

Eta-invariant and Pontrjagin duality in K -theory

Anton Savin* and Boris Sternin*

Moscow State University

e-mail: antonsavin@mtu-net.ru

e-mail: sternin@math.uni-potsdam.de

Abstract

The topological significance of the spectral Atiyah–Patodi–Singer η -invariant is investigated. We show that twice the fractional part of the invariant is computed by the linking pairing in K -theory with the orientation bundle of the manifold. The Pontrjagin duality implies the nondegeneracy of the linking form. An example of a nontrivial fractional part for an even-order operator is presented.

Keywords: eta-invariant, K -theory, Pontrjagin duality, linking coefficients, Atiyah–Patodi–Singer theory, modulo n index

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Introduction

P. Gilkey noticed [1] that the Atiyah–Patodi–Singer η -invariant [2] is rigid in the class of differential operators on closed manifolds when the following condition is satisfied

$$\text{ord}A + \dim M \equiv 1 \pmod{2}.$$

More precisely, in this case the fractional part of the spectral η -invariant of an elliptic self-adjoint differential operator A is determined by the principal symbol and it is a homotopy invariant of the principal symbol.

In this framework the η -invariant of geometric first-order operators on even-dimensional manifolds was studied in [3]. It turned out that the invariant takes only dyadic values. Moreover, nontrivial fractional parts not equal to $1/2$ appear only on nonorientable manifolds. As a typical example we should point out (self-adjoint) Dirac operator on manifolds with a pin^c structure in the tangent bundle. The operators of this form are defined, for example, on the projective spaces RP^{2n} . In this case, the fractional part of the η -invariant is an important pin^c -cobordism invariant [4]. The case of general differential operators remained unexplored. Even the problem of nontriviality of the η -invariant's fractional part for even-order operators on odd-dimensional manifolds remained open. We answer these questions in the present paper.

The following natural question appears, when we try to compute a fractional homotopy invariant: what terms could be used to express such an invariant? One of the

candidates is the expression of the invariant in terms of some *linking indices* [5, 6]. These indices define the bilinear pairing

$$\mathrm{Tor}H_i(M) \times \mathrm{Tor}H_{n-i-1}(M) \longrightarrow Q/Z$$

of torsion homology classes on a closed oriented n -dimensional manifold. The linking indices have numerous applications (see [7],[8],[9],[10],[11]).

In the present paper, we study a similar linking form in K -theory and elliptic theory (the latter appeared also in the paper [12]). The Pontrjagin duality in K -theory permits us to prove the nondegeneracy of the linking form. The main result of the paper is the equality of fractional part of twice the η -invariant with the linking index of the principal symbol of the elliptic operator with the orientation bundle of the manifold. Using this equality and the properties of the linking form, we prove the nontriviality of the fractional part of the η -invariant for some even order operators.

The proofs of the above results are based on a reduction of the spectral invariant under consideration to a homotopy invariant dimension functional of subspaces defined by pseudodifferential projections. This functional was introduced in [13, 14]. The index formula for elliptic operators which act in such subspaces makes it possible to express the fractional part of the dimension functional in terms of the index of an operator in elliptic theory with coefficients in a finite cyclic group Z_n . Together with the corresponding theorem for the index modulo n this expresses the fractional invariant under consideration in topological terms.

Let us briefly describe the contents of the paper. In the first section, we recall for the reader's convenience the relationship of the η -invariant to the dimension functional of subspaces defined by pseudodifferential projections. Here we briefly present the necessary results from [13, 14, 15]. We also express the fractional parts of the invariants in terms of the index of some elliptic operators in subspaces. At the beginning of the second section we describe the main results of [16] concerning index theory modulo n . The rest of the section is devoted to the proof of the Pontrjagin duality in K -theory. The linking pairing is introduced in Section 3. The fourth section contains a computation of the action of antipodal involution on K -groups of real vector bundles. Here we use the description [17] of these groups in terms of Clifford algebras. In the following section we state the main theorem, which expresses the fractional part of twice the dimension functional in terms of a linking index. Examples are presented. In particular, we construct an even subspace on which the dimension functional takes a value with a nontrivial fractional part by means of the "cross product" [18] of elliptic operators. This gives a positive answer to the question of P. Gilkey concerning the existence of even-order operators with nontrivial fractional part of the η -invariant.

1 Eta-invariant and index in subspaces

1. Let E be a vector bundle on a smooth manifold M . A linear subspace $\widehat{L} \subset C^\infty(M, E)$ is called *pseudodifferential*, if it is the range of a pseudodifferential projection P of order zero

$$\widehat{L} = \text{Im } P, \quad P : C^\infty(M, E) \rightarrow C^\infty(M, E).$$

The vector bundle $L = \text{Im } \sigma(P) \subset \pi^*E$ over the cospheres S^*M is called the *symbol* of a subspace.

Consider the antipodal involution

$$\alpha : T^*M \longrightarrow T^*M, \quad \alpha(x, \xi) = (x, -\xi)$$

of the cotangent bundle T^*M . A subspace $\widehat{L} \subset C^\infty(M, E)$ is *even (odd)*, if its symbol L is invariant (antiinvariant) under the involution:

$$L = \alpha^*L, \quad \text{or} \quad L \oplus \alpha^*L = \pi^*E. \quad (1)$$

Denote by $\widehat{\text{Even}}(M)$ ($\widehat{\text{Odd}}(M)$) the semigroups of all even (odd) pseudodifferential subspaces with respect to the direct sum.

Pseudodifferential subspaces can be also defined by means of self-adjoint elliptic operators. In this case the parity of the subspace corresponds to the parity of order for differential operators.

Proposition 1 *Let A be an elliptic self-adjoint operator of a nonnegative order. Then the subspace $\widehat{L}_+(A)$ generated by eigenvectors of A , which correspond to nonnegative eigenvalues, is pseudodifferential and its symbol $L_+(A)$ is equal to the nonnegative spectral subbundle of the principal symbol $\sigma(A)$:*

$$L_+(A) = L_+(\sigma(A)) \in \text{Vect}(S^*M).$$

Suppose that A is a differential operator or, more generally, it satisfies the following condition

$$\alpha^*\sigma(A) = \pm\sigma(A)$$

on its principal symbol. Then the subspace $\widehat{L}_+(A)$ is even (odd).

The proof of this proposition can be found in [19] or [14].

Even subspaces on odd-dimensional manifolds and odd subspaces on even-dimensional ones admit a homotopy invariant functional which is an analog of the notion of dimension of finite-dimensional vector spaces.

