

COORDINATE INVARIANCE OF THE CONE ALGEBRA WITH ASYMPTOTICS

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ABSTRACT. The cone algebra with discrete asymptotics on a manifold with conical singularities is shown to be invariant under natural coordinate changes, where the symbol structure (i.e., the Fuchsian interior symbol, conormal symbols of all orders) follows a corresponding transformation rule.

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INTRODUCTION

Pseudo-differential operators on a manifold B with set of conical singularities S form an algebra of operators acting in weighted Sobolev spaces on $B \setminus S$. A conical singularity is modelled on a cone $X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ with base X (here, X is a closed C^∞ -manifold) and the cone axis $\overline{\mathbb{R}}_+$. Operators and spaces are described in terms of a chosen splitting of coordinates $(t, x) \in \mathbb{R}_+ \times X$ with a controlled behaviour as $t \rightarrow 0$. Any other admissible choice $(r, y) \in \mathbb{R}_+ \times X$ is connected with (t, x) by a diffeomorphism $\chi : \overline{\mathbb{R}}_+ \times X \rightarrow \overline{\mathbb{R}}_+ \times X$, $(r, y) = \chi(t, x)$. With B we then associate a stretched manifold \mathbb{B} which is a C^∞ -manifold with C^∞ -boundary $\partial\mathbb{B} \cong \{\text{union of all base manifolds } X \text{ to points in } S\}$ and $B \setminus S \cong \mathbb{B} \setminus \partial\mathbb{B}$. In particular, for $B = X^\Delta$ with S consisting of the tip of the cone, we have $\mathbb{B} = \overline{\mathbb{R}}_+ \times X$. Thus coordinate changes for \mathbb{B} are in general transition diffeomorphisms that are smooth up to the boundary. A basic property of the operators in the algebra should be their invariance under such diffeomorphisms. The present paper settles this problem for the cone algebra of Schulze [17] with asymptotics that consists of Mellin pseudo-differential operators with

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meromorphic Mellin symbols plus so-called Green operators which map into subspaces with asymptotics as $t \rightarrow 0$. The meromorphic structure of the Mellin symbols reflects asymptotic information in the algebra. Thereby, asymptotics refer to the particular choice of coordinates. It is by no means evident that the rather fragile asymptotic data are respected under χ . The coarser case of the cone algebra without asymptotics has been treated by Schrohe in [14].

An algebra of pseudo-differential operators on a manifold with conical singularities which is naturally coordinate-invariant can be generated by the Lie algebra of totally-characteristic vector fields, cf. Melrose and Mendoza [12]. Relations between the different choices of “cone algebras” in the literature are expounded by Lauter and Seiler [9]. Let us also mention that the Mellin pseudo-differential algebra on the half-axis in Eskin’s book [5] (with Hilbert-Schmidt operators as residual elements) can be subsumed under the concept of cone algebras on the half-axis.

The smaller the algebra is, for instance, concerning the ideal of residual elements, the more involved is the question of invariance. Furthermore, it is interesting for various applications to single out sub-algebras characterised by specific asymptotic information, cf. the discussion in [17] concerning the transmission property in boundary value problems as an aspect of cone asymptotics and the investigations of Witt [24]. The boundary symbol calculus for pseudo-differential operators with the transmission property is a particularly simple example of a cone algebra on the half-axis. It is known to be invariant under diffeomorphisms of $\overline{\mathbb{R}}_+$. Another important class of invariant operators on manifolds with conical singularities is formed by the differential operators of Fuchs type. Parametrices to elliptic differential operators of Fuchs type, expressed with respect to different coordinate systems (t, x) and (r, y) , respectively, belong to the cone algebra with asymptotics. Hence also these special elements in the cone algebra are assumed to be coordinate-invariant. This suggests that the answer to the invariance question should be positive; finding a precise proof for this, however, turned out to be much deeper than originally expected.

Operators on manifolds with conical singularities are a basic ingredient for calculi on manifolds with higher (polyhedral) singularities, e.g., on manifolds with edges (especially, boundaries) and corners. The approach in the papers of Schulze [16], [19] and in the book of Egorov and Schulze [4] amounts to pseudo-differential machineries with cone operator-valued symbols. In this context, any change of the ideal of residual elements in the original cone algebra enormously changes the resulting algebra of operators of the higher singularity order. Such ideals may be indeed very rich. They can lead to full operator algebras on lower-dimensional skeleta of polyhedra which are generated by operator-valued symbols taking values in the residual elements of an algebra of smaller singularity order. For such symbols and associated operators, there is also the problem of invariance under adequate transition diffeomorphisms. The coordinate invariance of the cone algebra is then a necessary information.

The plan of the paper is as follows: In Chapter 1 we prove the invariance of the cone algebra with asymptotics on the half-axis, regarded as a space with conical singularity at the origin. The typical ideas already arise in this case that is also of independent interest. The cone algebra on the half-axis contains many interesting invariant sub-algebras, especially algebras with pseudo-differential symbols that are non-degenerate at the origin, as they occur in mixed elliptic problems in the calculus transversal to interfaces of the mixed conditions, cf. Schulze and Seiler [20]. We derive explicit transformation rules for the lower-order conormal symbols that apply to non-linear differential equations, cf. Liu and Witt [10], [11]. Chapter 2 solves the invariance problem for the general cone algebra with discrete asymptotics on a manifold with conical singularities, i.e., for the case of arbitrary base manifolds (of the local cones). Here, we systematically employ the kernel cut-off method in the higher-dimensional situation, combined with specific functional-analytic constructions on Fréchet spaces of meromorphic pseudo-differential operator-valued functions in the complex plane of the Mellin covariable.

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1. CONE OPERATORS ON THE HALF-AXIS

1.1. The cone algebra. Let $(Mu)(z) = \int_0^\infty t^{z-1}u(t)dt$ be the Mellin transform on the half-axis $\mathbb{R}_+ \ni t$, first defined for functions $u \in C_0^\infty(\mathbb{R}_+)$. The Mellin covariable $z \in \mathbb{C}$ varies on $\Gamma_\beta = \{z \in \mathbb{C}: \operatorname{Re} z = \beta\}$ for some $\beta \in \mathbb{R}$. For the inverse, we have $(M^{-1}g)(t) = (2\pi i)^{-1} \int_{\Gamma_\beta} t^{-z}g(z)dz$, which is actually independent of β for $u \in C_0^\infty(\mathbb{R}_+)$, where $g(z) = (Mu)(z) \in \mathcal{A}(\mathbb{C})$. Here $\mathcal{A}(U)$ for $U \subseteq \mathbb{C}$ open denotes the space of all holomorphic functions on U endowed with the Fréchet topology of uniform convergence on compact subsets. Let $S^\mu(\mathbb{R})$ for $\mu \in \mathbb{R}$ be the space of symbols of order μ with constants coefficients, i.e., the space of all $f(\tau) \in C^\infty(\mathbb{R})$ such that $|\partial_\tau^k f(\tau)| \leq c\langle \tau \rangle^{\mu-k}$ for all $k \in \mathbb{N}$, $\tau \in \mathbb{R}$, with certain constants $c = c(f, k) > 0$. Denote by $S_{\text{cl}}^\mu(\mathbb{R})$ the subspace of all classical symbols, i.e., the space of all symbols which have an asymptotic expansion $f(\tau) \sim \sum_{j=0}^\infty \pi(\tau)f_{(\mu-j)}(\tau)$ with certain $f_{(\mu-j)}(\tau) \in C^\infty(\mathbb{R} \setminus \{0\})$, $f_{(\mu-j)}(\lambda\tau) = \lambda^{\mu-j}f_{(\mu-j)}(\tau)$ for $\lambda \in \mathbb{R}_+$, $\tau \neq 0$, and any 0-excision function $\pi(\tau)$. Both $S^\mu(\mathbb{R})$ and $S_{\text{cl}}^\mu(\mathbb{R})$ are Fréchet spaces in a canonical way. Hence we also have the spaces $S_{\text{cl}}^\mu(\mathbb{R}_+ \times \mathbb{R}) = C^\infty(\mathbb{R}_+, S_{\text{cl}}^\mu(\mathbb{R}))$, $S_{\text{cl}}^\mu(\overline{\mathbb{R}_+} \times \mathbb{R}) = C^\infty(\overline{\mathbb{R}_+}, S_{\text{cl}}^\mu(\mathbb{R}))$ etc., and the same without subscript “cl”.

We also use notation like $S_{\text{cl}}^\mu(\Gamma_\beta)$ or $S_{\text{cl}}^\mu(\overline{\mathbb{R}_+} \times \Gamma_\beta)$ when the covariable τ plays the role of $\operatorname{Im} z$ on the line Γ_β . More generally, if $B(\mathbb{R})$ is a space of distributions on \mathbb{R} , then we have the corresponding space $B(\Gamma_\beta)$ for $\beta \in \mathbb{R}$. Further examples are: $L^2(\Gamma_\beta)$ – the space of square integrable functions with respect to $d\tau$, $\tau = \operatorname{Im} z$, $\mathcal{S}(\Gamma_\beta)$ – the Schwartz space.

Let $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ for $s, \gamma \in \mathbb{R}$ be the completion of $C_0^\infty(\mathbb{R}_+)$ with respect to the norm $\|\langle z \rangle^s (Mu)(z)|_{\Gamma_{\frac{1}{2}-\gamma}}\|_{L^2(\Gamma_{\frac{1}{2}-\gamma})}$, where $\langle z \rangle = (1 + |z|^2)^{1/2}$. Especially, $\mathcal{H}^{0,\gamma}(\mathbb{R}_+) = t^\gamma L^2(\mathbb{R}_+)$, with $L^2(\mathbb{R}_+)$ being taken with respect to dt . This definition shows that the weighted Mellin transform $M_\gamma: u \mapsto (Mu)|_{\Gamma_{\frac{1}{2}-\gamma}}$ extends from $C_0^\infty(\mathbb{R}_+)$ to an isomorphism $M_\gamma: \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \rightarrow \langle z \rangle^{-s} L^2(\Gamma_{\frac{1}{2}-\gamma})$.

With each $f(z) \in S^\mu(\Gamma_{\frac{1}{2}-\gamma})$, we associate the weighted Mellin pseudo-differential operator

$$\operatorname{op}_M^\gamma(f) = M_\gamma^{-1} f(z) M_\gamma: \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \rightarrow \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+)$$

that is continuous for all $s \in \mathbb{R}$. For $\gamma = 0$ we also write $\operatorname{op}_M(f)$. Similarly, we can form $\operatorname{op}_M^\gamma(f)$ for Mellin symbols $f(t, t', z) \in S^\mu(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma})$. For $f(t, t', z) \in S^\mu(\overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+} \times \Gamma_{\frac{1}{2}-\gamma})$, the operator

$$\omega \operatorname{op}_M^\gamma(f) \tilde{\omega}: \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \rightarrow \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+)$$

is continuous for all $s \in \mathbb{R}$, where $\omega(t)$, $\tilde{\omega}(t)$ are arbitrary cut-off functions, i.e., $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}_+})$ with $\omega(t) = \tilde{\omega}(t) = 1$ near $t = 0$.

We denote by $L_{\text{cl}}^\mu(\mathbb{R}_+)$ the space of all classical pseudo-differential operators on \mathbb{R}_+ , i.e., the space of all operators $F^{-1}p(t, \tau)F + C$, where $p(t, \tau) \in S_{\text{cl}}^\mu(\mathbb{R}_+ \times \mathbb{R})$ and $C \in L^{-\infty}(\mathbb{R}_+)$ is an operator with kernel in $C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$, with $F = F_{t \rightarrow \tau}$ being the Fourier transform on \mathbb{R} . Then $\operatorname{op}_M^\gamma(f) \in L_{\text{cl}}^\mu(\mathbb{R}_+)$ for $f(t, t', z) \in S_{\text{cl}}^\mu(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma})$.

Further we denote by $S^{\mu;\eta}(\mathbb{R}_+ \times \mathbb{R})$ the space of all symbols $p(t, \tau) \in S^\mu(\mathbb{R}_+ \times \mathbb{R})$ which fulfil the symbol estimates

$$|\partial_t^m \partial_\tau^k p(t, \tau)| \leq c \langle t \rangle^{\eta-m} \langle \tau \rangle^{\mu-k}$$

for all $m, k \in \mathbb{N}$ and $t \geq 1$, $\tau \in \mathbb{R}$, with constants $c = c(m, k) > 0$, cf. [13]. Then $L^{\mu;\eta}(\mathbb{R}_+)$ denotes the space of all operators of the form $F^{-1}p(t, \tau)F + C$, where $p(t, \tau) \in S^{\mu;\eta}(\mathbb{R}_+ \times \mathbb{R})$

and $C \in L^{-\infty;-\infty}(\mathbb{R}_+)$, i.e., $C \in L^{-\infty;-\infty}(\mathbb{R}_+)$ is an operator with kernel $C(t, t')$ such that $\pi(t)\pi(t')C \in \mathcal{S}(\mathbb{R} \times \mathbb{R})|_{\mathbb{R}_+ \times \mathbb{R}_+}$, for any 0-excision function π .

We now briefly introduce the cone algebra on \mathbb{R}_+ with discrete asymptotics. Roughly speaking, it is a certain subspace of $\{t^\delta \text{op}_M(f)t^{-\gamma} : f(t, t', z) \in S_{\text{cl}}^\mu(\overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+} \times \Gamma_{\frac{1}{2}})\}$, $\mu \in \mathbb{R}$, for some choice of weights $\gamma, \delta \in \mathbb{R}$. The weight factors $t^\delta, t^{-\gamma}$ are not always essential, so we will ignore them for a while. Formulations with the Mellin transform are of interest only near $t = 0$. Away from $t = 0$, we refer to standard pseudo-differential operators with the Fourier transform and additionally impose an exit behaviour as $t \rightarrow \infty$ in the sense of $L^{\mu;0}(\mathbb{R}_+)$ that guarantees continuity in Sobolev spaces globally up to infinity. For this reason we employ a mixture between the spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ and $H^s(\mathbb{R}_+)$, namely we set

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = \{\omega u + (1 - \omega)v : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+), v \in H^s(\mathbb{R}_+)\}$$

for a fixed cut-off function $\omega(t)$. (This definition is, of course, independent of ω .) Concerning a choice of the Hilbert space structure in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ and other simple properties of these spaces, cf., e.g., [17]. The scalar product in $L^2(\mathbb{R}_+) = \mathcal{K}^{0,0}(\mathbb{R}_+)$ gives rise to a non-degenerate sesquilinear pairing between $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ and $\mathcal{K}^{-s,-\gamma}(\mathbb{R}_+)$ for all $s, \gamma \in \mathbb{R}$; so we can also talk about formally adjoint operators.

Subspaces with discrete asymptotics of type P for a finite sequence $P = \{(p_j, m_j)\}_{j=1}^N \subset \mathbb{C} \times \mathbb{N}$ with $N = N(P)$ and $\pi_{\mathbb{C}}P = \bigcup_{j=1}^N \{p_j\} \subset \{z \in \mathbb{C} : \frac{1}{2} - \gamma + \vartheta < \text{Re } z < \frac{1}{2} - \gamma\}$ for a given $-\infty < \vartheta < 0$ are defined as follows: First we form the linear span $\mathcal{E}_P(\mathbb{R}_+)$ of all functions $\omega(t)t^{-p} \log^k t$ for arbitrary $(p, k) \in \mathbb{C} \times \mathbb{N}$, where $(p, m) \in P$, $0 \leq k \leq m$, with a fixed cut-off function $\omega(t)$. We then have $\mathcal{E}_P(\mathbb{R}_+) \subset \mathcal{K}^{\infty,\gamma}(\mathbb{R}_+)$. Moreover, we set $\Theta = (\vartheta, 0]$ (which is regarded as a weight interval) and $\mathcal{K}_{\Theta}^{s,\gamma}(\mathbb{R}_+) = \bigcap_{\varepsilon > 0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(\mathbb{R}_+)$ in its projective limit topology and define

$$(1.1.1) \quad \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) = \mathcal{K}_{\Theta}^{s,\gamma}(\mathbb{R}_+) + \mathcal{E}_P(\mathbb{R}_+)$$

to be the space of all $u = u_{\text{flat}} + u_{\text{sing}}$, where $u_{\text{flat}} \in \mathcal{K}_{\Theta}^{s,\gamma}(\mathbb{R}_+)$, $u_{\text{sing}} \in \mathcal{E}_P(\mathbb{R}_+)$. The space (1.1.1) is independent of ω , and it is a Fréchet space. The definition easily extends to the case of the infinite weight interval $\Theta = (-\infty, 0]$ and infinite sequences $P = \{(p_j, m_j)\}_{j=1}^{\infty}$ with $\pi_{\mathbb{C}}P \subset \{z \in \mathbb{C} : \text{Re } z < \frac{1}{2} - \gamma\}$, $\text{Re } p_j \rightarrow -\infty$ as $j \rightarrow \infty$, by taking a projective limit $\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) = \varprojlim_{k \in \mathbb{N}} \mathcal{K}_{P_k}^{s,\gamma}(\mathbb{R}_+)$, where P_k for $k \in \mathbb{N}$ is a sequence of finite asymptotic types with $\pi_{\mathbb{C}}P_k \subset \{z \in \mathbb{C} : \frac{1}{2} - \gamma - (k+1) < \text{Re } z < \frac{1}{2} - \gamma\}$ and $P = \bigcup_k P_k$. We call such sequences P discrete asymptotic types associated with the weight data (γ, Θ) for $\Theta = (\vartheta, 0]$, $-\infty \leq \vartheta < 0$. Define $\text{As}(\gamma, \Theta)$ to be the set of all those P . We will also use the Fréchet spaces

$$\mathcal{S}_P^{\gamma}(\mathbb{R}_+) = \{\omega u + (1 - \omega)v : u \in \mathcal{K}_P^{\infty,\gamma}(\mathbb{R}_+), v \in \mathcal{S}(\overline{\mathbb{R}_+})\},$$

where ω is a cut-off function.

An operator $G \in L^{-\infty}(\mathbb{R}_+)$ which induces continuous maps

$$(1.1.2) \quad G : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \rightarrow \mathcal{S}_P^{\delta}(\mathbb{R}_+), \quad G^* : \mathcal{K}^{s,-\delta}(\mathbb{R}_+) \rightarrow \mathcal{S}_Q^{-\gamma}(\mathbb{R}_+)$$

for all $s \in \mathbb{R}$, with G^* being the formal adjoint to G and discrete asymptotic types P and Q associated with the weight data (δ, Θ) and $(-\gamma, \Theta)$, respectively, is called a Green operator with discrete asymptotics. The space of all these operators is denoted by $C_G(\mathbb{R}_+, (\gamma, \delta, \Theta))$.

Green operators appear as residual elements in the calculus of Mellin pseudo-differential operators with meromorphic symbols. Consider a sequence $R = \{(r_j, n_j)\}_{j \in \mathbb{Z}}$ such that $\pi_{\mathbb{C}}R \cap \{z \in \mathbb{C} : c \leq \text{Re } z \leq c'\}$ is finite for all $c < c'$, where $\pi_{\mathbb{C}}R = \bigcup_{j \in \mathbb{Z}} \{r_j\}$. Such sequences will be called discrete asymptotic types for Mellin symbols. The set of all these sequences R is denoted by \mathbf{As} . Then $\mathcal{M}_R^{-\infty}$ denotes the space of all functions $f \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}}R)$ which are meromorphic with poles at r_j of multiplicities $n_j + 1$, $j \in \mathbb{Z}$, and which are such that

$\pi(z)f(z)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ for each $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$ for arbitrary $c < c'$. Here, π is a $\pi_{\mathbb{C}R}$ -excision function, i.e., $\pi(z) \in C^\infty(\mathbb{C})$, $\pi(z) = 0$ for $\text{dist}(z, \pi_{\mathbb{C}R}) < c_0$ and $\pi(z) = 1$ for $\text{dist}(z, \pi_{\mathbb{C}R}) > c$, for certain $0 < c_0 < c_1$.

Let $\mathcal{M}_{\mathcal{O}}^\mu$ for $\mu \in \mathbb{R}$ be the space of all $h \in \mathcal{A}(\mathbb{C})$ such that $h|_{\Gamma_\beta} \in S_{\text{cl}}^\mu(\Gamma_\beta)$ for each $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$ for arbitrary $c < c'$. The spaces $\mathcal{M}_R^{-\infty}$, $\mathcal{M}_{\mathcal{O}}^\mu$ and $\mathcal{M}_R^\mu = \mathcal{M}_{\mathcal{O}}^\mu + \mathcal{M}_R^{-\infty}$ are Fréchet spaces; natural semi-norm systems follow immediately from the definition. The union of the spaces \mathcal{M}_R^μ over $R \in \mathbf{As}$ will be denoted by $\mathcal{M}_{\text{as}}^\mu$.

Given $\gamma, \delta \in \mathbb{R}$ and a weight interval $\Theta = (-(k+1), 0]$ for $k \in \mathbb{N}$, we denote by $C_{M+G}(\mathbb{R}_+, (\gamma, \delta, \Theta))$ the space of all operators $M + G$, where $G \in C_G(\mathbb{R}_+, (\gamma, \delta, \Theta))$ and

$$(1.1.3) \quad M = \tilde{\omega}(t)t^{\delta-\gamma} \sum_{j=0}^k t^j \text{op}_M^{\varrho_j}(f_j)\tilde{\omega}_0(t)$$

for arbitrary (so-called conormal symbols) $f_j(z) \in \mathcal{M}_R^{-\infty}$ and $\varrho_j \in \mathbb{R}$, $\gamma - j \leq \varrho_j \leq \gamma$, where $\pi_{\mathbb{C}R_j} \cap \Gamma_{\frac{1}{2}-\varrho_j} = \emptyset$ for $j = 0, \dots, k$. The cut-off functions $\tilde{\omega}(t)$, $\tilde{\omega}_0(t)$ are also arbitrary. Given two operators of the form (1.1.3) with the same f_0, \dots, f_k , but different cut-off functions or shifts, their difference belongs to $C_G(\mathbb{R}_+, (\gamma, \delta, \Theta))$, cf. [17]. The cone algebra with discrete asymptotics is the union of all spaces in the following definition.

Definition 1.1.1 *The space $C^\mu(\mathbb{R}_+, \mathbf{g})$ of all cone operators of order $\mu \in \mathbb{R}$ on \mathbb{R}_+ with discrete asymptotics and weight data $\mathbf{g} = (\gamma, \delta, \Theta)$, $\gamma, \delta \in \mathbb{R}$, $\Theta = (-(k+1), 0]$, $k \in \mathbb{N}$, is defined as the set of all operators of the form*

$$(1.1.4) \quad A = \omega t^{\delta-\gamma} \text{op}_M^\gamma(h)\omega_0 + (1-\omega)A_\infty(1-\omega_1) + M + G$$

with cut-off functions $\omega, \omega_0, \omega_1$ satisfying $\omega\omega_0 = \omega$, $\omega\omega_1 = \omega_1$, and

- (i) $h(t, z) \in C^\infty(\overline{\mathbb{R}_+}, \mathcal{M}_{\mathcal{O}}^\mu)$,
- (ii) $M + G \in C_{M+G}(\mathbb{R}_+, \mathbf{g})$,
- (iii) $A_\infty \in L_{\text{cl}}^\mu(\mathbb{R}_+) \cap L^{\mu;0}(\mathbb{R}_+)$.

We have $C^\mu(\mathbb{R}_+, \mathbf{g}) \subset L_{\text{cl}}^{\mu;0}(\mathbb{R}_+)$. Setting $\mathbf{g}_k = (\gamma, \delta, (-(k+1), 0])$ it is also clear that $C^\mu(\mathbb{R}_+, \mathbf{g}_{k+1}) \subset C^\mu(\mathbb{R}_+, \mathbf{g}_k)$ for all $k \in \mathbb{N}$. For $\mathbf{g} = (\gamma, \delta, \Theta)$, $\Theta = (-\infty, 0]$, we then define $C^\mu(\mathbb{R}_+, \mathbf{g}) = \bigcap_k C^\mu(\mathbb{R}_+, \mathbf{g}_k)$.

An operator $A \in C^\mu(\mathbb{R}_+, \mathbf{g})$ for $\mathbf{g} = (\gamma, \delta, \Theta)$ induces continuous maps

$$(1.1.5) \quad A: \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \rightarrow \mathcal{K}^{s-\mu,\delta}(\mathbb{R}_+)$$

and

$$A: \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) \rightarrow \mathcal{K}_Q^{s-\mu,\delta}(\mathbb{R}_+)$$

for all $s \in \mathbb{R}$ and any asymptotic type $P \in \text{As}(\gamma, \Theta)$, with some resulting asymptotic type $Q = Q(P, A) \in \text{As}(\delta, \Theta)$.

The principal symbol of an operator $A \in C^\mu(\mathbb{R}_+, \mathbf{g})$ that is responsible for the ellipticity of A is a triple

$$(1.1.6) \quad \sigma(A) = (\tilde{\sigma}_\psi^\mu(A), \sigma_M^{\gamma-\delta}(A), \sigma_e^0(A)).$$

Here

$$(1.1.7) \quad \tilde{\sigma}_\psi^\mu(A)(t, \tilde{\tau}) = t^{\delta-\gamma} \sigma_\psi^\mu(A)(t, t^{-1}\tilde{\tau}),$$

with $\sigma_\psi^\mu(A)$ being the homogeneous principal symbol of $A \in L_{\text{cl}}^\mu(\mathbb{R}_+)$. The principal conormal symbol $\sigma_M^{\gamma-\delta}(A)$ of conormal order $\gamma - \delta$,

$$(1.1.8) \quad \sigma_M^{\gamma-\delta}(A)(z) = h(0, z) + f_0(z),$$

is defined in terms of the Mellin symbols $h(t, z)$ in (1.1.4) and $f_0(z)$ in (1.1.3). It satisfies

$$(1.1.9) \quad \sigma_\psi^\mu \sigma_M^\mu(A)(z) \Big|_{z=\frac{1}{2}-\gamma-i\tilde{\tau}} = \tilde{\sigma}_\psi^\mu(A)(0, \tilde{\tau}).$$

Finally, the exit symbol $\sigma_e^0(A)$ is the class of $p(t, \tau) \in S^{\mu;0}(\mathbb{R}_+ \times \mathbb{R})$ modulo $S^{\mu-1;-1}(\mathbb{R}_+ \times \mathbb{R})$. If $\sigma(A)$ vanishes (i.e., all its components are zero), then (1.1.4) is compact for all $s \in \mathbb{R}$. In addition, we have the conormal symbols $\sigma_M^{\gamma-\delta-j}(A)$ of lower order,

$$(1.1.10) \quad \sigma_M^{\gamma-\delta-j}(A)(z) = \frac{1}{j!} \frac{\partial^j}{\partial t^j} h(t, z) \Big|_{t=0} + f_j(z),$$

for $j = 1, \dots, k$. They belong to $\mathcal{M}_{R_j}^\mu$ for certain discrete asymptotic types R_j for Mellin symbols and are crucial in determining the asymptotics of solutions to elliptic equations. Let us finally recall that the principal symbol behaves multiplicatively under composition, i.e., $\sigma(AB) = \sigma(A)\sigma(B)$ for $A \in C^\mu(\mathbb{R}_+, (\gamma, \delta, \Theta))$, $B \in C^\nu(\mathbb{R}_+, (\beta, \gamma, \Theta))$ with the component-wise composition, where the principal conormal symbol is multiplied together with a shift in the first argument. More generally, the conormal symbols can be computed by the Mellin translation product

$$(1.1.11) \quad \sigma_M^{\beta-\delta-l}(AB) = \sum_{p+q=l} (T^{\beta-\gamma-q} \sigma_M^{\gamma-\delta-p}(A)) \sigma_M^{\beta-\gamma-q}(B)$$

for $l = 0, \dots, k$, where $(T^\varrho f)(z) = f(z + \varrho)$.

