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Chapter 2

Quantization of Lagrangian Modules

Contents

In this chapter we use the wave packet transform described in Chapter 1[1] to quantize extended classical states represented by so-called *Lagrangian sumbanifolds* of the phase space. Functions on a Lagrangian manifold form a module over the ring of classical Hamiltonian functions on the phase space (with respect to pointwise multiplication). The quantization procedure intertwines this multiplication with the action of the corresponding quantum Hamiltonians; hence we speak of quantization of Lagrangian modules. The semiclassical states obtained by this quantization procedure provide asymptotic solutions to differential equations with a small parameter. Locally, such solutions can be represented by WKB elements. Global solutions are given by Maslov's canonical operator [2]; also see, e.g., [3] and the references therein. Here the canonical operator is obtained in the framework of the universal quantization procedure provided by the wave packet transform. This procedure was suggested in [4] (see also the references there) and further developed in [5]; our exposition is in the spirit of these papers. Some further bibliographical remarks can be found in the beginning of Chapter 1.

2.0 Preliminaries

The notion of the wave packet transform is based on wave packets—the simplest semiclassical states whose oscillation front consists of a single point in the phase space. These wave packets can be treated as the *semiclassical quantization* of the classical states represented by points in the phase space. However, it is well known that semiclassical wave functions are not exhausted by those whose oscillation front is a singleton. A trivial example, of course, is given by a "discrete" linear combination (finite or infinite) of Gaussian wave packets: let (q_k, p_k), $k = 0, 1, \ldots,$ be a sequence of distinct points in the phase space $\mathbf{R}^{2n}$, and let a_k be a sequence of numbers sufficiently rapidly decreasing to zero. Consider the function

$$\psi(x) = \sum_k a_k G(q_k, p_k; x), \tag{2.1}$$

where $G(q_k, p_k; x)$ is the Gaussian wave packet centered at (q_k, p_k). The terms of the sum (2.1) are almost orthogonal in $L^2(\mathbf{R}^n)$ in the sense that

$$(G(q_k, p_k; x), G(q_l, p_l; x)) = O(h^\infty) \quad \text{for} \quad k \neq l. \tag{2.2}$$

Thus, it is pretty obvious for a finite sum (2.1) (and can be proved with little more effort for the case in which there are infinitely many terms) that the oscillation front of $\psi(x)$ has the form

$$OF(\psi) = \overline{\{(q_k, p_k)\}}. \tag{2.3}$$

[1] The preliminary version of Chapter 1 was published as the preprint [1].

Here the bar stands for the closure (which is, of course, unnecessary for the case of a finite sum). Thus, $\psi(x)$ is the simplest example of a pure semiclassical state (i.e., a semiclassical state represented by a wave function) for which the corresponding classical state is mixed rather than pure: it is given by the measure ν on $\mathbf{R}^{2n}_{(q,p)}$ of the form

$$\nu = \sum_k b_k \delta(p - p_k)\delta(q - q_k), \tag{2.4}$$

where the coefficients b_k are

$$b_k = \frac{|a_k|^2}{\sum\limits_l |a_l|^2}. \tag{2.5}$$

However, this example is somewhat artificial in that wave functions of that form rarely occur as physically meaningful solutions of the Schrödinger equation. In the simplest case, such solutions are represented by the so-called *WKB-elements* (where WKB stands for Wentzel, Kramers, and Brillouin) of the form

$$\psi(x, h) = e^{\frac{i}{h}S(x)}\varphi(x), \tag{2.6}$$

where $S(x)$ and $\varphi(x)$ are smooth functions, $S(x)$ real-valued and $\varphi(x)$ compactly supported. As was already mentioned in § 1.3.2 of Chapter 1, the oscillation front of the function (2.6) has the form

$$OF(\psi) = \Lambda \cap \{(p, q)\} \mid q \in \operatorname{supp} \varphi\}, \tag{2.7}$$

where Λ is the manifold in the phase space determined by the function S according to the formula

$$\Lambda = \left\{(q, p) \,\middle|\, p = \frac{\partial S}{\partial q}(q)\right\}. \tag{2.8}$$

The manifold (2.8) is *Lagrangian*, that is, it is n-dimensional and the restriction of the symplectic form

$$\omega^2 = dp \wedge dq \equiv \sum_{j=1}^{n} dp_j \wedge dq_j \tag{2.9}$$

vanishes on Λ:

$$i^*\omega^2 = 0, \tag{2.10}$$

where $i : \Lambda \to \mathbf{R}^{2n}_{p,q}$ is the embedding.

Thus, the WKB-element (2.6) is the simplest example of a pure semiclassical state for which the corresponding classical state is a mixed state concentrated on a smooth submanifold of the phase space. More precisely, this state is concentrated on Λ and is represented by the measure

$$\nu = |\varphi(q)|^2 \delta\left(p - \frac{\partial S}{\partial q}(q)\right)\, dq = |\varphi(q)|^2 \delta_{(\Lambda, dq)}(q, p), \tag{2.11}$$

where $\delta_{(\Lambda,dq)}$ is the Dirac delta function on Λ corresponding to the measure dq: the action of this function on an arbitrary test function $\chi(q,p)$ is given by

$$< \delta_{(\Lambda,dq)}, \chi > \stackrel{\text{def}}{=} \int\limits_{\Lambda} \chi(q,p)|_{\Lambda} dq. \tag{2.12}$$

To obtain (2.11), it suffices to recall that, by definition,

$$< \nu, H > = \lim_{h\to 0}(\psi, \hat{H}\psi) \tag{2.13}$$

for an arbitrary classical Hamiltonian $H(q,p)$, rewrite the right-hand side of (2.13) in the form of an oscillatory integral, and apply the stationary phase method.

We point out that the oscillation front of the state (2.6) is Lagrangian. This is in fact a very deep phenomenon, and we shall shortly see that, indeed, the oscillation front of a "nice" semiclassical state ψ cannot be arbitrary.

For now, our aim is to learn how to obtain semiclassical states with oscillation front a given submanifold $M \subset \mathbf{R}^{2n}_{q,p}$ in the phase space (satisfying certain restrictions). To this end, we shall try to generalize the idea underlying the "discrete" example (2.1), that is, construct the desired state as a superposition of states with one-point oscillation fronts, i.e., Gaussian wave packets. Let us show how this can be done for the WKB-element (2.6). Since the oscillation front Λ is a manifold, instead of a discrete sum we have an integral over the manifold. To obtain the desired decomposition, we proceed as follows. First, we use the "approximate δ-function"

$$\delta_h(x) = \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} e^{-\frac{x^2}{2h}}, \quad h \to 0. \tag{2.14}$$

We have

$$\delta_h(x) \to \delta(x), \quad h \to 0, \tag{2.15}$$

(weak convergence), since

$$\int\limits_{-\infty}^{\infty} \delta_h(x)\, dx = 1 \tag{2.16}$$

and $\delta_h(x) = O(h^\infty)$ on any closed set that does not contain the point $x = 0$. We use identity (2.16) and write

$$\psi(x,h) = \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} \int e^{\frac{i}{h}\left[S(x)+\frac{i}{2}(x-y)^2\right]}\varphi(x)\, dy. \tag{2.17}$$

Our subsequent computations will be carried out modulo $O(\sqrt{h})$ and are somewhat heuristic in that we do not fill in all the details; note that all this serves only as a motivation for our subsequent constructions. We would like to replace the argument x in $\varphi(x)$ and $S(x)$ on the right-hand side in (2.17) by y, making only a small error as $h \to 0$. Since $\varphi(x)$ is independent of h, this is possible for $\varphi(x)$ in view of (2.15). However, we must be much more careful with the exponent, which contains h in the denominator. Here the following assertion proves to be of use:

Proposition 1

$$\left(e^{\frac{i}{h}at^3} - 1\right) e^{-\frac{t^2}{2h}} = O(\sqrt{h}), \quad h \to 0, \tag{2.18}$$

uniformly for real a from any compact set.