Theorem 1 [13, 14] *There exists a unique additive homotopy invariant functional*

$$d : \widehat{\text{Even}}(M^{\text{odd}}) \rightarrow Z \left[\frac{1}{2} \right], \quad \text{and} \quad d : \widehat{\text{Odd}}(M^{\text{ev}}) \rightarrow Z \left[\frac{1}{2} \right],$$

with the properties:

1. (relative dimension)

$$d(\widehat{L} + L_0) - d(\widehat{L}) = \dim L_0$$

for a pair $\widehat{L} + L_0, \widehat{L}$ of subspaces, which differ by a finite-dimensional space L_0 ;

2. (complement)

$$d(\widehat{L}) + d(\widehat{L}^\perp) = 0, \tag{2}$$

here \widehat{L}^\perp denotes the orthogonal complement of \widehat{L} .

It turns out that the spectral Atiyah–Patodi–Singer η -invariant is equal to the dimension functional of subspaces.

Theorem 2 [13, 14] *Let $\widehat{L}_+(A)$ be the nonnegative spectral subspace of an elliptic self-adjoint differential operator A of positive order. Then the following equality holds*

$$\eta(A) = d(\widehat{L}_+(A)), \tag{3}$$

provided the order of the operator and the dimension of the manifold have opposite parities.

The equality (3) is also valid for *admissible* pseudodifferential operators in the sense of [1].

2. The dimension functional of pseudodifferential subspaces enters the index formula for elliptic operators in subspaces.

Indeed, consider two pseudodifferential subspaces $\widehat{L}_{1,2} \subset C^\infty(M, E_{1,2})$ and a pseudodifferential operator

$$D : C^\infty(M, E_1) \longrightarrow C^\infty(M, E_2).$$

Suppose that the following inclusion is valid $D\widehat{L}_1 \subset \widehat{L}_2$. Then the restriction

$$D : \widehat{L}_1 \longrightarrow \widehat{L}_2 \tag{4}$$

is called an *operator acting in subspaces*. The restriction

$$\sigma(D) : L_1 \longrightarrow L_2 \tag{5}$$

of the principal symbol $\sigma(D)$ to the symbols of subspaces over the cospheres S^*M is called the *symbol of operator in subspaces*. It can be shown that the closure of operator (4) with respect to Sobolev norm defines an operator with Fredholm property if and only if the symbol (5) is *elliptic*, i.e. it is a vector bundle isomorphism.

The following index formula for elliptic operators in subspaces was obtained in [13, 14].

Theorem 3 *Let*

$$\widehat{L}_{1,2} \in \widehat{\text{Even}}(M^{\text{odd}}) \quad \text{or} \quad \widehat{\text{Odd}}(M^{\text{ev}}). \quad (6)$$

Then the index of an elliptic operator $D : \widehat{L}_1 \rightarrow \widehat{L}_2$ is equal to

$$\text{ind}(D, \widehat{L}_1, \widehat{L}_2) = \frac{1}{2} \text{ind} \widetilde{D} + d(\widehat{L}_1) - d(\widehat{L}_2), \quad (7)$$

where in the case of odd subspaces

$$\widetilde{D} : C^\infty(M, E_1) \rightarrow C^\infty(M, E_2)$$

is the usual¹ elliptic operator with principal symbol $\sigma(\widetilde{D})$ equal to

$$\sigma(\widetilde{D}) = \sigma(D) \oplus \alpha^* \sigma(D) : L_1 \oplus \alpha^* L_1 \rightarrow L_2 \oplus \alpha^* L_2.$$

For even subspaces, the following formula is valid

$$\widetilde{D} : C^\infty(M, E_1) \rightarrow C^\infty(M, E_1),$$

where the usual elliptic operator with principal symbol $\sigma(\widetilde{D})$ is defined by

$$\sigma(\widetilde{D}) = [\alpha^* \sigma(D)]^{-1} \sigma(D) \oplus 1 : L_1 \oplus L_1^\perp \rightarrow L_1 \oplus L_1^\perp.$$

3. As a direct consequence of the index formula (7), we have the following corollary.

Corollary 1 *The fractional part of twice the functional d is determined by the principal symbol of the subspace \widehat{L} as an element of the group $K(S^*M)/K(M)$.*

Let us apply the index formula (7) for operators in subspaces to the computation of the fractional part of the invariant d . To this end let us recall the following property of even (odd) subspaces.

¹That is, this operator acts in spaces of vector bundle sections.

Theorem 4 *The symbol of a subspace \widehat{L} with parity conditions (6) defines a 2-torsion element in the group $K(S^*M)/K(M)$. In other words, for some number N and a vector bundle $F \in \text{Vect}(M)$ on the base there exists an isomorphism*

$$\sigma : 2^N L \longrightarrow \pi^* F, \quad 2^N L = \underbrace{L \oplus \dots \oplus L}_{2^N \text{ copies}}. \quad (8)$$

The proof of this theorem for even subspaces is contained in [1], for odd subspaces — in [14].

Consider the corresponding elliptic operator (in subspaces)

$$\widehat{\sigma} : 2^N \widehat{L} \longrightarrow C^\infty(M, F) \quad (9)$$

with symbol (8). By virtue of (2) the space of vector bundle sections has “dimension” zero: $d(C^\infty(M, F)) = 0$. Therefore, the index formula (7) gives an equality

$$\text{ind}(\widehat{\sigma}, 2^N \widehat{L}, C^\infty(M, F)) = \frac{1}{2} \text{ind} \widetilde{\sigma} + 2^N d(\widehat{L}). \quad (10)$$

Placing the operators $\widehat{\sigma}$ and $\widetilde{\sigma}$ under the index sign, we can obtain the following expression for fractional part of twice the value of the dimension functional (cf. [16])

$$\left\{ 2d(\widehat{L}) \right\} = \frac{1}{2^N} \text{mod } 2^N\text{-ind}[(1 \pm \alpha^*) \widehat{\sigma}], \quad (11)$$

where $\text{mod } 2^N\text{-ind } D \in \mathbb{Z}_{2^N}$ denotes the index of a Fredholm operator D reduced modulo 2^N , while $\alpha^* \widehat{\sigma}$ denotes operator in subspaces

$$\widehat{\alpha^* \sigma} : 2^N \widehat{\alpha^* L} \longrightarrow C^\infty(M, F).$$

Note that the index of operator (9) as well as the index in (11) as a residue modulo 2^N is determined by the principal symbol of the operator. In the next section we show how this index can be computed.