The main result of Chapter 1 is the invariance of the cone algebra $C^\mu(\mathbb{R}_+, \mathbf{g})$ under coordinate diffeomorphisms. Let $\chi: \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ be a diffeomorphism. Further we assume that

$$(1.1.12) \quad \left| \frac{d^j \chi}{dt^j}(t) \right| \leq C_j (1+t)^{1-j}$$

for all $j \in \mathbb{N}$ with certain constants $C_j > 0$ and

$$(1.1.13) \quad \chi(t) \geq c_0 t$$

for another constant $c_0 > 0$. Then we have the well-known push-forward of pseudo-differential operators $\chi_*: L^{\mu;0}(\mathbb{R}_+) \rightarrow L^{\mu;0}(\mathbb{R}_+)$ for each $\mu \in \mathbb{R}$.

Theorem 1.1.2 *The operator push-forward under χ induces an isomorphism*

$$\chi_*: C^\mu(\mathbb{R}_+, \mathbf{g}) \rightarrow C^\mu(\mathbb{R}_+, \mathbf{g})$$

for all $\mu \in \mathbb{R}$ and $\mathbf{g} = (\gamma, \delta, \Theta)$. Moreover, $\chi_* \sigma(A) = \sigma(\chi_* A)$ under a canonical push-forward χ_* on the symbol algebra.

The precise form of the symbol push-forward will be given below. Furthermore, we express the transformation rules for the lower-order conormal symbols.

1.2. Spaces with asymptotics and Green operators.

Proposition 1.2.1 *Let $\chi: \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ be a diffeomorphism satisfying (1.1.12), (1.1.13). Then the pull-back under χ induces an isomorphism $\chi^*: \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \rightarrow \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ for all $s, \gamma \in \mathbb{R}$.*

Proof. The invariance of the usual Sobolev spaces is well-known. The same is true for the spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$. \square

An asymptotic type $P \in \text{As}(\mathbf{g})$ for $\mathbf{g} = (\gamma, \Theta)$, $\gamma \in \mathbb{R}$, $\Theta = (-\infty, 0]$ is said to satisfy the shadow condition if

$$(1.2.1) \quad (p, m) \in P \implies (p-j, m(j)) \in P$$

for all $j \in \mathbb{N}$ and certain $m(j) \geq m$. For $\Theta = (\vartheta, 0]$ and $\vartheta > -\infty$ we talk about the shadow condition if (1.2.1) holds for all those $j \in \mathbb{N}$ such that $\text{Re } p - j > \frac{1}{2} - \gamma + \vartheta$.

Proposition 1.2.2 *Let $P \in \text{As}(\mathfrak{g})$ for $\mathfrak{g} = (\gamma, \Theta)$, $\gamma \in \mathbb{R}$, $\Theta = (\vartheta, 0]$ satisfy the shadow condition. Then the pull-back under χ induces an isomorphism*

$$(1.2.2) \quad \chi^*: \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) \rightarrow \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)$$

for all $s \in \mathbb{R}$.

Proof. It suffices to show the assertion for $\vartheta > -\infty$, since then it also follows for $\vartheta = -\infty$ in view of the projective limit involved in the definition. Moreover, by injectivity and surjectivity of the pull-back in Proposition 1.2.1, we only have to show that the pull-back χ^* in (1.2.2) induces the corresponding continuous operator. By definition we have

$$\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) = \mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+) + \mathcal{E}_P(\mathbb{R}_+).$$

Then it suffices to show that

$$\begin{aligned} \chi^*: \mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+) &\rightarrow \mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+), \\ \chi^*: \mathcal{E}_P(\mathbb{R}_+) &\rightarrow \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+). \end{aligned}$$

The first relation follows from Proposition 1.2.1, for the projective limits involved in the definition of $\mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+)$. The space $\mathcal{E}_P(\mathbb{R}_+)$ is spanned by functions of the form

$$u(r) = \omega(r)r^{-p} \log^k r, \quad p \in \mathbb{C}, \quad k \in \mathbb{N},$$

where $(p, m) \in P$, $0 \leq k \leq m$. First let $k = 0$ and set $r = \chi(t)$. Then $(\chi^*u)(t) = u(\chi(t)) = \omega(\chi(t))(\chi(t))^{-p}$. From the Taylor expansion of $\chi(t)$ at 0,

$$\chi(t) = \sum_{j=0}^N c_j t^j + t^{N+1} \chi_{(N+1)}(t),$$

where $c_0 = 0$ and $\chi_{(N+1)}(t)$ is smooth up to $t = 0$, we get

$$\chi(t)^{-p} = (c_1 t)^{-p} \left(1 + \sum_{j=2}^N \frac{c_j}{c_1} t^{j-1} + \frac{\chi_{(N+1)}(t)}{c_1} t^N \right)^{-p}.$$

To calculate $\left(1 + \sum_{j=2}^N \frac{c_j}{c_1} t^{j-1} + \frac{\chi_{(N+1)}(t)}{c_1} t^N \right)^{-p}$ we employ the well-known formula

$$(1.2.3) \quad (1+x)^z = \sum_{j=0}^{\infty} \binom{z}{j} x^j, \quad \binom{z}{j} = \frac{(-1)^j \Gamma(-z+j)}{j! \Gamma(-z)}, \quad z \in \mathbb{C},$$

where $\Gamma(z)$ is the Γ -function. Then, for every $N \in \mathbb{N}$, we obtain a representation

$$(1.2.4) \quad u(\chi(t)) = \omega(\chi(t)) t^{-p} \left(\sum_{j=0}^N d_j t^j + f_{(N+1)}(t) t^{N+1} \right)$$

with certain constants d_j and a function $f_{(N+1)}(t)$ smooth up to $t = 0$. Since the partial sum on the right of (1.2.4) belongs to $\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)$ and the remainder is in $\mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+)$ when N is large enough, we get $u(\chi(t)) \in \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)$.

The case $k > 0$ follows by differentiating the transformation rule with respect to p . \square

Let us now formulate the coordinate invariance of the Green operators.

Theorem 1.2.3 *The operator push-forward under χ induces an isomorphism*

$$\chi_*: C_G(\mathbb{R}_+, (\gamma, \delta, \Theta)) \rightarrow C_G(\mathbb{R}_+, (\gamma, \delta, \Theta)).$$

Proof. Let G be a Green operator with asymptotic types $P \in \text{As}(\delta, \Theta)$, $Q \in \text{As}(-\gamma, \Theta)$, cf. the definition in Section 1.1. Without loss of generality we may assume that P and Q satisfy the shadow condition. Theorem 1.2.2 implies that the pull-back under χ induces isomorphisms of the spaces involved in (1.1.2). Therefore, χ_*G is again a Green operator with asymptotic types P and Q . \square

1.3. Push-forward of Mellin operators. To prove the invariance of Mellin operators we need the following results.

Lemma 1.3.1 *Let $\chi: \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ be a diffeomorphism. Then the function*

$$a(t, t') = \frac{\log t - \log t'}{\log \chi(t) - \log \chi(t')}$$

is well-defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and $t'^k \partial_{t'}^k a(t, t')|_{t'=t}$ is smooth up to $t = 0$ for all $k \in \mathbb{N}$.

For the next proposition, cf. [18, Proposition 2.3.81].

Proposition 1.3.2 *Let $f(t, t', z) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \mathcal{M}_R^{-\infty})$ for some $R \in \mathbf{As}$ and $\gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}}R \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$. Then we have $\omega \text{op}_M^\gamma(f)\omega_0 \in C_{M+G}(\mathbb{R}_+, \mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma, \Theta)$, $\Theta = (-\infty, 0]$, and arbitrary cut-off functions ω, ω_0 .*

Theorem 1.3.3 *Let $\chi: \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ be a diffeomorphism and ω, ω_0 be cut-off functions. Furthermore, let $R \in \mathbf{As}$ with $\pi_{\mathbb{C}}R \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$ and $f(z) \in \mathcal{M}_R^{-\infty}$. Then the push-forward under χ^{-1} of the operator $\omega \text{op}_M^\gamma(f)\omega_0$ is a smoothing Mellin+Green operator, i.e., $\chi_*^{-1}(\omega \text{op}_M^\gamma(f)\omega_0) \in C_{M+G}(\mathbb{R}_+, (\gamma, \gamma, \Theta))$, $\Theta = (-\infty, 0]$.*

Proof. We may assume that $\gamma = \frac{1}{2}$; the case for arbitrary weights is completely analogous. Setting $r = \chi(t)$ we have

$$\begin{aligned} & \chi_*^{-1}(\omega(r) \text{op}_M^{\frac{1}{2}}(f)\omega_0(r))u(t) \\ &= \frac{\omega(\chi(t))}{2\pi i} \int_{\Gamma_0} \int_0^\infty \left(\frac{\chi(t)}{r'}\right)^{-z} f(z)\omega_0(r')u(\chi^{-1}(r'))\frac{dr'}{r'} dz \\ &= \frac{\omega(\chi(t))}{2\pi i} \int_{\Gamma_0} \int_0^\infty \left(\frac{\chi(t)}{\chi(t')}\right)^{-z} f(z)\omega_0(\chi(t'))u(t')\frac{\chi'(t')t'}{\chi(t')} \frac{dt'}{t'} dz \\ &= \frac{\tilde{\omega}(t)}{2\pi i} \int_{\Gamma_0} \int_0^\infty \left(\frac{t}{t'}\right)^{-z} \left(\frac{t'\chi(t)}{t\chi(t')}\right)^{-z} f(z)\frac{\chi'(t')t'}{\chi(t')}\tilde{\omega}_0(t')u(t')\frac{dt'}{t'} dz, \end{aligned}$$

where $\tilde{\omega}(t) = \omega(\chi(t))$, $\tilde{\omega}_0(t') = \omega_0(\chi(t'))$ are cut-off functions. Using Proposition 1.3.2, to complete the proof we only need to show that

$$(1.3.1) \quad \left(\frac{t'\chi(t)}{t\chi(t')}\right)^{-z} f(z) \frac{\chi'(t')t'}{\chi(t')} \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \mathcal{M}_R^{-\infty})$$

Since $\chi(t)$ can be rewritten as $\chi(t) = t\alpha(t)$ with $\alpha(t) \in C^\infty(\overline{\mathbb{R}}_+)$ and $\alpha(0) = \chi'(0) > 0$, we have

$$(1.3.2) \quad \varphi(t, t') = \frac{t'\chi(t)}{t\chi(t')} = \frac{\alpha(t)}{\alpha(t')} > 0$$

for all $t, t' \in \overline{\mathbb{R}}_+$ and $\varphi(t, t') \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$. Moreover, $e^{-z \log \varphi(t, t')} f(z) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \mathcal{M}_R^{-\infty})$, since $f(z) \in \mathcal{M}_R^{-\infty}$. Because of $\frac{\chi'(t')t'}{\chi(t')} \in C^\infty(\overline{\mathbb{R}}_+)$, we get (1.3.1). \square

Theorem 1.3.4 *Let $\chi: \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ be a diffeomorphism and $h(r, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_0^\mu)$, $\mu \in \mathbb{R}$. Then*

$$\chi_*^{-1}(\omega \text{op}_M^\gamma(h)\omega_0) = \tilde{\omega} \text{op}_M^\gamma(\tilde{h})\tilde{\omega}_0 + G_0$$

where $\tilde{h}(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_\mathcal{O}^\mu)$ and $G_0 \in C_G(\mathbb{R}_+, (\gamma, \gamma, \Theta))$, $\Theta = (-\infty, 0]$. Here $\omega, \omega_0, \tilde{\omega}, \tilde{\omega}_0$ are arbitrary cut-off functions.

Proof. As in the proof of Theorem 1.3.3 we may assume that $\gamma = \frac{1}{2}$. We may further assume that $h(r, z)$ is supported close to $r = 0$.

Let $\psi \in C_0^\infty(\mathbb{R}_+)$ be supported in a small neighbourhood of $\varrho = 1$ with $\psi(\varrho) = 1$ for ϱ close to 1. Then

$$(1.3.3) \quad \text{op}_M^{\frac{1}{2}} \left(\psi \left(\frac{r}{r'} \right) h(r, z) \right) = \text{op}_M^{\frac{1}{2}} ((H(\psi)h)(r, z))$$

and

$$(H(1 - \psi)h)(r, z) = h(r, z) - (H(\psi)h)(r, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_\mathcal{O}^{-\infty}),$$

where $H(\psi)$ is a kernel cut-off operator, cf. Dorschfeldt [2] and Schulze [17]. Analogously to the proof of Theorem 1.3.3 we get

$$(1.3.4) \quad \chi_*^{-1} \left(\omega \text{op}_M^{\frac{1}{2}}(h)\omega_0 - \omega \text{op}_M^{\frac{1}{2}} \left(\psi \left(\frac{r}{r'} \right) h \right) \omega_0 \right) \in C_{M+G} \left(\mathbb{R}_+, \left(\frac{1}{2}, \frac{1}{2}, \Theta \right) \right).$$

The operator in (1.3.4) can be written as $\tilde{\omega} \text{op}_M^{\frac{1}{2}}(h_1)\tilde{\omega}_0 + G_1$ for $h_1 \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_\mathcal{O}^{-\infty})$ and $G_1 \in C_G(\mathbb{R}_+, (\frac{1}{2}, \frac{1}{2}, \Theta))$. In fact, let $f_j(z) \in \sigma_M^{-j}(\chi_*^{-1}(\omega \text{op}_M^{\frac{1}{2}}((H(1 - \psi)h)(r, z))\omega_0))$ for all $j \in \mathbb{N}$. It is easily seen that $f_j(z) \in \mathcal{M}_\mathcal{O}^{-\infty}$ for all j . By Borel's summation method, we find an $h_1(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_\mathcal{O}^{-\infty})$ such that $\frac{1}{j!} \partial_t^j h_1(0, z) = f_j(z)$ for all j . Then $G_1 = \chi_*^{-1}(\omega \text{op}_M^{\frac{1}{2}}((H(1 - \psi)h)(r, z))\omega_0 - \tilde{\omega} \text{op}_M^{\frac{1}{2}}(h_1)\tilde{\omega}_0) \in C_{M+G}(\mathbb{R}_+, (\frac{1}{2}, \frac{1}{2}, \Theta))$ and, furthermore, $\sigma_M^{-j}(G_1) = 0$ for all $j \in \mathbb{N}$. But this implies that $G_1 \in C_G(\mathbb{R}_+, (\frac{1}{2}, \frac{1}{2}, \Theta))$.

Now we consider $\chi_*^{-1}(\omega \text{op}_M^{\frac{1}{2}}(\psi(\frac{r}{r'})h)\omega_0)$. We have

$$\begin{aligned} & \chi_*^{-1} \left(\omega \text{op}_M^{\frac{1}{2}} \left(\psi \left(\frac{r}{r'} \right) h \right) \omega_0 \right) u(t) \\ &= \frac{1}{2\pi i} \int_{\Gamma_0} \int_0^\infty \left(\frac{\chi(t)}{\chi(t')} \right)^{-z} \omega(\chi(t)) \psi \left(\frac{\chi(t)}{\chi(t')} \right) h(\chi(t), z) \frac{\chi'(t')t'}{\chi(t')} \omega_0(\chi(t')) u(t') \frac{dt'}{t'} dz \\ &= \frac{1}{2\pi i} \int_{\Gamma_0} \int_0^\infty \left(\frac{t}{t'} \right)^{-\frac{\log \chi(t) - \log \chi(t')}{\log t - \log t'} z} \omega(\chi(t)) \psi \left(\frac{\chi(t)}{\chi(t')} \right) h(\chi(t), z) \frac{\chi'(t')t'}{\chi(t')} \omega_0(\chi(t')) u(t') \frac{dt'}{t'} dz \\ &= \frac{\omega(\chi(t))}{2\pi i} \int_{\Gamma_0} \int_0^\infty \left(\frac{t}{t'} \right)^{-z} \psi \left(\frac{\chi(t)}{\chi(t')} \right) h(\chi(t), a(t, t')z) a(t, t') \frac{\chi'(t')t'}{\chi(t')} \omega_0(\chi(t')) u(t') \frac{dt'}{t'} dz \\ &= \tilde{\omega} \text{op}_M^{\frac{1}{2}}(g)\tilde{\omega}_0 u(t), \end{aligned}$$

where

$$g(t, t', z) = \psi \left(\frac{\chi(t)}{\chi(t')} \right) h(\chi(t), a(t, t')z) a(t, t') v(t'), \quad v(t') = \frac{\chi'(t')t'}{\chi(t')} \in C^\infty(\overline{\mathbb{R}}_+),$$

and $\tilde{\omega}(t) = \omega(\chi(t))$, $\tilde{\omega}_0(t') = \omega_0(\chi(t'))$ are cut-off functions.

Since $\frac{\chi(t)}{\chi(t')} = \frac{t}{t'} \varphi(t, t')$ and $a(t, t') = \frac{\log \frac{t}{t'}}{\log(\frac{t}{t'} \varphi(t, t'))}$, cf. (1.3.2), we get

$$\begin{aligned} k(g)(t, t', \varrho) &= (M_{\frac{1}{2}, z \rightarrow \varrho}^{-1} g)(t, t', \varrho) \\ &= \int_{-\infty}^\infty \varrho^{-i\tau} \psi \left(\frac{t}{t'} \varphi(t, t') \right) h \left(\chi(t), \frac{\log \frac{t}{t'}}{\log(\frac{t}{t'} \varphi(t, t'))} i\tau \right) v(t') \frac{\log \frac{t}{t'}}{\log(\frac{t}{t'} \varphi(t, t'))} d\tau, \end{aligned}$$

where $k(g)$ is interpreted in the distributional sense.

Now consider

(1.3.5)

$$\tilde{k}(g)(t, t', \varrho) = \int_{-\infty}^{\infty} \varrho^{-i\tau} \psi(\varrho\varphi(t, t/\varrho)) h\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i\tau\right) v(t') \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} d\tau.$$

Note that the expression $\varrho\varphi(t, t/\varrho)$ is strictly increasing in $\varrho \in \mathbb{R}_+$, since $\frac{\partial}{\partial \varrho}(\varrho\varphi(t, t/\varrho)) = \varphi(t, t/\varrho)v(t/\varrho) > 0$. Moreover, $\lim_{\varrho \rightarrow +0} \varrho\varphi(t, t/\varrho) = 0$ and $\lim_{\varrho \rightarrow \infty} \varrho\varphi(t, t/\varrho) = \infty$, since $d_1 \leq \varphi(t, t') \leq d_2$ for some $0 < d_1 < d_2 < \infty$. Thus, $\varrho = 1$ is the unique solution to the equation $\varrho\varphi(t, t/\varrho) = 1$ for each $t \in \overline{\mathbb{R}}_+$; in particular, $\frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))}$ is a smooth function of ϱ , and $\psi(\varrho\varphi(t, t/\varrho)) = 0$ except for ϱ in a small neighbourhood of 1 when t is close to 0.

Since $k(g)(t, t', \frac{t}{t'}) = \tilde{k}(g)(t, t', \frac{t}{t'})$, we get

$$\text{op}_{\frac{1}{2}}^M(g)u(t) = \int_0^{\infty} k(g)(t, t', \frac{t}{t'})u(t') \frac{dt'}{t'} = \int_0^{\infty} \tilde{k}(g)(t, t', \frac{t}{t'})u(t') \frac{dt'}{t'} = \text{op}_{\frac{1}{2}}^M(\tilde{g})u(t),$$

where $\tilde{g}(t, t', z) = (M_{\frac{1}{2}, \varrho \rightarrow z} \tilde{k}(g))(t, t', z)$. Thus $\tilde{g}(t, t', z) = g_0(t, z)v(t')$, where

$$(1.3.6) \quad g_0(t, z) = \int_0^{\infty} \int_{-\infty}^{\infty} \varrho^{z-i\tau} \psi(\varrho\varphi(t, t/\varrho)) h\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i\tau\right) \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} d\tau \frac{d\varrho}{\varrho}$$

and $g_0(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{A}(\mathbb{C}))$. We want to show that $g_0(t, \beta + i\sigma) \in C^\infty(\overline{\mathbb{R}}_+, S_{\text{cl}}^\mu(\mathbb{R}_\sigma))$, uniformly in $c_1 \leq \beta \leq c_2$ for arbitrary $c_1 < c_2$. We have

(1.3.7) $g_0(t, \beta + i\sigma)$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \int_0^{\infty} \varrho^{-1+i(\sigma-\tau)} \varrho^{\beta+1} \psi(\varrho\varphi(t, t/\varrho)) h\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i\tau\right) \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} \frac{d\varrho}{\varrho} d\tau \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \varrho^{-1-i\tau} \varrho^{\beta+1} \psi(\varrho\varphi(t, t/\varrho)) h\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i(\sigma + \tau)\right) \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} \frac{d\varrho}{\varrho} d\tau \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} (-1 - i\tau)^{-N} \varrho^{-1-i\tau} (-\varrho\partial_\varrho)^N \left\{ \varrho^{\beta+1} \psi(\varrho\varphi(t, t/\varrho)) \right. \\ &\quad \left. \times h\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i(\sigma + \tau)\right) \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} \right\} \frac{d\varrho}{\varrho} d\tau. \end{aligned}$$

The latter integral converges for $N \in \mathbb{N}$ large enough. Note that

$$\begin{aligned} &(-\varrho\partial_\varrho)^N \left\{ \varrho^{\beta+1} \psi(\varrho\varphi(t, t/\varrho)) h\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i(\sigma + \tau)\right) \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} \right\} \\ &= \sum_{k=0}^N c_k(t, \varrho) (\partial_\tau^k h)\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i(\sigma + \tau)\right) (\sigma + \tau)^k \end{aligned}$$

with certain coefficients $c_k(t, \varrho) \in C^\infty(\overline{\mathbb{R}}_+, C_0^\infty(\mathbb{R}_+))$. In order to show the symbol estimates for g_0 we first look at the derivatives of h under the integral, i.e.,

$$\begin{aligned} &\left| \partial_t^\alpha \partial_\sigma^l \left\{ (\partial_\tau^k h)\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i(\sigma + \tau)\right) (\sigma + \tau)^k \right\} \right| \\ &\leq \sum_{j=0}^l \binom{l}{j} \left| \partial_t^\alpha \partial_\sigma^j \left\{ (\partial_\tau^k h)\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i(\sigma + \tau)\right) \right\} \partial_\sigma^{l-j} (\sigma + \tau)^k \right| \\ &\leq \tilde{d}(t, \varrho) \langle \sigma + \tau \rangle^{\mu-l} \leq d(t, \varrho) \langle \sigma \rangle^{\mu-l} \langle \tau \rangle^{|\mu-l|} \end{aligned}$$

with certain $d(t, \varrho) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}_+)$. Thus from (1.3.7) we get

$$|\partial_t^\alpha \partial_\sigma^l g_0(t, \beta + i\sigma)| \leq \langle \sigma \rangle^{\mu-l} \int_{-\infty}^{\infty} \int_0^{\infty} |1 + i\tau|^{-N} |p(t, \varrho)| \langle \tau \rangle^{|\mu-l|} \frac{d\varrho}{\varrho} d\tau,$$

with $p(t, \varrho) \in C^\infty(\overline{\mathbb{R}}_+, C_0^\infty(\mathbb{R}_+))$, where p is a polynomial of derivatives of $\varrho^{\beta+1} \psi(\varrho\varphi(t, t/\varrho))$. If N is chosen sufficiently large, then the integral on the right-hand side converges absolutely. In view of the fact that $\partial_\varrho^k \{\varrho^{\beta+1} \psi(\varrho\varphi(t, t/\varrho))\}$ depends continuously on β for arbitrary $k \in \mathbb{N}$, we conclude that

$$(1.3.8) \quad g_0(t, \beta + i\sigma) \in C^\infty(\overline{\mathbb{R}}_+, S^\mu(\mathbb{R}_\sigma))$$

uniformly in $c_1 \leq \beta \leq c_2$ for arbitrary $c_1 < c_2$.