Indeed,

$$\left|e^{\frac{i}{h}at^3} - 1\right| = \left|\frac{it^3}{h}\int_0^a e^{\frac{i}{h}at^3 v}\,dv\right| \le \frac{|at^3|}{h}; \tag{2.19}$$

it remains to note that

$$\left|t^3 e^{-\frac{t^2}{2h}}\right| \le Ch^{\frac{3}{2}}, \tag{2.20}$$

where the constant C is independent of h.

Exercise. Prove the estimate (2.20).

Hint. Find the maximum over t of the left-hand side.

It follows from this proposition that it suffices to expand $S(x)$ in the Taylor series around $x = y$ and reject all terms of order ≥ 3. Then we obtain

$$\psi(x,h) \equiv \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} \int e^{\frac{i}{h}\left[S(y)+p(y)(x-y)+\frac{1}{2}<(Q(y)+iE)(x-y),x-y>\right]}\varphi(y)\,dy, \tag{2.21}$$

where $\equiv$ is congruence modulo $O(\sqrt{h})$, $p(y) = \frac{\partial S}{\partial y}(y)$, and $Q(y)$ is the matrix of the second derivatives of S at the point y.

The representation (2.21) can be viewed as an integral over the Lagrangian manifold Λ:

$$\psi(x,h) = \int_\Lambda \varphi(\alpha)\widetilde{G}(S(\alpha),Q(\alpha),q(\alpha),p(\alpha);x)\,dq(\alpha), \tag{2.22}$$

where $\widetilde{G}$ is the *modified Gaussian wave packet*

$$\widetilde{G}(S,Q,q,p;x) = \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} e^{\frac{i}{h}\left[S+p(x-q)+\frac{1}{2}<(Q+iE)(x-q),x-q>\right]}, \tag{2.23}$$

differing from the "standard" wave packet by the additional phase factor $e^{\frac{i}{h}S}$ and the additional quadratic form $\frac{1}{2} < Q(x-q), x-q >$ with real matrix Q in the exponent,

$q(\alpha)$ and $p(\alpha)$ are the equations of Λ in some local coordinate system, and $\varphi(\alpha)$, $S(\alpha)$, and $Q(\alpha)$ are the expressions of φ, S, and Q in that coordinate system.[2]

Although the expression (2.22) uses modified wave packets, it can be reduced to a form containing the usual wave packets[3]

$$G(q,p;x) = \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} e^{\frac{i}{h}\left[p(x-q)+\frac{i}{2}(x-q)^2\right]}. \tag{2.24}$$

Indeed, the factor $e^{\frac{i}{h}S}$ is independent of x and can be incorporated in the coefficients in the expansion (2.22). (Or otherwise, somebody may prefer to retain it in the definition of the Gaussian wave packet, for it is a unimodular constant factor, quite natural in quantum mechanics.) More interestingly, we can get rid of the quadratic term $\frac{1}{2} < Q(x-q), x-q >$ in the exponent at the expense of altering the coefficients $\varphi(\alpha)$. Namely, along with (2.22), let us consider the integral

$$\psi_1(x,h) = \int\limits_{\Lambda} e^{\frac{i}{h}S(\alpha)}\varphi_1(\alpha)G(q(\alpha),p(\alpha);x)\,dq(\alpha). \tag{2.25}$$

We claim that one can choose the amplitude $\varphi_1(\alpha)$ so that the integral (2.25), as well as (2.22), will represent the function (2.6). For clarity, let us explicitly write out both integrals as oscillatory integrals:

$$\psi(x,h) = \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} \int e^{\frac{i}{h}\left[S(q)+p(q)(x-q)+\frac{1}{2}<(Q(q)+iE)(x-q),(x-q)>\right]}\varphi(q)\,dq, \tag{2.26}$$

$$\psi_1(x,h) = \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} \int e^{\frac{i}{h}\left[S(q)+p(q)(x-q)+\frac{i}{2}(x-q)^2\right]}\varphi_1(q)\,dq. \tag{2.27}$$

The integral (2.26) is reduced to (2.6) by the stationary phase method; the stationary point of the phase satisfy the equations

$$q = x \tag{2.28}$$

for each x. Since the additional term $\frac{1}{2} < Q(q)(x-q), x-q >$ has a second-order zero at $q = x$, it follows that the integral (2.27) has the same stationary points and the same values of the phase function at these points as the integral (2.26). In both cases the stationary points are nondegenerate owing to the presence of the positive definite imaginary part $\frac{i}{2}(x-q)^2$ of the phase function. However, the Hessians of the phase are not the same; we obtain φ_1 as the product of φ by the square root of the ratio of these Hessians (with an appropriately chosen sign). The reader is encouraged to fill in the missing details.

[2]We avoid the more rigorous but awkward notation $\varphi(\alpha(q))$, $S(\alpha(q))$, etc.

[3]We use a normalization of wave packets different from that in Chapter 1.

We can now rewrite the representation (2.27) of the function $\psi \equiv \varphi_1$ in the form

$$\begin{aligned}\psi(x,h) &= \iint\limits_{\mathbf{R}^{2n}} G(q,p;x)e^{\frac{i}{h}S(q)}\varphi_1(q)\delta_{(\Lambda,dq)}dq\,dp \\ &= (\pi h)^{n/4}U^*\left[e^{\frac{i}{h}S(\alpha)}\varphi_1(\alpha)\delta_{(\Lambda,dq)}\right], \end{aligned} \tag{2.29}$$

where U^* is the adjoint of the wave packet transform U introduced in Chapter 1.

We see that the function ψ is obtained by an application of the adjoint transform U^* to a distribution (in fact, a measure) concentrated on the manifold $\Lambda = WF(\psi)$.

In our example, ψ was the simplest WKB-element in that the corresponding Lagrangian manifold is covered by the nonsingular chart (only the coordinate representation is needed, and there are no charts involving the mixed coordinate-momenta representation). However, the expression (2.29) is not dependent on the existence of the nonsingular coordinates, so that we can expect that this representation remains valid even in singular charts, that is, on the entire manifold. This will be studied in § 2, while in § 1 we introduce and recall some auxiliary material.

2.1 Auxiliary information

2.1.1 Properties of oscillation fronts and the wave packet transform

Both the notion of the oscillation front and the wave packet transform were introduced in Chapter 1. For the reader's convenience, here we recall some of these properties. We start from the definitions of oscillation support and oscillation front. Consider the space

$$H(\mathbf{R}^n_x) = \bigcap_s H^s(\mathbf{R}^n_x) \tag{2.30}$$

of semiclassical states, where $H^s(\mathbf{R}^n_x)$ is the *quantum Sobolev space* of functions $\psi(x,h)$ with finite norm

$$||\psi||_s = \sup_{h\in(0,1]} \left|\left|\left(1+x^2-h^2\frac{\partial^2}{\partial x^2}\right)^{s/2}\psi\right|\right|_{L^2(\mathbf{R}^n_x)}. \tag{2.31}$$

Let $\psi \in H^s(\mathbf{R}^n_x)$. We say that a point x_0 belongs to the oscillation support of ψ and write

$$x_0 \in \text{osc-supp}\,\psi$$

if $\varphi(x_0) = 0$ for every function $\varphi \in C_0^\infty(\mathbf{R}^n_x)$ such that $\varphi\psi = O(h^\infty)$. Next, we say that a point $(q_0,p_0) \in \mathbf{R}^{2n}$ belongs to the oscillation front of ψ and write

$$(q_0,p_0) \in OF(\psi)$$

if $H(q_0, p_0) = 0$ for every function $H \in C_0^\infty(\mathbf{R}^{2n})$ such that $\hat{H}\psi = O(h^\infty)$ (here $\hat{H}$ is the quantum Hamiltonian operator corresponding to the symbol H). Then

$$\pi(OF(\psi)) = \text{ osc-supp } \psi,$$

where

$$\pi : \mathbf{R}^{2n} \to \mathbf{R}^n, \qquad (q,p) \mapsto q,$$

is the natural projection. Moreover, for any Hamiltonian function $H(q,p)$, one has

$$OF(\hat{H}u) \subset OF(u) \subset OF(\hat{H}u) \cup \{(q,p) \,|\, H(q,p) = 0\}, \tag{2.32}$$

and if $H(q,p) = 0$ in a neighborhood of the point (q_0, p_0), then $(q_0, p_0) \neq OF(\hat{H}u)$.

