Elliptic Complexes of Pseudodifferential Operators on Manifolds with Edges

Bert-Wolfgang Schulze
Institut für Mathematik
Universität Potsdam
Postfach 60 15 53
14415 Potsdam
Germany

Nikolai Tarkhanov[1]
Institute of Physics
Russian Academy of Sciences
Akademgorodok
660036 Krasnoyarsk
Russia

June 12, 1998

[1]Supported by the Max-Planck Gesellschaft.

Abstract

On a compact closed manifold with edges live pseudodifferential operators which are block matrices of operators with additional edge conditions like boundary conditions in boundary value problems. They include Green, trace and potential operators along the edges, act in a kind of Sobolev spaces and form an algebra with a wealthy symbolic structure. We consider complexes of Fréchet spaces whose differentials are given by operators in this algebra. Since the algebra in question is a microlocalization of the Lie algebra of typical vector fields on a manifold with edges, such complexes are of great geometric interest. In particular, the de Rham and Dolbeault complexes on manifolds with edges fit into this framework. To each complex there correspond two sequences of symbols, one of the two controls the interior ellipticity while the other sequence controls the ellipticity at the edges. The elliptic complexes prove to be Fredholm, i.e., have a finite-dimensional cohomology. Using specific tools in the algebra of pseudodifferential operators we develop a Hodge theory for elliptic complexes and outline a few applications thereof.

AMS subject classification: primary: 58G05; secondary: 58A14, 58G10.

Key words and phrases: manifolds with singularities, pseudodifferential operators, elliptic complexes, Hodge theory.

Contents

Introduction

De Rham [dR55] was the first to give a representation of the singular homologies of a differentiable manifold by means of the cohomology of closed differential forms. He combined the theory of differential forms of E. Cartan with the idea of a distribution of Sobolev and Schwartz to identify singular cycles with closed currents on the manifold. When completed with the standard regularisation, this leads to a representation of singular cycles by smooth closed differential forms modulo coboundaries.

The de Rham theory highlighted the classical result of Hodge [Hod41] according to which the cohomology of the complex of differential forms on a smooth compact closed manifold can be represented via harmonic forms. Though being of analytical nature, the harmonic forms proved thus to bear information on the topology of the manifold. Afterwards this aspect of the de Rham theory entered into the Atiyah-Singer index theorem.

In 1970, Singer [Sin71] presents a program aimed at extending the theory of elliptic operators and their index to "non-smooth manifolds of special type and to a context where it is natural that *integer* (index) be replaced by *real number*."

A problem from Singer's program is to produce a Hodge theory on manifolds with singularities and on pseudomanifolds. The starting difficulty consists of the proper definition of differential forms, exterior derivatives and Laplacians under singular structures. The major contributions to Hodge theory are those of Teleman [Tel79, Tel83] by using a combinatorial method and of Cheeger [Che80, Che83] by using a geometric approach.

Unlike these techniques, Shaw [Sha83] developed a Hodge theory on domains with conical points on the boundary by studying an elliptic boundary value problem, so-called d-Neumann problem. Using arguments similar to those of Kondrat'ev [Kon67] he proved the existence and compactness of the Neumann operator that solves this problem. In addition, his set up yields information on the regularity of harmonic forms near singularities.

It was in the spirit of this idea that Schulze [Sch88a] studied complexes of Fréchet spaces on a compact closed manifold with conical points, whose differentials belong to an algebra of pseudodifferential operators living on the manifold. For the topological cone over a smooth compact closed manifold, this algebra was before introduced by Rempel and Schulze [RS86]. It contains typical differential operators on the cone which are of Fuchs type, and bears a symbol structure allowing parametrix construction on the symbol level. As order reductions, adjoints and parametrices to elliptic elements are available within the algebra, the matter is reduced to a single operator, the Laplacian of the complex. Schulze [Sch88a] proved a Hodge decomposition theorem for elliptic complexes and showed asymptotic expansions of harmonic sections close to conical points.

When applied to the de Rham complex on a manifold with conical points, this provides us with proper function spaces and highlights the nature of involved classes of pseudodifferential operators. The function spaces in question are weighted Sobolev spaces, the weights being powers of the distance function to the singular points. The connection of the cohomology of the de Rham complex in weighted Sobolev spaces with the singular homologies of the manifold is a more delicate problem.

Yet another approach to the analysis on manifolds with conical singularities is Melrose's calculus of totally characteristic pseudodifferential operators (cf. [Mel93]). These operators live on a smooth compact manifold with boundary obtained by blowing up a closed manifold with conical points at each singular point. The Fuchs-type differential operators are totally characteristic because the Fuchs derivative proves surprisingly to be tangential to the boundary. Melrose gave an exposition of Hodge theory in the analytic framework of his calculus and proved an analogue of the de Rham theorem (*ibid.*, 6.4).

As one might expect, the Hodge theory carries over to an arbitrary manifold with singularities had we a calculus for single operators so that all the formal manipulations such as compositions, adjoints and so on be controlled within the given class of operators and also on the symbol level. However, it seems to be difficult to establish a meaningful abstract Hodge theory because there is no unified approach to the calculus on singular varieties. Each algebra of pseudodifferential operators so far known proves to bear a specific symbolic structure which results in various definitions of ellipticity.

With this as our starting point, we examine in this paper elliptic complexes on a compact closed manifold with edges. Close to an edge, such a manifold is a fibre bundle over the edge whose fibre is the topological cone over a smooth compact closed manifold. In particular, the product of a smooth compact closed manifold and a compact closed manifold with conical points is a manifold with edges. This concept also encompasses manifolds with boundary, the boundary being an edge of the lowest codimension. An algebra of pseudodifferential boundary value problems without transmission property was first introduced by Rempel and Schulze [RS82b] who enriched considerably the results of Boutet de Monvel [BdM71] and Eskin [Esk73]. In 1990, Schulze [Sch92, Sch94] combined this theory with the calculus on the cone of [RS86] to define an algebra of pseudodifferential operators on a closed manifold with edges. It starts with typical differential operators on the manifold, such as the products of differential operators along edges and

Fuchs-type operators in the fibres, the Laplacians for warped wedge metrics, and so on. The operators in the algebra are block matrices similar to those in Boutet de Monvel's algebra, with the usual principal homogeneous symbol over the cotangent bundle to the manifold as well as an additional symbol over the cotangent bundle to each edge. This latter takes its values in the cone algebra in fibres and controls the generalised Lopatinskii condition. The ellipticity of a single operator refers to both the symbol levels and extends in a natural way to the definition of an elliptic complex. In this setting, we establish the Fredholm property of elliptic complexes in natural weighted Sobolev spaces and present a Hodge theory for them.

Let us finally mention that the school of Melrose [Mel96a] also develops general pseudodifferential calculi on manifolds with corners (stratified spaces). The definition of a manifold with corners is based on the model spaces which are products of half-lines and lines. It thus encompasses in particular manifolds with boundary and manifolds with simple conical points and edges on the boundary. The principal significance of the class of manifolds with corners lies in the fact that passage 'to the boundary' and 'to the product' stays within the class. This theory has some intersection with our operator algebra for edge singularities (or more generally with the corner algebra of Schulze [Sch89a, Sch92]), as far as it concerns the allowed classes of degenerate interior symbols or certain blow-up and invariance aspects. On the other hand, there are specific differences, in particular, with respect to the nature of weighted Sobolev spaces, the various ideals of smoothing operators with asymptotics, the additional conditions along the lower-dimensional skeletons, and so on. While a manifold with edges (even on the boundary) can be desingularised to a manifold with corners under a blow-up along each edge, our approach to the calculus involves much more analytical ingredients than that in [Mel96a].

Acknowledgments. The authors are indebted to B. Fedosov for numerous discussions which often opened new aspects of the theory.

Chapter 1

Manifolds with Edges

1.1 C^∞ *structures*

The concept of a manifold with edges starts with a model space bearing edges. As such we take the wedge $W = \mathbb{R}^q \times C_t(X)$, where $C_t(X)$ is the topological cone over a compact closed manifold X of dimension n, i.e.,

$$C_t(X) = \frac{\bar{\mathbb{R}}_+ \times X}{\{0\} \times X},$$

and $\mathbb{R}^q$ or, more precisely, $E = \mathbb{R}^q \times \{\text{tip of } C_t(X)\}$ stands for the edge. We give $C_t(X)$ the natural quotient topology, and W the product topology. Then, W is a Hausdorff topological space.

The important point to note here is the form of W close to the edge. In general, W is not a topological manifold of dimension $q+1+n$, as shows the fairly obvious example where $n = 0$ and X consists of at least three different points on the unit circle in $\mathbb{R}^2$.

A general continuous mapping has no relation to the edge of the wedge W it maps. To keep the wedge structure of W, we will consider only those continuous mappings between open sets Ω_1, Ω_2 in W which obey the pair (W, E). More precisely, we require $f\colon \Omega_1 \to \Omega_2$ to satisfy $f(\Omega_1 \cap E) \subset \Omega_2 \cap E$, provided that $\Omega_1 \cap E \neq \emptyset$. In particular, a mapping $f\colon \Omega_1 \to \Omega_2$ is said to be a *homeomorphism on the wedge* if it a homeomorphism and restricts to a homeomorphism of $\Omega_1 \cap E$ onto $\Omega_2 \cap E$.

Our next task is to endow W with a C^∞ structure with edges which amounts to defining C^∞ functions on W. The cone cross-section X is always assumed to be a C^∞ manifold, hence W bears a natural C^∞ structure away from the edge. Moreover, W has the standard C^∞ structure along the edge E. When looking for a C^∞ with edges on W, one wishes to take into account both the structures. However, there is no canonical way to do this unless W is embedded into an Euclidean space $\mathbb{R}^N$, smoothly away from the edge. In the sequel we think of W as being embedded.

If such is the case, by a C^∞ *function* on an open set $\Omega \subset W$ one might mean the restriction, to Ω, of a C^∞ function on a neighbourhood of Ω in

$\mathbb{R}^N$. If Ω does not meet the edge, this concept proves to agree with the purely intrinsic description in terms of local coordinates. For Ω intersecting the edge, there is no intrinsic description of $C^\infty(\Omega)$ in general though there may be such characterisations for particular embeddings $W \hookrightarrow \mathbb{R}^N$.

Having fixed a C^∞ structure with edges on W, we are in a position to define diffeomorphisms of W. Namely, if $\Omega_1, \Omega_2 \subset W$ are open, then a mapping $\delta\colon \Omega_1 \to \Omega_2$ is said to be a *diffeomorphism* if it is a homeomorphism on the wedge with inverse $\delta^{-1}\colon \Omega_2 \to \Omega_1$ and the components of δ and δ^{-1} are in $C^\infty(\Omega_1)$ and $C^\infty(\Omega_2)$ respectively.

It immediately follows that any diffeomorphism $\delta : \Omega_1 \to \Omega_2$ restricts to a diffeomorphism $\delta\colon \Omega_1 \cap E \to \Omega_2 \cap E$.

If δ is a diffeomorphism of open sets $\Omega_1, \Omega_2 \subset W$, then there exist open sets $\tilde{\Omega}_1, \tilde{\Omega}_2 \subset \mathbb{R}^N$ with $\Omega_1 = \tilde{\Omega}_1 \cap W$, $\Omega_2 = \tilde{\Omega}_2 \cap W$, and a diffeomorphism $\tilde{\delta}\colon \tilde{\Omega}_1 \to \tilde{\Omega}_2$ such that $\delta = \tilde{\delta}|_{\Omega_1}$ (cf. Lemma 1.5.1 in Melrose [Mel96a]).

Now, the definition of a *manifold with edges* follows precisely along classical lines.

Definition 1.1.1 *By a manifold with edges is meant a pair (M, S), M being a Hausdorff topological space and S a closed subspace of M, such that both $M \setminus S$ and S are manifolds and each point $p \in S$ has a neighbourhood O on M homeomorphic to an open set $\Omega \subset W$, the restriction of the homeomorphism to $O \cap S$ being onto $\Omega \cap E$.*

The space S is said to be the *set of edges* of M. It is allowed to consist of a finite number of connected components of various dimensions, $S = \cup S_i$, where S_i is a manifold of dimension q_i. The model wedge W does vary together with $p \in S$, however W is required to be the same along each component S_i. A familiar topological argument shows that this assumption involves no loss of generality.

We will abbreviate (M, S) to M when no confusion can arise, and call M the manifold with edges.

If $W_i = \mathbb{R}^{q_i} \times C_t(X_i)$ is the model wedge for S_i, then the number $q_i + 1 + \dim X_i$ is equal to the dimension of $M \setminus S$ and thus independent of i. This number is called the *dimension* of M.

Close to an edge S_i, the manifold M gives rise to a cone bundle whose base is S_i and whose fibre is $C_t(X_i)$.

As mentioned, M is not necessarily a topological manifold unless all the cone cross-sections X_i are homeomorphic to spheres. However, we call M a manifold for the analytical objects will be considered on $M \setminus S$.

The structure of *charts with edges* on a manifold with edges M is clear from the Definition 1.1.1. When speaking on charts $h\colon O \to W$ on M, one specifies the domains O of these. Namely, O is allowed to meet at most one of the edges S_i. If O intersects S_i, then h maps O to W_i and the image of $O \cap S_i$ by h lies on the edge of W_i. If O is away from the edges, then as target space for h we can take either any one of the wedges W_i or the Euclidean space $\mathbb{R}^{\dim M}$ (cf. Fig. 1.1).

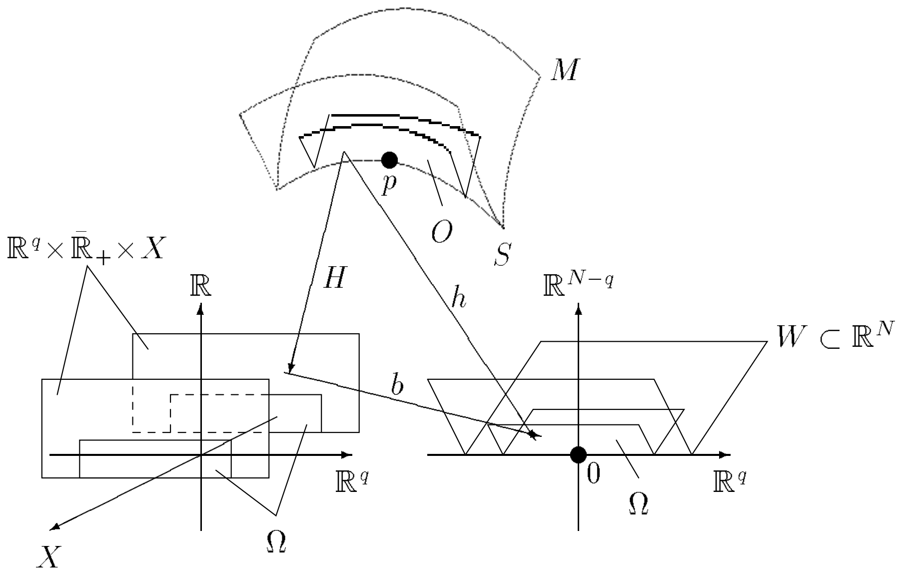

Fig. 1.1: A chart with edges.

Two charts with edges (h_1, O_1) and (h_2, O_2) are said to be *compatible* if either $O_1 \cap O_2 = \emptyset$ or $h_2 \circ h_1^{-1} \colon h_1(O_1 \cap O_2) \to h_2(O_1 \cap O_2)$ is a diffeomorphism of open sets on a model wedge or in $\mathbb{R}^{\dim M}$.

We emphasise that if $O_1 \cap O_2$ intersects S, then both O_2 and O_2 meet the same component of S, S_i say, and so $h_2 \circ h_1^{-1}$ is required to be a diffeomorphism of open sets on the wedge W_i. On the other hand, if $O_1 \cap O_2$ does not meet S, then O_1 and O_2 may intersect different components of S or lie away from S at all. In any case $h_2 \circ h_1^{-1}$ is a diffeomorphism of open sets lying outside the singularities.

An *atlas* on M is a system of charts $(h_\iota, O_\iota)_{\iota \in I}$ which are compatible in pairs and which cover M, i.e., $M = \cup_{\iota \in I} O_\iota$.

A C^∞ *structure with edges* on M is a maximal atlas, i.e., an atlas which contains any chart compatible with each element of the atlas.

By a *coordinate system* on a manifold M, with a C^∞ structure with edges, we shall mean a pointed chart, i.e., a chart (h, O) and a point $p \in O$ such that $h(p) = 0$.

If M has a C^∞ structure with edges, we denote by $C^\infty(M)$ the space of all functions $u \colon M \to \mathbb{C}$ such that $u \circ h^{-1}$ is C^∞ on $h(O)$, for each chart (h, O).

Definition 1.1.2 *A C^∞ manifold with edges is a pair $(M, \mathcal{F})$, where M is a paracompact manifold with edges and $\mathcal{F} = C^\infty(M)$ for some C^∞ structure with edges on M.*

The concept of a C^∞ manifold with edges encompasses in particular the class of smooth manifolds with boundary. Each boundary hypersurface is regarded as an edge of minimal codimension, with the inward-pointing normal bundle as cone bundle.

Definition 1.1.2 can be easily generalised to the case when the cone bases X_i are smooth compact manifolds with boundary. This would correspond to manifolds with edges on the boundary.

On the other hand, there may be fairly artificial C^∞ structures with edges.

Example 1.1.3 Let M be a C^∞ manifold of dimension $q+1+n$ and S a q-dimensional C^∞ submanifold of M. For each point $p \in S$, there are a neighbourhood O on M and a diffeomorphism h of O onto an open subset Ω of $\mathbb{R}^q \times B$, B being the unit ball of centre 0 in $\mathbb{R}^{1+n}$, such that $h(O \cap S) = \Omega \cap (\mathbb{R}^q \times \{0\})$. Since B can equally well be regarded as the topological cone over the unit sphere in $\mathbb{R}^{1+n}$, i.e. $B \cong C_t(\mathbb{S}^n)$, these diffeomorphisms h give rise to a C^∞ structure with edges on M, with S an edge. It is clear that this C^∞ structure with edges in fact coincides with the original C^∞ structure of M.

□

1.2 *Vector bundles*

Let M be a C^∞ manifold with edges. The differential geometry of M will be developed by straightforward extension from the singularityless case, which is first recalled succinctly.

We begin with the construction of the tangent bundle of a manifold with edges. In the singularityless case this construction is direct and is followed by that of the cotangent bundle via invoking the functor of the dual bundle. For manifolds with singularities, however, a direct construction of the cotangent bundle has proved to survive rather than that of the tangent bundle (cf. Melrose [Mel96a, 1.10]).

The manifold M comes equipped with its algebra of C^∞ functions, $C^\infty(M)$, of which we are going to make use. Localising this leads to the cotangent bundle. Thus, for $p \in M$, let $\mathcal{I}_p(M) = \{u \in C^\infty(M) \colon u(p) = 0\}$ be the ideal of smooth functions vanishing at p. We define the cotangent space at p by $T_p^*M = \mathcal{I}_p(M)/\mathcal{I}_p^2(M)$ where $\mathcal{I}_p^2(M)$ is the linear span of products of pairs of elements in $\mathcal{I}_p(M)$. It is clear that T_p^*M bears a natural linear structure inherited from $\mathcal{I}_p(M)$.

If p lies on the smooth part of M, then it is easy to see, by reference to local coordinates and use of Taylor's formula, that T_p^*M is a vector space of dimension equal to that of M. However, this is no longer true for singular points p, i.e. those lying on the edges of M. Clearly, the dimension of T_p^*M does not exceed N provided p belongs to an edge with the model wedge embedded to $\mathbb{R}^N$. The following example shows that the dimension of T_p^*M can in fact vary from $\dim M$ to N.

Example 1.2.1 Pick $N-q$ points a_i on the unit sphere $\mathbb{S}^{N-q-1}$ in $\mathbb{R}^{N-q}$, i.e.

$$a_i = (a_{i,1}, \dots, a_{i,N-q}), \quad i = 1, \dots, N-q,$$

and put $X = \{a_1, \dots, a_N\}$. Then, X is a manifold of dimension 0 and the topological cone over X allows an embedding to $\mathbb{R}^{N-q}$ by

$$C_t(X) = \{ta_i : t \in \bar{\mathbb{R}}_+,\ i = 1, \dots, N-q\}$$

(a *hedgehog-like* manifold with a conical point at the origin). The wedge $W = \mathbb{R}^q \times C_t(X)$ is thus embedded to $\mathbb{R}^N$ and has a natural C^∞ structure with edge $E = \mathbb{R}^q \times \{0\}$. It is a simple matter to see that

$$\dim T_p^* W = q + r \quad \text{for each} \quad p \in E,$$

where $r = \mathrm{rank}_{\mathbb{R}}\,(\mathrm{a_{i,j}})$. Hence it follows that the dimension of T_p^*W over the edge may be any integer in the interval $[q+1, N]$. There are essentially two cases where this dimension is precisely $q+1$. One of the two corresponds to X consisting of two antipodal points on the sphere $\mathbb{S}^{N-q-1}$ and thus to $W \overset{\text{dif.}}{\cong} \mathbb{R}^q \times (-1, 1)$ (the case of C^∞ manifolds). The other case corresponds to X consisting of a single point on $\mathbb{S}^{N-q-1}$ and thus to $W \overset{\text{dif.}}{\cong} \mathbb{R}^q \times \bar{\mathbb{R}}_+$ (the case of C^∞ manifolds with boundary).

□

The cotangent spaces T_p^*M over all $p \in M$ combine to give rise to the *cotangent bundle* of M,

$$T^*M = \bigcup_{p \in M} T_p^*M,$$

with the natural projection $\pi: T^*M \to M$ mapping T_p^*M to p. We check at once that T^*M is still a manifold with edges, though of subtler structures. Namely, to an edge S_i in M with model wedge $\mathbb{R}^{q_i} \times C_t(X_i)$ there corresponds an edge in T^*M with model wedge $\mathbb{R}^{q_i + \dim M} \times C_t(X_i \cup \mathbb{S}^{r_i - 2})$, where $r_i \geq 1$. Moreover, we have $r_i > 1$ unless the singularity of M along S_i is artificial.

If $u \in C^\infty(M)$ and $p \in M$, then $u - u(p) \in \mathcal{I}_p(M)$. Thus, the class of $u - u(p)$ in T_p^*M defines a section of T^*M, i.e., a mapping $du: M \to T^*M$ satisfying $\pi \circ du = \mathrm{Id}$.

Using these *explicit* sections of T^*M we introduce the following space of functions on T^*M,

$$\mathcal{F} = \{f : T^*M \to \mathbb{C} \text{ such that } f \circ du \in C^\infty(M) \text{ for all } u \in C^\infty(M)\}. \tag{1.2.1}$$

Proposition 1.2.2 *$(T^*M, \mathcal{F})$, with $\mathcal{F}$ defined by* (1.2.1)*, is a C^∞ manifold with edges.*

Proof. See Proposition 1.10.1 in Melrose [Mel96a].

□

It is easy to check that $\pi\colon T^*M \to M$ is a smooth mapping of manifolds with edges. If $h : O \to \mathbb{R}^{\dim M}$ is a non-singular chart on M and $p \in O$, then each $u \in \mathcal{I}_p(M)$ can be written as $u = \sum_{j=1}^{\dim M} c_j dh_j(p)$ modulo $\mathcal{I}_p^2(M)$, where $c_j \in \mathbb{C}$ are constant. Hence $(dh_j(p))$ is a basis in each cotangent space T_p^*M, $p \in O$. It varies smoothly in $p \in O$, which leads to a diffeomorphism $\delta : \pi^{-1}(O) \to O \times \mathbb{R}^{\dim}$ given by $(p, [u]) \mapsto (p, c)$, where $c = (c_j)$ are the coordinates in T_p^*M with respect to $(dh_j(p))$. Moreover, the restriction of δ to each fibre $\pi^{-1}(p) = T_p^*M$ is a linear mapping. However, the linear structures in the fibres $\pi^{-1}(p) \subset T^*M$ fail to vary smoothly close to edges where the dimension of $\pi^{-1}(p)$ jumps. Thus, the cotangent bundle to a manifold with edges does not fit in the usual concept of a vector bundle. The singular structure of M near edges forces the cotangent bundle to bear new features such as jumps of fibre dimensions. Rather than to modify the concept of a vector bundle to meet these pecularities we will blow up M along the edges, thus confining the discussion to C^∞ manifolds with boundary and standard vector bundles. The first approach would perhaps better suit to differential geometry while the second one is suggested by the analysis of partial differential operators.

