

**A General Index Formula  
on Toric Manifolds  
with Conical Points**

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July 7, 1999

**Abstract**

We solve the index problem for general elliptic pseudodifferential operators on toric manifolds with conical points.

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## Introduction

Elliptic theory for pseudodifferential operators on manifolds with conical singularities gives two sorts of principal symbols responsible for the Fredholm property of the operator in weighted Sobolev spaces (see, e.g., [Sch91]). If  $M$  is such a manifold with a smooth interior part  $M_{\text{int}} = M \setminus \{v_1, v_2, \dots, v_N\}$  where  $v_1, v_2, \dots, v_N$  are conical points, and  $A: H^{s, \bar{\gamma}}(M) \rightarrow H^{s-m, \bar{\gamma}}(M)$  a pseudodifferential operator of order  $m \in \mathbb{R}$  in weighted Sobolev spaces with weights  $\bar{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_N)$  at the conical points, then the following leading symbols are defined:

1. An *interior principal symbol*  $\sigma_{\text{int}}(A) = a(x, \xi)$  is a smooth function on  $T^*M_{\text{int}} \setminus \{0\}$  homogeneous of degree  $m$  in  $\xi$ .
2. For each conical point  $v_\nu$ , a *conormal symbol*  $\sigma_c^\nu(A) = A_c^\nu(\tau)$  is a parameter-dependent pseudodifferential operator of order  $m$  on a smooth manifold  $X_\nu$  which is the base of the cone at the point  $v_\nu$ .

Here  $\tau$  is a complex parameter varying in a strip around a weight line  $\Gamma_{\gamma_\nu} = \{\tau \in \mathbb{C}: \Im \tau = \gamma_\nu\}$ .

The operator is called elliptic if all the above mentioned symbols are invertible. Thus,  $\sigma_{\text{int}}(A) = a(x, \xi)$  is an invertible homomorphism of vector bundles, for each  $(x, \xi) \in T^*M_{\text{int}} \setminus \{0\}$ , and  $\sigma_c^\nu(A) = A_c^\nu(\tau)$  are invertible operators on  $X_\nu$ , for each  $\tau$  in the strip. It is well known that an elliptic operator is Fredholm and its index is completely defined by its interior and conormal principal symbols. In other words,  $\text{ind } A$  is a functional  $F(a, A_c^1(\tau), \dots, A_c^N(\tau))$  on the set of the leading interior and conormal symbols. This functional was found for many important particular cases, namely

1. for any 2-dimensional manifold  $M$ , cf. [FST97];
2. for Dirac type operators, cf. [APS75];
3. for operators satisfying some symmetry conditions, cf. [SSS98], [FST97].

Unfortunately, it is still unknown in general despite the efforts of many specialists.

The fact itself that the functional has not yet been found means that one can hardly expect an elegant and simple formula. One can rather hope for a complicated algorithm which is difficult to apply. Nevertheless, the general index formula is worth obtaining even for the sake of pure interest.

In this paper we solve the problem for general elliptic operators on toric manifolds. In other words, although no conditions (like symmetry) are imposed on operators, there are strong restrictions concerning the topology of

the manifold instead. Our manifold

$$\begin{aligned} M_{\text{int}} &= \mathbb{R} \times \mathbb{T}^n \\ &= \mathbb{R} \times \underbrace{\mathbb{S}^1 \times \dots \times \mathbb{S}^1}_n \end{aligned} \quad (0.1)$$

is the product of the real line  $\mathbb{R}$  and the  $n$ -dimensional torus with global coordinates

$$(t, x) = (t, x^1, \dots, x^n)$$

where  $t \in \mathbb{R}$  and  $x^j \in \mathbb{R} \pmod{2\pi}$ . The points  $t = \pm\infty$  are treated as conical points with base manifolds  $X_{\pm} = \mathbb{T}^n$ .

We consider only trivial vector bundles. So, we can deal with pseudodifferential operators using global coordinates and global frames. Even in this case our formula is far from being elegant and simple.

Just as in the 2-dimensional case the index formula consists of three summands. The first one is the Atiyah-Singer functional which under our assumptions has the form

$$\text{AS}(a) = \frac{(-1)^n n!}{(2n+1)! (2\pi i)^{n+1}} \int_{\mathbb{S}^*(M_{\text{int}})} \text{tr} (a^{-1} da)^{2n+1}. \quad (0.2)$$

We also assume that near  $t = \pm\infty$  the interior principal symbol  $a = a(t, \tau, x, \xi)$  is independent of  $t$ . In particular, the integrand in (0.2) vanishes for large  $|t|$ .

The next term is the so-called  $\eta$ -invariant. For  $t = +\infty$  its definition is as follows. Consider an operator on the torus  $\mathbb{T}^n$

$$D(\tau) = (A_c^+)^{-1} (\tau + i\gamma_+) (A_c^+)' (\tau + i\gamma_+)$$

depending on a parameter  $\tau$  in the strip  $|\Im\tau| < \varepsilon$  around the real axis. Let

$$D_N(\tau) = \sum_{k=0}^{N-1} \frac{1}{k!} D^{(k)}(\tau) (-i\gamma_+)^k \quad (0.3)$$

be a finite Taylor expansion for  $D(\tau - i\gamma_+)$  (the latter function may make no sense for real  $\tau$  but the expansion (0.3) does). We set

$$-\frac{1}{2} \eta(A_c^+) = \frac{1}{2\pi i} \overline{\text{Tr}} D_N(\tau) \quad (0.4)$$

where  $\overline{\text{Tr}}$  is the so-called *regularised trace*, cf. [Mel95], equal to the constant term of the asymptotic expansion

$$I_J(T) = \int_{-T}^T d\tau_J \int_0^{\tau_J} d\tau_{J-1} \dots \int_0^{\tau_1} \overline{\text{Tr}} D_N^{(J)}(\tau) d\tau$$

for  $T \rightarrow \infty$ . Note that the expression (0.4) is meaningful for  $J > n$  and is independent of  $J$  and  $N$ , provided both are greater than  $n$ .

Unfortunately, the term  $-(1/2)\eta(A_c^+)$  does not exhaust the contribution of the conical point  $t = +\infty$ . We denote the additional contribution by  $\Phi(A_c^+)$ , so that our index formula takes the form

$$\text{ind } A = \text{AS}(a) - \frac{1}{2}\eta(A_c^+) + \Phi(A_c^+) - \frac{1}{2}\eta(A_c^-) + \Phi(A_c^-). \quad (0.5)$$

The main purpose of the present paper is to find a formula for  $\Phi(A_c^+)$  as explicit as possible. This will be done by means of the *Odd Periodicity Theorem* for formal symbols which may be considered as an odd version of the periodicity theorem of [Fed78]. In contrast to the latter here we need not only the fact of coincidence of two differential 1-forms up to an exact form but rather the corresponding primitive function.

The paper is divided into three parts. In Section 1 a variation formula for the functional  $\Phi$  is obtained. In the next section we reduce the problem of finding the primitive functional to the periodicity theorem proved in Section 3. In the course of the proof we obtain the desired expression for the primitive functional. In conclusion we discuss possible generalisations of our method.

## 1 Variation formula

We start with the following algebraic index formula

$$\begin{aligned} \text{ind } A &= \text{Tr } \rho_0(1 - \bar{r} \circ \bar{a})|_N - \text{Tr } \rho_0(1 - \bar{a} \circ \bar{r})|_N \\ &\quad - \frac{1}{2}\eta(A_c^+) - \frac{1}{2\pi i} \overline{\text{Tr}} \text{Op}_{\mathbb{T}^n} \bar{r}(+\infty, \tau) \circ \bar{a}'(+\infty, \tau)|_{N-1} \\ &\quad - \frac{1}{2}\eta(A_c^-) + \frac{1}{2\pi i} \overline{\text{Tr}} \text{Op}_{\mathbb{T}^n} \bar{r}(-\infty, \tau) \circ \bar{a}'(-\infty, \tau)|_{N-1} \end{aligned} \quad (1.1)$$

obtained in [FST97], the sign “+” in the last summand being due to the fact that we use the same global variable  $t \in \mathbb{R}$  along both cone axes. Here  $\rho_0(t)$  is a cut-off function vanishing near  $t = \pm\infty$ , and

$$\bar{a} = \sum_{k=0}^{\infty} h^k a_k(t, \tau, x, \xi) \quad (1.2)$$

is a formal symbol on  $M = \mathbb{R} \times \mathbb{T}^n$ , with  $t \in \mathbb{R}$  and  $x \in \mathbb{T}^n$ , such that the difference

$$A - \text{Op}_M(\bar{a}|_N) := A - \text{Op}_M\left(\sum_{k=0}^{N-1} a_k\right)$$

is a pseudodifferential operator of order  $m - N$ .