The property that is possibly most important to us is the behavior of the oscillation front under the wave packet transform. We state this property in the form of a theorem.

Theorem 2 *The wave packet transform has the following properties:*

(i) $OF(\psi) =$ osc-supp $(U\psi)$ *for every* $\psi \in H(\mathbf{R}^n_x)$.

(ii) $OF(U^*f) \subset$ osc-supp f *for every* $f \in H(\mathbf{R}^{2n}_{q,p})$.

Remark 1 The inclusion (ii) becomes an equality whenever $f \in \mathcal{F}H(\mathbf{R}^{2n}_{q,p})$, where $\mathcal{F}H(\mathbf{R}^{2n}_{q,p})$ is the range of U in $H(\mathbf{R}^{2n}_{q,p})$. This follows from assertion (i) of the theorem.

The rather technical proof of both assertions is based on the stationary phase method, and we omit it.

2.1.2 Complex pseudocoordinates

In what follows we shall frequently use *complex pseudocoordinates* on the phase space and on submanifolds of it. Let us give the corresponding definitions. Let M be a real manifold of dimension k. A k-tuple $(z_1, \ldots, z_k)$ of smooth complex-valued functions defined in a neighborhood of a point $\alpha_0 \in M$ is called a *pseudocoordinate system* on M near α_0 if $\det \frac{\partial z}{\partial \alpha} \neq 0$ in this neighborhood, where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is an arbitrary coordinate system on M near α_0. If $z = (z_1, \ldots, z_k)$ is a pseudocoordinate system on M, then we introduce the derivatives

$$\frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_k} \right)$$

of an arbitrary function $f \in C^\infty(M)$ by the formula

$$\frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial \alpha} \right)^{-1} \frac{\partial f}{\partial \alpha}, \tag{2.33}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is an arbitrary coordinate system on M. One can readily see that this is well-defined (i.e. independent of the choice of α), and it follows that if $z = (\beta_1, \dots, \beta_n)$ is a system of real-valued functions, then

$$\frac{\partial f}{\partial z} = \left(\frac{\partial \beta}{\partial \alpha}\right)^{-1} \frac{\partial f}{\partial \alpha} = \frac{\partial f}{\partial \beta}, \tag{2.34}$$

that is, we obtain the usual partial derivatives. The derivatives (2.33) possess most of the properties of usual partial derivatives. In particular, we have

$$\frac{\partial^2}{\partial z_k \partial z_l} = \frac{\partial^2}{\partial z_l \partial z_k}, \tag{2.35}$$

the usual expression

$$df = \frac{\partial f}{\partial z} dz = \sum_{j=1}^{k} \frac{\partial f}{\partial z_j} dz_j \tag{2.36}$$

for the differential holds, and similarly, if

$$\omega = \sum \omega_{i_1 \dots i_k} dz_{i_1} \wedge \dots \wedge dz_{i_k}, \tag{2.37}$$

then

$$d\omega = \sum \frac{\partial \omega_{i_1 \dots i_k}}{\partial z_s} dz_s \wedge dz_{i_1} \wedge \dots \wedge dz_{i_k}. \tag{2.38}$$

Next, one has an analogue of the Taylor series expansion:

$$f(\alpha) - f(\alpha_0) = \sum_{|\gamma|=0}^{N=1} \frac{(z(\alpha) - z(\alpha_0))^\gamma}{\gamma!} \frac{\partial^\gamma f}{\partial z^\gamma}(\alpha_0) + O(|\alpha - \alpha_0|^N). \tag{2.39}$$

Here $\gamma = (\gamma_1, \dots, \gamma)$ is a multiindex; instead of $O(|\alpha - \alpha_0|^N)$ one can safely write $O(|z - z_0|^N)$. We encourage the reader to prove all these properties by way of exercise. We shall use special pseudocoordinates on the phase space $\mathbf{R}^{2n}_{q,p}$ and on Lagrangian submanifolds $\Lambda \subset \mathbf{R}^{2n}_{q,p}$. Let

$$z_j = q_j + ip_j, \quad \overline{z}_j = q_j - ip_j, \quad j = 1, \dots, n. \tag{2.40}$$

Thus, in effect we have equipped the phase space $\mathbf{R}^{2n}$ with a complex structure.

Lemma 3 *The $2n$-tuple*

$$(z, \overline{z}) = (z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_n) \tag{2.41}$$

is a pseudocoordinate system on $\mathbf{R}^{2n}_{q,p}$.

Proof. One has

$$\frac{\partial(z,\overline{z})}{\partial(q,p)} = \begin{pmatrix} E & iE \\ E & -iE \end{pmatrix}.$$

By subtracting the first row from the second, we obtain the matrix

$$\begin{pmatrix} E & iE \\ 0 & -2iE \end{pmatrix},$$

which is obviously nondegenerate.

Lemma 4 *Let $\Lambda \subset \mathbf{R}^{2n}$ be a Lagrangian manifold. Then the restrictions of the functions $z = (z_1, \ldots, z_n)$ to Λ (which will be denoted by the same letters) form a pseudo-coordinate system on Λ.*

Proof. We must prove that

$$\frac{D(q+ip)}{D\alpha} \equiv \det \frac{\partial(q+ip)}{\partial\alpha} \neq 0. \tag{2.42}$$

For brevity, we write

$$A = \frac{\partial q}{\partial \alpha}, \quad B = \frac{\partial p}{\partial \alpha}.$$

Since Λ is a submanifold, it follows that the mapping

$$\alpha \mapsto (q(\alpha), p(\alpha))$$

is an embedding. Consequently, the matrix

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

is full rank, whence

$$\det(A^*A + B^*B) \neq 0. \tag{2.43}$$

Next, we use the Lagrangian property of Λ. We have $dp \wedge dq|_\Lambda = 0$, or

$$(B d\alpha) \wedge (A d\alpha) = 0. \tag{2.44}$$

In more detail,

$$0 = \sum_{j,m,s=1}^{n} B_{jm} A_{js} d\alpha_m \wedge d\alpha_s = \sum_{j=1}^{n} \sum_{1 \le m < s \le n} [B_{jm} A_{js} - B_{js} A_{jm}]\, d\alpha_m \wedge d\alpha_s \tag{2.45}$$

(we have used the antisymmetry of the exterior product). Since the products $d\alpha_m \wedge d\alpha_s$ with $m < s$ form a basis in $\Lambda^2(\Lambda)$, it follows that

$$B_{jm}A_{js} - B_{js}A_{jm} = 0 \tag{2.46}$$

for $1 \le m < s \le n$. But then (2.46) is true for $s > m$ by antisymmetry, while for $s = m$ it is valid automatically. All in all, (2.46) holds for any s and m, so that in the matrix notation we have

$$B^*A - A^*B = 0. \tag{2.47}$$

It remains to note that

$$(A^* - iB^*)(A + iB) = A^*A + B^*B + i(A^*B - B^*A) = A^*A + B^*B, \tag{2.48}$$

whence it follows from (2.43) that (2.42) holds. The proof is complete.

Let us present the expressions of some useful quantities in complex pseudocoordinates.