In a similar manner, we obtain that

$$(1.3.9) \quad \left(g_0 - \left(\frac{\chi(t)}{t\chi'(t)} \right)^{\mu+1} H(\psi)(\pi(\tau)h_{(\mu)}(\chi(t), i\tau)) \right) (t, \beta + i\sigma) \in C^\infty(\overline{\mathbb{R}}_+, S^{\mu-1}(\mathbb{R}_\sigma)),$$

holds uniformly in $c_1 \leq \beta \leq c_2$ for arbitrary $c_1 < c_2$, where $\pi(\tau)$ is a 0-excision function and $h_{(\mu)}(r, i\tau)$ is the leading homogeneous component of $h(r, z)$. In fact, we especially have

$$h(r, i\tau) = \pi(\tau)h_{(\mu)}(r, i\tau) + h_{\mu-1}(r, i\tau),$$

where $h_{\mu-1}(r, i\tau) \in C^\infty(\overline{\mathbb{R}}_+, S_{cl}^{\mu-1}(\mathbb{R}_\tau))$. Denoting the function in (1.3.9) by $g_1(t, z)$, where $z = \beta + i\sigma$, we have

$$\begin{aligned} g_1(t, z) &= \int_{-\infty}^{\infty} \int_0^{\infty} \varrho^{z-i\tau} \left(\psi(\varrho\varphi(t, t/\varrho)) h\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i\tau\right) \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} \right. \\ &\quad \left. - \psi(\varrho) \left(\frac{\chi(t)}{t\chi'(t)} \right)^{\mu+1} \pi(\tau) h_{(\mu)}(\chi(t), i\tau) \right) \frac{d\varrho}{\varrho} d\tau \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \varrho^{z-i\tau} \psi(\varrho) \left(\left(\frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} \right)^{\mu+1} - \left(\frac{\chi(t)}{t\chi'(t)} \right)^{\mu+1} \right) \pi(\tau) h_{(\mu)}(\chi(t), i\tau) \frac{d\varrho}{\varrho} d\tau \\ &\quad + \int_{-\infty}^{\infty} \int_0^{\infty} \varrho^{z-i\tau} \psi(\varrho\varphi(t, t/\varrho)) h_{\mu-1}\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i\tau\right) \\ &\quad \times \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} \frac{d\varrho}{\varrho} d\tau \bmod C^\infty(\overline{\mathbb{R}}_+; \mathcal{M}_{\mathcal{O}}^{-\infty}) \\ &= (k(t, z+1) - k(t, z)) + \int_{-\infty}^{\infty} \int_0^{\infty} \varrho^{z-i\tau} \psi(\varrho\varphi(t, t/\varrho)) h_{\mu-1}\left(\chi(t), \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} i\tau\right) \\ &\quad \times \frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} \frac{d\varrho}{\varrho} d\tau \bmod C^\infty(\overline{\mathbb{R}}_+; \mathcal{M}_{\mathcal{O}}^{-\infty}). \end{aligned}$$

The second summand is of the same quality as (1.3.6) with $\mu - 1$ substituted for μ , hence it is analytic in z and belongs to $C^\infty(\overline{\mathbb{R}}_+, S^{\mu-1}(\mathbb{R}_\sigma))$ uniformly in $c_1 \leq \beta \leq c_2$ for arbitrary $c_1 < c_2$, by (1.3.8). The function $k(t, z)$ in the first summand equals

$$(1.3.10) \quad k(t, z) = \int_{-\infty}^{\infty} \int_0^{\infty} \varrho^{z-i\tau} \psi(\varrho) m(t, \varrho) \pi(\tau) h_{(\mu)}(\chi(t), i\tau) \frac{d\varrho}{\varrho} d\tau,$$

where

$$\left(\frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} \right)^{\mu+1} - \left(\frac{\chi(t)}{t\chi'(t)} \right)^{\mu+1} = (\varrho - 1)m(t, \varrho).$$

For that note that $\frac{\log \varrho}{\log(\varrho\varphi(t, t/\varrho))} \Big|_{\varrho=1} = \frac{\chi(t)}{t\chi'(t)}$, hence $m(t, \varrho)$ is a smooth function. From (1.3.10) it follows that $k(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^\mu)$ by the kernel cut-off technique. Now both $k(t, z)$ and

$k(t, z + 1)$ have the same principal symbol, hence

$$k(t, z + 1) - k(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{\mu-1}).$$

A repeated application of this argument eventually yields $g_0(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^\mu)$.

Thus we have found that $\tilde{g}(t, t', z) = g_0(t, z)v(t') \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^\mu)$ and

$$\chi_*^{-1} \left(\omega \operatorname{op}_M^{\frac{1}{2}} \left(\psi \left(\frac{r}{r'} \right) h \right) \omega_0 \right) = \tilde{\omega} \operatorname{op}_M^{\frac{1}{2}}(g)\tilde{\omega}_0 = \tilde{\omega} \operatorname{op}_M^{\frac{1}{2}}(\tilde{g})\tilde{\omega}_0.$$

In view of the next lemma the proof is complete. \square

Lemma 1.3.5 *Let $g(t, t', z) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^\mu)$, $\mu \in \mathbb{R}$. Then there exists an $h(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^\mu)$ such that*

$$\omega \operatorname{op}_M^\gamma(g)\omega_0 = \omega \operatorname{op}_M^\gamma(h)\omega_0 + G$$

with some $G \in C_G(\mathbb{R}_+, \mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma, \Theta)$, $\Theta = (-\infty, 0]$.

Proof. First there is an $h_0 \in S_{\text{cl}}^\mu(\overline{\mathbb{R}}_+ \times \Gamma_{\frac{1}{2}-\gamma})$ such that $G_0, G_0^*: \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \rightarrow \mathcal{K}^{\infty,\gamma}(\mathbb{R}_+)$ for all $s \in \mathbb{R}$, where $G_0 = \omega \operatorname{op}_M^0(g)\omega_0 - \omega \operatorname{op}_M^0(h_0)\omega_0$. This is a consequence of the fact that Mellin actions with (t, t') -dependent symbols can be turned into Mellin actions with t -dependent symbols modulo smoothing remainders in the cone algebra without asymptotics, cf. [18, Theorem 2.2.31]. Applying kernel cut-off, we get $h_1 = H(\psi)h_0 \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^\mu)$, where $G_1 = \omega \operatorname{op}_M^0((H(1-\psi)h_0))\omega_0 + G_0$ is so that $G_1, G_1^*: \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \rightarrow \mathcal{K}^{\infty,\gamma}(\mathbb{R}_+)$ for all $s \in \mathbb{R}$, since $(H(1-\psi)h_0)(t, z) \in S^{-\infty}(\overline{\mathbb{R}}_+ \times \Gamma_{\frac{1}{2}-\gamma})$. We have both $G_1 \in C^\mu(\mathbb{R}_+, \mathbf{g})$ and G_1 is smoothing, i.e., $G_1 \in C_{M+G}(\mathbb{R}_+, \mathbf{g})$. Moreover, $\sigma_M^{-j}(G_1)(z) \in \mathcal{M}_{\mathcal{O}}^{-\infty}$ for all $j \in \mathbb{N}$, since the conormal symbols of the operators $\omega \operatorname{op}_M^\gamma(g)\omega_0, \omega \operatorname{op}_M^\gamma(h_1)\omega_0$ are holomorphic. As in the beginning of the proof of Theorem 1.3.4, we can construct an $h_2(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{-\infty})$ such that $G = G_1 - \omega \operatorname{op}_M^0(h_2)\omega_0 \in C_G(\mathbb{R}_+, \mathbf{g})$. It remains to set $h(t, z) = h_1(t, z) + h_2(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^\mu)$. \square

1.4. Invariance of the cone algebra. Let $\chi: \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ be a C^∞ -diffeomorphism as above satisfying (1.1.12), (1.1.13).

Theorem 1.4.1 *The operator push-forward under χ induces an isomorphism*

$$\chi_*: C^\mu(\mathbb{R}_+, \mathbf{g}) \rightarrow C^\mu(\mathbb{R}_+, \mathbf{g})$$

for all $\mu \in \mathbb{R}$ and $\mathbf{g} = (\gamma, \delta, \Theta)$. Moreover, we have

- (i) $\tilde{\sigma}_\psi^\mu(\chi_* A)(r, \tilde{\varrho})|_{r=\chi(t), \tilde{\varrho}=\frac{\chi(t)}{\chi'(t)}\tilde{\tau}} = \left(\frac{\chi(t)}{t}\right)^{\gamma-\delta} \tilde{\sigma}_\psi^\mu(A)(t, \tilde{\tau}),$
- (ii) $\sigma_M^{\gamma-\delta}(\chi_* A)(z) = \chi'(0)^{\gamma-\delta} \sigma_M^{\gamma-\delta}(A)(z),$
- (iii) $\sigma_e(\chi_* A)(r, \varrho)|_{r=\chi(t), \varrho=\chi'(t)^{-1}\tau} = \sigma_e(A)(t, \tau) \bmod S^{\mu-1, -1}(\mathbb{R}_+ \times \mathbb{R}).$

Proof. In view of the well-known fact that the (interior) pseudo-differential calculus is coordinate invariant and also that the factor $t^{\delta-\gamma}$ in (1.1.3), (1.1.4) can be ignored, for it causes a contribution likewise known to be invariant, the results of Sections 1.2, 1.3 immediately give the coordinate invariance of the cone algebra with asymptotics.

Further, (i), (iii) are well-known; (i) follows from the transformation rule for $\sigma_\psi^\mu(A)$ and the compatibility condition (1.1.7), while (iii) can be found, e.g., in [13]. Finally, since $a(t, t')|_{t'=t=0} = 1$, cf. Lemma 1.3.1, and $\frac{\chi'(t')t'}{\chi(t)}|_{t'=0} = 1$, (ii) can be easily derived from the proofs of Theorem 1.3.3 and Theorem 1.3.4. (Cf. also the proof of Theorem 2.4.1 in a more general situation below.) \square

The following facts are used in the derivation of the transformation rules for the conormal symbols $\sigma_M^{\gamma-\delta-j}(\chi_* A)(z)$ in Subsection 1.5.

Lemma 1.4.2 *Let $A \in C^\mu(\mathbb{R}_+, \mathbf{g})$ for $\mathbf{g} = (\gamma, \delta, \Theta)$, $\Theta = -(k+1), 0]$, $k \in \mathbb{N} \cup \{\infty\}$, be such that $\sigma_M^{\gamma-\delta-j}(A)(z) = 0$ for all $j = 0, 1, \dots, k$. Then $\sigma_M^{\gamma-\delta-j}(\chi_* A)(z) = 0$ for all $j = 0, 1, \dots, k$.*

Proof. If $\sigma_M^{\gamma-\delta-j}(A)(z) = 0$ for $A \in C^\mu(\mathbb{R}_+, \mathbf{g})$ and all $j = 0, 1, \dots, k$, then A can be written as an operator in $C^\mu(\mathbb{R}_+, \mathbf{g})$ without the smoothing Mellin part M and with a certain holomorphic Mellin symbol, say $\tilde{h}(r, z)$, such that $\partial_r^j \tilde{h}(0, z) = 0$ for all $j = 0, 1, \dots, k$. Applying formula (1.5.29) below to the resulting holomorphic Mellin symbol \tilde{g} of $\chi_*^{-1}A$, cf. the proof of Theorem 1.3.4, we obtain $\frac{1}{j!j'}(\partial_t^j \partial_{t'}^{j'} \tilde{g})(0, 0, z - j') = 0$ for all $j, j' \in \mathbb{N}$ with $j + j' \leq k$. \square

Lemma 1.4.3 *Let $A \in C^\mu(\mathbb{R}_+, \mathbf{g})$ for $\mathbf{g} = (\gamma, \delta, \Theta)$, $\Theta = -(k+1), 0]$, $k \in \mathbb{N} \cup \{\infty\}$, be such that $\sigma_M^{\gamma-\delta-j}(A)(z) \in \mathcal{M}_{\mathcal{O}}^\mu$ for all $j = 0, 1, \dots, k$. Then $\sigma_M^{\gamma-\delta-j}(\chi_* A)(z) \in \mathcal{M}_{\mathcal{O}}^\mu$ for all $j = 0, 1, \dots, k$.*

Proof. This follows from Theorem 1.3.4, since an operator $A \in C^\mu(\mathbb{R}_+, \mathbf{g})$ with holomorphic conormal symbols can be written as a cone operator without the smoothing Mellin part. \square

Remark 1.4.4 *For $\sigma_j(z) \in \mathcal{M}_{\text{as}}^\mu$, $j = 0, 1, \dots, k$, with $\sigma_0(z) \in \mathcal{M}_P^\mu$ for a certain $P \in \mathbf{As}$, $\pi_{\mathbb{C}}P \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$, there is an operator $A \in C^\mu(\mathbb{R}_+, \mathbf{g})$, where $\mathbf{g} = (\gamma, \delta, \Theta)$, $\Theta = -(k+1), 0]$, such that*

$$\sigma_M^{\gamma-\delta-j}(A)(z) = \sigma_j(z)$$

for all $j = 0, 1, \dots, k$, cf. [17, Remark 1.2.9].

Appendix to Section 1.5. An intrinsic interpretation of the principal symbol. We shall interpret the various components of the principal symbol $\sigma(A) = (\tilde{\sigma}_\psi^\mu(A), \sigma_M^{\gamma-\delta}(A), \sigma_e^0(A))$, cf. (1.1.6), as a single, naturally defined continuous section of a line bundle $\mathbb{E}^{\gamma-\delta}$ on a certain topological space \mathbb{T} . Thereby, in contrast to the rest of the paper, we consider operators in $C^\mu(\mathbb{R}_+, \mathbf{g})$ exhibiting classical exit behaviour as $t \rightarrow \infty$. That means that instead of the space $C^\mu(\mathbb{R}_+, \mathbf{g})$, as defined in Definition 1.1.1, we solely consider the space $C^\mu(\mathbb{R}_+, \mathbf{g}) \cap L_{\text{cl}}^{\mu;0}(\mathbb{R}_+)$. Equivalently, the requirement $A_\infty \in L_{\text{cl}}^\mu(\mathbb{R}_+) \cap L^{\mu;0}(\mathbb{R}_+)$ in (iii) is strengthened to $A_\infty \in L_{\text{cl}}^{\mu;0}(\mathbb{R}_+)$. (For details see [18].)

In coordinates $(t, \tau) \in \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$, $\sigma_\psi^\mu(A)(t, \tau)$ is then a classical symbol of order 0 in t as well. Furthermore, the exit symbol $\sigma_e^0(A) \in S^{\mu;0}(\mathbb{R}_+ \times \mathbb{R})/S^{\mu-1;-1}(\mathbb{R}_+ \times \mathbb{R})$ can be represented by a function $\sigma_e^0(A)(t, \tau)$ in $(t, \tau) \in \mathbb{R}_+ \times \mathbb{R}$ which is homogeneous of degree 0 in t and a classical symbol of order μ in τ . The basic relations (1.1.7), (1.1.9), i.e.,

$$\begin{aligned} \sigma_\psi^\mu(A)(t, \tau) &= t^{\gamma-\delta} \tilde{\sigma}_\psi^\mu(t, \tilde{\tau}) \Big|_{\tilde{\tau}=t\tau}, \\ \sigma_\psi^\mu \sigma_M^{\gamma-\delta}(A)(z) \Big|_{z=\frac{1}{2}-\gamma-i\tilde{\tau}} &= \tilde{\sigma}_\psi^\mu(A)(t, \tilde{\tau}) \Big|_{t=0}, \end{aligned}$$

leading to the invariance discussion, are completed by the compatibility condition

$$(1.4.1) \quad \sigma_\psi^\mu \sigma_e^0(A)(t, \tau) = \sigma_e^0 \sigma_\psi^\mu(A)(t, \tau).$$

The diffeomorphism $\chi: \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ is now assumed to satisfy $\chi \in S_{\text{cl}}^1(\overline{\mathbb{R}}_+)$ instead of $\chi \in S^1(\overline{\mathbb{R}}_+)$. That means that $\chi(t)$ possesses an asymptotic expansion $\chi(t) \sim \sum_{j \geq 0} \psi(t) \chi_{(1-j)}(t)$ as $t \rightarrow \infty$, with $\psi(t)$ an excision function, into homogeneous components $\chi_{(1-j)}(t)$, i.e., $\chi_{(1-j)}(\lambda t) = \lambda^{1-j} \chi_{(1-j)}(t)$ for $\lambda > 0$.

In complete analogy to Theorem 1.1.2 we have:

Theorem 1.4.5 *The operator push-forward under χ induces an isomorphism*

$$(1.4.2) \quad \chi_*: C^\mu(\mathbb{R}_+, \mathbf{g}) \cap L_{\text{cl}}^{\mu;0}(\mathbb{R}_+) \rightarrow C^\mu(\mathbb{R}_+, \mathbf{g}) \cap L_{\text{cl}}^{\mu;0}(\mathbb{R}_+)$$

for all $\mu \in \mathbb{R}$ and $\mathbf{g} = (\gamma, \delta, \Theta)$. Moreover, $\chi_*\sigma(A) = \sigma(\chi_*A)$ as before, where $\sigma_M^{\gamma-\delta}(A)$, $\tilde{\sigma}_\psi^\mu(A)$ are transformed according to (i), (ii) of Theorem 1.4.1 and

$$(1.4.3) \quad \sigma_e^0(A_\chi)(\hat{r}, \hat{\rho})|_{\hat{r}=\chi_{(1)}(t), \hat{\rho}=\chi'_{(1)}(t)^{-1}\tau} = \sigma_e^0(A)(t, \tau).$$

Here $\chi_{(1)}(t)$ is the homogeneous principal part of $\chi(t)$.

Note that $\chi_{(1)}(t) = at$ for some $a > 0$ and then $r = at(1 + o(1))$, $\rho = a^{-1}\tau(1 + o(1))$ as $t \rightarrow \infty$ when $(r, \rho) = (\chi(t), \chi'(t)^{-1}\tau)$, i.e., in the limit $t \rightarrow \infty$, $(t, \tau) \mapsto (\hat{r}, \hat{\rho})$ in (1.4.3) transforms like a cotangent vector.

The basic difficulty is the circumstance that there are two principal symbols, namely $\sigma_\psi^\mu(A)$ and $\tilde{\sigma}_\psi^\mu(A)$, both providing the same kind of information. Thereby, the behaviour of $\tilde{\sigma}_\psi^\mu(A)$ as $t \rightarrow 0$ gives a description at $t = 0$, while $\sigma_\psi^\mu(A)$ reflects the behaviour as $t \rightarrow \infty$. Thus both symbols must be taken into account.

We comprise the information provided by $\sigma_\psi^\mu(A)$, $\tilde{\sigma}_\psi^\mu(A)$ in (1.4.4) and (1.4.5), respectively, into a single inclusion in (1.4.9) below. The cotangent bundle $T^*\mathbb{R}_+$ is canonically trivial, i.e., $T^*\mathbb{R}_+ \cong \mathbb{R}_+ \times \mathbb{R}$, with global coordinates $(t, \tau) \in \mathbb{R}_+ \times \mathbb{R}$, τ being the covariable to t . Thus it may be extended to a vector bundle $T^*(\mathbb{R}_+ \cup \{\infty\}) \cong (\mathbb{R}_+ \cup \{\infty\}) \times \mathbb{R}$ on the half-line \mathbb{R}_+ (partially) compactified by one point at $t = \infty$. Then we have

$$(1.4.4) \quad \sigma_\psi^\mu(A) \in S^{(\mu)}(T^*(\mathbb{R}_+ \cup \{\infty\}) \setminus 0; \mathbb{C}),$$

for $\sigma_\psi^\mu(A)(t, \tau) \rightarrow \sigma_e^0\sigma_\psi^\mu(A)(1, \tau)$ as $t \rightarrow \infty$ in $S^{(\mu)}(\mathbb{R}_\tau \setminus 0)$.

On the other hand, recall that the compressed cotangent bundle $\tilde{T}^*\overline{\mathbb{R}}_+$ is defined via the transition functions $U \cap V \ni u \mapsto rt^{-1}J_\chi^{-1}$, where $U, V \subseteq \overline{\mathbb{R}}_+$ are open, t and r are coordinates of u in U and V , respectively, which are connected by the diffeomorphism $r = \chi(t)$, and J_χ is the Jacobian of χ at u . Let $\tilde{\pi}: \tilde{T}^*\overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ be the projection.

For $\alpha \in \mathbb{R}$, let $\mathcal{L}^\alpha \rightarrow \overline{\mathbb{R}}_+$ be the line bundle defined by the transition functions $U \cap V \ni u \mapsto (r/t)^\alpha$, where U, V, t , and $r = \chi(t)$ have the same meaning as above. Then $\tilde{\pi}^*\mathcal{L}^{\gamma-\delta}$ is a line bundle on $\tilde{T}^*\overline{\mathbb{R}}_+$. By the compatibility condition (1.1.7) and (1.4.4), we see that

$$(1.4.5) \quad \tilde{\sigma}_\psi^\mu(A) \in S^{(\mu)}(\tilde{T}^*\overline{\mathbb{R}}_+ \setminus 0; \tilde{\pi}^*\mathcal{L}^{\gamma-\delta}).$$

We now introduce a new line bundle, $E^{\gamma-\delta} \rightarrow \tilde{T}^*(\overline{\mathbb{R}}_+ \cup \{\infty\})$, by glueing the bundles $\tilde{\pi}^*\mathcal{L}^{\gamma-\delta} \rightarrow \tilde{T}^*\overline{\mathbb{R}}_+$ and $T^*(\mathbb{R}_+ \cup \{\infty\}) \times \mathbb{C} \rightarrow T^*(\mathbb{R}_+ \cup \{\infty\})$ on \mathbb{R}_+ .

Lemma 1.4.6 *There is a natural bundle isomorphism*

$$(1.4.6) \quad \begin{array}{ccc} \mathcal{L}^{\gamma-\delta}|_{\tilde{T}^*\overline{\mathbb{R}}_+} & \xrightarrow{\Phi^{\gamma-\delta}} & T^*\mathbb{R}_+ \times \mathbb{C} \\ \downarrow & & \downarrow \\ \tilde{T}^*\overline{\mathbb{R}}_+ & \xrightarrow{\varphi} & T^*\mathbb{R}_+ \end{array}$$

defined in local coordinates by $\Phi^{\gamma-\delta}(t, \tilde{\tau}, \tilde{a}) = (t, t^{-1}\tilde{\tau}, t^{\delta-\gamma}\tilde{a})$.

Proof. Under a coordinate change, i.e., $r = \chi(t)$ as above, we get the diagram

$$\begin{array}{ccc} (t, \tilde{\tau}, \tilde{a}) & \xrightarrow{\Phi^{\gamma-\delta}} & (t, t^{-1}\tilde{\tau}, t^{\delta-\gamma}\tilde{a}) \\ \downarrow & & \downarrow \\ (r, rt^{-1}J_\chi^{-1}\tilde{\tau}, r^{\gamma-\delta}t^{\delta-\gamma}\tilde{a}) & \xrightarrow{\Phi^{\gamma-\delta}} & (r, t^{-1}J_\chi^{-1}\tilde{\tau}, t^{\delta-\gamma}\tilde{a}) \end{array}$$

which commutes. Here the vertical lines are the isomorphisms induced by χ on $\mathcal{L}^{\gamma-\delta}|_{\tilde{T}^*(U \cap V)}$ and $T^*(U \cap V) \times \mathbb{C}$, respectively. Therefore, the definition of $\Phi^{\gamma-\delta}$ is compatible with coordinate changes. \square

Definition 1.4.7 Set $\bar{T}^*(\overline{\mathbb{R}}_+ \cup \{\infty\}) = \tilde{T}^*\overline{\mathbb{R}}_+ \cup_{\varphi} T^*(\mathbb{R}_+ \cup \{\infty\})$,

$$(1.4.7) \quad E^{\gamma-\delta} = \tilde{\pi}_* \mathcal{L}^{\gamma-\delta} \cup_{\Phi^{\gamma-\delta}} (T^*(\mathbb{R}_+ \cup \{\infty\}) \times \mathbb{C}).$$

On $\bar{T}^*(\overline{\mathbb{R}}_+ \cup \{\infty\})$, there are global coordinates $(t, \bar{\tau})$, where $t \in [0, \infty]$ and $\bar{\tau} = \tilde{\tau}/(1+t) = \tau/(1+t^{-1})$, with (t, τ) the global coordinates on $T^*(\mathbb{R}_+ \cup \{\infty\})$ mentioned above and $\tilde{\tau}$ and τ are related by $\tilde{\tau} = t\tau$. Accordingly, there is a metric on $\bar{T}^*(\overline{\mathbb{R}}_+ \cup \{\infty\})$ in which $|\bar{\tau}| = 1$ if $|\tau| = 1+t^{-1}$ or, equivalently, $|\bar{\tau}| = 1+t$. Note that $\bar{\tau} = \tilde{\tau}$ at $t = 0$ and $\bar{\tau} = \tau$ at $t = \infty$.

Definition 1.4.8 For $A \in C^\mu(\mathbb{R}_+, \mathfrak{g}) \cap L_{\text{cl}}^{\mu;0}(\mathbb{R}_+)$, we define

$$(1.4.8) \quad \bar{\sigma}_\psi^\mu(A)(t, \bar{\tau}) = \begin{cases} \tilde{\sigma}_\psi^\mu(A)(t, \tilde{\tau}) & \text{if } t < \infty, \\ \sigma_\psi^\mu(A)(t, \tau) & \text{if } t > 0. \end{cases}$$

Lemma 1.4.9 For $A \in C^\mu(\mathbb{R}_+, \mathfrak{g}) \cap L_{\text{cl}}^{\mu;0}(\mathbb{R}_+)$, $\bar{\sigma}_\psi^\mu(A)(t, \bar{\tau})$ is a well-defined section of the bundle $E^{\gamma-\delta}$ on $\bar{T}^*(\overline{\mathbb{R}}_+ \cup \{\infty\}) \setminus 0$, i.e.,

$$(1.4.9) \quad \bar{\sigma}_\psi^\mu(A) \in S^{(\mu)}(\bar{T}^*(\overline{\mathbb{R}}_+ \cup \{\infty\}) \setminus 0; E^{\gamma-\delta}).$$

Proof. This is an immediate consequence of (1.1.7) and the construction of the bundle $E^{\gamma-\delta}$. \square

By (1.4.9), $\bar{\sigma}_\psi^\mu(A)$ can likewise be considered as being defined on the sphere bundle $\bar{S}^*(\overline{\mathbb{R}}_+ \cup \{\infty\}) = \{(t, \bar{\tau}) \in \bar{T}^*(\overline{\mathbb{R}}_+ \cup \{\infty\}); |\bar{\tau}| = 1\}$, i.e.,

$$(1.4.10) \quad \bar{\sigma}_\psi^\mu(A) \in C^\infty(\bar{S}^*(\overline{\mathbb{R}}_+ \cup \{\infty\}); E^{\gamma-\delta}).$$

Remark 1.4.10 The choice of the metric on $\bar{T}^*(\overline{\mathbb{R}}_+ \cup \{\infty\})$ is the only non-canonical part in the construction. For $\mu = 0$, however, we can choose $\bar{S}^*(\overline{\mathbb{R}}_+ \cup \{\infty\})$ to be the sphere bundle of $\bar{T}^*(\overline{\mathbb{R}}_+ \cup \{\infty\})$ to get all canonically defined.

Note that $\bar{S}^*(\overline{\mathbb{R}}_+ \cup \{\infty\})$ has two connected components both diffeomorphic to the interval $[0, \infty]$, i.e., $\bar{S}^*(\overline{\mathbb{R}}_+ \cup \{\infty\}) = \bar{S}_+ \cup \bar{S}_-$, defined by $\bar{\tau} > 0$ and $\bar{\tau} < 0$, respectively.