The formal symbol  $\bar{r}$  is a parametrix of  $\bar{a}$ , so that

$$\begin{aligned} 1 - \bar{r} \circ \bar{a}, \\ 1 - \bar{a} \circ \bar{r} \end{aligned}$$

are formal symbols with compact supports,  $\circ$  meaning the symbol multiplication

$$\bar{a} \circ \bar{b} = \sum_{\alpha \in \mathbb{Z}_+^{n+1}} \frac{(-ih)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \bar{a} \partial_x^\alpha \bar{b}.$$

We also emphasise that the formal symbols

$$\bar{a}(\pm\infty, \tau) = \sum_{k=0}^{\infty} h^k a_k(\pm\infty, \tau, x, \xi)$$

are completely defined by parameter-dependent operators  $A_c^\pm(\tau + i\gamma_\pm)$ . The complete symbols of these operators are the sums of homogeneous components  $a_k^\pm(\tau, x, \xi)$  of degree  $m - k$  in the covariables  $(\tau, \xi) \in \mathbb{R}^{1+n}$ . Then the following *compatibility condition* holds

$$\bar{a}(\pm\infty, \tau) = \pm \sum_{k=0}^{\infty} h^k a_k^\pm(\tau - ih\gamma_\pm, x, \xi),$$

the right-hand side being understood as a formal Taylor expansion

$$\pm \sum_{k,l=0}^{\infty} \frac{h^{k+l}}{l!} (\partial/\partial\tau)^l a_k^\pm(\tau, x, \xi) (-i\gamma_\pm)^l. \quad (1.3)$$

The sign “ $-$ ” in front of (1.3) for  $t = -\infty$  is due to the change of orientation of the  $t$ -axis when  $t = -\infty$  is treated as a conical point.

We would like to modify Equality (1.1) by replacing the interior terms in (1.1) by the Atiyah-Singer functional (0.2) with  $a = a_0(t, \tau, x, \xi)$  and keeping the  $\eta$ -invariant terms. This leads to the functional  $F(\bar{a})$  on formal symbols given by

$$\begin{aligned} F(\bar{a}) + \text{AS}(a) &= \text{Tr } \rho_0 (1 - \bar{r} \circ \bar{a})|_N - \text{Tr } \rho_0 (1 - \bar{a} \circ \bar{r})|_N \\ &\quad - \frac{1}{2\pi i} \overline{\text{Tr}} \text{Op}_{\mathbb{T}^n} \bar{r}(+\infty, \tau) \circ \bar{a}'(+\infty, \tau)|_{N-1} \\ &\quad + \frac{1}{2\pi i} \overline{\text{Tr}} \text{Op}_{\mathbb{T}^n} \bar{r}(-\infty, \tau) \circ \bar{a}'(-\infty, \tau)|_{N-1} \\ &:= G(\bar{a}). \end{aligned} \quad (1.4)$$

We first obtain a variation formula for this functional. To formulate the result, we need the notion of a *formal trace* which is also due to Melrose [Mel95].

Let  $A(\tau)$  be a parameter-dependent pseudodifferential operator on the torus  $\mathbb{T}^n$  with a complete symbol

$$a(\tau, x, \xi) \sim \sum a_k(\tau, x, \xi)$$

where  $a_k$  are homogeneous functions of degree  $m - k$  in  $(\tau, \xi) \in \mathbb{R}^{1+n}$ . We define

$$\widetilde{\text{Tr}} A(\tau) := \overline{\text{Tr}} \frac{\partial}{\partial \tau} A(\tau).$$

It turns out, cf. [Mel95], that this functional is local. More precisely,

$$\widetilde{\text{Tr}} A(\tau) = \frac{1}{(2\pi)^n} \int_{T^*(\mathbb{T}^n)} (\text{tr } a_k(1, x, \xi) - \text{tr } a_k(-1, x, \xi)) dx d\xi \quad (1.5)$$

where  $m - k = -n$  (if there are no homogeneous components of degree  $-n$ , then  $\widetilde{\text{Tr}} A(\tau) = 0$ ). This formula allows us to define a formal trace even on formal symbols

$$\bar{a}(\tau) = \sum_{k=0}^{\infty} h^k a_k(\tau, x, \xi)$$

with homogeneous coefficients  $a_k(\tau, x, \xi)$ . We simply set  $\widetilde{\text{Tr}} \bar{a}(\tau)$  to be equal to the right-hand side of (1.5). Clearly, we have

$$\begin{aligned} \widetilde{\text{Tr}} \frac{\partial}{\partial \xi_j} \bar{a}(\tau) &= \widetilde{\text{Tr}} \frac{\partial}{\partial x^j} \bar{a}(\tau) \\ &= 0. \end{aligned}$$

In particular, it implies that  $\widetilde{\text{Tr}}$  vanishes on commutators, i.e.,

$$\widetilde{\text{Tr}} \bar{a} \circ \bar{b} = \widetilde{\text{Tr}} \bar{b} \circ \bar{a}.$$

Consider now the functional  $G(\bar{a})$  on formal symbols (1.2) depending on some extra parameters. Denoting by  $d$  the differential with respect to these extra parameters, we have the following formula.

**Lemma 1.1**

$$-dG(\bar{a}) = \frac{1}{2\pi i} \widetilde{\text{Tr}} (\bar{r}(\infty, \tau) \circ d\bar{a}(\infty, \tau)) - \frac{1}{2\pi i} \widetilde{\text{Tr}} (\bar{r}(-\infty, \tau) \circ d\bar{a}(-\infty, \tau)). \quad (1.6)$$

**Remark 1.2** The index formula for an elliptic operator (1.1) may be rewritten as

$$\text{ind } A = G(\bar{a}) - \frac{1}{2} \eta(A_c^+) - \frac{1}{2} \eta(A_c^-).$$

Then Equality (1.6) follows from the variation formula for the  $\eta$ -invariant since  $d \text{ind } A = 0$ . Our lemma means that this formula is still valid for formal symbols, no matter if there exist invertible conormal symbols  $A_c^\pm$  compatible with  $\bar{a}$ .

**Proof.** Consider

$$d \overline{\text{Tr}} \text{Op}_{\mathbb{T}^n} \bar{r}(+\infty, \tau) \circ \bar{a}'(+\infty, \tau)|_{N-1}.$$

Omitting the subscript  $\mathbb{T}^n$  and the argument  $+\infty$ , we have

$$\begin{aligned} & d \overline{\text{Tr}} \text{Op} \bar{r}(\tau) \circ \bar{a}'(\tau)|_{N-1} \\ &= \overline{\text{Tr}} \text{Op} (d\bar{r}(\tau) \circ \bar{a}'(\tau) + \bar{r}(\tau) \circ d\bar{a}'(\tau))|_{N-1} \\ &= \overline{\text{Tr}} \frac{\partial}{\partial \tau} \text{Op} \bar{r}(\tau) \circ d\bar{a}(\tau)|_{N-1} + \overline{\text{Tr}} \text{Op} (d\bar{r}(\tau) \circ \bar{a}'(\tau) - \bar{r}'(\tau) \circ d\bar{a}(\tau))|_{N-1}. \end{aligned}$$

The first term is by definition

$$\widetilde{\text{Tr}} \text{Op} \bar{r}(\tau) \circ d\bar{a}(\tau)|_{N-1}$$

or, by (1.5),

$$\widetilde{\text{Tr}} \bar{r}(\tau) \circ d\bar{a}(\tau).$$

The second term may be transformed further to

$$\begin{aligned} & \overline{\text{Tr}} \text{Op} (d\bar{r}(\tau) + \bar{r}(\tau) \circ d\bar{a}(\tau) \circ \bar{r}(\tau)) \circ \bar{a}'(\tau)|_{N-1} \\ & \quad - \overline{\text{Tr}} \text{Op} (\bar{r}'(\tau) + \bar{r}(\tau) \circ \bar{a}'(\tau) \circ \bar{r}(\tau)) \circ d\bar{a}(\tau)|_{N-1} \\ & \quad + \overline{\text{Tr}} \text{Op} [\bar{r}(\tau) \circ d\bar{a}(\tau), \bar{r}(\tau) \circ \bar{a}'(\tau)]|_{N-1} \end{aligned}$$

with the last term vanishing since  $\overline{\text{Tr}}$  vanishes on commutators. Now the symbols

$$\begin{aligned} & d\bar{r}(\tau) + \bar{r}(\tau) \circ d\bar{a}(\tau) \circ \bar{r}(\tau), \\ & \bar{r}'(\tau) + \bar{r}(\tau) \circ \bar{a}'(\tau) \circ \bar{r}(\tau) \end{aligned}$$

have compact supports in  $\tau, \xi$ , so all the regularisations needed to define  $\overline{\text{Tr}}$  may be omitted.

For a symbol  $\bar{b}(\tau)$  with compact support we have

$$\overline{\text{Tr}} \text{Op} (\bar{b}(\tau)|_{N-1}) = \int_{-\infty}^{\infty} \text{Tr} \text{Op} (\bar{b}(\tau)|_{N-1}) d\tau$$

where  $\text{Tr}$  stands for the trace of the operator on  $\mathbb{T}^n$ . Finally,

$$\begin{aligned} & \frac{1}{2\pi i} d \overline{\text{Tr}} \text{Op} \bar{r}(\tau) \circ \bar{a}'(\tau)|_{N-1} = \frac{1}{2\pi i} \widetilde{\text{Tr}} \bar{r}(\tau) \circ d\bar{a}(\tau) \\ & \quad + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \text{Tr} \text{Op} ((d\bar{r}(\tau) + \bar{r}(\tau) \circ d\bar{r}(\tau) \circ \bar{r}(\tau)) \circ \bar{a}'(\tau)|_{N-1}) d\tau \\ & \quad - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \text{Tr} \text{Op} ((\bar{r}'(\tau) + \bar{r}(\tau) \circ \bar{a}'(\tau) \circ \bar{r}(\tau)) \circ d\bar{a}(\tau)|_{N-1}) d\tau. \quad (1.7) \end{aligned}$$

A similar expression is valid for  $t = -\infty$ .

