Elliptic Operators in Subspaces and the Eta Invariant

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Abstract

The paper deals with the calculation of the fractional part of the η-invariant for elliptic self-adjoint operators in topological terms. The method used to obtain the corresponding formula is based on the index theorem for elliptic operators in subspaces obtained in [1], [2]. It also utilizes K-theory with coefficients $\mathbf{Z}_n$. In particular, it is shown that the group $K(T^*M, \mathbf{Z}_n)$ is realized by elliptic operators (symbols) acting in appropriate subspaces.

Keywords: index of elliptic operators in subspaces, K-theory, eta-invariant, mod k index, Atiyah–Patodi–Singer theory

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Contents

Introduction

The spectral η-invariant of a self-adjoint elliptic operator on a closed manifold was introduced by Atiyah, Patodi, and Singer [3]. It appeared as a nonlocal contribution to an index formula for manifolds with boundary obtained via the heat equation technique. From the very moment of its introduction, it was clear that this spectral invariant in the general case is neither an invariant of the principal symbol of the operator nor a homotopy invariant of the operator itself. More precisely, for a generic smooth one-parameter family A_t of elliptic self-adjoint pseudodifferential operators, the function $\eta(A_t)$ is a piecewise smooth function of t with jumps occurring at the parameter values for which some eigenvalue of the operator in the family changes its sign.

P. Gilkey [4] observed that for differential operators satisfying the parity condition

$$\operatorname{ord} A + \dim M \equiv 1 \pmod 2, \tag{1}$$

the η-invariant of a one-parameter family is a piecewise constant function. This implies, in particular, that in this case the fractional part of the spectral η-invariant of a self-adjoint elliptic operator is in fact a homotopy invariant depending on the principal symbol of the operator alone . Thus, we arrive at the problem of computing the fractional part of the η-invariant. This fractional invariant was successfully applied in several problems of topology and differential geometry (e.g., see [5, 6, 7, 8]).

Our approach to this problem is based on the following observation [1]: the η-invariant of an operator A satisfying condition (1) is completely determined by the nonnegative spectral subspace $\widehat{L}_+(A)$ of this operator. The fractional part of the invariant, in turn, is determined by the so-called *symbol of the subspace*. This is a vector bundle on the cospheres S^*M over the manifold generated by the positive eigensubspaces of the self-adjoint symbol $\sigma(A)$ of the operator A. First, this allows us to identify the η-invariant of self-adjoint elliptic operators with a dimension-type functional on the corresponding spectral subspaces introduced in [1]. This implies that the η-invariant in this case takes only dyadic values,

$$\eta(A) \in \mathbf{Z}\left[\frac{1}{2}\right].$$

Second, we can apply the index formula for elliptic operators in subspaces [1, 2]. The index formula gives an expression for the fractional part of the η-invariant for the case in which the operator A defines a trivial element in the group $K^1(T^*M)$. In the general case, however, the index formula in subspaces alone does not compute the η-invariant. Nevertheless, it reduces the computation of the fractional part to the "index modulo n" problem for operators in subspaces. The term "modulo n" here expresses the fact that in this case the index of an elliptic operator in subspaces, being reduced modulo n, becomes an invariant of the principal symbol of the operator. It turns out that such elliptic operators on a closed manifold define the K-theory with coefficients in $\mathbf{Z}_n$. In particular, the index of an operator is computed modulo n by the direct image in K-theory.

Let us briefly describe the contents of the paper. In the first section we recall the main results of [1] and [2]: the necessary definitions are reproduced, and theorems on the "dimension-type" functional, as well as index formulas for operators acting in subspaces are written out. A theorem expressing the dimension type functional in terms of the η-invariant, is stated. It should be noted that elliptic operators in subspaces are a generalization of Wiener-Hopf operators (see, e.g. [9]). A different class of operators in subspaces, the so-called *Toeplitz operators*, is considered in [10, 11].

In the following section, the direct computation of the η-invariant via the index formula is carried out in a special situation. Examples are given. It is also shown in this section that the computation of the η-invariant in the general case is reduced to the index modulo n computation for an elliptic operator acting in subspaces of a special form. This computation is carried out in Section 3, where we generalize a method applied in [12] to the computation of fractional parts of η-invariants of operators with coefficients in flat bundles. In the next section we show that the stable homotopy classes of elliptic operators modulo n are classified by the K-theory group $K(T^*M, \mathbf{Z}_n)$ with coefficients $\mathbf{Z}_n$. Our final formula for the fractional parts of the invariants, obtained in the last section, can be interpreted in the following way (cf. [12]).

Theorem 1 *The principal symbol of an elliptic self-adjoint operator A whose order satisfies* (1) *defines an element in K-theory with coefficients*

$$[\sigma(A)] \in K\left(T^*M, \mathbf{Z}\left[\frac{1}{2}\right]\Big/\mathbf{Z}\right).$$

The fractional part $\{\eta(A)\}$ of the η-invariant of A is computed by the direct image homomorphism

$$\{\eta(A)\} = p_![\sigma(A)] \in K\left(pt, \mathbf{Z}\left[\frac{1}{2}\right]\Big/\mathbf{Z}\right) = \mathbf{Z}\left[\frac{1}{2}\right]\Big/\mathbf{Z},$$

for the map $p: M \to pt$.

Two appendixes are placed at the end of the paper. In the first appendix we give a description of the odd K-theory group $K^1(T^*M)$ in terms of elliptic self-adjoint operators as well as in terms of subspaces defined by pseudodifferential projections. In the second appendix, the description of the K-theory with $\mathbf{Z}_n$ coefficients is given. The necessary material from this theory, used in the main part of the paper, is explained.

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1 Subspaces and the index formulas

1. Spaces defined by pseudodifferential projections on a smooth closed manifold M were considered in [1, 2]. More precisely, a subspace

$$\widehat{L} \subset C^\infty(M, E)$$

in the space of smooth sections of a vector bundle E on M is said to be *admissible* if

$$\widehat{L} = \operatorname{Im} P, \quad P: C^\infty(M, E) \to C^\infty(M, E)$$

for some pseudodifferential projection P of order zero. In this case the vector bundle (or, more precisely, the subbundle)

$$L = \operatorname{Im} \sigma(P) \subset \pi^* E \in \operatorname{Vect}(S^* M) \qquad (2)$$

is called the *(principal) symbol of the subspace.* Here $\pi : S^*M \to M$ is the projection for the cosphere bundle of M.

For a symbol L as in (2), one can always construct a subspace $\widehat{L} \subset C^\infty(M, E)$. This statement is based on the construction (see [14] for details) of a pseudodifferential projection P from the principal symbol-projection $\sigma(P)$ by a Cauchy-type integral

$$P = -\frac{1}{2\pi i} \int\limits_{|\lambda-1|=\varepsilon} (P_0 - \lambda I)^{-1} \, d\lambda,$$

where P_0 is a zero order pseudodifferential operator with principal symbol $\sigma(P)$ and ε is a small number such that the circle $|\lambda - 1| = \varepsilon$ does not contain any eigenvalues of the operator P_0.

On the total space of the cotangent bundle T^*M we consider the antipodal involution

$$\alpha : T^*M \longrightarrow T^*M, \quad \alpha(x, \xi) = (x, -\xi).$$

A subspace $\widehat{L} \subset C^\infty(M, E)$ is said to be *even (odd)* with respect to the involution α if the principal symbol L is invariant (antiinvariant) under the involution:

$$L = \alpha^* L, \quad \text{or} \quad L \oplus \alpha^* L = \pi^* E. \qquad (3)$$

We point out that both equalities in this formula are equalities of subbundles in the ambient bundle $\pi^* E$. Denote the semigroups of even(odd) subspaces by $\widehat{\operatorname{Even}}(M)$ $\left(\widehat{\operatorname{Odd}}(M)\right)$. The symbols of even (odd) subspaces will be referred to as even (odd) bundles for brevity.

It turns out that if the parities of the subspaces and of the dimension of M are opposite, then the subspaces have a homotopy invariant that is an analog of the notion of dimension of a finite-dimensional vector space. More precisely, the following theorem holds [1, 2].

Theorem 2 *There is a unique additive functional*

$$d : \widehat{\operatorname{Even}}\left(M^{odd}\right) \to \mathbf{Z}\left[\frac{1}{2}\right], \quad \text{or} \quad d : \widehat{\operatorname{Odd}}\left(M^{ev}\right) \to \mathbf{Z}\left[\frac{1}{2}\right]$$

with the following properties:

1. *(invariance)* $d\left(U\widehat{L}\right)=d\left(\widehat{L}\right)$ *for all invertible pseudodifferential operators* U *with even principal symbol:* $\alpha^*\sigma(U)=\sigma(U)$;

2. *(relative index)* $d\left(\widehat{L}_1\right)-d\left(\widehat{L}_2\right)=\operatorname{ind}\left(\widehat{L}_1,\widehat{L}_2\right)$ *for two subspaces with equal principal symbols;*[1]

3. *(complement)* $d\left(\widehat{L}\right)+d\left(\widehat{L}^{\perp}\right)=0,$ *where* $\widehat{L}^{\perp}$ *denotes the orthogonal complement of* $\widehat{L}$.

Corollary 1 *The functional* d *is a homotopy invariant of the subspace, while its fractional part is an invariant of the principal symbol of the subspace.*

Indeed, the homotopy invariance follows from the invariance property. Moreover, it follows from the relative index property that the fractional part is determined by the principal symbol of the subspace.