Definition 1.4.11 The topological space \mathbb{T} assigned to $C^\mu(\mathbb{R}_+, \mathfrak{g}) \cap L_{\text{cl}}^{\mu;0}(\mathbb{R}_+)$ is

$$(1.4.11) \quad \mathbb{T} = (\bar{\Gamma}_{\frac{1}{2}-\gamma} \cup \bar{S}_+ \cup \bar{\mathbb{R}}_\tau \cup \bar{S}_-) / \sim,$$

where $\bar{\Gamma}_{\frac{1}{2}-\gamma} = [\frac{1}{2} - \gamma - i\infty, \frac{1}{2} - \gamma + i\infty]_z$, $\bar{S}_\pm = [0, \infty]_t \times \{\pm\infty\}_{\bar{\tau}}$, and $\bar{\mathbb{R}}_\tau = [-\infty, \infty]_\tau$. Moreover, the points $\bar{\Gamma}_{\frac{1}{2}-\gamma} \ni \frac{1}{2} - \gamma \pm i\infty \sim (0, \mp\infty) \in \bar{S}_\mp$ and $\bar{S}_\pm \ni (\infty, \pm\infty) \sim \pm\infty \in \bar{\mathbb{R}}_\tau$ are identified.

A basis for the topology of \mathbb{T} is given by all sets of the form $U \cup V$, where $U \subseteq \bar{\Gamma}_{\frac{1}{2}-\gamma}$ is open, $V \subseteq \mathbb{T} \setminus \bar{\Gamma}_{\frac{1}{2}-\gamma}$ is open (here $\mathbb{T} \setminus \bar{\Gamma}_{\frac{1}{2}-\gamma}$ is equipped with the topology of an open interval), and if $U \neq \emptyset$ then $V = \mathbb{T} \setminus \bar{\Gamma}_{\frac{1}{2}-\gamma}$.

We write $\bar{\tau} = \pm\infty$ on \bar{S}_\pm having in mind that

$$\bar{\sigma}_\psi^\mu(A)(t, \pm 1) = \lim_{\bar{\tau} \rightarrow \pm\infty} \langle \bar{\tau} \rangle^{-\mu} \bar{\sigma}_\psi^\mu(A)(t, \bar{\tau})$$

due to homogeneity.

Definition 1.4.12 The space $\mathbb{E}^{\gamma-\delta}$ is defined by glueing (without torsion in the fibres) the trivial bundle $\bar{\Gamma}_{\frac{1}{2}-\gamma} \times \mathbb{C}$, the bundle $E^{\gamma-\delta}$ on $\bar{S}^*(\overline{\mathbb{R}}_+ \cup \{\infty\}) = \bar{S}_+ \cup \bar{S}_-$, and the trivial bundle $\bar{\mathbb{R}}_\tau \times \mathbb{C}$, according to the identification \sim of points in Definition 1.4.11. $\mathbb{E}^{\gamma-\delta}$ is then equipped with the weakest topology converting it into a complex line bundle on \mathbb{T} , i.e., with the weakest

topology such that the canonical projection $\mathbb{E}^{\gamma-\delta} \rightarrow \mathbb{T}$ is continuous and, moreover, the fibre over each point of \mathbb{T} is homeomorphic to \mathbb{C} .

Theorem 1.4.13 *The triple $(\tilde{\sigma}_\psi^\mu(A), \sigma_M^{\gamma-\delta}(A), \sigma_e^0(A))$ of principal symbols for an operator $A \in C^\mu(\mathbb{R}_+, \mathbf{g}) \cap L_{\text{cl}}^{\mu;0}(\mathbb{R}_+)$ give rise to a continuous section $\sigma(A)$ of the line bundle $\mathbb{E}^{\gamma-\delta}$, i.e.,*

$$(1.4.12) \quad \sigma(A) \in C^0(\mathbb{T}, \mathbb{E}^{\gamma-\delta}).$$

Proof. We define

$$(1.4.13) \quad \sigma(A)(\zeta) = \begin{cases} \langle z \rangle^{-\mu} \sigma_M^{\gamma-\delta}(A)(z) & \text{if } \zeta = z \in \Gamma_{\frac{1}{2}-\gamma}, \\ \tilde{\sigma}_\psi^\mu(A)(\zeta) & \text{if } \zeta \in \bar{S}_+ \cup \bar{S}_-, \\ \langle \tau \rangle^{-\mu} \sigma_e^0(A)(1, \tau) & \text{if } \zeta = \tau \in \mathbb{R}_\tau. \end{cases}$$

Then $\sigma(A)$ is a well-defined continuous section of the bundle $\mathbb{E}^{\gamma-\delta}$ on \mathbb{T} . In fact, it is quite obvious that $\sigma(A)$ is well-defined and continuous on $\mathbb{T} \setminus \Gamma_{\frac{1}{2}-\gamma}$. Moreover, changing coordinates locally at $t = 0$, i.e., $r = \chi(t)$, in view of $rt^{-1}|_{t=0} = \chi'(0)$, the compressed principal symbol $\tilde{\sigma}_\psi^\mu(A)(t, \tilde{\tau})$ is multiplied by $\chi'(0)^{\gamma-\delta}$ at $t = 0$. Therefore, due to the definition of the topology of \mathbb{T} and the compatibility condition (1.1.9), the Mellin symbol $\sigma_M^{\gamma-\delta}(A)(z)$ is also multiplied by $\chi'(0)^{\gamma-\delta}$, while $\tilde{\tau}$ at $t = 0$, i.e., $z \in \Gamma_{\frac{1}{2}-\gamma}$, remains fixed. But $\chi'(0)^{\gamma-\delta}$ is the factor appearing in the transformation formula (ii) of Theorem 1.4.1. \square

We conclude with the following observation: Let \mathbb{T}_{st} be the space \mathbb{T} equipped with the topology of the circle S^1 and $i: \mathbb{T}_{\text{st}} \rightarrow \mathbb{T}$ be the identity map which is continuous. Further let $\mathbb{E}_{\text{st}}^{\gamma-\delta} = i^* \mathbb{E}^{\gamma-\delta}$ be the pull-back of $\mathbb{E}^{\gamma-\delta}$ under i . Then

$$i^*: C^0(\mathbb{T}, \mathbb{E}^{\gamma-\delta}) \rightarrow C^0(\mathbb{T}_{\text{st}}, \mathbb{E}_{\text{st}}^{\gamma-\delta})$$

and

$$(1.4.14) \quad \mathbb{E}_{\text{st}}^{\gamma-\delta} \cong S^1 \times \mathbb{C}.$$

In particular, having fixed the trivialisation in (1.4.14), sections of the bundle $\mathbb{E}_{\text{st}}^{\gamma-\delta}$ can be identified with functions on S^1 .

Now let $A \in C^\mu(\mathbb{R}_+, \mathbf{g}) \cap L_{\text{cl}}^{\mu;0}(\mathbb{R}_+)$ be elliptic, i.e., its symbol $\sigma(A)$, as defined in Theorem 1.4.13, is everywhere invertible. Then $i^* \sigma(A)$ can be viewed as a map

$$i^* \sigma(A): S^1 \rightarrow \mathbb{C} \setminus 0.$$

Theorem 1.4.14 *For an elliptic operator $A \in C^\mu(\mathbb{R}_+, \mathbf{g}) \cap L_{\text{cl}}^{\mu;0}(\mathbb{R}_+)$, the index of the corresponding Fredholm operator (1.1.5) is given by*

$$(1.4.15) \quad \text{ind } A = \text{deg} \left(\frac{i^* \sigma(A)}{|i^* \sigma(A)|} \right),$$

where deg denotes the mapping degree (the winding number) of maps $S^1 \rightarrow S^1$.

1.5. Symbolic rules. Here we provide formulas describing the behaviour of the conormal symbols under coordinate changes. Thereby, we restrict ourselves to the case that $\gamma - \delta = \mu$. Throughout, let a diffeomorphism $\chi: \bar{\mathbb{R}}_+ \rightarrow \bar{\mathbb{R}}_+$, $\chi(0) = 0$, $\chi'(0) > 0$, be fixed.

Theorem 1.5.1 *There are universal polynomials $q_j(y_1, y_2, \dots, y_j)$, $j = 1, 2, \dots$, such that, for each operator $A \in C^\mu(\mathbb{R}_+; \mathbf{g}_k)$, where $\mathbf{g}_k = (\gamma, \gamma - \mu, (-k - 1, 0])$,*

$$(1.5.1) \quad \sigma_M^{\mu-j}(\chi_* A) = \chi'(0)^{\mu-j} \left\{ \sigma_M^{\mu-j}(A) + \sum_{r=0}^{j-1} \sum_{a_1+2a_2+\dots+(j-r)a_{j-r}=j-r} \frac{\bar{q}_1^{a_1} \bar{q}_2^{a_2} \dots \bar{q}_{j-r}^{a_{j-r}}}{a_1! a_2! \dots a_{j-r}!} \Delta[\mu-r; a_1, a_2, \dots, a_{j-r}] \sigma_M^{\mu-r}(A) \right\}$$

for $j = 0, 1, \dots, k$. Here

$$(1.5.2) \quad \bar{q}_j(\chi) = q_j \left(\frac{\chi''(0)}{\chi'(0)}, \frac{\chi^{(3)}(0)}{\chi'(0)}, \dots, \frac{\chi^{(j+1)}(0)}{\chi'(0)} \right).$$

The linear operators $\Delta[\mu; a_1, a_2, \dots, a_j]$, acting on meromorphic functions, are defined in (1.5.15) below.

Remark 1.5.2 (a) *The first few of the polynomials $q_j(y_1, y_2, \dots, y_j)$ are*

$$\begin{aligned} q_1 &= \frac{1}{2} y_1, & q_2 &= -\frac{1}{4} y_1^2 + \frac{1}{6} y_2, & q_3 &= \frac{1}{8} y_1^3 - \frac{1}{6} y_1 y_2 + \frac{1}{24} y_3, \\ q_4 &= -\frac{3}{32} y_1^4 - \frac{1}{24} y_1 y_3 + \frac{1}{6} y_1^2 y_2 - \frac{1}{24} y_2^2 + \frac{1}{120} y_4, \\ q_5 &= \frac{1}{16} y_1^5 - \frac{7}{48} y_1^3 y_2 + \frac{5}{72} y_1 y_2^2 + \frac{1}{24} y_1^2 y_3 - \frac{1}{48} y_2 y_3 - \frac{1}{120} y_1 y_4 + \frac{1}{720} y_5. \end{aligned}$$

For a possibility to calculate these polynomials, see (1.5.41).

(b) *In general, the polynomial $q_j(y_1, y_2, \dots, y_j)$ fulfills the homogeneity relation*

$$(1.5.3) \quad q_j(\lambda y_1, \lambda^2 y_2, \dots, \lambda^j y_j) = \lambda^j q_j(y_1, y_2, \dots, y_j), \quad \lambda \in \mathbb{C}.$$

Moreover, it is seen from (1.5.41) that the factor in front of y_j equals $\frac{1}{(j+1)!}$. Consequently, $\bar{q}_1, \bar{q}_2, \bar{q}_3, \dots$ can be any sequences of real numbers, by an appropriate choice of the diffeomorphism χ .

Remark 1.5.3 *To unify notation, we set $q_0 = 1$ and*

$$\sum_{a_1+2a_2+\dots+ja_j=j} \frac{\bar{q}_1^{a_1} \bar{q}_2^{a_2} \dots \bar{q}_j^{a_j}}{a_1! a_2! \dots a_j!} \Delta[\mu; a_1, a_2, \dots, a_j] = 1$$

when $j = 0$.

Theorem 1.5.1 is proved in a series of lemmas.

Lemma 1.5.4 *For $j, r \in \mathbb{N}$, $j \geq r$, there are linear operators $T_r^j(\mu, \chi)$, acting on meromorphic Mellin symbols, such that, for each $A \in C^\mu(\mathbb{R}_+; \mathbf{g}_k)$, where $\mathbf{g}_k = (\gamma, \gamma - \mu, (-k - 1, 0])$,*

$$(1.5.4) \quad \sigma_M^{\mu-j}(\chi_* A) = \chi'(0)^{\mu-j} \sum_{r=0}^j T_r^j(\mu, \chi) \sigma_M^{\mu-r}(A)$$

for $j = 0, 1, \dots, k$. In particular, the operators $T_r^j(\mu, \chi)$ are independent of k .

Proof. Consider the composition of linear maps

$$A \mapsto \chi_* A \mapsto (\sigma_M^\mu(\chi_* A), \sigma_M^{\mu-1}(\chi_* A), \dots, \sigma_M^{\mu-k}(\chi_* A)).$$

By Lemma 1.4.2, this map only depends on $\sigma_M^\mu(A), \sigma_M^{\mu-1}(A), \dots, \sigma_M^{\mu-k}(A)$. Thus it descends to a linear map

$$(\sigma_M^\mu(A), \sigma_M^{\mu-1}(A), \dots, \sigma_M^{\mu-k}(A)) \mapsto (\sigma_M^\mu(\chi_* A), \sigma_M^{\mu-1}(\chi_* A), \dots, \sigma_M^{\mu-k}(\chi_* A)),$$

and (1.5.4) follows. The independence of the operators $T_r^j(\mu, \chi)$ from k follows from the embedding $C^\mu(\mathbb{R}_+, \mathbf{g}_{k+1}) \subset C^\mu(\mathbb{R}_+, \mathbf{g}_k)$. \square

Notice that

$$(1.5.5) \quad T_r^j(\mu, \chi): \mathcal{M}_{\text{as}}^\mu \rightarrow \mathcal{M}_{\text{as}}^\mu.$$

Moreover, Lemma 1.4.3 gives us that $T_r^j(\mu, \chi): \mathcal{M}_O^\mu \rightarrow \mathcal{M}_O^\mu$.

Lemma 1.5.5 *For $j, r \in \mathbb{N}$, $j \geq r$, we have*

$$(1.5.6) \quad T_r^j(\mu, \chi) = T_0^{j-r}(\mu - r, \chi).$$

Proof. This is a consequence of the embedding $C^{\mu-r}(\mathbb{R}_+, \mathbf{g}'_{k-r}) \subset C^\mu(\mathbb{R}_+, \mathbf{g}_k)$, where $\mathbf{g}'_{k-r} = (\gamma, \gamma - (\mu - r), (-(k - r) - 1, 0])$, which holds modulo Green operators and the fact that this embedding is compatible with coordinate changes. \square

In particular, by (ii) of Theorem 1.4.1,

$$(1.5.7) \quad T_j^j(\mu, \chi) = 1$$

for all $j \in \mathbb{N}$. By Lemma 1.5.5, it remains to calculate the operators $T_0^j(\mu, \chi)$ for $j > 0$. We first employ the invariance of the Mellin translation product (1.1.11) under the operator push-forward with respect to χ .

Lemma 1.5.6 *The operators $T_0^j(\mu, \chi)$ satisfy the relations*

$$(1.5.8) \quad T_0^l(\mu + \rho, \chi) \{ \sigma_0(z + \rho) \sigma_1(z) \} = \sum_{j+k=l} (T_0^j(\mu, \chi) \sigma_0)(z + \rho - k) T_0^k(\rho, \chi) \sigma_1(z)$$

for all $\sigma_0(z) \in \mathcal{M}_{\text{as}}^\mu$, $\sigma_1(z) \in \mathcal{M}_{\text{as}}^\rho$.

Proof. The Mellin translation product gives

$$\sigma_M^{\mu+\rho-l}(\chi_*(AB))(z) = \sum_{j+k=l} \sigma_M^{\mu-j}(\chi_*A)(z + \rho - k) \sigma_M^{\rho-k}(\chi_*B)(z).$$

Inserting (1.5.4) into this equation yields

$$\begin{aligned} \sum_{u=0}^l T_u^l(\mu + \rho, \chi) \left\{ \sum_{r+s=u} \sigma_M^{\mu-r}(A)(z + \rho - s) \sigma_M^{\rho-s}(B)(z) \right\} = \\ = \sum_{j+k=l} \left\{ \sum_{r=0}^j T_r^j(\mu, \chi) \sigma_M^{\mu-r}(A) \right\} (z + \rho - k) \left\{ \sum_{s=0}^k T_s^k(\rho, \chi) \sigma_M^{\rho-s}(B)(z) \right\}. \end{aligned}$$

Since the functions $\sigma_M^{\mu-r}(A)(z)$, $\sigma_M^{\rho-s}(B)(z)$ for $0 \leq r \leq l$, $0 \leq s \leq l$ can be chosen independently of each other, cf. Remark 1.4.4, we set $\sigma_M^{\mu-r}(A)(z) = \sigma_M^{\rho-s}(B)(z) = 0$ for $0 < r \leq l$, $0 < s \leq l$, and find

$$\begin{aligned} T_0^l(\mu + \rho, \chi) \{ \sigma_M^\mu(A)(z + \rho) \sigma_M^\rho(B)(z) \} = \\ = \sum_{j+k=l} \left\{ T_0^j(\mu, \chi) \sigma_M^\mu(A) \right\} (z + \rho - k) T_0^k(\rho, \chi) \sigma_M^\rho(B)(z). \end{aligned}$$

This is relation (1.5.8). \square

Note that all the other relations, i.e., these for $(r, s) \neq (0, 0)$, which also follow from the latter proof are consequences of (1.5.6), (1.5.8).

Next we are going to define the operators $\Delta[\mu; a_1, a_2, \dots, a_j]$ appearing in Theorem 1.5.1.

For $j \in \mathbb{N}$, we first define

$$(1.5.9) \quad \Delta_j \sigma(z) = z(\sigma(z) - \sigma(z - j)),$$

where $\sigma(z)$ is an arbitrary meromorphic function on \mathbb{C} . Note that

$$(1.5.10) \quad \Delta_j: \mathcal{M}_{\text{as}}^\mu \rightarrow \mathcal{M}_{\text{as}}^\mu,$$

since $\sigma(z), \sigma(z - j)$ possess the same parameter-dependent principal symbol, cf. the remark before Lemma 1.5.13 below.

Lemma 1.5.7 *For $j \in \mathbb{N}$, $j \geq 1$, we have*

$$(1.5.11) \quad \begin{aligned} (\Delta_j + \mu + \rho)\{\sigma_0(z + \rho)\sigma_1(z)\} &= \\ &= ((\Delta_j + \mu)\sigma_0)(z + \rho)\sigma_1(z) + \sigma_0(z + \rho - j)(\Delta_j + \rho)\sigma_1(z) \end{aligned}$$

for all meromorphic functions $\sigma_0(z), \sigma_1(z)$ on \mathbb{C} .

Proof. Indeed,

$$\begin{aligned} (\Delta_j + \mu + \rho)\{\sigma_0(z + \rho)\sigma_1(z)\} &= \\ &= (z + \mu + \rho)\sigma_0(z + \rho)\sigma_1(z) - z\sigma_0(z + \rho - j)\sigma_1(z - j) \\ &= (z + \mu + \rho)\sigma_0(z + \rho)\sigma_1(z) - (z + \rho)\sigma_0(z + \rho - j)\sigma_1(z) \\ &\quad + (z + \rho)\sigma_0(z + \rho - j)\sigma_1(z) - z\sigma_0(z + \rho - j)\sigma_1(z - j) \\ &= ((\Delta_j + \mu)\sigma_0)(z + \rho)\sigma_1(z) + \sigma_0(z + \rho - j)(\Delta_j + \rho)\sigma_1(z) \end{aligned}$$

for all meromorphic functions $\sigma_0(z), \sigma_1(z)$. □

Then, for $j_1, j_2, \dots, j_m \in \mathbb{N}$, $j_1, j_2, \dots, j_m \geq 1$, we define

$$(1.5.12) \quad \begin{aligned} \Delta(\mu; j_1, j_2, \dots, j_m) &= (\Delta_{j_1} + \mu - (j_2 + j_3 + \dots + j_m)) \\ &\quad (\Delta_{j_2} + \mu - (j_3 + \dots + j_m)) \dots (\Delta_{j_{m-1}} + \mu - j_m)(\Delta_{j_m} + \mu) \end{aligned}$$

and $\Delta(\mu; j_1, j_2, \dots, j_m) = 1$ if $m = 0$. (Notice the particular use of parentheses instead of brackets as in (1.5.1).) As an immediate consequence of (1.5.11) we get:

Lemma 1.5.8 *For all $j_1, j_2, \dots, j_m \in \mathbb{N}$, $j_1, j_2, \dots, j_m \geq 1$, we have*

$$(1.5.13) \quad \begin{aligned} \Delta(\mu + \rho; j_1, \dots, j_m)\{\sigma_0(z + \rho)\sigma_1(z)\} &= \\ &\quad \sum_{\substack{\{i_1, \dots, i_r\} \cup \{k_1, \dots, k_s\} = \{1, \dots, m\}, \\ \{i_1, \dots, i_r\} \cap \{k_1, \dots, k_s\} = \emptyset, \\ i_1 < \dots < i_r, k_1 < \dots < k_s}} \\ &\quad (\Delta(\mu; j_{i_1}, \dots, j_{i_r})\sigma_0)(z + \rho - \sum_{1 \leq h \leq s} j_{k_h})\Delta(\rho; j_{k_1}, \dots, j_{k_s})\sigma_1(z). \end{aligned}$$

Proof. We proceed by induction on m . Relation (1.5.13) for $m = 1$ is nothing but (1.5.11). The inductive step $m \rightarrow m + 1$ then follows from the following calculation:

$$\Delta(\mu + \rho; j_0, j_1, \dots, j_m) = (\Delta_{j_0} + \mu + \rho - (j_1 + \dots + j_m)) \Delta(\mu + \rho; j_1, \dots, j_m)$$

and

$$\begin{aligned}
& (\Delta_{j_0 + \mu + \rho - (j_1 + \dots + j_m)}) \left((\Delta(\mu; j_{i_1}, \dots, j_{i_r}) \sigma_0) (z + \rho - \sum_{1 \leq h \leq s} j_{k_h}) \times \right. \\
& \quad \left. \Delta(\rho; j_{k_1}, \dots, j_{k_s}) \sigma_1(z) \right) = \\
& = \left((\Delta_{j_0 + \mu - \sum_{1 \leq \tilde{h} \leq r} j_{i_{\tilde{h}}}) \Delta(\mu; j_{i_1}, \dots, j_{i_r}) \sigma_0 \right) (z + \rho - \sum_{1 \leq h \leq s} j_{k_h}) \times \\
& \quad \Delta(\rho; j_{k_1}, \dots, j_{k_s}) \sigma_1(z) + (\Delta(\mu; j_{i_1}, \dots, j_{i_r}) \sigma_0) (z + \rho - j_0 - \sum_{1 \leq h \leq s} j_{k_h}) \times \\
& \quad (\Delta_{j_0 + \rho - \sum_{1 \leq h \leq s} j_{k_h}) \Delta(\rho; j_{k_1}, \dots, j_{k_s}) \sigma_1(z) \\
& = (\Delta(\mu; j_0, j_{i_1}, \dots, j_{i_r}) \sigma_0) (z + \rho - \sum_{1 \leq h \leq s} j_{k_h}) \Delta(\rho; j_{k_1}, \dots, j_{k_s}) \sigma_1(z) + \\
& \quad + (\Delta(\mu; j_{i_1}, \dots, j_{i_r}) \sigma_0) (z + \rho - j_0 - \sum_{1 \leq h \leq s} j_{k_h}) \Delta(\rho; j_0, j_{k_1}, \dots, j_{k_s}) \sigma_1(z),
\end{aligned}$$

where the latter holds by (1.5.11). \square

With $\sum_{1 \leq h \leq s} j_{k_h} = k$ in (1.5.13), this relation is basically (1.5.8). However, we still have to take into account that the operators $\Delta(\mu; j_1, \dots, j_m)$ are not linearly independent.

Lemma 1.5.9 *For $j, k \in \mathbb{N}$, $j, k \geq 1$, we have*

$$(1.5.14) \quad (\Delta_j + \mu - k)(\Delta_k + \mu) - (\Delta_k + \mu - j)(\Delta_j + \mu) = (j - k)(\Delta_{j+k} + \mu).$$

Proof. We have

$$\begin{aligned}
& (\Delta_j + \mu - k)(\Delta_k + \mu) \sigma(z) - (\Delta_k + \mu - j)(\Delta_j + \mu) \sigma(z) \\
& = (\Delta_j + \mu - k) [(z + \mu) \sigma(z) - z \sigma(z - k)] \\
& \quad - (\Delta_k + \mu - j) [(z + \mu) \sigma(z) - z \sigma(z - j)] \\
& = (z + \mu - k) [(z + \mu) \sigma(z) - z \sigma(z - k)] \\
& \quad - z [(z + \mu - j) \sigma(z - j) - (z - j) \sigma(z - j - k)] \\
& \quad - (z + \mu - j) [(z + \mu) \sigma(z) - z \sigma(z - j)] \\
& \quad + z [(z + \mu - k) \sigma(z - k) - (z - k) \sigma(z - j - k)] \\
& = (j - k) [(z + \mu) \sigma(z) - \sigma(z - j - k)] = (j - k)(\Delta_{j+k} + \mu) \sigma(z)
\end{aligned}$$

for any meromorphic function $\sigma(z)$. \square

Lemma 1.5.9 allows us to change freely the order of the parameters j_1, j_2, \dots, j_m in defining the operators $\Delta(\mu; j_1, j_2, \dots, j_m)$, up to linear combinations of operators of exactly the same kind. In particular, we can assume that $j_1 \leq j_2 \leq \dots \leq j_m$. Accordingly, we finally introduce

$$(1.5.15) \quad \Delta[\mu; a_1, a_2, \dots, a_j] = \Delta(\mu; \underbrace{1, \dots, 1}_{a_1 \text{ times}}, \underbrace{2, \dots, 2}_{a_2 \text{ times}}, \dots, \underbrace{j, \dots, j}_{a_j \text{ times}}).$$

Proposition 1.5.10 *For any sequence $\bar{q}_1, \bar{q}_2, \bar{q}_3, \dots$ of complex numbers, the operators*

$$(1.5.16) \quad T^l(\mu) = \sum_{a_1 + 2a_2 + \dots + la_l = l} \frac{\bar{q}_1^{a_1} \bar{q}_2^{a_2} \dots \bar{q}_l^{a_l}}{a_1! a_2! \dots a_l!} \Delta[\mu; a_1, a_2, \dots, a_l],$$

where $l \in \mathbb{N}$, $\mu \in \mathbb{R}$, solve the functional equation (1.5.8), with the operators $T_0^l(\mu, \chi)$ replaced with $T^l(\mu)$.