Now, consider the variation of the interior terms in  $G(\bar{a})$ , namely

$$\begin{aligned}
& d \operatorname{Tr} \rho_0 (1 - \bar{r} \circ \bar{a})|_N - d \operatorname{Tr} \rho_0 (1 - \bar{a} \circ \bar{r})|_N \\
&= -\operatorname{Tr} \rho_0 (d\bar{r} \circ \bar{a} + \bar{r} \circ d\bar{a})|_N + \operatorname{Tr} \rho_0 (d\bar{a} \circ \bar{r} + \bar{a} \circ d\bar{r})|_N \\
&= -\operatorname{Tr} \rho_0 ((d\bar{r} + \bar{r} \circ d\bar{a} \circ \bar{r}) \circ \bar{a})|_N + \operatorname{Tr} \rho_0 (\bar{a} \circ (d\bar{r} + \bar{r} \circ d\bar{a} \circ \bar{r}))|_N \\
&\quad - \operatorname{Tr} \rho_0 (\bar{r} \circ d\bar{a} \circ (1 - \bar{r} \circ \bar{a}))|_N + \operatorname{Tr} \rho_0 ((1 - \bar{a} \circ \bar{r}) \circ d\bar{a} \circ \bar{r})|_N.
\end{aligned} \tag{1.8}$$

Again we have symbols with compact supports, hence we may change the order of symbols under the trace sign.

Denoting  $d\bar{r} + \bar{r} \circ d\bar{a} \circ \bar{r}$  by  $\bar{b}$ , we rewrite the first two terms in (1.8) as

$$\begin{aligned}
-\operatorname{Tr} [\bar{a}, \rho_0] \circ \bar{b}|_N &= -\operatorname{Tr} \left( \sum_{k=1}^{\infty} \frac{(-ih)^k}{k!} \partial_{\tau}^k \bar{a} \partial_t^k \rho_0 \right) \circ \bar{b}|_N \\
&= -\frac{i}{2\pi} \int_{-\infty}^{\infty} \operatorname{Tr}_{\mathbb{T}^n} (\bar{a}' \circ \bar{b})|_{N-1} d\tau.
\end{aligned} \tag{1.9}$$

Here we made use of the fact that, for  $k > 1$ ,

$$\int_0^{\infty} (\partial/\partial t)^k \rho_0(t) dt = 0$$

and, for  $k = 1$ , this integral is equal to  $-1$ .

The last two terms in (1.8) transform as follows

$$\begin{aligned}
& \operatorname{Tr} [\bar{r}, \rho_0] \circ (1 - \bar{a} \circ \bar{r}) \circ d\bar{a}|_N \\
&\quad - \operatorname{Tr} \rho_0 \circ \bar{r} \circ d\bar{a} \circ (1 - \bar{r} \circ \bar{a})|_N + \operatorname{Tr} \rho_0 \circ (1 - \bar{r} \circ \bar{a}) \circ \bar{r} \circ d\bar{a}|_N \\
&= \operatorname{Tr} [\bar{r}, \rho_0] \circ (1 - \bar{a} \circ \bar{r}) \circ d\bar{a}|_N + \operatorname{Tr} [\bar{r} \circ d\bar{a}, \rho_0] \circ (1 - \bar{r} \circ \bar{a})|_N.
\end{aligned}$$

Reasoning similarly to (1.9), we get

$$\begin{aligned}
& \frac{i}{2\pi} \int_{-\infty}^{\infty} \operatorname{Tr}_{\mathbb{T}^n} \bar{r}' \circ (1 - \bar{a} \circ \bar{r}) \circ d\bar{a}|_{N-1} d\tau + \frac{i}{2\pi} \int_{-\infty}^{\infty} \operatorname{Tr}_{\mathbb{T}^n} (\bar{r} \circ d\bar{a})' \circ (1 - \bar{r} \circ \bar{a})|_{N-1} d\tau \\
&= \frac{i}{2\pi} \int_{-\infty}^{\infty} \operatorname{Tr}_{\mathbb{T}^n} (\bar{r}' \circ (1 - \bar{a} \circ \bar{r}) \circ d\bar{a} + (\bar{r} \circ \bar{a})' \circ (\bar{r} \circ d\bar{a}))|_{N-1} d\tau \\
&= \frac{i}{2\pi} \int_{-\infty}^{\infty} \operatorname{Tr}_{\mathbb{T}^n} (\bar{r}' + \bar{r} \circ \bar{a}' \circ \bar{r}) \circ d\bar{a}|_{N-1} d\tau.
\end{aligned} \tag{1.10}$$

Gathering (1.7), (1.9) and (1.10), we come to (1.6).  $\square$

Recall that the symbol  $\bar{a}$  is invertible for  $|\tau|^2 + |\xi|^2 > 1$  and in this domain the parametrix  $\bar{r}$  coincides with the inverse of  $\bar{a}$  with respect to the  $\circ$ -product

of symbols. We denote this inverse by  $\bar{a}^{\circ(-1)}$ . So, the variation formula for the functional  $G(\bar{a})$  may be written in the form

$$-dG(\bar{a}) = \frac{1}{2\pi i} \widetilde{\text{Tr}} \bar{a}^{\circ(-1)} \circ d\bar{a} \Big|_{t=-\infty}^{t=+\infty}. \quad (1.11)$$

Consider now the Atiyah-Singer functional (0.2). The factor in front of the integral in (0.2) which is inessential so far will be denoted by  $c_n$ . The function  $a$  depending on  $t, \tau, x, \xi$  and some extra parameters is supposed to be homogeneous in  $\tau, \xi$  and independent of  $t$  near  $t = \pm\infty$ , and  $da$  means the differential with respect to total variables including extra parameters. We represent the variation of (0.2) in the form similar to (1.11), with a properly defined functional  $\widetilde{\text{Tr}}$  on the differential form  $(a^{-1} da)^{\wedge(2n+1)}$ . Set

$$\widetilde{\text{Tr}} (a^{-1} da)^{\wedge(2n+1)} = \int_{\mathbb{R}_x^n} \int_{\mathbb{R}_\xi^n} \text{tr} (a^{-1} da)^{\wedge(2n+1)} \Big|_{\tau=-1}^{\tau=1}. \quad (1.12)$$

This partial integration gives us an one-form, the free differentials are supposed to be the first ones in the integrand. The integral (1.12) converges because the integrand is a form homogeneous of degree 0 in  $\tau, \xi$ . This implies that

$$(a^{-1} da)^{\wedge(2n+1)} = \sigma(\tau, \xi) \wedge d\xi_1 \wedge \dots \wedge d\xi_n + \dots$$

with  $\sigma$  homogeneous of degree  $-n$  in  $\tau, \xi$ , where the dots denote the terms where at least one of the differentials  $d\xi_1, \dots, d\xi_n$  is missing. Then

$$\begin{aligned} & \int_{\mathbb{R}_\xi^n} \text{tr} (a^{-1} da)^{\wedge(2n+1)} \Big|_{\tau=-1}^{\tau=1} \\ &= \int_{\mathbb{R}_\xi^n} (\text{tr} \sigma(1, \xi) - \text{tr} \sigma(-1, \xi)) \wedge d\xi_1 \wedge \dots \wedge d\xi_n \end{aligned}$$

where the integrand may be estimated as  $O(|\xi|^{-(n+1)})$  whence the convergence follows.

The  $(2n+1)$ -form  $\text{tr} (a^{-1} da)^{\wedge(2n+1)}$  is closed implying that (1.12) is a closed one-form. Integrating it over the axis  $\mathbb{R}_t$ , we come to the following representation of the Atiyah-Singer functional.

**Lemma 1.3**

$$\int_{S^*(M_{\text{int}})} \text{tr} (a^{-1} da)^{\wedge(2n+1)} = - \int_{\mathbb{R}_t} \widetilde{\text{Tr}} (a^{-1} da)^{\wedge(2n+1)}. \quad (1.13)$$

**Proof.** The integrand is identically 0 near  $t = \pm\infty$ , so, in fact, we may integrate over a finite interval  $-T < t < T$ . Consider the intersection of two domains

$$\begin{aligned} D_1 &= \{-T < t < T, \quad |\tau|^2 + |\xi|^2 > 1\}, \\ D_2 &= \{-T < t < T, \quad -1 < \tau < 1\} \end{aligned}$$

in the space  $\mathbb{R}^{2+2n} = \mathbb{R}_t \times \mathbb{R}_\tau \times \mathbb{R}_x^n \times \mathbb{R}_\xi^n$ .