Remark 1 Roughly speaking, the functional d measures the deviation of the subspace $\widehat{L}$ from a space of sections of a vector bundle, since the third property implies

$$d\left(C^\infty(M,E)\right)=0.$$

2. Let $\widehat{L}_{1,2}\subset C^\infty(M,E_{1,2})$ be two subspaces. Consider a pseudodifferential operator

$$D:C^\infty(M,E_1)\longrightarrow C^\infty(M,E_2)$$

in the ambient spaces. Suppose that this operator preserves the subspaces: $D\widehat{L}_1\subset\widehat{L}_2$. The restriction

$$D:\widehat{L}_1\longrightarrow\widehat{L}_2 \tag{4}$$

is called an *operator acting in subspaces.* In this case the principal symbol $\sigma(D)$ can also be restricted to a homomorphism

$$\sigma(D):L_1\longrightarrow L_2 \tag{5}$$

of vector bundles on S^*M. The homomorphism (5) is called the *principal symbol of the operator in subspaces.* Conversely, it is easily seen that for any symbol (5)

[1]The relative index of subspaces $\operatorname{ind}\left(\widehat{L}_1,\widehat{L}_2\right)$ is computed in terms of the projections by the formula [15]

$$\operatorname{ind}\left(\widehat{L}_1,\widehat{L}_2\right)=\operatorname{ind}\left(P_2:\operatorname{Im}P_1\to\operatorname{Im}P_2\right),\quad \widehat{L}_{1,2}=\operatorname{Im}P_{1,2}.$$

One can check that the operator on the right-hand side of the formula is Fredholm.

it is always possible to construct an operator in subspaces. Indeed, let us take an arbitrary pseudodifferential operator $D_0 : C^\infty(M, E_1) \to C^\infty(M, E_2)$ with principal symbol $\sigma(D_0)$ (on the complement of L_1 we can specify the principal symbol in arbitrary way); then the operator

$$D = P_2 D_0,$$

where P_2 is a pseudodifferential projection on the subspace $\widehat{L}_2$, acts in the pair of subspaces $\widehat{L}_1, \widehat{L}_2$ and has the principal symbol (5).

It is proved in [16] that the closure

$$D : H^s(M, E_1) \supset \widehat{L}_1 \longrightarrow \widehat{L}_2 \subset H^{s-m}(M, E_2), \qquad m = \mathrm{ord} D,$$

of the operator (4) in Sobolev spaces is a Fredholm operator if and only if the principal symbol (5) is *elliptic*, i.e., an isomorphism of vector bundles.

For elliptic operators in subspaces the following index formulas were obtained in [1, 2].

Theorem 3 *Let $D : \widehat{L}_1 \longrightarrow \widehat{L}_2$, $\widehat{L}_{1,2} \subset C^\infty(M, E_{1,2})$ be an elliptic operator in subspaces of the same parity*

$$\widehat{L}_{1,2} \in \widehat{\mathrm{Even}}\left(M^{odd}\right) \quad \text{or} \quad \widehat{\mathrm{Odd}}\left(M^{ev}\right).$$

Then the index of D is equal to

$$\mathrm{ind}\left(D, \widehat{L}_1, \widehat{L}_2\right) = \frac{1}{2}\mathrm{ind}\widetilde{D} + d\left(\widehat{L}_1\right) - d\left(\widehat{L}_2\right), \tag{6}$$

where

$$\widetilde{D} : C^\infty(M, E_1) \longrightarrow C^\infty(M, E_2)$$

is an elliptic operator with principal symbol

$$\sigma\left(\widetilde{D}\right) = \sigma(D) \oplus \alpha^* \sigma(D) : L_1 \oplus \alpha^* L_1 \longrightarrow L_2 \oplus \alpha^* L_2$$

for odd subspaces. In the case of even subspaces, the operator is

$$\widetilde{D} : C^\infty(M, E_1) \longrightarrow C^\infty(M, E_1),$$
$$\sigma\left(\widetilde{D}\right) = [\alpha^* \sigma(D)]^{-1} \sigma(D) \oplus 1 : L_1 \oplus L_1^\perp \longrightarrow L_1 \oplus L_1^\perp.$$

3. The functional d of subspaces, occurring in this theory, can be expressed in terms of the η-invariant of Atiyah–Patodi–Singer.

Namely, for an elliptic self-adjoint operator

$$A : C^\infty (M, E) \longrightarrow C^\infty (M, E),$$

consider the subspace $\widehat{L}_+ (A) \subset C^\infty (M, E)$, generated by the eigenvectors of A corresponding to nonnegative eigenvalues. It is well-known (e.g., see [3]) that the spectral projection $P_+ (A)$ on this subspace is a pseudodifferential operator of order zero. Thus, the subspace $\widehat{L}_+ (A)$ is admissible. The symbol $L_+ (A)$ of the subspace can be calculated explicitly (see Appendix A).

Moreover, if A is a differential operator, then the subspace $\widehat{L}_+ (A)$ is even or odd, according to the order of the operator A. The same property holds for a class of pseudodifferential operators introduced in [4]: these are pseudodifferential operators with classical complete symbols such that the homogeneous terms in the asymptotic expansion of the symbol possess the $\mathbf{R}_*$-invariance (cf. [17]):

$$\sigma (A) (x, \xi) \sim \sum_{j=o}^{\infty} a_{d-j} (x, \xi), \qquad a_k (x, -\xi) = (-1)^k a_k (x, \xi), \quad \text{for all } k \leq d. \quad (7)$$

For this class of operators the functional d of the spectral subspace is equal to the η-invariant [1, 2]. Namely, the following theorem is valid.

Theorem 4 *The nonnegative spectral subspace $\widehat{L}$ of a self-adjoint elliptic operator A under the condition* (7) *satisfies*

$$d \left(\widehat{L} \right) = \eta (A) \qquad (8)$$

provided that the order of A and the dimension of the manifold have opposite parities.

Remark 2 It was shown in [1] that an even subspace can be realized as a spectral subspace of this kind if and only if the orthogonal projection on it satisfies (7). In the odd situation there is a similar restriction [2]: the orthogonal projection must satisfy (7) with $(-1)^k$ replaced by $(-1)^{k+1}$for $k < 0$.

Equation (8) makes two computations equivalent, one of the fractional part of the functional d in terms of the principal symbol of the subspace and the other of the η-invariant via the principal symbol of a self-adjoint operator. In the present paper we use the terminology of subspaces and the corresponding index theory.

2 The fractional part via an index formula

1. On a smooth manifold M consider an admissible subspace $\widehat{L} \subset C^\infty(M,E)$. In this section we compute the fractional part of $d(\widehat{L})$ for the case in which the principal symbol

$$L \subset \pi^* E \in \mathrm{Vect}\,(S^*M), \quad \pi : S^*M \to M,$$

of $\widehat{L}$ on the cosphere bundle S^*M is lifted from the base manifold M, i.e., for some vector bundle $F \in \mathrm{Vect}\,(M)$ there is an isomorphism of vector bundles

$$\sigma : L \longrightarrow \pi^* F. \tag{9}$$

Consider an elliptic operator in subspaces with principal symbol (9). The index formula (6) for this operator gives

$$\mathrm{ind}\left(\widehat{\sigma}, \widehat{L}, C^\infty(M,F)\right) = \frac{1}{2}\mathrm{ind}\,\widehat{\widetilde{\sigma}} + d\left(\widehat{L}\right), \tag{10}$$

$\widetilde{\sigma} = \sigma(\widetilde{D})$. It is clear from this formula that the fractional part of $d(\widehat{L})$ is either 1/2 or 0. Furthermore, it can be computed by the formula

$$\{d\left(\widehat{L}\right)\} = \left\{\frac{1}{2}\mathrm{ind}\widehat{\widetilde{\sigma}}\right\}. \tag{11}$$

Let us consider two applications of this formula. In the first application formula (11) implies an integrality theorem for the functional d, while the second example presents a nontrivial fractional part of the functional d.

2. Examples. On a smooth oriented closed Riemannian manifold M we consider an elliptic self-adjoint differential operator of the second order

$$A = d\delta - \delta d : C^\infty\left(M, \Lambda^1(M)\right) \longrightarrow C^\infty\left(M, \Lambda^1(M)\right)$$

in the spaces of complexified smooth sections of the bundle of exterior 1-forms; here d is the exterior derivative d and δ is the adjoint operator with respect to the Riemannian metric. Let us calculate the fractional part of the functional d on the nonnegative spectral subspace $\widehat{L}$ of the operator A. The principal symbol of A is

$$\sigma(A)(\xi) = \xi \wedge \xi\rfloor - \xi\rfloor\xi\wedge : \pi^*\Lambda^1(M) \longrightarrow \pi^*\Lambda^1(M),$$

where $\xi\wedge$ is the exterior product by a covector ξ and $\xi\rfloor$ is the inner product by the same covector with respect to the Riemannian metric (e.g., see [18]). For an arbitrary point $(x,\xi) \in S^*_x M$ of the cosphere bundle, the symbol L of the spectral subspace coincides with the line spanned by the covector ξ. Hence, $L\subset\pi^*\Lambda^1(M)$ is an even subbundle. The line bundle L is trivial. We choose the trivialization

$$\begin{array}{rcl}\sigma : L & \to & \pi^*\mathbf{C}, \\ \sigma(x,\xi)\eta & = & \langle\xi,\eta\rangle_x\end{array}, \qquad (x,\xi)\in S_x^*M,\ \eta\in L, \tag{12}$$

where $\langle\xi,\eta\rangle_x$ denotes the Hermitian inner product of two covectors with respect to the Riemannian metric at the point x. For the corresponding pseudodifferential operator

$$\widehat{\sigma}:\widehat{L}\longrightarrow C^\infty(M)$$

in the subspaces, the index formula (6) implies

$$\operatorname{ind}\left(\widehat{\sigma},\widehat{L},C^\infty(M)\right)=\frac{1}{2}\operatorname{ind}\widehat{\widetilde{\sigma}}+d\left(\widehat{L}\right).$$

It follows from (12) that the symbol $\widetilde{\sigma}$ is constant:

$$\begin{array}{c}\widetilde{\sigma}:\pi^*\Lambda^1(M)\to\pi^*\Lambda^1(M),\\ \widetilde{\sigma}(\xi)=\sigma^{-1}(-\xi)\,\sigma(\xi)\oplus 1=-1\oplus 1:L\oplus L^\perp\to L\oplus L^\perp.\end{array}$$

Hence, we obtain the integrality result for the functional d:

$$d\left(\widehat{L}\right)=\operatorname{ind}\left(\widehat{\sigma},\widehat{L},C^\infty(M)\right)\in\mathbf{Z}.$$

The corresponding statement about the η-invariant was proved in [4].