Proof. This is a reformulation of Lemma 1.5.8. \square

Now we are going to show that the solutions to (1.5.8) supplied in Proposition 1.5.10 are the only ones within a certain class of operators to which the operators $T_0^l(\mu, \chi)$ belong.

Definition 1.5.11 For $l \in \mathbb{N}$, the class \mathfrak{M}_l consists of all linear operators T , acting on meromorphic functions $\sigma(z)$ on \mathbb{C} , having the form

$$(1.5.17) \quad T\sigma(z) = \sum_{r=0}^l p_r(z)\sigma(z-r),$$

where the $p_r(z)$ are certain polynomials in z of degree l , $0 \leq r \leq l$, such that

$$(1.5.18) \quad T: \mathcal{M}_{\text{as}}^\mu \rightarrow \mathcal{M}_{\text{as}}^\mu$$

holds for all $\mu \in \mathbb{R}$.

Remark 1.5.12 For an operator T of the form (1.5.17) to belong to \mathfrak{M}_l it is sufficient to demand $T: \mathcal{M}_{\mathcal{O}}^\mu \rightarrow \mathcal{M}_{\mathcal{O}}^\mu$ for some $\mu \in \mathbb{R}$ instead of (1.5.18) for all $\mu \in \mathbb{R}$, cf. the proof of Lemma 1.5.15.

In the following lemma, a generalisation of the well-known fact that the principal pseudo-differential symbol of $h|_{\Gamma_\beta}$ for $h \in \mathcal{M}_{\text{as}}^\mu$ is independent of $\beta \in \mathbb{R}$ is stated. We shall write

$$(1.5.19) \quad \sigma_\psi^{\mu-j}(h)(\beta + i\tau) = \sigma_\psi^{\mu-j}(h|_{\Gamma_\beta})(\tau),$$

where $\sigma_\psi^{\mu-j}(h|_{\Gamma_\beta})$ is the j th homogeneous component of $h|_{\Gamma_\beta}$. Thereby, the meaning of the right-hand side of (1.5.19) is obvious if $h \in \mathcal{M}_{\mathcal{O}}^\mu$, while for general $h \in \mathcal{M}_{\text{as}}^\mu$ we write $h = h_0 + h_1$ with $h_0 \in \mathcal{M}_{\mathcal{O}}^\mu$, $h_1 \in \mathcal{M}_{\text{as}}^{-\infty}$ and define the right-hand side of (1.5.19) to be $\sigma_\psi^{\mu-j}(h_0|_{\Gamma_\beta})$. Notice that $\sigma_\psi^{\mu-j}(h) \in C^\infty(\mathbb{R}_\beta; S^{(\mu-j)}(\mathbb{R}_\tau \setminus \{0\}))$.

Lemma 1.5.13 Let $h \in \mathcal{M}_{\text{as}}^\mu$. Further let $h_k(i\tau) = \sigma_\psi^{\mu-k}(h|_{\Gamma_0}) \in S^{(\mu-k)}(\mathbb{R}_\tau \setminus \{0\})$, $k = 0, 1, 2, \dots$. Then

$$(1.5.20) \quad \sigma_\psi^{\mu-j}(h)(\beta + i\tau) = \sum_{r=0}^j \binom{\mu-j+r}{r} (i\tau)^{-r} h_{j-r}(i\tau) \beta^r.$$

In particular, $\sigma_\psi^{\mu-j}(h)(\beta + i\tau)$ is a polynomial in β of degree j with coefficients belonging to $S^{(\mu-j)}(\mathbb{R}_\tau \setminus \{0\})$.

Proof. We may assume that $h \in \mathcal{M}_{\mathcal{O}}^\mu$. From the Cauchy-Riemann equation, we inductively infer that $\partial^j h / \partial \beta^j = (-i)^j \partial^j h / \partial \tau^j$ for any $j = 0, 1, 2, \dots$, for $\partial_{\bar{z}} \partial_z^j h = 0$, while Euler's homogeneity relation yields that

$$\frac{1}{j!} \frac{\partial^j h_k}{\partial \tau^j}(i\tau) = \binom{\mu-k}{j} (i\tau)^{-j} h_k(i\tau).$$

Now, employing Taylor's formula, we get

$$(1.5.21) \quad h(\beta + i\tau) = \sum_{r=0}^j \frac{\beta^r}{r!} \frac{\partial^r h}{\partial \beta^r}(i\tau) + \frac{\beta^{j+1}}{j!} \int_0^1 (1-\sigma)^j \frac{\partial^{j+1} h}{\partial \beta^{j+1}}(\sigma\beta + i\tau) d\sigma.$$

For fixed β , the integral on the right-hand side of (1.5.21) belongs to $S_{\text{cl}}^{\mu-j-1}(\mathbb{R}_\tau)$. Thus we obtain

$$\begin{aligned} \sigma_\psi^{\mu-j}(h|_{\Gamma_\beta}) &= \sum_{r=0}^j \frac{1}{r!} \sigma_\psi^{\mu-j} \left(\frac{\partial^r h}{\partial \beta^r}(i\tau) \right) \beta^r = \sum_{r=0}^j \frac{(-i)^r}{r!} \sigma_\psi^{\mu-j} \left(\frac{\partial^r h}{\partial \tau^r}(i\tau) \right) \beta^r \\ &= \sum_{r=0}^j \frac{1}{r!} \frac{\partial^r h_{j-r}}{\partial \tau^r}(i\tau) \beta^r = \sum_{r=0}^j \binom{\mu-j+r}{r} (i\tau)^{-r} h_{j-r}(i\tau) \beta^r. \quad \square \end{aligned}$$

Remark 1.5.14 *If functions $h_k \in S^{(\mu-k)}(\mathbb{R} \setminus \{0\})$ for $k = 0, 1, 2, \dots$ are given, then there exists an $h \in \mathcal{M}_{\text{as}}^\mu$ such that*

$$(1.5.22) \quad h_k(i\tau) = \sigma_\psi^{\mu-k}(h|_{\Gamma_0})$$

holds for all k . In fact, let $g \in S_{\text{cl}}^\mu(\Gamma_0)$ be such that $g(i\tau) \sim \sum_{k=0}^\infty h_k(i\tau)$ in the sense of asymptotic summation in $S_{\text{cl}}^\mu(\Gamma_0)$. Then $h(z) = H(\psi)g \in \mathcal{M}_{\mathcal{O}}^\mu$, where $H(\psi)$ is the kernel cut-off operator, cf. (1.3.3), possesses property (1.5.22), since $h|_{\Gamma_0} - g \in S^{-\infty}(\Gamma_0)$. In particular, (1.5.20) holds for such an h .

For an operator T of the form (1.5.17), we write

$$p_r(z) = \alpha_{0r}z^l + \alpha_{1r}z^{l-1} + \dots + \alpha_{l-1r}z + \alpha_{lr}$$

with uniquely determined coefficients $\alpha_{kr} \in \mathbb{C}$, $0 \leq k \leq l$.

Lemma 1.5.15 *An operator T of the form (1.5.17) belongs to \mathfrak{M}_l if and only if, for each k , $0 \leq k < l$,*

$$(1.5.23) \quad \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & l+1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{l-k-1} & \dots & (l+1)^{l-k-1} \end{pmatrix} \begin{pmatrix} \alpha_{k0} \\ \alpha_{k1} \\ \vdots \\ \alpha_{kl} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Proof. Condition (1.5.23) is obviously equivalent to

$$(1.5.24) \quad \sum_{r=0}^l (\beta - r)^s \alpha_{kr} = 0$$

for all $\beta \in \mathbb{R}$, $s \in \mathbb{N}$, $0 \leq s < l - k$.

Now let $h \in \mathcal{M}_{\text{as}}^\mu$,

$$\sigma_\psi^{\mu-j}(h) = \sum_{s=0}^j b_{js}(i\tau) \beta^s, \quad j = 0, 1, 2, \dots$$

with $b_{js} \in S^{(\mu-j)}(\mathbb{R} \setminus \{0\})$, $b_{js}(i\tau) = \binom{\mu-j+s}{s} (i\tau)^{-s} d_{j-s}(i\tau)$, $0 \leq s \leq j$, for certain $d_s \in S^{(\mu-s)}(\mathbb{R} \setminus \{0\})$, according to Lemma 1.5.13. Then direct computation gives

$$\begin{aligned} \sigma_\psi^{\mu+l-j}(Th) &= \sum_{k=0}^{\max\{j,l\}} \sum_{s=0}^{j-k} \sum_{v=0}^{j-k-s} C_{kv}(\beta) \left[\sum_{r=0}^l (\beta - r)^s \alpha_{kr} \right] b_{j-k-v,s}(i\tau) (i\tau)^{l-k-v} \\ &= \sum_{k=0}^{\max\{j,l\}} \sum_{s=0}^{j-k} \sum_{v=0}^{j-k-s} \tilde{C}_{jksv}(\beta) \left[\sum_{r=0}^l (\beta - r)^s \alpha_{kr} \right] d_{j-k-v-s}(i\tau) (i\tau)^{l-k-v-s}, \end{aligned}$$

where

$$C_{kv}(\beta) = \binom{l-k}{v} \beta^v, \quad \tilde{C}_{jksv}(\beta) = \binom{\mu-j+k+v+s}{s} C_{kv}(\beta).$$

By Remark 1.5.14, $\sigma_\psi^{\mu+l-j}(Th)$ vanishes for all $0 \leq j < l$ and all $h \in \mathcal{M}_{\text{as}}^\mu$, i.e., $Th \in \mathcal{M}_{\text{as}}^\mu$ for all $h \in \mathcal{M}_{\text{as}}^\mu$, if and only if (1.5.24) is valid for all $0 \leq k < l$. \square

Lemma 1.5.16 *For a fixed k , $0 \leq k < l$, the general solution to the system (1.5.23) is*

$$(1.5.25) \quad \alpha_{kr} = (-1)^r \sum_{s=0}^{l-k} \binom{l-k}{s} \gamma_{k,r-s}, \quad 0 \leq r \leq l,$$

where $\gamma_{k0}, \gamma_{k1}, \dots, \gamma_{kk} \in \mathbb{C}$ are arbitrary numbers. Thereby, in (1.5.25) we have set $\gamma_{kr} = 0$ if $r \notin \{0, 1, \dots, k\}$.

Proof. First of all, observe that

$$(1.5.26) \quad \sum_{r=0}^l (-1)^r (a+r)^h \binom{l}{r} = 0$$

for all $a \in \mathbb{Z}$, $h \in \mathbb{N}$, and $0 \leq h < l$. Indeed, since

$$\sum_{r=0}^l (-1)^r (a+r)^h \binom{l}{r} = \sum_{s=0}^h \left[\sum_{r=0}^l (-1)^r r^s \binom{l}{r} \right] a^{h-s} \binom{h}{s},$$

it suffices to prove (1.5.26) for $a = 0$ and all h , $0 \leq h < l$. For $h = 0$, (1.5.26) holds, for

$$\sum_{r=0}^l (-1)^r \binom{l}{r} = (1-1)^l = 0.$$

For $h > 0$, there are integers δ_s , $s = 0, \dots, h-1$, such that

$$\begin{aligned} \sum_{r=0}^l (-1)^r r^h \binom{l}{r} + \sum_{s=0}^{h-1} \delta_s \left[\sum_{r=0}^l (-1)^r r^s \binom{l}{r} \right] &= \sum_{r=0}^l (-1)^r \left[r^h + \sum_{s=0}^{h-1} \delta_s r^s \right] \binom{l}{r} \\ &= \sum_{r=0}^l (-1)^r r(r-1)\dots(r-h+1) \binom{l}{r} = \sum_{r=h}^l (-1)^r \frac{r!}{(r-h)!} \binom{l}{r} \\ &= (-1)^h \frac{l!}{(l-h)!} \sum_{r=h}^l (-1)^{r-h} \binom{l-h}{r-h} = 0. \end{aligned}$$

Thus (1.5.26) follows by induction on h .

Now choosing, for some $s \in \{0, 1, \dots, k\}$, $\gamma_{kr} = \delta_{rs} \gamma_{ks}$, where δ_{rs} is the Kronecker symbol, (1.5.26) shows that (1.5.25) is the superposition of $k+1$ linearly independent solutions to (1.5.23), i.e., (1.5.25) is a solution to (1.5.23) in which the $k+1$ parameters $\gamma_{k0}, \gamma_{k1}, \dots, \gamma_{kk}$ enter freely.

But (1.5.23) is an under-determined linear system in the unknowns $\alpha_{k0}, \alpha_{k1}, \dots, \alpha_{kl}$ whose coefficient matrix has rank $l-k$ (Vandermond's determinant). Thus its general solution contains $k+1$ free parameters. \square

Theorem 1.5.17 *For $j, r \in \mathbb{N}$, $j \geq r$, the operators $T_r^j(\mu, \kappa)$ belong to \mathfrak{M}_{j-r} .*

Beginning of proof of Theorem 1.5.17. Since we already know $T_r^j(\mu, \kappa): \mathcal{M}_{\text{as}}^\mu \rightarrow \mathcal{M}_{\text{as}}^\mu$, it suffices to prove that $T_r^j(\mu, \kappa)$ possesses the form (1.5.17). We proceed formally and consider an operator of the form $t^{-\mu} \text{op}_M^\gamma(f)$, where $f = f(t, z)$. A strict justification can be given by repeating the arguments in the proofs of Theorem 1.3.3 and Theorem 1.3.4.

The operator push-forward of the operator $t^{-\mu} \text{op}_M^\gamma(f)$ under χ is $r^{-\mu} \text{op}_M^\gamma(\tilde{f})$ with the double symbol

$$(1.5.27) \quad \tilde{f}(r, r', z) = \alpha(t)^\mu \left(\frac{\alpha(t)}{\alpha(t')} \right)^z f(t, z) \frac{\alpha(t')}{\chi'(t')} \Big|_{\substack{t=\chi^{-1}(r) \\ t'=\chi^{-1}(r')}}$$

where $\chi(t) = t\alpha(t)$. In particular, $\chi^{(j)}(t) = j\alpha^{(j-1)}(t) + t\alpha^{(j)}(t)$, $j \geq 1$, and

$$(1.5.28) \quad \alpha^{(j)}(0) = \frac{\chi^{(j+1)}(0)}{j+1}$$

for $j = 0, 1, 2, \dots$. The l th Mellin symbol of $r^{-\mu} \text{op}_M^\gamma(\tilde{f})$ equals

$$(1.5.29) \quad \sigma_M^{\mu-l}(r^{-\mu} \text{op}_M^\gamma(\tilde{f})) = \sum_{j+k=l} \frac{1}{j!k!} \frac{\partial^{j+k} \tilde{f}}{\partial r^j \partial r'^k}(0, 0, z-k).$$

We evaluate (1.5.27), (1.5.29) in a series of additional lemmas and remarks. \square

Definition 1.5.18 (a) Let \mathcal{R} denote the ring of all polynomials with rational coefficients in the commuting variables $x_0, x_1, x_2, \dots, y_1, y_2, \dots$. The group of variables x_0, x_1, x_2, \dots is also denoted by x , i.e., $x = (x_0, x_1, x_2, \dots)$, while the group of variables y_1, y_2, \dots is denoted by y , i.e., $y = (y_1, y_2, \dots)$. \mathcal{R} is equipped with a grading defined as follows: $q(x, y) \in \mathcal{R}_j$ for $j = 0, 1, 2, \dots$ if and only if $q(x, y)$ is a finite sum of monomials

$$\alpha x_0^{a_0} x_1^{a_1} \dots x_j^{a_j} y_1^{b_1} \dots y_j^{b_j},$$

where $\alpha \in \mathbb{Q}$ and $(a_1 + b_1) + 2(a_2 + b_2) + \dots + j(a_j + b_j) = j$. The graded derivation ∂ on \mathcal{R} , $\partial: \mathcal{R}_j \rightarrow \mathcal{R}_{j+1}$, is defined on generators by

$$\begin{aligned} x_j &\mapsto x_{j+1} - x_j y_1, & j = 0, 1, 2, \dots, \\ y_j &\mapsto y_{j+1} - x_j y_1, & j = 1, 2, \dots \end{aligned}$$

(b) \mathcal{R}' , that is now a ring of polynomials in the variables $x' = (x'_0, x'_1, x'_2, \dots)$, $y' = (y'_1, y'_2, \dots)$, is a second copy of \mathcal{R} . It is equipped with the graded derivation ∂' . Further we consider the tensor product $\tilde{\mathcal{R}} = \mathcal{R} \otimes \mathcal{R}'$ which is a bi-graded ring with the natural bi-grading $\tilde{\mathcal{R}}_{jk} = \mathcal{R}_j \otimes \mathcal{R}'_k$, $j, k = 0, 1, 2, \dots$

The reader should keep in mind that in the following computations the variable x_j stands for $\frac{\alpha^{(j)}(t)}{\chi'(t)}$, while y_j stands for $\frac{\chi^{(j+1)}(t)}{\chi'(t)}$. The consideration of \mathcal{R}' is necessary, since in the computation of $\frac{\partial^{j+k} \tilde{f}}{\partial r^j \partial r'^k}(r, r', z)$ quotients $\frac{\alpha^{(j)}(t')}{\chi'(t')}$, $\frac{\chi^{(j+1)}(t')}{\chi'(t')}$ are also encountered. Notice that with respect to the bi-grading of $\tilde{\mathcal{R}}$ the derivation ∂ is of type $(1, 0)$, while ∂' is of type $(0, 1)$.

Lemma 1.5.19 For any $q(x, y) \in \mathcal{R}$,

$$(1.5.30) \quad \begin{aligned} \frac{d}{dt} q \left(\frac{\alpha(t)}{\chi'(t)}, \dots, \frac{\alpha^{(j-1)}(t)}{\chi'(t)}, \frac{\chi''(t)}{\chi'(t)}, \dots, \frac{\chi^{(j)}(t)}{\chi'(t)} \right) &= \\ &= (\partial q) \left(\frac{\alpha(t)}{\chi'(t)}, \dots, \frac{\alpha^{(j)}(t)}{\chi'(t)}, \frac{\chi''(t)}{\chi'(t)}, \dots, \frac{\chi^{(j+1)}(t)}{\chi'(t)} \right) \end{aligned}$$

A similar statement holds for polynomials in \mathcal{R}' , the derivation ∂' , and derivatives with respect to t' .

Proof. This follows from

$$\begin{aligned}\frac{d}{dt} \left(\frac{\alpha^{(j)}(t)}{\chi'(t)} \right) &= \frac{\alpha^{(j+1)}(t)}{\chi'(t)} - \frac{\alpha^{(j)}(t)\chi''(t)}{\chi'(t)^2}, \\ \frac{d}{dt} \left(\frac{\chi^{(j+1)}(t)}{\chi'(t)} \right) &= \frac{\chi^{(j+2)}(t)}{\chi'(t)} - \frac{\chi^{(j+1)}(t)\chi''(t)}{\chi'(t)^2}\end{aligned}$$

and the definition of ∂ . □

Lemma 1.5.20 *We have*

$$(1.5.31) \quad \frac{\partial^{j+k} \tilde{f}}{\partial r^j \partial r'^k}(r, r', z) = \alpha(t)^{\mu-j-k} \left(\frac{\alpha(t)}{\alpha(t')} \right)^{z+k} \times \\ \sum_{0 \leq r \leq j} q_{jkr} \left(z + \mu, z; \frac{\alpha(t)}{\chi'(t)}, \dots, \frac{\chi''(t)}{\chi'(t)}, \dots, \frac{\alpha(t')}{\chi'(t')}, \dots, \frac{\chi''(t')}{\chi'(t')}, \dots \right) \frac{\partial^r f}{\partial t^r}(t, z)$$

for certain polynomials $q_{jkr}(\mu, z; x, y, x', y') \in \mathbb{Z}[\mu, z] \otimes \tilde{\mathcal{R}}_{j-r, k}$, where $t = \chi^{-1}(r)$, $t' = \chi^{-1}(r')$. Moreover, $q_{jkr}(\mu, z)$ is of degree j in μ and of degree k in z .

Proof. Define $q_{000}(\mu, z) = x'_0$,

$$q_{j+1kr}(\mu, z) = \begin{cases} (\mu - j) x_1 q_{jk0}(\mu, z) + x_0 (\partial q_{jk0})(\mu, z) & \text{if } r = 0, \\ (\mu - j) x_1 q_{jkr}(\mu, z) + x_0 (\partial q_{jkr})(\mu, z) + x_0 q_{jkr-1}(\mu, z) & \text{if } 1 \leq r \leq j, \\ x_0 q_{jkj}(\mu, z) & \text{if } r = j + 1, \end{cases}$$

and

$$q_{jk+1r}(\mu, z) = -(z + k) x'_1 q_{jkr}(\mu, z) + x'_0 (\partial' q_{jkr})(\mu, z).$$

The polynomials $q_{jkr}(\mu, z)$ are well-defined, since the operators

$$(\mu - j)x_1 + x_0\partial, \quad -(z + k)x'_1 + x'_0\partial'$$

and

$$x_0, \quad -(z + k)x'_1 + x'_0\partial'$$

commute. Moreover, $q_{jkr}(\mu, z) \in \mathbb{Z}[\mu, z] \otimes \tilde{\mathcal{R}}_{j-r, k}$, and $q_{jkr}(\mu, z)$ is of degree j in μ and of degree k in z .

Then (1.5.31) holds for $j = k = 0$, and its validity for general j, k follows from an inductive argument: we have $\frac{\partial}{\partial r} = \frac{1}{\chi'(t)} \frac{\partial}{\partial t}$, $\frac{\partial}{\partial r'} = \frac{1}{\chi'(t')} \frac{\partial}{\partial t'}$ as well as

$$\begin{aligned}\frac{1}{\chi'(t)} \frac{\partial}{\partial t} \left(\alpha(t)^{\mu-j-k} \left(\frac{\alpha(t)}{\alpha(t')} \right)^{z+k} q_{jkr}(z + \mu, z) \frac{\partial^r f}{\partial t^r}(t, z) \right) \\ = (\mu - j - k) \alpha(t)^{\mu-j-k-1} \frac{\alpha'(t)}{\chi'(t)} \left(\frac{\alpha(t)}{\alpha(t')} \right)^{z+k} q_{jkr}(z + \mu, z) \frac{\partial^r f}{\partial t^r}(t, z)\end{aligned}$$

$$\begin{aligned}
& + \alpha(t)^{\mu-j-k-1}(z+k) \left(\frac{\alpha(t)}{\alpha(t')} \right)^{z+k} \frac{\alpha'(t)}{\chi'(t)} q_{jkr}(z+\mu, z) \frac{\partial^r f}{\partial t^r}(t, z) \\
& + \alpha(t)^{\mu-j-k-1} \left(\frac{\alpha(t)}{\alpha(t')} \right)^{z+k} \frac{\alpha(t)}{\chi'(t)} (\partial q_{jkr})(z+\mu, z) \frac{\partial^r f}{\partial t^r}(t, z) \\
& + \alpha(t)^{\mu-j-k-1} \left(\frac{\alpha(t)}{\alpha(t')} \right)^{z+k} q_{jkr}(z+\mu, z) \frac{\alpha(t)}{\chi'(t)} \frac{\partial^{r+1} f}{\partial t^{r+1}}(t, z) \\
& = \alpha(t)^{\mu-j-k-1} \left(\frac{\alpha(t)}{\alpha(t')} \right)^{z+k} \left\{ \left((z+\mu-j) \frac{\alpha'(t)}{\chi'(t)} q_{jkr}(z+\mu, z) \right. \right. \\
& \quad \left. \left. + \frac{\alpha(t)}{\chi'(t)} (\partial q_{jkr})(z+\mu, z) \right) \frac{\partial^r f}{\partial t^r}(t, z) + q_{jkr}(z+\mu, z) \frac{\alpha(t)}{\chi'(t)} \frac{\partial^{r+1} f}{\partial t^{r+1}}(t, z) \right\}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{\chi'(t')} \frac{\partial}{\partial t'} \left(\alpha(t)^{\mu-j-k} \left(\frac{\alpha(t)}{\alpha(t')} \right)^{z+k} q_{jkr}(z+\mu, z) \frac{\partial^r f}{\partial t^r}(t, z) \right) \\
& = -\alpha(t)^{\mu-j-k-1}(z+k) \left(\frac{\alpha(t)}{\alpha(t')} \right)^{z+k+1} \frac{\alpha'(t')}{\chi'(t')} q_{jkr}(z+\mu, z) \frac{\partial^r f}{\partial t^r}(t, z) \\
& \quad + \alpha(t)^{\mu-j-k-1} \left(\frac{\alpha(t)}{\alpha(t')} \right)^{z+k+1} \frac{\alpha(t')}{\chi'(t')} (\partial' q_{jkr})(z+\mu, z) \frac{\partial^r f}{\partial t^r}(t, z) \\
& = \alpha(t)^{\mu-j-k-1}(z+k) \left(\frac{\alpha(t)}{\alpha(t')} \right)^{z+k+1} \left(-(z+k) \frac{\alpha'(t')}{\chi'(t')} q_{jkr}(z+\mu, z) \right. \\
& \quad \left. + \frac{\alpha(t')}{\chi'(t')} (\partial' q_{jkr})(z+\mu, z) \right) \frac{\partial^r f}{\partial t^r}(t, z),
\end{aligned}$$

which concludes the proof. \square

End of proof of Theorem 1.5.17. As a consequence of the foregoing lemma we obtain that

$$\begin{aligned}
(1.5.32) \quad & \frac{1}{j!k!} \frac{\partial^{j+k} \tilde{f}}{\partial r^j \partial r'^k}(0, 0, z) = \\
& = \chi'(0)^{\mu-j-k} \sum_{0 \leq r \leq j} p_{jkr} \left(z+\mu, z; \frac{\chi''(0)}{\chi'(0)}, \dots, \frac{\chi^{(j+k-r+1)}(0)}{\chi'(0)} \right) \frac{1}{r!} \frac{\partial^r f}{\partial t^r}(0, z),
\end{aligned}$$

where

$$(1.5.33) \quad p_{jkr}(\mu, z; y) = \frac{r!}{j!k!} q_{jkr}(\mu, z; x, y, x', y') \Big|_{\substack{x_0=x'_0=1, \\ x_p=x'_p=\frac{y_p}{p+1}, p \geq 1, \\ y'_p=y_p}},$$

regarding (1.5.28). In particular, $p_{jkr}(\mu, z; y) \in \mathbb{Z}[\mu, z] \otimes \mathcal{R}_{j-r+k}$, $p_{jkr}(\mu, z; y)$ is of degree j in μ and of degree k in z , and p_{j00} is independent of z , i.e.,

$$(1.5.34) \quad p_{j00}(\mu, z; y) = p_{j00}(\mu; y).$$

Furthermore, by (1.5.29)

$$(1.5.35) \quad T_r^l(\mu, \chi)\sigma(z) = \sum_{\substack{j+k=l, \\ j \geq r}} p_{jkr} \left(z+\mu-k, z-k; \frac{\chi''(0)}{\chi'(0)}, \dots, \frac{\chi^{(j+k-r+1)}(0)}{\chi'(0)} \right) \sigma(z-k)$$

showing that $T_r^l(\mu, \chi)$ actually possesses the form (1.5.17). \square

We still need the following facts.