The difference between the left-hand side and the right-hand side of (1.13) may be considered as the integral over  $\partial(D_1 \cap D_2)$ . Indeed, the boundary of  $D_1 \cap D_2$  consists of the cylinder

$$\{-T < t < T, \quad |\tau|^2 + |\xi|^2 = 1\}$$

which is  $\mathbb{S}^*(M_{\text{int}})$  with the opposite orientation, two hyperplanes

$$\{-T < t < T, \quad \tau = \pm 1\}$$

(the integral over them gives just the right-hand side of (1.13)), and two hyperplane domains

$$\{t = \pm T, \quad -1 < \tau < 1, \quad |\tau|^2 + |\xi|^2 > 1\}.$$

There is also an infinite part of the boundary

$$\{-T < t < T, \quad -1 < \tau < 1, \quad |\xi| = R\},$$

with  $R \rightarrow \infty$ , but the integral over this part tends to 0.

Now observe that the integral

$$\int_D \text{tr}(a^{-1} da)^{\wedge(2n+1)}$$

over any domain  $D$  in the hyperplane  $t = \text{const}$  is equal to 0. Indeed, setting  $\rho = (\tau^2 + |\xi|^2)^{1/2}$ , we may write, using the homogeneity in  $\tau, \xi$ ,

$$a = \rho^m a_0$$

where  $a_0$  means the restriction of  $a$  to the hypersurface  $\tau^2 + |\xi|^2 = 1$ . Thus,

$$a^{-1} da = m \rho^{-1} d\rho + a_0^{-1} da_0$$

and

$$\text{tr}(a^{-1} da)^{\wedge(2n+1)} = \text{tr}(a_0^{-1} da_0)^{\wedge(2n+1)} + (2n+1) m \rho^{-1} d\rho \wedge \text{tr}(a_0^{-1} da_0)^{\wedge 2n}.$$

The second term is identically 0, it means that the form

$$\text{tr}(a^{-1} da)^{\wedge(2n+1)}$$

is lifted from the hypersurface  $\tau^2 + |\xi|^2 = 1$  by the radial projection

$$\pi : (\tau, \xi) \mapsto \left( \frac{\tau}{\rho}, \frac{\xi}{\rho} \right).$$

This projection is degenerate on any  $(2n - 1)$ -dimensional domain  $D$  in the hyperplane  $t = \text{const}$ , hence the integral over the domain vanishes. For a closed form the integral over the boundary  $\partial(D_1 \cap D_2)$  vanishes, by the Stokes theorem, proving the lemma.  $\square$

Now it is easy to get the following variation formula for the Atiyah-Singer functional.

**Lemma 1.4**

$$d \text{AS}(a) = -c_n \widetilde{\text{Tr}}(a^{-1} da)^{\wedge(2n+1)} \Big|_{t=-\infty}^{t=\infty}. \quad (1.14)$$

**Proof.** Suppose, we have one extra parameter  $s$ . Then

$$- \widetilde{\text{Tr}}(a^{-1} da)^{\wedge(2n+1)}$$

is a closed one-form in  $s$  and  $t$ , so it may be written in the form

$$p(s, t) ds + q(s, t) dt$$

where

$$(\partial/\partial t)p(s, t) \equiv (\partial/\partial s)q(s, t).$$

Using (1.13), we get

$$\text{AS}(a) = c_n \int_{-\infty}^{\infty} q(s, t) dt,$$

so that

$$\begin{aligned} d \text{AS}(a) &= c_n ds \int_{-\infty}^{\infty} (\partial/\partial s)q(s, t) dt \\ &= c_n ds \int_{-\infty}^{\infty} (\partial/\partial t)p(s, t) dt \\ &= c_n (p(s, \infty) ds - p(s, -\infty) ds), \end{aligned}$$

which is precisely (1.14).  $\square$

The result of our considerations is summarised as a corollary.

**Corollary 1.5** *The following variation formula holds*

$$- d F(\bar{a}) = \left( c_0 \widetilde{\text{Tr}} \bar{a}^{\circ(-1)} \circ d\bar{a} - c_n \widetilde{\text{Tr}}(a^{-1} da)^{\wedge(2n+1)} \right) \Big|_{t=-\infty}^{t=+\infty} \quad (1.15)$$

where  $a = a_0$  and

$$\begin{aligned} c_0 &= \frac{1}{2\pi i}; \\ c_n &= \frac{(-1)^n n!}{(2n+1)! (2\pi i)^{n+1}}. \end{aligned}$$

Our next aim is to show that the expression in parenthesis in (1.15) admits a primitive functional. This means that there exists a functional  $\Phi(\bar{b})$  defined on elliptic symbols

$$\bar{b} = \sum_{k=0}^{\infty} h^k b_k(\tau, x, \xi)$$

depending on the variables  $(\tau, x, \xi) \in \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}^n$ , with  $b_k(\tau, x, \xi)$  homogeneous of order  $m - k$  in  $\tau, \xi$ , such that

$$-d\Phi(\bar{b}) = c_0 \widetilde{\text{Tr}} \bar{b}^{\circ(-1)} \circ d\bar{b} - c_n \widetilde{\text{Tr}} (b^{-1} db)^{\wedge(2n+1)}.$$

Having granted this, we may rewrite (1.15) as

$$dF(\bar{a}) = d\Phi(\bar{a}|_{t=+\infty}) - d\Phi(\bar{a}|_{t=-\infty}) \quad (1.16)$$

and we arrive at the following theorem.

**Theorem 1.6** *The functional  $F(\bar{a})$  on symbols  $\bar{a} = \bar{a}(t, \tau, x, \xi)$  depends only on the restrictions  $\bar{a}|_{t=\pm\infty} = \bar{a}(\pm\infty, \tau, x, \xi)$ :*

$$F(\bar{a}) = \Phi(\bar{a}|_{t=+\infty}) - \Phi(\bar{a}|_{t=-\infty}). \quad (1.17)$$

**Proof.** Clearly, (1.17) holds for symbols  $\bar{a}(t, \tau, x, \xi)$  which are, in fact, independent of  $t$ . For an arbitrary symbol  $\bar{a}(t) = \bar{a}(t, \tau, x, \xi)$  we construct a path  $\bar{a}(s, t)$ ,  $s \in [0, 1]$ , connecting the given symbol with one independent of  $t$ , that is

$$\begin{aligned} \bar{a}(1, t) &= \bar{a}(t), \\ \bar{a}(0, t) &= \bar{a} \end{aligned}$$

$\bar{a}$  being independent of  $t$ . This path may be constructed as follows. Let  $\bar{a}(t)$  be independent of  $t$  for  $t < -T$  and  $t > T$ . Then we define

$$\bar{a}(s, t) = \begin{cases} \bar{a}(t) & \text{if } t \leq (2s-1)T; \\ \bar{a}((2s-1)T) & \text{if } t \geq (2s-1)T \end{cases}$$

and apply any standard smoothing procedure to obtain a smooth path

$$\bar{a}(s, t) = \bar{a}(s, t, \tau, x, \xi).$$

Applying (1.16) to this family, where  $d$  is taken with respect to the parameter  $s$ , and integrating over  $s$  from 0 to 1, we come to (1.17). □

## 2 Reduction to the Periodicity Theorem

We start with the algebra of formal symbols on the torus  $\mathbb{T}^n$  described in the previous section. It consists of formal expansions

$$\begin{aligned}\bar{a} &= \bar{a}(\tau, x, \xi) \\ &= \sum_{k=0}^{\infty} h^k a_k(\tau, x, \xi)\end{aligned}$$

where  $\tau \in \mathbb{R}$  and  $x \in \mathbb{T}^n$ ,  $\xi \in \mathbb{R}^n$ , the functions  $a_k$  being homogeneous of order  $m - k$  in  $\tau, \xi$ , with a given  $m \in \mathbb{R}$ . The product denoted by  $\circ$  is defined by

$$\bar{a} \circ \bar{b} = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{(-ih)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \bar{a} \partial_x^\alpha \bar{b}.$$

There is a trace  $\widetilde{\text{Tr}}$  defined on the *whole algebra*,

$$\widetilde{\text{Tr}} \bar{a} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n \times \mathbb{R}^n} \text{tr} a_k \Big|_{\tau=-1}^{\tau=1} dx d\xi$$

with  $m - k = -n$  (if there are no components of degree  $-n$ , then  $\widetilde{\text{Tr}}$  is zero). This algebra will be denoted by  $\mathcal{A}$  or, in more detail,

$$\left( \mathcal{A}, \circ, \widetilde{\text{Tr}} \right).$$

Let  $\bar{a} \in \mathcal{A}$  be an elliptic formal symbol. It means that its leading term  $a = a_0(\tau, x, \xi)$  is an invertible function for  $(\tau, \xi) \neq 0$ . In this case there exists a unique inverse for  $\bar{a}$  in the algebra  $\mathcal{A}$ , given by

$$\bar{a}^{\circ(-1)} = a^{-1} \circ \sum_{k=0}^{\infty} (1 - \bar{a} \circ a^{-1})^{\circ k}. \quad (2.1)$$

Here  $\circ k$  means the  $k$ -th power with respect to the  $\circ$ -product. Thus,  $\circ(-1)$  means the inversion operation in  $\mathcal{A}$  while  $a^{-1}$  is simply the inverse matrix-valued function considered as a symbol consisting of the leading term only.

Given an elliptic symbol  $\bar{a} \in \mathcal{A}$  depending on some parameters, we consider the functional  $\widetilde{\text{Tr}} \bar{a}^{\circ(-1)} \circ d\bar{a}$  and reduce it successively to a simpler form.