Similarly the dual spaces $T_pM = \left(T_p^*M\right)^*$ fit together to form the *tangent bundle* $TM = \cup_{p\in M} T_pM$, which is also naturally a C^∞ manifold with edges. For a manifold with edges, the elements of $C^\infty(M, TM)$ are called smooth *vector fields*. Each element $v \in C^\infty(M, TM)$ acts on $C^\infty(M)$, essentially by definition,

$$vu\,(p) = \langle v(p), du\,(p)\rangle, \quad u \in C^\infty(M), \tag{1.2.2}$$

$\langle\cdot,\cdot\rangle$ being the pairing between T_pM and T_p^*M. In particular, $vu \equiv 0$ if u is constant. Moreover this action shows that $C^\infty(M, TM)$ is a Lie algebra with Poisson brackets $[v_1, v_2]u\,(p) = (v_1v_2 - v_2v_1)u\,(p)$, for $u \in C^\infty(M)$.

We finish this section by recalling the standard notion of a vector bundle over a C^∞ manifold with edges. As mentioned, it relies on the C^∞ structure of M induced by an embedding, and so it is not sufficient to cover natural geometric bundles unless the singularities have been blown up.

Note that a mapping $f\colon M_1 \to M_2$ between C^∞ manifolds with edges is *smooth*, i.e. C^∞, if the pull-back $f^*u = u \circ f$ belongs to $C^\infty(M_1)$ whenever $u \in C^\infty(M_2)$.

For open sets $O_1 \subset M_1$ and $O_2 \subset M_2$, we call a mapping $\delta\colon O_1 \to O_2$ the *diffeomorphism* if it is C^∞, has a C^∞, two-sided, inverse and restricts to a diffeomorphism of the pieces of O_1 and O_2 along the edges.

Definition 1.2.3 *Let $\pi\colon V \to M$ be a smooth mapping of manifolds with edges. We say that (V, π) is a fibre bundle with typical fibre V_f if there are a covering of M by open sets O_ι and diffeomorphisms $\delta_\iota\colon \pi^{-1}(O_\iota) \to O_\iota \times V_f$ such that $\pi\colon \pi^{-1}(O_\iota) \to O_\iota$ is the composition of δ_ι with projection onto the first factor O_ι in $O_\iota \times V_f$.*

It follows from this definition that $\pi^{-1}(p)$ is diffeomorphic to V_f for all $p \in M$; we will call V_f "the" fibre of V. We require V_f to be a C^∞ manifold, and so $M \times V_f$ is a C^∞ manifold with edges $\{\text{edges of } M\} \times V_f$.

A section of a fibre bundle V over M is a mapping $s : M \to V$ such that $\pi s(p) = p$ for all $p \in M$.

We denote by $C^\infty(M, V)$ the set of all smooth, i.e. C^∞, sections of V over M. It does not bear any structure but those induced pointwise by hypothetical structures in the fibres $\pi^{-1}(p)$, $p \in M$. Such is the case, in particular, for vector bundles V, where $C^\infty(M, V)$ is a vector space.

Consider the diffeomorphism $\delta_j \circ \delta_\iota^{-1}$ of $(O_\iota \cap O_j) \times V_f$ which we obtain from the definition of a fibre bundle. It is a mapping from $O_\iota \cap O_j$ to the group of diffeomorphisms of the fibre V_f.

Definition 1.2.4 *A fibre bundle $\pi: V \to M$ is said to be a vector bundle if its typical fibre is a vector space V_f and if the diffeomorphisms δ_ι may be chosen in such a way that the diffeomorphisms $\delta_j \circ \delta_\iota^{-1}: \{p\} \times V_f \to \{p\} \times V_f$ are invertible linear mappings for all $p \in O_\iota \cap O_j$.*

If $f: M_1 \to M_2$ is a smooth mapping of manifolds with edges and V is a fibre bundle over M_2, we denote by f^*V the fibre bundle over M_1 obtained by pulling back V. More precisely, f^*V is the smooth fibre bundle given by restricting the fibre bundle $M_1 \times V$ over $M_1 \times M_2$ to the graph Γ_f of f, so that

$$\begin{aligned} f^*V &= (\mathrm{Id} \times \pi)^{-1}\Gamma_{\mathrm{f}} \\ &= \{(p, v) \in M_1 \times V : \pi(v) = f(p)\}. \end{aligned}$$

The induced mapping on sections, $f^*u = u \circ f$, will be referred to as the *pull-back* mapping; it maps $C^\infty(M_2, V)$ to $C^\infty(M_1, f^*V)$.

1.3 *Blowing up edges*

In order to simplify various analytic constructions it proves to be helpful to blow up the manifold M along edges.

The main idea of blowing up a manifold at a singular set S is to replace the manifold by a bigger set in which some information concerning the direction of approach to S is included. Directions are associated to curves with only their endpoint on S and which are not "tangent" to S. The set of equivalence classes of such curves is in many cases specified as the inward-pointing spherical normal bundle to S. For more details we refer the reader to Melrose [Mel96a, Ch. 5].

Let S_i be a connected component of the set of edges of M. By Definition 1.1.1, this amounts to saying that each point $p \in S_i$ has a neighbourhood O in M which bears the structure of a fibre bundle over $O \cap S_i$, with $C_t(X_i)$ as a typical fibre. Indeed, having fixed a diffeomorphism $h: O \to W_i$, one can set $\pi(p) = h^{-1}(y)$, for $p \in O$, where y is the projection of $h(p)$ on the edge in W_i. However, there need not exist any neighbourhood O of the entire edge S_i which is a fibre bundle over S_i whose typical fibre is $C_t(X_i)$. While this is the case for Riemannian manifolds with edges, in general the existence of such a neighbourhood should be postulated.

Recall that we think of $C_t(X_i)$ as being embedded as a conic set in an Euclidean space $\mathbb{R}^{N-q_i}$ where q_i stands for the dimension of S_i. The usual way to achieve such an embedding is as follows. Let $e\colon X_i \hookrightarrow \mathbb{R}^{N-q_i-1}$ be an embedding of X_i as a compact submanifold in $\mathbb{R}^{N-q_i-1}$, with N large enough, which does exist by the Whitney theorem. Then,

$$C_t(X_i) = \{t\,\mathrm{SP}\,(\mathrm{e(x)}) \in \mathbb{R}^{\mathrm{N-q_i}}\colon\ \mathrm{t} \in \bar{\mathbb{R}}_+,\ \mathrm{x} \in \mathrm{X_i}\}, \tag{1.3.1}$$

SP being the stereographic projection of $\mathbb{R}^{N-q_i-1}$ onto the upper hemisphere $\mathbb{S}_+^{N-q_i-1}$ in $\mathbb{R}^{N-q_i}$,

$$\mathrm{SP}\,(\mathrm{z}) = \frac{(\mathrm{z}, 1)}{\sqrt{|\mathrm{z}|^2 + 1}}, \quad \mathrm{z} \in \mathbb{R}^{\mathrm{N-q_i-1}}.$$

Blowing up the tip 0 in $C_t(X_i)$ simply amounts to the introduction of polar coordinates (t, x). Namely, we define $C_t(X_i)$ blown up at 0 to be $[C_t(X_i); \{0\}] = \bar{\mathbb{R}}_+ \times X_i$ (cf. the notation of Melrose [Mel96a, 5.1]) together with the associated blow-down mapping $b\colon [C_t(X_i); \{0\}] \to C_t(X_i)$ given by

$$b(t, x) = t\,\mathrm{SP}\,(\mathrm{e(x)}). \tag{1.3.2}$$

This is a diffeomorphism of $[C_t(X_i); \{0\}] \setminus (\{0\} \times X_i)$ onto $C_t(X_i) \setminus \{0\}$ and has rank 1 at the boundary $\partial[C_t(X_i); \{0\}] = \{0\} \times X_i$, which projects to the tip 0.

We blow up M along the edge S_i by blowing up the tip at each fibre of M over S_i, close to S_i. Namely, let $\pi\colon O \to S_i$ be a fibre bundle of some neighbourhood O of S_i in M, with typical fibre $C_t(X_i)$. Given any point $p \in S_i$, the fiber $\pi^{-1}(p)$ is diffeomorphic to $C_t(X_i)$, the diffeomorphism being locally C^∞ in p. We define the fibre $\pi^{-1}(p)$ blown up at the tip to be $\bar{\mathbb{R}}_+ \times X_i$ along with the associated blow-down mapping $\bar{\mathbb{R}}_+ \times X_i \to \pi^{-1}(p)$ obtained by composing (1.3.2) and the inverse of the diffeomorphism $\pi^{-1}(p) \to C_t(X_i)$. Once again this is equivalent to introducing the polar coordinates at each fibre of O over S_i.

Denote by $[M; S_i]$ the topological space constructed from M by replacing the neighbourhood O by $S_i \times \bar{\mathbb{R}}_+ \times X_i$ via gluing with any one of the above diffeomorphisms. What we have done is we attached a *cylindrical end* along the edge S_i.

We now apply this construction again, thus blowing up M along each of the edges. The resulting topological space is denoted by

$$[M; S] = [\ldots[[M; S_1]; S_2]\ldots];$$

it is immaterial which order of edges we choose to define $[M; S]$ as long as the edges are disjoint.

Remark 1.3.1 *The space* $[M;S]$ *is referred to as the 'stretched manifold' associated to the manifold with edges* M *(cf. Schulze* [Sch98, 3.1.1]*).*

From the construction of $[M;S]$ it is immediate that $[M;S]$ is a C^∞ manifold with boundary. Moreover, the corresponding blow-down mapping $b\colon [M;S] \to M$ is C^∞ and restricts to a diffeomorphism of $[M;S] \setminus \partial[M;S]$ onto $M \setminus S$.

Note that the boundary of $[M;S]$ has as many connected components as the number of edges of M. In fact,

$$\partial[M;S] = \cup_i \, S_i \times \{0\} \times X_i$$

under the identification above. The component $S_i \times \{0\} \times X_i$ of the boundary has the structure of a fibre bundle over S_i whose typical fibre is X_i, a C^∞ compact closed manifold. Thus, the boundary of $[M;S]$ is a fibre bundle over the set of edges,

$$\begin{array}{c} \partial[M;S] \\ \big\downarrow{\scriptstyle b} \\ S \end{array} \tag{1.3.3}$$

the typical fibre varying along with the component of S.

The analysis on a manifold with edges has been proved to refer to the associated stretched manifold.

We finish this section with a brief discussion of the pull-backs of vector bundles over M under the blow-down mapping $b\colon [M;S] \to M$. If V is a C^∞ vector bundle over M, then the fibre of b^*V over a point $p \in \partial[M;S]$ is $V_{b(p)}$. As b blows down entire fibres of (1.3.3), it follows that b^*V should be "constant" along the fibres of the boundary fibration. Hence, vector bundles of interest over $[M;S]$ seem at first sight to obey the fibration (1.3.3). The cotangent bundle T^*M to a manifold with edges, however, gives some suggestive evidence to the contrary. Indeed, as defined in Section 1.2, T^*M does not meet Definition 1.2.3 while being of great importance in the analysis on M. As a substitute for the pull-back of T^*M under the blow-down mapping $b\colon [M;S] \to M$ one would like to have just the cotangent bundle to $[M;S]$. Since this latter need not be "constant" along the fibres of (1.3.3), we have thus to permit arbitrary C^∞ vector bundles over $[M;S]$, i.e., including those which do not respect the boundary fibration.

1.4 *Distributions on manifolds with edges*

Suppose (M,S) is a C^∞ manifold with edges. On the non-singular part $M \setminus S$ of M all the usual function spaces are well-defined. So, we restrict our attention to the behaviour of functions close to edges.

If $u \in C^\infty(M)$, then the pull-back of u under the blow-down mapping $[M;S] \to M$ belongs to $C^\infty([M;S])$. The converse is not true because a function $u \in C([M;S])$ can be lifted to a continuous function on M if and only if it is constant along the fibres of (1.3.3). Thus the store of C^∞ functions on $[M;S]$ is much richer than that on M.

On the other hand, denote by $\dot{C}^\infty(M)$ the subspace of $C^\infty(M)$ consisting of functions which are *flat* on S, i.e., vanish up to the infinite order on S. Then, the pull-back of $\dot{C}^\infty(M)$ under the blow-down mapping $[M;S] \to M$ coincides with the subspace of $C^\infty([M;S])$ consisting of functions which are flat on the boundary of $[M;S]$.

Let $\dot{C}^\infty([M;S],\Omega)$ stand for the space of C^∞ densities on $[M;S]$ which are flat on $\partial[M;S]$. From what has already been said, we deduce that the dual space

$$C^{-\infty}([M;S]) = \left(\dot{C}^\infty([M;S],\Omega)\right)' \tag{1.4.1}$$

is of great importance in the analysis on manifolds with edges.

Proposition 1.4.1 *The elements of* $C^{-\infty}([M;S])$ *are extendible distributions in the sense that, given any* C^∞ *manifold* $\mathcal{M}$ *extending* $[M;S]$*, the restriction mapping* $\mathcal{E}'_{[M;S]}(\mathcal{M}) \to C^{-\infty}([M;S])$ *is surjective with null space consisting of distributions supported on* $\partial[M;S]$*.*

Proof. Indeed, by the Hahn-Banach theorem, each continuous linear functional on $\dot{C}^\infty([M;S],\Omega)$ extends to the space $C^\infty([M;S],\Omega)$, of which the dual is just $\mathcal{E}'_{[M;S]}(\mathcal{M})$. Hence the surjectivity of the restriction mapping $\mathcal{E}'_{[M;S]}(\mathcal{M}) \to C^{-\infty}([M;S])$ is immediate. On the other hand, the characterisation of the null space of this mapping follows from the fact that $\dot{C}^\infty([M;S],\Omega)$ is the closure in $C^\infty([M;S],\Omega)$ of the subspace of C^∞ densities supported away from the boundary in $[M;S]$. This completes the proof.

□

In the next chapter we introduce a scale of weighted Sobolev spaces on $[M;S]$, with the distance to the boundary as a weight factor, which "connects" $\dot{C}^\infty([M;S])$ and $C^{-\infty}([M;S])$.

Note that similar arguments apply to define generalised sections of a vector bundle V over $[M;S]$, to be denoted $\dot{C}^\infty([M;S],V)$, $C^{-\infty}([M;S],V)$, etc.

Chapter 2

Sobolev Spaces with Asymptotics

2.1 *"Twisted" Sobolev spaces*

To organise a calculus of pseudodifferential operators on a manifold with edges, it proves to be helpful to reformulate the standard Sobolev spaces on the smooth part of the manifold in anisotropic terms, close to edges. Namely, when restricted to a fibred neighbourhood of an edge, they can be described as Sobolev spaces along the edge of functions taking their values in suitable Sobolev spaces in fibres. Moreover, some actions on the cotangent spaces to the edges turn out to be involved, a fact clarifying the use of "twisted" in the specification of the spaces.

We begin with a general concept of a "twisted" Sobolev space. For this purpose, let S be a C^∞ compact closed manifold of dimension q and $V \to S$ a Banach bundle over S, with typical fibre V_f.

Fix a strictly positive C^∞ function $\eta \to \langle \eta \rangle$ on $\mathbb{R}^q$, such that $\langle \eta \rangle = |\eta|$ for all $|\eta| \geq c$ with some $c > 0$. In this way we obtain what is known as a 'smoothed norm function'. The following property of such a function proves to be extremely useful in the sequel.

Lemma 2.1.1 (Peetre's inequality) *There is a constant $C > 0$ such that, given any $s \in \mathbb{R}$, we have*

$$\langle \eta \rangle^s \leq C^{|s|} \langle \eta - \theta \rangle^{|s|} \langle \theta \rangle^s \quad \textit{for all} \quad \eta, \theta \in \mathbb{R}^q.$$

Proof. The proof is elementary.

□

Denote by $\pi : T^*S \to S$ the cotangent bundle of S and by π^*V the pull-back of V under π. As defined in Section 1.2, π^*V is a Banach bundle over T^*S and the fibre of π^*V over a point $(p, \eta) \in T^*S$ is V_p, the fibre of V over $p \in S$. Though being independent of $\eta \in T^*_p(S)$ as a Banach space, the fibre H_p might bear a Banach structure varying along with η. Our next concern will be the behaviour of Banach structures allowed under varying η.

Proposition 2.1.2 *The following are equivalent:*

1) There is a family $(\|\cdot\|_\eta)_{\eta\in\mathbb{R}^q}$ of norms on V_f such that, given any non-zero $v \in V_f$, we have

$$\frac{1}{C}\left(\min \frac{\langle\eta\rangle}{\langle\theta\rangle}, \frac{\langle\theta\rangle}{\langle\eta\rangle}\right)^\epsilon \leq \frac{\|v\|_\eta}{\|v\|_\theta} \leq C\left(\max \frac{\langle\eta\rangle}{\langle\theta\rangle}, \frac{\langle\theta\rangle}{\langle\eta\rangle}\right)^\epsilon \quad \text{for all} \quad \eta, \theta \in \mathbb{R}^q,$$

with constants ϵ and C independent of v and η, θ.

2) There is a family $(\lambda(\eta))_{\eta\in\mathbb{R}^q}$ of isomorphisms of V_f such that

$$\|\lambda(\eta)\lambda^{-1}(\theta)\|_{L(V_f)} \leq C\left(\max \frac{\langle\eta\rangle}{\langle\theta\rangle}, \frac{\langle\theta\rangle}{\langle\eta\rangle}\right)^\epsilon \quad \text{for all} \quad \eta, \theta \in \mathbb{R}^q,$$

with constants ϵ and C independent of η, θ.

Proof. 1) $\Rightarrow$ 2) Pick an arbitrary isomorphism T of V_f. Given $\eta \in \mathbb{R}^q$, let

$$\lambda(\eta)v = \frac{\|v\|_\eta}{\|Tv\|}\, Tv \quad \text{for} \quad v \in V_f \setminus \{0\} \tag{2.1.1}$$

and $\lambda(\eta)0 = 0$. It is clear that $\lambda(\eta)$ is an isomorphism of V_f and

$$\lambda^{-1}(\eta)g = \frac{\|g\|}{\|T^{-1}g\|_\eta}\, T^{-1}g$$

unless $g \neq 0$. Hence it follows that

$$\|\lambda(\eta)\lambda^{-1}(\theta)g\| = \frac{\|T^{-1}g\|_\eta}{\|g\|}\, \frac{\|g\|}{\|T^{-1}g\|_\theta}\, \|g\|$$

for $v \neq 0$, and consequently

$$\|\lambda(\eta)\lambda^{-1}(\theta)\|_{L(V_f)} \leq \sup_{v\in V_f} \frac{\|v\|_\eta}{\|v\|_\theta}$$

showing 2).

2) $\Rightarrow$ 1) Given any $\eta \in \mathbb{R}^q$, set

$$\|v\|_\eta = \|\lambda(\eta)v\| \quad \text{for} \quad v \in V_f,$$

as follows from (2.1.1). If $\eta, \theta \in \mathbb{R}^q$, then

$$\begin{aligned} \frac{\|v\|_\eta}{\|v\|_\theta} &= \frac{\|\lambda(\eta)\lambda^{-1}(\theta)\lambda(\theta)v\|}{\|\lambda(\theta)v\|} \\ &\leq \|\lambda(\eta)\lambda^{-1}(\theta)\|_{L(V_f)} \end{aligned}$$

which establishes the right estimate in 1). Interchanging η and θ yields immediately the left estimate in 1), as desired.

□

Note that the family of isomorphisms $(\lambda(\eta))_{\eta\in\mathbb{R}^q}$ in 2) also satisfies

$$\|\lambda(\eta+\theta)\lambda^{-1}(\eta)\|_{L(V_f)} \leq c\,\langle\eta\rangle^{\epsilon} \quad \text{for all} \quad \eta,\theta\in\mathbb{R}^q, \tag{2.1.2}$$

as is easy to see from Lemma 2.1.1 (cf. [RST97]).

Example 2.1.3 As but one example of $(\lambda(\eta))_{\eta\in\mathbb{R}^q}$ satisfying 2) we show $\lambda(\eta)=\kappa^{-1}_{\langle\eta\rangle}$, $\eta\in\mathbb{R}^q$, where $(\kappa_\theta)_{\theta>0}$ is a group action on V_f.

□

We are now able to introduce "twisted" Sobolev spaces $H^s(S,\pi^*V)$, $s\in\mathbb{R}$, on S. Let $(\lambda(\eta))_{\eta\in\mathbb{R}^q}$ be a family of isomorphisms on the typical fibre V_f of V, fulfilling condition 2) of Proposition 2.1.2. This family goes to be involved into the definition of "twisted" Sobolev spaces. For this reason we speak on sections of the induced bundle π^*V over S rather than on sections of V. Though π^*V is a Banach bundle over T^*S, we may identify S with the zero section of T^*S and hence the designation $H^s(S,\pi^*V)$ will cause no perplexity.

We begin with a local situation where a section $u\in\mathcal{D}'(S,V)$ is supported in the domain of some chart on S and V is trivial over the domain.

Definition 2.1.4 *For $s\in\mathbb{R}$, the space $H^s(\mathbb{R}^q,\pi^*V)$ is defined to consist of all $u\in\mathcal{S}'(\mathbb{R}^q,V)$ such that $\mathcal{F}u\in L^1_{\mathrm{loc}}(\mathbb{R}^q,V)$ and*

$$\|u\|_{H^s(\mathbb{R}^q,\pi^*V)}=\left(\int_{\mathbb{R}^q}\langle\eta\rangle^{2s}\|\lambda(\eta)\mathcal{F}_{y\mapsto\eta}u\|^2_{V_f}d\eta\right)^{1/2}<\infty.$$

If we replace $\|\cdot\|_{V_f}$ by another equivalent norm on V_f, then we get an equivalent norm on $H^s(\mathbb{R}^q,\pi^*V)$. Furthermore, this norm is independent of the concrete choice of $\eta\mapsto\langle\eta\rangle$ modulo equivalence of norms.

The space $H^s(\mathbb{R}^q,\pi^*V)$ is easily checked to be a Banach space and even a Hilbert space provided V_f is Hilbert. For further properties of these spaces we refer the reader to [ST95], here confining ourselves merely to a motivation of "twisted" Sobolev spaces.

Example 2.1.5 Let $V_f=H^s(\mathbb{R}^{N-q})$. Consider the group action $(\kappa_\theta)_{\theta>0}$ on V_f given by

$$\kappa_\theta u(z)=\theta^{\frac{N-q}{2}}u(\theta z), \qquad u\in H^s(\mathbb{R}^{N-q}),$$

and set $\lambda(\eta)=\kappa^{-1}_{\langle\eta\rangle}$, for $\eta\in\mathbb{R}^q$. Then $H^s(\mathbb{R}^q,\pi^*H^s(\mathbb{R}^{N-q}))=H^s(\mathbb{R}^N)$ (cf. Schulze [Sch91, 3.1.1]). In order to extend this equality to Sobolev spaces on product manifolds, it is necessary to fall outside the limits of group actions on fibres and allow general actions meeting the conditions of Proposition 2.1.2.

□

To carry over the definition of "twisted" Sobolev spaces to the case of Banach bundles over S, we need the following lemma.

Lemma 2.1.6 *Suppose* $(\lambda(\eta))_{\eta\in\mathbb{R}^q}$ *is a family of isomorphisms on* V_f *satisfying*

$$\sup_{\eta\in\mathbb{R}^q} \|\lambda(\eta)T\lambda^{-1}(\eta)\|_{L(V_f)} < \infty \quad \text{for all} \quad T \in L(V_f). \tag{2.1.3}$$

Then, for each $T \in C^\infty(\mathbb{R}^q, L(V_f))$ *with values in invertible operators on* V_f*, the norms of* u *and* Tu *in* $H^s(\mathbb{R}^q, \pi^*V)$ *are equivalent on sections* u *supported in a fixed compact subset of* $\mathbb{R}^q$*.*

Proof. Indeed, the condition (2.1.3) implies that both the operator-valued symbol $a(y,\eta) = T(y)$ and its pointwise inverse belong to the symbol space $\mathcal{S}^0(T^*\mathbb{R}^q; \pi^*V)$ (cf. Section 3.2 below). As a pseudodifferential operator with a symbol in this space extends to a continuous mapping $H^s_{\text{comp}}(\mathbb{R}^q, \pi^*V) \to H^s_{\text{loc}}(\mathbb{R}^q, \pi^*V)$, for each $s \in \mathbb{R}$, the lemma follows. See also Proposition 5 in [Sch91, 3.1.2].

□

It is worth pointing out that the action of Example 2.1.5 fulfils condition (2.1.3) while its inverse does not.

Pick a covering (O'_ι) of S by open sets, each O'_ι lying in the domain of some chart $y = h'_\iota(p)$ on S and V being trivial over O'_ι, and choose a subordinate partition (φ_ι) of unity on S. Moreover, fix trivialisations $\delta_\iota \colon V|_{O'_\iota} \to O'_\iota \times V_f$ of the bundle V over O'_ι and denote by t_ι the composition of δ_ι with the projection to the second factor in $O'_\iota \times V_f$.

