}) \breve{f}_2(h) \, dh$$

is the standard convolution in the group algebra, but the problem is that $\breve{f}_1 * \breve{f}_2$ need not be the group Fourier transform of any function $\varphi(t_1, \ldots, t_n)$.

However, the convolution can be expressed as

$$\breve{f}_1 * \breve{f}_2 = f(\overset{1}{v}_1, \ldots, \overset{n}{v}_n)(\breve{f}_2), \quad (2.116)$$

where $v_1, \ldots, v_n$ are the right-invariant vector fields on G corresponding to the basis elements $a_1, \ldots, a_n$ of the Lie algebra.

The main conclusions concerning ordered representations of Lie commutation relations are as follows.

(1) For nilpotent commutation relations, the technique of noncommutative analysis applies in full strength and produces left ordered representation in the form of differential operators that are S^∞-generators in S^∞. Accordingly, the functions of $(\overset{1}{A}_1, \ldots, \overset{n}{A}_n)$ with symbols in S^∞ are in general well defined and form an algebra.

(2) For soluble relations, the global ordered representation still exists, but is given by pseudodifferential operators whose one-parameter groups may exhibit exponential growth. The class of admissible symbols is usually restricted by some severe analyticity conditions.

(3) Finally, for general Lie commutation relations that are neither nilpotent nor soluble, there is usually no globally defined ordered representation, and so the calculus of functions of such operators is, in a sense, "crippled." An adequate theory in this case is representation theory of Lie groups.

Fortunately, in applications of noncommutative analysis one usually deals with nilpotent or at least soluble relations (in any case, the absence of explicit formulas makes noncommutative analysis less useful for general commutation relations). In the following section, we shall consider a class of nonlinear relations close to that of nilpotent Lie algebras.

2.3 Non-Lie (nonlinear) commutation relations

In this section, which is final in this chapter, we consider two classes of commutation relations which in a way generalize the Lie commutation relations but still are soluble under appropriate conditions. The first of these classes was introduced in [3]; the second class was mentioned in [1] and [3] and the computations were carried out in [2]. Our exposition mainly follows [2].

1°. **"Matrix commutativity" relations**. We consider commutation relations of the form

$$A_k A_j = \sum_{r=1}^{n} A_r F_{kj}(A_k), \quad j, k = 1, \ldots, n, \qquad (2.117)$$

where the $F_{kl}^s(x) \in \mathcal{F}$ are given symbols. The notation of noncommutative analysis is strong enough to admit rewriting these relations in matrix form:

$$A_k \vec{A} = \vec{A} F_k(A_k), \quad k = 1, \ldots, n, \qquad (2.118)$$

where $\vec{A}$ is the row vector $\vec{A} = (A_1, \ldots, A_n)$ and $F_k(x)$ is the matrix function with entries

$$(F_k(x))_{nj} = F_{kj}^r(x). \qquad (2.119)$$

Clearly, relations (2.117) generalize Lie commutation relations, which are obtained from the former in the special case in which $F_k(x) = Ex + c_k$, E being the identity matrix and c_k being the kth matrix of the adjoint representation.

We impose some conditions on relations (2.117) in order to ensure that the corresponding representation operators can be evaluated in a closed form. These conditions generalize the nilpotency condition for Lie algebras. They are as follows.

Condition A. Each matrix $F_k(x)$ is lower triangular, that is, $F_{kj}^r(x) = 0$ for $r < j$.

Condition B. The diagonal entries $F_{kj}^j(x)$ of the matrix function $F_k(x)$ have inverse functions $\varphi_{kj}(x) \in \mathcal{F}$:

$$F_{kj}^j(\varphi_{kj}(x)) = \varphi_{kj}(F_{kj}^j(x)) = x, \quad k, j = 1, \ldots, n.$$

Both conditions are satisfied for nilpotent (and even soluble) Lie algebras (in an appropriate basis); the first condition is just the condition that the algebra be soluble; the second condition is obviously satisfied, since for Lie algebras the diagonal entries of the matrix $F_k(x)$ have the form

$$F_{kj}^j(x) = x + c_{kj}^j, \qquad (2.120)$$

whence $\varphi_{kj}(x) = x - c_{kj}^j$.