**1°** At the first step we pass to the Weyl algebra of formal symbols

$$\mathcal{A}_W = \left( \mathcal{A}, *, \widetilde{\text{Tr}} \right).$$

It consists of the same set of symbols with the same trace functional but the product  $*$  is different. Denoting

$$D = -\frac{ih}{2} \left( \frac{\partial}{\partial \xi_j} \frac{\partial}{\partial y^j} - \frac{\partial}{\partial x^j} \frac{\partial}{\partial \eta_j} \right)$$

(with the Einstein summation convention), we define

$$\bar{a} * \bar{b} = e^D \bar{a}(\tau, x, \xi) \bar{b}(\tau, y, \eta) \Big|_{\substack{y=x \\ \eta=\xi}},$$

where on the right-hand side we have the usual product of formal power series. It is known [BS91] that the product  $*$  is associative and  $\widetilde{\text{Tr}}$  vanishes on commutators. Moreover,

$$\widetilde{\text{Tr}} \bar{a} * \bar{b} = \widetilde{\text{Tr}} \bar{a} \bar{b}.$$

There exists an isomorphism

$$\mathcal{I} : \mathcal{A} \rightarrow \mathcal{A}_W$$

given by

$$\mathcal{I} \bar{a} = \sum_{\alpha \in \mathbb{Z}_+^n} \left( \frac{i\hbar}{2} \right)^{|\alpha|} \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \bar{a}$$

with the inverse

$$\mathcal{I}^{-1} \bar{a} = \sum_{\alpha \in \mathbb{Z}_+^n} \left( -\frac{i\hbar}{2} \right)^{|\alpha|} \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \bar{a}.$$

Observe that  $\widetilde{\text{Tr}} \bar{a} = \widetilde{\text{Tr}} \mathcal{I} \bar{a}$  since the functional  $\widetilde{\text{Tr}}$  vanishes on derivatives. Using these properties, we rewrite

$$\begin{aligned} \widetilde{\text{Tr}} \bar{a}^{\circ(-1)} \circ d\bar{a} &= \widetilde{\text{Tr}} \mathcal{I} (\bar{a}^{\circ(-1)} \circ d\bar{a}) \\ &= \widetilde{\text{Tr}} (\mathcal{I} \bar{a})^{*(-1)} * d(\mathcal{I} \bar{a}) \\ &= \widetilde{\text{Tr}} (\mathcal{I} \bar{a})^{*(-1)} d(\mathcal{I} \bar{a}). \end{aligned}$$

**2°** At the next step we get rid of higher-order terms. This procedure is based on the following lemma.

**Lemma 2.1** *Let  $d$  and  $\delta$  be commuting differentials of the algebra  $\mathcal{A}_W$ . Then*

$$\delta \widetilde{\text{Tr}} \bar{a}^{*(-1)} * d\bar{a} = d \widetilde{\text{Tr}} \bar{a}^{*(-1)} * \delta \bar{a}.$$

**Proof.** Using that

$$\delta \bar{a}^{*(-1)} = -\bar{a}^{*(-1)} * \delta \bar{a} * \bar{a}^{*(-1)}$$

we obtain on the left-hand side

$$- \widetilde{\text{Tr}} \bar{a}^{*(-1)} * \delta \bar{a} * \bar{a}^{*(-1)} * d\bar{a} + \widetilde{\text{Tr}} \bar{a}^{*(-1)} * \delta d\bar{a}.$$

Similarly, on the right-hand side we get

$$- \widetilde{\text{Tr}} \bar{a}^{*(-1)} * d\bar{a} * \bar{a}^{*(-1)} * \delta \bar{a} + \widetilde{\text{Tr}} \bar{a}^{*(-1)} * d\delta \bar{a}.$$

The first terms differ only by the order of factors under  $\widetilde{\text{Tr}}$ , so, they are equal. The second terms are also equal since  $d\delta = \delta d$ .  $\square$

We apply this lemma to the symbol  $\mathcal{I}\bar{a}$  depending on parameters,  $d$  being the differential with respect to these parameters. Consider the leading term  $a = a_0$  of this symbol and a family

$$\begin{aligned}\bar{b} &= a + t(\mathcal{I}\bar{a} - a) \\ &= a + t\bar{c}.\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}\widetilde{\text{Tr}}(\mathcal{I}\bar{a})^{*(-1)} * d(\mathcal{I}\bar{a}) &= \widetilde{\text{Tr}} a^{*(-1)} * da + \int_0^1 \delta \widetilde{\text{Tr}} \bar{b}^{*(-1)} * d\bar{b} \\ &= \widetilde{\text{Tr}} a^{*(-1)} * da + d \int_0^1 \widetilde{\text{Tr}} \bar{b}^{*(-1)} * \delta\bar{b}.\end{aligned}$$

Here the function  $a(\tau, x, \xi)$  is treated as a symbol consisting of the leading term only, and  $\delta$  means the differential with respect to  $t$ . The integral may be easily calculated:

$$\begin{aligned}\int_0^1 \widetilde{\text{Tr}}(a + t\bar{c})^{*(-1)} * \bar{c} dt &= \widetilde{\text{Tr}} \int_0^1 (1 + t a^{*(-1)} * \bar{c})^{*(-1)} * a^{*(-1)} * \bar{c} dt \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \widetilde{\text{Tr}}(a^{*(-1)} * \bar{c})^{*(j+1)} \\ &= \widetilde{\text{Tr}} \ln(1 + a^{*(-1)} * \bar{c}).\end{aligned}$$

Thus, the result of the first two steps may be written in the form

$$\widetilde{\text{Tr}} \bar{a}^{\circ(-1)} \circ d\bar{a} = \widetilde{\text{Tr}} a^{*(-1)} * da + d \widetilde{\text{Tr}} \ln(1 + a^{*(-1)} * (\mathcal{I}\bar{a} - a)) \quad (2.2)$$

where the series for the logarithm is taken with respect to the  $*$ -product.

**Example 2.2** For the case  $n = 1$  the functional under the differential in (2.2) is

$$\widetilde{\text{Tr}} a^{-1} \left( a_1 + \frac{1}{2} \frac{\partial^2 a}{\partial x \partial \xi} \right)$$

which is precisely the additional term in the index formula for surfaces with conical points, cf. [FST97]. This example clarifies the reason of the passage to the Weyl symbols.

**3°** For a given  $i = 1, 2, \dots, n$ , let  $(\mathcal{A}_W)_i$  denote the  $*$ -product algebra on the torus  $T^{n-1} \subset \mathbb{T}^n$  obtained by fixing  $x^i$  (and  $\xi_i$ ). Thus,  $(\bar{a}^{*(-1)} * d\bar{a})_i$  means that in the expression  $\bar{a}^{*(-1)} * d\bar{a}$  we keep the terms which do not contain derivatives in  $x^i$  and  $\xi_i$ . Similarly,  $(\bar{a}^{*(-1)} * d\bar{a})_{ij}$  with  $i > j$  means that we consider the  $*$ -product algebra  $(\mathcal{A}_W)_{ij}$  on the torus  $\mathbb{T}^{n-2} \subset \mathbb{T}^n$  obtained by fixing  $x^i, x^j$  (and  $\xi_i, \xi_j$ ), and so on.

**Lemma 2.3** For  $i_1 < i_2 < \dots < i_J$ , the expression

$$\widetilde{\text{Tr}} (\bar{a}^{*(-1)} * d\bar{a})_{i_1 i_2 \dots i_J}$$

is exact with a primitive functional

$$\Phi = -\widetilde{\text{Tr}} \left( \bar{a}^{*(-1)} * \xi_{i_1} \frac{\partial \bar{a}}{\partial \xi_{i_1}} \right)_{i_1 i_2 \dots i_J} \quad (2.3)$$

(no summation over  $i_1$ ).

**Proof.** Clearly,

$$\delta = \xi_{i_1} \frac{\partial}{\partial \xi_{i_1}}$$

is a derivation of the algebra  $(\mathcal{A}_W)_{i_1}$  as well as of the algebra  $(\mathcal{A}_W)_{i_1 i_2 \dots i_J}$ , since the  $*$ -product does not contain derivatives in  $\xi_{i_1}$ . Thus, applying Lemma 2.1 yields

$$\widetilde{\text{Tr}} \delta (\bar{a}^{*(-1)} * d\bar{a})_{i_1 i_2 \dots i_J} = d \text{Tr} \left( \bar{a}^{*(-1)} * \xi_{i_1} \frac{\partial \bar{a}}{\partial \xi_{i_1}} \right)_{i_1 i_2 \dots i_J}.$$

On the other hand,

$$\begin{aligned} & \widetilde{\text{Tr}} \xi_{i_1} \frac{\partial}{\partial \xi_{i_1}} (\bar{a}^{*(-1)} * d\bar{a})_{i_1 i_2 \dots i_J} \\ &= \widetilde{\text{Tr}} \frac{\partial}{\partial \xi_{i_1}} (\xi_{i_1} \bar{a}^{*(-1)} * d\bar{a})_{i_1 i_2 \dots i_J} - \widetilde{\text{Tr}} (\bar{a}^{*(-1)} * d\bar{a})_{i_1 i_2 \dots i_J} \end{aligned}$$

yields (2.3), since  $\widetilde{\text{Tr}}$  vanishes on complete derivatives. □