The example just described can be generalized to the case of operators A with coefficients in a vector bundle $E\in\operatorname{Vect}(M)$. To this end, one must replace the exterior derivative d and its adjoint by a covariant differential ∇ and the corresponding adjoint operator for the vector bundle E. In this case one also obtains the integrality result.

Moreover, the operator $d\delta$-δd can be considered for exterior forms of arbitrary degree. For such a generalization the integrality of the functional d remains valid.

This result can be obtained along similar lines.

In the second example, following [19], we construct a self-adjoint elliptic operator of order 1 on the (nonorientable) projective space $\mathbf{RP}^{2n}$ with a nontrivial fractional part of the η-invariant.

On the even-dimensional sphere S^{2n} we consider the self-adjoint elliptic operator

$$d+\delta:C^\infty\left(S^{2n},\Lambda\left(S^{2n}\right)\right)\longrightarrow C^\infty\left(S^{2n},\Lambda\left(S^{2n}\right)\right),$$

acting on exterior differential forms. The vector bundle $\Lambda(S^{2n})$ is trivial. Let us choose the following trivialization

$$\Lambda\left(S^{2n}\right)\xrightarrow{\gamma}\Lambda^{ev}\left(\mathbf{R}^{2n+1}\right),$$

where $\Lambda^{ev}(\mathbf{R}^{2n+1})$ is the restriction to the sphere of even-degree exterior forms on the space $\mathbf{R}^{2n+1}$, given by the following formula with respect to the decomposition of $\Lambda(S^{2n})$:

$$\gamma_\theta(\omega^e, \omega^o) = \omega^e + d\theta \wedge \omega^o,$$

where $\theta \in S^{2n}$ and $\omega^{e/o} \in \Lambda^{ev/odd}(S^{2n})$. The isomorphism γ takes the operator $d+\delta$ to the operator

$$A = \gamma(d+\delta)\gamma^{-1} : C^\infty\left(S^{2n}, \Lambda^{ev}\left(\mathbf{R}^{2n+1}\right)\right) \longrightarrow C^\infty\left(S^{2n}, \Lambda^{ev}\left(\mathbf{R}^{2n+1}\right)\right)$$

in the bundle $\Lambda^{ev}(\mathbf{R}^{2n+1})$. This operator is known as the *tangential operator of the de Rham complex* (see [19]). It can be shown that its principal symbol is equal to

$$\sigma(A)(\theta,\xi) = ie(\theta)\,e(\xi) : \pi^*\Lambda^{ev}\left(\mathbf{R}^{2n+1}\right) \to \pi^*\Lambda^{ev}\left(\mathbf{R}^{2n+1}\right), \tag{13}$$

where the coordinates (θ, ξ) on the cosphere bundle S^*S^{2n} are given by

$$(\theta,\xi) \in T^*S^{2n} = \left\{(\theta,\xi) \in S^{2n} \times \mathbf{R}^{2n+1} \middle| \, \xi \perp \theta\right\}$$

and we use the notation

$$e(v) = i\left(v \wedge - v \rfloor\right) : \Lambda^{ev}\left(\mathbf{R}^{2n+1}\right) \to \Lambda^{odd}\left(\mathbf{R}^{2n+1}\right).$$

Formula (13) has the following merit: the bundles where the operator A acts, as well as the expression for the operator itself, are invariant under the antipodal involution of the sphere:

$$\alpha : S^{2n} \to S^{2n}, \qquad \alpha(\theta) = -\theta.$$

Thus, A is actually the pull-back of an elliptic self-adjoint operator

$$A_0 : C^\infty\left(\mathbf{RP}^{2n}, \Lambda^{ev}\left(\mathbf{R}^{2n+1}\right)\right) \longrightarrow C^\infty\left(\mathbf{RP}^{2n}, \Lambda^{ev}\left(\mathbf{R}^{2n+1}\right)\right)$$

on the quotient space $S^{2n}/\alpha = \mathbf{RP}^{2n}$. Let us compute the fractional parts of the η-invariants of the operators A and A_0.

Consider the nonnegative spectral subspaces $\widehat{L}_+(A)$ and $\widehat{L}_+(A_0)$ for both operators. It follows from (13) that the projections π_+ and $\pi_{+,0}$ for the symbols $L_+(A)$ and $L_+(A_0)$ of these subspaces have the form

$$\pi_+(\theta,\xi) = \frac{1+ie(\theta)\,e(\xi)}{2} : \Lambda^{ev}\left(\mathbf{R}^{2n+1}\right) \to \Lambda^{ev}\left(\mathbf{R}^{2n+1}\right) \qquad \text{on } S^{2n},$$

$$\pi_{+,0}(\theta,\xi) = \frac{1+ie(\theta)\,e(\xi)}{2} : \Lambda^{ev}\left(\mathbf{R}^{2n+1}\right) \to \Lambda^{ev}\left(\mathbf{R}^{2n+1}\right) \qquad \text{on } \mathbf{RP}^{2n}$$

(this follows from the skew-commutativity $e(\theta)\, e(\xi) = -e(\xi)\, e(\theta)$ and the equality $e(v)\, e(v) = |v|^2$). The operator $e(\theta)$ contains the multiplication by the unit normal vector θ. It follows that the projections π_+ and $\pi_{+,0}$ establish isomorphisms between the bundle of even exterior forms on the sphere and on the projective space, respectively, and the subbundles $L_+(A)$ and $L_+(A_0)$:

$$\pi_+ : \pi^*\Lambda^{ev}\left(S^{2n}\right) \longrightarrow L_+(A),\ \pi_{+,0} : \pi^*\Lambda^{ev}\left(\mathbf{RP}^{2n}\right) \longrightarrow L_+(A_0). \tag{14}$$

Thus, we obtain two elliptic operators in subspaces, namely,

$$\begin{aligned} \widehat{\pi}_+ &: C^\infty\left(S^{2n}, \Lambda^{ev}\left(S^{2n}\right)\right) \longrightarrow \widehat{L}_+(A), \\ \widehat{\pi}_{+,0} &: C^\infty\left(\mathbf{RP}^{2n}, \Lambda^{ev}\left(\mathbf{RP}^{2n}\right)\right) \longrightarrow \widehat{L}_+(A_0). \end{aligned}$$

Let us apply the index formula for operators in subspaces. It gives

$$\operatorname{ind}\widehat{\pi}_+ = \frac{1}{2}\operatorname{ind}\widehat{\widehat{\pi}}_+ - \eta(A), \quad \operatorname{ind}\widehat{\pi}_{+,0} = \frac{1}{2}\operatorname{ind}\widehat{\widehat{\pi}}_{+,0} - \eta(A_0), \tag{15}$$

where the principal symbols of elliptic operators

$$\begin{aligned} \widehat{\widehat{\pi}}_+ &: C^\infty\left(S^{2n}, \Lambda^{ev}\left(S^{2n}\right) \oplus \Lambda^{ev}\left(S^{2n}\right)\right) \to C^\infty\left(S^{2n}, \Lambda^{ev}\left(\mathbf{R}^{2n+1}\right)\right), \\ \widehat{\widehat{\pi}}_{+,0} &: C^\infty\left(\mathbf{RP}^{2n}, \Lambda^{ev}\left(\mathbf{RP}^{2n}\right) \oplus \Lambda^{ev}\left(\mathbf{RP}^{2n}\right)\right) \to C^\infty\left(\mathbf{RP}^{2n}, \Lambda^{ev}\left(\mathbf{R}^{2n+1}\right)\right) \end{aligned}$$

are given by the formula

$$\widetilde{\pi}_+(\omega_1, \omega_2) = \frac{1}{2}\left(\omega_1 + \omega_2 + ie(\theta)\, e(\xi)\,(\omega_1 - \omega_2)\right) \quad \text{at } (\theta, \xi) \in S^*S^{2n}, \tag{16}$$

and the same expression holds for $\widetilde{\pi}_{+,0}$ on $S^*\left(\mathbf{RP}^{2n}\right)$. Since the pull-back of the symbol $\widetilde{\pi}_{+,0}$ to S^*S^{2n} is equal to $\widetilde{\pi}_+$, from the locality of the index of elliptic operators we obtain

$$\operatorname{ind}\widehat{\widehat{\pi}}_{+,0} = \frac{1}{2}\operatorname{ind}\widehat{\widehat{\pi}}_+.$$

Let us compute the index of $\widehat{\widehat{\pi}}_+$ on the sphere. It is clear from (16) that the symbol of this operator is equivalent to the direct sum of the identity operator

$$\omega_1 \overset{1}{\longmapsto} \omega_1$$

and an elliptic symbol

$$e(\xi) : \pi^*\Lambda^{ev}\left(S^{2n}\right) \to \pi^*\Lambda^{odd}\left(S^{2n}\right),$$

which is equal to the symbol of the de Rham operator on the sphere. The index of the de Rham operator is equal to the Euler characteristic. Hence, on an even-dimensional sphere we have

$$\mathrm{ind}\widehat{\widehat{\pi}}_{+} = 2.$$

Thus $\mathrm{ind}\widehat{\widehat{\pi}}_{+,0} = 1$.