Lemma 1.5.21 *Let $T(\mu) \in \mathfrak{M}_l$, $\mu \in \mathbb{R}$, be a solution to the functional equation*

$$(1.5.36) \quad T(\mu + \rho) \{ \sigma_0(z + \rho) \sigma_1(z) \} = (T(\mu) \sigma_0)(z + \rho) \sigma_1(z) + \sigma_0(z + \rho - l) (T(\rho) \sigma_1)(z),$$

which should hold for all meromorphic $\sigma_0(z), \sigma_1(z)$ and all $\mu, \rho \in \mathbb{R}$. Then there are uniquely determined constants $\bar{\gamma}, \bar{\beta} \in \mathbb{C}$ such that

$$(1.5.37) \quad T(\mu) = \bar{\gamma} (\Delta_l + \mu) + S(\bar{\beta})$$

for all $\mu \in \mathbb{R}$, where $S(\bar{\beta})\sigma(z) = \bar{\beta}(\sigma(z) - \sigma(z - l))$.

Proof. By Lemma 1.5.7, $\bar{\gamma} (\Delta_l + \mu)$ is a solution to (1.5.36). Then adding $\bar{\beta} \sigma_0(z + \rho) \sigma_1(z) - \bar{\beta} \sigma_0(z + \rho - l) \sigma_1(z - l)$ to both sides of (1.5.11), we see that also (1.5.37) is a solution to (1.5.36).

We show the reverse direction. By (1.5.17), we may write

$$T(\mu) \sigma(z) = \sum_{r=0}^l p_r(\mu, z) \sigma(z - r),$$

where $p_r(\mu, z)$ are certain polynomials in z of degree l depending on the additional parameter μ . Therefore, (1.5.36) gives

$$\begin{aligned} \sum_{r=0}^l p_r(\mu + \rho, z) \sigma_0(z + \rho - r) \sigma_1(z - r) &= \\ &= \sum_{r=0}^l p_r(\mu, z + \rho) \sigma_0(z + \rho - r) \sigma_1(z) + \sum_{r=0}^l p_r(\rho, z) \sigma_0(z + \rho - l) \sigma_1(z - r). \end{aligned}$$

Comparison of coefficients immediately yields that in the sums in this equality all summands with $0 < r < l$ disappear. Thus

$$\begin{aligned} p_0(\mu + \rho, z) \sigma_0(z + \rho) \sigma_1(z) + p_l(\mu + \rho, z) \sigma_0(z + \rho - l) \sigma_1(z - l) &= \\ = p_0(\mu, z + \rho) \sigma_0(z + \rho) \sigma_1(z) + p_l(\mu, z + \rho) \sigma_0(z + \rho - l) \sigma_1(z) + \\ + p_0(\rho, z) \sigma_0(z + \rho - l) \sigma_1(z) + p_l(\rho, z) \sigma_0(z + \rho - l) \sigma_1(z - l) \end{aligned}$$

and

$$\begin{aligned} p_0(\mu + \rho, z) &= p_0(\mu, z + \rho), \\ p_l(\mu + \rho, z) &= p_l(\rho, z), \\ p_l(\mu, z + \rho) &= -p_0(\rho, z). \end{aligned}$$

We infer that $p_l(\mu, z)$ is actually independent of μ , i.e.,

$$(1.5.38) \quad p_0(\mu, z) = \bar{p}(z + \mu), \quad p_l(\mu, z) = -\bar{p}(z)$$

with a uniquely determined polynomial $\bar{p}(z)$ of degree at most l . The aim is to show that $\bar{p}(z) = \bar{\gamma}z + \bar{\beta}$ with uniquely determined constants $\bar{\gamma}, \bar{\beta} \in \mathbb{C}$.

By Lemma 1.5.16, there are constants $\gamma_{00}(\mu), \gamma_{10}(\mu), \gamma_{11}(\mu), \dots, \gamma_{l-1,0}(\mu), \dots, \gamma_{l-1,l-1}(\mu) \in \mathbb{C}$ and constants $\beta_0(\mu), \dots, \beta_l(\mu) \in \mathbb{C}$ such that

$$p_r(\mu, z) = (-1)^r \sum_{k=0}^{l-1} \left(\sum_{s=0}^{l-k} \binom{l-k}{s} \gamma_{k,r-s}(\mu) \right) z^{l-k} + \beta_r(\mu),$$

where again $\gamma_{ks}(\mu) = 0$ if $s \notin \{0, 1, \dots, k\}$. But $p_r(\mu, z) = 0$ for $1 \leq r \leq l-1$, therefore,

$$\sum_{s=0}^{l-k} \binom{l-k}{s} \gamma_{k, r-s}(\mu) = 0, \quad 1 \leq r \leq l-1,$$

for $0 \leq k < l$. This, however, is a linear system in the unknowns $\gamma_{k0}(\mu), \gamma_{k1}(\mu), \dots, \gamma_{kk}(\mu)$, with its coefficient matrix having rank $\min\{k+1, l-1\}$. Thus we conclude that

$$\gamma_{k0}(\mu) = \gamma_{k1}(\mu) = \dots = \gamma_{kk}(\mu) = 0$$

if $k < l-1$ and, moreover,

$$\gamma_{l-1, r}(\mu) = (-1)^r \bar{\gamma}(\mu), \quad 0 \leq r \leq l-1,$$

for a certain $\bar{\gamma}(\mu) \in \mathbb{C}$. In particular,

$$p_0(\mu, z) = \bar{\gamma}(\mu)z + \beta_0(\mu), \quad p_l(\mu, z) = -\bar{\gamma}(\mu)z + \beta_l(\mu).$$

By (1.5.38), $\bar{\gamma}(\mu) = \bar{\gamma}$ and $\beta_l(\mu) = -\bar{\beta}$ are independent of μ , and $\beta_0(\mu) = \bar{\gamma}\mu + \bar{\beta}$, i.e.,

$$p_0(\mu, z) = \bar{\gamma}(z + \mu) + \bar{\beta}, \quad p_l(\mu, z) = -\bar{\gamma}z - \bar{\beta},$$

and $T(\mu) = \bar{\gamma}(\Delta_l + \mu) + S(\bar{\beta})$. \square

Remark 1.5.22 For $T(\mu)$, $\mu \in \mathbb{R}$, as in Lemma 1.5.21, we have $T(\mu) = \bar{\gamma}(\Delta_l + \mu)$, i.e., $S(\bar{\beta}) = 0$, if and only if, for some $\mu \in \mathbb{R}$, $T(\mu)z$ is a polynomial in z of degree 1 without constant term.

Lemma 1.5.23 Let $\mu \in \mathbb{N}$. Then, for the operator $\omega(t)t^{-\mu} \in C^\mu(\mathbb{R}_+, \mathbf{g})$, $\mathbf{g} = (\gamma, \gamma - \mu, (-\infty, 0])$, we have

$$(1.5.39) \quad \sigma_M^{\mu-j}(\chi_*(\omega(t)t^{-\mu})) = p_{j00} \left(\mu; \frac{\chi''(0)}{\chi'(0)}, \frac{\chi^{(3)}(0)}{\chi'(0)}, \dots, \frac{\chi^{(j+1)}(0)}{\chi'(0)} \right)$$

for $j = 0, 1, 2, \dots$, where $p_{j00}(\mu; y_1, \dots, y_j) \in \mathcal{R}_j$ is the polynomial defined in (1.5.33), (1.5.34).

Proof. We use the calculations leading to the proof of Theorem 1.5.17. The operator $\chi_*(\omega(t)t^{-\mu})$ equals $\omega(t)r^{-\mu}\alpha(t)^\mu$, where $r = \chi(t)$, $\chi(t) = t\alpha(t)$. Hence

$$\sigma_M^{\mu-j}(\chi_*(\omega(t)t^{-\mu})) = \frac{1}{j!} \frac{d^j}{dr^j} (\alpha(t)^\mu) \Big|_{r=0}.$$

But

$$\frac{d^j}{dr^j} (\alpha^\mu(t)) = \alpha(t)^{\mu-j} q_{j00} \left(\mu; \frac{\alpha(t)}{\chi'(t)}, \dots, \frac{\alpha^{(j-1)}(t)}{\chi'(t)}, \frac{\chi''(t)}{\chi'(t)}, \dots, \frac{\chi^{(j)}(t)}{\chi'(t)} \right)$$

for any $j = 0, 1, 2, \dots$. Indeed, this is true for $j = 0$. Moreover,

$$\begin{aligned} & \frac{1}{\chi'(t)} \frac{d}{dt} \left(\alpha(t)^{\mu-j} q_{j00} \left(\mu; \frac{\alpha(t)}{\chi'(t)}, \dots, \frac{\alpha^{(j-1)}(t)}{\chi'(t)}, \frac{\chi''(t)}{\chi'(t)}, \dots, \frac{\chi^{(j)}(t)}{\chi'(t)} \right) \right) \\ &= (\mu-j)\alpha(t)^{\mu-j-1} \frac{\alpha'(t)}{\chi'(t)} q_{j00} \left(\mu; \frac{\alpha(t)}{\chi'(t)}, \dots, \frac{\alpha^{(j-1)}(t)}{\chi'(t)}, \frac{\chi''(t)}{\chi'(t)}, \dots, \frac{\chi^{(j)}(t)}{\chi'(t)} \right) \\ & \quad + \alpha(t)^{\mu-j-1} \frac{\alpha(t)}{\chi'(t)} (\partial q_{j00}) \left(\mu; \frac{\alpha(t)}{\chi'(t)}, \dots, \frac{\alpha^{(j)}(t)}{\chi'(t)}, \frac{\chi''(t)}{\chi'(t)}, \dots, \frac{\chi^{(j+1)}(t)}{\chi'(t)} \right) \\ &= \alpha(t)^{\mu-j-1} q_{j+1,00} \left(\mu; \frac{\alpha(t)}{\chi'(t)}, \dots, \frac{\alpha^{(j)}(t)}{\chi'(t)}, \frac{\chi''(t)}{\chi'(t)}, \dots, \frac{\chi^{(j+1)}(t)}{\chi'(t)} \right), \end{aligned}$$

and (1.5.39) follows. \square

Lemma 1.5.24 For $-\partial_t \in C^1(\mathbb{R}_+, \mathbf{g})$, $\mathbf{g} = (\gamma, \gamma - 1, (-\infty, 0])$, i.e.,

$$(1.5.40) \quad \sigma_M^{1-j}(-\partial_t)(z) = \begin{cases} z & \text{if } j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

we have that, for each $j \in \mathbb{N}$, $\sigma_M^{1-j}(\chi_*(-\partial_t))(z)$ is a polynomial in z of degree 1 without constant term.

Proof. We simply have $\chi_*(-\partial_t) = -\chi'(t)\partial_r = r^{-1}(\chi'(t)(-r\partial_r))$, where $r = \chi(t)$. Hence

$$\sigma_M^{1-j}(\chi_*(-\partial_t))(z) = \frac{1}{j!} \frac{d^j}{dr^j} (\chi'(t)) \Big|_{r=0} z$$

for all $j = 0, 1, 2, \dots$ □

Proof of Theorem 1.5.1. First we show that there are uniquely determined coefficients $\bar{q}_1, \bar{q}_2, \bar{q}_3, \dots$ such that (1.5.1), with j replaced by l , holds for all $l \in \mathbb{N}$. The dependence of \bar{q}_j on χ is determined afterwards.

We proceed by induction on l . For $l = 0$, we are already done by (ii) of Theorem 1.4.1.

Assume that the proof has been supplied up to $l - 1$, for some $l \geq 1$. Especially, the coefficients $\bar{q}_1, \bar{q}_2, \dots, \bar{q}_{l-1}$ have been calculated. We then consider the operators

$$\tilde{T}_0^l(\mu, \chi) = T_0^l(\mu, \chi) - \sum_{a_1+2a_2+\dots+(l-1)a_{l-1}=l} \frac{\bar{q}_1^{a_1} \bar{q}_2^{a_2} \dots \bar{q}_{l-1}^{a_{l-1}}}{a_1! a_2! \dots a_{l-1}!} \Delta[\mu; a_1, a_2, \dots, a_{l-1}],$$

$\mu \in \mathbb{R}$, i.e., we subtract from $T_0^l(\mu, \chi)$ the operators corresponding to the choice $\bar{q}_l = 0$ in Proposition 1.5.10. In particular, $\tilde{T}_0^l(\mu, \chi) \in \mathfrak{M}_l$ and

$$\begin{aligned} \tilde{T}_0^l(\mu + \rho, \chi) \{ \sigma_0(z + \rho) \sigma_1(z) \} &= \sum_{j+k=l} (T_0^j(\mu, \chi) \sigma_0)(z + \rho - k) (T_0^k(\rho, \chi) \sigma_1)(z) \\ &- \sum_{j+k=l} \left\{ \sum_{\substack{a_1+2a_2+\dots+ja_j=j, \\ a_j=0 \text{ if } j=l}} \frac{\bar{q}_1^{a_1} \bar{q}_2^{a_2} \dots \bar{q}_j^{a_j}}{a_1! a_2! \dots a_j!} \Delta[\mu; a_1, a_2, \dots, a_j] \sigma_0(z + \rho - k) \right\} \\ &\quad \times \left\{ \sum_{\substack{b_1+2b_2+\dots+kb_k=k, \\ b_k=0 \text{ if } k=l}} \frac{\bar{q}_1^{b_1} \bar{q}_2^{b_2} \dots \bar{q}_k^{b_k}}{b_1! b_2! \dots b_k!} \Delta[\rho; b_1, b_2, \dots, b_k] \sigma_1(z) \right\} \end{aligned}$$

by Lemma 1.5.6 and Proposition 1.5.10, respectively,

$$= (\tilde{T}_0^l(\mu, \chi) \sigma_0)(z + \rho) \sigma_1(z) + \sigma_0(z + \rho - l) (\tilde{T}_0^l(\rho, \chi) \sigma_1)(z)$$

by the inductive hypothesis. Thus $\tilde{T}_0^l(\mu, \chi)$ solves (1.5.36) and, therefore, there exist uniquely determined constants $\bar{q}_l = \bar{q}_l(\chi) \in \mathbb{C}$, $\bar{\beta}_l = \bar{\beta}_l(\chi) \in \mathbb{C}$, such that

$$\tilde{T}_0^l(\mu, \chi) = \bar{q}_l(\Delta_l + \mu) + S(\bar{\beta}_l)$$

by Lemma 1.5.21. By Lemma 1.5.24 and Remark 1.5.22, however, $\bar{\beta}_l(\chi) = 0$ showing that $\tilde{T}_0^l(\mu, \chi) = \bar{q}_l(\Delta_l + \mu)$. This proves (1.5.1).

In order to conclude the general form of $\bar{q}_l(\chi)$ we invoke Lemma 1.5.23. This yields

$$\sigma_M^{\mu-l}(\chi_*(\omega(t)t^{-\mu}))(z) = T_0^l(\mu, \chi)1,$$

since

$$\sigma_M^{\mu-j}(\omega(t)t^{-\mu})(z) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

i.e.,

$$(1.5.41) \quad p_{l00} \left(\mu; \frac{\chi''(0)}{\chi'(0)}, \dots, \frac{\chi^{(l+1)}}{\chi'(0)} \right) = \mu \bar{q}_l(\chi) \\ + \sum_{a_1+2a_2+\dots+(l-1)a_{l-1}=l} \frac{\bar{q}_1(\chi)^{a_1} \bar{q}_2(\chi)^{a_2} \dots \bar{q}_{l-1}(\chi)^{a_{l-1}}}{a_1! a_2! \dots a_{l-1}!} \Delta[\mu; a_1, a_2, \dots, a_{l-1}] 1$$

showing that $\bar{q}_l(\chi)$ has the desired form, i.e.,

$$\bar{q}_l(\chi) = q_l \left(\frac{\chi''(0)}{\chi'(0)}, \dots, \frac{\chi^{(l+1)}}{\chi'(0)} \right)$$

for a certain $q_l(y_1, \dots, y_l) \in \mathcal{R}_l$, for (1.5.2) is known to be true for $\bar{q}_1(\chi), \dots, \bar{q}_{l-1}(\chi)$. Note that $\Delta[\mu; a_1, a_2, \dots, a_{l-1}] 1$ is a constant depending on μ , but $\bar{q}_l(\chi)$ is independent of μ .

This completes the proof of Theorem 1.5.1. \square

2. OPERATORS ON HIGHER-DIMENSIONAL CONES

2.1. The cone algebra. The cone algebra with discrete asymptotics on the stretched cone $X^\wedge = \mathbb{R}_+ \times X$ with base X of dimension n , where X is a closed C^∞ -manifold, is motivated as follows: Let

$$(2.1.1) \quad A = t^{\delta-\gamma} \sum_{j=0}^{\mu} a_j(t) \left(-t \frac{\partial}{\partial t} \right)^j$$

be a differential operator of Fuchs type, where $a_j(t) \in C^\infty(\overline{\mathbb{R}}_+, \text{Diff}^{\mu-j}(X))$. Here $\text{Diff}^{\mu-j}(X)$ is the space of all differential operators of order $\mu-j$ on X with smooth coefficients. Assume that A is elliptic (with respect to the symbols that are defined below). The pseudodifferential cone algebra in the sense of Definition 2.1.1 below solves the problem of expressing a parametrix of A and the asymptotics of solutions as an element of elliptic regularity, cf. [17]. The answer in terms of an operator algebra on X^\wedge is also necessary for applications to analogous problems on manifolds with edges and higher corners. In this context the cone algebra is the range of operator-valued symbols, where compositions and inverses are controlled within the calculus, cf. [16], [18]. Global constructions require coordinate changes that preserve the piecewise smooth geometry.

For the cone we consider diffeomorphisms of the form $\chi : \overline{\mathbb{R}}_+ \times X \rightarrow \overline{\mathbb{R}}_+ \times X$. The asymptotic data that are generated by operators A refer to a chosen splitting (t, x) of variables and depend on the global ‘‘spectral’’ behaviour along X of the so-called conormal symbols. These structures turn out to be rather sensitive under changes of coordinates. It is an important point to characterise the transformation rules for the ingredients of the cone algebra that govern asymptotics.

Asymptotics of solutions $u(t, x)$ to an equation $Au = f$, with f having asymptotics in a weighted Sobolev space, cf. the definitions below, and u belonging to a space with weight $\gamma \in \mathbb{R}$, have the form

$$(2.1.2) \quad u(t, x) \sim \sum_j \sum_{k=0}^{m_j} c_{jk}(x) t^{-p_j} \log^k t$$

as $t \rightarrow 0$, with a sequence

$$(2.1.3) \quad P = \{(p_j, m_j, L_j)\}_{0 \leq j \leq N}$$

for an $N = N(P) \leq \infty$, $\pi_{\mathbb{C}}P = \bigcup_{0 \leq j \leq N} \{p_j\} \subset \mathbb{C}$, $\operatorname{Re} p_j < \frac{n+1}{2} - \gamma$, $\operatorname{Re} p_j \rightarrow -\infty$ as $j \rightarrow \infty$ if $N(P) = \infty$, $m_j \in \mathbb{N}$, and finite-dimensional subspaces $L_j \subset C^\infty(X)$ with $e_{jk} \in L_j$ for all $0 \leq k \leq m_j$. Given a weight interval $\Theta = (\vartheta, 0]$, $-\infty \leq \vartheta < 0$, we denote by $\operatorname{As}(X, \mathbf{g})$ for $\mathbf{g} = (\gamma, \Theta)$ the set of all P with $\pi_{\mathbb{C}}P \subset \{z: \frac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} z < \frac{n+1}{2} - \gamma\}$.

Asymptotics of solutions to elliptic differential operators of Fuchs type are characterised in Kondrat'ev [8] (for boundary value problems). In this case the expansions (2.1.2) can be derived directly in terms of the non-bijectivity points of the principal conormal symbol

$$(2.1.4) \quad \sigma_M^{\gamma-\delta}(A)(z) = \sum_{j=0}^{\mu} a_j(0)z^j: H^s(X) \rightarrow H^{s-\mu}(X),$$

with z varying in \mathbb{C} and $H^s(X)$ being the standard Sobolev space on X of smoothness $s \in \mathbb{R}$.

Denote by $L_{\text{cl}}^\mu(X)$ the space of all classical pseudo-differential operators on X of order μ . Further let $L_{\text{cl}}^\mu(X; \mathbb{R}^l)$ be the space of all classical parameter-dependent pseudo-differential operators on X of order μ , with parameter $\lambda \in \mathbb{R}^l$, $l \in \mathbb{N}$. We also write $L_{\text{cl}}^\mu(X; \Gamma_\beta)$ when \mathbb{R} is replaced by Γ_β . The spaces $L_{\text{cl}}^\mu(X)$, $L_{\text{cl}}^\mu(X; \mathbb{R}^l)$ are endowed with natural Fréchet topologies.

Let $\mathcal{M}_{\mathcal{O}}^\mu(X)$ be the space of all $h(z) \in \mathcal{A}(\mathbb{C}, L_{\text{cl}}^\mu(X))$ such that $h|_{\Gamma_\beta} \in L_{\text{cl}}^\mu(X; \Gamma_\beta)$ for all $\beta \in \mathbb{R}$, uniformly for β in compact intervals. Inverses to elliptic elements in $\mathcal{M}_{\mathcal{O}}^\mu(X)$, cf. the definition below, are meromorphic, where poles, multiplicities, and Laurent coefficients are described by sequences

$$R = \{(r_j, n_j, H_j)\}_{j \in \mathbb{Z}}$$

with $\pi_{\mathbb{C}}R = \bigcup_{j \in \mathbb{Z}} \{r_j\} \subset \mathbb{C}$, $\operatorname{Re} r_j \rightarrow \mp\infty$ as $j \rightarrow \pm\infty$, $n_j \in \mathbb{N}$, and finite-dimensional subspaces $H_j \subset L^{-\infty}(X)$ of finite-rank operators. The set of all such R is denoted by $\mathbf{As}(X)$ and the elements are called discrete asymptotic types of Mellin symbols.

Given an $R \in \mathbf{As}(X)$, we define $\mathcal{M}_R^{-\infty}(X)$ to be the space of all meromorphic operator-valued functions $f(z) \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}}R, L^{-\infty}(X))$ with poles in r_j of multiplicities $n_j + 1$ and Laurent coefficients at $(z - r_j)^{-(k+1)}$ in H_j for all $j \in \mathbb{Z}$, $0 \leq k \leq n_j$, and $\pi(z)h(z)|_{\Gamma_\beta} \in L^{-\infty}(X; \Gamma_\beta)$ for all $\beta \in \mathbb{R}$ uniformly for β in compact intervals, and any $\pi_{\mathbb{C}}R$ -excision function $\pi(z)$. Furthermore, we set

$$\mathcal{M}_R^\mu(X) = \mathcal{M}_{\mathcal{O}}^\mu(X) + \mathcal{M}_R^{-\infty}(X).$$

All these spaces are Fréchet spaces in a natural way, cf. [18].

Choose a parameter-dependent elliptic element $R^s(\tau) \in L_{\text{cl}}^s(X; \mathbb{R})$, $s \in \mathbb{R}$, which induces isomorphisms $R^s(\tau): H^r(X) \rightarrow H^{r-s}(X)$ for all $r, \tau \in \mathbb{R}$. Then the weighted space $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s, \gamma \in \mathbb{R}$ is defined as the completion of $C_0^\infty(X^\wedge)$ with respect to the norm $\{(2\pi i)^{-1} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(\operatorname{Im} z)(Mu)(z)\|_{L^2(X)}^2 dz\}^{1/2}$, where the Mellin transform M refers to $t \in \mathbb{R}_+$ and u is regarded as an element in $C_0^\infty(\mathbb{R}_+, C^\infty(X))$.

For each $f(t, t', z) \in C^\infty(\overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+}, L_{\text{cl}}^\mu(X; \Gamma_{\frac{n+1}{2}-\gamma}))$ we can pass to continuous operators

$$\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega}: \mathcal{H}^{s,\gamma}(X^\wedge) \rightarrow \mathcal{H}^{s-\mu,\gamma}(X^\wedge),$$

$s \in \mathbb{R}$, with cut-off functions $\omega(t)$, $\tilde{\omega}(t)$. Set

$$\mathcal{K}^{s,\gamma}(X^\wedge) = \omega \mathcal{H}^{s,\gamma}(X^\wedge) + (1 - \omega)H_{\text{cone}}^s(X^\wedge),$$

where the space $H_{\text{cone}}^s(X^\wedge)$, away from $t = 0$, is modelled on the standard Sobolev spaces in \mathbb{R}_x^{n+1} , such that, in particular, $(1 - \omega)H_{\text{cone}}^s(\mathbb{R}_+ \times S^n) = (1 - \omega)H^s(\mathbb{R}_x^{n+1})$ for any cut-off function $\omega(t)$, where $t = |\tilde{x}|$, $x = \frac{\tilde{x}}{|\tilde{x}|}$. The straightforward extension of this definition for

arbitrary X can be found in [18]. We are mainly interested in the behaviour of operators near zero. However, to have a convenient algebra with ellipticity up to infinity we take a class $L^{\mu;\eta}(X^\wedge)$ with exit behaviour as $t \rightarrow \infty$ which is defined similarly to that in the one-dimensional case treated above, with $\mu \in \mathbb{R}$ being the order and $\eta \in \mathbb{R}$ being a power weight at infinity. The precise definition can be found in Schrohe [13], cf. also Dorschfeldt, Grieme, and Schulze [3]. Operators $A \in L^{\mu;0}(X^\wedge)$ give rise to continuous maps $(1 - \omega)A(1 - \tilde{\omega}) : \mathcal{K}^{s,\gamma}(X^\wedge) \rightarrow \mathcal{K}^{s-\mu,\delta}(X^\wedge)$ for arbitrary cut-off functions $\omega, \tilde{\omega}$, and $s, \gamma, \delta \in \mathbb{R}$.