Introducing

$$\begin{aligned} \text{Tr} \bar{a}^{*(-1)} * d\bar{a} &:= \widetilde{\text{Tr}} \bar{a}^{*(-1)} * d\bar{a} \\ &\quad - \sum_i \widetilde{\text{Tr}} (\bar{a}^{*(-1)} * d\bar{a})_i \\ &\quad + \sum_{i < j} \widetilde{\text{Tr}} (\bar{a}^{*(-1)} * d\bar{a})_{ij} \\ &\quad - \sum_{i < j < k} \widetilde{\text{Tr}} (\bar{a}^{*(-1)} * d\bar{a})_{ijk} \\ &\quad + \dots, \end{aligned}$$

we see that the expressions  $\text{Tr } \bar{a}^{*(-1)} * d\bar{a}$  and  $\widetilde{\text{Tr}} \bar{a}^{*(-1)} * d\bar{a}$  differ by an exact term with a primitive

$$\begin{aligned}
\Phi &= \sum_i \widetilde{\text{Tr}} \left( \bar{a}^{*(-1)} * \xi_i \frac{\partial \bar{a}}{\partial \xi_i} \right)_i \\
&\quad - \sum_{i < j} \widetilde{\text{Tr}} \left( \bar{a}^{*(-1)} * \xi_i \frac{\partial \bar{a}}{\partial \xi_i} \right)_{ij} \\
&\quad + \sum_{i < j < k} \widetilde{\text{Tr}} \left( \bar{a}^{*(-1)} * \xi_i \frac{\partial \bar{a}}{\partial \xi_i} \right)_{ijk} \\
&\quad - \dots
\end{aligned} \tag{2.4}$$

It is worth pointing out that the functional  $\text{Tr } \bar{a}^{*(-1)} * d\bar{a}$  is the most essential part of  $\widetilde{\text{Tr}} \bar{a}^{*(-1)} * d\bar{a}$ , for the remaining terms come from  $*$ -product algebras on lower-dimensional tori and, as we have seen, they are exact. On the other hand, the expression  $\text{Tr } \bar{a}^{*(-1)} * d\bar{a}$  is more convenient than  $\widetilde{\text{Tr}} \bar{a}^{*(-1)} * d\bar{a}$  since all the terms in  $\text{Tr } \bar{a}^{*(-1)} * d\bar{a}$  contain precisely one derivative in each variable  $x^i$  and  $\xi_i$ , for  $i = 1, 2, \dots, n$ . This gives us a possibility to calculate the functional  $\text{Tr } \bar{a}^{*(-1)} * d\bar{a}$  by induction treating the algebra  $\mathcal{A}_W$  on the torus  $\mathbb{T}^n$  as an algebra on the one-dimensional torus  $\mathbb{T}^1$  with values in the algebra  $\mathcal{A}_W$  on  $\mathbb{T}^{n-1}$ .

### 3 Odd Periodicity Theorem

Consider the algebra  $\mathcal{A}_W$  on the one-dimensional torus  $\mathbb{T}^1$ . Denote

$$a_i b_j \omega^{ij} := \frac{\partial a}{\partial \xi} \frac{\partial b}{\partial x} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial \xi},$$

where

$$\begin{aligned}
a_1 &= \frac{\partial a}{\partial \xi}, \\
a_2 &= \frac{\partial a}{\partial x},
\end{aligned}$$

then

$$a * b = ab - \frac{ih}{2} a_i b_j \omega^{ij} + \dots$$

where the dots mean higher degree terms in  $h$ . Having a function  $a = a(\tau, x, \xi)$  depending on  $2k + 1$  extra parameters, we consider a differential  $(2k + 1)$ -form

$$\text{Tr} \left( a^{*(-1)} * da \right)^{\wedge(2k+1)} = \text{Tr} \underbrace{\left( a^{*(-1)} * da \right) * \dots * \left( a^{*(-1)} * da \right)}_{2k+1}.$$

Here  $da$  means the differential with respect to the parameters, the differentials of the parameters being multiplied by the wedge product.

Recall that the functional  $\text{Tr}$  for the one-dimensional torus may be written in the form

$$\text{Tr} (a^{*(-1)} * da)^{\wedge(2k+1)} = \frac{1}{2\pi} \int_{\mathbb{T}^1 \times \mathbb{R}} \text{tr} \kappa_1 \Big|_{\tau=-1}^{\tau=1} dx d\xi$$

where  $\kappa_1$  is the coefficient of  $h$  in the expansion

$$(a^{*(-1)} * da)^{\wedge(2k+1)} = \kappa_0 + h \kappa_1 + h^2 \kappa_2 + \dots$$

We abbreviate the notation  $\text{tr} \kappa_1 \Big|_{\tau=-1}^{\tau=1}$  to  $\tilde{\text{tr}} \kappa_1$ , this functional on the coefficient algebra (the algebra of matrices in the simplest case) vanishes on commutators.

Our purpose is the following theorem which connects the differential forms  $\text{Tr} (a^{*(-1)} * da)^{\wedge(2k+1)}$  and  $\tilde{\text{tr}} (a^{-1} da)^{\wedge(2k+3)}$ , the differentials in the latter expression are understood with respect to the whole set of variables, i.e.,  $x$ ,  $\xi$  and the extra parameters. We call it the *Odd Periodicity Theorem* since it applies to forms of an odd degree, in contrast to a periodicity theorem of [Fed78] for forms of an even degree. Loosely speaking, this latter result corresponds to the periodicity

$$K^0(X) \cong K^{-2}(X) \cong K^{-4}(X) \cong \dots$$

while Theorem 3.1 corresponds to

$$K^{-1}(X) \cong K^{-3}(X) \cong K^{-5}(X) \cong \dots$$

**Theorem 3.1** *The identity holds*

$$c_k \text{Tr} (a^{*(-1)} * da)^{\wedge(2k+1)} = c_{k+1} \int_{\mathbb{T}^1 \times \mathbb{R}^1} \tilde{\text{tr}} (a^{-1} da)^{\wedge(2k+3)} - d \int_{\mathbb{T}^1 \times \mathbb{R}^1} \tilde{\text{tr}} \varphi_k.$$

**Proof.** We perform calculations in the algebra  $\mathcal{A}_W$  modulo  $h^2$  since we need only coefficients at  $h$  to calculate the trace. Using Equality (2.1) for  $a^{*(-1)}$ , we get

$$a^{*(-1)} = a^{-1} - \frac{ih}{2} a^{-1} a_i a^{-1} a_j a^{-1} \omega^{ij} + \dots$$

and

$$\begin{aligned} a^{*(-1)} * da &= a^{-1} da - \frac{ih}{2} (a^{-1})_i (da)_j \omega^{ij} - \frac{ih}{2} a^{-1} a_i a^{-1} a_j a^{-1} da \omega^{ij} + \dots \\ &= a^{-1} da + \frac{ih}{2} a^{-1} a_i (a^{-1} da)_j \omega^{ij} + \dots \end{aligned}$$

By induction we obtain further

$$\begin{aligned} \mathrm{Tr} (a^{*(-1)} * da)^{\wedge(2k+1)} &= (2k+1) \frac{i}{2} \mathrm{Tr} a^{-1} a_i (a^{-1} da)_j (a^{-1} da)^{2k} \omega^{ij} \\ &\quad - \frac{i}{2} \sum_{\nu=0}^{2k-1} \mathrm{Tr} (a^{-1} da)^\nu (a^{-1} da)_i ((a^{-1} da)^{2k-\nu})_j \omega^{ij}. \end{aligned} \quad (3.1)$$

We omit the sign  $\wedge$  for the exterior product of differential forms and use the fact that cyclic permutations of factors under the  $\mathrm{Tr}$  sign are possible. Let us transform the second summand in (3.1) substituting

$$(a^{-1} da)_i = d(a^{-1} a_i) + a^{-1} da a^{-1} a_i - a^{-1} a_i a^{-1} da. \quad (3.2)$$

It gives

$$\begin{aligned} & - \frac{i}{2} \sum_{\nu=0}^{2k-1} \mathrm{Tr} (a^{-1} da)^\nu d(a^{-1} a_i) ((a^{-1} da)^{2k-\nu})_j \omega^{ij} \\ & - \frac{i}{2} \sum_{\nu=0}^{2k-1} \mathrm{Tr} (a^{-1} da)^{\nu+1} a^{-1} a_i ((a^{-1} da)^{2k-\nu})_j \omega^{ij} \\ & + \frac{i}{2} \sum_{\nu=0}^{2k-1} \mathrm{Tr} (a^{-1} da)^\nu a^{-1} a_i (a^{-1} da) ((a^{-1} da)^{2k-\nu})_j \omega^{ij}. \end{aligned} \quad (3.3)$$

In the third line we have

$$a^{-1} da ((a^{-1} da)^{2k-\nu})_j = ((a^{-1} da)^{2k-\nu+1})_j - (a^{-1} da)_j (a^{-1} da)^{2k-\nu}.$$