2 Index modulo n and Pontrjagin duality in K -theory

Let us recall the main results of index theory modulo n from [16].

1. Let n be a natural number. *Elliptic operator modulo n* is an elliptic operator of the form

$$D : n\widehat{L}_1 \oplus C^\infty(M, E_1) \rightarrow n\widehat{L}_2 \oplus C^\infty(M, F_1), \quad (12)$$

where

$$\widehat{L}_1 \subset C^\infty(M, E), \widehat{L}_2 \subset C^\infty(M, F)$$

are pseudodifferential subspaces. The direct sums

$$n \left(\widehat{L}_1 \oplus C^\infty(M, E_1) \right) \xrightarrow{nD} n \left(\widehat{L}_2 \oplus C^\infty(M, F_1) \right) \quad (13)$$

are called *trivial* operators modulo n . The group of stable homotopy classes of elliptic operators modulo trivial ones is denoted by $\text{Ell}(M, Z_n)$. In [16] it is shown that this group defines K -theory with coefficients in Z_n .

Theorem 5 *There is an isomorphism of groups*

$$\text{Ell}(M, Z_n) \xrightarrow{\chi_n} K_c(T^*M, Z_n).$$

Here K_c denotes K -theory with compact supports.

Let us give an explicit formula for the isomorphism χ_n . Recall that K -theory with coefficients Z_n is defined by means of the so-called Moore space M_n . This is a topological space with K -groups equal to

$$\widetilde{K}^0(M_n) = Z_n, \quad K^1(M_n) = 0.$$

For instance, as a Moore space we can take a two-dimensional complex obtained from the disk D^2 by identifying the points on its boundary under the action of the group Z_n :

$$M_n = \left\{ D^2 \subset C \mid |z| \leq 1 \right\} / \left\{ e^{i\varphi} \sim e^{i(\varphi + \frac{2\pi k}{n})} \right\}.$$

Denote the vector bundle corresponding to the generator of the group $\widetilde{K}^0(M_n) = Z_n$ by γ_n . Let us also fix a trivialization β of the direct sum $n\gamma_n$

$$n\gamma_n \xrightarrow{\beta} C^n.$$

For a topological space X , its K -groups with coefficients Z_n are defined according to the formula

$$K^*(X, Z_n) = K^*(X \times M_n, X \times pt). \quad (14)$$

It is shown in [16] that an arbitrary operator (12) is stably homotopic to some operator of the form

$$n\widehat{L} \xrightarrow{D} C^\infty(M, F). \quad (15)$$

An operator of this type defines a family of usual elliptic symbols on M (here we use the difference construction for elliptic families)

$$\chi_n[D] = \left[\pi^*F \xrightarrow{\sigma^{-1}(D)} nL \xrightarrow{\beta^{-1} \otimes 1_L} \gamma_n \otimes nL \xrightarrow{1_\gamma \otimes \sigma(D)} \gamma_n \otimes \pi^*F \right] \in K_c(T^*M \times M_n, T^*M \times pt) \quad (16)$$

The family is parametrized by the Moore space M_n .

Note that the index of an elliptic operator modulo n is determined by its principal symbol as a residue

$$\text{mod } n\text{-ind} \left(n\hat{L} \xrightarrow{D} C^\infty(M, F) \right) \in Z_n.$$

This index-residue can be computed topologically, the corresponding index theorem is equivalent to the commutativity of the triangle

$$\begin{array}{ccc} & \text{Ell}(M, Z_n) & \\ \swarrow \chi_n & & \searrow \text{mod } n\text{-ind} \\ K_c(T^*M, Z_n) & \xrightarrow{p!} & Z_n, \end{array} \quad (17)$$

where $p! : K_c(T^*M, Z_n) \rightarrow Z_n$ is the direct image mapping in K -theory with coefficients \mathbf{Z}_n .

We will use later an exact sequence relating elliptic operators modulo n to the usual elliptic operators.

To this end let us denote the group of stable homotopy classes of elliptic operators by $\text{Ell}(M)$ (see [18]), and a similar stable homotopy group for pseudodifferential subspaces by $\text{Ell}_1(M)$ (see [16]).

The mappings

$$\text{Ell}(M) \xrightarrow{\chi_0} K_c(T^*M), \quad \text{Ell}_1(M) \xrightarrow{\chi_1} K_c^1(T^*M),$$

which associate principal symbols to operators and subspaces, define isomorphisms with the corresponding K -groups (recall (see [2] or [16]) that the second mapping is defined as the composition

$$\chi_1 : \text{Ell}_1(M) \rightarrow K(S^*M) / K(M) \xrightarrow{\delta} K_c^1(T^*M),$$

where the first mapping associates symbol to a subspace. Then we apply the isomorphism δ induced from the coboundary mapping in K -theory:

$$\delta : K(S^*M) \rightarrow K_c^1(T^*M).$$

The following diagram is commutative

$$\begin{array}{ccccccccc} \text{Ell}(M) & \xrightarrow{\chi_0} & \text{Ell}(M) & \xrightarrow{i} & \text{Ell}(M, Z_n) & \xrightarrow{\partial} & \text{Ell}_1(M) & \xrightarrow{\chi_1} & \text{Ell}_1(M) \\ \downarrow \chi_0 & & \downarrow \chi_0 & & \downarrow \chi_n & & \downarrow \chi_1 & & \downarrow \chi_1 \\ K_c(T^*M) & \xrightarrow{\chi_0} & K_c(T^*M) & \xrightarrow{i'} & K_c(T^*M, Z_n) & \xrightarrow{\partial'} & K_c^1(T^*M) & \xrightarrow{\chi_1} & K_c^1(T^*M), \end{array} \quad (18)$$

here the mapping $\times n$ sends an element x of an abelian group to its multiple nx , the mapping i is induced by the embedding of usual elliptic operators into the set of elliptic operators modulo n , while the Bokstein homomorphism ∂ is defined by the formula

$$\partial \left[n\hat{L}_1 \oplus C^\infty(M, E_1) \xrightarrow{D} n\hat{L}_2 \oplus C^\infty(M, F_1) \right] = [\hat{L}_1] - [\hat{L}_2].$$