We start from the expressions for partial derivatives in $\mathbf{R}^{2n}_{p,q}$. It obviously follows from (2.40) that

$$\frac{\partial}{\partial q} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \quad \frac{\partial}{\partial p} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right), \tag{2.49}$$

and consequently,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial q} - i\frac{\partial}{\partial p}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial q} + i\frac{\partial}{\partial p}\right). \tag{2.50}$$

Next, we proceed to differential forms. We have

$$dz = dq + idp, \quad d\overline{z} = dq - idp, \tag{2.51}$$

and accordingly,

$$dq = \frac{1}{2}(dz + d\overline{z}), \quad dp = \frac{1}{2i}(dz - d\overline{z}). \tag{2.52}$$

Now the fundamental 1-form $p\,dq$ in the complex pseudocoordinates has the expression

$$\omega^1 \equiv p\,dq = \frac{1}{4i}(z - \overline{z})(dz + d\overline{z}), \tag{2.53}$$

and the symplectic form is given by the expression

$$\omega^2 \equiv dp \wedge dq = \frac{1}{2i}dz \wedge d\overline{z}. \tag{2.54}$$

Now let us write out the expression for the Hamiltonian vector field. The Hamiltonian vector field $V(H)$ of a Hamiltonian function H has the characteristic property

$$V(H)\rfloor\omega^2 = -dH. \tag{2.55}$$

(Here $\rfloor$ stands for inner multiplication). Let

$$V(H) = a\frac{\partial}{\partial z} + b\frac{\partial}{\partial \overline{z}}. \tag{2.56}$$

Substituting this into the left-hand side of (2.55), we obtain, with regard to (2.54)

$$-dH = \frac{1}{2i}a\,d\overline{z} - \frac{1}{2i}b\,dz,$$

whence

$$a = -2i\frac{\partial H}{\partial \overline{z}}, \quad b = 2i\frac{\partial H}{\partial z}.$$

Finally, we see that

$$V(H) = 2i\left(\frac{\partial H}{\partial z}\frac{\partial}{\partial \overline{z}} - \frac{\partial H}{\partial \overline{z}}\frac{\partial}{\partial z}\right). \tag{2.57}$$

Now suppose that we are given a Lagrangian manifold $\Lambda \subset \mathbf{R}^{2n}$ and a Hamiltonian vector field $V(H)$. We are especially interested in the case in which Λ is invariant with respect to $V(H)$, that is, $V(H)$ is tangent to Λ. The following lemma gives necessary and sufficient conditions for this.

Lemma 5 *The Lagrangian manifold Λ is invariant with respect to the Hamiltonian vector field $V(H)$ if and only if $H|_\Lambda = \mathrm{const}$.*

Proof. *Necessity.* Suppose that Λ is invariant. Then

$$dH(\xi) = -\omega^2(V(H), \xi) = 0$$

for any tangent vector $\xi \in T\Lambda$, since Λ is Lagrangian. It follows that H is (locally) constant on Λ.

Sufficiency. Let H be (locally) constant on Λ. Then

$$\omega^2(V(H)_\alpha, \xi) = -dH(\xi) = 0$$

for any tangent vector $\xi \in T_\alpha\Lambda$, that is, $V(H)_\alpha$ belongs to the skew-orthogonal complement of $T_\alpha\Lambda$, which coincides with $T_\alpha\Lambda$, since $T_\alpha\Lambda$ is Lagrangian. The proof of Lemma 5 is complete.

Now let $V(H)$ be tangent to Λ, and let φ be a smooth function on Λ. Let us compute $V(H)\varphi$ at an arbitrary point $\alpha_0 \in \Lambda$. It suffices to assume that Λ is a plane passing through the point $\alpha_0 = (0,0)$ and that φ is linear: $\varphi(z) = bz$ in the pseudocoordinates z on Λ. We extend φ to the entire phase space by the same formula. Now, to compute $V(H)\varphi$, we can apply formula (2.57). Then we obtain

$$V(H)\varphi = -2i\frac{\partial H}{\partial \overline{z}}\frac{\partial \varphi}{\partial z}, \tag{2.58}$$

or

$$V(H)|_\Lambda = -2i\frac{\partial H}{\partial \overline{z}}\frac{\partial}{\partial z}. \tag{2.59}$$

Note the difference between the expressions (2.57) and (2.59): the former gives the Hamiltonian vector field in the phase space, while the latter gives the restriction of $V(H)$ to the invariant manifold Λ.

The formulas obtained here will be used in the sequel.

2.1.3 Functions of noncommuting operators

Throughout the chapter, we shall freely use the notation and techniques of noncommutative analysis (functions of noncommuting operators), which can be found in the books [6, 7] as well as in Appendix[4] of the present book. We recall that while various precautions and assumptions are needed when one deals with general functions of noncommuting operators, everything is almost automatic as long as only polynomial symbols are considered. Then virtually no functional-analytic assumptions on the operators (like self-adjointness, normality, etc.) are necessary, and the theory is essentially algebraic.

2.2 Oscillation fronts and Lagrangian manifolds

The example considered in § 0 suggests a method for constructing a function with a prescribed wave front $M \subset \mathbf{R}^{2n}_{q,p}$, where M is a manifold: we take the distribution

$$\chi = e^{\frac{i}{h}S(\alpha)}\varphi(\alpha)\delta_{(M,d\mu)}, \tag{2.60}$$

where $\delta_{(M,d\mu)}$ is the δ-function on M with some measure $d\mu$ on M and S, φ are some smooth functions on M, and set

$$\psi = U^*\chi, \tag{2.61}$$

where U^* is the adjoint of the wave packet transform; then one can expect that

$$OF(\psi) = \overline{\{\alpha \in M \mid \varphi(\alpha) \neq 0\}} \tag{2.62}$$

[4]To be written yet.

if the phase $S(\alpha)$ is chosen appropriately. Actually, the inclusion

$$OF(\psi) \subset \overline{\{\alpha \in M \mid \varphi(\alpha) \neq 0\}} \tag{2.63}$$

is easy to establish. (We leave this to the reader as an exercise.) But the opposite inclusion cannot be achieved in general. This will be explained in detail in Subsection 2.1. We also note that along with (2.61), there is another natural method of constructing a function whose oscillation front is a given manifold M. Namely, we take a function $\chi \in \mathcal{F}H(\mathbf{R}^{2n}_{q,p})$ such that osc-supp $\chi = M$ and then set

$$\psi = U^{-1}\chi; \tag{2.64}$$

by Theorem 2, (i) we then have $OF(\psi) = M$, so it might seem that this method would provide a wider supply of possible oscillation fronts. However, this is not the case; we shall see in Subsection 2.2 that the inclusion $\chi \in \mathcal{F}H(\mathbf{R}^{2n}_{q,p})$ imposes serious restrictions on the possible oscillation support of χ, at least in the class of WKB-type semiclassical functions. That is why in subsequent sections we only use the first method of quantization of states, related to formula (2.61).

2.2.1 Quantization of states by continuous superposition (the adjoint wave packet transform)

Thus, we seek quantum states corresponding to classical states concentrated on a submanifold M in the form (2.61), assuming that S and φ are smooth functions, φ is compactly supported, the imaginary part of S is nonnegative, and $d\mu$ is some smooth measure (volume form) on M (possibly, with complex coefficients). Let us rewrite the function (2.61) explicitly in the form of an oscillatory integral:

$$\psi(x,h) = \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} \int\limits_M e^{\frac{i}{h}\Phi(x,\alpha)}\varphi(\alpha)\,d\mu(\alpha), \tag{2.65}$$

where

$$\Phi(x,\alpha) = S(\alpha) + p(\alpha)(x - q(\alpha)) + \frac{i}{2}(x - q(\alpha))^2. \tag{2.66}$$

Let us study the oscillation front of the function (2.65). We take an arbitrary point $(q_0, p_0) \in \mathbf{R}^{2n}_{q,p}$. The question is, when does (q_0, p_0) belong to the oscillation front on ψ? Let us take a smooth function $H(q,p)$ supported in a small neighborhood of the point (q_0, p_0). We compute $H\psi$:

$$\begin{aligned} H\psi &\equiv H(x,p)\psi \qquad\qquad (2.67)\\ &= \left(\frac{1}{2\pi h}\right)^{\frac{3n}{2}} \iiint e^{\frac{i}{h}[p(x-y)+\Phi(y,\alpha)]} H(x,p)\varphi(\alpha)\,d\mu(\alpha)\,dy\,dp. \end{aligned}$$