So, using once again cyclic permutations under the trace sign, we reduce (3.3) to

$$\begin{aligned} & - \frac{i}{2} \sum_{\nu=0}^{2k-1} \mathrm{Tr} d(a^{-1} a_i) ((a^{-1} da)^{2k-\nu})_j (a^{-1} da)^\nu \omega^{ij} \\ & - \frac{i}{2} \mathrm{Tr} (a^{-1} da)^{2k} a^{-1} a_i (a^{-1} da)_j \omega^{ij} \\ & + \frac{i}{2} \mathrm{Tr} a^{-1} a_i ((a^{-1} da)^{2k+1})_j \omega^{ij} - ik \mathrm{Tr} a^{-1} a_i (a^{-1} da)_j (a^{-1} da)^{2k} \omega^{ij}. \end{aligned}$$

Together with the first summand in (3.1) we get

$$\begin{aligned} & \frac{i}{2} \mathrm{Tr} a^{-1} a_i ((a^{-1} da)^{2k+1})_j \omega^{ij} \\ & - \frac{i}{2} \sum_{\nu=0}^{2k-1} \mathrm{Tr} d(a^{-1} a_i) ((a^{-1} da)^{2k-\nu})_j (a^{-1} da)^\nu \omega^{ij}. \end{aligned} \quad (3.4)$$

In the first summand we integrate by parts with respect to the  $j$ -th variable. Since  $\omega^{ij}$  is antisymmetric, we come to

$$-\frac{i}{2} \operatorname{Tr} a^{-1} a_i a^{-1} a_j (a^{-1} da)^{2k+1} \omega^{ij}.$$

In the second summand we integrate by parts with respect to  $d$ , thus obtaining

$$\begin{aligned} & -\frac{i}{2} d \sum_{\nu=0}^{2k-1} \operatorname{Tr} a^{-1} a_i ((a^{-1} da)^{2k-\nu})_j (a^{-1} da)^\nu \omega^{ij} \\ & + \frac{i}{2} \sum_{\nu=0}^{2k-1} \operatorname{Tr} a^{-1} a_i d \left( ((a^{-1} da)^{2k-\nu})_j (a^{-1} da)^\nu \right) \omega^{ij}. \end{aligned}$$

Clearly,

$$d (a^{-1} da)^\nu = 0,$$

for even  $\nu$ , and

$$d (a^{-1} da)^\nu = - (a^{-1} da)^{\nu+1},$$

for  $\nu$  odd. Thus, taking  $\nu = 2l - 1$ , for  $l = 1, 2, \dots, k$ , we get

$$\begin{aligned} & d \left( ((a^{-1} da)^{2k-2l+1})_j (a^{-1} da)^{2l-1} \right) \\ & = - ((a^{-1} da)^{2k-2l+2})_j (a^{-1} da)^{2l-1} + ((a^{-1} da)^{2k-2l+1})_j (a^{-1} da)^{2l} \\ & = - (a^{-1} da)^{2k-2l+1} (a^{-1} da)_j (a^{-1} da)^{2l-1} \end{aligned}$$

or, using the identity (3.2) once more and gathering all the terms, we rewrite (3.4) in the form

$$\begin{aligned} & -\frac{i}{2} \operatorname{Tr} a^{-1} a_i a^{-1} a_j (a^{-1} da)^{2k+1} \omega^{ij} \\ & -\frac{i}{2} d \sum_{\nu=0}^{2k-1} \operatorname{Tr} a^{-1} a_i ((a^{-1} da)^{2k-\nu})_j (a^{-1} da)^\nu \omega^{ij} \\ & -\frac{i}{2} \sum_{l=1}^k \operatorname{Tr} a^{-1} a_i (a^{-1} da)^{2k-2l+1} d (a^{-1} a_j) (a^{-1} da)^{2l-1} \omega^{ij} \\ & -\frac{i}{2} \sum_{l=1}^k \operatorname{Tr} a^{-1} a_i (a^{-1} da)^{2k-2l+2} a^{-1} a_j (a^{-1} da)^{2l-1} \omega^{ij} \\ & + \frac{i}{2} \sum_{l=1}^k \operatorname{Tr} a^{-1} a_i (a^{-1} da)^{2k-2l+1} a^{-1} a_j (a^{-1} da)^{2l} \omega^{ij}. \end{aligned} \quad (3.5)$$

Denoting the sum in the third line of (3.5) by  $\Sigma$  and integrating by parts with respect to  $d$ , we get

$$\begin{aligned}
\Sigma &= \sum_{l=1}^k \text{Tr} a^{-1} a_i (a^{-1} da)^{2k-2l+1} d(a^{-1} a_j) (a^{-1} da)^{2l-1} \omega^{ij} \\
&= -d \sum_{l=1}^k \text{Tr} a^{-1} a_i (a^{-1} da)^{2k-2l+1} a^{-1} a_j (a^{-1} da)^{2l-1} \omega^{ij} \\
&\quad + \sum_{l=1}^k \text{Tr} d(a^{-1} a_i) (a^{-1} da)^{2k-2l+1} a^{-1} a_j (a^{-1} da)^{2l-1} \omega^{ij} \\
&\quad - \sum_{l=1}^k \text{Tr} a^{-1} a_i (a^{-1} da)^{2k-2l+2} a^{-1} a_j (a^{-1} da)^{2l-1} \omega^{ij} \\
&\quad + \sum_{l=1}^k \text{Tr} a^{-1} a_i (a^{-1} da)^{2k-2l+1} a^{-1} a_j (a^{-1} da)^{2l} \omega^{ij}.
\end{aligned}$$

Now

$$\sum_{l=1}^k \text{Tr} d(a^{-1} a_i) (a^{-1} da)^{2k-2l+1} a^{-1} a_j (a^{-1} da)^{2l-1} \omega^{ij} = -\Sigma$$

which can be seen by cyclic permutations under the  $\text{Tr}$  sign. This gives the following expression for  $\Sigma$

$$\begin{aligned}
\Sigma &= -\frac{1}{2} d \sum_{l=1}^k \text{Tr} a^{-1} a_i (a^{-1} da)^{2k-2l+1} a^{-1} a_j (a^{-1} da)^{2l-1} \omega^{ij} \\
&\quad - \frac{1}{2} \sum_{l=1}^k \text{Tr} a^{-1} a_i (a^{-1} da)^{2k-2l+2} a^{-1} a_j (a^{-1} da)^{2l-1} \omega^{ij} \\
&\quad + \frac{1}{2} \sum_{l=1}^k \text{Tr} a^{-1} a_i (a^{-1} da)^{2k-2l+1} a^{-1} a_j (a^{-1} da)^{2l} \omega^{ij}
\end{aligned}$$

and, finally,

$$\begin{aligned}
&\text{Tr} (a^{*(-1)} * da)^{\wedge(2k+1)} \\
&= -\frac{i}{4} \sum_{\nu=0}^{2k+1} (-1)^\nu \text{Tr} a^{-1} a_i (a^{-1} da)^\nu a^{-1} a_j (a^{-1} da)^{2k-\nu+1} \omega^{ij} \\
&\quad - \frac{i}{2} d \sum_{\nu=0}^{2k-1} \text{Tr} a^{-1} a_i ((a^{-1} da)^{2k-\nu})_j (a^{-1} da)^\nu \omega^{ij} \\
&\quad + \frac{i}{4} d \sum_{l=1}^k \text{Tr} a^{-1} a_i (a^{-1} da)^{2k-2l+1} a^{-1} a_j (a^{-1} da)^{2l-1} \omega^{ij} \quad (3.6)
\end{aligned}$$

giving the desired identity. The density  $\varphi_k$  may be read off from (3.6).  $\square$

We have considered the case when the functions  $a_k(\tau, x, \xi, \dots)$  take values in the matrix algebra with the trace  $\tilde{\text{tr}}$  defined by

$$\tilde{\text{tr}} a = \text{tr} a \Big|_{\tau=-1}^{\tau=1}.$$

But an important observation is that the theorem is still valid in a more general case when the functions  $a_k$  take values in an associative algebra with a trace functional  $\tilde{\text{tr}}$  vanishing on commutators. In particular, we apply this theorem to formal symbols on the one-dimensional torus  $\mathbb{T}^1$  with coefficients in the algebra  $\mathcal{A}_W$  on the torus  $\mathbb{T}^{n-1}$  with the trace functional  $\text{Tr}$  (instead of  $\tilde{\text{tr}}$ ).