Consider the natural inclusions $Z_n \subset Z_{mn}$ for a pair of natural numbers n, m . The direct limit

$$\lim_{n \rightarrow \infty} \mathbf{Z}_n = Q/Z$$

permits us to define K - and Ell-groups with coefficients in Q/Z :

$$\begin{aligned} K_c(T^*M, Q/Z) &= \varinjlim K_c(T^*M, Z_n), \\ \text{Ell}(M, Q/Z) &= \varinjlim \text{Ell}(M, Z_n). \end{aligned}$$

Moreover, (18) transforms into the diagram with exact rows

$$\begin{array}{ccccccc} \text{Ell}(M) & \rightarrow & \text{Ell}(M) \otimes Q & \xrightarrow{i} & \text{Ell}(M, Q/Z) & \xrightarrow{\partial} & \text{Ell}_1(M) \rightarrow \text{Ell}_1(M) \otimes Q \\ \downarrow \chi_0 & & \downarrow \chi_0 \otimes 1 & & \downarrow \chi_{Q/Z} & & \downarrow \chi_1 \quad \downarrow \chi_1 \otimes 1 \\ K_c(T^*M) & \rightarrow & K_c(T^*M) \otimes Q & \xrightarrow{i'} & K_c(T^*M, Q/Z) & \xrightarrow{\partial'} & K_c^1(T^*M) \rightarrow K_c^1(T^*M) \otimes Q. \end{array} \quad (19)$$

2. Consider the intersection form

$$K_c^i(T^*M, Q/Z) \times K_c^i(M) \longrightarrow K_c^0(T^*M, Q/Z) \xrightarrow{p_i} Q/Z, \quad (20)$$

which is induced by the product and the direct image mapping p_i in K -theory. The intersection of elements x and y is denoted by $x \cap y$.

Recall that a pairing

$$\langle \cdot, \cdot \rangle : G_1 \times G_2 \rightarrow G_3$$

of abelian groups $G_{1,2}$ with values in abelian group G_3 is called *nondegenerate* if the mappings

$$\langle x, \cdot \rangle : G_2 \rightarrow G_3 \quad \text{and} \quad \langle \cdot, y \rangle : G_1 \rightarrow G_3,$$

obtained by fixing the values of one of the arguments are zero only for $x = 0$ (correspondingly $y = 0$).

Theorem 6 (Pontrjagin duality) *The pairing (20) is nondegenerate. In addition, fixing its first argument, we obtain an isomorphism*

$$K_c^i(T^*M, Q/Z) \simeq \text{Hom} \left(K^i(M), Q/Z \right). \quad (21)$$

Proof. It can be easily shown that the isomorphism (21) implies the nondegeneracy of the pairing.

Similarly to Poincaré duality (e.g., see [20]), the Pontrjagin duality can be proved by means of the Mayer–Vietoris principle. For an arbitrary open subset $U \subset M$, let us consider the mapping

$$K_c^i(T^*U, Q/Z) \longrightarrow \text{Hom}(K^i(U), Q/Z). \quad (22)$$

We want to prove that this mapping is an isomorphism for $U = M$. According to the Mayer–Vietoris principle it suffices a) to verify it for a contractible subset U ; b) to prove the inductive statement: if the mapping is an isomorphism for two open subsets U, V and for their intersection $U \cap V$ then it is an isomorphism for the union.

Consider a contractible set U . Then

$$K_c^*(T^*U, Q/Z) = K^*(pt) = Q/Z \oplus 0$$

and

$$K^*(U) = Z \oplus 0,$$

while the corresponding mapping

$$Q/Z \rightarrow \text{Hom}(Z, Q/Z)$$

is an isomorphism. The validity of condition a) is thereby proved.

To prove b) consider the diagram

$$\begin{array}{ccccccc} \rightarrow K_c^{i-1}(T^*(U \cap V), Q/Z) & \rightarrow & K_c^i(T^*(U \cup V), Q/Z) & \rightarrow & K_c^i(T^*(U \sqcup V), Q/Z) & \rightarrow & \\ & & \downarrow & & \downarrow & & \downarrow \\ \rightarrow \text{Hom}(K^{i-1}(U \cap V), Q/Z) & \rightarrow & \text{Hom}(K^i(U \cup V), Q/Z) & \rightarrow & \text{Hom}(K^i(U \sqcup V), Q/Z) & \rightarrow & . \end{array}$$

Here the upper row is the Mayer–Vietoris exact sequence, the lower row is obtained from a similar sequence applying the functor $\text{Hom}(\cdot, Q/Z)$ (this functor preserves the exactness of sequences). Vertical mappings are induced by the intersection pairing. This diagram is commutative up to sign. Suppose that the mapping (22) is an isomorphism for U, V and their intersection $U \cap V$. In accordance with the 5-lemma we obtain the isomorphism for the union $U \cup V$. Thus, we establish condition b).

Therefore, both conditions of the Mayer–Vietoris principle are satisfied. The theorem is proved.

Remark 1 For K -theory with coefficients in a topological group R/Z (its definition can be found in [2]), one can obtain by the same method the Pontrjagin duality of groups $K_c^i(T^*M, R/Z)$ and $K^i(M)$:

$$K_c^i(T^*M, R/Z) \simeq \text{Hom}(K^i(M), R/Z), \quad K^i(M) \simeq \text{Hom}(K_c^i(T^*M, R/Z), R/Z),$$

i.e. both groups are character groups of each other.

Remark 2 By a similar method one can prove the Pontrjagin duality for an arbitrary K -oriented closed manifold or manifold with boundary (i.e., a manifold with a $spin^c$ structure in the tangent bundle). From this more general viewpoint, Theorem 6 establishes the Pontrjagin–Lefschetz duality in K -theory for an almost-complex manifold T^*M of groups with compact supports and absolute groups.

Remark 3 One could also prove the nondegeneracy of the pairing

$$K_c^i(T^*M) \times K^i(M, Q/Z) \longrightarrow K_c^0(T^*M, Q/Z) \xrightarrow{p} Q/Z.$$

Just like in (co)homology theory (see [5]), the Pontrjagin duality implies Poincaré duality for torsion subgroups. The bilinear form defining this duality is called the *linking form*.

3 Linking form in K -theory

1. Consider the Bokstein homomorphism

$$\partial : K_c^i(T^*M, Q/Z) \rightarrow K_c^{i-1}(T^*M)$$

(see diagram (19)). The range of this mapping consists of finite order elements. Denote by $\text{Tor}G$ the torsion subgroup of an abelian group G .