The stationary point equations for the phase function

$$\Psi(p, x, y, \alpha) = p(x - y) + \Phi(y, \alpha) \tag{2.68}$$

of the integral (2.67) read

$$\begin{cases} \dfrac{\partial \Psi}{\partial p} = x - y = 0, \\ \dfrac{\partial \Psi}{\partial y} = -p + p(\alpha) + i(y - q(\alpha)) = 0, \\ \dfrac{\partial \Psi}{\partial \alpha} = \dfrac{\partial p}{\partial \alpha}(y - q(\alpha)) + \dfrac{\partial S(\alpha)}{\partial \alpha} - p(\alpha)\dfrac{\partial q(\alpha)}{\partial \alpha} + i\dfrac{\partial q(\alpha)}{\partial \alpha}(q(\alpha) - y) = 0. \end{cases} \tag{2.69}$$

Thus, for given x, at the stationary points one has

$$\begin{aligned} &q(\alpha) = y = x, \\ &p = p(\alpha), \end{aligned} \tag{2.70}$$

$$dS(\alpha) = p(\alpha)\, dq(\alpha). \tag{2.71}$$

Suppose that $(q_0, p_0) \notin M$. Then the equations

$$x = q_0 = q(\alpha) \quad \text{and} \quad p = p_0 = p(\alpha) \tag{2.72}$$

are inconsistent, and one of the first two equations in (2.70) is necessary violated on supp $H(x, p)$ (provided that the support is sufficiently small), so that $\hat{H}\psi = O(h^\infty)$. If $(q_0, p_0) \in M$, then let α_0 be the coordinate tuple on M such that

$$q(\alpha_0) = q_0, \quad p(\alpha_0) = p_0. \tag{2.73}$$

We see that again $\hat{H}\psi = O(h^\infty)$ unless

$$dS(\alpha_0) = p(\alpha_0)\, dq(\alpha_0). \tag{2.74}$$

We have thereby proved the following assertion.

Lemma 6 *If a point (q_0, p_0) belongs to $OF(\psi)$, then $(q_0, p_0) \in M$, $(q_0, p_0) \in \operatorname{supp} \varphi$, and relation* (2.74) *holds.*

Exercise. Prove that the assumptions of the lemma are not only necessary, but also sufficient for the inclusion $(q_0, p_0) \in OF(\psi)$.

Now if we wish that

$$OF(\psi) = M \cap \operatorname{supp} \varphi \tag{2.75}$$

for any completely supported amplitude function φ, then we must require (2.74) for any $\alpha_0 \in M$, that is,

$$dS = p\,dq|_M. \tag{2.76}$$

In particular, it follows by applying the exterior differential to both sides of (2.76) that

$$\omega^2|_M \equiv dp \wedge dq|_M = 0, \tag{2.77}$$

that is, M is *isotropic*. In particular, the dimension of M does not exceed n. Indeed, Eq. (2.77) just means that M is contained in the skew-orthogonal complement $M^\perp$ of itself with respect to the form ω^2 : $M \subset M^\perp$. Since the symplectic form ω^2 is nondegenerate, it follows that dim $M^\perp = 2n - \dim M$. Hence

$$\dim M \le 2n - \dim M, \tag{2.78}$$

or dim $M \le n$, as desired.

In this book we shall only consider the simplest case in which M is of maximal dimension M. In this case, M is called a *Lagrangian* manifold.

2.2.2 Quantization by the inverse wave packet transform

Now we try a different possibility, namely, formula (2.64). In this case, χ must be a function from $\mathcal{F}H(\mathbf{R}^{2n}_{q,p})$ with osc-supp$(\chi) = M$. We shall consider the simplest case in which the function χ itself has the WKB form

$$\chi(q,p) = e^{\frac{i}{h}\Phi(q,p)}a(q,p,h) + O(h^N), \tag{2.79}$$

where N can be chosen as large as desired. Here

$$\Phi(q,p) = \Phi_1(q,p) + i\Phi_2(q,p) \tag{2.80}$$

is complex-valued and satisfies $\Phi_2(q,p) \ge 0$, and a is regular in h. Clearly, we have

$$\text{osc-supp}\,\chi = \{(q,p)\,|\,\Phi_2(q,p) = 0\} \cap \operatorname{supp} a, \tag{2.81}$$

and if we wish to have osc-supp $\chi = M$, we must require that M be the set of zeros of $\Phi_2(q,p)$.

Now we exploit the condition $\chi \in \mathcal{F}H(\mathbf{R}^{2n}_{q,p})$.

Lemma 7 *On the support of a, one has*

$$\frac{\partial \Phi}{\partial z} = \frac{p}{2} + O(\Phi_2^\infty) \tag{2.82}$$

(here we use the pseudocoordinates $(z,\overline{z})$ on $\mathbf{R}^{2n}_{q,p}$ introduced in § 1.)

Proof. The condition for χ to belong to $\mathcal{F}H(\mathbf{R}^{2n}_{q,p})$ reads

$$2h\frac{\partial\chi}{\partial z} - ip\chi = 0 \tag{2.83}$$

(see Chapter 1). Substituting (2.79) into (2.83) and matching the coefficients of h^0, we obtain

$$e^{\frac{i}{h}\Phi(q,p)}\left(2i\frac{\partial\Phi}{\partial z} - ip\right)a(q,p,0) = O(h), \tag{2.84}$$

whence it follows that

$$\frac{\partial\Phi}{\partial z} = \frac{p}{2} + O(\Phi_2) \tag{2.85}$$

(see [6, Lemma 4.1, p. 470]). To obtain arbitrarily high power of Φ_2 in the estimate, one has to consider subsequent terms in the expansion in h; we omit this procedure to avoid cumbersome calculations.

Lemma 8 *The manifold M is isotropic on the support of a.*

Proof. We use (2.85). On M we have

$$\Phi_2 = 0, \quad d\Phi_2 = 0, \tag{2.86}$$

since Φ_2 is nonnegative. Consequently,

$$\frac{\partial\Phi_1}{\partial z} = \frac{1}{2}\frac{\partial\Phi_1}{\partial q} - \frac{i}{2}\frac{\partial\Phi_1}{\partial p} = \frac{p}{2} \quad \text{on} \quad M, \tag{2.87}$$

whence

$$\left.\frac{\partial\Phi_1}{\partial q}\right|_M = p, \quad \left.\frac{\partial\Phi_1}{\partial p}\right|_M = 0. \tag{2.88}$$

Differentiating (2.88), we obtain the following equations on the tangent space $T_\alpha M$ at an arbitrary point $\alpha \in M$:

$$\begin{array}{l} \dfrac{\partial^2\Phi_1}{\partial p\partial q}\,dq + \dfrac{\partial^2\Phi_1}{\partial p^2}\,dp = 0, \\ \dfrac{\partial^2\Phi_1}{\partial q\partial p}\,dp + \dfrac{\partial^2\Phi_1}{\partial q^2}\,dq = dp. \end{array} \tag{2.89}$$

It follows that

$$\begin{aligned} dp\wedge dq &= \frac{\partial^2\Phi_1}{\partial q\partial p}dp\wedge dq + \frac{\partial^2\Phi_1}{\partial q^2}dq\wedge dq \\ &= -\frac{\partial^2\Phi_1}{\partial p^2}dp\wedge dp \\ &= 0 \end{aligned} \tag{2.90}$$

on M, that is, M is isotropic. The proof of the lemma is complete.

We see that the method based on the use of U^{-1} does not provide a wider supply of oscillation fronts than the method based on the use of U^*. In what follows we shall only use the adjoint transform U^*, for this approach results in much simpler formulas.

2.3 Maslov's canonical operator. The special construction

We take a Lagrangian submanifold Λ of the phase space $\mathbf{R}^{2n}_{q,p}$. Our aim in the present section is to construct a mapping taking each smooth compactly supported function $\varphi \in C_0^\infty(\Lambda)$ to a semiclassical wave function ψ with $OF(\psi) = \Lambda \cap \operatorname{supp} \varphi$ with the following properties:

(1) The mapping
$$\varphi \mapsto \psi = K\varphi \in H(\mathbf{R}^n_x)$$
is linear.