Given $s \in \mathbb{R}$, the space $H^s(S, \pi^*V)$ is defined to consist of all sections $u \in \mathcal{D}'(S, V)$ such that $(h'_\iota)_* t_\iota \varphi_\iota u \in H^s(\mathbb{R}^q, \pi^*(\mathbb{R}^q \times V_f))$ for each ι. The norm in this space is defined by

$$\|u\|_{H^s(S,\pi^*V)} = \left(\sum_\iota \int \langle\eta\rangle^{2s} \|\lambda(\eta)\mathcal{F}_{y\mapsto\eta}(h'_\iota)_* t_\iota \varphi_\iota u\|^2_{V_f} d\eta \right)^{1/2}.$$

Proposition 2.1.7 *If* $(\lambda(\eta))_{\eta\in\mathbb{R}^q}$ *satisfies condition* (2.1.3)*, then the space* $H^s(S, \pi^*V)$ *is independent of which partition of unity on* S*, local coordinates on* S *and trivialisations of* V *we choose to define it.*

Proof. It suffices to use Lemma 2.1.6 together with the fact that the space $H^s_{\text{loc}}(\mathbb{R}^q, \pi^*(\mathbb{R}^q \times V_f))$ is invariant under changes of coordinates in $\mathbb{R}^q$ (cf. Behm [Beh95, 1.2.7]).

□

As is shown in [FST98b], we have $H^s(S, \pi^*H^s(F)) = H^s(S \times F)$ for a suitable family $(\lambda(\eta))_{\eta\in\mathbb{R}^q}$ of isomorphisms on $H^s(F)$, F being a C^∞ compact closed manifold.

2.2 *Sobolev spaces on manifolds with edges*

Let (M,S) be a C^∞ closed manifold with edges. By abuse of language we introduce Sobolev spaces on M in case S consists of merely one connected component of dimension q. Generalisation to the case of a finite number of components is fairly straightforward.

As mentioned above, the analysis will take place on an associated stretched manifold $\mathcal{M} = [M;S]$ rather than on M. This latter is a smooth manifold with boundary, the boundary being the total space of a fibration $b\colon \partial\mathcal{M} \to S$ with a compact typical fibre X. Namely,

$$\begin{array}{ccc} b^{-1}(O'_\iota) & \xrightarrow{\delta_\iota} & O'_\iota \times X \\ \Big\downarrow b & \swarrow \text{proj} & \\ O'_\iota & & \end{array}$$

(cf. (1.3.3)), for a finite covering (O'_ι) of S.

We now invoke the fact that each manifold with compact boundary has a 'collar', i.e., the boundary $\partial\mathcal{M}$ possesses a neighbourhood O in $\mathcal{M}$ which is diffeomorphic to $\partial\mathcal{M} \times \bar{\mathbb{R}}_+$. When combined with boundary fibration (2.2.1), this gives rise to a fibration of the whole manifold $\mathcal{M}$ near the boundary. Namely, letting δ denote a diffeomorphism of a neighbourhood of $\partial\mathcal{M}$ onto $\partial\mathcal{M} \times \bar{\mathbb{R}}_+$, we arrive at a fibre bundle

$$\begin{array}{ccc} \delta^{-1}\left(b^{-1}(O'_\iota) \times \bar{\mathbb{R}}_+\right) & \xrightarrow{\Delta_\iota} & O'_\iota \times \left(\bar{\mathbb{R}}_+ \times X\right) \\ \Big\downarrow F & \swarrow \text{proj} & \\ O'_\iota & & \end{array} \qquad (2.2.1)$$

with base S and typical fibre $\bar{\mathbb{R}}_+ \times X$. Here,

$$\begin{array}{rcl} F & = & b \circ \text{proj} \circ \delta; \\ \Delta_\iota & = & (\delta_\iota \times 1) \circ \delta. \end{array}$$

In turn fibre bundle (2.2.1) enables us to introduce various Banach bundles over S relevant to function spaces on $\mathcal{M}$. Namely, given a function space V_f on the semicylinder $\bar{\mathbb{R}}_+ \times X$, we might consider a fibre bundle V over S whose fibre over a point $p \in O'_\iota$ consists of all functions u on the fibre $F^{-1}(p)$, such that the push-forward $(\Delta_\iota)_* u$ of u under Δ_ι belongs to V_f. Were the space V_f invariant under the diffeomorphisms $\Delta_{j\iota}(p)$ of $\bar{\mathbb{R}}_+ \times X$ induced by $\Delta_j \Delta_\iota^{-1}$, this would allow us to conclude that V is a fibre bundle over S with typical fibre V_f.

We thus turn to Sobolev spaces on $\bar{\mathbb{R}}_+ \times X$ to be used. These are $H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)$, for $s, \gamma \in \mathbb{R}$. We begin with the case of non-negative integer s.

Let $s \in \mathbb{Z}_+$ and $\gamma \in \mathbb{R}$. Denote by $\mathcal{H}^{s,\gamma,0}(\bar{\mathbb{R}}_+ \times X)$ the set of all distributions u on $\bar{\mathbb{R}}_+ \times X$ whose derivatives up to order s are locally integrable and satisfy

$$\|u\|^2_{\mathcal{H}^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X)} := \int_0^\infty \frac{1}{t^{2\gamma}} \sum_{j+A\le s} \|(tD_t)^j u\|^2_{H^A(X)} \frac{dt}{t} < \infty.$$

This norm is intended to control the behaviour of functions near $t=0$. On the other hand, away from $t=0$ we wish to model the pull-back of the usual Sobolev spaces under polar coordinates (1.3.2). It proves to be helpful to incorporate an additional weight factor t^δ, $\delta\in\mathbb{R}$, into the norm expression. Thus, for $s\in\mathbb{Z}_+$ and $\delta\in\mathbb{R}$, we define $\mathcal{H}^{s,0,\delta}(\bar{\mathbb{R}}_+\times X)$ to consist of all distributions u on $\mathbb{R}_+\times X$ whose derivatives up to order s are locally integrable and make finite the norm

$$\|u\|^2_{\mathcal{H}^{s,0,\delta}(\bar{\mathbb{R}}_+\times X)} := \int_0^\infty t^{2\delta} \sum_{j+A\le s} \frac{1}{t^{2(j+A)}} \|(tD_t)^j u\|^2_{H^A(X)} t^n dt,$$

n being the dimension of X.

The definitions then extend to arbitrary $s\in\mathbb{R}$ by interpolation and duality (cf. [ST95]).

The spaces $\mathcal{H}^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X)$ allow an equivalent formulation in terms of the Mellin transform along $\mathbb{R}_+$. The analysis on the cone employs the Mellin transform and weight factors only near the vertex which corresponds to a neighbourhood of $t=0$. The weight factor $t^{-2\gamma}$ affects the space also for $t\to\infty$. It will be advantageous to introduce another variant of spaces on $\bar{\mathbb{R}}_+\times X$ that refers to the Mellin transform and to weight factors only close to $t=0$. For this purpose, we multiply $\mathcal{H}^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X)$ by a cut-off function $\omega\in C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+)$ and add $(1-\omega)\mathcal{H}^{s,0,\delta}(\bar{\mathbb{R}}_+\times X)$. Thus, for $s,\gamma,\delta\in\mathbb{R}$, we set

$$H^{s,\gamma,\delta}(\bar{\mathbb{R}}_+\times X) = \omega\,\mathcal{H}^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X) + (1-\omega)\,\mathcal{H}^{s,0,\delta}(\bar{\mathbb{R}}_+\times X), \qquad (2.2.2)$$

where the right-hand side is regarded as a non-direct sum of Fréchet spaces. For $\delta=0$, we abbreviate $H^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X)$ to $H^{s,\gamma}(\bar{\mathbb{R}}_+\times X)$.

The most important property of spaces (2.2.2) is that C^∞ functions of compact support are dense in $H^{s,\gamma,\delta}(\bar{\mathbb{R}}_+\times X)$, for each $s,\gamma,\delta\in\mathbb{R}$.

It is easy to see that the spaces $H^{s,\gamma,\delta}(\bar{\mathbb{R}}_+\times X)$ are invariant under those diffeomorphisms of $\bar{\mathbb{R}}_+\times X$ which behave properly at $t=\infty$ (cf. Schrohe [Sch96]). Hence we may apply the above scheme to the space $V_f = H^{s,\gamma,\delta}(\bar{\mathbb{R}}_+\times X)$, thus arriving at a Banach bundle over S to be denoted $H^{s,\gamma,\delta}(F^{-1}(\cdot))$.

We endow the fibre $H^{s,\gamma,\delta}(\bar{\mathbb{R}}_+\times X)$ with the group action $(\kappa_\theta)_{\theta>0}$ given by $\kappa_\theta u\,(t,x) = \theta^{\frac{1+n}{2}} u(\theta t,x)$, and set $\lambda(\eta)=\kappa^{-1}_{\langle\eta\rangle}$, for $\eta\in\mathbb{R}^q$ (cf. Example 2.1.5).

The 'twisted' Sobolev space $H^s(S,\pi^* H^{s,\gamma}(F^{-1}(\cdot)))$ bears informations on the behaviour of functions close to the edge S on M. It gives rise to weighted Sobolev spaces on M.

Proposition 2.2.1 *Assume that u is a distribution with a compact support in $O \setminus \partial\mathcal{M}$, O being a collar neighbourhood of the boundary. Then $u \in H^s(S, \pi^* H^{s,\gamma,\delta}(F^{-1}(\cdot)))$, for $s, \gamma, \delta \in \mathbb{R}$, if and only if $u \in H^s(\mathcal{M})$.*

Proof. It suffices to combine Propositions 2.1.4 and 2.1.20 in [ST95]. □

From this proposition we deduce that the localisation of the space $H^s(S, \pi^* H^{s,\gamma}(F^{-1}(\cdot)))$ to each compact subset of $O \setminus \partial\mathcal{M}$ coincides with that of the usual Sobolev space $H^s(\mathcal{M})$. Hence we can paste together these spaces just in the same way as in (2.2.2).

We say that a distribution on $\mathcal{M} \setminus \partial\mathcal{M}$ is supported close to the boundary of $\mathcal{M}$ if it vanishes away from a compact subset of O. Fix a C^∞ function ω on $\mathcal{M}$ that is supported close to the boundary and is equal to 1 in a smaller neighbourhood of ∂M. Then, each distribution u in the interior of $\mathcal{M}$ can be written as $u = u_1 + u_2$ where $u_1 = \omega u$ is supported close to the boundary and the support of $u_2 = (1 - \omega)u$ does not meet the boundary.

Definition 2.2.2 *For $s, \gamma \in \mathbb{R}$, the space $H^{s,\gamma}(\mathcal{M})$ is defined to consist of all distributions u on $\mathcal{M} \setminus \partial\mathcal{M}$ such that $\omega u \in H^s(S, \pi^* H^{s,\gamma}(F^{-1}(\cdot)))$ and $(1 - \omega)u \in H^s(\mathcal{M})$.*

By Proposition 2.2.1, the space $H^{s,\gamma}(\mathcal{M})$ is independent of which cut-off function ω we choose to define it. It is a Banach space and even a Hilbert space under the norm

$$\|u\|_{H^{s,\gamma}(\mathcal{M})} = \left(\|\omega u\|^2_{H^s(S, \pi^* H^{s,\gamma}(F^{-1}(\cdot)))} + \|(1 - \omega)u\|^2_{H^s(\mathcal{M})} \right)^{1/2} \quad (2.2.3)$$

which does depend on ω modulo equivalence.

The crucial point of Definition 2.2.2 is the particular choice of the family of isomorphisms $(\lambda(\eta))_{\eta \in \mathbb{R}^q}$ on the typical fibre $H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)$. For the definition to be correct, this family should be so chosen that the 'twisted' Sobolev space $H^s(S, \pi^* H^{s,\gamma}(F^{-1}(\cdot)))$ agree with the usual Sobolev space $H^s(\mathcal{M})$ on compact subsets of $O \setminus \partial\mathcal{M}$. No further conditions are required. Yet another choice of this family is discussed in Section 2.4.

As the spaces $H^{s,\gamma}(\mathcal{M})$ are invariant under multiplication by C^∞ functions, the definition of weighted Sobolev spaces on $\mathcal{M}$ carries over to sections of smooth vector bundles on $\mathcal{M}$. We will write $H^{s,\gamma}(\mathcal{M}, V)$, etc. for the corresponding spaces, V being a C^∞ bundle over $\mathcal{M}$.

2.3 *The nature of asymptotics*

When constructing a parametrix to an elliptic operator close to an edge on M, one reformulates the operator as a Fourier pseudodifferential operator along the edge with a symbol taking its values in the 'cone algebras' on the fibres. Then, the parametrix construction starts by looking for a precise parametrix in each fibre. Recall that the fibres are locally identified with the semicylinder $\mathbb{R}_+ \times X$ and the operators on the fibres are of Fuchs-type in $t \in \mathbb{R}_+$. The parametrices of such operators are known to bear so-called 'conormal' asymptotic expansions at $t = 0$. The asymptotics of solutions to the Fuchs-type differential equations on the semiaxis are simply pull-backs of Euler exponential solutions to ordinary differential equations with constant coefficients under the mapping $\mathbb{R}_+ \to \mathbb{R}$ given by $t \mapsto r = -\log t$. When having given Sobolev spaces with asymptotics in the fibres of M close to S, we may define Sobolev spaces with asymptotics on M just in the same way as in Definition 2.2.2. We therefore restrict our attention to spaces with asymptotics on the semicylinder $\mathbb{R}_+ \times X$.

In the Mellin transform picture, the asymptotics of solutions to elliptic equations of Fuchs type on $\mathbb{R}_+ \times X$ are related to analytic functionals of the form

$$u \mapsto \sum_{\mu=1}^{M} \sum_{j=0}^{j_\mu} f_{\mu j} \frac{\partial^j u}{\partial z^j}(p_\mu), \quad u \in \mathcal{A}((p_\mu)),$$

where (p_μ) are some points in the complex plane, j_μ non-negative integers, and $f_{\mu j}$ smooth functions on X. Such functionals are carried by discrete sets (p_μ) which, in turn, prove to coincide with the spectrums of the conormal symbols of the corresponding equations. The spectrum in a fibre may vary along the edge S which causes serious difficulties when one works in the framework of discrete asymptotic types. However, the 'general position' corresponds to the case where the spectrum varies within a closed set away from the weight line in the complex plane, whose intersection with each horizontal strip of finite width remains compact while the fibre varies along the edge. To handle such asymptotics, Schulze [Sch88b] introduced continuous asymptotic types.

For a compact set $K \subset \mathbb{C}$, we denote by $\mathcal{A}'_K$ the space of all analytic functionals having K as a *carrier*. If the complement of K is connected, then $\mathcal{A}'_K \cong \mathcal{A}(K)'$, $\mathcal{A}(K)$ being the space of analytic functions on K with the usual inductive limit topology.

The space $\mathcal{A}'_K$ is known to be a nuclear Fréchet space. Therefore, we may consider the space $\mathcal{A}'_K(\mathbb{C}, C^\infty(X)) = \mathcal{A}'_K \otimes_\pi C^\infty(X)$ of analytic functionals with values in $C^\infty(X)$ carried by K.

Pick a cut-off function $\omega \in C^\infty_{\text{comp}}(\bar{\mathbb{R}}_+)$, i.e., any function with $\omega(t) \equiv 1$ near $t = 0$.

Suppose K lies below from a weight line $\Gamma_{-\gamma} = \{z \in \mathbb{C} : \Im z = -\gamma\}$, i.e., $\Im z < -\gamma$ for all $z \in K$. Then, for any $f \in \mathcal{A}'_K(\mathbb{C}, C^\infty(X))$, the potential $\omega(t)\langle f(x), t^{iz}\rangle$ is easily verified to be in $H^{\infty,\gamma}(\bar{\mathbb{R}}_+ \times X)$.

We want to introduce subspaces of $H^{s,\gamma}(\mathbb{R}_+ \times X)$ consisting of the functions which have a gain in the weight up to elements of certain spaces of potentials. For this purpose, we need some preliminaries.

A *weight datum* $w = (\gamma, (-l, 0])$ consists of a number $\gamma \in \mathbb{R}$ and an interval $(-l, 0]$ on the real axis, with $0 < l \leq \infty$. By abuse of language, we will consider only weight data with finite weight intervals, i.e., $l < \infty$, referring the reader to [ST98a] for the general case.

A set $\sigma \subset \mathbb{C}$ is called a *carrier of asymptotics* if σ is closed, has connected complement and if the portion of σ in each strip $\{c' \leq \Im z \leq c''\}$ is compact.

By an *asymptotic type* related to a weight datum $w = (\gamma, (-l, 0])$ is meant any pair as $= (\sigma, \Sigma)$, where

- σ is a carrier of asymptotics contained in the strip $-\gamma - l \leq \Im z < -\gamma$; and
- Σ is a closed subspace of $\mathcal{A}'_\sigma(\mathbb{C}, C^\infty(X))$.

Let us denote by As(w) the set of all asymptotic types related to w. We are now in a position to introduce our spaces with asymptotics.

Definition 2.3.1 *For an asymptotic type* as $= (\sigma, \Sigma)$ *related to a weight datum* $w = (\gamma, (-l, 0])$*, the space* $H^{s,\gamma}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ *is defined to consist of all* $u \in H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)$ *such that* $u - \omega \langle f, t^{iz} \rangle \in H^{s,\gamma+l-0}(\bar{\mathbb{R}}_+ \times X)$*, for some* $f \in \Sigma$.

The elements of

$$H^{s,\gamma+l-0}(\bar{\mathbb{R}}_+ \times X) = \bigcap_{\epsilon > 0} H^{s,\gamma+l-\epsilon}(\bar{\mathbb{R}}_+ \times X)$$

may be regarded as being *flat* of order $l - 0$ relative to the weight γ.

Note that Definition 2.3.1 is independent of the particular choice of the cut-off function ω.

To topologise the space $H^{s,\gamma}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$, we denote by $\mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ the space of all potentials $u(t,x) = \omega(t)\langle f(x), t^{iz}\rangle$ in $H^{\infty,\gamma}(\bar{\mathbb{R}}_+ \times X)$, such that $f \in \Sigma$.

By the Köthe-Grothendieck duality, $\mathcal{A}'_\sigma(\mathbb{C}, C^\infty(X)) \overset{\text{top.}}{\cong} \mathcal{A}(\hat{\mathbb{C}} \backslash \sigma, C^\infty(X))$ has a Fréchet topology which is nuclear. Then, we endow the subspace Σ by the induced topology. Moreover, the mapping $\Sigma \to \mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ given by $f \mapsto \omega(t)\langle f(x), t^{iz}\rangle$ is easily checked to be injective and surjective. Thus, we can give $\mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ the topology induced by this algebraic isomorphism.

We make $H^{s,\gamma}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ a Fréchet space by endowing it with the topology of the non-direct sum of Fréchet spaces,

$$H^{s,\gamma}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X) = \mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X) + H^{s,\gamma+l-0}(\bar{\mathbb{R}}_+ \times X). \tag{2.3.1}$$

Moreover, if $\sigma \cap \Gamma_{-\gamma-l} = \emptyset$, then the sum (2.3.1) is direct.

If $u(t,x) = \omega(t)\,\langle f(x), t^{iz}\rangle$ is an element of $\mathcal{A}_{as}(\bar{\mathbb{R}}_+ \times X)$, with as $= (\sigma, \Sigma)$, then

$$\kappa_\theta u\,(t,x) = \omega(\theta t)\,\langle \theta^{\frac{1+n}{2}+iz} f(x), t^{iz}\rangle, \quad \theta > 0.$$

For each fixed $\theta > 0$, the function $\omega(\theta t)$ is again of cut-off nature and so does not affect the property of being in $\mathcal{A}_{as}(\bar{\mathbb{R}}_+ \times X)$ modulo elements of $H^{\infty,\gamma+l-0}(\bar{\mathbb{R}}_+ \times X)$. Further,

$$\theta^{\frac{1+n}{2}+iz} = \exp\left(\left(\frac{1+n}{2} + iz\right)\log\theta\right)$$

is an entire function of z, hence the product $\theta^{\frac{1+n}{2}+iz} f(x)$ is again an analytic functional with values in $C^\infty(X)$ and carrier σ. Whether this functional belongs to Σ or not, would be a property of Σ itself. A kind of this property is that Σ is invariant under multiplication by entire functions. If such is the case, then $\kappa_\theta u \in \mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ modulo $H^{\infty,\gamma+l-0}(\bar{\mathbb{R}}_+ \times X)$, for all $\theta > 0$. Hence it follows that the space $H^{s,\gamma}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ is invariant under the group action $(\kappa_\theta)_{\theta>0}$.

On the other hand, it is obvious that the representation of a function in the form $u(t,x) = \omega(t)\,\langle f(x), t^{iz}\rangle$ modulo $H^{s,\gamma+l-0}(\bar{\mathbb{R}}_+ \times X)$ depends on the particular choice of coordinates. We next look for conditions on an asymptotic type as $= (\sigma,\Sigma)$, such that the space $H^{s,\gamma}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ is invariant under diffeomorphisms of $\bar{\mathbb{R}}_+ \times X$.

First, for any $p \in \mathbb{C}$, we have

$$\begin{aligned} t^p\, u(t,x) &= \omega(t)\,\langle f(x), t^{i(z-ip)}\rangle \\ &= \omega(t)\,\langle (w \mapsto w+ip)^* f(x),\, t^{iw}\rangle \end{aligned}$$

where $(w \mapsto w+ip)^* f$ is the pull-back of f under the biholomorphism $w \mapsto w+ip$ of the complex plane. Thus, the product is of asymptotic type $(w \mapsto w+ip)^*\mathrm{as} = (\sigma - \mathrm{ip},\, (\mathrm{w} \mapsto \mathrm{w}+\mathrm{ip})^*\Sigma)$.

We say that an asymptotic type as $= (\sigma,\Sigma)$ satisfies the *shadow condition* if, for each $\nu = 0,1,\ldots$, the intersection of $\sigma - i\nu$ with the strip $-\gamma - l \le \Im z < -\gamma$ belongs to σ and the restriction of $(w \mapsto w+i\nu)^*\Sigma$ to this intersection is a subspace of Σ.

If 'as' satisfies the shadow condition, then the space $H^{s,\gamma}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ is invariant under multiplication by functions of $C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+)$. However, this condition is not yet sufficient to ensure the invariance under changes of coordinates. Indeed, a diffeomorphism of $\bar{\mathbb{R}}_+ \times X$ is of the form

$$\begin{cases} \tau = \tau(t,x), \\ \chi = \chi(t,x) \end{cases}$$

in a neighbourhood of $t = 0$, where $\tau(t,x) = te^{\phi(t,x)}$ for some C^∞ function ϕ on $[0,R) \times X$. Hence a potential $\omega(\tau)\langle f(\chi),\, \tau^{iz}\rangle$ pulls back to

$$\begin{aligned} &\omega\left(te^{\phi(t,x)}\right)\left\langle e^{iz\phi(t,x)} f(\chi(t,x)),\, t^{iz}\right\rangle \\ &\quad = \omega\left(te^{\phi(t,x)}\right)\sum_{\nu=0}^{N-1} t^\nu \left\langle \frac{1}{\nu!}\,\partial_t^\nu\, e^{iz\phi(t,x)} f(\chi(t,x))\Big|_{t=0},\, t^{iz}\right\rangle \end{aligned}$$

modulo $H^{\infty,\gamma+N}(\bar{\mathbb{R}}_+ \times X)$, for any $N = 0,1,\ldots$. Note that the function $\omega\left(te^{\phi(t,x)}\right)$ is of cut-off nature near $t = 0$. Thus, when taking $N \ge l$, we should be able to carry out the operations $f \mapsto \partial_t^\nu \, e^{iz\phi(t,x)} f(\chi(t,x)) \,|_{t=0}$ within the space Σ. Such a property of Σ has to be postulated.

Example 2.3.2 Let Σ' be a closed subspace of $\mathcal{A}'_\sigma$ invariant under multiplication by entire functions. Then, $\Sigma = \Sigma' \otimes_\pi C^\infty(X)$ meets all the conditions above.

□

Applying the general construction of Section 2.1 to $V_f = H^{s,\gamma}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ yields 'twisted' Sobolev spaces with asymptotics $H^s(S, \pi^* H^{s,\gamma}_{\mathrm{as}}(F^{-1}(\cdot)))$, for $s, \gamma \in \mathbb{R}$ and an asymptotic type 'as'. In turn, gluing together these spaces with the usual Sobolev spaces $H^s(\mathcal{M})$ away from the boundary leads us to Sobolev spaces with asymptotics on M, $H^{s,\gamma}_{\mathrm{as}}(\mathcal{M})$ (cf. Definition 2.2.2).