Definition 1 *The linking form* is the pairing

$$\begin{aligned} \cap : \text{Tor}K_c^{i-1}(T^*M) \times \text{Tor}K^i(M) &\longrightarrow Q/Z, \\ (x, y) &\mapsto x' \cap y, \end{aligned}$$

where $x' \in K_c^i(T^*M, Q/Z)$ is an arbitrary element of K -theory with coefficients such that $\partial x' = x$, where $x' \cap y \in Q/Z$ is the intersection index from the previous section.

Similarly to the intersection form, the linking form is defined by a product

$$\begin{aligned} \text{Tor}K_c^{i-1}(T^*M) \times \text{Tor}K^i(M) &\longrightarrow K_c(T^*M, Q/Z), \\ (x, y) &\mapsto x'y, \quad \partial x' = x \end{aligned} \tag{23}$$

Lemma 1 *The product (23) and the linking form are well-defined.*

Proof. We need to show that the indeterminacy of the choice of x' does not affect the product $x'y$. Let $x = \partial x''$. For the difference $x' - x''$ we obtain $\partial(x' - x'') = 0$. Thus, $x' - x'' = i(z)$ for some $z \in K_c^i(T^*M) \otimes Q$. However, the product $zy \in K_c^0(T^*M) \otimes Q$ is a torsion element. Hence, $zy = 0$. We have proved that $x'y = x''y$. Lemma is proved.

The linking form can be defined similarly using the torsion property of the second argument. To this end consider the Bokstein homomorphism ∂ in the coefficient sequence

$$\dots \rightarrow K^{i-1}(M, Q/Z) \xrightarrow{\partial} K^i(M) \rightarrow K^i(M) \otimes Q \rightarrow \dots$$

As a linking index of elements $x \in \text{Tor}K_c^{i-1}(T^*M), y \in \text{Tor}K^i(M)$ let us put

$$x \cap' y = x \cap y'.$$

It turns out that both methods define linking pairings which coincide up to sign. Namely, the following equality is valid

$$x \cap' y = (-1)^{\deg x + 1} x \cap y.$$

This formula follows from the proposition.

Proposition 2 *Consider $x' \in K_c^i(T^*M, Q/Z), y' \in K^j(M, Q/Z)$. Then the following equality holds*

$$\partial x' y' = (-1)^{\deg x' + 1} x' \partial y'.$$

Proof. Let us assume that x', y' are induced from K -groups with coefficients Z_n for n large enough:

$$x' = I_* x_0, y' = n_* y_0, x_0 \in K_c^i(T^*M, Z_n), y_0 \in K^j(M, Z_n), \quad (24)$$

where I denotes the embedding $Z_n \subset Q/Z$.

Consider the exact sequence

$$0 \rightarrow Z_n \xrightarrow{\times n} Z_{n^2} \xrightarrow{\text{mod } n} Z_n \rightarrow 0$$

and the corresponding sequence in K -theory

$$\dots \rightarrow K_c^{i+j}(T^*M, Z_n) \xrightarrow{\partial''} K_c^{i+j+1}(T^*M, Z_n) \xrightarrow{\times n} K_c^{i+j+1}(T^*M, Z_{n^2}) \rightarrow \dots$$

The Bokstein homomorphism satisfies the Leibniz rule

$$\partial''(x_0 y_0) = \partial x_0 y_0 + (-1)^{\deg x_0} x_0 \partial y_0. \quad (25)$$

On the other hand, by virtue of the exactness, we have $\times n \circ \partial''(x_0 y_0) = 0$. Hence,

$$I_* \partial''(x_0 y_0) = 0. \quad (26)$$

The expressions (24), (25), (26) together imply the desired

$$\partial x' y' + (-1)^{\deg x'} x' \partial y' = 0.$$

Theorem 7 (Poincaré duality for torsion groups) *The linking form*

$$\mathrm{Tor}K_c^{i-1}(T^*M) \times \mathrm{Tor}K^i(M) \longrightarrow Q/Z$$

is nondegenerate. In particular, it defines isomorphisms

$$\mathrm{Tor}K_c^{i-1}(T^*M) \simeq \mathrm{Hom}(\mathrm{Tor}K^i(M), Q/Z),$$

$$\mathrm{Tor}K^i(M) \simeq \mathrm{Hom}(\mathrm{Tor}K_c^{i-1}(T^*M), Q/Z).$$

Corollary 2 *The groups $\mathrm{Tor}K_c^{i-1}(T^*M)$ and $\mathrm{Tor}K^i(M)$ are isomorphic.*

This follows from (noncanonical) isomorphism $G \simeq \mathrm{Hom}(G, Q/Z)$ for a finite abelian group G .

Proof of Theorem 7. Let us first prove the nondegeneracy with respect to the second argument. Suppose that $x \cap y = 0$ for an arbitrary $x \in \mathrm{Tor}K_c^{i-1}(T^*M)$. Therefore, for any $x' \in K_c^i(T^*M, Q/Z)$ we also have $x' \cap y = 0$. The Pontrjagin duality implies that $y = 0$.

The nondegeneracy of the pairing with respect to the first argument follows from the Pontrjagin duality corresponding to the intersection form

$$K_c^{i-1}(T^*M) \times K^i(M, Q/Z) \rightarrow Q/Z$$

(see Remark 3).

The second claim of the theorem follows from the finiteness of the torsion subgroups. The theorem is thereby proved.

2. The isomorphism of elliptic theory and K -theory from the first part of the previous section permits us to define the linking form in terms of elliptic operators.

Definition 2 *The linking form in elliptic theory* is the bilinear pairing

$$\begin{aligned} \mathrm{TorEll}_1(M) \times \mathrm{Tor}K^0(M) &\longrightarrow Q/Z, \\ (x, y) &\mapsto \mathrm{ind}x'y, \quad \partial x' = x. \end{aligned}$$

Proposition 3 *The linking forms in elliptic and K -theory are isomorphic, i.e. the following diagram commutes*

$$\begin{array}{ccc} \mathrm{TorEll}_1(M) \times \mathrm{Tor}K^0(M) & \rightarrow & Q/Z \\ \downarrow \chi_1 \times 1 & & \downarrow \mathrm{Id} \\ \mathrm{Tor}K_c^1(T^*M) \times \mathrm{Tor}K^0(M) & \rightarrow & Q/Z. \end{array}$$

Proof of the proposition follows from the commutative diagram

$$\begin{array}{ccccc}
\mathrm{TorEll}_1(M) \times \mathrm{Tor}K^0(M) & \rightarrow & \mathrm{Ell}(M, Q/Z) & \xrightarrow{\mathrm{ind}} & Q/Z \\
\downarrow & & \downarrow & & \parallel \\
\mathrm{Tor}K_c^1(T^*M) \times \mathrm{Tor}K^0(M) & \rightarrow & K_c(T^*M, Q/Z) & \xrightarrow{p_*} & Q/Z.o
\end{array}$$

The left square commutes due to the isomorphism of coefficient sequences in K -theory and in elliptic theory (see (19)), while the right square commutes by virtue of the modulo n index formula (see (17)). The proposition is proved.