(2) If $H(q,p)$ is a Hamiltonian function, then the following diagram commutes modulo $O(h)$:

$$\begin{array}{ccc} C_0^\infty(\Lambda) & \xrightarrow{H(q,p)|_\Lambda} & C_0^\infty(\Lambda) \\ \Big\downarrow K & & \Big\downarrow K \\ H(\mathbf{R}^n) & \xrightarrow{\hat{H}=H(q,\hat{p})} & H(\mathbf{R}^n). \end{array} \tag{2.91}$$

The commutativity of diagram (2.91) can be interpreted as follows. The space $C_0^\infty(\Lambda)$ of smooth compactly supported functions on the Lagrangian manifold Λ is a module over the ring $S^\infty(\mathbf{R}^{2n})$ of Hamiltonian functions $H(q,p)$: the multiplication is given by

$$H \cdot \varphi = H_\Lambda \varphi, \tag{2.92}$$

where on the right-hand side the usual multiplication of functions is used. The space $C^\infty(\Lambda)$ equipped with this structure will be referred to as a *Lagrangian module*.

On the other hand, the space $H(\mathbf{R}^n)$ has the structure of a left module over quantum observables (h-pseudodifferential operators); the mapping K (modulo $O(h)$) is an *intertwining operator* between these two structures. This explains why the title of this chapter involves the words "quantization of Lagrangian modules." We try to define the mapping K by formula (2.61) with χ given by (2.60), where $\varphi \in C_0^\infty(\Lambda)$ is the function to which the mapping K is applied and the other elements in the formula (namely, S and $d\mu$) are yet to be defined.

We start from $d\mu$. This must be a measure (volume form) on Λ. Although there are countless possibilities to choose this form, we take advantage of the special pseudocoordinates $(z_1, \ldots, z_n)$ on M, guaranteed by Lemma 4. Namely, we take

$$d\mu = sz \equiv dz_1 \wedge \ldots \wedge dz_n|_\Lambda. \tag{2.93}$$

This form will be referred to as the *special measure* on Λ.

Next, we must find a function S on Λ such that

$$dS = p\, dq|_\Lambda. \tag{2.94}$$

The manifold Λ is Lagrangian, i.e. $dp \wedge dq|_\Lambda = 0$. Hence the form $p\, dq$ is locally exact (Eq. (2.94) is locally solvable). However, we need a global solution, which does not necessarily exist. A necessary and sufficient condition for the existence of a global solution is that the cohomology class of the form $p\, dq|_\Lambda$ is zero, that is,

$$[p\, dq|_\Lambda] = O \in H^1(\Lambda, \mathbf{R}). \tag{2.95}$$

Condition (2.95) is known as the *first quantization condition.* We shall assume for now that this condition is satisfied and fix some solution S of the Pfaff equation (2.94).

(If Λ is connected, then a solution S can be singled out by choosing a "*marked point*" $\alpha_0 \in \Lambda$ and by requiring $S(\alpha_0) = 0$.)

Once we have fixed the function S, our operator K is determined. We introduce the corresponding definition.

Definition 9 *Maslov's special canonical operator* on Λ is the operator acting by the formula

$$\begin{aligned} K_\Lambda \quad &: \quad C_0^\infty(\Lambda) \to H(\mathbf{R}^n_x) \qquad (2.96)\\ K_\Lambda \varphi \quad &= \quad \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} \int\limits_\Lambda e^{\frac{i}{h}\left[S(\alpha)+p(\alpha)(x-q(\alpha))+\frac{i}{2}(x-q(\alpha))^2\right]} \varphi(\alpha)\, dz(\alpha). \end{aligned}$$

Needless to say, the canonical operator depends on the choice of the marked point (of the function S). The function S will be referred to as the *action* on Λ.

Of course, one must also verify that the operator K_Λ is well-defined, that is, $K_\Lambda\varphi \in H(\mathbf{R}^n_x)$ whenever $\varphi \in C_0^\infty(\Lambda)$. We leave this easy exercise to the reader.

Remark 2 In Definition 9 (Eq. (2.96)), one might wish to give a wider definition by allowing the amplitude $\varphi(\alpha)$ depend also on x : $\varphi = \varphi(x, \alpha)$. However (at least, in the leading term, and actually, modulo an arbitrary power of h) this does not change the class of functions representable in the form $K_\Lambda\varphi$. Indeed, let us represent $\varphi(x,\alpha)$ by the Taylor series in powers of $x-q(\alpha)$:

$$\varphi(x,\alpha)=\varphi(q(\alpha),\alpha)+\sum_{|\gamma=0}^{N-1}F_\gamma(\alpha)(x-q(\alpha))^\gamma+\text{remainder}. \tag{2.97}$$

One has

$$\frac{\partial\Phi(x,\alpha)}{\partial\alpha}=-i(x-q(\alpha))\frac{\partial z(\alpha)}{\partial\alpha}, \tag{2.98}$$

where $\Phi(x,\alpha)$ is the phase function of the integral (2.96), or

$$x-q(\alpha)=i\frac{\partial\Phi}{\partial\alpha}(x,\alpha)\left(\frac{\partial z(\alpha)}{\partial\alpha}\right)^{-1}. \tag{2.99}$$

Thus, we can rewrite (2.97) in the form

$$\varphi(x,\alpha)=\varphi(q(\alpha),\alpha)+\sum_{|\gamma|=0}^{N-1}\widetilde{F}_\gamma(\alpha)\left(\frac{\partial\Phi}{\partial\alpha}\right)^\gamma+\text{remainder}, \tag{2.100}$$

where the new remainder contains products $\left(\frac{\partial\Phi}{\partial\alpha}\right)^\gamma$ with $|\gamma|=N$. Thus, all terms on the right-hand side in (2.97) except for the first belong to the so-called *gradient ideal* $\mathcal{J}(\Phi)$ generated by the partial derivatives $\frac{\partial\Phi(x,\alpha)}{\partial\alpha_j}$, $j=1,\dots,n$. Substituting the expression (2.97) for φ into (2.96), we see that integration by parts is possible in all these terms and that the corresponding integrals are $O(h^N)$, where N is the number of integrations by part.

Next, we need to prove that Maslov's special canonical operator thus defined provides the mod $O(h)$-commutativity of the diagram (2.91). In doing so, we at the same time will try to learn how to solve pseudodifferential equations in the class of semiclassical wave functions. This will be done in the next section.

2.4 Commutation with quantum Hamiltonians

In this section we deal with the following problem. Given a semiclassical wave function of the form $K_\Lambda\varphi$ (2.96) and a quantum Hamiltonian $\hat{H}$, how to compute $\hat{H}K_\Lambda\varphi$? Once we know the answer, we shall be able to solve equations of the form

$$\hat{H}u\equiv H(x,\hat{p})u=0 \tag{2.101}$$

modulo high powers of h. As a by-product, we shall also prove the commutativity of diagram (2.91). For simplicity, we assume that the operator $H(x,\hat{p})$ is *differential*, that is,

$$H(q,p)=\sum_{|\gamma|\leq m} a_\gamma(q)p^\gamma \tag{2.102}$$

is polynomial. (This requirement is actually unessential.) The following theorem holds.

Theorem 10 (the first term in the commutation formula with the Hamiltonian) *Under the above assumptions, one has*

$$\hat{H}K_\Lambda\varphi = K_\Lambda(H(q,p)|_\Lambda\varphi)+O(h) \tag{2.103}$$

for any function $\varphi\in C_0^\infty(\Lambda)$.