The fractional parts of the η-invariants are now obtained from (15). Namely,

$$\{\eta(A)\} = 0, \qquad \{\eta(A_0)\} = \frac{1}{2}.$$

3. In the general situation the pull-back of the form (9) for the subspaces considered always exists provided that we take sufficiently many copies of the original subspace $\widehat{L}$. More precisely, the following theorem is valid.

Theorem 5 *For a subspace $\widehat{L} \in \widehat{\mathrm{Even}}\left(M^{odd}\right)$ or $\widehat{\mathrm{Odd}}\left(M^{ev}\right)$ with principal symbol L, there exists a positive integer N such that the vector bundle $2^N L$ on S^*M can be lifted from the base M. That is, for some vector bundle $F \in \mathrm{Vect}(M)$ there exists an isomorphism*

$$\sigma : 2^N L \longrightarrow \pi^* F, \qquad 2^N L = \underbrace{L \oplus \ldots \oplus L}_{2^N\, copies}. \tag{17}$$

Remark 3 In terms of the difference construction (see Appendix A), this theorem states that the subspaces under consideration define torsion elements of order a power of 2 in the group $K^1(T^*M)$. In the following these elements will be called 2-torsion elements for short.

Proof. The first part of the theorem, concerning even subspaces, follows from [4], where it is shown that for an odd-dimensional manifold M the bundle

$$P^*M = S^*M / \{(x,\xi) \sim (x,-\xi)\}$$

of projective spaces has the same K-theory groups as M except for the 2-torsion. This isomorphism modulo 2-torsion is established by the natural projection

$$\pi_P^* : K(M) \longrightarrow K(P^*M), \qquad \pi_P : P^*M \to M. \tag{18}$$

More precisely, $\ker \pi_P^* = 0$, and $\mathrm{coker}\pi_P^*$ is a 2-torsion group.

On the other hand, it is shown in [2], where odd subspaces are studied, that for an odd vector bundle L on S^*M there exists a positive integer N such that the bundle $2^N L$ and its complement $2^N \alpha^* L$ are isomorphic,

$$\sigma_0 : 2^N L \longrightarrow 2^N \alpha^* L.$$

This follows from the observation that the projection $S^*M \to P^*M$ for even-dimensional manifolds induces an isomorphism

$$K(P^*M) \to K(S^*M)$$

in K-theory modulo 2-torsion. Using this isomorphism, we can construct the required pull-back (17) by the formula

$$\sigma : 2^{N+1}L \xrightarrow{1\oplus\sigma_0} 2^N L \oplus 2^N \alpha^* L = 2^N \pi^* E.$$

This completes the proof of the theorem.

Let us consider an elliptic operator in subspaces

$$\widehat{\sigma} : 2^N \widehat{L} \longrightarrow C^\infty(M, F)$$

with principal symbol (17). Just as above, we write out the index formula for this operator:

$$\operatorname{ind}\left(\widehat{\sigma}, 2^N \widehat{L}, C^\infty(M,F)\right) = \frac{1}{2}\operatorname{ind}\widetilde{\sigma} + 2^N d\left(\widehat{L}\right). \qquad (19)$$

This formula, along with the integrality of the index, implies that the functional d is dyadic rational and has at most 2^{N+1} as the denominator. However, there is an essential difference between formulas (19) and (10). While the left- and right-hand sides of (10) are determined by the principal symbol of the operator in subspaces only modulo $\mathbf{Z}$,[2] both sides of (19) are determined by the principal symbol of the operator modulo a multiple of 2^N. To show this, let us consider another operator in subspaces with the same principal symbol:

$$\widehat{\sigma}' : 2^N \widehat{L}' \longrightarrow C^\infty(M,F), \quad \text{where } \sigma' = \sigma \text{ and } L' = L.$$

By the multiplicative property of the index, we obtain

$$\operatorname{ind}\left(\widehat{\sigma}, 2^N \widehat{L}, C^\infty(M,F)\right) - \operatorname{ind}\left(\widehat{\sigma}', 2^N \widehat{L}', C^\infty(M,F)\right) = 2^N \operatorname{ind}\left(\widehat{L}, \widehat{L}'\right) \equiv 0 \pmod{2^N}.$$

Moreover, to calculate the fractional part of the functional $d\left(\widehat{L}\right)$, one just has to calculate the right-hand side of (19) modulo 2^N.

Thus, the computation of the fractional part of the functional d is reduced to the modulo 2^N index problem for the elliptic operator in subspaces

$$\widehat{\sigma} : 2^N \widehat{L} \longrightarrow C^\infty(M,F).$$

This problem is solved in the next section.

[2]In other words, for operators in subspaces with the same principal symbol both sides of the formula can take arbitrary integer values.

3 Index theory modulo n

For a given positive integer $n \geq 2$, we consider elliptic operators in subspaces of a special form:

$$D : n\widehat{L} \longrightarrow C^{\infty}(M, F). \tag{20}$$

We point out that here the subspace $\widehat{L}$ need not satisfy any parity conditions, as was assumed in the preceding section. The mod n-ind D — index of D modulo n is determined by the principal symbol of the operator

$$\sigma(D) : nL \longrightarrow \pi^* F,$$

as a residue:

$$\text{mod } n\text{-ind}\, D = f(\sigma(D)) \in \mathbf{Z}_n.$$

We recall now that in the usual elliptic theory the integer index is determined and computed by the difference construction for the principal symbol of the operator:

$$\text{ind} D \in \mathbf{Z}, \qquad [\sigma(D)] \in K(T^*M) = K(T^*M, \mathbf{Z}),$$

where the K-theory is the usual one, namely, with integer coefficients (e.g., see [20]). Hence, it is natural to expect that to compute the index with values in $\mathbf{Z}_n$, one has to obtain a difference construction with values in K-theory with $\mathbf{Z}_n$ coefficients:

$$\text{mod } n\text{-ind}\, D \in \mathbf{Z}_n, \qquad [\sigma(D)] \in K(T^*M, \mathbf{Z}_n).$$

The necessary information about this theory is provided for the reader's convenience in Appendix B.

Let us define this difference construction. First of all, we rewrite the group $K(T^*M, \mathbf{Z}_n)$ in terms of the usual K-theory. We have

$$K(T^*M, \mathbf{Z}_n) = K(T^*M \times M_n, T^*M \times pt), \tag{21}$$

where M_n is the so-called *Moore space* (see Appendix B). It readily follows from (21) that the elements of the group $K(T^*M, \mathbf{Z}_n)$ can be represented as families of elliptic symbols[3] on the manifold M, where the Moore space serves as the parameter

[3] Here we utilize the natural construction [21] that assigns an element

$$[\sigma] \in K(T^*M \times X)$$

of the K-theory group to each family $\sigma(x)$, $x \in X$, of elliptic symbols on the manifold M with the parameter space X:

$$\sigma(x) : \pi^* E \longrightarrow \pi^* F, \quad E, F \in \text{Vect}(M \times X), \ \pi : S^*M \times X \to M \times X.$$

space for the family. We construct the corresponding family of elliptic symbols as the following composition of elliptic families in subspaces:

$$[\sigma(D)] = \left[\pi^*F \overset{\sigma^{-1}(D)}{\longrightarrow} nL \overset{\beta^{-1}\otimes 1_L}{\longrightarrow} \gamma_n \otimes nL \overset{1_\gamma\otimes\sigma(D)}{\longrightarrow} \gamma_n \otimes \pi^*F\right] \in K(T^*M \times M_n, T^*M \times pt), \tag{22}$$

where γ_n is the one-dimensional vector bundle generating the reduced K-group $K(M_n, pt)$ of the Moore space and β is the trivialization

$$\beta : n\gamma_n \longrightarrow \mathbf{C}^n.$$

In terms of this difference construction, we can state a formula for the index modulo n.

Theorem 6 *For an elliptic operator D in subspaces,*

$$D : n\widehat{L} \longrightarrow C^\infty(M, F),$$

the index modulo n is computed by the formula

$$\bmod n\text{-ind}\, D = p_! [\sigma(D)] \tag{23}$$

for the direct image

$$p_! : K(T^*M, \mathbf{Z}_n) \longrightarrow K(pt, \mathbf{Z}_n) = \mathbf{Z}_n,$$

in K-theory (with coefficients) corresponding to the map $p : M \longrightarrow pt$.

Proof. We consider three families of elliptic operators in subspaces, parametrized by the Moore space M_n. These families correspond to the symbols in the formula (22) and are as follows:

$$C^\infty(M, F) \overset{D^{-1}}{\longrightarrow} n\widehat{L},$$
$$n\widehat{L} \overset{\beta^{-1}\otimes 1_{\widehat{L}}}{\longrightarrow} \gamma_n \otimes n\widehat{L},$$
$$\gamma_n \otimes n\widehat{L} \overset{1_\gamma\otimes D}{\longrightarrow} \gamma_n \otimes C^\infty(M, F)$$

(here by D^{-1} we denote an almost inverse, i.e. inverse up to compact operators, of D). The first family is constant, and hence, its index is just a number. The second family consists of isomorphisms, and consequently, it has trivial index. The third family is merely the tensor product of the original operator D in subspaces by the vector bundle γ_n over the parameter space. Hence, the index of the composition of these three families is equal to

$$\mathrm{ind}\left([1_\gamma \otimes D] \circ \left[\beta^{-1} \otimes 1_{\widehat{L}}\right] \circ D^{-1}\right) = [\gamma_n]\,\mathrm{ind}D + 0 - \mathrm{ind}D \in K\left(M_n\right). \qquad (24)$$

On the other hand, the index of the family of elliptic operators

$$[1_\gamma \otimes D] \circ \left[\beta^{-1} \otimes 1_{\widehat{L}}\right] \circ D^{-1} : C^\infty\left(M, F\right) \longrightarrow \gamma_n \otimes C^\infty\left(M, F\right)$$

is calculated by the Atiyah-Singer index formula for families (see [21]). Thus, in the notation of (22) this gives

$$\mathrm{ind}\left([1_\gamma \otimes D] \circ \left[\beta^{-1} \otimes 1_{\widehat{L}}\right] \circ D^{-1}\right) = p_!\left[\sigma\left(D\right)\right] \in K\left(M_n\right). \qquad (25)$$

On the other hand, taking into account the isomorphism

$$K\left(M_n\right) = \mathbf{Z} \oplus \mathbf{Z}_n \qquad \text{with } [\gamma_n] - 1 \text{ as generator of the torsion part } \mathbf{Z}_n,$$

we obtain, by comparing the expression (25) with formula (24),

$$\mathrm{mod}\ n\text{-ind}\, D = p_!\left[\sigma\left(D\right)\right].$$

The index formula is thereby proved.