Let us write an explicit formula for the linking form in elliptic theory. Let $x = [\widehat{L}]$ be a pseudodifferential subspace and $y = [G_1] - [G_2]$ be a difference of vector bundles $G_{1,2} \in \mathrm{Vect}(M)$. According to the first definition of the linking form for some element $x' = \left[n\widehat{L} \xrightarrow{\widehat{\sigma}} C^\infty(M, F) \right] \in \mathrm{Ell}(M, Z_n)$ the linking coefficient is equal to

$$x \cap y = \mathrm{ind}x'y = \frac{1}{n} \mathrm{mod} \, n\text{-ind}(\widehat{\sigma} \otimes 1_{G_1 - G_2}), \quad (27)$$

where $\widehat{\sigma} \otimes 1_{G_1 - G_2}$ denotes operator $\widehat{\sigma}$ with coefficients in $G_1 - G_2$. By the second definition of the linking form for some $y' = [mG_1 \xrightarrow{g} mG_2] \in K^1(M, Z_m)$ (see [2]) this index is defined as

$$x \cap y = \mathrm{ind}xy' = \frac{1}{m} \mathrm{mod} \, m\text{-ind}(1_{\widehat{L}} \otimes g),$$

where the operator in subspaces $1_{\widehat{L}} \otimes g$ has principal symbol $mL \otimes G_1 \xrightarrow{1_L \otimes g} mL \otimes G_2$.

Remark 4 Applying the Poincaré isomorphism in complex K -theory to the manifold T^*M (see [21])

$$K_c^1(T^*M) \simeq K_1(M),$$

the linking pairing can be considered as a *nondegenerate* pairing of K -homology with K -cohomology of the manifold M :

$$\mathrm{Tor}K_1(M) \times \mathrm{Tor}K^0(M) \xrightarrow{\cap} Q/Z.$$

Let us also note the similarity of expression (27) for the linking index and (11) for the fractional part of the dimension functional. Unlike the linking index, the latter formula contains the action of the antipodal involution instead of the product with some bundle. We show in the following section that the antipodal involution acts on K -groups as a tensor product with the orientation bundle of the manifold (cf. [22] in the orientable case).

4 Antipodal involution and orientability

Let V be a real vector bundle over a compact space X . Consider the antipodal involution

$$\begin{aligned} \alpha : V &\longrightarrow V, \\ v &\longmapsto -v. \end{aligned}$$

Theorem 8 *The induced involution α^* in K -theory is equal to*

$$\alpha^* = (-1)^n \Lambda^n(V) : K_c^*(V) \longrightarrow K_c^*(V), \quad (28)$$

where $\Lambda^n(V)$ for $n = \dim V$ is the orientation bundle of V .

Proof. Without loss of generality we can assume that V is even-dimensional: $n = \dim V = 2k$. We can also consider only the action of α^* on even K -groups $K_c^0(V)$ (the odd case reduces to this situation taking a sum with a one-dimensional trivial bundle, since in this case both sides of the equality (28) change by -1).

M. Karoubi in [17, 22] found a description of the groups $K_c^0(V)$ in terms of Clifford algebras. Let us recall the basic definitions. On the space X , consider the bundle $Cl(V)$ of Clifford algebras of the vector bundle V . Over a point x of the base X the Clifford algebra $Cl(V_x)$ is multiplicatively generated by the vector space V_x with the following relations

$$v_1 v_2 + v_2 v_1 = 2 \langle v_1, v_2 \rangle$$

for some scalar product \langle, \rangle in V .

Let us consider quadruples (E, c, f_1, f_2) , where E is a complex vector bundle over X , c is a homomorphism

$$c : Cl(V) \longrightarrow \text{Hom}(E, E)$$

of algebra bundles. We say that c defines a Clifford module structure on E . The involutions $f_{1,2}$ of E are supposed to skew commute with the Clifford multiplication:

$$f_i c + c f_i = 0, \quad i = 1, 2.$$

Stable homotopy equivalence relation is defined on the set of quadruples (E, c, f_1, f_2) . The corresponding group of equivalence classes is denoted by $K^V(X)$. It is proved in [17] that this group is isomorphic to $K_c^0(V)$. The isomorphism

$$t : K^V(X) \longrightarrow K_c^0(V)$$

is given by the following explicit formula

$$t[E, c, f_1, f_2] = \left[\pi^* \ker(f_1 - 1) \xrightarrow{(1-c(v)f_2)(1+c(v)f_1)} \pi^* \ker(f_2 - 1) \right], \quad \text{where } \pi : SV \rightarrow X. \quad (29)$$

Here SV is the sphere bundle of V , and the element in the left hand side of the equality is understood in the sense of difference construction (e.g., see [23]) for the relative group $K(BV, SV) \simeq K_c(V)$.

The formula (29) implies that the antipodal involution α^* acts on the quadruples according to the formula

$$\alpha^* [E, c, f_1, f_2] = [E, -c, f_1, f_2].$$

Let us show that $[E, -c, f_1, f_2]$ differs from the quadruple $[E, c, f_1, f_2] \otimes \Lambda^{2k}(V)$ by a vector bundle isomorphism (cf. [3]).

To this end, consider the local orthonormal base e_1, e_2, \dots, e_{2k} of V . Let us define the element

$$\beta = i^k c(e_1) \dots c(e_{2k}).$$

One verifies easily that $\beta^2 = 1$ and β skew commutes with the Clifford action c and commutes with the involutions $f_{1,2}$

$$\beta c + c\beta = 0, \quad f_i \beta = \beta f_i. \quad (30)$$

A direct computation shows that the change of the orthonormal base changes β by the sign of the transition matrix determinant. Therefore, globally this element defines a vector bundle isomorphism

$$\beta : E \longrightarrow E \otimes \Lambda^{2k}(V)$$

(for brevity we denote it by the same symbol). At the same time the commutation relations (30) transform to

$$\beta^{-1} (c \otimes 1_{\Lambda^{2k}(V)}) \beta = -c, \quad \beta^{-1} (f_i \otimes 1_{\Lambda^{2k}(V)}) \beta = f_i.$$

Hence, the quadruples

$$\alpha^* [E, c, f_1, f_2] \quad \text{and} \quad [E, c, f_1, f_2] \otimes \Lambda^{2k}(V)$$

are isomorphic. Theorem is thereby proved.