We introduce the matrix operators

$$U_k = \exp\left\{ \left(F_k(\overset{2}{x}_k) - E\,\overset{2}{x}_k\right) \frac{\overset{1}{\partial}}{\partial x_k} \right\} \qquad (2.121)$$

in $\mathcal{F}_n$ and the scalar operators

$$\begin{aligned}
D_{sj} &= [U_n U_{n-1} \ldots U_{j+1}]_{sj}, \quad j < n, \\
D_{sn} &= \delta_{sn}, \qquad\qquad\qquad s = 1, \ldots, n,
\end{aligned} \qquad (2.122)$$

where δ_{sn} is the Kronecker delta.

Next, let $R_j : \mathcal{F}_n \to \mathcal{F}_n$ be the operator acting by the following formulas:

$$(R_j f)(x_1, \ldots, x_n) = \begin{cases} f(x_1, \ldots, x_j, \varphi_{j+1,j}(x_{j+1}), \ldots, \varphi_{nj}(x_n)), \\ f(x_1, \ldots, x_n), \qquad\qquad\qquad\qquad j = n. \end{cases}$$

We refer the reader to [2] for the proof of the following two theorems.

Theorem 8 *The left ordered representation operators of the commutation relations* (2.117) *satisfying conditions* **A** *and* **B** *exist* (*and, of course, are unique if the Jacobi condition is satisfied*). *They can be chosen to satisfy the system of equations*

$$\sum_{s=1}^{n} l_s D_{sj} = x_j, \quad j = 1, \ldots, n. \qquad (2.123)$$

We do not give the proof here, since it is completely similar to that used in the analysis of Lie commutation relations in § 2.2.

Now let us define operators

$$M_{sj} : \mathcal{F}_n \to \mathcal{F}_n, \quad s, j = 1, \ldots, n,$$

by setting, for each $j = 1, \ldots, n$,

$$M_{sj} = 0, \quad s < j; \qquad (2.124)$$

$$M_{jj} = R_j, \qquad (2.125)$$

$$M_{sj} = -R_s \sum_{l=1}^{s-1} D_{sl} M_{lj}, \quad s > j. \qquad (2.126)$$

(Relations (2.126) are applied consecutively for $s = j + 1, j + 2, \ldots, n$. Each time, the right-hand side is already known.)

Theorem 9 *The operators*

$$l_s = \sum_{k=1}^{n} x_k M_{ks} \tag{2.127}$$

are the operators of left ordered representation of the system of commutation relations (2.117).

Again, we omit the proof, which amounts to the boresome substitution of the operators (2.127), with regard for the recursion relations (2.124)–(2.126), into Eqs. (2.123) and is in fact an exercise in solving triangular systems of equations.

$2°$. **Lie commutation relations with variable coefficients**. The second class of commutation relations considered here generalizes Lie commutation relations in a somewhat different but very natural direction.

Suppose that we are given a tuple of operators $(A_1, \ldots, A_n)$ that satisfy the commutation relations

$$[A_j, A_k] = -i \sum_{r=1}^{n} c_{jk}^r (B_1, \ldots, B_m) A_r, \quad j, k = 1, \ldots, n, \tag{2.128}$$

where $B_1, \ldots, B_m$ are some other operators commuting with one another:

$$[B_l, B_s] = 0, \quad s, l = 1, \ldots, m. \tag{2.129}$$

An example is given by a suitable system of vector fields $x_1 = iA_1, \ldots, x_n = iA_n$ in the space[4] $\mathbf{R}^k$, so that the commutator of any two fields is expressed as a linear combination of these fields with coefficients depending on x:

$$[X_j, X_k] = \sum c_{jk}^r (x) X_r. \tag{2.130}$$

---

[4]Generalizations to manifolds are also possible; see [3], § 3.5 for some discussion of the topic.