Remark 4 A similar technique yields the so-called "mod n" index formula for boundary value problems [22], [23].

4 Elliptic theory with $\mathbf{Z_n}$ coefficients

The difference construction (22) is not an entirely computational trick involved in the modulo n-index calculation above. In the present section we show that, by analogy with the usual difference construction, the map (22) establishes an isomorphism between the K-theory $K\left(T^*M, \mathbf{Z}_n\right)$ with $\mathbf{Z}_n$ coefficients and the group of stable homotopy classes of elliptic operators in subspaces of the form (20).

We consider elliptic operators of the form

$$D = n\widehat{L}_1 \oplus C^\infty\left(M, E_1\right) \to n\widehat{L}_2 \oplus C^\infty\left(M, F_1\right), \ \ \widehat{L}_1 \subset C^\infty\left(M, E\right), \widehat{L}_2 \subset C^\infty\left(M, F\right). \qquad (26)$$

This is slightly different from (20); the difference is motivated by the requirement that the inverse operator be of the same structure. Let us state the stable homotopy classification problem for such operators. First, we introduce *trivial* operators. These are

1. Operators of multiplication by an isomorphism $g : E_1 \to F_1$ of vector bundles

$$C^\infty(M, E_1) \xrightarrow{g_*} C^\infty(M, F_1). \tag{27}$$

2. Direct sums of n copies of an elliptic operator in subspaces

$$n\left(\widehat{L}_1 \oplus C^\infty(M, E_1)\right) \xrightarrow{nD} n\left(\widehat{L}_2 \oplus C^\infty(M, F_1)\right). \tag{28}$$

We identify operators of the form (26) that differ by isomorphisms of the corresponding vector bundles E, F, E_1, F_1. Two elliptic operators D_1 and D_2 are *stably homotopic* if they become homotopic after we add some trivial operators to each of them. The abelian group formed by the classes of stably homotopic elliptic operators is denoted by $\mathrm{Ell}(M, \mathbf{Z}_n)$.

Lemma 1 *The operator* (26) *is equivalent to an operator of the form*

$$n\widehat{L} \xrightarrow{D} C^\infty(M, F), \tag{29}$$

discussed in the previous section (20).

Proof. The space $C^\infty(M, E_1)$ can be eliminated in (26) by adding the trivial operator

$$(n-1)\left(C^\infty(M, E_1) \xrightarrow{1} C^\infty(M, E_1)\right).$$

The subspace $\widehat{L}_2$ on the right-hand side of the formula can be eliminated in the following way. Let us add the trivial operator

$$n\left(\widehat{L}_2^\perp \xrightarrow{1} \widehat{L}_2^\perp\right)$$

to the operator D. Then we obtain an operator of the form

$$n\left(\widehat{L}_1 \oplus \widehat{L}_2^\perp\right) \longrightarrow n\left(\widehat{L}_2 \oplus \widehat{L}_2^\perp\right) \oplus C^\infty(M, F_1).$$

To complete the proof of the lemma, it suffices to show that the resulting subspace

$$\widehat{L}_2 \oplus \widehat{L}_2^\perp \subset C^\infty(M, F_1) \oplus C^\infty(M, F_1)$$

is homotopic to the subspace

$$C^\infty(M, F_1) \oplus 0 \subset C^\infty(M, F_1) \oplus C^\infty(M, F_1),$$

since a homotopy of subspaces can be lifted to a homotopy of elliptic operators in subspaces. The required homotopy of subspaces is given in terms of the projection P on the subspace $\widehat{L}_2$ by the formula

$$\widehat{L}_\varphi = \operatorname{Im} P_\varphi,\ P_\varphi = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} + (1-P)\begin{pmatrix} \sin^2\varphi & \cos\varphi\sin\varphi \\ \cos\varphi\sin\varphi & \cos^2\varphi \end{pmatrix}.$$

Here $\widehat{L}_\varphi \subset C^\infty(M, F_1) \oplus C^\infty(M, F_1)$ and φ varies from 0 to $\pi/2$. The proof is complete.

Elliptic theory with coefficients $\mathbf{Z}_n$, realized as the group $\operatorname{Ell}(M, \mathbf{Z}_n)$, is related to the usual group $\operatorname{Ell}(M)$ of classes of stably homotopic elliptic operators and the group $\operatorname{Ell}_1(M)$ of stably homotopic subspaces (concerning the latter group, see Appendix A at the end of the paper). More precisely, there is an exact sequence

$$\operatorname{Ell}(M) \xrightarrow{\times n} \operatorname{Ell}(M) \xrightarrow{i} \operatorname{Ell}(M, \mathbf{Z}_n) \xrightarrow{j} \operatorname{Ell}_1(M) \xrightarrow{\times n} \operatorname{Ell}_1(M), \tag{30}$$

where by $\times n$ we denote multiplication by n, the map i is induced by the inclusion of the set of elliptic operators into elliptic operators of the form (26), and the homomorphism j is given by the formula

$$j\left[n\widehat{L}_1 \oplus C^\infty(M, E_1) \xrightarrow{D} n\widehat{L}_2 \oplus C^\infty(M, F_1)\right] = \left[\widehat{L}_1\right] - \left[\widehat{L}_2\right].$$

Proposition 1 The sequence (30) is exact.

Proof. It is straightforward to check that (30) is a complex. Let us verify the exactness.

Let $[D] \in \ker j$. By Lemma 1 we can suppose that the operator D has the form (29). Since $\left[\widehat{L}_1\right] = 0 \in \operatorname{Ell}_1(M)$, it follows that the subspace $\widehat{L}$ is homotopic[4] to the space of sections of a subbundle $E' \subset E$. Hence, the operator D is homotopic to an elliptic operator in the spaces of sections of bundles E' and F. Consequently, we obtain

$$[D] \in \operatorname{Im} i.$$

Let $\left[\widehat{L}\right] \in \ker\{\times n\} \bigcap \operatorname{Ell}_1(M)$. This implies that the subspace $n\widehat{L}$ is homotopic to the space of sections of a vector bundle. Consequently, there exists an elliptic operator in subspaces

$$n\widehat{L} \xrightarrow{D} C^\infty(M, F).$$

[4]Here and in what follows we omit the standard considerations concerning the stabilization of elements in the Grothendieck group.

Hence

$$\left[\widehat{L}\right] = j\,[D]\,,$$

as desired. The remaining case $[D] \in \ker i$ can be treated along similar lines and is left to the reader.

The difference construction

$$D \longmapsto [\sigma(D)] \in K(T^*M, \mathbf{Z}_n) \tag{31}$$

from the preceding section (see (22)) extends to a homomorphism of groups

$$\mathrm{Ell}(M, \mathbf{Z}_n) \longrightarrow K(T^*M, \mathbf{Z}_n)\,,$$

since the map (22) sends the trivial operators (27) and (28) to zero in K-theory.

The difference constructions in $\mathbf{Z}_n$-theory and the one in the usual elliptic theory are related by the diagram

$$\begin{array}{ccccccccc}
\mathrm{Ell}(M) & \stackrel{\times n}{\to} & \mathrm{Ell}(M) & \stackrel{i}{\to} & \mathrm{Ell}(M, \mathbf{Z}_n) & \stackrel{j}{\to} & \mathrm{Ell}_1(M) & \stackrel{\times n}{\to} & \mathrm{Ell}_1(M) \\
\downarrow \chi_0 & & \downarrow \chi_0 & & \downarrow \chi_n & & \downarrow \chi_1 & & \downarrow \chi_1 \\
K(T^*M) & \stackrel{\times n}{\to} & K(T^*M) & \stackrel{i'}{\to} & K(T^*M, \mathbf{Z}_n) & \stackrel{j'}{\to} & K^1(T^*M) & \stackrel{\times n}{\to} & K^1(T^*M)\,.
\end{array} \tag{32}$$

Here the symbol χ with subscripts denotes difference constructions, and the lower row in the diagram is part of the exact coefficient sequence in K-theory (see Appendix B): i' is the reduction modulo n and j' is the Bokstein homomorphism.

Theorem 7 *The diagram* (32) *is commutative.*

Corollary 2 *The difference homomorphism*

$$\mathrm{Ell}(M, \mathbf{Z}_n) \xrightarrow{\chi_n} K(T^*M, \mathbf{Z}_n)$$

is an isomorphism of groups.

The usual difference homomorphisms

$$\mathrm{Ell}(M) \stackrel{\chi_0}{\to} K(T^*M) \quad \text{and} \quad \mathrm{Ell}_1(M) \stackrel{\chi_1}{\to} K^1(T^*M)$$

are isomorphisms, so the corollary follows by applying the 5-lemma to the commutative diagram (32).