For natural asymptotic types 'as', the spaces $H^{s,\gamma}_{\mathrm{as}}(\mathcal{M})$ are invariant under multiplication by smooth functions on $\mathcal{M}$. A familiar argument then shows that they extend as well to sections of smooth vector bundles over $\mathcal{M}$. In the sequel, $H^{s,\gamma}_{\mathrm{as}}(\mathcal{M}, V)$ stands for a Sobolev space with asymptotics of sections of a C^∞ bundle V over $\mathcal{M}$.

2.4 *Invariance*

Let (M, S) be a C^∞ closed manifold with an edge S and let $\mathcal{M} = [M; S]$ be an associated stretched manifold. As described above, the boundary of $\mathcal{M}$ has the structure of a fibre bundle over S, with a blow-down mapping $b : \partial\mathcal{M} \to S$. The typical fibre of b is a C^∞ compact closed manifold X. It follows that $\mathcal{M}$ bears a distinguished class of smooth vector fields which are unrestricted in the interior and which lie tangent at the boundary to the leaves of the fibration. If denoting this space by $\mathcal{V}_b$, b standing for the blow-down mapping, we will be aimed at finding an appropriate microlocalisation of $\mathcal{V}_b$ which contains all differential operators on $\mathcal{M}$ manufactured from elements of $\mathcal{V}_b$, and parametrices to elliptic differential operators. The operators in question are intended to act continuously on the weighted Sobolev spaces $H^{s,\gamma}(\mathcal{M})$ introduced in Section 2.3. From this we deduce that, when discussing the invariance of the spaces $H^{s,\gamma}(\mathcal{M})$ under changes of coordinates, we must restrict our attention to those mappings $f : \mathcal{M} \to \mathcal{M}$ which preserve the class $\mathcal{V}_b$. In other words, these mappings should obey the boundary fibration $b \colon \partial\mathcal{M} \to S$ in the sense that there is a commutative diagram

$$\begin{array}{ccc} \partial\mathcal{M} & \xrightarrow{f} & \partial\mathcal{M} \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} \\ S & \xrightarrow{f_S} & S \end{array} \qquad (2.4.1)$$

with f_S a mapping of S. This diagram means simply that f preserves the fibres of b and thus restricts to a mapping of S. Evidently, if f is smooth, then so is f_S.

Let us now turn to describing the vector fields of $\mathcal{V}_b$ and mappings f with property (2.4.1) in local coordinates of $\partial\mathcal{M}$. We restrict ourselves to a collar neighbourhood O of $\partial\mathcal{M}$ which is diffeomorphic to $\partial\mathcal{M} \times \bar{\mathbb{R}}_+$. Then, we may take $(y, t, x) \in S \times \bar{\mathbb{R}}_+ \times X$ as local coordinates in O, where y restricts to $\partial\mathcal{M}$ to coordinates lifted from S, t is a defining function for $\partial\mathcal{M}$ and x restricts to $\partial\mathcal{M}$ to coordinates on each fibre $b^{-1}(y) = X$. In terms of these coordinates, $\mathcal{V}_b$ is a complete natural Lie algebra on $\mathcal{M}$, with a local basis at a point of O, as a C^∞ module,

$$tD_y, \ tD_t, \ D_x. \tag{2.4.2}$$

Given open sets $O_1, O_2 \in O$, a C^∞ mapping $f : O_1 \to O_2$ can be written in terms of the coordinates as $(y, t, x) \mapsto (v, \tau, \chi)$ where v, τ and χ are smooth mappings of $(y, t, x) \in O_1$.

Lemma 2.4.1 *Each mapping* $f : O_1 \to O_2$ *meeting* (2.4.1) *is of the form*

$$\begin{cases} v &= f_S(y) + t\psi(y, t, x), \\ \tau &= t \exp(\phi(y, t, x)), \\ \chi &= \chi(y, t, x), \end{cases} \tag{2.4.3}$$

with f_S, ψ, ϕ *and* χ *smooth for* $t \in [0, R)$, $R > 0$.

Proof. Indeed, if f maps $O_1 \cap \partial\mathcal{M}$ to $\partial\mathcal{M}$, then a simple analysis yields $\tau(y, t, x) = te^{\phi(y,t,x)}$, where ϕ is a C^∞ function of $(y, t, x) \in O_1$ and t bounded. On the other hand, if f preserves the fibres of the blow-down mapping b, then $v(y, t, x) = f_S(y) + t\psi(y, t, x)$, with f_S and ψ smooth on the same set, as is clear from the Taylor formula. This completes the proof. □

By the above, the pull-back of a differential operator in τD_v, D_χ and τD_τ under a diffeomorphism $O_1 \to O_2$ of the form (2.4.3) is a differential operator in the vector fields (2.4.2). In other words, the algebra $\mathcal{V}_b$ is invariant under those diffeomorphisms of $\mathcal{M}$ which satisfy (2.4.1).

To control the behaviour of functions u on O under actions of vector fields (2.4.2), the natural weighted Sobolev spaces are described as follows. We first take u with a support which projects to a coordinate patch on S. When identifying this patch with an open set $\Omega \subset \mathbb{R}^q$, we are thus reduced to functions given on the stretched wedge $\mathcal{W} = \mathbb{R}^q \times \bar{\mathbb{R}}_+ \times X$.

Suppose $s \in \mathbb{Z}_+$ and $\gamma \in \mathbb{R}$. Denote by $\mathcal{H}^{s,\gamma,0}(\mathcal{W})$ the set of all distributions u on $\mathbb{R}^q \times \mathbb{R}_+ \times X$ whose derivatives up to order s are locally integrable and satisfy

$$\|u\|^2_{\mathcal{H}^{s,\gamma,0}(\mathcal{W})} := \int_{\mathbb{R}^q} \int_0^\infty \frac{1}{t^{2\gamma}} \sum_{|\beta|+j+A \le s} \|(tD_y)^\beta (tD_t)^j u\|^2_{H^A(X)} \frac{dt}{t} \frac{dy}{t^q} < \infty.$$

The factor t^{-q} in front of dy is included by purely aesthetic reasons to keep in mind the relation to the usual Sobolev spaces under the change of variables

$$\begin{array}{rcl} t & \mapsto & -\log t, \\ x & \mapsto & x, \\ y & \mapsto & t^{-1}y. \end{array}$$

The norms $\|\cdot\|_{\mathcal{H}^{s,\gamma,0}(\mathcal{W})}$ allow one to describe the behaviour of functions close to $t=0$. However, the weight factor $t^{-2\gamma}$ affects the space also for $t\to\infty$. For this reason, we single out the contribution of such a factor at $t=\infty$ and glue together resulting spaces, just as in (2.2.2).

Namely, for $s\in\mathbb{Z}_+$ and $\delta\in\mathbb{R}$, let $\mathcal{H}^{s,0,\delta}(\mathcal{W})$ consist of all distributions u on $\mathcal{W}$, such that the derivatives of u up to order s are locally integrable and make finite the norm

$$\|u\|^2_{\mathcal{H}^{s,0,\delta}(\mathcal{W})} := \int_{\mathbb{R}^q}\int_0^\infty t^{2\delta} \sum_{|\beta|+j+A\le s} \frac{1}{t^{2(|\beta|+j+A)}} \|(tD_y)^\beta (tD_t)^j u\|^2_{H^A(X)} t^n dt dy,$$

n being the dimension of X.

To integer $s<0$, the definitions of $\mathcal{H}^{s,\gamma,0}(\mathcal{W})$ and $\mathcal{H}^{s,0,\delta}(\mathcal{W})$ extend by duality and then, to fractional s, by complex interpolation.

We now fix a cut-off function $\omega\in C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+)$ near $t=0$. For $s,\gamma,\delta\in\mathbb{R}$, set

$$H^{s,\gamma,\delta}(\mathcal{W}) = \omega\,\mathcal{H}^{s,\gamma,0}(\mathcal{W}) + (1-\omega)\,\mathcal{H}^{s,0,\delta}(\mathcal{W}) \qquad (2.4.4)$$

where the right side is thought of as a non-direct sum of Fréchet spaces. Moreover, we abbreviate $H^{s,\gamma,0}(\mathcal{W})$ to $H^{s,\gamma}(\mathcal{W})$.

Once again C^∞ functions of compact support are dense in $H^{s,\gamma,\delta}(\mathcal{W})$, for each $s,\gamma,\delta\in\mathbb{R}$.

Lemma 2.4.2 *Given any $s,\gamma,\delta\in\mathbb{R}$, the norm in $H^{s,\gamma,\delta}(\mathcal{W})$ is equivalent to*

$$\|u\|_{H^{s,\gamma,\delta}(\mathcal{W})} \sim \|\omega u\|_{\mathcal{H}^{s,\gamma,0}(\mathcal{W})} + \|(1-\omega)u\|_{\mathcal{H}^{s,0,\delta}(\mathcal{W})}.$$

Proof. Cf. (2.2.1) in [ST95].

□

We will touch only a few aspects of the theory of spaces $H^{s,\gamma,\delta}(\mathcal{W})$, for $\delta=0$. A remarkable feature of these spaces is the invariance under diffeomorphisms of the wedge $\mathcal{W}$ which fulfil (2.4.1).

Proposition 2.4.3 *Let δ be a diffeomorphism of $\mathcal{W}$ of the form* (2.4.3). *Then $\delta^* u\in H^{s,\gamma}(\mathcal{W})$ for each $u\in H^{s,\gamma}(\mathcal{W})$ vanishing away from a compact subset of $\mathcal{W}$.*

Proof. The proof consists of a straightforward verification as in Theorem 1.2.3 of [ST95].

□

From (2.4.4) and Proposition 2.2.1 we deduce immediately that the spaces $H^s(\mathbb{R}^q,\pi^*H^{s,\gamma}(\bar{\mathbb{R}}_+\times X))$ and $H^{s,\gamma}(\mathcal{W})$ agree on compact sets away from the boundary in $\mathcal{W}$. Moreover, this is still true on all subsets of $\mathcal{W}$

of the form $\{t \geq \epsilon\}$, where $\epsilon > 0$ (cf. Proposition 2.1.20 *ibid.*). As for the behaviour of functions at $t = 0$, simple examples show that the spaces $H^s(\mathbb{R}^q, \pi^* H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X))$ and $H^{s,\gamma}(\mathcal{W})$ are different. Our next objective is to rewrite spaces (2.4.4) as 'twisted' Sobolev spaces on the edge $S = \mathbb{R}^q$.

Proposition 2.4.4 *For each $s \in \mathbb{Z}_+$ and $\gamma, \delta \in \mathbb{R}$, it follows that*

$$H^{s,\gamma,\delta}(\mathcal{W}) \overset{\text{top.}}{\cong} H^s(\mathbb{R}^q, \pi^* H^{s,\gamma+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+ \times X)),$$

the Banach structure in fibres being

$$\|v\|_\eta^2 = \sum_{|\beta| \leq s} \frac{\eta^{2\beta}}{\langle\eta\rangle^{2s}} \|v\|^2_{H^{s-|\beta|,\gamma+\frac{q}{2}-|\beta|,\delta}(\bar{\mathbb{R}}_+ \times X)}, \quad \eta \in \mathbb{R}^q. \tag{2.4.5}$$

Proof. We have

$$\begin{aligned} \|u\|^2_{H^{s,\gamma,\delta}(\mathcal{W})} &\sim \|\omega u\|^2_{\mathcal{H}^{s,\gamma,0}(\mathcal{W})} + \|(1-\omega)u\|^2_{\mathcal{H}^{s,0,\delta}(\mathcal{W})} \\ &= \int_{\mathbb{R}^q} \sum_{|\beta| \leq s} \|\omega (tD_y)^\beta u\|^2_{H^{s-|\beta|,\gamma+\frac{q}{2},0}(\bar{\mathbb{R}}_+ \times X)} dy \\ &\quad + \int_{\mathbb{R}^q} \sum_{|\beta| \leq s} \|(1-\omega)(tD_y)^\beta u\|^2_{H^{s-|\beta|,0,\delta-|\beta|}(\bar{\mathbb{R}}_+ \times X)} dy, \end{aligned}$$

the first equality being a consequence of Lemma 2.4.2. The same reasoning applies to (2.2.2), thus giving

$$\begin{aligned} \|u\|^2_{H^{s,\gamma,\delta}(\mathcal{W})} &\sim \int_{\mathbb{R}^q} \sum_{|\beta| \leq s} \|(tD_y)^\beta u\|^2_{H^{s-|\beta|,\gamma+\frac{q}{2},\delta-|\beta|}(\bar{\mathbb{R}}_+ \times X)} dy \\ &\sim \int_{\mathbb{R}^q} \sum_{|\beta| \leq s} \|D_y^\beta u\|^2_{H^{s-|\beta|,\gamma+\frac{q}{2}-|\beta|,\delta}(\bar{\mathbb{R}}_+ \times X)} dy. \end{aligned}$$

Since $H^{\tilde{s},\tilde{\gamma},\delta}(\bar{\mathbb{R}}_+ \times X)$ is a Hilbert space, we may invoke Parseval's formula to obtain

$$\begin{aligned} \|u\|^2_{H^{s,\gamma,\delta}(\mathcal{W})} &\sim \int_{\mathbb{R}^q} \sum_{|\beta| \leq s} \eta^{2\beta} \|\mathcal{F}_{y \mapsto \eta} u\|^2_{H^{s-|\beta|,\gamma+\frac{q}{2}-|\beta|,\delta}(\bar{\mathbb{R}}_+ \times X)} d\eta \\ &= \int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\mathcal{F}_{y \mapsto \eta} u\|^2_\eta d\eta. \end{aligned}$$

Finally, from the elementary inequality

$$\frac{1}{s!} \left(1 + |\eta|^2\right)^s \leq \sum_{|\beta| \leq s} \eta^{2\beta} \leq \left(1 + |\eta|^2\right)^s \tag{2.4.6}$$

we conclude that

$$\frac{1}{\langle\eta\rangle^s} \|v\|_{H^{s,\gamma+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+ \times X)} \leq \|v\|_\eta \leq \text{const}\, \|v\|_{H^{s,\gamma+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+ \times X)},$$

where the constant is independent of $v \in H^{s,\gamma+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+ \times X)$ and $\eta \in \mathbb{R}^q$. Hence (2.4.5) is really a family of norms on $H^{s,\gamma+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+ \times X)$, and the proposition follows.

□

We next claim that the family $(\|\cdot\|_\eta)_{\eta\in\mathbb{R}^q}$ fulfils the first item of Proposition 2.1.2.

Lemma 2.4.5 *Given any $\eta, \theta \in \mathbb{R}^q$, we have*

$$\|v\|_\eta \leq C \left(\max \frac{\langle\eta\rangle}{\langle\theta\rangle}, \frac{\langle\theta\rangle}{\langle\eta\rangle}\right)^s \|v\|_\theta \quad \textit{for all} \quad v \in H^{s,\gamma+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+ \times X),$$

with C a constant independent of v and η, θ.

Proof. Indeed,

$$\begin{aligned} \|v\|_\eta^2 &= \sum_{B=0}^{s} \sum_{|\beta|=B} \frac{\eta^{2\beta}}{\langle\eta\rangle^{2s}} \|v\|^2_{H^{s-|\beta|,\gamma+\frac{q}{2}-|\beta|,\delta}(\bar{\mathbb{R}}_+\times X)} \\ &\sim \sum_{B=0}^{s} \left(\sum_{|\beta|\leq B} \frac{\eta^{2\beta}}{\langle\eta\rangle^{2s}}\right) \|v\|^2_{H^{s-B,\gamma+\frac{q}{2}-B,\delta}(\bar{\mathbb{R}}_+\times X)}, \end{aligned}$$

the latter equality being due to the fact that

$$\|v\|_{H^{\tilde{s}-1,\tilde{\gamma}+\frac{q}{2}-1,\delta}(\bar{\mathbb{R}}_+\times X)} \leq C \ \|v\|_{H^{\tilde{s},\tilde{\gamma}+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+\times X)},$$

where the constant C depends only on ω ($C = 1$ if $\operatorname{supp}\omega \subset [0,1]$). Combining this with (2.4.6) we arrive at the following equivalent expression for the family (2.4.5):

$$\|v\|_\eta^2 \sim \sum_{B=0}^{s} \frac{1}{\langle\eta\rangle^{2(s-B)}} \|v\|^2_{H^{s-B,\gamma+\frac{q}{2}-B,\delta}(\bar{\mathbb{R}}_+\times X)}.$$

Now it is easy to see that

$$\|v\|_\eta^2 \leq \mathrm{const} \left(\max_{\mathrm{B}=0,\ldots,\mathrm{s}} \frac{\langle\theta\rangle^{2(\mathrm{s-B})}}{\langle\eta\rangle^{2(\mathrm{s-B})}}\right) \|\mathrm{v}\|_\theta^2$$

for all $\eta, \theta \in \mathbb{R}^q$, whence the lemma follows.

□

Applying the general construction (2.1.1), we get a family of isomorphisms $(\lambda(\eta))_{\eta\in\mathbb{R}^q}$ in $H^{s,\gamma+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+ \times X)$, such that

$$\|v\|_\eta = \|\lambda(\eta)v\|_{H^{s,\gamma+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+\times X)}$$

for all $v \in H^{s,\gamma+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+ \times X)$. Moreover, this family satisfies estimates like those in the second item of Proposition 2.1.2. However, it does depend on many parameters entering into the definition of the space $H^{s,\gamma+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+\times X)$, such as s, γ, δ, etc. To highlight the family $(\lambda(\eta))_{\eta\in\mathbb{R}^q}$ we need a modification of the space $\mathcal{H}^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X)$ which does not affect the behaviour of function near $t=0$, but only $t=\infty$. Namely, we replace the norm $\|\cdot\|_{\mathcal{H}^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X)}$ by the norm

$$\|u\|^2_{\tilde{\mathcal{H}}^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X)} := \int_0^\infty \frac{1}{t^{2\gamma}} \sum_{j+A\le s} (1+t^2)^{s-j-A} \|(tD_t)^j u\|^2_{H^A(X)} \frac{dt}{t},$$

for $s\in\mathbb{Z}_+$. It is a simple matter to see that equality (2.2.2) remains valid with $\mathcal{H}^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X)$ replaced by $\tilde{\mathcal{H}}^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X)$.

Lemma 2.4.6 *If $s\in\mathbb{Z}_+$ and $\gamma,\delta\in\mathbb{R}$, then*

$$\begin{aligned}\|v\|_\eta \;\sim\;& \left\|\langle\eta\rangle^{-s+\gamma+\frac{q}{2}}(\omega u)\left(\tfrac{t}{\langle\eta\rangle},x\right)\right\|_{\tilde{\mathcal{H}}^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X)}\\ &+\left\|(1-\omega(t))\langle\eta\rangle^{-\delta-\frac{1+n}{2}}u\left(\tfrac{t}{\langle\eta\rangle},x\right)\right\|_{\mathcal{H}^{s,0,\delta}(\bar{\mathbb{R}}_+\times X)},\end{aligned}$$

the constants of the equivalence estimates being independent of $\eta\in\mathbb{R}^q$.

Note that this lemma enables us to extend the definition of the family $(\|\cdot\|_\eta)_{\eta\in\mathbb{R}^q}$ to all real values s.

Proof. By a variant of Lemma 2.4.2,

$$\|v\|^2_\eta \sim \sum_{|\beta|\le s}\frac{\eta^{2\beta}}{\langle\eta\rangle^{2s}}\left(\|\omega v\|^2_{\mathcal{H}^{s-|\beta|,\gamma+\frac{q}{2}-|\beta|,0}(\bar{\mathbb{R}}_+\times X)} + \|(1-\omega)v\|^2_{\mathcal{H}^{s-|\beta|,0,\delta}(\bar{\mathbb{R}}_+\times X)}\right)$$

and so

$$\begin{aligned}\|v\|^2_\eta \;\sim\;& \sum_{|\beta|\le s}\frac{\eta^{2\beta}}{\langle\eta\rangle^{2s}}\int_0^\infty\frac{1}{t^{2\gamma}}\sum_{j+A\le s-|\beta|} t^{2|\beta|}\|(tD_t)^j\omega u\|^2_{H^A(X)}\frac{dt}{t^{1+q}}\\ &+\sum_{|\beta|\le s}\frac{\eta^{2\beta}}{\langle\eta\rangle^{2s}}\int_0^\infty t^{2\delta}\sum_{j+A\le s-|\beta|}\frac{1}{t^{2(j+A)}}\|(tD_t)^j(1-\omega)u\|^2_{H^A(X)}\,t^n dt.\end{aligned}$$

Interchanging the sums in β and j, A and taking into account that

$$\sum_{|\beta|\le s-j-A}\eta^{2\beta}t^{2|\beta|}\sim\langle t\eta\rangle^{2(s-j-A)}$$

(cf. (2.4.6)), we obtain

$$\begin{aligned}\|v\|^2_\eta \;\sim\;& \int_0^\infty\frac{1}{t^{2\gamma}}\sum_{j+A\le s}\frac{\langle t\eta\rangle^{2(s-j-A)}}{\langle\eta\rangle^{2s}}\|(tD_t)^j\omega u\|^2_{H^A(X)}\frac{dt}{t^{1+q}}\\ &+\int_0^\infty t^{2\delta}\sum_{j+A\le s}\frac{1}{(t\langle\eta\rangle)^{2(j+A)}}\|(tD_t)^j(1-\omega)u\|^2_{H^A(X)}\,t^n dt.\end{aligned}$$

Let us make use of the change of variables $\vartheta = t\langle\eta\rangle$. To this end we note that $tD_t = \vartheta D_\vartheta$ and

$$\begin{aligned}\langle\vartheta\frac{\eta}{\langle\eta\rangle}\rangle &\sim \left(1+\vartheta^2\left(\frac{|\eta|}{\langle\eta\rangle}\right)^2\right)^{\frac{1}{2}}\\ &\sim (1+\vartheta^2)^{\frac{1}{2}}\end{aligned}$$

on the support of $\omega\left(\frac{\vartheta}{\langle\eta\rangle}\right)$. Indeed, $\langle\eta\rangle = |\eta|$ for $|\eta| \ge c$ and, moreover, the set $\{\vartheta\colon \frac{\vartheta}{\langle\eta\rangle} \in \operatorname{supp}\omega\}$ is bounded uniformly in $|\eta| \le c$. Hence it follows that

$$\begin{aligned}\|v\|_\eta \sim\ & \left\|\langle\eta\rangle^{-s+\gamma+\frac{q}{2}}(\omega u)\left(\frac{t}{\langle\eta\rangle},x\right)\right\|_{\tilde{\mathcal{H}}^{s,\gamma,0}(\bar{\mathbb{R}}_+\times X)}\\ &+\left\|\langle\eta\rangle^{-\delta-\frac{1+n}{2}}((1-\omega)u)\left(\frac{t}{\langle\eta\rangle},x\right)\right\|_{\mathcal{H}^{s,0,\delta}(\bar{\mathbb{R}}_+\times X)}.\end{aligned}$$

To complete the proof, it suffices to observe that the support of the function $(1-\omega)\left(\frac{\vartheta}{\langle\eta\rangle}\right)$ is bounded away from $\vartheta = 0$ uniformly in $\eta \in \mathbb{R}^q$. Thus, we may replace it by $1-\omega(\vartheta)$ up to an equivalent norm.

□

Were it possible to replace also $\omega\left(\frac{\vartheta}{\langle\eta\rangle}\right)$ by $\omega(\vartheta)$ modulo an equivalent norm, Lemma 2.4.6 give us $\|v\|_\eta \sim \|\lambda(\eta)v\|_{H^{s,\gamma+\frac{q}{2},\delta}(\bar{\mathbb{R}}_+\times X)}$, provided that $-\delta-\frac{1+n}{2} = \gamma+\frac{q}{2}-s$, where

$$\lambda(\eta)v\,(t,x) = \langle\eta\rangle^{-\delta-\frac{1+n}{2}}v\left(\frac{t}{\langle\eta\rangle},x\right).$$

This latter family of isomorphisms enters into the definition of 'twisted' Sobolev spaces $H^s(S,\pi^*H^{s,\gamma}(F^{-1}(\cdot)))$ of Section 2.2. In contrast to Proposition 2.4.3 it is unknown at present whether the spaces $H^s(S,\pi^*H^{s,\gamma}(F^{-1}(\cdot)))$ are invariant under diffeomorphisms of $\mathcal{M}$ of the form (2.4.3). In [ST95] this question is reduced to an L^2-estimate for Calderòn-Zygmund operators (cf. Lemma 2.2.12 *ibid.*). This estimate is easily verified for those diffeomorphisms (2.4.3) which fulfil $\psi \equiv 0$. Thus, the spaces $H^{s,\gamma}(\mathcal{M})$ are invariant under the diffeomorphisms of $\mathcal{M}$, which obey the fibration (2.2.1) of $\mathcal{M}$ close to the boundary.