Let us now describe the asymptotic contribution in the cone algebra on X^\wedge . Set $\mathcal{K}_\Theta^{s,\gamma}(X^\wedge) = \varprojlim_{\varepsilon > 0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(X^\wedge)$ for $\Theta = (\vartheta, 0]$, $-\infty \leq \vartheta < 0$. Especially, $\mathcal{K}_{(-\infty,0]}^{s,\gamma}(X^\wedge) = \mathcal{K}^{s,\infty}(X^\wedge)$. Moreover, for finite Θ and $P \in \text{As}(X, \mathbf{g})$, $\mathbf{g} = (\gamma, \Theta)$, we define $\mathcal{E}_P(X^\wedge)$ to be the linear span of all functions of the form $\omega(t)c_{jk}(x)t^{-pj} \log^k t$ for $j = 0, \dots, N$, $0 \leq k \leq m_j$, and $c_{jk} \in L_j$ for all j, k , cf. (2.1.3), and a fixed cut-off function ω . The space $\mathcal{E}_P(X^\wedge)$ is finite-dimensional. Set

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{E}_P(X^\wedge)$$

and equip this space with its natural Fréchet topology. This definition extends to $P \in \text{As}(X, \mathbf{g})$ for infinite Θ in a manner analogous to the one-dimensional case treated above. Furthermore, we set

$$\mathcal{S}_P^\gamma(X^\wedge) = \{\omega u + (1 - \omega)v; u \in \mathcal{K}_P^{\infty,\gamma}(X^\wedge), v \in \mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X))\}$$

which is also a Fréchet space.

An element $G \in L^{-\infty}(X^\wedge)$ is called a Green operator with discrete asymptotics if it induces continuous operators

$$(2.1.5) \quad G : \mathcal{K}^{s,\gamma}(X^\wedge) \rightarrow \mathcal{S}_P^\delta(X^\wedge), \quad G^* : \mathcal{K}^{s,-\delta}(X^\wedge) \rightarrow \mathcal{S}_Q^{-\gamma}(X^\wedge)$$

for arbitrary $s \in \mathbb{R}$ and certain asymptotic types $P \in \text{As}(X, (\delta, \Theta))$, $Q \in \text{As}(X, (-\gamma, \Theta))$ depending on G , with G^* being the formal adjoint with respect to the chosen scalar product in $\mathcal{K}^{0,0}(X^\wedge) = t^{-n/2}L^2(\mathbb{R}_+ \times X)$. The space of all operators of this form is denoted by $C_G(X^\wedge, (\gamma, \delta, \Theta))$. Moreover, given $\gamma, \delta \in \mathbb{R}$ and a weight interval $\Theta = (-(k+1), 0]$, $k \in \mathbb{N}$, we denote by $C_{M+G}(X^\wedge, (\gamma, \delta, \Theta))$ the space of all operators $M + G$, where $G \in C_G(X^\wedge, (\gamma, \delta, \Theta))$ and

$$(2.1.6) \quad M = \tilde{\omega} t^{\delta-\gamma} \sum_{j=0}^k t^j \text{op}_M^{\varrho_j - \frac{n}{2}}(f_j) \tilde{\omega}_0$$

for arbitrary $f_j(z) \in \mathcal{M}_{R_j}^{-\infty}(X)$ and $\varrho_j \in \mathbb{R}$, $\gamma - j \leq \varrho_j \leq \gamma$, with $\pi_{\mathbb{C}} R_j \cap \Gamma_{\frac{n+1}{2} - \varrho_j} = \emptyset$ for all j . Given two operators of the form (2.1.6) with the same conormal symbols f_j , $0 \leq j \leq k$, but different cut-off functions $\tilde{\omega}, \tilde{\omega}_0$ or shifts ϱ_j , their difference belongs to $C_G(X^\wedge, (\gamma, \delta, \Theta))$. The cone algebra with discrete asymptotics is the union of all spaces in the following definition.

Definition 2.1.1 *The space $C^\mu(X^\wedge, \mathbf{g})$ of all cone operators of order $\mu \in \mathbb{R}$ on $X^\wedge = \mathbb{R}_+ \times X$ with discrete asymptotics and weight data $\mathbf{g} = (\gamma, \delta, \Theta)$ for $\gamma, \delta \in \mathbb{R}$, $\Theta = (-(k+1), 0]$, $k \in \mathbb{N}$, is defined to be the set of all operators of the form*

$$(2.1.7) \quad A = \omega t^{\delta-\gamma} \text{op}_M^{\gamma - \frac{n}{2}}(h) \omega_0 + (1 - \omega)A_\infty(1 - \omega_1) + M + G$$

with cut-off functions $\omega, \omega_0, \omega_1$ satisfying $\omega \omega_0 = \omega$, $\omega \omega_1 = \omega_1$, and

- (i) $h(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_\Theta^\mu(X))$,
- (ii) $M + G \in C_{M+G}(X^\wedge, \mathbf{g})$,
- (iii) $A_\infty \in L_{\text{cl}}^\mu(\mathbb{R}_+ \times X) \cap L^{\mu;0}(\mathbb{R}_+ \times X)$.

The extension of this definition to the infinite weight interval Θ is analogous to the case of the half-line \mathbb{R}_+ .

An operator $A \in C^\mu(X^\wedge, \mathbf{g})$ for $\mathbf{g} = (\gamma, \delta, \Theta)$ induces continuous maps

$$A: \mathcal{K}^{s, \gamma}(X^\wedge) \rightarrow \mathcal{K}^{s-\mu, \delta}(X^\wedge)$$

and

$$A: \mathcal{K}_P^{s, \gamma}(X^\wedge) \rightarrow \mathcal{K}_Q^{s-\mu, \delta}(X^\wedge)$$

for all $s \in \mathbb{R}$ and $P \in \text{As}(X, (\gamma, \Theta))$ with some resulting $Q \in \text{As}(X, (\delta, \Theta))$.

The principal symbol structure of operators $A \in C^\mu(X^\wedge, \mathbf{g})$ is given by triples $\sigma(A) = (\tilde{\sigma}_\psi^\mu(A), \sigma_M^{\gamma-\delta}(A), \sigma_e^0(A))$. Here $\tilde{\sigma}_\psi^\mu(A)(t, x, \tilde{\tau}, \xi) = t^{\gamma-\delta} \sigma_\psi^\mu(A)(t, x, t^{-1}\tilde{\tau}, \xi)$, with $\sigma_\psi^\mu(A)$ being the homogeneous principal symbol of $A \in L_{\text{cl}}^\mu(X^\wedge)$. The principal conormal symbol of conormal order $\gamma-\delta$ is defined as above, cf. (1.1.8). The exit symbol $\sigma_e^0(A)$ stems from the calculus in $L^{\mu;0}(\mathbb{R} \times X)$ as $t \rightarrow \infty$, similarly to the one-dimensional case.

There are also lower order conormal symbols, given by the same expressions as in (1.1.10). Compositions of operators A, B in the cone algebra belong to the cone algebra again, where $\sigma(AB) = \sigma(A)\sigma(B)$, with component-wise composition of symbols. The lower order conormal symbols of compositions are given by (1.1.11).

Let $\chi: \overline{\mathbb{R}}_+ \times X \rightarrow \overline{\mathbb{R}}_+ \times X$ be a diffeomorphism and suppose that

$$(2.1.8) \quad |\partial_t^j \partial_x^\alpha \chi(t, x)| \leq C_{j\alpha} (1+t)^{1-j}$$

for all $j \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, in local coordinates on $x \in X$, with certain constants $C_{j\alpha} > 0$ and

$$(2.1.9) \quad |\chi(t, x)| \geq c_0 t$$

for some constant $c_0 > 0$. Then applying the push-forward under χ of standard pseudo-differential operators on X^\wedge , i.e., $\chi_*: L^{\mu;0}(X^\wedge) \rightarrow L^{\mu;0}(X^\wedge)$, we can ask for the invariance of the subspace $C^\mu(X^\wedge, \mathbf{g})$.

Theorem 2.1.2 *The operator push-forward under χ induces an isomorphism*

$$\chi_*: C^\mu(X^\wedge, \mathbf{g}) \rightarrow C^\mu(X^\wedge, \mathbf{g})$$

for all $\mu \in \mathbb{R}$ and $\mathbf{g} = (\gamma, \delta, \Theta)$. Moreover, we have $\chi_*\sigma(A) = \sigma(\chi_*A)$ under a canonical push-forward χ_* on the symbol algebra.

2.2. Spaces with asymptotics and Green operators. Let $\chi: \overline{\mathbb{R}}_+ \times X \rightarrow \overline{\mathbb{R}}_+ \times X$ be a diffeomorphism. Write

$$(2.2.1) \quad (r, y) = \chi(t, x) = (\sigma(t, x), \kappa(t, x)) = (\sigma(t, x), \kappa_1(t, x), \dots, \kappa_n(t, x));$$

then $\sigma(0, x) = 0$ and $\partial_t \sigma(0, x) > 0$ for all $x \in X$. Moreover, $\underline{\kappa}: X \rightarrow X$ defined by $\underline{\kappa}(x) = \kappa(0, x)$ is a diffeomorphism.

Proposition 2.2.1 *Let $\chi: \overline{\mathbb{R}}_+ \times X \rightarrow \overline{\mathbb{R}}_+ \times X$ be a diffeomorphism satisfying (2.1.8), (2.1.9). Then the pull-back under χ induces an isomorphism $\chi^*: \mathcal{K}^{s, \gamma}(X^\wedge) \rightarrow \mathcal{K}^{s, \gamma}(X^\wedge)$ for all $s, \gamma \in \mathbb{R}$.*

Proof. See the proof of Proposition 1.2.1. □

Proposition 2.2.2 *Let $P \in \text{As}(X, \mathbf{g})$ for $\mathbf{g} = (\gamma, \Theta)$, $\gamma \in \mathbb{R}$, $\Theta = (\vartheta, 0]$, $-\infty \leq \vartheta < 0$. Then there is a $\tilde{P} \in \text{As}(X, \mathbf{g})$ such that, for all $s \in \mathbb{R}$, the pull-back under χ induces an isomorphism*

$$(2.2.2) \quad \chi^*: \mathcal{K}_P^{s, \gamma}(X^\wedge) \rightarrow \mathcal{K}_{\tilde{P}}^{s, \gamma}(X^\wedge)$$

onto its (finite-codimensional if $\vartheta > -\infty$) range.

Proof. Similarly to the proof of Theorem 1.2.2 we may assume that $\vartheta > -\infty$. Further we may assume that P satisfies the shadow condition, i.e., $P = \{(p_{jl}, m_{jl}, L_{jl})\}_{\substack{0 \leq j \leq N, \\ 0 \leq l \leq n_j}}$, where

$p_{jl} = p_{j0} - l$, $m_{jl} \leq m_{j'}$, $L_{jl} \subseteq L_{j'}$ for $0 \leq l \leq l' \leq n_j$, and n_j is the largest integer less than $\operatorname{Re} p_{j0} - \left(\frac{n+1}{2} - \gamma + \vartheta\right)$. Moreover, $p_{j0} - p_{j'0} \notin \mathbb{Z}$ for $j \neq j'$.

Since $\chi^*: \mathcal{K}_{\Theta}^{s,\gamma}(X^\wedge) \rightarrow \mathcal{K}_{\tilde{P}}^{s,\gamma}(X^\wedge)$ by Proposition 2.2.1, we still have to construct $\tilde{P} \in \operatorname{As}(X, \mathbf{g})$ such that

$$\chi^*: \mathcal{E}_P(X^\wedge) \rightarrow \mathcal{K}_{\tilde{P}}^{s,\gamma}(X^\wedge).$$

The space $\mathcal{E}_P(X^\wedge)$ is spanned by functions of the form

$$u(r, y) = \omega(r) c_{jlk}(y) r^{-p} \log^k r|_{p=p_{jl}}, \quad (p_{jl}, m_{jl}, L_{jl}) \in P, \quad 0 \leq k \leq m_{jl}, \quad c_{jlk}(y) \in L_{jl}.$$

Let us assume $k = 0$; the assertion for $k > 0$ then follows by differentiating the transformation rule derived below with respect to p . We have

$$(\chi^* u)(t, x) = u(\chi(t, x)) = \omega(\sigma(t, x)) c_{j0}(\kappa(t, x)) \sigma(t, x)^{-p_{jl}}.$$

Using Taylor expansion of σ at $t = 0$,

$$(2.2.3) \quad \sigma(t, x) = \sum_{h=0}^N c_h(x) t^h + \sigma_{(N+1)}(t, x) t^{N+1},$$

with $c_0(x) = \sigma(0, x) = 0$, $c_1(x) = \partial_t \sigma(0, x) > 0$, $c_h(x) \in C^\infty(X)$, $1 \leq j \leq N$, and $\sigma_{(N+1)}(t, x)$ is smooth up to $t = 0$, we get

$$\sigma(t, x) = c_1(x) t \left(1 + \sum_{h=2}^N \frac{c_h(x)}{c_1(x)} t^{h-1} + \frac{\sigma_{(N+1)}(t, x)}{c_1(x)} t^N \right).$$

For every $N' \in \mathbb{N}$ we find a $N \in \mathbb{N}$ sufficiently large such that formula (1.2.3) yields

$$(2.2.4) \quad \sigma(t, x)^{-p_{jl}} = t^{-p_{jl}} \left(\sum_{h=0}^{N'} d_h(x) t^h + f_{(N'+1)}(t, x) t^{N'+1} \right),$$

with certain $d_h(x) \in C^\infty(X)$ and $f_{(N'+1)}(t, x)$ smooth up to $t = 0$.

Note that $\omega(\sigma(t, x))$ can be rewritten as $\omega(\sigma(t, x)) = \tilde{\omega}(t) \omega_1(t, x)$, with a cut-off function $\tilde{\omega}(t)$ satisfying $\tilde{\omega}(t) = 1$ for $0 \leq t \leq c$ and a sufficiently large constant $c > 0$, and $\omega_1(t, x)$ is smooth and equals 1 close to $t = 0$. Using formula (2.2.4) for $N' \in \mathbb{N}$ sufficiently large and Taylor expansion of $c_{j0}(\kappa(t, x))$ at $t = 0$, we obtain the representation

$$(2.2.5) \quad u(\chi(t, x)) = \tilde{\omega}(t) t^{-p_{jl}} \left(\sum_{h=0}^{N''} \tilde{c}_h(x) t^h + g_{(N''+1)}(t, x) t^{N''+1} \right)$$

with certain $\tilde{c}_h(x) \in C^\infty(X)$ and $g_{(N''+1)}(t, x)$ that is smooth up to $t = 0$.

From this construction it is seen that the coefficients $\tilde{c}_h(x)$ belong to a finite-dimensional subspace $\tilde{L}_{jl} \subset C^\infty(X)$ which can be expressed in terms of L_{j0}, \dots, L_{jl} , some first coefficients $d_h(x)$ from (2.2.4), with l in (2.2.4) varying from 0 up to the l under consideration, and some first derivatives of $\kappa(t, x)$ with respect to t at $t = 0$. In particular, in case $m_{j0} = m_{j1} = \dots = 0$, we get

$$(2.2.6) \quad \tilde{L}_{j0} = c_1(x)^{-p_{j0}} \underline{\kappa}^* L_{j0}, \quad \tilde{L}_{j1} = c_1(x)^{-p_{j1}} \underline{\kappa}^* L_{j1} + c_1(x)^{-p_{j0}} H, \quad \dots,$$

where $H = \left\{ -p \frac{c_2(x)}{c_1(x)} \underline{\kappa}^* c_{j0} + \langle \underline{\kappa}^* d c_{j0}, \partial_t \kappa|_{t=0} \rangle; c_{j0} \in L_{j0} \right\}$ and $c_1(x) = \partial_t \sigma(0, x)$, $c_2(x) = \frac{1}{2} \partial_t^2 \sigma(0, x), \dots$ are as in (2.2.3). Similar statements remain true when logarithms are involved.

Finally, we put $\tilde{P} = \{(p_{jl}, m_{jl}, \tilde{L}_{jl})\}_{\substack{0 \leq j \leq N, \\ 0 \leq l \leq n_j}}$, with the finite-dimensional subspaces $\tilde{L}_{jl} \subset C^\infty(X)$ having just been calculated. \square

Remark 2.2.3 (a) In general, $\tilde{P} \in \text{As}(X, \mathfrak{g})$ cannot be chosen such that χ^* in (2.2.2) is surjective. For example, the choice of \tilde{L}_{j_1} in (2.2.6) is the best possible one (under the assumption $m_{j_1} = 0$), and, given L_{j_0}, L_{j_1} with $\dim L_{j_0} \geq 1$, one easily constructs a diffeomorphism χ such that $\dim \tilde{L}_{j_1} > \dim L_{j_1}$. This means, in particular, that asymptotic types as defined above do not have a coordinate-invariant meaning when $\dim X > 0$.

(b) Under the natural assumption that the shadow condition is satisfied, however, weak asymptotic types, i.e., only sequences $\{(p_j, m_j)\}_{j \in \mathbb{N}} \subset \mathbb{C} \times \mathbb{N}$ are prescribed, cf. Schulze [18], are coordinate-invariant. The same holds for a refined notion of asymptotic type, where additionally linear relations between the various coefficients $c_{jk}(x) \in L_j$, even for different j , are taken into account, cf. Liu and Witt [11].

As a consequence of Proposition 2.2.2 we get the coordinate invariance of Green operators with discrete asymptotics.

Theorem 2.2.4 The operator push-forward under χ induces an isomorphism

$$\chi_*: C_G(X^\wedge, (\gamma, \delta, \Theta)) \rightarrow C_G(X^\wedge, (\gamma, \delta, \Theta)).$$

Proof. The arguments are analogous to those in the proof of Theorem 1.2.3. \square

2.3. Push-forward of Mellin operators. To show the invariance of Mellin operators modulo Green operator it is sufficient to consider coordinates changes in a neighbourhood of $\{0\} \times X$. Let $\varepsilon > 0$ be small and such that

$$(2.3.1) \quad |\partial_x \sigma(t, x)| < \varepsilon$$

for all $(t, x) \in U$ (because of $\sigma(0, x) = 0$), where $U = [0, \varepsilon_1) \times V$ for some open coordinate neighbourhood $V \subset X$ and $\varepsilon_1 > 0$ is small.

For the following three lemmas, cf. Schrohe [14, Lemma 2.4, Lemma 2.6, Proposition 2.9].

Lemma 2.3.1 The function $|\det D\chi(t, x)| \frac{t}{\sigma(t, x)}$, together with all its derivatives, is bounded on U . Moreover, this function is bounded away from zero provided that the constant $\varepsilon > 0$ in (2.3.1) is sufficiently small.

Lemma 2.3.2 We have

$$\left(\frac{\sigma(t, x)}{\sigma(t', x')} \right)^{-i\rho} e^{i(\kappa(t, x) - \kappa(t', x'))\eta} = \left(\frac{t}{t'} \right)^{-i(B_1 T(t, t')\rho + B_2 T(t, t')\eta)} e^{i(x - x')(B_3 \rho + B_4 \eta)},$$

where

$$\begin{aligned} B_1(t, t', x, x') &= \int_0^1 \partial_t(\log \sigma)(t' + \vartheta(t - t'), x' + \vartheta(x - x')) d\vartheta, \\ B_2^t(t, t', x, x') &= - \int_0^1 \partial_t \kappa(t' + \vartheta(t - t'), x' + \vartheta(x - x')) d\vartheta, \\ B_3^t(t, t', x, x') &= - \int_0^1 \partial_x(\log \sigma)(t' + \vartheta(t - t'), x' + \vartheta(x - x')) d\vartheta, \\ B_4^t(t, t', x, x') &= \int_0^1 \partial_x \kappa(t' + \vartheta(t - t'), x' + \vartheta(x - x')) d\vartheta. \end{aligned}$$

Here $T(t, t') = \frac{t-t'}{\log t - \log t'}$, B_1, B_2, B_3 , and B_4 are matrix functions of sizes 1×1 , $1 \times n$, $n \times 1$, and $n \times n$, respectively, and superscript t denotes matrix transposition.

Lemma 2.3.3 Let B be the $(n+1) \times (n+1)$ matrix

$$B(t, t', x, x') = \begin{pmatrix} B_1(t, t', x, x')T(t, t') & B_2(t, t', x, x')T(t, t') \\ B_3(t, t', x, x') & B_4(t, t', x, x') \end{pmatrix}$$

with the matrices B_1, B_2, B_3, B_4 being defined in Lemma 2.3.2. Suppose that $|x - x'|, |\frac{t}{t'} - 1|, t$, and t' are sufficiently small. Then $B(t, t', x, x')$ is invertible, and with $A(t, t', x, x') =$

$B(t, t', x, x')^{-1}$, the norm $\|t'^k \partial_{t'}^k D_x^\alpha D_{x'}^{\alpha'} A(t, t', x, x')\|_{\mathcal{L}(\mathbb{C}^{n+1})}$ is bounded for all $k \in \mathbb{N}$ and all multi-indices $\alpha, \alpha' \in \mathbb{N}^n$, and the matrix function

$$t'^k \partial_{t'}^k D_x^\alpha D_{x'}^{\alpha'} A(t, t', x, x')|_{t'=t}$$

is smooth up to $t = 0$.

For the next proposition, cf. [18, Proposition 2.3.81].

Proposition 2.3.4 *Let $f(t, t', z) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \mathcal{M}_R^{-\infty}(X))$ for some $R \in \mathbf{As}(X)$. Let $\gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}} R \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$. Then we have*

$$\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(f) \tilde{\omega} \in C_{M+G}(X^\wedge, \mathbf{g})$$

for $\mathbf{g} = (\gamma, \gamma, \Theta)$, $\Theta = (-\infty, 0]$, and arbitrary cut-off functions $\omega, \tilde{\omega}$.

Theorem 2.3.5 *Let $\chi: \overline{\mathbb{R}}_+ \times X \rightarrow \overline{\mathbb{R}}_+ \times X$ be a diffeomorphism and ω, ω_0 be cut-off functions. Further let $R \in \mathbf{As}(X)$ with $\pi_{\mathbb{C}} R \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$ and $f(z) \in \mathcal{M}_R^{-\infty}(X)$. Then the push-forward under χ^{-1} of the operator $\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(f) \omega_0$ is a smoothing Mellin + Green operator, i.e.,*

$$\chi_*^{-1}(\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(f) \omega_0) \in C_{M+G}(X^\wedge, \mathbf{g}),$$

for $\mathbf{g} = (\gamma, \gamma, \Theta)$, $\Theta = (-\infty, 0]$.

Proof. Analogously to the proof of Theorem 1.3.3 we may assume that $\gamma = \frac{n+1}{2}$. Let $\{V_1, \dots, V_N\}$ be a covering of X by coordinate neighbourhoods. In view of the compactness of X , we may assume that this covering is chosen in such a way that, for all $1 \leq i, j \leq N$, $V_i \cup V_j$ is also a coordinate neighbourhood. Let $\{\varphi_1, \dots, \varphi_N\}$ be a subordinate partition of unity. Then $\omega \operatorname{op}_M^{\frac{1}{2}}(f) \omega_0 = \sum_{i,j=1}^N \omega \operatorname{op}_M^{\frac{1}{2}}(f_{ij}) \omega_0$, where $f_{ij} = \varphi_i f \varphi_j$. The terms $\omega \operatorname{op}_M^{\frac{1}{2}}(f_{ij}) \omega_0$ can be treated separately for all i, j . Thus, without loss of generality, we can replace f_{ij} by f which is now localised in the coordinate neighbourhood $V_i \cup V_j$, from now on denoted by V . Then

$$\begin{aligned} \chi_*^{-1}(\omega \operatorname{op}_M^{\frac{1}{2}}(f) \omega_0) u(t, x) &= \frac{1}{2\pi i} \int_{\Gamma_0} \int_0^\infty \iint \omega(\sigma(t, x)) e^{i(\kappa(t, x) - \kappa(t', x')) \eta} \left(\frac{\sigma(t, x)}{\sigma(t', x')} \right)^{-z} \\ &\quad \times f(\kappa(t, x), z, \eta) \omega_0(\sigma(t', x')) u(t', x') \frac{|\det D\chi(t', x')|}{\sigma(t', x')} t' dx' d\eta \frac{dt'}{t'} dz. \end{aligned}$$

We have $\exp(i(\kappa(t, x) - \kappa(t', x')) \eta) = \exp(i(\kappa(0, x) - \kappa(0, x')) \eta) \exp(i(\kappa(t, x) - \kappa(0, x) - \kappa(t', x') + \kappa(0, x')) \eta) = \exp(i(x - x')(\alpha(x, x') \eta)) \exp(i\beta(t, t', x, x') \eta)$, where $\alpha(x, x') = B_4(0, 0, x, x')$ is a smooth non-degenerate $n \times n$ matrix function on $V \times V$ and $\beta(t, t', x, x') = \kappa(t, x) - \kappa(0, x) - \kappa(t', x') + \kappa(0, x')$ is smooth on $U \times U$, i.e., up to $t, t' = 0$.