Proof of Theorem 7. The commutativity of the leftmost and rightmost squares of the diagram is clear (since the difference homomorphism is, at least, a homomorphism of groups). Let us consider the second square of the diagram:

$$\begin{array}{ccc} \mathrm{Ell}\,(M) & \stackrel{i}{\longrightarrow} & \mathrm{Ell}\,(M,\mathbf{Z}_n) \\ \downarrow \chi_0 & & \downarrow \chi_n \\ K\,(T^*M) & \stackrel{i'}{\longrightarrow} & K\,(T^*M,\mathbf{Z}_n)\,. \end{array}$$

For an elliptic operator $[D] \in \mathrm{Ell}\,(M)$,

$$D : C^\infty\,(M,E) \longrightarrow C^\infty\,(M,F)\,,$$

an explicit calculation shows that by passing through the upper right corner of the square we obtain

$$\chi_n i\,[D] = [\sigma\,(D)]\,([\gamma_n]-1) \in K\,(T^*M \times M_n, T^*M \times pt)\,;$$

here $[\sigma\,(D)] = \chi_0\,[D] \in K\,(T^*M)$ is the usual difference homomorphism. On the other hand, the reduction modulo n map i' is exactly the multiplication by the element $[\gamma_n]-1$. Thus, the second square is commutative.

Finally, let us check the commutativity of the remaining third square

$$\begin{array}{ccc} \mathrm{Ell}\,(M,\mathbf{Z}_n) & \stackrel{j}{\longrightarrow} & \mathrm{Ell}_1\,(M) \\ \downarrow \chi_n & & \downarrow \chi_1 \\ K\,(T^*M,\mathbf{Z}_n) & \stackrel{j'}{\longrightarrow} & K^1\,(T^*M)\,. \end{array}$$

For an elliptic operator

$$n\widehat{L} \stackrel{D}{\longrightarrow} C^\infty\,(M,F)\,, \qquad [D] \in \mathrm{Ell}\,(M,\mathbf{Z}_n)\,,$$

on the one hand, we obtain

$$\chi_1\,(j\,[D]) = [L] \in K^1\,(T^*M)\,.$$

On the other hand, the difference construction for the operator D gives

$$\chi_n\,[D] = \left[\pi^*F \stackrel{\sigma^{-1}(D)}{\longrightarrow} nL \stackrel{\beta^{-1}\otimes 1_L}{\longrightarrow} \gamma_n \otimes nL \stackrel{1_\gamma\otimes\sigma(D)}{\longrightarrow} \gamma_n \otimes \pi^*F\right] \in K\,(T^*M,\mathbf{Z}_n)\,. \quad (33)$$

The Bokstein map j'

$$j' : K\,(T^*M,\mathbf{Z}_n) \longrightarrow K^1\,(T^*M)$$

with respect to the identifications

$$\begin{aligned} K\left(T^*M, \mathbf{Z}_n\right) &= K\left(T^*M \times M_n, T^*M \times pt\right), \\ K^1\left(T^*M\right) &= K\left(T^*M \times S_1, T^*M \times pt\right), \end{aligned}$$

is induced by the inclusion

$$S^1 \overset{i_0}{\subset} M_n,$$

so that

$$j' = \left(1_{T^*M} \times i_0\right)^*.$$

Let us compute the family of elliptic symbols (33) on the circle $S^1 \subset M_n$. By choosing a polar coordinate ζ,

$$S^1 = \left\{\zeta \mid \zeta = e^{i\varphi}, 0 \le \varphi < 2\pi\right\},$$

we represent the family (33) in the form

$$\pi^*F \xrightarrow{\sigma^{-1}} nL \xrightarrow{\zeta\oplus 1} nL \xrightarrow{\sigma} \pi^*F \tag{34}$$

with respect to the natural trivialization of the bundle γ_n on S^1 (here for brevity the principal symbol of the operator D is denoted by σ). The diagonal operator $\zeta \oplus 1$ acts according to the formula

$$\left(\zeta \oplus 1\right)\left(u_1, u_2, \ldots, u_n\right) = \left(\zeta u_1, u_2, \ldots, u_n\right).$$

Let us rewrite the symbol σ also in block matrix form:

$$\sigma = \left(\sigma_1, \ldots, \sigma_n\right), \qquad \sigma_i : L \longrightarrow \pi^*F.$$

It follows from the ellipticity of the symbol σ that the components σ_i are monomorphic. Consider also the inverse symbol σ^{-1}

$$\sigma^{-1} = \begin{pmatrix} \sigma^1 \\ \vdots \\ \sigma^n \end{pmatrix}, \qquad \sigma^i : \pi^*F \longrightarrow L.$$

We readily obtain the identities

$$\sum_{i=1}^n \sigma_i \sigma^i = 1, \qquad \sigma^i \sigma_j = \delta^i_j$$

(which are the component representations of the formulas $\sigma^{-1}\sigma = 1$ and $\sigma\sigma^{-1} = 1$). This implies, in particular, that the operator $\sigma_1\sigma^1$ is a projection onto a subbundle isomorphic to the original bundle L

$$\operatorname{Im}\sigma_1\sigma^1 \xrightarrow{\sigma_1} L.$$

Hence, the family (34), after we multiply the matrices in (34), acquires the form

$$\sigma \circ (\zeta \oplus 1) \circ \sigma^{-1} = \zeta\sigma_1\sigma^1 + \left(1 - \sigma_1\sigma^1\right).$$

This completes the proof of the theorem, since the family of elliptic symbols

$$\zeta\sigma_1\sigma^1 + \left(1 - \sigma_1\sigma^1\right) \in K\left(T^*M \times S^1, T^*M \times pt\right) = K^1\left(T^*M\right),$$

parametrized by the circle, coincides with the difference construction for the subspace $\widehat{L}$ (see Appendix A).

5 A formula for the fractional part

Let us return to the original object of our study, the fractional part of the functional d. Namely, for an admissible subspace $\widehat{L} \subset C^\infty(M, E)$ and the pull-back of the principal symbol from the manifold M by means of an isomorphism σ,

$$2^N L \xrightarrow{\sigma} \pi^* F, \qquad F \in \operatorname{Vect}(M), \pi : S^*M \to M, \tag{35}$$

in Section 2 the following formula was obtained for the functional d:

$$d\left(\widehat{L}\right) = \frac{1}{2^N}\left(\operatorname{ind}\left(\widehat{\sigma}, 2^N\widehat{L}, C^\infty(M, F)\right) - \frac{1}{2}\operatorname{ind}\widehat{\widehat{\sigma}}\right) \in \mathbf{Z}\left[\frac{1}{2}\right].$$

For the fractional part of d this formula gives

$$\{d\left(\widehat{L}\right)\} = \frac{1}{2^N}\left(\operatorname{mod} 2^N\text{-ind}\left(\widehat{\sigma}, 2^N\widehat{L}, C^\infty(M, F)\right) - \frac{1}{2}\left(\operatorname{mod} 2^{N+1}\text{-ind}\widehat{\widehat{\sigma}}\right)\right).$$

In addition, the first term can be computed by the modulo 2^N-index formula, and the second one by the Atiyah-Singer formula. The two terms, in fact, can be composed. Namely, a straightforward computation shows that in the even (odd) cases the resulting formulas are, respectively,

$$\begin{gathered}\{d\left(\widehat{L}\right)\} = \\ \tfrac{1}{2^{N+1}}\operatorname{mod} 2^{N+1}\text{-ind}\left(2^{N+1}\widehat{L} \xrightarrow{\widehat{\sigma}\oplus\widehat{\alpha^*\sigma}} C^\infty(M, F \oplus F)\right), \quad \widehat{L} \in \widehat{\operatorname{Even}}\left(M^{odd}\right), \\ \tfrac{1}{2^{N+1}}\operatorname{mod} 2^{N+1}\text{-ind}\left(2^{N+1}\widehat{L} \xrightarrow{1\oplus\alpha^*\widehat{[\sigma^{-1}]}\widehat{\sigma}} C^\infty\left(M, 2^N E\right)\right), \quad \widehat{L} \in \widehat{\operatorname{Odd}}\left(M^{ev}\right).\end{gathered} \tag{36}$$

Concerning the difference construction of Section 4, the two formulas show that the fractional part of the functional d is computed by the direct image in K-theory:

$$2^{N+1}\{d(\widehat{L})\} = p_!\,[L] \in \mathbf{Z}_{2^{N+1}}, \quad [L] \in K\left(T^*M, \mathbf{Z}_{2^{N+1}}\right),$$

where the element $[L]$ is equal to the difference construction for the operators in (36)

$$[L] = \left[2^{N+1}L \stackrel{\sigma\oplus\alpha^*\sigma}{\longrightarrow} \pi^*F \oplus \pi^*F\right] \quad \text{or} \quad \left[2^{N+1}L \stackrel{1\oplus\alpha^*\left[\sigma^{-1}\right]\sigma}{\longrightarrow} 2^N\pi^*E\right]. \qquad (37)$$

In the remaining part of the section we shall show that even though the construction of the element $[L]$ depends on the number N and the isomorphism σ, in the limit as $N \to \infty$ the result becomes in a certain sense unique, that is, independent of σ. Let us proceed to precise statements.

As the number N grows, consider an increasing sequence of groups

$$\mathbf{Z}_2 \subset \mathbf{Z}_4 \subset \ldots \subset \mathbf{Z}_{2^{N'}} \subset \ldots$$

with the fractional parts of dyadic numbers as its injective limit:

$$\varinjlim_{N' \to \infty} \mathbf{Z}_{2^{N'}} = \mathbf{Z}\left[\frac{1}{2}\right] \Big/ \mathbf{Z}.$$

Consider also the corresponding sequence in K-theory:

$$K\left(T^*M, \mathbf{Z}\left[\frac{1}{2}\right] \Big/ \mathbf{Z}\right) = \varinjlim_{N' \to \infty} K\left(T^*M, \mathbf{Z}_{2^{N'}}\right). \qquad (38)$$

In this notation, we prove the following theorem, which shows the uniqueness of the element $[L]$.