Proof. We have

$$\hat{p}e^{\frac{i}{h}\Phi(x,\alpha)}=e^{\frac{i}{h}\Phi(x,\alpha)}\left(\hat{p}+\frac{\partial\Phi}{\partial x}(x,\alpha)\right), \tag{2.104}$$

where $\Phi(x,\alpha)$ is the phase function of the integral (2.96). Let us use the following theorem of noncommutative analysis (e.g., see [7]): if

$$AC=CB, \tag{2.105}$$

then

$$f(A)C=Cf(B) \tag{2.106}$$

for any symbol $f(y)$ provided that A and B are generators in the relevant symbol class. In our case, $H(q,p)$ is a polynomial in p, so that all operators are generators, and from (2.104) we obtain

$$H(x,\hat{p})e^{\frac{i}{h}\Phi(x,\alpha)}=e^{\frac{i}{h}\Phi(x,\alpha)}H\left(\overset{2}{x},\overset{1}{\overline{\hat{p}+\frac{\partial\Phi}{\partial x}(x,\alpha)}}\right). \tag{2.107}$$

(The numbers over the operators on the right-hand side in (2.107) are Feynman numbers; they indicate the order in which these operators act).

Now we use the following formula (e.g., see [8]):

$$f\left(\hat{p}+\frac{\partial S}{\partial x}\right)=f\left(\overset{2}{\hat{p}}+\frac{\delta S}{\delta x}\left(\overset{1}{x},\overset{3}{x}\right)\right), \tag{2.108}$$

where $\frac{\delta S}{\delta x}$ is the difference derivative,

$$\frac{\delta S}{\delta x}(x,y)=\frac{S(x)-S(y)}{x-y}. \tag{2.109}$$

Substituting this into (2.107) with $f = H$ and $S = \Phi$, we obtain, by expanding in Taylor series and commuting,

$$\begin{aligned} H\left(\overset{2}{x}, \overset{1}{\overline{\hat{p} + \frac{\partial \Phi}{\partial x}}}\right) &= H\left(x, \frac{\partial \Phi}{\partial x}\right) - ihH_p\left(x, \frac{\partial \Phi}{\partial x}\right)\frac{\partial}{\partial x} \\ &\quad -ihH_{pp}\left(x, \frac{\partial \Phi}{\partial x}\right)\frac{\partial^2 \Phi}{\partial x^2} + O(h^2). \end{aligned} \tag{2.110}$$

Next, we expand $H\left(x, \frac{\partial \Phi}{\partial x}\right)$ in power of $x - q(\alpha)$:

$$H\left(x, \frac{\partial \Phi}{\partial x}\right) = H(q(\alpha), p(\alpha)) + (x - q(\alpha))F(x, \alpha), \tag{2.111}$$

where

$$F(x, \alpha) = \frac{\delta H}{\delta x}(x, q(\alpha); p(\alpha)) + i\frac{\delta H}{\delta p}\left(x; p(\alpha), \frac{\partial \Phi}{\partial x}\right). \tag{2.112}$$

As was already mentioned in the preceding (see Remark 2), terms of the form $(x - q(\alpha))F(x, \alpha)$ belong to the gradient ideal $\mathcal{J}(\Phi)$ and hence produce $O(h)$ in the integral. Thus we have

$$\begin{aligned} \hat{H}(K_\Lambda \varphi)(x) &= \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} \int\limits_\Lambda H(x, \hat{p}) e^{\frac{i}{h}\Phi(x,\alpha)} \varphi(\alpha)\, d\mu(\alpha) \\ &= \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} \int\limits_\Lambda e^{\frac{i}{h}\Phi(x,\alpha)} (H(q(\alpha), p(\alpha)) \\ &\quad + (x - q(\alpha))F(x, \alpha))\varphi(\alpha)\, d\mu(\alpha) + O(h) \\ &= \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} \int\limits_\Lambda e^{\frac{i}{h}\Phi(x,\alpha)} (H|_\Lambda \varphi)(\alpha)\, d\mu(\alpha) + O(h). \end{aligned} \tag{2.113}$$

The proof of the theorem is complete.

Remark 3 The assertion of the theorem remains valid for nonpolynomial symbols $H(x, p)$. However, the proof is entirely different and is based on stationary phase computations like that used for the integrals (2.26), (2.27). We omit these cumbersome computations.

Corollary 11 *The diagram* (2.91) *commutes modulo* $O(h)$.

Now suppose that we intend to find an asymptotic solution of the equation

$$\hat{H}\psi = 0 \tag{2.114}$$

in the form

$$\psi = K_\Lambda \varphi. \tag{2.115}$$

It follows from Theorem 10 that Λ must be a Lagrangian manifold lying on the level surface $H = 0$ of the Hamiltonian function. This guarantees us that

$$\hat{H}\psi = K_\Lambda(H|_\Lambda \varphi) + O(h) = O(h). \tag{2.116}$$

However, it is well known that the estimate (2.116) does not guarantee that ψ is close to an asymptotic solution even if $\hat{H}$ has certain nice properties (i.e., is self-adjoint, has discrete spectrum, etc.) The minimum meaningful requirement is

$$\hat{H}\psi = O(h^2). \tag{2.117}$$

The ensure this, we must find the second term in the commutation formula.

Theorem 12 (the second term in the commutation formula) *Suppose that the Lagrangian manifold Λ lies on the zero level of the Hamiltonian function $H(q,p)$:*

$$H(q,p)|_\Lambda = 0. \tag{2.118}$$

Then the following commutation formula holds:

$$\hat{H} K_\Lambda \varphi = -ih K_\Lambda(\mathcal{P}\varphi) + O(h^2) \tag{2.119}$$

for every $\varphi \in C_0^\infty(\Lambda)$, where $\mathcal{P}$ is the first-order transport operator *on Λ:*

$$\mathcal{P} = V(H) - \frac{1}{2}\mathrm{tr}\,\frac{\partial^2 H}{\partial x \partial p} + \frac{1}{2}\frac{\mathcal{L}_{V(H)} dz(\alpha)}{dz(\alpha)}. \tag{2.120}$$

Here $\mathcal{L}_{V(H)}$ is the Lie derivative along the field $V(H)$.

The complete proof can be found in [4]; here we only sketch it. To obtain the desired formula, one has to expand $H\left(x, \frac{\partial \Phi}{\partial x}\right)$ in Taylor series in powers of $x - q(\alpha)$ up to the second order (terms of order ≥ 3 produce $O(h^2)$ in the result) and then use integration by parts to obtain

$$\begin{aligned} \mathcal{P} \;=\; & -2i\frac{\partial H}{\partial \bar{z}}\frac{\partial}{\partial z} - 2\mathrm{tr}\left(\frac{\partial^2 H}{\partial \bar{z}\partial \bar{z}}\frac{\partial^2 \Phi}{\partial z \partial z}\right) - i\mathrm{tr}\frac{\partial^2 H}{\partial z \partial \bar{z}} \\ & + \frac{i}{2}\left(\mathrm{tr}\,\frac{\partial^2 H}{\partial \bar{z}\partial \bar{z}} + \mathrm{tr}\,\frac{\partial^2 H}{\partial z \partial z}\right). \end{aligned} \tag{2.121}$$

Exercise. Verify (2.12) by straightforward computation.

Now we take advantage of the following fact.

Lemma 13

$$\mathcal{L}_{V(H)}dz(\alpha) = \left\{2i\mathrm{tr}\frac{\partial^2 H}{\partial\overline{z}\partial\overline{z}} - 2i\mathrm{tr}\frac{\partial^2 H}{\partial\overline{z}\partial z} - i\mathrm{tr}\left(\frac{\partial^2 H}{\partial\overline{z}\partial\overline{z}}\frac{\partial^2\Phi}{\partial z\partial z}\right)\right\} dz(\alpha). \tag{2.122}$$

The proof is by straightforward computation. Substituting (2.122) into (2.121) and using (2.49), (2.50), and (2.59), we obtain the desired result.