After the change of variables $\eta = \alpha(x, x')^{-1} \xi$ we obtain

$$\begin{aligned} &\chi_*^{-1}(\omega \operatorname{op}_M^{\frac{1}{2}}(f) \omega_0) u(t, x) \\ &= \frac{1}{2\pi i} \int_{\Gamma_0} \int_0^\infty \iint \omega(\sigma(t, x)) e^{i(x-x') \xi} \left(\frac{t}{t'} \right)^{-z} \left(\frac{t' \sigma(t, x)}{t \sigma(t', x')} \right)^{-z} e^{i\beta(t, t', x, x') \alpha(x, x')^{-1} \xi} \\ &\quad \times f(\kappa(t, x), z, \alpha(x, x')^{-1} \xi) \omega_0(\sigma(t', x')) u(t', x') \det |\alpha(x, x')|^{-1} \frac{|\det D\chi(t', x')|}{\sigma(t', x')} t' dx' d\xi \frac{dt'}{t'} dz \\ &= \frac{\tilde{\omega}(t)}{2\pi i} \int_{\Gamma_0} \int_0^\infty \iint e^{i(x-x') \xi} \left(\frac{t}{t'} \right)^{-z} g(t, t', x, x', z, \xi) \tilde{\omega}_0(t') u(t', x') dx' d\xi \frac{dt'}{t'} dz, \end{aligned}$$

where

$$(2.3.2) \quad g(t, t', x, x', z, \xi) = \omega_1(t, x) \left(\frac{t' \sigma(t, x)}{t \sigma(t', x')} \right)^{-z} e^{i\beta(t, t', x, x') \alpha(x, x')^{-1} \xi} \\ \times f(\kappa(t, x), z, \alpha(x, x')^{-1} \xi) \omega_2(t', x') |\det \alpha(x, x')|^{-1} \frac{|\det D\chi(t', x')|}{\sigma(t', x')} t'$$

and $\tilde{\omega}$, $\tilde{\omega}_0$ are cut-off functions, with $\omega(\sigma(t, x)) = \tilde{\omega}(t)\omega_1(t, x)$, $\omega_0(\sigma(t, x)) = \tilde{\omega}_0(t)\omega_2(t', x')$ and ω_1 , ω_2 are equal to 1 close to $t = 0$ and $t' = 0$, respectively. By Proposition 2.3.4 we only have to show that

$$(2.3.3) \quad g(t, t', x, x', z, \xi) \in C^\infty([0, \varepsilon] \times [0, \varepsilon], M_R^{-\infty}(V))$$

for a sufficiently small $\varepsilon > 0$. (Here, $M_R^{-\infty}(V)$ means the subspace of all elements of $M_R^{-\infty}(X)$ supported in V in the above-mentioned sense.) Now $\sigma(t, x)$ can be rewritten as $\sigma(t, x) = t\partial_t\sigma(0, x) + t^2\tilde{\sigma}(t, x)$ (recall $\sigma(0, x) = 0$ and $\partial_t\sigma(0, x) > 0$ for all x), with $\tilde{\sigma}(t, x)$ smooth up to $t = 0$. Since X is compact, we get

$$(2.3.4) \quad \varphi(t, t', x, x') = \frac{t' \sigma(t, x)}{t \sigma(t', x')} = \frac{\partial_t \sigma(0, x) + t \tilde{\sigma}(t, x)}{\partial_t \sigma(0, x') + t' \tilde{\sigma}(t', x')} > 0 \quad \text{for } t, t' \text{ small,}$$

and $\varphi(t, t', x, x') \in C^\infty([0, \varepsilon] \times [0, \varepsilon] \times X \times X)$ for a sufficiently small $\varepsilon > 0$. It is also clear that φ is a bounded function in all its arguments for t, t' small. Hence

$$\left(\frac{t' \sigma(t, x)}{t \sigma(t', x')} \right)^{-z} = \exp(-z \log \varphi(t, t', x, x'))$$

and because of $f \in \mathcal{M}_R^{-\infty}(X)$ we obtain, for any $\pi_{\mathbb{C}R}$ -excision function $\pi(z)$,

$$\pi(\beta + i\tau) \left(\frac{t' \sigma(t, x)}{t \sigma(t', x')} \right)^{-\beta - i\tau} f(\beta + i\tau, \kappa(t, x), \alpha(x, x')^{-1} \xi) \\ \in C^\infty([0, \varepsilon] \times [0, \varepsilon], S^{-\infty}(V \times V \times \mathbb{R}_{\xi, \tau}^{n+1})),$$

uniformly in $c \leq \beta \leq c'$ for arbitrary $c < c'$. This yields (2.3.3), since the other terms in (2.3.2) are smooth, cf. Lemma 2.3.1, and do not affect the symbol estimates. \square

Theorem 2.3.6 *Let $\chi: \overline{\mathbb{R}}_+ \times X \rightarrow \overline{\mathbb{R}}_+ \times X$ be a diffeomorphism and $h \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_O^\mu(X))$, $\mu \in \mathbb{R}$. Then*

$$\chi_*^{-1}(\omega \operatorname{op}_M^{\gamma - \frac{n}{2}}(h)\omega_0) = \tilde{\omega} \operatorname{op}_M^{\gamma - \frac{n}{2}}(\tilde{h})\tilde{\omega}_0 + G_0,$$

where $\tilde{h} \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_O^\mu(X))$ and $G_0 \in C_G(X^\wedge, (\gamma, \gamma, \Theta))$, $\Theta = (-\infty, 0]$; ω , ω_0 , $\tilde{\omega}$, $\tilde{\omega}_0$ are cut-off functions.

Proof. We again assume that $\gamma = \frac{n+1}{2}$. Let $\{V_1, \dots, V_N\}$ be an open covering of X by coordinate neighbourhoods and $\{\varphi_1, \dots, \varphi_N\}$ be a subordinate partition of unity. As in the foregoing proof we may assume that, for all $1 \leq i, j \leq N$, $V_i \cup V_j$ is also a coordinate neighbourhood. We then write $\omega \operatorname{op}_M^{\frac{1}{2}}(h)\omega_0$ as the sum of the operators $\omega \operatorname{op}_M^{\frac{1}{2}}(h_{ij})\omega_0$, where $h_{ij} = \varphi_i h \varphi_j$, and treat each of terms $\omega \operatorname{op}_M^{\frac{1}{2}}(h_{ij})\omega_0$ for all i, j separately. So, without loss of generality, we replace h_{ij} by an h the kernel of which is localised in $\overline{\mathbb{R}}_+ \times V \times V$.

Let $\psi \in C_0^\infty(\overline{\mathbb{R}}_+)$ be supported in a small neighbourhood of $\varrho = 1$ with $\psi(\varrho) = 1$ for ϱ close 1. Then

$$\omega \operatorname{op}_M^{\frac{1}{2}}(h)\omega_0 - \omega \operatorname{op}_M^{\frac{1}{2}}(\psi(r/r')h)\omega_0 \in C_{M+G}\left(X^\wedge, \left(\frac{n+1}{2}, \frac{n+1}{2}, \Theta\right)\right),$$

since $h(r, z) - H(\psi)h(r, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_O^{-\infty}(X))$, cf. [2], [17]. By virtue of the invariance of $C_{M+G}(X^\wedge, (\frac{n+1}{2}, \frac{n+1}{2}, \Theta))$, that we know from Theorem 2.3.5, we have $\chi_*^{-1}(\omega \operatorname{op}_M^{\frac{1}{2}}(h)\omega_0 -$

$\omega \operatorname{op}_M^{\frac{1}{2}}(\psi(r/r')h)\omega_0 \in C_{M+G}(X^\wedge, (\frac{n+1}{2}, \frac{n+1}{2}, \Theta))$, where additionally all conormal symbols of the latter operator vanish. Therefore, as in the proof of Theorem 1.3.4, the operator $\chi_*^{-1}(\omega \operatorname{op}_M^{\frac{1}{2}}(h)\omega_0 - \omega \operatorname{op}_M^{\frac{1}{2}}(\psi(r/r')h)\omega_0)$ can be given the form $\tilde{\omega} \operatorname{op}_M^{\frac{1}{2}}(h_1)\tilde{\omega}_0 + G_1$, where $h_1(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{-\infty}(X))$ and $G_1 \in C_G(X^\wedge, (\frac{n+1}{2}, \frac{n+1}{2}, \Theta))$.

Thus we still have to consider the operator $\chi_*^{-1}(\omega \operatorname{op}_M^{\frac{1}{2}}(\psi(\frac{r}{r'})h)\omega_0)$. We have, after the change of variables $\begin{pmatrix} \rho \\ \eta \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \begin{pmatrix} \tau \\ \xi \end{pmatrix}$ and using Lemmas 2.3.2, 2.3.3, where $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, A_1 is an $1 \times (n+1)$ matrix, and A_2 is an $n \times (n+1)$ matrix,

$$\begin{aligned} & \chi_*^{-1}(\omega \operatorname{op}_M^{\frac{1}{2}}(\psi(r/r')h)\omega_0)u(t, x) \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \iint \omega(\sigma(t, x)) \left(\frac{\sigma(t, x)}{\sigma(t', x')} \right)^{-i\rho} e^{i(\kappa(t, x) - \kappa(t', x'))\eta} \psi \left(\frac{\sigma(t, x)}{\sigma(t', x')} \right) \\ & \quad \times h(\sigma(t, x), \kappa(t, x), i\rho, \eta) \omega_0(\sigma(t', x')) u(t', x') \frac{|\det D\chi(t', x')|}{\sigma(t', x')} t' dx' d\eta \frac{dt'}{t'} d\rho \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \iint \omega(\sigma(t, x)) \left(\frac{t}{t'} \right)^{-i(B_1 T(t, t')\rho + B_2 T(t, t')\eta)} e^{i(x-x')(B_3 \rho + B_4 \eta)} \psi \left(\frac{\sigma(t, x)}{\sigma(t', x')} \right) \\ & \quad \times h(\sigma(t, x), \kappa(t, x), i\rho, \eta) \omega_0(\sigma(t', x')) u(t', x') \frac{|\det D\chi(t', x')|}{\sigma(t', x')} t' dx' d\eta \frac{dt'}{t'} d\rho \\ &= \tilde{\omega} \operatorname{op}_M^{\frac{1}{2}}(g)\tilde{\omega}_0 u(t, x), \end{aligned}$$

where

$$\begin{aligned} g(t, t', x, x', i\tau, \xi) &= \omega_1(t, x) \psi \left(\frac{\sigma(t, x)}{\sigma(t', x')} \right) h \left(\sigma(t, x), \kappa(t, x), iA_1 \begin{pmatrix} \tau \\ \xi \end{pmatrix}, A_2 \begin{pmatrix} \tau \\ \xi \end{pmatrix} \right) \\ & \quad \times \omega_2(t', x') \frac{|\det D\chi(t', x')|}{\sigma(t', x')} t' |\det A(t, t', x, x')| \end{aligned}$$

and $\omega(\sigma(t, x)) = \tilde{\omega}(t)\omega_1(t, x)$, $\omega_0(\sigma(t', x')) = \tilde{\omega}_0(t')\omega_2(t', x')$, with $\tilde{\omega}$, $\tilde{\omega}_0$ being cut-off functions and ω_1, ω_2 are equal to 1 close to $t = 0$ and $t' = 0$, respectively.

Proceeding as in the proof of Theorem 1.3.4 we switch $\frac{t}{t'}$ -dependence to z -dependence. (The following argument corresponds to the replacement of $\varphi(t, t')$ in (1.3.5) with $\varphi(\varrho t', t')$ instead of $\varphi(t, t/\varrho)$, which likewise works.) For $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, this technique applies for $T(t, t')$ which can be written as $T(t, t') = t' \frac{\frac{t}{t'} - 1}{\log \frac{t}{t'}}$. More precisely, let

$$\begin{aligned} \tilde{B}_1(t, t', \varrho, x, x') &= t' \int_0^1 \partial_t(\log \sigma) \left(t' + t'\vartheta(\varrho - 1), x' + \vartheta(x - x') \right) d\vartheta, \\ \tilde{B}_2^t(t, t', \varrho, x, x') &= -t' \int_0^1 \partial_t \kappa \left(t' + t'\vartheta(\varrho - 1), x' + \vartheta(x - x') \right) d\vartheta \end{aligned}$$

(multiplication by t' and replacement of $\frac{t}{t'}$ by ϱ) and $\tilde{B}(t, t', \varrho, x, x') = \begin{pmatrix} \tilde{B}_1 \frac{\varrho-1}{\log \varrho} & \tilde{B}_2 \frac{\varrho-1}{\log \varrho} \\ B_3 & B_4 \end{pmatrix}$,

$$\tilde{A}(t, t', \varrho, x, x') = \tilde{B}(t, t', \varrho, x, x')^{-1} = \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix},$$

where \tilde{A}_1, \tilde{A}_2 are $1 \times (n+1)$ and $n \times (n+1)$ matrices, respectively. Analogously to the proof of Theorem 1.3.4 we get a new symbol, $\tilde{g}(t, t', x, x', z, \xi)$, which is holomorphic in z and

smooth up to $t, t' = 0$:

$$\begin{aligned} \tilde{g}(t, t', x, x', z, \xi) &= \int_0^\infty \int_{-\infty}^\infty \varrho^{z-i\tau} \omega_1(t, x) \psi(\varrho \varphi(\varrho t', t', x, x')) \\ &\quad \times h \left(\sigma(t, x), \kappa(t, x), i\tilde{A}_1(t, t', \varrho, x, x') \begin{pmatrix} \tau \\ \xi \end{pmatrix}, \tilde{A}_2(t, t', \varrho, x, x') \begin{pmatrix} \tau \\ \xi \end{pmatrix} \right) \\ &\quad \times \omega_2(t', x') \frac{|\det D\chi(t', x')|}{\sigma(t', x')} t' |\det \tilde{A}(t, t', \varrho, x, x')| d\tau \frac{d\varrho}{\varrho}, \end{aligned}$$

cf. (2.3.4). Furthermore, analogously to the proof of Theorem 1.3.4 we can show that $\tilde{g}(t, t', x, x', \beta + i\tau, \xi) \in C^\infty([0, \varepsilon] \times [0, \varepsilon], S_{\text{cl}}^\mu(V \times V \times \mathbb{R}_{\tau, \xi}^{n+1}))$ uniformly for $c_1 \leq \beta \leq c_2$ for arbitrary $c_1 < c_2$, where we clearly mean symbols in local coordinates.

Then we get

$$\chi_*^{-1}(\omega \text{op}_M^{\frac{1}{2}}(\psi(r/r')h)\omega_0) = \tilde{\omega} \text{op}_M^{\frac{1}{2}}(g)\tilde{\omega}_0 = \tilde{\omega} \text{op}_M^{\frac{1}{2}}(\tilde{g})\tilde{\omega}_0,$$

where $\tilde{g} \in C^\infty([0, \varepsilon] \times [0, \varepsilon], \mathcal{M}_\mathcal{O}^\mu(X))$.

Finally, we apply the analogue of Lemma 1.3.5 for the higher-dimensional case. \square

2.4. Invariance of the cone algebra. Recall from Section 2.2 that we are considering diffeomorphisms $\chi: \overline{\mathbb{R}}_+ \times X \rightarrow \overline{\mathbb{R}}_+ \times X$, where

$$(r, y) = \chi(t, x) = (\sigma(t, x), \kappa(t, x)).$$

Then $\sigma(0, x) = 0$, $\partial_t \sigma(0, x) > 0$ for all $x \in X$, and $\underline{\kappa}: X \rightarrow X$ defined by $\underline{\kappa}(x) = \kappa(0, x)$ is a C^∞ -diffeomorphism. Moreover, χ fulfils (2.1.8), (2.1.9) so that $\chi_*: L^{\mu;0}(X^\wedge) \rightarrow L^{\mu;0}(X^\wedge)$.

Theorem 2.4.1 *The operator push-forward under χ induces an isomorphism*

$$\chi_*: C^\mu(X^\wedge, \mathbf{g}) \rightarrow C^\mu(X^\wedge, \mathbf{g})$$

for all $\mu \in \mathbb{R}$ and $\mathbf{g} = (\gamma, \delta, \Theta)$, $\Theta = (-(k+1), 0]$, $k \in \mathbb{N} \cup \{\infty\}$, and we have

- (i) $\tilde{\sigma}_\psi^\mu(\chi_* A)(r, y, \tilde{\varrho}, \eta) \Big|_{\substack{(r,y)=\chi(t,x), \\ (\tilde{\varrho},\eta)=\frac{\sigma(t)}{t}(D\chi(t,x)^t)^{-1}(\tilde{\tau},\xi)}} = \left(\frac{\sigma(t)}{t}\right)^{\gamma-\delta} \tilde{\sigma}_\psi^\mu(A)(t, x, \tilde{\tau}, \xi),$
- (ii) $\sigma_M^{\gamma-\delta}(\chi_* A)(z) = \underline{\kappa}_* \left(\partial_t \sigma(0, x)^{z+\gamma-\delta} \sigma_M^{\gamma-\delta}(A)(z) \partial_t \sigma(0, x)^{-z} \right),$
- (iii) $\sigma_e(\chi_* A)(r, y, \varrho, \eta) \Big|_{\substack{(r,y)=\chi(t,x), \\ (\varrho,\eta)=(D\chi(t,x)^t)^{-1}(\tau,\xi)}} = \sigma_e(A)(t, x, \tau, \xi) \text{ mod } S^{\mu-1, -1}(\mathbb{R}_+ \times \mathbb{R}).$

In (ii), $\partial_t \sigma(0, x)$ is regarded as multiplication operator on $C^\infty(X)$, $\partial_t \sigma(0, x) > 0$ for all $x \in X$, and the expression in parenthesis on the right-hand side is understood as the composition of three z -dependent pseudo-differential operators on X . Furthermore, $\underline{\kappa}_*$ is the operator push-forward under $\underline{\kappa}$.

Proof. As in the proof of Theorem 1.4.1 the results of Sections 2.2, 2.3 immediately give the coordinate invariance of the cone algebra with asymptotics. Further, (i), (iii) are well-known; (i) follows from the transformation rule for $\sigma_\psi^\mu(A)$ and a compatibility condition between $\tilde{\sigma}_\psi^\mu(A)$ and $\sigma_\psi^\mu(A)$ analogously to that in (1.1.9), while (iii) can be found, e.g., in [13].

Thus it remains to prove (ii). Again we assume $\gamma = \frac{n+1}{2}$. Moreover, $\sigma_M^{\gamma-\delta}(\omega(t)t^{\delta-\gamma})(z) = 1$ and $\sigma_M^{\gamma-\delta}(\chi_*(\omega(t)t^{\delta-\gamma}))(z) = \underline{\kappa}_* (\partial_t \sigma(0, x)^{\gamma-\delta})$ so that we can further assume that $\gamma = \delta$ employing the Mellin translation product (1.1.11). Then we have to look at the proofs of Theorems 2.3.5, 2.3.6.

First let $f(z) \in \mathcal{M}_P^{-\infty}(X)$ be as in the proof of Theorem 2.3.5. Recall that the kernel of $f(z)$ is supported in $V \times V$, where V is a coordinate neighbourhood of X . We have

$\sigma_M^0(\omega \operatorname{op}_M^{\frac{1}{2}}(f)\omega_0)(z) = f(z)$ and

$$\left(\sigma_M^0(\chi_*^{-1}(\omega \operatorname{op}_M^{\frac{1}{2}}(f)\omega_0))(z) \right) v(x) = \iint e^{i(x-x')\xi} g(0, 0, x, x', z, \xi) v(x') dx' d\xi,$$

where

$$g(0, 0, x, x', z, \xi) = \left(\frac{\partial_t \sigma(0, x)}{\partial_t \sigma(0, x')} \right)^{-z} f(\underline{\kappa}(x), z, \alpha(x, x')^{-1} \xi) |\det \alpha(x, x')|^{-1} |\det D\underline{\kappa}(x')|$$

by (2.3.2). In fact, $\frac{t'\sigma(t, x)}{t\sigma(t', x')} \Big|_{t'=t=0} = \frac{\partial_t \sigma(0, x)}{\partial_t \sigma(0, x')}$, $\beta(0, 0, x, x') = 0$, and

$$\frac{|\det D\chi(0, x')|}{\sigma(t', x')} t' \Big|_{t'=0} = \left| \det \begin{pmatrix} \partial_t \sigma(0, x') & 0 \\ \partial_t \kappa(0, x') & \partial_x \kappa(0, x') \end{pmatrix} \right| \partial_t \sigma(0, x')^{-1} = |\det D\underline{\kappa}(x')|.$$

Thus we get

$$\begin{aligned} \left(\sigma_M^0(\chi_*^{-1}(\omega \operatorname{op}_M^{\frac{1}{2}}(f)\omega_0))(z) \right) v(x) &= \partial_t \sigma(0, x)^{-z} \iint e^{i(x-x')\xi} f(\underline{\kappa}(x), z, \alpha(x, x')^{-1} \xi) \\ &\quad |\det \alpha(x, x')|^{-1} |\det D\underline{\kappa}(x')| (\partial_t \sigma(0, x')^z v(x')) dx' d\xi \\ &= \underline{\kappa}_*^{-1} \left(\partial_t \sigma(0, x)^{-z} \sigma_M^0(\omega \operatorname{op}_M^{\frac{1}{2}}(f)\omega_0)(z) \partial_t \sigma(0, x)^z \right) v(x), \end{aligned}$$

since $e^{i(\underline{\kappa}(x) - \underline{\kappa}(x'))\eta} = e^{i(x-x')\alpha(x, x')\eta}$, which entails (ii) in this case.

Next let $h(t, z) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_\mathcal{O}^\mu(X))$ be as in the proof of Theorem 2.3.6. In particular, the kernel of $h(t, z)$ is supported in $\overline{\mathbb{R}}_+ \times V \times V$, where V is a coordinate neighbourhood of X . The transformation rule (ii) for $\omega \operatorname{op}_M^{\frac{1}{2}}(H(1 - \psi)h)\omega_0$ follows analogously to the first part of the proof. Thus it remains to deal with the operator $\omega \operatorname{op}_M^{\frac{1}{2}}(\psi(\frac{r}{r'})h)\omega_0$. We have $\sigma_M^0(\omega \operatorname{op}_M^{\frac{1}{2}}(\psi(\frac{r}{r'})h)\omega_0)(z) = H(\psi)h(0, z)$. Furthermore,

$$\left(\sigma_M^0(\chi_*^{-1}(\omega \operatorname{op}_M^{\frac{1}{2}}(\psi(\frac{r}{r'})h)\omega_0))(z) \right) v(x) = \iint e^{i(x-x')\xi} \tilde{g}(0, 0, x, x', z, \xi) v(x') dx' d\xi,$$

where

$$\begin{aligned} \tilde{g}(0, 0, x, x', z, \xi) &= \int_0^\infty \int_{-\infty}^\infty \varrho^{z-i\tau} \psi \left(\varrho \frac{\partial_t \sigma(0, x)}{\partial_t \sigma(0, x')} \right) h \left(0, \underline{\kappa}(x), i\tau, \tilde{A}_2(0, 0, \varrho, x, x') \begin{pmatrix} \tau \\ \xi \end{pmatrix} \right) \\ &\quad \times |\det D\underline{\kappa}(x')| |\det \tilde{A}(0, 0, \varrho, x, x')| d\tau \frac{d\varrho}{\varrho}, \end{aligned}$$

since $\tilde{B}_1(0, 0, \varrho, x, x') = \frac{\log \varrho}{\varrho - 1}$, $\tilde{B}_2(0, 0, \varrho, x, x') = 0$. In particular, $\tilde{A}(0, 0, \varrho, x, x')$ is independent of ϱ and $\tilde{A}_1(0, 0, \varrho, x, x') = (1, 0, \dots, 0)$, $\tilde{A}_2(0, 0, \varrho, x, x') = (-B_4^{-1}B_3, B_4^{-1})$, where

$$\begin{aligned} B_3^t(0, 0, x, x')(x - x') &= - \int_0^1 \partial_x (\log \partial_t \sigma)(0, x' + \vartheta(x - x')) d\vartheta (x - x') \\ &= - (\log \partial_t \sigma(0, x) - \log \partial_t \sigma(0, x')), \\ B_4^t(0, 0, x, x')(x - x') &= \int_0^1 \partial_x \underline{\kappa}(x' + \vartheta(x - x')) d\vartheta (x - x') \\ &= \underline{\kappa}(x) - \underline{\kappa}(x'). \end{aligned}$$

We get

$$\begin{aligned} \tilde{g}(0, 0, x, x', z, \xi) &= \left(\frac{\partial_t \sigma(0, x)}{\partial_t \sigma(0, x')} \right)^{-z} \int_{-\infty}^{\infty} e^{i \log \left(\frac{\partial_t \sigma(0, x)}{\partial_t \sigma(0, x')} \right) \tau} (M\psi)(z - i\tau) \\ &\quad \times h(0, \underline{\kappa}(x), i\tau, -B_4^{-1} B_3 \tau + B_4^{-1} \xi) \, d\tau \, |\det D\underline{\kappa}(x')| \, |\det B_4(0, 0, x, x')|^{-1} \end{aligned}$$

and, therefore,

$$\begin{aligned} \left(\sigma_M^0(\chi_*^{-1}(\omega \operatorname{op}_M^{\frac{1}{2}}(\psi(\frac{r}{r'})h)\omega_0))(z) \right) v(x) &= \iint \int_{-\infty}^{\infty} \left(\frac{\partial_t \sigma(0, x)}{\partial_t \sigma(0, x')} \right)^{-z} \\ &\quad \times e^{i \log \left(\frac{\partial_t \sigma(0, x)}{\partial_t \sigma(0, x')} \right) \tau + i(x-x')\xi} (M\psi)(z - i\tau) h(0, \underline{\kappa}(x), i\tau, -B_4^{-1} B_3 \tau + B_4^{-1} \xi) \\ &\quad \times |\det D\underline{\kappa}(x')| \, |\det B_4(0, 0, x, x')|^{-1} v(x') \, d\tau \, dx' \, d\xi \\ &= \iint \int_{-\infty}^{\infty} \left(\frac{\partial_t \sigma(0, x)}{\partial_t \sigma(0, x')} \right)^{-z} e^{i(\underline{\kappa}(x) - \underline{\kappa}(x'))\eta} (M\psi)(z - i\rho) h(0, \underline{\kappa}(x), i\rho, \eta) \\ &\quad \times |\det D\underline{\kappa}(x')| v(x') \, d\rho \, dx' \, d\eta \end{aligned}$$

after the change of variables $\begin{pmatrix} \rho \\ \eta \end{pmatrix} = \begin{pmatrix} -B_4^{-1} B_3 & 0 \\ -B_4^{-1} & B_4^{-1} \end{pmatrix} \begin{pmatrix} \tau \\ \xi \end{pmatrix}$,

$$= \iint \int_{-\infty}^{\infty} \left(\frac{\partial_t \sigma(0, x)}{\partial_t \sigma(0, x')} \right)^{-z} e^{i(\underline{\kappa}(x) - \underline{\kappa}(x'))\eta} (H(\psi)h)(0, \underline{\kappa}(x), z, \eta) |\det D\underline{\kappa}(x')| v(x') \, dx' \, d\eta$$

that gives us (ii) also in this case. \square

Remark 2.4.2 (a) *As in Lemmas 1.4.2, 1.4.3 it can be shown that both the “holomorphic” sub-calculus, i.e., the calculus with $\sigma_M^{\mu-j}(A)(z) \in \mathcal{M}_O^\mu(X)$ for all $j = 0, 1, \dots, k$, as well as the “flat” sub-calculus, i.e., the calculus with $\sigma_M^{\mu-j}(A)(z) = 0$ for all $j = 0, 1, \dots, k$, are coordinate-invariant.*

(b) *Using an analogue of Lemma 1.4.2 it can further be shown that transformation formulae for the lower-order conormal symbols are also available, cf. the proof of Lemma 1.5.4. In the present situation, however, the resulting formulae will be considerably more complicated – due to inherent non-commutativity, cf. the transformation formula for the principal conormal symbol in Theorem 2.4.1 (ii).*

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