Here the B_j are just the operators of multiplication by independent variables. This example also has another important property: the commutators $[X_j, x_l]$ are just functions of x:

$$[X_j, x_l] = X_j(x_l) = \varphi_{jl}(x). \tag{2.131}$$

We also require that this general property be valid for our system of operators. Thus, we consider the following "Lie commutation relations with variable coefficients":

$$[A_j, A_k] = -i \sum_{r=1}^{n} c_{jk}^r(B_1, \dots, B_m) A_r, \; j, k = 1, \dots, n; \tag{2.132}$$

$$[A_j, B_s] = -i\varphi_{js}(B_1, \dots, B_m), \; j = 1, \dots, n, \; s = 1, \dots, m; \tag{2.133}$$

$$[B_j, B_s] = 0, \quad j, s = 1, \dots, n. \tag{2.134}$$

By virtue of the last group of relations, we need not introduce distinct Feynman indices for the arguments $B_1, \dots, B_m$ of functions occurring in (2.132)–(2.133).

We shall outline a method for constructing the left ordered representation of the commutation relations (2.132)–(2.134). Our line of argument follows [2]. We refer the reader to [2] for the proofs of a series of rather technical lemmas involved in this argument.

Thus, we intend to construct the left ordered representation of an operator tuple $(\overset{1}{A}_1, \dots, \overset{n}{A}_n, \overset{n+1}{B}_1, \dots, \overset{n+1}{B}_m)$ satisfying the commutation relations (2.132)–(2.134). First, we accomplish the following auxiliary task. Consider the exponential symbol

$$\chi(t, \tau, x, y) e^{i(t_1 x_1 + \dots + t_n x_n + \tau_1 y + \dots + \tau_m y_m)} \in \mathcal{F}_{n+m}$$

of $n + m$ arguments $(x_1, \dots, x_n, y_1, \dots, y_m)$ depending on real parameters $(t_1, \dots, t_n, \tau_1, \dots, \tau_m)$ and introduce the corresponding operator

$$
\begin{aligned}
U(t, \tau) &= \chi(t, \tau, \overset{1}{A}_1, \dots, \overset{n}{A}_n, \overset{n+1}{B}) \\
&= e^{i(\tau_m B_m + \dots + \tau_1 B_1)} e^{i t_n A_n} \cdot \dots \cdot e^{i t_1 A_1}. \tag{2.135}
\end{aligned}
$$

Needless to say, this is just the corresponding product of exponentials. We shall find an operator $\hat{Q}_j = \hat{Q}_j \left(\overset{2}{t}, \tau, -i \overset{1}{\frac{\partial}{\partial t}}, -i \overset{1}{\frac{\partial}{\partial \tau}} \right)$ such that

$$\hat{Q}_j U(t, \tau) = A_j U(t, \tau), \quad j = 1, \dots, n. \tag{2.136}$$

Lemma 10 *One has*

$$-i\frac{\partial}{\partial t_j}U(t,\tau) \;=\; \left[\sum_{s=1}^{n}\psi_j^s(B,t)A_s + \omega_j(B,t,\tau)\right]U(-t,\tau),$$

$$j = 1,\dots,n, \tag{2.137}$$

$$-i\frac{\partial}{\partial \tau_j}U(t,\tau) \;=\; B_jU(t,\tau), \qquad j = 1,\dots,m, \tag{2.138}$$

where

$$\omega_j(y,t,\tau) = -\sum_{r,l=1}^{m}\tau_l\psi_i^r(y,t)\varphi_{rl}(y) \tag{2.139}$$

and the functions $\psi_j^s(y,t)$ are obtained by successively solving the systems of ordinary differential equations

$$\left[\frac{\partial}{\partial t_k} - \sum_{s=1}^{m}\varphi_{ks}(y)\frac{\partial}{\partial y_s}\right]\Omega(t_1,\dots,t_k,0,\dots,0,y)$$

$$= C_k(y)\Omega(t_1,\dots,t_k,0,\dots,0,y), \quad k = 1,\dots,n, \tag{2.140}$$

$$\Omega(0,y) = E \tag{2.141}$$

and by setting

$$\psi_j^s(y,t) = \Omega_j^s(y,0,\dots,0,t_{j+1},\dots,t_n). \tag{2.142}$$

In (2.140), the matrices $C_k(y)$ are given by

$$(C_k(y))_{js} = c_{kj}^s(y) \tag{2.143}$$

(see (2.132)).