Corollary 14 *Suppose that the Hamiltonian function depends on h in a regular way:*

$$H(q,p,h) = H_0(q,p) - ihH_1(q,p) + \dots. \tag{2.123}$$

If $H_0(q,p)|_\Lambda = 0$, then the commutation formula

$$\hat{H}K_\Lambda\varphi = -ihK_\Lambda\mathcal{P}\varphi + O(h^2) \tag{2.124}$$

holds, where the transport operator $\mathcal{P}$ has the form

$$\mathcal{P} = V(H_0) + \frac{1}{2}\frac{\mathcal{L}_{V(H_0)}dz}{dz} + H_{sub}; \tag{2.125}$$

here

$$H_{sub} = H_1(q,p) - \frac{1}{2}\mathrm{tr}\frac{\partial^2 H_0}{\partial x\partial p} \tag{2.126}$$

is the subprincipal symbol *of the operator $\hat{H}$.*

2.5 Solution of the transport equation and Maslov's canonical operator with arbitrary measure. The second quantization condition

It follows from Theorem 12 and the subsequent corollary that for the function $\psi = K_\Lambda\varphi$ to satisfy the equation $\hat{H}\psi = 0$ modulo $O(h^2)$, it is necessary and sufficient that $H_0 = 0$ on Λ and that the amplitude φ satisfy the transport equation

$$\mathcal{P}\varphi \equiv \left[V(H_0) + H_{sub} + \frac{1}{2}\frac{\mathcal{L}_{V(H_0)}dz}{dz}\right]\varphi = 0. \tag{2.127}$$

Equation (2.127) is a first-order linear ordinary differential equation on the Lagrangian manifold Λ. In the general case, it is hard to say anything about nontrivial solutions of this equation (for example, little can be said if the trajectories of $V(H_0)$ behave badly, say, form an irrational winding of the torus Λ). However, the situation is much simpler if we can reduce our equation to the form $V(H_0)u = 0$, i.e., get rid of the zero-order terms. In this case, there is a nontrivial solution $\varphi = \text{const}$ and we obtain a nontrivial solution of the original equation. When is this possible? Suppose that $\hat{H}$ is a self-adjoint operator and the functions H_0 and H_1 are real-valued; then $H_{sub} = 0$ and the transport operator acquires the form

$$\mathcal{P} = V(H_0) + \frac{1}{2}\frac{\mathcal{L}_{V(H_0)}dz}{dz}. \tag{2.128}$$

Next, let $d\sigma$ be a measure on Λ invariant with respect to the Hamiltonian vector field:

$$\mathcal{L}_{V(H_0)}(d\sigma) = 0. \tag{2.129}$$

Then

$$\begin{aligned} V(H_0)\left(\frac{d\sigma}{dz}\right) &= \mathcal{L}_{V(H_0)}\left(\frac{d\sigma}{dz}\right) \\ &= \frac{\mathcal{L}_{V(H_0)}d\sigma}{d\sigma} - \frac{\mathcal{L}_{V(H_0)}dz}{dz} = -\frac{\mathcal{L}_{V(H_0)}dz}{dz}. \end{aligned} \tag{2.130}$$

Such measures $d\sigma$ can often be found from certain physical considerations. Set

$$f(\alpha) = \sqrt{\frac{d\sigma(\alpha)}{dz(\alpha)}} \tag{2.131}$$

(this is always possible locally). Then

$$V(H_0)f = -\frac{f}{2}\frac{\mathcal{L}_{V(H_0)}dz}{dz}, \tag{2.132}$$

and if we set $\varphi = fu$, then we obtain

$$\begin{aligned} \mathcal{P}fu &= -\frac{f}{2}\frac{\mathcal{L}_{V(H_0)}dz}{dz}u + fV(H_0)u + \frac{f}{2}\frac{\mathcal{L}_{V(H_0)}dz}{dz}u \\ &= fV(H_0)u; \end{aligned} \tag{2.133}$$

in other words,

$$f^{-1}\mathcal{P}f = V(H_0), \tag{2.134}$$

and we have locally reduced the transport operator to the pure Hamiltonian vector field. This is possible globally only if there is a continuous branch of the function (2.131) on the entire manifold Λ.

Definition 15 A measure $d\sigma$ on the Lagrangian manifold Λ is said to be *quantized* if the function

$$f(\alpha) = \sqrt{\frac{d\sigma(\alpha)}{dz(\alpha)}}$$

has a continuous (and hence smooth) branch on Λ. This condition is called the *second quantization condition.*

Definition 16 Let Λ be a Lagrangian manifold in $\mathbf{R}^{2n}$ satisfying the first quantization condition (2.95). Let $d\sigma$ be a measure on Λ satisfying the second quantization condition. *Maslov's general canonical operator* corresponding to the pair $(\Lambda, d\sigma)$ is the mapping

$$K_{(\Lambda,d\sigma)} : C_0^\infty(\Lambda) \to H(\mathbf{R}^n_x) \tag{2.135}$$

given by the formula

$$K_{(\Lambda,d\sigma)}\varphi = \left(\frac{1}{2\pi h}\right)^{\frac{n}{2}} \int\limits_\Lambda e^{\frac{i}{h}\Phi(x,\alpha)}\varphi(\alpha)\sqrt{\frac{d\sigma(\alpha)}{dz(\alpha)}}dz(\alpha). \tag{2.136}$$

Here the branch of the square root is chosen in some way.

The following theorem is obvious from the preceding considerations.

Theorem 17 *Suppose that the Hamiltonian function* $H(q,p,h)$ (2.123) *satisfies*

$$H_0(q,p)|_\Lambda = 0.$$

Then

$$\hat{H}K_{(\Lambda,d\sigma)}\varphi = -ihK_{(\Lambda,d\sigma)}\mathcal{P}\varphi + O(h^2),$$

where

$$\mathcal{P} = V(H_0) + H_{sub} + \frac{1}{2}\frac{\mathcal{L}_{V(H_0)}d\sigma(\alpha)}{d\sigma(\alpha)}. \tag{2.137}$$

For the case in which $H_{sub} = 0$ and the measure is invariant under the Hamiltonian vector field, we obtain $\mathcal{P} = V(H_0)$.

Final Remark We conclude this chapter with the following remark. The canonical operator was constructed in this chapter (and will be constructed in the next chapter) under the assumption that *two quantization conditions* are satisfied. In practical applications, however, this is almost never the case. Both conditions are usually violated, but their "sum" is satisfied for certain discrete energy levels, or (putting this mathematically rather than physically) for certain "admissible" values of the parameter h. This will be discussed in detail in the chapter devoted to the eigenvalue problems.

Bibliography

[1] V. Nazaikinskii, B.-W. Schulze, and B. Sternin. *Quantization and the Wave Packet Transform.* Univ. Potsdam, Institut für Mathematik, Potsdam, Mai 1999. Preprint N 99/8.

[2] V.P. Maslov. *Théorie des Perturbations et Méthods Asymptotiques.* Dunod, Paris, 1972. French transl. from the Russian 1965 edition.

[3] A. Mishchenko, V. Shatalov, and B. Sternin. *Lagrangian Manifolds and the Maslov Operator.* Springer-Verlag, Berlin–Heidelberg, 1990.

[4] B. Sternin and V. Shatalov. *Ten Lectures on Quantization*, volume 22. Université de Nice-Sophia Antipolis, Nice, Juin 1994. Preprint.

[5] V. Nazaikinskii and B. Sternin. Wave packet transform in symplectic geometry and asymptotic quantization. In B.P. Komrakov, I.S. Krasil'shchik, G.L. Litvinov, and A.B. Sossinsky, editors, *Lie Groups and Lie Algebras. Their Representations, Generalizations and Applications*, number 433 in Mathematics and Its Applications, 1998, pages 47–70, Dordrecht–Boston–London. Kluwer Academic Publishers.

[6] V.P. Maslov. *Operator Methods.* Nauka, Moscow, 1973. English transl.: *Operational Methods*, Mir, Moscow, 1976.

[7] V. Nazaikinskii, B. Sternin, and V. Shatalov. *Methods of Noncommutative Analysis. Theory and Applications.* Mathematical Studies. Walter de Gruyter Publishers, Berlin–New York, 1995.

[8] M. V. Karasev and V. P. Maslov. *Nonlinear Poisson Brackets. Geometry and Quantization.* Nauka, Moscow, 1991. English transl.: Nonlinear Poisson Brackets. Geometry and quantization. Translations of Mathematical Monographs, vol. **119** AMS, Providence, Rhode Island, 1993.