Chapter 3

Operator Algebras on a Manifold with Edges

The theory of elliptic differential operators on compact manifolds without boundary is well understood in the setting of standard pseudodifferential operators. We have a concept of ellipticity which says, e.g., for classical pseudodifferential operators, that when the principal homogeneous symbol is bijective outside the zero section of the cotangent bundle, then A is a Fredholm operator between the corresponding Sobolev spaces (the ellipticity is even necessary). And we find a parametrix within pseudodifferential operators belonging to the inverse of the principal homogeneous symbol. The analysis of differential operators on spaces with piecewise smooth geometry, which is necessary in various applications in physics and engineering, leads to the necessity to do the same for manifolds with singularities, e.g., for manifolds with edges. As before, we need the calculus for single operators in such a way that all the formal manipulations such as compositions, adjoints, and so on be controlled within the given class of operators and also on symbolic level. An algebra of this sort was constructed in Schulze [Sch92, Sch91]. Unfortunately, the theory of operators on wedges is not so standard as the "usual" calculus of pseudodifferential operators which may be found in classical papers of Hörmander and other authors and in a number of monographs. The recent development for operators with singularities has created various schools that emphasise rather different aspects. This chapter gives a brief introduction to the theory of Schulze [Sch92, Sch91], where we arrange a number of things in a more transparent way and also formulate some new results.

3.1 *Typical symbols*

Let (M, S) be a C^∞ manifold with edges. Throughout this chapter we assume that S consists of the only component which is a C^∞ closed compact manifold of dimension q. All the results carry over to the case of S consisting

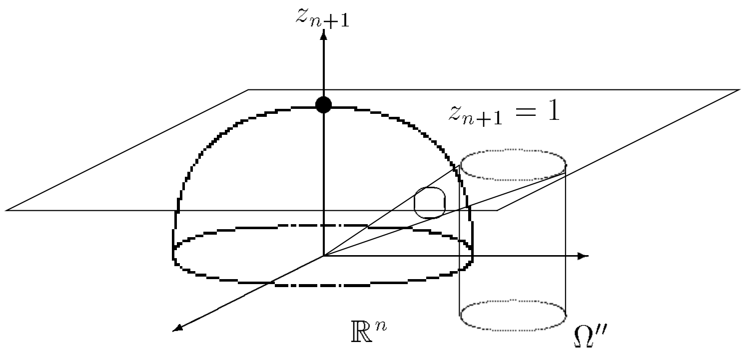

Fig. 3.1: Stereographic projection.

of a finite number of such components, the extension being straightforward.

As M is smooth apart from S, we restrict our attention to what happens close to S. By Definition 1.1.1, each point $p \in S$ lies in the domain of some chart $h\colon O \to W$ on M, with $W = \mathbb{R}^q \times C_t(X)$ a wedge. We regard $C_t(X)$ as being embedded in an Euclidean space $\mathbb{R}^{N-q}$ as in (1.3.1). Moreover, given a sufficiently small coordinate patch Ω'' on X, we may always assume that

$$e\colon \Omega'' \hookrightarrow \{z \in \mathbb{R}^{N-q-1}\colon z_{n+1} = \ldots = z_{N-q-1} = 0\}$$

where n stands for the dimension of X. Indeed, $C_t(X)$ is defined up to a diffeomorphism in $\mathbb{R}^{N-q}$ and we can organise such a diffeomorphism so that the above condition is fulfilled. When thinking of the first n components of z as being local coordinates in Ω'', we obtain a diffeomorphism of the part $\mathbb{R}^q \times C_t(\Omega'')$ of W onto an open set in $\mathbb{R}^{q+1+n}$, where

$$C_t(\Omega'') = \{t\,\mathrm{SP}\,(\mathrm{x}, 0)\colon \mathrm{t} \in \bar{\mathbb{R}}_+,\ \mathrm{x} \in \mathrm{e}(\Omega'')\}$$

(cf. Fig. 3.1).

More precisely, the restriction of $\mathrm{SP}\colon \mathbb{R}^{\mathrm{N-q-1}} \to \mathbb{S}_+^{\mathrm{N-q-1}}$ to the subspace $\mathbb{R}^n \times \{0\}$ is naturally identified with the stereographic projection of $\mathbb{R}^n$ onto the upper hemisphere $\mathbb{S}^n_+$ in $\mathbb{R}^{1+n}$. Thus, we can take $(y, z') \in \mathbb{R}^q \times \mathbb{R}^{1+n}$ as local coordinates in $\mathbb{R}^q \times C_t(\Omega'')$, with $z' = t\,\mathrm{SP}\,(\mathrm{x})$. To express the derivatives in z' in terms of polar coordinates $(t, x) \in \bar{\mathbb{R}}_+ \times \Omega''$, we need the following lemma.

Lemma 3.1.1 *Let* SP *be the stereographic projection of* $\mathbb{R}^n$ *onto* $\mathbb{S}^n_+$ *and let* $z = t\,\mathrm{SP}\,(\mathrm{x})$. *Then,*

$$\det \frac{\partial z}{\partial(t, x)} = (-1)^n \frac{t^n}{(|x|^2 + 1)^{\frac{1+n}{2}}}.$$

Proof. Immediate.

□

From this lemma it follows that

$$\frac{\partial}{\partial z} = \mathrm{SP}\,(\mathrm{x})\,\frac{\partial}{\partial \mathrm{t}} + \sqrt{|\mathrm{x}|^2+1}\,\mathcal{P}(\mathrm{x})\,\frac{1}{\mathrm{t}}\frac{\partial}{\partial \mathrm{x}}, \tag{3.1.1}$$

where $\mathcal{P}(x)$ is an $((n+1)\times n)$-matrix of polynomials in $\mathbb{R}^n$. Hence, smooth vector fields close to the edge of M are lifted to vector fields on the stretched manifold $\mathcal{M} = [M; S]$ spanned by

$$D_y,\ D_t,\ \frac{1}{t}D_x$$

in local coordinates close to the boundary of $\mathcal{M}$. The universal enveloping algebra of these vector fields consists of those differential operators on $\mathcal{M}$ which may be expressed locally as finite sums of their products, i.e.,

$$\begin{aligned} A &= \sum_{|\beta|+j+|\alpha|\le m} \tilde{A}_{\beta,j,\alpha}(y,t,x)\,D_y^\beta D_t^j \left(\frac{1}{t}D_x\right)^\alpha \\ &= \frac{1}{t^m}\sum_{|\beta|+j+|\alpha|\le m} A_{\beta,j,\alpha}(y,t,x)\,(tD_y)^\beta (tD_t)^j D_x^\alpha, \end{aligned} \tag{3.1.2}$$

the coefficients $A_{\beta,j,\alpha}$ being smooth up to $t=0$. In this way we obtain what will be referred to as 'typical' differential operators on a manifold with edges.

Since the multiplicative singular factor t^{-m} might be traced back to the Sobolev norm, we are left with the task of studying the universal enveloping algebra of the Lie algebra $\mathcal{V}_b$ on $\mathcal{M}$ (cf. (2.4.2)). To this end, the concept of a *compressed cotangent bundle* ${}^bT^*\mathcal{M}$ of $\mathcal{M}$ proves to be of use (cf. Melrose [Mel96b]). Namely, the maximality and independence of the set (2.4.2) means that there is a bundle ${}^bT\mathcal{M}$ naturally associated to $\mathcal{V}_b$, such that $\mathcal{V}_b = C^\infty(\mathcal{M}, {}^bT\mathcal{M})$. This bundle comes equipped with a mapping to the ordinary tangent bundle $\iota_b\colon {}^bT\mathcal{M} \to T\mathcal{M}$ induced by the proper inclusion $\mathcal{V}_b \hookrightarrow C^\infty(\mathcal{M}, T\mathcal{M})$. Evidently, ι_b is an isomorphism over the interior of $\mathcal{M}$ because $\mathcal{V}_b$ consists of all smooth vector fields there, but is neither injective nor surjective on the boundary where it has range of rank n, the fibre dimension of b, in $T\partial\mathcal{M}$. Then, ${}^bT\mathcal{M}$ is defined by simply demanding that the vector fields (2.4.2) be a spanning set of sections. Namely, any $v \in \mathcal{V}_b$ may be uniquely expressed as

$$v = \sum_{\iota=1}^{q} c_{\iota,0,0}\,t\partial_{y_\iota} + c_{0,1,0}\,t\partial_t + \sum_{j=1}^{n} c_{0,0,j}\,\partial_{x_j},$$

and then the coefficients $c_{\iota,0,0}$, $c_{0,1,0}$ and $c_{0,0,j}$ evaluated at $p \in \mathcal{M}$ are linear coordinates in the fibre ${}^bT_p\mathcal{M}$. The dual to ${}^bT\mathcal{M}$, denoted by ${}^bT^*\mathcal{M}$, is spanned locally by the 1-forms

$$\frac{dy_\iota}{t},\ \frac{dt}{t},\ dx_j \tag{3.1.3}$$

which are singular as forms in the usual sense but smooth as sections of ${}^bT^*\mathcal{M}$.

There is a symbol mapping on the set of typical differential operators on $\mathcal{M}$. For A as in (3.1.2) set

$$ {}^b\sigma^m(A)(y,t,x;\tilde{\eta},\tilde{\tau},\xi) = \sum_{|\beta|+j+|\alpha|=m} A_{\beta,j,\alpha}(y,t,x)\,\tilde{\eta}^\beta\tilde{\tau}^j\xi^\alpha. \tag{3.1.4} $$

It is easy to see that ${}^b\sigma^m(A)$ is an invariantly defined homogeneous polynomial of degree m on the fibres of ${}^bT^*\mathcal{M}$. As usual, A is said to be *elliptic* if ${}^b\sigma^m(A)$ does not vanish away from the zero section of ${}^bT^*\mathcal{M}$. In this global context, by t is meant a defining function of the boundary of $\mathcal{M}$.

The key examples of typical differential operators on $\mathcal{M}$ are the Laplacians of metrics in the class

$$ \{g \in C^\infty(\mathrm{Sym}^2({}^\mathrm{b}\mathrm{T}^*\mathcal{M})\colon \mathrm{g} \gg 0 \text{ on } {}^\mathrm{b}\mathrm{T}^*\mathcal{M}\}. $$

The following is due to Mazzeo [Maz91].

Proposition 3.1.2 *For any metric g of the above class, the Laplacian Δ_g is elliptic.*

Proof. This is easy to check.

□

We next strengthen the above argument by looking at the pull-backs of the usual pseudodifferential operators near the edge of M under the blow-down mapping b.

Consider a geometric wedge

$$ W = \Omega' \times \{t\,\mathrm{SP}\,(\mathrm{x}) \in \mathbb{R}^{1+\mathrm{n}}\colon \mathrm{t} \geq 0,\ \mathrm{x} \in \Omega''\} $$

in $\mathbb{R}^{q+1+n}$, where Ω' is an open set in $\mathbb{R}^q$ and Ω'' stands for a coordinate patch on X which we identify with an open subset of $\mathbb{R}^n$. Write $w = (y,z)$ for the coordinates in $\mathbb{R}^{q+1+n}$, where y runs over $\mathbb{R}^q$ and z runs over $\mathbb{R}^{1+n}$.

Let $m \in \mathbb{R}$. Our objective will be the behaviour of $\Psi^\mathrm{m}_\mathrm{cl}(\mathrm{W})$ under the change of coordinates

$$ \left\{ \begin{array}{rcl} y & \mapsto & y, \\ (t,x) & \mapsto & z = t\,\mathrm{SP}\,(\mathrm{x}) \end{array} \right. \tag{3.1.5} $$

defining a smooth mapping $b\colon \mathcal{W} \to W$, where $\mathcal{W} = \Omega' \times \bar{\mathbb{R}}_+ \times \Omega''$ is the associated stretched wedge.

Given any $A \in \Psi^\mathrm{m}_\mathrm{cl}(\mathrm{W})$, this leads to the diagram

$$ \begin{array}{ccc} C^\infty_{\mathrm{comp}}(W) & \xrightarrow{A} & C^\infty_{\mathrm{loc}}(W) \\ \downarrow b^* & & \downarrow b^* \\ C^\infty_{\mathrm{comp}}(\mathcal{W}) & \xrightarrow{b^\sharp A} & C^\infty_{\mathrm{loc}}(\mathcal{W}), \end{array} $$

b^* meaning the pull-back operator on functions. The *pull-back* $b^\sharp A$ of A under b is defined so as to make this diagram commutative, i.e., $b^\sharp A = b^* A b_*$. As b restricts to a diffeomorphism of the interior of $\mathcal{W}$ onto the interior of W, a standard result is that b induces a bijection of the spaces of pseudodifferential operators in the interiors of W and $\mathcal{W}$.

Lemma 3.1.3 *For each* $a \in \mathcal{S}^m_{\mathrm{cl}}(W \times \mathbb{R}^{q+1+n})$*, there exists a symbol* $\tilde{a} \in \mathcal{S}^m_{\mathrm{cl}}(\mathcal{W} \times \mathbb{R}^{q+1+n})$ *such that*

$$b^\sharp \operatorname{op}(\mathrm{a}) = \frac{1}{\mathrm{t}^{\mathrm{m}}} \operatorname{op}(\tilde{\mathrm{a}}(\mathrm{y}, \mathrm{t}, \mathrm{x}; \mathrm{t}\eta, \mathrm{t}\tau, \xi)) \mod \Psi^{-\infty}(\Omega' \times \mathbb{R}_+ \times \Omega''). \quad (3.1.6)$$

Proof. Indeed, we have the following asymptotic formula for the complete symbol $b^\sharp a$ of $b^\sharp \operatorname{op}(\mathrm{a})$:

$$b^\sharp a\,(y,t,x;\eta,\tau,\xi)\,|_{(y,t,x)=b^{-1}(w)}$$
$$\sim \sum_\gamma \frac{1}{\gamma!} \partial^\gamma_\omega a\left(w, \left(\frac{\partial b^{-1}}{\partial w}\right)'(\eta,\tau,\xi)\right) D^\gamma_{w'}\left(e^{i\langle b^{-1}(w')-b^{-1}(w)-\frac{\partial b^{-1}}{\partial w}(w'-w),\,(\eta,\tau,\xi)\rangle}\right)|_{w'=w},$$

where ω stands for the covariable of w and $\left(\frac{\partial b^{-1}}{\partial w}\right)'$ is the transpose of the tangent mapping $\frac{\partial b^{-1}}{\partial w}$ (see [Hör85, 18.1.17]).

By induction in $|\gamma|$, we can assert that

$$D^\gamma_{w'}\left(e^{i\langle b^{-1}(w')-b^{-1}(w)-\frac{\partial b^{-1}}{\partial w}(w'-w),\,(\eta,\tau,\xi)\rangle}\right)|_{w'=w} = t^{-|\gamma|}\, p_\gamma(x;t\tau,\xi)\,|_{(y,t,x)=b^{-1}(w)},$$

where p_γ is a polynomial of degree $\le |\gamma|/2$ in $(t\tau,\xi)$ and coefficients in $C^\infty_{loc}(\Omega'')$. On the other hand,

$$\begin{aligned}
\left(\frac{\partial b^{-1}}{\partial w}\right)'(\eta,\tau,\xi) &= \left(\frac{\partial b}{\partial(y,t,x)}\right)^{-1'}|_{(y,t,x)=b^{-1}(w)} \begin{pmatrix}\eta\\ \tau\\ \xi\end{pmatrix} \\
&= \begin{pmatrix} I_q & 0 \\ 0 & (\mathrm{SP}\,(\mathrm{x}))' \\ 0 & \frac{1}{t}\left(\frac{\partial \mathrm{SP}\,(\mathrm{x})}{\partial x}\right)^{-1} \end{pmatrix}' |_{(y,t,x)=b^{-1}(w)} \begin{pmatrix}\eta\\ \tau\\ \xi\end{pmatrix} \\
&= \frac{1}{t}\begin{pmatrix} I_q & 0 & 0 \\ 0 & \mathrm{SP}\,(\mathrm{x}) & \left(\frac{\partial \mathrm{SP}\,(\mathrm{x})}{\partial x}\right)^{-1'} \end{pmatrix} \begin{pmatrix}t\eta\\ t\tau\\ \xi\end{pmatrix} |_{(y,t,x)=b^{-1}(w)};
\end{aligned}$$

here I_q is the identity $(q\times q)$-matrix and $\left(\frac{\partial \mathrm{SP}\,(\mathrm{x})}{\partial x}\right)^{-1}$ is the left inverse matrix for $\frac{\partial \mathrm{SP}\,(\mathrm{x})}{\partial x}$ (it is of the form $\sqrt{|x|^2+1}\,\mathcal{P}(x)$, cf. (3.1.1)).

Write

$$a(w,\omega) \sim \sum_{j=0}^\infty a_{m-j}(w,\omega),$$

with $a_{m-j} \in C^\infty_{\mathrm{loc}}(W \times \mathbb{R}^{q+1+n})$ homogeneous of degree $m-j$ in ω away from a ball. Pick an excision function χ on $\mathbb{R}^{q+1+n}$. By the above,

$$\begin{aligned} & b^\sharp a\,(y,t,x;\eta,\tau,\xi) \\ & \quad \sim \sum_\gamma \sum_j \frac{1}{\gamma!} \partial^\gamma_\omega a_{m-j}\Big(w,\ \Big(\frac{\partial b^{-1}}{\partial w}\Big)'(\eta,\tau,\xi)\Big)|_{w=b(y,t,x)}\, t^{-|\gamma|}\, p_\gamma(x;t\tau,\xi) \\ & \quad \sim \frac{1}{t^m} \sum_\gamma \sum_j \chi(\eta,\tau,\xi) \frac{1}{\gamma!}\, t^j\, \tilde{a}^{(\gamma)}_{m-j} \left((y, t\,\mathrm{SP}\,(\mathrm{x})),\ \mathrm{T(x)} \begin{pmatrix} t\eta \\ t\tau \\ \xi \end{pmatrix} \right) p_\gamma(x,t\tau,\xi), \end{aligned}$$

where $\tilde{a}^{(\gamma)}_{m-j}(w,\omega)$ is the unique homogeneous extension of $\partial^\gamma_\omega a_{m-j}(w,\omega)$ from large ω to $\omega \neq 0$, and

$$T(x) = \begin{pmatrix} I_q & 0 & 0 \\ 0 & \mathrm{SP}\,(\mathrm{x}) & \left(\frac{\partial \mathrm{SP}\,(\mathrm{x})}{\partial x}\right)^{-1'} \end{pmatrix}.$$

Rearranging the summands on the right-hand side with respect to the degree of homogeneity, we get

$$b^\sharp a\,(y,t,x;\eta,\tau,\xi) \sim \frac{1}{t^m} \sum_{\iota=0}^{\infty} \chi(\eta,\tau,\xi)\, \tilde{a}_{m-\iota}(y,t,x;t\eta,t\tau,\xi), \qquad (3.1.7)$$

with $\tilde{a}_{m-\iota}(y,t,x;\tilde{\eta},\tilde{\tau},\xi) \in C^\infty_{\mathrm{loc}}(\mathcal{W} \times (\mathbb{R}^{q+1+n} \setminus \{0\}))$ homogeneous of degree $m-\iota$ in $(\tilde{\eta},\tilde{\tau},\xi)$.

By a familiar argument from the theory of pseudodifferential operators, there exists a symbol $\tilde{a}(y,t,x;\tilde{\eta},\tilde{\tau},\xi) \in C^\infty_{\mathrm{loc}}(\mathcal{W} \times \mathbb{R}^{q+1+n})$ with the property that $\tilde{a} \sim \sum_{\iota=0}^\infty \chi\, \tilde{a}_{m-\iota}$. Combining this with (3.1.7) yields (3.1.6), and the lemma follows.

□

Equality (3.1.6) gives rise to a class $\mathcal{S}^m({}^bT^*\mathcal{M})$ of symbols on $\mathcal{M}$ which bears information on the boundary fibration. More precisely, it consists of those symbols of order $m \in \mathbb{R}$ on $\mathcal{M}$ which are of the form

$$a(y,t,x;\eta,\tau,\xi) = \tilde{a}(y,t,x;t\eta,t\tau,\xi) \qquad (3.1.8)$$

in local coordinates close to the boundary of $\mathcal{M}$, where $\tilde{a}(y,t,x;\tilde{\eta},\tilde{\tau},\xi)$ is a symbol of order m smooth up to $t=0$. Symbols (3.1.8) are said to be *typical* in the analysis on manifolds with edges.

Thus, $\mathcal{S}^m({}^bT^*\mathcal{M})$ is a subspace of $\mathcal{S}^m(T^*\mathcal{M})$; we endow it with a canonical Fréchet topology.

We use the symbol $\mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{M})$ to denote the subspace of $\mathcal{S}^m({}^bT^*\mathcal{M})$ which originates from $\mathcal{S}^m_{\mathrm{cl}}(T^*\mathcal{M})$ in the same way. As usually, we set

$$\mathcal{S}^{-\infty}({}^bT^*\mathcal{M}) = \bigcap_{m\in\mathbb{R}} \mathcal{S}^m_{cl}({}^bT^*\mathcal{M}).$$

Whether the multiplicative singular factor t^{-m} can be handled separately by attributing it to the Sobolev norm or not, does depend on the weighted spaces we use. Such is certainly the case for spaces (2.4.4) but not for those of Definition 2.2.2. For this reason, we introduce also symbol spaces $t^{-m}\,\mathcal{S}^m({}^bT^*\mathcal{M})$ and those consisting of classical symbols, for $m \in \mathbb{R}$. As mentioned, in this global context t stands for a defining function of $\partial\mathcal{M}$.

Lemma 3.1.3 is nothing but the statement that the pull-back of any symbol $a \in \mathcal{S}^m_{\mathrm{cl}}(T^*W)$ under the blow-down mapping $b\colon \mathcal{W} \to W$ belongs to $t^{-m}\,\mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{W})$ modulo smoothing operators in the interior of $\mathcal{W}$ which are anyway negligible. The important point to note here is that the symbols of $t^{-m}\,\mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{M})$ are much more general than those induced by the blow-down mapping.

In the sequel we will restrict our discussion to classical symbols in $t^{-m}\,\mathcal{S}^m({}^bT^*\mathcal{M})$. In local coordinates close to the boundary, each symbol $a \in t^{-m}\,\mathcal{S}^m_{\mathrm{cl}}(\mathcal{W} \times \mathbb{R}^{q+1+n})$ may be written in the form

$$a(y,t,x;\eta,\tau,\xi) = \frac{1}{t^m}\,\tilde{a}(y,t,x;t\eta,t\tau,\xi)$$

with $\tilde{a} \in \mathcal{S}^m_{\mathrm{cl}}(\mathcal{W} \times \mathbb{R}^{q+1+n})$. Write $\tilde{a} \sim \sum_j \chi\,\tilde{a}_{m-j}$, where χ is an excision function and $\tilde{a}_{m-j} \in C^\infty_{\mathrm{loc}}(\mathcal{W} \times (\mathbb{R}^{q+1+n} \setminus \{0\}))$ are homogeneous of degree $m-j$. Set

$$ {}^b\sigma^m(a)(y,t,x;\tilde{\eta},\tilde{\tau},\xi) = \tilde{a}_m(y,t,x;\tilde{\eta},\tilde{\tau},\xi) \qquad (3.1.9)$$

(cf. (3.1.4)).

Though defined in local coordinates on $\mathcal{M}$, the principal homogeneous symbol (3.1.9) behaves like a function on the compressed cotangent bundle ${}^bT^*\mathcal{M}$. Thus, the following definition makes sense

Definition 3.1.4 *A symbol $a \in t^{-m}\,\mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{M})$ is called elliptic if*

$${}^b\sigma^m(a)(y,t,x;\tilde{\eta},\tilde{\tau},\xi) \neq 0 \quad \text{for all } (y,t,x;\tilde{\eta},\tilde{\tau},\xi) \in {}^bT^*\mathcal{M} \setminus \{0\}.$$

The ellipticity of a 'typical' symbol subtends the non-singularity of $t^m a$ up to the boundary of $\mathcal{M}$. As t is different from zero away from the boundary of $\mathcal{M}$, each 'typical' symbol elliptic in the sense of Definition 3.1.4 is so in the usual sense in the interior of $\mathcal{M}$.