Corollary 3 *The same formula holds for K -theory with coefficients \mathbf{Z}_n*

$$\alpha^* = (-1)^n \Lambda^n(V) : K_c^*(V, \mathbf{Z}_n) \longrightarrow K_c^*(V, \mathbf{Z}_n).$$

Indeed, K -theory with coefficients is defined by means of the Moore space M_n by the formula

$$K_c^*(V, \mathbf{Z}_n) = K_c^*(V \times M_n, V \times pt).$$

Let us apply our theorem to the space $X \times M_n$ and the pull-back of bundle V . This gives the desired formula for K -theory with coefficients.

Remark 5 In the case when X is a smooth manifold with the cotangent bundle $V = T^*X$, (29) defines a class of elliptic symbols, such that an arbitrary symbol reduces to a symbol of this form by a stable homotopy.

5 The main theorem

Theorem 9 *Fractional part $\{2d(\hat{L})\}$ of twice the value of the dimension functional d on subspace \hat{L} is equal to the linking index of the subspace with the orientation bundle of the manifold $\Lambda^n(M)$, $n = \dim M$:*

$$\{2d(\hat{L})\} = [L] \cap (1 - [\Lambda^n(M)]) \in Z \left[\frac{1}{2} \right] / Z.$$

Proof. Recall the expression (11) of the fractional part of the dimension functional d :

$$\{2d(\hat{L})\} = \frac{1}{2^N} \text{mod } 2^N\text{-ind} [(1 \pm \alpha^*) \hat{\sigma}], \quad [\sigma] \in K_c^0(T^*M, Z_{2^N}),$$

where $\sigma : 2^N L \rightarrow \pi^* F$ is a vector bundle isomorphism. By Theorem 8 of the previous section we have

$$(1 \pm \alpha^*) [\sigma] = [1 - \Lambda^n(M)] [\sigma].$$

Hence,

$$\{2d(\hat{L})\} = \frac{1}{2^N} \text{mod } 2^N\text{-ind} [(1 - \Lambda^n(M)) \hat{\sigma}].$$

This coincides with the definition of the linking index in Section 3, see (27).

Note also that $[L] \in K_c^1(T^*M)$ is a torsion element by virtue of Theorem 4, the torsion property for the difference $[1 - \Lambda^n(M)] \in K(M)$ is proved in Proposition 4. The theorem is proved.

Corollary 4 *The dimension functional takes half-integer values*

$$\{2d(\hat{L})\} = 0$$

when the manifold is orientable.

Proposition 4 *Let M be a nonorientable manifold of dimension $2k$ or $2k + 1$. Then the following integrality estimate of the invariant d is valid:*

$$\{2^{k+1}d(\hat{L})\} = 0. \tag{31}$$

The same statements hold for the η -invariant. It is shown in the next section that the above estimate is exact for $k \geq 1$.

Proof of Proposition 4. The orientation bundle $\Lambda^n(M^n)$ is a one-dimensional bundle with structure group Z_2 . Hence, it is the pull-back of the universal bundle from the classifying space $BZ_2 = RP^\infty$, i.e. there is a vector bundle isomorphism

$$\Lambda^n(M^n) \simeq f^* \gamma, \quad \text{for some mapping } f : M^n \rightarrow RP^N.$$

Here γ is the line bundle over the projective space RP^N . By the approximation theorem we can suppose that $N = n$. The reduced K -groups of the projective spaces are well known

$$\widetilde{K}(RP^{2k}) \simeq \widetilde{K}(RP^{2k+1}) \simeq Z_{2^k}.$$

Thus, we have

$$2^k (1 - [\Lambda^n(M^n)]) = 0.$$

Hence, we obtain the desired

$$\{2^{k+1}d(\widehat{L})\} = [L] \cap [2^k (1 - \Lambda^n(M^n))] = [L] \cap 0 = 0.$$

6 Examples

1. Consider the even-dimensional real projective space RP^{2n} . The reduced K -group of this manifold is cyclic

$$\widetilde{K}(RP^{2n}) \simeq Z_{2^n}.$$

The generator is given by the orientation bundle

$$1 - [\Lambda^{2n}(RP^{2n})] \in \widetilde{K}(RP^{2n}).$$

On the other hand, the projective space RP^{2n} has a pin^c structure, while the principal symbol of the self-adjoint pin^c Dirac operator D on it (this operator was constructed in [3]) is a generator of the isomorphic group

$$[\sigma(D)] \in K_c^1(T^*RP^{2n}) = \text{Tor}K_c^1(T^*RP^{2n}) \simeq Z_{2^n}.$$

The nondegeneracy of the linking form implies that these generators have a nontrivial linking index

$$2^{n-1} [\sigma(D)] \cap [1 - [\Lambda^{2n}(RP^{2n})]] = \frac{1}{2}.$$

Hence, the η -invariant of the pin^c Dirac operator D has a large fractional part:

$$\{2^n \eta(D)\} = 2^{n-1} [\sigma(D)] \cap [1 - [\Lambda^{2n}(RP^{2n})]] = \frac{1}{2}.$$

This example shows that the estimate (31) is precise on even-dimensional manifolds.

2. Let us construct an operator on an odd-dimensional manifold with a nontrivial fractional part of the η -invariant. To this end we apply the cross product of elliptic operators (cf. [18]).

Let D_1 be an elliptic self-adjoint operator on an even-dimensional manifold M_1 with odd symbol:

$$\sigma(D_1)(x, -\xi) = -\sigma(D_1)(x, \xi),$$

and D_2 be an elliptic operator on an odd-dimensional M_2 with the symbol satisfying

$$\sigma(D_2)(x, -\xi) = \sigma(D_2)^*(x, \xi) \quad (32)$$

(we suppose that D_2 is an endomorphism). Denote by M the Cartesian product of the manifolds $M_1 \times M_2$. Consider the cross product

$$[\sigma(D_1)] \times [\sigma(D_2)] \in \text{Tor}K_c^1(T^*M)$$

of the corresponding elliptic symbols. Here

$$[\sigma(D_1)] \in K_c^1(T^*M_1), [\sigma(D_2)] \in K_c^0(T^*M_2).$$

For the (self-adjoint) symbol $\sigma(D)$ of the product the following formula is valid

$$\sigma(D) = \begin{pmatrix} \sigma(D_1) \otimes 1 & 1 \otimes \sigma(D_2)^* \\ 1 \otimes \sigma(D_2) & -\sigma(D_1) \otimes 1 \end{pmatrix}. \quad (33)$$

Let us compute the linking index of $[\sigma(D_1)] \times [\sigma(D_2)]$ with the orientation bundle of $M_1 \times M_2$.