This functional admits an *inductive definition*. To formulate it, we need to slightly modify the  $*$ -product introducing a set  $h = (h_1, h_2, \dots, h_n)$  of formal parameters for the torus  $\mathbb{T}^n$ , for each pair  $(x^j, \xi_j)$  we consider its own parameter  $h_j$ . In multi-index notation  $h^\alpha = h_1^{\alpha_1} h_2^{\alpha_2} \dots h_n^{\alpha_n}$  the elements of our algebra  $\mathcal{A}_n$  are formal series

$$\bar{a} = \sum_{\alpha \in \mathbb{Z}_+^n} h^\alpha a_\alpha(\tau, \xi, x, \dots)$$

with  $a_\alpha$  being homogeneous of order  $-|\alpha|$  in  $(\tau, \xi)$ . The multiplication (now denoted by  $\star$ ) is similar to the  $*$ -product but with different parameters  $h_j$  for each pair  $x^j, \xi_j$ . More precisely, introduce an operator

$$D = -\frac{i}{2} \sum_{j=1}^n h_j \left( \frac{\partial}{\partial \xi_j} \frac{\partial}{\partial y^j} - \frac{\partial}{\partial x^j} \frac{\partial}{\partial \eta_j} \right).$$

Then

$$\bar{a} \star \bar{b} = e^D \bar{a}(\tau, x, \xi, \dots) \bar{b}(\tau, y, \eta, \dots) \Big|_{\substack{y=x \\ \eta=\xi}}.$$

We define  $\text{Tr} = \text{Tr}_n$  on the algebra  $\mathcal{A}_n$  by setting

$$\text{Tr}_n \bar{a} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n \times \mathbb{R}^n} \tilde{\text{tr}} a_{1,1,\dots,1} dx d\xi. \quad (3.7)$$

Observe that the functional  $\tilde{\text{Tr}}$  on the algebra  $\mathcal{A}_W$  may be written as

$$\tilde{\text{Tr}} \bar{a} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n \times \mathbb{R}^n} \tilde{\text{tr}} \sum_{|\alpha|=n} a_\alpha dx d\xi. \quad (3.8)$$

Thus, (3.7) corresponds only to one summand in (3.8), namely to that with  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ .

The algebra  $\mathcal{A}_n$  on the  $n$ -dimensional torus  $\mathbb{T}^n$  may be thought of as an algebra  $\mathcal{A}_1$  on an one-dimensional torus with values in the algebra  $\mathcal{A}_{n-1}$ . Indeed, a great advantage of the functional  $\text{Tr}$  compared to the functional  $\widetilde{\text{Tr}}$  consists in the following *chain rule*

$$\text{Tr}_n \bar{a} = \frac{1}{2\pi} \int_{\mathbb{T}^1 \times \mathbb{R}^1} \text{Tr}_{n-1} a_1 dx^1 d\xi_1$$

giving a possibility to treat the algebra  $(\mathcal{A}_n, \star_n, \text{Tr}_n)$  as an algebra on the one-dimensional torus  $(\mathcal{A}_1, \star_1, \text{Tr}_1)$  with values in the algebra  $(\mathcal{A}_{n-1}, \star_{n-1}, \text{Tr}_{n-1})$ .

In particular, Theorem 3.1 may be applied to  $(\mathcal{A}_1, \star_1, \text{Tr}_1)$  with values in  $(\mathcal{A}_{n-1}, \star_{n-1}, \text{Tr}_{n-1})$  resulting in the following theorem which is the aim of this paper.

### Theorem 3.2

$$c_0 \widetilde{\text{Tr}} \bar{a}^{*(-1)} * d\bar{a} = c_n \widetilde{\text{Tr}} (a^{-1} da)^{\wedge(2n+1)} - d\Phi \quad (3.9)$$

where

$$\Phi = \int_{\mathbb{T}^n \times \mathbb{R}^n} \text{tr} \varphi \Big|_{\tau=-1}^{\tau=1} dx d\xi$$

with a polynomial  $\varphi$  depending on  $a^{-1}$ , partial derivatives of  $a$  and  $\xi_j$ .

**Proof.** First we replace  $\widetilde{\text{Tr}}$  by  $\text{Tr} = \text{Tr}_n$  by Lemma 2.3. The difference has the form  $-d\Phi$  with  $\Phi$  defined by (2.4). Next, we apply step by step the Periodicity Theorem. Let us explain the first step in more detail. We write

$$\text{Tr} \bar{a}^{*(-1)} * d\bar{a} := \text{Tr}_n \bar{a}^{\star_n(-1)} \star_n d\bar{a}$$

and then, representing the algebra  $\mathcal{A}_n$  as the algebra  $\mathcal{A}_1$  with values in  $\mathcal{A}_{n-1}$  and using Theorem 3.1,

$$\text{Tr}_n \bar{a}^{\star_n(-1)} \star_n d\bar{a} = c_1 \frac{1}{2\pi} \int_{\mathbb{T}^1 \times \mathbb{R}^1} \text{Tr}_{n-1} (\bar{a}^{\star_{n-1}(-1)} \star_{n-1} d\bar{a})^{\wedge 3} \quad (3.10)$$

(at the first step we have no  $d\Phi$ ). Next, we again apply Theorem 3.1 to the integrand on the right of (3.10). This gives us

$$c_2 \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2 \times \mathbb{R}^2} \text{Tr}_{n-2} (\bar{a}^{\star_{n-2}(-1)} \star_{n-2} d\bar{a})^{\wedge 5} + d\Phi_2$$

(this time  $\Phi_2$  is non-zero), and so on. After  $n$  steps we arrive at (3.9).  $\square$

## 4 Concluding remarks

In this section we discuss possible generalisations of our methods.

1° First let us observe that the Atiyah-Singer functional and  $\eta$ -invariant terms are objects of a very different nature. The Atiyah-Singer functional is defined purely in terms of differential geometry (and even differential topology) while the  $\eta$ -invariant is defined in terms of operator theory. Clearly, adjusting such different in nature objects requires an “ugly” functional like our functional  $\Phi$ .

One encounters the same difficulty even in classical boundary value problems. Since then a possible way out is to deform our problem to one where the contributions of the objects of different nature are separated (do not interact). In our case this deformation is very simple, it is described in Theorem 1.6.

In the general case one could proceed as follows. We take a direct sum of  $N$  copies of the operator  $A$  on a manifold with a conical point  $v_1$  (we assume for simplicity that there is only one conical point) and try to deform the interior part  $A_{\text{int}}$  of our operator to a Dirac-type operator (preserving the product structure near the conical point). Such a homotopy is always possible for smooth manifolds  $M$  with properly chosen  $N$ , hopefully it is also possible in the presence of conical points. Whenever such a path is constructed, we can use it to describe additional terms in the index formula by means of variations along the path.

More precisely, let  $A_{\text{int}}(s)$ ,  $0 \leq s \leq 1$ , be the needed deformation, so that

$$\begin{aligned} A_{\text{int}}(0) &= A_{\text{int}}, \\ A_{\text{int}}(1) &= D_{\text{int}}, \end{aligned}$$

where  $D$  is a Dirac-type operator. Everything is considered on  $M_{\text{int}}$  with a product structure on the cylindrical end representing the conical point. At the beginning we have also a conormal symbol  $A_c(\tau)$  compatible with  $A_{\text{int}}$ . At the end we take any conormal symbol  $D_c(\tau)$  compatible with  $D_{\text{int}}$  and invertible on some weight line  $\Im\tau = \text{const}$  (such a symbol can be found). By the algebraic index theorem for operators  $A$  and  $D$  we have

$$\text{ind } A = G(A_{\text{int}}) - \frac{1}{2} \eta(A_c(\tau)),$$

where  $G(A_{\text{int}})$  is defined similarly to (1.4), and

$$\text{ind } D = G(D_{\text{int}}) - \frac{1}{2} \eta(D_c(\tau)).$$

On the other hand, by the Atiyah-Patodi-Singer Theorem we have

$$\text{ind } D = \text{AS}(\sigma_{\text{int}}(D)) - \frac{1}{2} \eta(D_c(\tau)),$$

whence

$$\begin{aligned} F(D_{\text{int}}) &:= G(D_{\text{int}}) - \text{AS}(\sigma_{\text{int}}(D)) \\ &= 0. \end{aligned}$$

Now, using the homotopy, we may write

$$F(A_{\text{int}}) = - \int_0^1 d_s F(A_{\text{int}}(s)),$$

and we come to the formula

$$\text{ind } A = \text{AS}(\sigma_{\text{int}}(A)) - \frac{1}{2} \eta(A_c(\tau)) - \int_0^1 d_s F(A_{\text{int}}(s)).$$

**2°** The next step, which also seems to be very easy, is to find a good variation formula for the functional  $F(A_{\text{int}}(s))$ . For the summand  $G(A_{\text{int}}(s))$  the variation formula is similar to (1.6), namely

$$-dG = \frac{1}{2\pi i} \widetilde{\text{Tr}}(\bar{r}(\infty, \tau) \circ d\bar{a}(\infty, \tau))$$

with the only difference that formal symbols on manifolds are more complicated objects than on tori.

So far everything goes in much the same way as in the case of a torus. We encounter the first difficulty when studying the variation of the Atiyah-Singer functional. In general this functional has a more complicated structure, so the variation formula (1.15) fails. Nevertheless, a local formula for the variation of the type

$$dF(A_{\text{int}}(s)) = \widetilde{\text{Tr}} f(A_{\text{int}}(s), \dot{A}_{\text{int}}(s))$$

still may be found, since for the Atiyah-Singer functional we have an explicit formula.

**3°** A real failure comes when we try to find a primitive functional  $\Phi$ . All we can do here is to formulate a conjecture

$$dF(A_{\text{int}}(s)) = d\widetilde{\text{Tr}} \varphi(\bar{a}(\tau))$$

where  $\bar{a}(\tau)$  is a parameter-dependent formal symbol on  $X$  (the base of the cone) obtained by the restriction of the interior formal symbol  $\bar{a}_{\text{int}}$  to  $t = \infty$ . The local density  $\varphi$  depends on the coefficients of  $\bar{a}(\tau)$  and the inverse  $a_0^{-1}(\tau)$  polynomially. The present paper may be considered as the first example in favour of this conjecture.

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