The proof, rather technical, can be found in [2], pp. 128–131.

Now suppose that the symbol matrix $\psi(y,t)$ with entries $\psi_j^s(y,t)$ is invertible in $\mathcal{F}_m$, that is, the inverse matrix exists and its entries belong to $\mathcal{F}_m$ (with respect to the variable y). Then we can set

$$Q(t,\tau,x,y) = \psi^{-1}(y,t)(x - \omega(y,t,\tau)). \tag{2.144}$$

We claim that

$$Q_j \left(\overset{2}{t}, \overset{2}{\tau}, -i \, \frac{\overset{1}{\partial}}{\partial t}, -i \, \frac{\overset{1}{\partial}}{\partial \tau} \right) U(t,\tau) = A_j U(t,\tau), \quad j = 1, \dots, n. \quad (2.145)$$

The proof is based on the following general assertion, which will also prove useful in the sequel.

Lemma 11 *Let $f \in \mathcal{F}$. Under suitable functional-analytic conditions (see [2]),*

$$f \left(-i \frac{\partial}{\partial t} \right) e^{iAt} = f(A) e^{iAt}. \quad (2.146)$$

Proof. We have the relation

$$L_A L_{e^{itA}} = -i \frac{\partial}{\partial t} L_{e^{itA}}$$

on the space of operator families independent of t. Hence

$$f(L_A) L_{e^{itA}} = f \left(-i \frac{\partial}{\partial t} \right) L_{e^{itA}} \quad (2.147)$$

on the same space. Applying both sides of (2.147) to the identity element, we obtain (2.146).

The corresponding version of (2.146) holds for functions of several operators. Namely, if $f(x_1, \dots, x_n) \in \mathcal{F}_n$, then

$$f \left(-i \frac{\partial}{\partial t_1}, \dots, -i \frac{\partial}{\partial t_n} \right) e^{iA_n t_n} \times \dots \times e^{iA_1 t_1}$$

$$= f(\overset{1}{A}_1, \dots, \overset{n}{A}_n) e^{i\overset{n}{A}_n t_n} \times \dots \times e^{i\overset{1}{A}_1 t_1}. \quad (2.148)$$

Now (2.145) is obvious. Indeed, we have

$$Q \left(\overset{2}{t}, \overset{2}{\tau}, -i \, \frac{\overset{1}{\partial}}{\partial t}, -i \, \frac{\overset{1}{\partial}}{\partial \tau}, \right) U(t,\tau) \quad (2.149)$$

$$= \psi^{-1} \left(-i \, \frac{\overset{2}{\partial}}{\partial \tau}, \overset{1}{t} \right) \left(-i \frac{\partial U(t,\tau)}{\partial t} - \omega \left(-i \frac{\partial}{\partial \tau}, t, \tau \right) U(t,\tau) \right).$$

It remains to substitute the expression (2.137) for $\frac{\partial U}{\partial t}$ and use (2.148) for differentiations with respect to τ with regard for the fact that the operators B_j act last and commute with each other.

Now we are in a position to compute the left ordered representation operators. Let $f \in \mathcal{F}_{n+m}$ be an arbitrary symbol.