Theorem 8 *The element* $[L]$ *given by* (37) *and viewed as an element in the injective limit* (38) *is well defined*

$$[L] \in K\left(T^*M, \mathbf{Z}\left[\frac{1}{2}\right] \Big/ \mathbf{Z}\right),$$

i.e. independent of the choice of the isomorphism σ.

Proof. For two isomorphisms

$$2^N L \stackrel{\sigma}{\longrightarrow} \pi^*F, \quad \text{and} \quad 2^N L \stackrel{\sigma'}{\longrightarrow} \pi^*F'$$

let us compute the difference of the corresponding K-theory elements in (37). An explicit computation shows that the difference is equal to

$$\sigma\sigma'^{-1} \oplus \alpha^* \left[\sigma\sigma'^{-1}\right] \qquad \text{on an odd-dimensional manifold } M,$$
$$\sigma\sigma'^{-1} \oplus \alpha^* \left[\sigma'\sigma^{-1}\right] \qquad \text{on an even-dimensional } M.$$

From these formulas we see that the difference in question is equal to

$$[\sigma_0] \pm [\alpha^*\sigma_0] \in K(T^*M)$$

for the elliptic symbol $\sigma_0 = \sigma\sigma'^{-1}$, where the sign $\pm$ is opposite to the parity of the dimension of the manifold M.

In order to prove the theorem, it suffices to show that this elliptic symbol defines a 2-torsion element in the group $K(T^*M)$, since for the increasing numbers N' (see formula (38)) we take a direct sum of $2^{N'}$ copies of this operator. Let us prove the corresponding statement.

Theorem 9 *The involution*

$$\alpha^* : K(T^*M) \longrightarrow K(T^*M)$$

induced in K-theory of the spaces by the antipodal involution $\alpha : T^*M \longrightarrow T^*M$ is equal to $(-1)^{\dim M}$ modulo 2-torsion. More precisely, for an arbitrary element $x \in K(T^*M)$ and for sufficiently large positive integer N we have

$$\alpha^*\left(2^N x\right) = (-1)^{\dim M} 2^N x. \tag{39}$$

Proof. The idea is to apply the Mayer–Vietoris principle. We shall prove the equality for the entire K-theory (both even and odd).

Let us first check (39) over a point. For $x \in M$, we have

$$K^*(T_x^*M) \simeq K^*\left(\mathbf{R}^{\dim M}\right) = \mathbf{Z}.$$

An explicit formula for the generator of this group shows that the involution α^* is equal to $(-1)^{\dim M}$ in this case, as desired.

To apply the Mayer-Vietoris principle, we have to show the following. Given two open subsets

$$U, V \subset M$$

and the fact that property (39) is valid for U, V, and their intersection $U \bigcap V$, we must check that the same property holds for the union $U \bigcup V$. Let us write out a part of the Mayer-Vietoris exact sequence (for brevity, we consider here the property for the even K-group):

$$\begin{array}{ccccc} K^1\left(T^*\left(U\bigcap V\right)\right) & \to & K\left(T^*\left(U\bigcup V\right)\right) & \to & K\left(T^*U\right)\oplus K\left(T^*V\right) \\ \downarrow\alpha^* & & \downarrow\alpha^* & & \downarrow\alpha^*\oplus\alpha^* \\ K^1\left(T^*\left(U\bigcap V\right)\right) & \to & K\left(T^*\left(U\bigcup V\right)\right) & \to & K\left(T^*U\right)\oplus K\left(T^*V\right). \end{array}$$

Let us take an element $w\in K\left(T^*\left(U\bigcup V\right)\right)$. For this element we must show that the difference

$$\alpha^*w\pm w$$

is a 2-torsion element. The remaining of the proof is a standard diagram search argument and is omitted.

Now by applying the Mayer-Vietoris principle (see [24]), we prove (39) in general.

Thus, we have established the following theorem, which implies Theorem 1 of the introduction, concerning the formula for the η-invariant.

Theorem 10 *The principal symbol of the subspace* $\widehat{L}\in\widehat{\mathrm{Even}}\left(M^{odd}\right)$ *or* $\widehat{\mathrm{Odd}}\left(M^{ev}\right)$ *defines an element (see formula* (37)*)*

$$[L]\in K\left(T^*M,\mathbf{Z}\left[\frac{1}{2}\right]\Big/\mathbf{Z}\right),$$

such that the fractional part of the functional d is expressed via the direct image in K-theory as follows:

$$\left\{d\left(\widehat{L}\right)\right\}=p_![L]\in K\left(pt,\mathbf{Z}\left[\frac{1}{2}\right]\Big/\mathbf{Z}\right)=\mathbf{Z}\left[\frac{1}{2}\right]\Big/\mathbf{Z}$$

for the map $p: M\to pt$.

Appendixes

A Subspaces and $\mathbf{K^1}\left(\mathbf{T^*M}\right)$

It is well-known in elliptic theory on closed manifolds that the group of stably homotopic elliptic operators can be described in terms of K-theory of the total space of the cotangent bundle T^*M:

$$\mathrm{Ell}\left(M\right)\simeq K\left(T^*M\right). \tag{40}$$

For an elliptic operator

$$D : C^{\infty}(M,E) \longrightarrow C^{\infty}(M,F)$$

there is a difference construction defined in terms of the principal symbol of the operator,

$$\sigma(D) : \pi^* E \to \pi^* F, \qquad [\sigma(D)] \in K(T^*M), \quad \pi : S^*M \to M. \tag{41}$$

The isomorphism (40) can be interpreted from a rather different point of view. Formula (40) expresses the topological group $K(T^*M)$ via elliptic operators in analytic terms. There is a question as to whether a similar analytic realization is possible for the remaining group $K^1(T^*M)$ of the 2-periodic complex K-theory. This realization was established in the paper [12] in terms of self-adjoint elliptic operators. Let us state this result more precisely.

On a smooth manifold M we consider self-adjoint elliptic operators of a certain nonnegative order. Operators having only finitely many eigenvalues on one of the half-lines $(-\infty, 0)$ or $(0, +\infty)$ are called *trivial*. Two operators A_1 and A_2 are said to be *stably homotopic* if they become homotopic after the addition of some trivial operators to each of them. The abelian group of classes of stably homotopic operators is denoted by $\mathrm{Ell}_1(M)$. The analog of the difference construction, establishing an isomorphism

$$\mathrm{Ell}_1(M) \simeq K^1(T^*M),$$

is defined as follows.

For an elliptic self-adjoint operator

$$A : C^{\infty}(M,E) \longrightarrow C^{\infty}(M,E),$$

consider the subspace $\widehat{L}_+(A) \subset C^{\infty}(M,E)$ generated by the eigenspaces corresponding to nonnegative eigenvalues. It can be shown (e.g., see, [3]) that the spectral projection $P_+(A)$ on this subspace is a pseudodifferential operator of order zero, while the principal symbol $\sigma(P_+(A))$ of the projection is equal to the spectral projection for the principal symbol $\sigma(A)$ of the original operator A

$$\sigma(P_+(A)) = P_+(\sigma(A)) : \pi^*E \longrightarrow \pi^*E.$$

In particular, the range of this projection is a smooth vector bundle, denoted by

$$L_+(A) = \operatorname{Im} \sigma(P_+(A)) \subset \pi^*E \in \operatorname{Vect}(S^*M).$$

An analog of the difference construction (41) obtained in this situation by Atiyah, Patodi, and Singer can be written in the form

$$A \longrightarrow [L_+(A)] \in K(S^*M)/K(M) \overset{\delta}{\simeq} K^1(T^*M), \tag{42}$$

where the isomorphism δ is induced by the coboundary operator in the exact sequence of the pair (B^*M, S^*M) [5]

$$\ldots \to K(M) \to K(S^*M) \overset{\delta}{\to} K^1(T^*M) \to \ldots$$

In [12] it is proved that the map (42) induces the desired isomorphism

$$\mathrm{Ell}_1(M) \longrightarrow K^1(T^*M).$$

From this construction it becomes clear that the basis for this isomorphism is, actually, a more simple object — the subspace, determined by a pseudodifferential operator. Let us recall the corresponding definition (see [14]).

A linear subspace $\widehat{L} \subset C^\infty(M, E)$ is called *admissible* if it can be defined as the range of a pseudodifferential projection of order zero

$$\widehat{L} = \mathrm{Im}\, P, \quad P : C^\infty(M, E) \to C^\infty(M, E).$$

The subbundle $L = \mathrm{Im}\, \sigma(P) \subset \pi^* E$ is called the (principal) *symbol* of the subspace $\widehat{L}$.

Let us restate the definition of the group $\mathrm{Ell}_1(M)$ introduced above in terms of subspaces. More precisely, of all admissible subspaces, the spaces of sections $C^\infty(M, E)$, as well as the finite dimensional subspaces, are called *trivial.* Furthermore, two subspaces $\widehat{L}_1$ and $\widehat{L}_2$ are called *stably homotopic* if they become homotopic[6], after we add some trivial subspaces to each of them. Let us show that the resulting group $\mathrm{Ell}_1'(M)$ of classes of equivalent subspaces is naturally isomorphic to the group $\mathrm{Ell}_1(M)$ defined previously. To this end, to each self-adjoint elliptic operator we assign its spectral subspace

$$A \overset{j}{\longmapsto} \widehat{L}_+(A),$$

and, conversely, to each subspace we assign a self-adjoint elliptic operator by the formula

$$\widehat{L} \overset{j^{-1}}{\longmapsto} P - (1 - P), \qquad \text{for } \widehat{L} = \mathrm{Im}\, P. \tag{43}$$

Let us verify that the two mutually inverse maps

$$\mathrm{Ell}_1(M) \overset{j}{\longrightarrow} \mathrm{Ell}_1'(M) \quad \text{and} \quad \mathrm{Ell}_1'(M) \overset{j^{-1}}{\longrightarrow} \mathrm{Ell}_1(M)$$

are well-defined, i.e., they preserve the equivalence relations for both groups.