From the point of view of parametrix construction, the crucial property of typical symbols on $\mathcal{M}$ is that the Leibniz inverse to an elliptic typical symbol is again of the same type. To explain this in detail, let us write $\mathring{\mathcal{W}} = \Omega' \times \mathbb{R}_+ \times \Omega''$ for the interior of $\mathcal{W}$. Recall that by a *Leibniz inverse* for a symbol $a \in \mathcal{S}^m(\mathring{\mathcal{W}} \times \mathbb{R}^{q+1+n})$ is meant any symbol $p \in \mathcal{S}^{-m}(\mathring{\mathcal{W}} \times \mathbb{R}^{q+1+n})$ with the property that $p \circ a = a \circ p = 1$ modulo $\mathcal{S}^{-\infty}(\mathring{\mathcal{W}} \times \mathbb{R}^{q+1+n})$, where $a \circ b$ denotes the Leibniz product of symbols a and b in local coordinates of $\mathcal{W}$. The standard theory of pseudodifferential operators states that a symbol $a \in \mathcal{S}^m_{\mathrm{cl}}(\mathring{\mathcal{W}} \times \mathbb{R}^{q+1+n})$ possesses a Leibniz inverse if and only if a is elliptic. Moreover, this Leibniz inverse is unique modulo $\mathcal{S}^{-\infty}(\overset{\circ}{\mathcal{W}} \times \mathbb{R}^{q+1+n})$ (see for instance [Hör85, 18.1.9]). It is customary to write $a^{(-1)}$ for the Leibniz inverse of a symbol a.

Lemma 3.1.5 *Suppose $a \in \mathcal{S}^m_{\mathrm{cl}}(\overset{\circ}{\mathcal{W}} \times \mathbb{R}^{q+1+n})$ is elliptic. Then the Leibniz inverse of a has the asymptotic expansion*

$$a^{(-1)} \sim \sum_{\nu=0}^{\infty} (1 - p \circ a)^{\circ\nu} \circ p,$$

where $p = \chi\, \sigma^m(a)^{-1}$, with χ an excision function on $\mathbb{R}^{q+1+n}$, and the exponent $\circ\nu$ means the νth power with respect to the Leibniz product.

Proof. See *ibid.*

□

Having disposed of this preliminary step, we can now return to typical symbols (3.1.8). From what has already been said it follows that each elliptic 'typical' symbol $a \in t^{-m}\, \mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{W})$ has a Leibniz inverse $a^{(-1)}$ in $\mathcal{S}^{-m}_{\mathrm{cl}}(\overset{\circ}{\mathcal{W}} \times \mathbb{R}^{q+1+n})$.

Proposition 3.1.6 *For any elliptic symbol $a \in t^{-m}\, \mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{W})$, it follows that $a^{(-1)} \in t^m\, \mathcal{S}^{-m}_{\mathrm{cl}}({}^bT^*\mathcal{W})$.*

Proof. Since a is elliptic, the symbol

$$\begin{aligned} p(y,t,x;\eta,\tau,\xi) &= \chi(\eta,\tau,\xi)\,(\sigma^m(a)(y,t,x;\eta,\tau,\xi))^{-1} \\ &= t^m\, \chi(\eta,\tau,\xi)\,\left({}^b\sigma^m(a)(y,t,x;t\eta,t\tau,\xi)\right)^{-1} \end{aligned}$$

is in $t^m\, \mathcal{S}^{-m}_{\mathrm{cl}}({}^bT^*\mathcal{W})$.

By Lemma 3.1.5, we shall have established the proposition if we prove that, given any

$$\begin{aligned} a_1 &\in t^{-m_1}\, \mathcal{S}^{m_1}_{\mathrm{cl}}({}^bT^*\mathcal{W}), \\ a_2 &\in t^{-m_2}\, \mathcal{S}^{m_2}_{\mathrm{cl}}({}^bT^*\mathcal{W}), \end{aligned}$$

the Leibniz product of a_1 and a_2 belongs to $t^{-m_1-m_2}\, \mathcal{S}^{m_1+m_2}_{\mathrm{cl}}({}^bT^*\mathcal{W})$ modulo $\mathcal{S}^{-\infty}(\overset{\circ}{\mathcal{W}} \times \mathbb{R}^{q+1+n})$. Indeed, the Leibniz product itself is defined up to elements of this space.

For $i = 1, 2$, write $a_i(y,t,x;\eta,\tau,\xi) = \frac{1}{t^{m_i}}\, \tilde{a}_i(y,t,x;t\eta,t\tau,\xi)$ with suitable symbols $\tilde{a}_\nu \in \mathcal{S}^{m_i}_{\mathrm{cl}}({}^bT^*\mathcal{W})$. Then

$$\begin{aligned} & a_1 \circ a_2\,(y,t,x;\eta,\tau,\xi) \\ & \quad \sim \sum_{\beta,j,\alpha} \frac{1}{\beta! j! \alpha!}\, \partial_\eta^\beta \partial_\tau^j \partial_\xi^\alpha \left(\frac{1}{t^{m_1}}\, \tilde{a}_1\right) D_y^\beta D_t^j D_x^\alpha \left(\frac{1}{t^{m_2}}\, \tilde{a}_2\right). \end{aligned}$$

Let us examine the t- and η-, τ-powers of the summands on the right-hand side. The derivative $\partial_\eta^\beta \partial_\tau^j \partial_\xi^\alpha \left(\frac{1}{t^{m_1}}\, \tilde{a}_1\right)$ is of the form

$$t^{-m_1+|\beta|+j}\, \tilde{a}_1^{(\beta,j,\alpha)}(y,t,x;t\eta,t\tau,\xi),$$

with $\tilde{a}_1^{(\beta,j,\alpha)} \in \mathcal{S}_{\mathrm{cl}}^{m_1-|\beta|-j-|\alpha|}(\mathcal{W} \times \mathbb{R}^{q+1+n})$. On the other hand, the derivative $D_y^\beta D_t^j D_x^\alpha \left(\frac{1}{t^{m_2}}\,\tilde{a}_2\right)$ produces a sum of symbols of the form

$$t^{-m_2-k}\,\eta^\gamma\,\tau^\iota\,\tilde{a}_2^{(\gamma,\iota,\alpha,)}(y,t,x;t\eta,t\tau,\xi) \quad (|\gamma|+\iota \le j-k),$$

where $\tilde{a}_2^{(\gamma,\iota,\alpha)} \in \mathcal{S}_{\mathrm{cl}}^{m_2-|\gamma|-\iota}(\mathcal{W} \times \mathbb{R}^{q+1+n})$. What remains is $\frac{1}{t^{m_1+m_2}}\,(t\eta)^\gamma\,(t\tau)^\iota$ up to a multiplicative factor t^N, with N a non-negative integer.

Summarizing, we can assert that

$$\begin{aligned} & a_1 \circ a_2\,(y,t,x;\eta,\tau,\xi) \\ & \quad \sim \frac{1}{t^{m_1+m_2}} \sum_{\beta,j,\alpha} \tilde{a}^{(\beta,j,\alpha)}(y,t,x;t\eta,t\tau,\xi), \end{aligned}$$

where $\tilde{a}^{(\beta,j,\alpha)}(y,t,x;\tilde{\eta},\tilde{\tau},\xi) \in \mathcal{S}_{\mathrm{cl}}^{m_1+m_2-|\beta|-j-|\alpha|}(\mathcal{W} \times \mathbb{R}^{q+1+n})$. Finally, let us find a symbol $\tilde{a}$ in $\mathcal{S}_{\mathrm{cl}}^{m_1+m_2}(\mathcal{W} \times \mathbb{R}^{q+1+n}_{\tilde{\eta},\tilde{\tau},\xi})$, whose asymptotic expansion is $\sum_{\beta,j,\alpha} \tilde{a}^{(\beta,j,\alpha)}$. Then

$$a_1 \circ a_2\,(y,t,x;\eta,\tau,\xi) = \frac{1}{t^{m_1+m_2}}\,\tilde{a}(y,t,x;t\eta,t\tau,\xi) \mod \mathcal{S}^{-\infty}(\mathring{\mathcal{W}} \times \mathbb{R}^{q+1+n}),$$

which gives the desired conclusion.

□

It is worth pointing out that the proposition does not assert, for an elliptic typical symbol a, that *every* Leibniz inverse of a in the symbol algebra on the wedge is a typical symbol. In fact, $a^{(-1)}$ itself is defined up to elements of $\mathcal{S}^{-\infty}(\mathring{\mathcal{W}} \times \mathbb{R}^{q+1+n})$. Thus, Proposition 3.1.6 ensures only the existence of a typical symbol *within the equivalence class* of $a^{(-1)}$ in $\mathcal{S}^{-m}(\mathring{\mathcal{W}} \times \mathbb{R}^{q+1+n})$.

The condition that a is elliptic with respect to the symbol ${}^b\sigma^m$ (i.e., up to the boundary) is essential to the proof. It cannot be weakened because otherwise the "crude" parametrix $p = \chi\,\sigma^m(a)^{-1}$ is no longer in $t^m\,\mathcal{S}^{-m}({}^bT^*\mathcal{W})$.

If V, $\tilde{V}$ are C^∞ vector bundles over $\mathcal{M}$, then the space of typical symbols taking their values in bundle homomorphisms $V \to \tilde{V}$ can be defined by

$$\mathcal{S}_{\mathrm{cl}}^m({}^bT^*\mathcal{M}, \mathrm{Hom}(\mathrm{V},\tilde{\mathrm{V}})) = \mathcal{S}_{\mathrm{cl}}^{\mathrm{m}}({}^{\mathrm{b}}\mathrm{T}^*\mathcal{M}) \otimes_{\mathrm{C}^\infty(\mathcal{M})} \mathrm{C}^\infty(\mathcal{M}, \mathrm{Hom}(\mathrm{V},\tilde{\mathrm{V}})),$$

where $\mathrm{Hom}(\mathrm{V},\tilde{\mathrm{V}})$ is the bundle over $\mathcal{M}$ with fibre $\mathrm{Hom}(\mathrm{V_p},\tilde{\mathrm{V}}_{\mathrm{p}})$.

3.2 *Quantisation*

As described in Section 3.1, the space

$$t^{-\cdot}\,\mathcal{S}_{\mathrm{cl}}^{\cdot}({}^bT^*\mathcal{M}) = \bigcup_{m\in\mathbb{R}} t^{-m}\,\mathcal{S}_{\mathrm{cl}}^m({}^bT^*\mathcal{M})$$

of all typical symbols on $\mathcal{M}$ is closed under the Leibniz product on coordinate neighbourhoods in $\mathcal{M}$. In this section we are interested in finding a suitable *quantisation* of the typical symbols, i.e., in showing how each typical symbol on $\mathcal{M}$ defines a bounded operator in weighted Sobolev spaces on $\mathcal{M}$. There is no canonical way to do this; in particular, the quantisation does depend on the spaces we use. If these are the spaces $H^{s,\gamma}(\mathcal{M})$ modelled on (2.4.4) close to the boundary, the standard way of assigning an operator $H^{s,\gamma}(\mathcal{M}, V) \to H^{s-m,\gamma}(\mathcal{M}, \tilde{V})$ to a symbol $a \in \mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{M}, \mathrm{Hom}(\mathrm{V}, \tilde{\mathrm{V}}))$ is to invoke a partition of unity on $\mathcal{M}$ and oscillatory integrals in local coordinates. This still works for the symbols slowly oscillating at the boundary (cf. [RST97]), i.e., those of the form (3.1.8) with $\tilde{a} \in \mathcal{S}^m_{\mathrm{cl}}(T^*\mathring{\mathcal{M}}, \mathrm{Hom}(\mathrm{V}, \tilde{\mathrm{V}}))$ meeting the symbol estimates

$$\left|(tD_y)^\beta (tD_t)^j D^\alpha_x D^\gamma_{\tilde{\eta},\tilde{\tau},\xi} a(y,t,x;\tilde{\eta},\tilde{\tau},\xi)\right| \le c_{\beta,j,\alpha,\gamma} \, \langle(\tilde{\eta},\tilde{\tau},\xi)\rangle^{m-|\gamma|}$$

uniformly in (y,t,x) on compact subsets of coordinate neighbourhoods close to the boundary, for all multi-indices β, j, α and γ. While this is a natural way in the case of spaces (2.4.4), it is no longer so for the spaces of Definition 2.2.2. These are given from the very beginning as 'twisted' spaces along the edge, and so the natural way of quantising a typical symbol as an operator on the spaces in question is to assign a pseudodifferential operator on the edge to this symbol. This idea was developed by the first author in [Sch89b, Sch90]; the corresponding calculus of pseudodifferential operators with operator-valued symbols on the edge is known as 'edge calculus'. The core of the approach is to reformulate a given typical symbol close to $\partial\mathcal{M}$ as a symbol along the edge taking its values in the symbol algebras in fibres of $\mathcal{M}$ over S.

We begin by localising the problem of quantisation to a collar neighbourhood of the boundary. To this end, pick such a neighbourhood $O \cong \partial\mathcal{M} \times \mathbb{R}_+$ in $\mathcal{M}$. Let $\varphi_b \in C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+)$ be a cut-off function at $t = 0$. Via the above identification we may regard φ_b as being defined on the whole manifold $\mathcal{M}$. Set $\varphi_i = 1 - \varphi_b$, then $\varphi_i \in C^\infty(\mathcal{M})$ vanishes in a neighbourhood of $\partial\mathcal{M}$. We now choose C^∞ functions ψ_b and ψ_i on $\mathcal{M}$, such that $\mathrm{supp}\,\psi_{\mathrm{b}} \subset \mathrm{O}$, $\mathrm{supp}\,\psi_{\mathrm{i}} \subset \mathcal{M} \setminus \partial\mathcal{M}$ and ψ_ν "covers" φ_ν, i.e., $\psi_\nu \equiv 1$ on the support of φ_ν. To each symbol $a \in t^{-m}\,\mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{M}, \mathrm{Hom}(\mathrm{V}, \tilde{\mathrm{V}}))$ we can assign a classical pseudodifferential operator $\mathrm{op(a)}$ of order m in the interior of $\mathcal{M}$, as usually. In fact, $\mathrm{op(a)}$ is determined uniquely up to a smoothing operator in the interior of $\mathcal{M}$. It is easy to see from the pseudolocality property of pseudodifferential operators that $\mathrm{op(a)} = \varphi_{\mathrm{b}}\mathrm{op(a)}\psi_{\mathrm{b}} + \varphi_{\mathrm{i}}\mathrm{op(a)}\psi_{\mathrm{i}}$ holds modulo smoothing operators on $\mathring{\mathcal{M}}$. Indeed,

$$\mathrm{op(a)} - \varphi_{\mathrm{b}}\mathrm{op(a)}\psi_{\mathrm{b}} - \varphi_{\mathrm{i}}\mathrm{op(a)}\psi_{\mathrm{i}} = \varphi_{\mathrm{b}}\mathrm{op(a)}(1-\psi_{\mathrm{b}}) + \varphi_{\mathrm{i}}\mathrm{op(a)}(1-\psi_{\mathrm{i}}),$$

and for the proof it suffices to note that the supports of φ_ν and $(1-\psi_\nu)$ are disjoint. As the supports of φ_i and ψ_i do not meet the boundary, the operator $\varphi_i\mathrm{op(a)}\psi_{\mathrm{i}}$ is well-defined on the standard Sobolev spaces and thus extends to a continuous mapping $H^s_{\mathrm{loc}}(\mathring{\mathcal{M}}, V) \to H^s_{\mathrm{comp}}(\mathring{\mathcal{M}}, \tilde{V})$, for each $s \in \mathbb{R}$. The operator $\varphi_b \mathrm{op(a)} \psi_b$ has in general no extension to the weighted Sobolev spaces $H^{s,\gamma}(\mathcal{M}, V)$. However, it is supported in the collar neighbourhood of $\partial\mathcal{M}$ where the manifold $\mathcal{M}$ is stratified over S, with $\bar{\mathbb{R}}_+ \times X$ as a typical fibre. We make use of this fibration to reformulate, modulo smoothing operators in the interior of O, the restriction of op(a) to O as a pseudodifferential operator along S with a symbol taking its values in the cone algebras in the fibres of $\mathcal{M}$ over S. To this end, we observe from the very beginning that the action $\varphi_b \mathrm{op(a)} \psi_b$ does not include the values of the symbol a for t large enough, with the exception of those entering into the covariables $t\tau$ and $t\eta$. We can therefore modify a in t away from the support of φ_b without affecting the operator $\varphi_b \mathrm{op(a)} \psi_b$. In the sequel we often assume that the symbols in question vanish for $t > 0$ large enough. As explained, this assumption actually contains no loss of generality. Moreover, we restrict our attention to scalar symbols; the extension to the general case is straightforward.

We first demonstrate these techniques by those typical symbols which are polynomials in τ. Let A be a typical differential operator of order m in the stretched wedge $\mathcal{W} = \Omega' \times \bar{\mathbb{R}}_+ \times X$, i.e.,

$$A = \frac{1}{t^m} \sum_{|\beta|+j \le m} A_{\beta,j}(y,t)\,(tD_y)^\beta (tD_t)^j$$

where $A_{\beta,j} \in C^\infty_{\mathrm{loc}}(\Omega' \times \bar{\mathbb{R}}_+, \mathrm{Diff}^{m-|\beta|-j}(X))$.

In this case,

$$\begin{array}{rcl} \sigma^m(A)(y,t,x;\eta,\tau,\xi) &=& t^{-m} \sum\limits_{|\beta|+j\le m} \sigma^{m-|\beta|-j}(A_{\beta,j})(y,t,x;\xi)\,(t\eta)^\beta (t\tau)^j, \\ {}^b\sigma^m(A)(y,t,x;\tilde{\eta},\tilde{\tau},\xi) &=& \sum\limits_{|\beta|+j\le m} \sigma^{m-|\beta|-j}(A_{\beta,j})(y,t,x;\xi)\,\tilde{\eta}^\beta \tilde{\tau}^j. \end{array} \tag{3.2.1}$$

Given a symbol $a \in \mathcal{S}^m_{\mathrm{cl}}(\mathring{\mathcal{W}} \times \mathbb{R}^{q+1+n})$, we may assign a pseudodifferential operator to a, which acts in several of the variables t, x and y. In order to indicate the variables to which the pseudodifferential action refers with respect to the Fourier transform, we write $\mathrm{op}_{\Psi(t,x)}(a)$, $\mathrm{op}_{\Psi(t,x,y)}(a)$, and so on.

Write $A = \mathrm{op}(\mathrm{a}(\mathrm{y}, \eta))$, where

$$a(y,\eta) = \frac{1}{t^m} \sum_{|\beta|+j \le m} A_{\beta,j}(y,t)\,(t\eta)^\beta (tD_t)^j.$$

It is easy to check, for each $s, \gamma \in \mathbb{R}$, that $a(y,\eta)$ induces a family of continuous mappings $a(y,\eta)\colon H^{s,\gamma}_{\mathrm{loc}}(\bar{\mathbb{R}}_+ \times X) \to H^{s-m,\gamma-m}_{\mathrm{loc}}(\bar{\mathbb{R}}_+ \times X)$ parametrised by $(y,\eta) \in \Omega' \times \mathbb{R}^q$. Moreover, if the coefficients $A_{\beta,j}$ are independent of t for $t > 0$ large enough, then

$$a(y,\eta) \in \mathcal{S}^m(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X)))$$

for each $s, \gamma \in \mathbb{R}$, the symbol spaces in question being defined in Section 3.3.

Fix a cut-off function ω on $\bar{\mathbb{R}}_+$, so that $\omega(t) = 1$ for $t \le r$ and $\omega(t) = 0$ for $t \ge R$, where $0 < r < R < \infty$. Then $\varphi_0 = \omega$ and $\varphi_\infty = 1 - \omega$ give the partition of unity on the semiaxis subordinate to the covering $I_0 = [0, 2R)$, $I_\infty = (\frac{1}{2}r, \infty)$.

We now choose $\psi_0, \psi_\infty \in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+)$ such that $\operatorname{supp} \psi_\nu \subset \mathrm{I}_\nu$ and $\psi_\nu = 1$ near $\operatorname{supp} \varphi_\nu$.

Lemma 3.2.1 *Under the above assumptions, it follows that* $a = a_0 + a_\infty$, *where*

$$\begin{aligned} a_0(y,\eta) &= \varphi_0(t\langle\eta\rangle)\, a(y,\eta)\, \psi_0(t\langle\eta\rangle), \\ a_\infty(y,\eta) &= \varphi_\infty(t\langle\eta\rangle)\, a(y,\eta)\, \psi_\infty(t\langle\eta\rangle). \end{aligned}$$

Proof. Indeed,

$$\begin{aligned} a(y,\eta) &= \varphi_0(t\langle\eta\rangle)\, a(y,\eta) + \varphi_\infty(t\langle\eta\rangle)\, a(y,\eta) \\ &= \varphi_0(t\langle\eta\rangle)\, a(y,\eta)\, \psi_0(t\langle\eta\rangle) + \varphi_0(t\langle\eta\rangle)\, a(y,\eta)\, (1 - \psi_0(t\langle\eta\rangle)) \\ &\quad + \varphi_\infty(t\langle\eta\rangle)\, a(y,\eta)\, \psi_\infty(t\langle\eta\rangle) + \varphi_\infty(t\langle\eta\rangle)\, a(y,\eta)\, (1 - \psi_\infty(t\langle\eta\rangle)), \end{aligned}$$

which gives the desired equality because $a(y,\eta)$ is differential, and so local, in t.

□

For each $s, \gamma \in \mathbb{R}$, we still have

$$\begin{aligned} a_0(y,\eta) &\in \mathcal{S}^m(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X))), \\ a_\infty(y,\eta) &\in \mathcal{S}^m(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X))), \end{aligned}$$

provided that the coefficients $A_{\beta,j}$ are independent of t for $t > 0$ large enough.

The next task is to rewrite $a(y,\eta)$ as a pseudodifferential operator in t with respect to the Mellin transform, thus making $a_0(y,\eta)$ more prepared to act in the cone Sobolev spaces close to the singularity $t = 0$. Recall that by a Mellin operator with a symbol $s(t,z) \in C^\infty(\mathbb{R}_+, \mathcal{L}^m(X; \Gamma_{-\gamma}))$ we mean

$$\mathrm{op}_{\mathcal{M},\gamma}(\mathrm{s})\mathrm{u}\,(\mathrm{t}) = \frac{1}{2\pi} \int_{\Gamma_{-\gamma}} \mathrm{t}^{\mathrm{iz}} \mathrm{s}(\mathrm{t},\mathrm{z}) \mathcal{M}\mathrm{u}(\mathrm{z})\, \mathrm{dz}, \quad \mathrm{t} > 0, \tag{3.2.2}$$

defined on functions $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}_+, C^\infty(X))$. Here,

$$\mathcal{M}u(z) = \int_0^\infty t^{-iz} u(t) \frac{dt}{t}$$

stands for the Mellin transform of u.

Lemma 3.2.2 *For every* $\gamma \in \mathbb{R}$, *we have*

$$a(y,\eta) = \frac{1}{t^m}\, \mathrm{op}_{\mathcal{M},\gamma} \left(\tilde{\mathrm{h}}(\mathrm{y}, \mathrm{t}; \mathrm{t}\eta, \mathrm{z}) \right) \quad \textit{on} \quad \mathrm{C}^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times \mathrm{X}),$$

where $\tilde{h}(y,t;\tilde{\eta},z) = \sum_{|\beta|+j \le m} A_{\beta,j}(y,t)\, \tilde{\eta}^\beta z^j$.

Proof. If $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X)$, then by the properties of the Mellin transform

$$a(y,\eta)\,u = \frac{1}{t^m}\,\mathrm{op}_{\mathcal{M}}\left(\tilde{\mathrm{h}}(\mathrm{y},\mathrm{t};\mathrm{t}\eta,\mathrm{z})\right)\mathrm{u},$$

with $\tilde{h}$ given in the lemma. This corresponds to our statement for $\gamma = 0$.

On the other hand, from $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X)$ it follows that the Mellin transform of u is an entire function in the complex plane, satisfying

$$\sup_{z\in\Gamma_{-\gamma}} (1+|z|)^\nu\,|\mathcal{M}u\,(z,x)| < \infty, \quad \nu = 0,1,\ldots,$$

uniformly in γ on finite intervals of $\mathbb{R}$. (We fix $x \in X$ here.) Since $\tilde{h}$ is a polynomial in z, the *Cauchy theorem* shows that

$$\mathrm{op}_{\mathcal{M},\gamma}(\tilde{\mathrm{h}})\mathrm{u}\,(\mathrm{t},\mathrm{x}) = \frac{1}{2\pi}\int_{\Gamma_{-\gamma}} \mathrm{t}^{\mathrm{iz}}\tilde{\mathrm{h}}(\mathrm{y},\mathrm{t};\mathrm{t}\eta,\mathrm{z})\mathcal{M}\mathrm{u}(\mathrm{z},\mathrm{x})\,\mathrm{dz}$$

is independent of γ. This completes the proof.

□

Combining Lemmas 3.2.1 and 3.2.2 yields a precise representation

$$a(y,\eta) = a_0(y,\eta) + a_\infty(y,\eta) \quad \text{on} \quad C^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X), \tag{3.2.3}$$

with

$$\begin{aligned} a_0(y,\eta) &= t^{-m}\,\varphi_0(t\langle\eta\rangle)\,\mathrm{op}_{\mathcal{M},\gamma}\left(\tilde{\mathrm{h}}(\mathrm{y},\mathrm{t};\mathrm{t}\eta,\mathrm{z})\right)\,\psi_0(\mathrm{t}\langle\eta\rangle),\\ a_\infty(y,\eta) &= \varphi_\infty(t\langle\eta\rangle)\,a(y,\eta)\,\psi_\infty(t\langle\eta\rangle) \end{aligned}$$

and $\tilde{h}$ a polynomial in z.