We obviously have

$$
B_j[\![f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}_1, \ldots, \overset{n+1}{B}_n)]\!] = \overset{n+1}{B}_j \ f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}_1, \ldots, \overset{n+1}{B}_n),
$$

and so

$$
l_{B_j} = y_j, \quad j = 1, \ldots, n. \tag{2.150}
$$

Next, let us compute the left ordered representation operators for A_j, $j = 1, \ldots, n$. We have, with regard for (2.148),

$$
A_j[\![f(\overset{1}{A}_1, \ldots, \overset{n}{A}_n, \overset{n+1}{B}_1, \ldots, \overset{n+1}{B}_n)]\!] \tag{2.151}
$$

$$
= \left\{ A_j \left(f \left(-i\frac{\partial}{\partial t_1}, \ldots, -i\frac{\partial}{\partial t_n}, -i\frac{\partial}{\partial \tau_1}, \ldots, -i\frac{\partial}{\partial \tau_m} \right) \right) U(t, \tau) \right\} \Bigg|_{t=0, \tau=0}.
$$

Since the left multiplication by A_j commutes with the differentiations with respect to the parameters t and τ, we can continue (2.143) as

$$
\left\{ f \left(-i\frac{\partial}{\partial t_1}, \ldots, -i\frac{\partial}{\partial t_n}, -i\frac{\partial}{\partial \tau_1}, \ldots, -i\frac{\partial}{\partial \tau_m} \right) A_j U(t, \tau) \right\} \Bigg|_{t=0, \tau=0}
$$

$$
\tag{2.152}
$$

$$
= \left\{ [\![f \left(-i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial \tau} \right)]\!] \ [\![Q_j \left(\overset{2}{t}, \overset{2}{\tau}, -i\overset{1}{\frac{\partial}{\partial t}}, -i\overset{1}{\frac{\partial}{\partial \tau}} \right)]\!] U(t, \tau) \right\} \Bigg|_{t=0, \tau=0}.
$$

We compute the product of two pseudodifferential operators on the right-hand side of (2.144) using the already known right ordered representation for the tuple $\left(-i\overset{1}{\frac{\partial}{\partial t}}, -i\overset{1}{\frac{\partial}{\partial \tau}}, \overset{2}{\tau}, \overset{2}{t} \right)$.

Specifically, we have

$$
[\![f \left(-i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial \tau} \right)]\!] \ [\![Q_j \left(\overset{2}{t}, \overset{2}{\tau}, -i\overset{1}{\frac{\partial}{\partial t}}, -i\overset{1}{\frac{\partial}{\partial \tau}} \right)]\!]
$$

$$= g \left(-i \, \overset{2}{\frac{\partial}{\partial t}}, -i \, \overset{1}{\frac{\partial}{\partial \tau}}, \overset{2}{t}, \overset{2}{\tau} \right), \qquad (2.153)$$

where

$$g(x, y, t, \tau) = Q_j \left(t - i \, \overset{1}{\frac{\partial}{\partial x}}, \tau - i \, \overset{1}{\frac{\partial}{\partial y}}, \overset{2}{x}, \overset{2}{y} \right) (f(x, y)). \qquad (2.154)$$

By substituting this into (2.152) and by setting $t = \tau = 0$, we obtain

$$A_j[\![f(\overset{1}{A_1}, \dots, \overset{n}{A_n}, \overset{n+1}{B}{}_1, \dots, \overset{n+1}{B}{}_m)]\!] = (l_{A_j} f)(\overset{1}{A_1}, \dots, \overset{n}{A_n}, \overset{n+1}{B}{}_1, \dots, \overset{n+1}{B}{}_m), \qquad (2.155)$$

where

$$l_{A_j} = Q_j \left(-i \, \overset{1}{\frac{\partial}{\partial x}}, -i \, \overset{1}{\frac{\partial}{\partial y}}, \overset{2}{x}, \overset{2}{y} \right). \qquad (2.156)$$

Taking into account the form of the function Q_j, we see that the left ordered representation operators are linear in x, as is the case with Lie algebras.

Remark 2 The invertibility condition for the matrix $\psi(y, t)$ holds if one imposes "nilpotency conditions" similar to those for Lie algebras.

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Moscow–Potsdam