[5]Here B^*M denotes the unit ball bundle of the manifold M; $S^*M = \partial(B^*M)$.

[6]By a homotopy of subspaces $\widehat{L}_t$ we mean a family such that the corresponding family of orthogonal pseudodifferential projections P_t is smooth in the operator L^2 norm.

The operator j^{-1} preserves the equivalences, since a homotopy of subspaces implies the homotopy of the corresponding self-adjoint operators in formula (43).

The transformation j also preserves the equivalence relation. The check of this fact meets the following obstacle: for a smooth homotopy of self-adjoint operators A_t the corresponding family of spectral subspaces $\widehat{L}_+(A_t)$ varies in a discontinuous way. In the case of general position, the corresponding discontinuities arise for those values of the parameter t at which an eigenvalue of the operator A_t changes its sign[7]. In particular, when passing through the point of discontinuity, the subspaces $\widehat{L}_+(A_t)$ change by a finite-dimensional subspace, and in this way the equivalence class of the subspace $\widehat{L}_+(A_t)$ does not change during the homotopy.

Hence, we have established the equivalence of the groups $\mathrm{Ell}_1(M)$ and $\mathrm{Ell}_1'(M)$. That is why in the following the two groups will be denoted for brevity by $\mathrm{Ell}_1(M)$.

Corollary 3 *Two subspaces $\widehat{L}_{1,2}$ are stably homotopic if and only if their symbols define equal elements in the group $K(S^*M)/K(M)$.*

To conclude the section, let us present an explicit formula for the element of the difference construction (42) in the elliptic theory $\mathrm{Ell}_1(M)$. Namely, to each subspace $\widehat{L} = \operatorname{Im} P$ with symbol denoted by $L = \operatorname{Im}\sigma(P)$ we assign an elliptic family of symbols on the manifold M with parameter space $S^1 \ni z$:

$$z\sigma(P) + (1 - \sigma(P)) : \pi^*E \longrightarrow \pi^*E.$$

By the usual difference construction for elliptic families, this defines an element

$$[z\sigma(P) + (1 - \sigma(P))] \in K\left(T^*M \times S^1, T^*M \times pt\right) \equiv K^1(T^*M),$$

which, by the explicit Bott periodicity formula written out in [12], coincides with the construction (42).

B K-theory with $\mathbf{Z}_n$ coefficients

In this section we define the K-theory with $\mathbf{Z}_n$ coefficients and obtain its main properties that are used in the present paper. We use the results of the paper [25].

Let us choose a positive integer $n \geq 2$. To define the K-theory with coefficients $\mathbf{Z}_n$, we introduce the 2-dimensional complex M_n obtained from the unit 2-dimensional disk D^2 by the identification of the points on its boundary under the action of the group $\mathbf{Z}_n$:

$$M_n = \left\{ D^2 \subset \mathbf{C} \middle| \, |z| \leq 1 \right\} \Big/ \left\{ e^{i\varphi} \sim e^{i\left(\varphi + \frac{2\pi k}{n}\right)} \right\}.$$

[7]This is the well-known phenomenon of the spectral flow.

The space M_n is called the *Moore space.* For $n = 2$ it is exactly the real projective plane

$$M_2 = \mathbf{RP}^2.$$

This is the only case in which the Moore space is a smooth manifold.

For a topological space X, the K-theory with coefficients $\mathbf{Z}_n$ is defined in terms of the usual (integral) K-theory by the formula

$$K^*(X, \mathbf{Z}_n) = K^*(X \times M_n, X \times pt), \tag{44}$$

the grading $*$ here stands for 0 or 1, as usual in K-theory. The resulting groups form a generalized cohomology theory.

Let us compute the corresponding groups for the point:

$$K^0(pt, \mathbf{Z}_n) \quad \text{and} \quad K^1(pt, \mathbf{Z}_n).$$

It follows from definition (44) that these groups are isomorphic to the usual K-theory of the Moore space M_n,

$$K^*(pt, \mathbf{Z}_n) \equiv K^*(M_n, pt).$$

Thus, let us compute the usual K-groups for the Moore space. The computation relies upon the following embedding of S^1 in this space:

$$S^1 = \left\{ e^{i\varphi} \middle| 0 \le \varphi \le \frac{2\pi}{n} \right\} \subset M_n,$$

so that the quotient space is homeomorphic to the 2-sphere,

$$M_n / S^1 = S^2.$$

Hence, the exact sequence of the pair (M_n, S^1) in K-theory is

$$\begin{array}{ccccccccccc} \ldots \to \widetilde{K}(M_n) & \to & \widetilde{K}(S^1) & \to & K^1(S^2) & \to & K^1(M_n) & \to & K^1(S^1) & \stackrel{\delta}{\to} & \widetilde{K}(S^2) \to \ldots \\ & & \| & & \| & & & & \| & & \| \\ & & 0 & & 0 & & & & \mathbf{Z} & & \mathbf{Z}. \end{array}$$

Here the tilde sign $\tilde{}$ means that all groups are reduced, i.e. formed by vector bundles of (virtual) dimension zero and the coboundary map δ

$$K^1\left(S^1\right) \stackrel{\delta}{\to} \widetilde{K}\left(S^2\right)$$

is simply the multiplication by n in the groups $K^1(S^1) \simeq \widetilde{K}(S^2) \simeq \mathbf{Z}$. We obtain

$$K^1(M_n) = 0, K(M_n) = \mathbf{Z} \oplus \mathbf{Z}_n,$$

where the generator of the torsion part $\mathbf{Z}_n$ is

$$[\gamma_n] - 1 \in K(M_n)$$

for a line bundle $\gamma_n \in \mathrm{Vect}(M_n)$. Let us construct this bundle from the trivial line bundles on the elements of the partition of the Moore space

$$M_n = D_{1/2} \bigcup (M_n \setminus \{0\})$$

(here $D_{1/2}$ denotes the disk with radius equal to 1/2) by the transition function z.

The Whitney sum $n\gamma_n$ is a trivial vector bundle. In fact, its transition function is equal to $(z, z, ..., z)$. This transition function is homotopic to $(z^n, 1, ..., 1)$. Hence, it is easy to produce a trivialization for the corresponding bundle

$$n\gamma_n \quad \xrightarrow{\beta} \quad \mathbf{C}^n.$$

Namely, the trivialization β is identity on the disk $D_{1/2}$, while on the second element of the partition $M_n \setminus \{0\}$ it is the multiplication by the transition function $(z^n, 1, ..., 1)$.

Now we construct the exact coefficient sequence in K-theory corresponding to the groups $\mathbf{Z}$ and $\mathbf{Z}_n$.

Proposition 2 *The short exact sequence of groups*

$$0 \to \mathbf{Z} \stackrel{\times n}{\to} \mathbf{Z} \to \mathbf{Z}_n \to 0$$

induces an exact sequence in K-theory with coefficients

$$\to K(X) \stackrel{\times n}{\to} K(X) \to K(X, \mathbf{Z}_n) \to K^1(X) \stackrel{\times n}{\to} K^1(X) \to K^1(X, \mathbf{Z}_n) \to \qquad (45)$$

Proof. The sequence (45) is in fact a corollary of the formula for the Moore space

$$S^1 \subset M_n \to M_n / S^1 = S^2.$$

Namely, the sequence of the pair $(X \times M_n, X \times S^1)$ implies the exact sequence

$$\begin{aligned} K\left(X \times S^2, X \times pt\right) &\to K(X \times M_n, X \times pt) \to K\left(X \times S^1, X \times pt\right) \to \\ K^1\left(X \times S^2, X \times pt\right) &\to K^1(X \times M_n, X \times pt) \to K^1\left(X \times S^1, X \times pt\right) \to, \end{aligned}$$

Using Bott periodicity, we rewrite the corresponding terms in the sequence, obtaining the desired formula (45):

$$\to K(X) \to K(X, \mathbf{Z}_n) \to K^1(X) \stackrel{\times n}{\to} K^1(X) \to K^1(X, \mathbf{Z}_n) \to K(X) \stackrel{\times n}{\to}.$$

Remark 5 The geometric construction of the exact sequence (45) makes it possible to say that the "reduction modulo n" maps

$$K(X) \to K(X, \mathbf{Z}_n) \quad \text{and} \quad K^1(X) \to K^1(X, \mathbf{Z}_n)$$

are actually tensor products with the element $[\gamma_n] - 1 \in K(M_n, pt)$, while the Bokstein maps

$$K(X, \mathbf{Z}_n) \to K^1(X) \quad \text{and} \quad K^1(X, \mathbf{Z}_n) \to K(X)$$

are merely the induced maps for the inclusion

$$S^1 \subset M_n.$$

The direct image in K-theory with coefficients enjoys the same properties as in the usual theory, since for a smooth map of manifolds

$$f : X_1 \longrightarrow X_2$$

there is a usual direct image map "with parameters" (see [21])

$$f_! : K(T^*X_1 \times M_n) \longrightarrow K(T^*X_2 \times M_n)$$

that restricts to the map in K-theory with $\mathbf{Z}_n$ coefficients

$$f_! : K(T^*X_1, \mathbf{Z}_n) \longrightarrow K(T^*X_2, \mathbf{Z}_n).$$

In the "trivial" case, when f is a constant map to a point pt, we obtain a residue-valued direct image

$$f_! : K(T^*X_1, \mathbf{Z}_n) \longrightarrow K(pt, \mathbf{Z}_n) = \mathbf{Z}_n.$$

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