Conversely, from such a representation for a family $a(y,\eta)$ we could easily conclude that $a(y,\eta)$ is a typical symbol with values in the cone algebra on $\bar{\mathbb{R}}_+ \times X$. In fact, equality (3.2.3) allows one to extend the family $a(y,\eta)$, first defined on functions supported away from $t=0$, to the weighted spaces $H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)$, $s \in \mathbb{R}$. Recall that the definition of $H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)$ close to $t=0$ refers just to the Mellin transform on the weight line $\Gamma_{-\gamma}$. Therefore, in the sequel we shall derive a similar representation for arbitrary typical symbols.

For arbitrary typical symbols, it is no longer possible to obtain a precise representation like (3.2.3). When studying arbitrary symbols we should look for such a representation modulo “smoothing” operators.

Let $a \in t^{-m}\,\mathcal{S}^m({}^bT^*\mathcal{W})$ be a typical symbol of order m in a stretched wedge $\mathcal{W} = \Omega' \times \bar{\mathbb{R}}_+ \times X$, i.e.,

$$a(y,t,x;\eta,\tau,\xi) = \frac{1}{t^m}\,\tilde{a}(y,t,x;t\eta,t\tau,\xi)$$

with some $\tilde{a}(y,t,x;\tilde{\eta},\tilde{\tau},\xi) \in \mathcal{S}^m(\mathcal{W} \times \mathbb{R}^{q+1+n})$.

We may assign a pseudodifferential operator to a, which acts in several of the variables y, t and x. For indicating the variables to which the pseudodifferential action refers with respect to the Fourier transform, we write $\mathrm{op}_{\mathcal{F}_{t,x}}(a)$, $\mathrm{op}_{\mathcal{F}_x}(a)$, and so on. Thus,

$$\mathrm{op}_{\mathcal{F}_{t,x}}(a)\,(y,\eta) = \frac{1}{t^m}\,\mathcal{F}^{-1}_{(\tau,\xi)\mapsto(t,x)}\tilde{a}(y,t,x;t\eta,t\tau,\xi)\mathcal{F}_{(t,x)\mapsto(\tau,\xi)},$$

the operator family belonging to $C^\infty_{\mathrm{loc}}(\Omega', \Psi^m(\mathbb{R}_+\times X;\mathbb{R}^q_\eta))$.

Lemma 3.2.3 *Under the above assumptions, we have*

$$\mathrm{op}_{\mathcal{F}_{t,x}}(a) = a_0 + a_\infty \mod C^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+\times X;\mathbb{R}^q_\eta)),$$

where

$$\begin{aligned} a_0(y,\eta) &= \varphi_0(t\langle\eta\rangle)\,\mathrm{op}_{\mathcal{F}_{t,x}}(a)\,(y,\eta)\,\psi_0(t\langle\eta\rangle), \\ a_\infty(y,\eta) &= \varphi_\infty(t\langle\eta\rangle)\,\mathrm{op}_{\mathcal{F}_{t,x}}(a)\,(y,\eta)\,\psi_\infty(t\langle\eta\rangle). \end{aligned}$$

Proof. It is sufficient to imitate the proof of Lemma 3.2.1 and take into account that

$$\begin{aligned} \varphi_0(t\langle\eta\rangle)\,\mathrm{op}_{\mathcal{F}_{t,x}}(a)\,(y,\eta)\,(1-\psi_0(t\langle\eta\rangle)) &\in C^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+\times X;\mathbb{R}^q_\eta)), \\ \varphi_\infty(t\langle\eta\rangle)\,\mathrm{op}_{\mathcal{F}_{t,x}}(a)\,(y,\eta)\,(1-\psi_\infty(t\langle\eta\rangle)) &\in C^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+\times X;\mathbb{R}^q_\eta)) \end{aligned}$$

because the supports of $\varphi_\nu(t\langle\eta\rangle)$ and $1-\psi_\nu(t\langle\eta\rangle)$ are disjoint.

□

The task is now to find a suitable Mellin reformulation of the operator family $a_0(y,\eta)$. To this end, we make use of the following result which will be referred to as the *Mellin quantisation.*

Theorem 3.2.4 *For every $a \in \mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{W})$ there exists an entire function $\tilde{f}(y,t,x;\tilde{\eta},z,\xi)$ of $z\in\mathbb{C}$ with values in $\mathcal{S}^m_{\mathrm{cl}}(\mathcal{W}\times\mathbb{R}^{q+n}_{\tilde{\eta},\xi})$, such that*

1) $\tilde{f}(y,t,x;\tilde{\eta},\tau-i\gamma,\xi) \in \mathcal{S}^m_{\mathrm{cl}}(\mathcal{W}\times\mathbb{R}^{q+1+n}_{\tilde{\eta},\tau,\xi})$ uniformly in γ on finite intervals of $\mathbb{R}$; and

2) set $f(y,t,x;\eta,z,\xi) = \tilde{f}(y,t,x;t\eta,z,\xi)$, then, for each $\gamma\in\mathbb{R}$, we have the mixed Mellin-Fourier representation

$$\mathrm{op}_{\mathcal{F}_{t,x}}(a)\,(y,\eta) = \mathrm{op}_{\mathcal{M},\gamma}\left(\mathrm{op}_{\mathcal{F}_x}(f)\right)(y,\eta) \mod C^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+\times X;\mathbb{R}^q_\eta)). \tag{3.2.4}$$

Proof. Let us assume for a moment that (3.2.4) holds for some *single* $\gamma\in\mathbb{R}$. As $f(y,t,x;\eta,z,\xi)$ is holomorphic in $z\in\mathbb{C}$ and of polynomial growth in each strip $c' \le \Im z \le c''$ with finite $c' < c''$, it follows from the *Cauchy theorem* that

$$\mathrm{op}_{\mathcal{M},\gamma}\left(\mathrm{op}_{\mathcal{F}_x}(f)\right) = \mathrm{op}_{\mathcal{M},\delta}\left(\mathrm{op}_{\mathcal{F}_x}(f)\right)$$

for all $\delta\in\mathbb{R}$. Hence it is sufficient to prove (3.2.4) for a convenient weight γ. We take $\gamma = 0$.

Given a symbol $a \in \mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{W})$, we find an $\tilde{a} \in \mathcal{S}^m_{\mathrm{cl}}(\mathcal{W} \times \mathbb{R}^{q+1+n})$ with the property that $a(y,t,x;\eta,\tau,\xi) = \tilde{a}(y,t,x;t\eta,t\tau,\xi)$.

Set
$$s_m(y,t,x;\tilde{\eta},\tau,\xi) = \tilde{a}(y,t,x;\tilde{\eta},\tau,\xi);$$
then $s_m \in \mathcal{S}^m_{\mathrm{cl}}(\mathcal{W}\times\mathbb{R}^{q+1+n})$ and a straightforward analysis shows that

$$\begin{aligned} &\mathrm{op}_{\mathcal{F}_{t,x}}\left(\tilde{a}(y,t,x;\tilde{\eta},t\tau,\xi)\right) - \mathrm{op}_{\mathcal{M}}\left(\mathrm{op}_{\mathcal{F}_x}\left(s_m(y,t,x;\tilde{\eta},\tau,\xi)\right)\right) \\ &\quad = \mathrm{op}_{\mathcal{F}_{t,x}}\left(\tilde{a}_{m-1}(y,t,x;\tilde{\eta},t\tau,\xi)\right) \quad \mathrm{mod}\ \ C^\infty_{\mathrm{loc}}(\Omega',\Psi^{-\infty}(\mathbb{R}_+\times X;\mathbb{R}^q_{\tilde{\eta}})) \end{aligned}$$

for some $\tilde{a}_{m-1} \in \mathcal{S}^{m-1}_{\mathrm{cl}}(\mathcal{W}\times\mathbb{R}^{q+1+n}_{\tilde{\eta},\tilde{\tau},\xi})$.

We now proceed by induction. For $j = 1,2,\ldots$, there is a symbol $s_{m-j} \in \mathcal{S}^{m-j}_{\mathrm{cl}}(\mathcal{W}\times\mathbb{R}^{q+1+n})$ such that

$$\begin{aligned} &\mathrm{op}_{\mathcal{F}_{t,x}}\left(\tilde{a}_{m-j}(y,t,x;\tilde{\eta},t\tau,\xi)\right) - \mathrm{op}_{\mathcal{M}}\left(\mathrm{op}_{\mathcal{F}_x}\left(s_{m-j}(y,t,x;\tilde{\eta},\tau,\xi)\right)\right) \\ &\quad = \mathrm{op}_{\mathcal{F}_{t,x}}\left(\tilde{a}_{m-j-1}(y,t,x;\tilde{\eta},t\tau,\xi)\right) \quad \mathrm{mod}\ \ C^\infty_{\mathrm{loc}}(\Omega',\Psi^{-\infty}(\mathbb{R}_+\times X;\mathbb{R}^q_{\tilde{\eta}})) \end{aligned} \tag{3.2.5}$$

with some $\tilde{a}_{m-j-1} \in \mathcal{S}^{m-j-1}_{\mathrm{cl}}(\mathcal{W}\times\mathbb{R}^{q+1+n}_{\tilde{\eta},\tilde{\tau},\xi})$.

For $\tilde{a}_m := \tilde{a}$, we thus obtain the sequences $(\tilde{a}_{m-j})$ and (s_{m-j}), as above. Let
$$s(y,t,x;\tilde{\eta},\tau,\xi) \sim \sum_{j=0}^{\infty} s_{m-j}(y,t,x;\tilde{\eta},\tau,\xi),$$
the asymptotic sum being taken in $\mathcal{S}^m_{\mathrm{cl}}(\mathcal{W}\times\times\mathbb{R}^{q+1+n})$.

It follows from (3.2.5) that

$$\begin{aligned} &\mathrm{op}_{\mathcal{F}_{t,x}}\left(\tilde{a}(y,t,x;\tilde{\eta},t\tau,\xi)\right) \\ &\quad = \sum_{j=0}^{J-1} \mathrm{op}_{\mathcal{M}}\left(\mathrm{op}_{\mathcal{F}_x}\left(s_{m-j}(y,t,x;\tilde{\eta},\tau,\xi)\right)\right) + \mathrm{op}_{\mathcal{F}_{t,x}}\left(\tilde{a}_{m-J}(y,t,x;\tilde{\eta},t\tau,\xi)\right) \end{aligned}$$

modulo $C^\infty_{\mathrm{loc}}(\Omega',\Psi^{-\infty}(\mathbb{R}_+\times X;\mathbb{R}^q_{\tilde{\eta}}))$, for each $J = 1,2,\ldots$, whence

$$\begin{aligned} &\mathrm{op}_{\mathcal{F}_{t,x}}\left(\tilde{a}(y,t,x;\tilde{\eta},t\tau,\xi)\right) \\ &\quad = \ \mathrm{op}_{\mathcal{M}}\left(\mathrm{op}_{\mathcal{F}_x}\left(s(y,t,x;\tilde{\eta},\tau,\xi)\right)\right) \quad \mathrm{mod}\ \ C^\infty_{\mathrm{loc}}(\Omega',\Psi^{-\infty}(\mathbb{R}_+\times X;\mathbb{R}^q_{\tilde{\eta}})). \end{aligned}$$

We now invoke a *kernel cut-off* construction of Schulze (see for instance [Sch98, 2.2.2]) which ensures that every Mellin pseudodifferential operator can be written, modulo smoothing remainders, as a Mellin operator with a symbol which extends to an entire function in the covariable. Thus, there is a new function $\tilde{f}(y,t,x;\tilde{\eta},z,\xi) \in \mathcal{A}(\mathbb{C}_z,\mathcal{S}^m_{\mathrm{cl}}(\mathcal{W}\times\mathbb{R}^{q+n}_{\tilde{\eta},\xi}))$ such that

$$\tilde{f}(y,t,x;\tilde{\eta},\tau,\xi) = s(y,t,x;\tilde{\eta},\tau,\xi) \quad \mathrm{mod}\ \ \mathcal{S}^{-\infty}(\mathcal{W}\times\mathbb{R}^{q+1+n}_{\tilde{\eta},\tau,\xi}).$$

Hence

$$\mathrm{op}_{\mathcal{M}}\left(\mathrm{op}_{\mathcal{F}_x}\left(s(y,t,x;\tilde{\eta},\tau,\xi)\right)\right) = \mathrm{op}_{\mathcal{M}}\left(\mathrm{op}_{\mathcal{F}_x}\left(\tilde{f}(y,t,x;\tilde{\eta},z,\xi)\right)\right)$$

modulo $C^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+ \times \mathrm{X}; \mathbb{R}^{\mathrm{q}}_{\tilde\eta})$.

Finally, substituting $\tilde\eta = t\eta$ and interpreting t as an action from the left we get

$$\begin{aligned}
\mathrm{op}_{\mathcal{F}_{t,x}}(\mathrm{a})(\mathrm{y},\eta) &= \mathrm{op}_{\mathcal{F}_{t,x}}(\tilde{\mathrm{a}}(\mathrm{y},\mathrm{t},\mathrm{x};\mathrm{t}\eta,\mathrm{t}\tau,\xi)) \\
&= \mathrm{op}_{\mathcal{M}}\left(\mathrm{op}_{\mathcal{F}_x}(\mathrm{s}(\mathrm{y},\mathrm{t},\mathrm{x};\mathrm{t}\eta,\tau,\xi))\right) \quad \mathrm{mod} \;\; \mathrm{C}^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+ \times \mathrm{X}; \mathbb{R}^{\mathrm{q}}_{\eta})) \\
&= \mathrm{op}_{\mathcal{M}}\left(\mathrm{op}_{\mathcal{F}_x}\left(\tilde{\mathrm{f}}(\mathrm{y},\mathrm{t},\mathrm{x};\mathrm{t}\eta,\mathrm{z},\xi)\right)\right) \quad \mathrm{mod} \;\; \mathrm{C}^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+ \times \mathrm{X}; \mathbb{R}^{\mathrm{q}}_{\eta})),
\end{aligned}$$

which completes the proof.

□

The theorem is still true if we drop the assumption that a is classical. For a refined proof in the case of non-classical symbols we refer the reader to Gil, Schulze and Seiler [GSS97]. They even show explicit formulas for the holomorphic symbol f and for the remainder, thus arriving at a topological isomorphism between the symbol classes involved.

We can now return to reformulating the operator family a_0 of Lemma 3.2.3. Denote by $\mathcal{M}(\mathbb{C}, \Psi^{\mathrm{m}}(\mathrm{X}; \mathbb{R}^{\mathrm{q}}_{\tilde\eta}))$ the subspace of $\mathcal{A}(\mathbb{C}, \Psi^{\mathrm{m}}(\mathrm{X}; \mathbb{R}^{\mathrm{q}}_{\tilde\eta}))$ consisting of all functions $h(z)$ such that $h(\tau - i\gamma) \in \Psi^{\mathrm{m}}(\mathrm{X}; \mathbb{R}^{\mathrm{q}+1}_{\tilde\eta,\tau})$ uniformly in γ on finite intervals of $\mathbb{R}$.

Corollary 3.2.5 *Given any $a \in t^{-m}\,\mathcal{S}^m({}^bT^*\mathcal{W})$, there is a C^∞ function $\tilde h(y,t;\tilde\eta,z)$ of $(y,t) \in \Omega' \times \bar{\mathbb{R}}_+$ with values in $\mathcal{M}(\mathbb{C}, \Psi^{\mathrm{m}}(\mathrm{X}; \mathbb{R}^{\mathrm{q}}_{\tilde\eta}))$, such that*

$$\mathrm{op}_{\mathcal{F}_{t,x}}(\mathrm{a})(\mathrm{y},\eta) = \frac{1}{\mathrm{t}^{\mathrm{m}}}\,\mathrm{op}_{\mathcal{M},\gamma}(\mathrm{h})(\mathrm{y},\eta) \quad \mathrm{mod} \;\; \mathrm{C}^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+ \times \mathrm{X}; \mathbb{R}^{\mathrm{q}}_{\eta}))$$

for all $\gamma \in \mathbb{R}$, where $h(y,t;\eta,z) = \tilde h(y,t;t\eta,z)$.

Proof. From Theorem 3.2.4 it immediately follows that there is an $\tilde f(y,t,x;\tilde\eta,z,\xi) \in \mathcal{A}(\mathbb{C}, \mathcal{S}^m(\mathcal{W} \times \mathbb{R}^{q+n}))$ such that

- $\tilde f(y,t,x;\tilde\eta,\tau - i\gamma,\xi) \in \mathcal{S}^m(\mathcal{W} \times \mathbb{R}^{q+1+n}_{\tilde\eta,\tau,\xi})$ uniformly in γ on finite segments in $\mathbb{R}$; and
- for each $\gamma \in \mathbb{R}$, we have

  $$\mathrm{op}_{\mathcal{F}_{t,x}}(\mathrm{a})(\mathrm{y},\eta) = \frac{1}{\mathrm{t}^{\mathrm{m}}}\,\mathrm{op}_{\mathcal{M},\gamma}\left(\mathrm{op}_{\mathcal{F}_x}(\tilde{\mathrm{f}}(\mathrm{y},\mathrm{t},\mathrm{x};\mathrm{t}\eta,\mathrm{z},\xi))\right)$$

  modulo $C^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+ \times \mathrm{X}; \mathbb{R}^{\mathrm{q}}_{\eta}))$.

The lemma now follows with

$$\tilde h(y,t;\tilde\eta,z) = \mathrm{op}_{\mathcal{F}_x}(\tilde{\mathrm{f}}(\mathrm{y},\mathrm{t},\mathrm{x};\tilde\eta,\mathrm{z},\xi)).$$

□

Of course, the reader has recognised the explicit formulas of the proof of Lemma 3.2.2 beyond the abstract framework of the proof of Corollary 3.2.5 .

Combining Lemma 3.2.3 and Corollary 3.2.5 we can assert that

$$\mathrm{op}_{\mathcal{F}_{t,x}}(\mathrm{a}) = \mathrm{a}_0 + \mathrm{a}_\infty \quad \mathrm{mod} \;\; \mathrm{C}^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+ \times \mathrm{X}; \mathbb{R}^q_\eta)), \tag{3.2.6}$$

where

$$\begin{aligned} a_0(y,\eta) &= t^{-m}\, \varphi_0(t\langle\eta\rangle)\, \mathrm{op}_{\mathcal{M},\gamma}(\tilde{\mathrm{h}}(\mathrm{y},\mathrm{t};\mathrm{t}\eta,\mathrm{z}))\, \psi_0(\mathrm{t}\langle\eta\rangle), \\ a_\infty(y,\eta) &= \varphi_\infty(t\langle\eta\rangle)\, \mathrm{op}_{\mathcal{F}_{t,x}}(\mathrm{a})\,(\mathrm{y},\eta)\, \psi_\infty(\mathrm{t}\langle\eta\rangle). \end{aligned}$$

Proposition 3.2.6 *Assume that a vanishes for $t > 0$ large enough. For each $s, \gamma \in \mathbb{R}$, it follows that*

$$\begin{aligned} a_0(y,\eta) &\in \mathcal{S}^m\left(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X))\right), \\ a_\infty(y,\eta) &\in \mathcal{S}^m\left(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X))\right). \end{aligned}$$

Note that the restriction on the support of a in t is needed only in the proof of symbol estimates for $a_\infty(y,\eta)$.

Proof. We give the proof only for $a_0(y,\eta)$; the proof for $a_\infty(y,\eta)$ is similar (cf. [FST98a]).

Pick $u \in H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)$. Our task is to find a suitable estimate for the norm of $\kappa^{-1}_{\langle\eta\rangle}\left(D^\alpha_y D^\beta_\eta a_0(y,\eta)\right)\kappa_{\langle\eta\rangle} u$ in $H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X)$. By duality and interpolation, we might even assume that $s-m$ is a non-negative integer.

Since $\langle\eta\rangle$ coincides with $|\eta|$ for η away from a ball, a homogeneity argument shows that

$$|D^\gamma \langle\eta\rangle| \le c_\gamma \langle\eta\rangle^{1-|\gamma|} \quad \text{for all} \quad \eta \in \mathbb{R}^q, \tag{3.2.7}$$

the constant c_γ being independent of η. Hence $D^\alpha_y D^\beta_\eta a_0(y,\eta)$ is a sum of terms of the form

$$\frac{1}{t^m}\, \langle\eta\rangle^{-|\beta|}\, \varphi(t\langle\eta\rangle)\, \mathrm{op}_{\mathcal{M},\gamma}\left(\tilde{\mathrm{h}}(\mathrm{y},\mathrm{t};\mathrm{t}\eta,\mathrm{z})\right)\, \psi(\mathrm{t}\langle\eta\rangle)$$

modulo bounded factors depending only on η, where $\varphi \in C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+)$ is supported on the support of φ_0, $\psi \in C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+)$ is supported on the support of ψ_0 and $\tilde{h}(y,t;\tilde{\eta},z)$ is a C^∞ function of $(y,t) \in \Omega' \times \bar{\mathbb{R}}_+$ with values in $\mathcal{M}(\mathbb{C}, \Psi^{\mathrm{m}}(\mathrm{X}; \mathbb{R}^q_{\tilde{\eta}})))$.

We are thus reduced to proving the desired estimate for $\alpha = \beta = 0$. To do this, we note that

$$\begin{aligned} \mathcal{M}_{t\mapsto z}\left(\psi_0(t\langle\eta\rangle)\, \kappa_{\langle\eta\rangle} u\right) &= \mathcal{M}_{t\mapsto z}\left(\kappa_{\langle\eta\rangle}(\psi_0 u)\right) \\ &= \langle\eta\rangle^{\frac{1+n}{2}+iz}\, \mathcal{M}_{t\mapsto z}(\psi_0 u), \end{aligned}$$

and so

$$\begin{aligned} &\left(\kappa^{-1}_{\langle\eta\rangle}\, a_0(y,\eta)\, \kappa_{\langle\eta\rangle} u\right)(t) \\ &\quad = \langle\eta\rangle^m\, \varphi_0(t)\, \frac{1}{t^m}\, \frac{1}{2\pi} \int_{\Gamma_{-\gamma}} t^{iz}\, h\left(y, \frac{t}{\langle\eta\rangle}; t\frac{\eta}{\langle\eta\rangle}, z\right)\, \mathcal{M}_{t\mapsto z}(\psi_0 u)\, dz. \end{aligned}$$

Using this equality, we check at once that

$$\|\kappa_{\langle\eta\rangle}^{-1} a_0(y,\eta)\kappa_{\langle\eta\rangle} u\|_{\mathcal{H}^{s-m,\gamma-m}(\bar{\mathbb{R}}_+\times X)} \leq c\, \langle\eta\rangle^m \, \|\psi_0 u\|_{\mathcal{H}^{s,\gamma}(\bar{\mathbb{R}}_+\times X)}, \quad \eta \in \mathbb{R}^q, \tag{3.2.8}$$

uniformly in y on compact subsets of Ω', the constant c being independent of η and u.

By (2.2.2), the norms $\|u\|_{H^{s,\gamma}(\bar{\mathbb{R}}_+\times X)}$ and $\|u\|_{\mathcal{H}^{s,\gamma}(\bar{\mathbb{R}}_+\times X)}$ are equivalent on functions u supported in $t \leq R'$, for any fixed $R' < \infty$. Hence (3.2.8) implies

$$\begin{aligned}
\|\kappa_{\langle\eta\rangle}^{-1} a_0(y,\eta)\kappa_{\langle\eta\rangle} u\|_{H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+\times X)} &\sim \|\kappa_{\langle\eta\rangle}^{-1} a_0(y,\eta)\kappa_{\langle\eta\rangle} u\|_{\mathcal{H}^{s-m,\gamma-m}(\bar{\mathbb{R}}_+\times X)} \\
&\leq c\, \langle\eta\rangle^m \, \|\psi_0 u\|_{\mathcal{H}^{s,\gamma}(\bar{\mathbb{R}}_+\times X)} \\
&\sim c\, \langle\eta\rangle^m \, \|\psi_0 u\|_{H^{s,\gamma}(\bar{\mathbb{R}}_+\times X)} \\
&\leq C\, \langle\eta\rangle^m \, \|u\|_{H^{s,\gamma}(\bar{\mathbb{R}}_+\times X)}, \qquad \eta \in \mathbb{R}^q,
\end{aligned}$$

uniformly in y on compact subsets of Ω', with C a constant independent of η and u.