Proposition 5 *Suppose that M_2 is orientable. Then the following equality holds*

$$([\sigma(D_1)] \times [\sigma(D_2)]) \cap (1 - \Lambda^n(M^n)) = [\sigma(D_1)] \cap (1 - \Lambda^k(M_1)) \text{ind} D_2, \quad \dim M_1 = k. \quad (34)$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} K_c^0(T^*M_1, Q/Z) \times K_c^0(T^*M_2) & \rightarrow & K_c^0(T^*M, Q/Z) \\ \uparrow \cap (1 - \Lambda^k(M_1)) \times 1 & & \uparrow \cap (1 - \Lambda^k(M_1)) \\ \text{Tor}K_c^1(T^*M_1) \times K_c^0(T^*M_2) & \rightarrow & \text{Tor}K_c^1(T^*M). \end{array}$$

The horizontal mappings here are induced by products in K -theory. By virtue of the orientability of M_2 we have

$$1 - [\Lambda^k(M_1)] = 1 - [\Lambda^n(M)].$$

Thence, the desired (34) follows from the last diagram when we apply the direct image mapping

$$p_! : K_c^0(T^*M, Q/Z) \rightarrow Q/Z.$$

The proposition is proved.

Remark 6 Formula (34) is similar to the well-known property of the η -invariant (see [2]): η -invariant of a cross product is equal to the product of the η -invariant of the first factor and the index of the second operator.

Unfortunately, the bundle $L_+(\sigma(D))$ is not an even one. Indeed, the symbol $\sigma(D)$ satisfies the equality

$$\alpha^* \sigma(D) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma(D) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (35)$$

However, this shows that the spectral subspace transforms according to

$$\alpha^* L_+(\sigma(D)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L_+(\sigma(D)).$$

It turns out that the bundle $L_+(\sigma(D))$ is isomorphic to an even bundle.

Proposition 6 *There exists a vector bundle $L \in \text{Vect}(P^*M)$ on the projective spaces such that the pull-back p^*L to the cospheres S^*M under the projection $p : S^*M \rightarrow P^*M$ is isomorphic to $L_+(\sigma(D))$.*

Proof. Let us fix a nonsingular vector field on the odd-dimensional manifold M . Consider the corresponding splitting of the cotangent bundle $T^*M = V \oplus 1$ (with respect to some Riemannian metric). This implies for the cosphere bundle

$$S^*M = S(V \oplus 1).$$

The projectivization P^*M is diffeomorphic to the ball bundle

$$BV \subset S(V \oplus 1)$$

of V with the identification of antipodal points on the boundary:

$$P^*M = BV / \{v \sim -v \mid |v| = 1\}$$

(see Fig. 1). Then the bundle L on the projectivization P^*M is constructed similarly from the bundle $L_+(\sigma(D))$ on the ball bundle BV by means of the identification of fibers over antipodes $\pm v$ with respect to the involution $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The isomorphism of the pull-back of L to the spheres S^*M and the original bundle $L_+(\sigma(D))$ is checked straightforwardly. This proves the proposition.

From the preceding two propositions we immediately obtain a corollary.

Figure 1: Projective space bundle P^*M .

Corollary 5 *The subspace \widehat{L} corresponding to the constructed symbol L satisfies the equality*

$$\{2d(\widehat{L})\} = \{2d(\widehat{L}_+(D_1))\} \text{ind}D_2. \quad (36)$$

3. Let us apply the obtained formula in the following situation. Let $M_1 = RP^{2k}$ and D_1 be the pin^c Dirac operator from the first example. As a second factor take the circle $M_2 = S^1$ with a pseudodifferential operator D_2 on it with the following principal symbol

$$\sigma(D_2)(\varphi, \xi) = \begin{cases} e^{i\varphi}, & \xi > 0, \\ e^{-i\varphi}, & \xi < 0. \end{cases}$$

We obtain: $\text{ind}D_2 = 2$. The dimension functional on the subspace \widehat{L} of Proposition 6 has a nontrivial fractional part, more precisely

$$\{2^{k-1}d(\widehat{L})\} = \frac{1}{2}.$$

Let us improve this result replacing D_2 by an operator on the circle with the index equal to 1 (see [18]). Consider the symbol

$$\sigma(D_2)(\varphi, \xi) = \begin{cases} e^{i\varphi}, & \xi > 0, \\ 1, & \xi < 0. \end{cases}$$

The corresponding pseudodifferential operator

$$D_2 : C^\infty(S^1) \longrightarrow C^\infty(S^1),$$

acts on functions. The symbol under consideration satisfies the equality

$$\alpha^* \sigma(D_2) = e^{2i\varphi} \sigma(D_2)^*,$$

which is more general than (32). Nonetheless, Propositions 6, 5 can be generalized to cover this case. In the corresponding proofs, we only have to replace the involution $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ by $\begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}$, since the symbol $\sigma(D)$ of the cross product of D_1 and D_2 satisfies (cf. (35))

$$\alpha^* \sigma(D) = \begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix} \sigma(D) \begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}.$$

When we apply (36) to operator D , we obtain an even subspace \widehat{L} , whose dimension is a fractional number

$$\{2^k d(\widehat{L})\} = \frac{1}{2}.$$

Let us also note one corollary. It gives an answer to the question posed in [1].

Corollary 6 *The estimate (31) of the denominator of the fractional part of the η -invariant and of the dimension functional on odd-dimensional manifolds is precise. In particular, there exist even-order differential operators on odd-dimensional manifolds with a nontrivial fractional part of the η -invariant.*

4. It might seem exotic to consider differential operators of orders higher than one in index theory. However, there are geometric second-order operators with interesting spectral properties (see [1]). Let us also mention that the Hirzebruch operator on an oriented manifold is equivalent to a second order operator. In particular, the signature of an orientable manifold is equal to the index of a second order operator. This observation was used by A. Connes, D. Sullivan and N. Teleman [24] to express the signature of Lipschitz and quasiconformal manifolds as an index of a bounded Fredholm operator.

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