Summarizing we deduce that, for each compact set $K \subset \Omega'$ and multi-indices $\alpha, \beta \in \mathbb{Z}_+^q$, there is a constant $c_{K,\alpha,\beta}$ such that

$$\left\|\kappa_{\langle\eta\rangle}^{-1} \left(D_y^\alpha D_\eta^\beta a_0(y,\eta)\right) \kappa_{\langle\eta\rangle}\right\|_{\mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+\times X), H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+\times X))} \leq c_{K,\alpha,\beta}\, \langle\eta\rangle^{m-|\beta|}$$

whenever $y \in K$ and $\eta \in \mathbb{R}^q$. This is the desired conclusion.

□

Up to now we have tacitly assumed that the support of the symbol $a \in t^{-m}\, \mathcal{S}^m({}^bT^*\mathcal{W})$ with respect to x lies in the domain of some chart on X. We will now show how to dispense with this assumption. For this purpose we fix a finite atlas $(h_j'', O_j'')_{j\in J}$ on X, h_j'' being a diffeomorphism of O_j'' onto an open set Ω_j'' in $\mathbb{R}^n$. Let $(\varphi_j'')_{j\in J}$ be a partition of unity on X subordinate to this covering. For each j, we choose a $\psi_j'' \in C^\infty_{\mathrm{comp}}(O_j'')$ which is equal to 1 on the support of φ_j''. Let $a_j \in t^{-m}\, \mathcal{S}^m({}^bT^*(\Omega' \times \bar{\mathbb{R}}_+ \times \Omega_j''))$ be a local representation of a over O_j''. For every a_j, we find, by Corollary 3.2.5, a C^∞ function $\tilde{h}_j(y,t;\tilde{\eta},z)$ of $(y,t) \in \Omega' \times \bar{\mathbb{R}}_+$ with values in $\mathcal{M}(\mathbb{C}, \Psi^m(\Omega_j''; \mathbb{R}^q_{\tilde{\eta}}))$, such that

$$\mathrm{op}_{\mathcal{F}_{t,x}}(a_j)\,(y,\eta) = \frac{1}{t^m}\, \mathrm{op}_{\mathcal{M},\gamma}(\tilde{h}_j(y,t;t\eta,z)) \mod C^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+\times\Omega_j''; \mathbb{R}^q_\eta))$$

for all $\gamma \in \mathbb{R}$. Set

$$\begin{aligned}
\tilde{h}(y,t;\tilde{\eta},z) &= \textstyle\sum_{j\in J} \varphi_j''\, (h_j'')^* \tilde{h}_j(y,t;\tilde{\eta},z)(h_j'')_*\, \psi_j'', \\
h(y,t;\eta,z) &= \tilde{h}(y,t;t\eta,z).
\end{aligned}$$

It is a simple matter to see that $\tilde{h}$ is a C^∞ function of $(y,t) \in \Omega' \times \bar{\mathbb{R}}_+$ with values in $\mathcal{M}(\mathbb{C}, \Psi^m(X; \mathbb{R}^q_{\tilde{\eta}}))$. Moreover,

$$\begin{aligned}
a(y,\eta) &:= \sum_{j\in J} \varphi_j''\, (1 \times h_j'')^* \mathrm{op}_{\mathcal{F}_{t,x}}(a_j(y,t,x;\eta,\tau,\xi))\, (1 \times h_j'')_*\, \psi_j'' \\
&= \frac{1}{t^m}\, \mathrm{op}_{\mathcal{M},\gamma}(h)\,(y,\eta)
\end{aligned}$$

modulo $C^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+ \times \mathrm{X}; \mathbb{R}^q_\eta))$, for each $\gamma \in \mathbb{R}$. Thus, h is the desired Mellin quantisation of a.

Evidently, equality (3.2.6) and Proposition 3.2.6 remain valid under this more general setting. Yet another point to note here is that the extension to the case of symbols $a \in t^{-m}\,\mathcal{S}^m({}^bT\mathcal{W}, \mathrm{Hom}(\mathrm{V}, \tilde{\mathrm{V}}))$ is straighforward.

3.3 *Green operators*

By Proposition 3.2.6 we can assert that each typical interior symbol in a collar neighbourhood of $\partial\mathcal{M}$ gives rise to an operator-valued symbol on the edge S. This latter symbol fulfils certain 'twisted' symbol estimates rather than the usual ones; they include specific families of isomorphisms in fibres. Our next concern will be a brief exposition of the theory of corresponding pseudodifferential operators along S. This suggests a reformulation of the standard calculus in *anisotropic terms*, according to the anisotropic description of the Sobolev spaces on $\mathbb{R}^{q+1+n}$ as $H^s(\mathbb{R}^{q+1+n}) = H^s(\mathbb{R}^q, \pi^* H^s(\mathbb{R}^{1+n}))$. An abstract theory should contain both the calculus of pseudodifferential operators with operator-valued symbols and a correspondence to some "isotropic" background. Such a theory can be formulated independently and has also applications to more complicated singularities. In particular, we introduce *Green operators* on a manifold with edges, which are defined via the mapping properties of their 'edge' symbols.

The theory of pseudodifferential operators with operator-valued symbols is a natural extension of the scalar case, and the reader can recognise the basic ideas of the ordinary calculus.

Let V and $\tilde{V}$ be Banach spaces endowed with families of isomorphisms $(\lambda(\eta))_{\eta\in\mathbb{R}^q}$ and $\left(\tilde{\lambda}(\eta)\right)_{\eta\in\mathbb{R}^q}$, respectively. We assume that these families meet condition 2) of Proposition 2.1.2. In particular, we can take

$$\begin{aligned} \lambda(\eta) &= \kappa^{-1}_{\langle\eta\rangle}, \\ \tilde{\lambda}(\eta) &= \tilde{\kappa}^{-1}_{\langle\eta\rangle}, \end{aligned}$$

where $(\kappa_\theta)_{\theta>0}$ and $(\tilde{\kappa}_\theta)_{\theta>0}$ are group actions on V and $\tilde{V}$, respectively (see Example 2.1.3).

In the sequel, the objects under consideration will depend on the concrete choice of these families of isomorphisms, but they are fixed once and for all in any concrete case. For this reason, we suppress indices 'λ' and '$\tilde{\lambda}$' in the notation when no confusion can arise.

Definition 3.3.1 *Let O be an open set in $\mathbb{R}^Q$ and $s \in \mathbb{R}$. Denote by $\mathcal{S}^m(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ the space of all $a \in C^\infty_{\mathrm{loc}}(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ with the property that, for each $K \subset\subset O$ and $\alpha \in \mathbb{Z}^Q_+$, $\beta \in \mathbb{Z}^q_+$, there is a constant $c_{K,\alpha,\beta}$ such that*

$$\|\tilde{\lambda}(\eta)\,(D^\alpha_y D^\beta_\eta a(y,\eta))\,\lambda^{-1}(\eta)\|_{\mathcal{L}(V,\tilde{V})} \le c_{K,\alpha,\beta}\,\langle\eta\rangle^{m-|\beta|} \quad \text{for } y \in K,\ \eta \in \mathbb{R}^q. \tag{3.3.1}$$

The elements of $\mathcal{S}^m(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ are called 'twisted' operator-valued symbols of order m.

The best constants $c_{K,\alpha,\beta}$ in (3.3.1) form a system of seminorms on $\mathcal{S}^m(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ under which this space is Fréchet.

Denote by $\mathcal{S}^m(\mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ the subspace of $\mathcal{S}^m(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ consisting of the elements which are independent of y. Obviously, this subspace is closed.

Proposition 3.3.2 *For each $m \in \mathbb{R}$, it follows that*

$$\mathcal{S}^m(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V})) = C^\infty_{\mathrm{loc}}(O, \mathcal{S}^m(\mathbb{R}^q, \mathcal{L}(V, \tilde{V}))).$$

Proof. The proof is straightforward while being rather long. We refer the reader to Treves [Tre80].

□

This result applies in an evident way to introduce the spaces of symbols defined in y on 'poor' subsets of $\mathbb{R}^Q$.

Corollary 3.3.3 *Let $m \in \mathbb{R}$. Assume that σ is a Borel subset of $\mathbb{R}^Q$. Then*

$$\mathcal{S}^m(\sigma \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V})) = C^\infty_{\mathrm{loc}}(\sigma) \otimes_\pi \mathcal{S}^m(\mathbb{R}^q, \mathcal{L}(V, \tilde{V})).$$

Proof. This follows from Theorem 6 of Grothendieck [Gro55, Ch.2] once we notice that $C^\infty_{\mathrm{loc}}(\sigma)$ is a nuclear Fréchet space.

□

Many elements of the scalar theory may be obtained analogously also in the operator-valued case. In particular, we mention that the asymptotic sums of symbols of decreasing orders can be carried out within the symbol classes modulo

$$\mathcal{S}^{-\infty}(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V})) = \bigcap_m \mathcal{S}^m(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V})).$$

One can see that $\mathcal{S}^{-\infty}(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V})) = C^\infty_{\mathrm{loc}}(O, \mathcal{S}(\mathbb{R}^q, \mathcal{L}(V, \tilde{V})))$ where $\mathcal{S}$ stands for the space of rapidly decreasing functions. Thus, the space $\mathcal{S}^{-\infty}(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ is independent of the particular choice of the families of isomorphisms.

Proposition 3.3.4 *For each sequence $a_j \in \mathcal{S}^{m_j}(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ with $m_j \searrow -\infty$, there is a symbol a of order m_0 such that $a - \sum_{j=0}^{J-1} a_j$ is of order m_J, for all $J = 1, 2, \ldots$.*

Proof. Pick compact sets K_ν, $\nu = 1, 2, \ldots$, such that $K_\nu \subset K_{\nu+1}$ and $\cup_\nu K_\nu = O$. Fix $\chi \in C^\infty_{\mathrm{loc}}(\mathbb{R}^Q)$ with the property that $\chi(\eta) = 0$ for $|\eta| \leq \frac{1}{2}$ and $\chi(\eta) = 1$ for $|\eta| \geq 1$. Take a of the form $a = \sum_{j=0}^\infty \chi(\epsilon_j \eta) a_j$, where ϵ_j are chosen so small that

$$\|\tilde{\lambda}(\eta)\,(D_y^\alpha D_\eta^\beta \chi(\epsilon_j \eta) a_j(y, \eta))\,\lambda^{-1}(\eta)\|_{\mathcal{L}(V,\tilde{V})} \leq 2^{-j}\ \langle \eta \rangle^{m-|\beta|+1}$$

for $y \in K_\nu$ and $|\alpha| + |\beta| + \nu \le j$. It is easy to verify that the series defining a is convergent and that a possesses the desired property.

□

Obviously, the symbol a is unique modulo $\mathcal{S}^{-\infty}(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$. We write $a \sim \sum_{j=0}^{\infty} a_j$.

Example 3.3.5 As but one example of operator-valued symbols satisfying 'twisted' symbol estimates we show those of Proposition 3.2.6. In this case both $\lambda(\eta)$ and $\tilde{\lambda}(\eta)$ are given by the group action $\kappa^{-1}_{\langle\eta\rangle}$, where $\kappa_\theta u\,(t,x) = \theta^{\frac{1+n}{2}} u(\theta t, x)$ for $\theta > 0$.

□

Now we define the subspace of classical symbols of order m, which have asymptotic expansions into functions homogeneous in η of decreasing degrees $m - j$, $j \in \mathbb{Z}_+$. The concept of homogeneity may be introduced in the framework of arbitrary families of isomorphisms. Namely, a C^∞ function $a(y,\eta)$ on $O \times (\mathbb{R}^q \setminus \{0\})$ with values in $\mathcal{L}(V, \tilde{V})$ is called *homogeneous of degree* m for large η if

$$a(y,\theta\eta) = \theta^m\, \tilde{\lambda}^{-1}(\theta\eta)\tilde{\lambda}(\eta)\, a(y,\eta)\, (\lambda^{-1}(\theta\eta)\lambda(\eta))^{-1} \tag{3.3.2}$$

for all $|\eta| \ge c$ and $\theta \ge 1$, c being a positive constant. However, a function $a \in C^\infty_{\mathrm{loc}}(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ homogeneous of degree m for large η fails to fulfil symbol estimates (3.3.1) with $\beta \neq 0$, unless both $\lambda^{-1}(\theta\eta)\lambda(\eta)$ and $\tilde{\lambda}^{-1}(\theta\eta)\tilde{\lambda}(\eta)$ are independent of η for $|\eta| \ge c$. This makes definition (3.3.2) inefficient. In case $\lambda(\eta)$ and $\tilde{\lambda}(\eta)$ correspond to group actions on V and $\tilde{V}$ we have

$$\begin{aligned} \lambda^{-1}(\theta\eta)\lambda(\eta) &= \kappa_\theta, \\ \tilde{\lambda}^{-1}(\theta\eta)\tilde{\lambda}(\eta) &= \tilde{\kappa}_\theta \end{aligned}$$

provided $|\eta| \ge c$ and $\theta \ge 1$, and so the above condition is satisfied. This is just a relevant choice of $\lambda(\eta)$ and $\tilde{\lambda}(\eta)$ in the calculus of pseudodifferential operators on a manifold with edges, as is developed by the first author [Sch92, Sch94]. We thus confine our attention to the model case of group actions.

Proposition 3.3.6 *Let $a \in C^\infty_{\mathrm{loc}}(O \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(V, \tilde{V}))$ be homogeneous of degree m in $\eta \neq 0$, and let $\chi(\eta)$ be an excision function in $\mathbb{R}^q$. Then $\chi(\eta)a(y,\eta) \in \mathcal{S}^m(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$.*

Proof. Differentiating the equality $a(y,\theta\eta) = \theta^m \tilde{\kappa}_\theta a(y,\eta)\kappa_\theta^{-1}$ in η we deduce that a derivative of $a(y,\eta)$ in η is homogeneous of degree $m-1$. Hence it suffices to verify (3.3.1) for $\beta = 0$ in which case the proof is straightforward.

□

A symbol $a \in \mathcal{S}^m(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ is said to be *classical* if there are functions $a_{m-j} \in C^\infty_{\mathrm{loc}}(O \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(V, \tilde{V}))$ homogeneous of degree $m - j$

in $\eta \neq 0$, such that $a \sim \sum_{j=0}^{\infty} \chi \, a_{m-j}$ for any excision function χ. We denote by $\mathcal{S}^m_{\mathrm{cl}}(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ the space of all classical symbols. The components a_{m-j} are uniquely determined by a. In fact,

$$a_m(y, \eta) = \lim_{\theta \to \infty} \theta^{-m} \, \tilde{\kappa}_\theta^{-1} a(y, \theta\eta) \kappa_\theta$$

in the operator norm of $\mathcal{L}(V, \tilde{V})$. By iteration we then get $a_{m-j}(y, \eta)$ for all $j > 0$. As usually, for $a \in \mathcal{S}^m_{\mathrm{cl}}(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$, the component $\sigma^m_{edge}(a)\,(y, \eta) = a_m(y, \eta)$ is called the *principal symbol* of a.

Example 3.3.7 The symbol $a(y, \eta)$ of a typical differential operator A of order m in a stretched wedge is classical if the coefficients of A do not depend on t. In this case $a(y, \eta)$ is homogeneous of degree m, and so $\sigma^m_{\mathrm{edge}}(a)\,(y, \eta) = a(y, \eta)$, as is easy to check.

□

Our next example demonstrates rather strikingly that the property of being classical is too strong in applications.

Example 3.3.8 Let $V = \tilde{V} = L^2(\bar{\mathbb{R}}_+)$ and let $a(y, \eta) \colon u \mapsto \varphi u$ be the operator of multiplication by a non-zero function $\varphi \in C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+)$. Then $a \in \mathcal{S}^0(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ is not classical.

□

More appropriate is the class of symbols which possess a principal symbol only. An $a \in \mathcal{S}^m(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ has principal symbol if it is of the form $a = a' + a''$, where

$$\begin{aligned} a' &\in \mathcal{S}^m_{\mathrm{cl}}(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V})), \\ a'' &\in \mathcal{S}^{m''}(O \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V})) \end{aligned}$$

with some $m'' < m$. Then $\sigma^m_{\mathrm{edge}}(a)(y, \eta) = \sigma^m_{\mathrm{edge}}(a')(y, \eta)$ is called the *principal symbol* of a. By the above, the classical symbols meet this condition.

Let Ω' be an open set in $\mathbb{R}^q$. To any symbol $a \in \mathcal{S}^m(\Omega' \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$ we assign an operator $A = \mathrm{op(a)}$ by

$$Au\,(y) = \frac{1}{(2\pi)^q} \int e^{i\langle y, \eta \rangle} a(y, \eta) \mathcal{F}u(\eta)\, d\eta$$

first defined on $u \in C^\infty_{\mathrm{comp}}(\Omega', V)$. It is a simple matter to see that $\mathrm{op(a)}$ induces a continuous mapping $C^\infty_{\mathrm{comp}}(\Omega', V) \to C^\infty_{\mathrm{loc}}(\Omega', \tilde{V})$.

For $m \in \mathbb{R}$, we denote by $\Psi^{\mathrm{m}}(\Omega'; \mathrm{V}, \tilde{\mathrm{V}})$ the space of all operators $\mathrm{op(a)}$ corresponding to symbols $a(y, \eta) \in \mathcal{S}^m(\Omega' \times \mathbb{R}^q, \mathcal{L}(V, \tilde{V}))$. The elements of $\Psi^{\mathrm{m}}(\Omega'; \mathrm{V}, \tilde{\mathrm{V}})$ are said to be *pseudodifferential operators* of order $\leq m$ with operator-valued symbols. We write $\Psi^{\mathrm{m}}_{\mathrm{cl}}(\Omega'; \mathrm{V}, \tilde{\mathrm{V}})$ for the subspace of $\Psi^{\mathrm{m}}(\Omega'; \mathrm{V}, \tilde{\mathrm{V}})$ determined by classical symbols $a(y, \eta)$. As in the scalar theory, $\Psi^{-\infty}(\Omega'; \mathrm{V}, \tilde{\mathrm{V}})$ coincides with the space of all integral operators whose kernels are in $C^\infty_{\mathrm{loc}}(\Omega' \times \Omega', \mathcal{L}(V, \tilde{V}))$.

Theorem 3.3.9 *Every operator* $\mathrm{op}(a) \in \Psi^{\mathrm{m}}(\Omega'; \mathrm{V}, \tilde{\mathrm{V}})$ *extends to a continuous mapping* $H^s_{\mathrm{comp}}(\Omega', \pi^* V) \to H^{s-m}_{\mathrm{loc}}(\Omega', \pi^* \tilde{V})$*, for any* $s \in \mathbb{R}$.

Proof. For a proof in the case of group actions we refer the reader to Theorem 1.3.51 in Schulze [Sch98]. For a treatment of the general case use (2.1.2) and Proposition 1.6.3 of [RST97].

□

The basic elements of scalar pseudodifferential calculus have suitable analogues in the operator-valued case. This concerns, in particular, compositions with the Leibniz product on the symbolic level, etc., everything modulo $\Psi^{-\infty}(\Omega'; \mathrm{V}, \tilde{\mathrm{V}})$.

A more complete theory in the case of Fréshet spaces V and $\tilde{V}$ may be obtained by introducing a so-called *weak symbol topology* on symbol classes. This topology behaves well under tensor products and it allows one to prove continuity of pseudodifferential operators in various situations by invoking function-analytic properties of V and $\tilde{V}$. For a deeper discussion of the approach the reader may consult the original papers of Witt [Wit96b, Wit96a].

We are now in a position to introduce Green operators on a stretched wedge $\mathcal{W} = \Omega' \times \bar{\mathbb{R}}_+ \times X$ to be simply classical pseudodifferential operators along the edge Ω' whose symbols take their values in continuous mappings of Sobolev spaces with asymptotics in the fiber $\bar{\mathbb{R}}_+ \times X$. The designation 'Green operator' is motivated by the structure of the Green function of a classical elliptic boundary value problem. Such a function is, up to a fundamental solution of the elliptic operator, a pseudodifferential operator with a special Green symbol along the boundary. In this interpretation the boundary corresponds to the edge and the inner normal of the boundary to the model cone of the wedge.

Fix a weight datum $w = (\delta, (-l, 0])$. For an asymptotic type as related to w, the space $H^{s-m,\delta}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ can be written as a projective limit of Banach spaces invariant under κ_θ. This is also the case for $H^{\infty,\delta}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$, which gives us symbol spaces $\mathcal{S}^m_{\mathrm{cl}}(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{\infty,\delta}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X))$.

For any element $a \in \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{\infty,\delta}(\bar{\mathbb{R}}_+ \times X))$, we can define the formal adjoint a^* as an element of $\mathcal{L}(H^{-\infty,-\delta}(\bar{\mathbb{R}}_+ \times X), H^{-s,-\gamma}(\bar{\mathbb{R}}_+ \times X))$ via the non-degenerate pairings $H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \times H^{-s,-\gamma}(\bar{\mathbb{R}}_+ \times X) \to \mathbb{C}$ induced by the inner product in $H^{0,0}(\bar{\mathbb{R}}_+ \times X)$. Namely, we set

$$(a\, u, v)_{H^{0,0}(\bar{\mathbb{R}}_+ \times X)} = (u, a^*\, v)_{H^{0,0}(\bar{\mathbb{R}}_+ \times X)} \quad \text{for} \quad u, v \in C^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times X).$$

Thus, to any $a(y,\eta) \in \mathcal{S}^m_{\mathrm{cl}}(\Omega \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{\infty,\delta}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X))$ there corresponds pointwise a formal adjoint $a^*(y,\eta)$ and we may require this operator-valued function to be in $\mathcal{S}^m_{\mathrm{cl}}(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,-\delta}(\bar{\mathbb{R}}_+ \times X), H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(\bar{\mathbb{R}}_+ \times X))$ for another asymptotic type $\tilde{\mathrm{as}}$ related to a weight datum $\tilde{w} = (-\gamma, (-l, 0])$.

Since we are aimed at the analysis near $t = 0$, we shall replace both $H^{\infty,\delta}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ and $H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(\bar{\mathbb{R}}_+ \times X)$ by the subspaces

$$\begin{aligned} \mathcal{S}^{\delta}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X) &= \omega\, H^{\infty,\delta}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X) + (1-\omega)\, \mathcal{S}(\bar{\mathbb{R}}_+ \times X), \\ \mathcal{S}^{-\gamma}_{\tilde{\mathrm{as}}}(\bar{\mathbb{R}}_+ \times X) &= \omega\, H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(\bar{\mathbb{R}}_+ \times X) + (1-\omega)\, \mathcal{S}(\bar{\mathbb{R}}_+ \times X), \end{aligned}$$

respectively, where ω is a cut-off function and $\mathcal{S}(\bar{\mathbb{R}}_+ \times X) = \mathcal{S}(\bar{\mathbb{R}}_+, C^\infty(X))$. It is easily seen that $\mathcal{S}^\delta_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ and $\mathcal{S}^{-\gamma}_{\tilde{\mathrm{as}}}(\bar{\mathbb{R}}_+ \times X)$ are actually independent of the concrete choice of ω.

Definition 3.3.10 *An operator-valued function $g(y,\eta)$ on $\Omega' \times \mathbb{R}^q$ is called a Green edge symbol of order m with asymptotics if there are asymptotic types*

$$\begin{array}{rcl} \mathrm{as} & \in & \mathrm{As}(\delta, (-l, 0]), \\ \tilde{\mathrm{as}} & \in & \mathrm{As}(-\gamma, (-l, 0]) \end{array} \tag{3.3.3}$$

such that

$$\begin{array}{rcl} g(y,\eta) & \in & \cap_{s\in\mathbb{R}}\, \mathcal{S}^m_{\mathrm{cl}}(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), \mathcal{S}^\delta_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X))), \\ g^*(y,\eta) & \in & \cap_{s\in\mathbb{R}}\, \mathcal{S}^m_{\mathrm{cl}}(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,-\delta}(\bar{\mathbb{R}}_+ \times X), \mathcal{S}^{-\gamma}_{\tilde{\mathrm{as}}}(\bar{\mathbb{R}}_+ \times X))). \end{array}$$

It is worth pointing out here that the space of Green edge symbols depends on the particular choice of the scalar product in $H^{0,0}(\bar{\mathbb{R}}_+ \times X)$. Other allowed scalar products may be obtained by different Riemannian metrics on $\bar{\mathbb{R}}_+ \times X$ related to the original one by diffeomorphisms which are smooth up to $t = 0$.

To specify Green edge symbols with asymptotics one introduces *double* weight data which are triples $w = (\gamma, \delta, (-l, 0])$ consisting of real numbers γ and δ and an interval $(-l, 0]$ with $0 < l \le \infty$. For such a weight datum w, denote by $\mathcal{S}^m_G(\Omega' \times \mathbb{R}^q; w)$ the set of all Green edge symbols of order m and with asymptotic types satisfying (3.3.3).

Theorem 3.3.9 shows that the operator op(g) corresponding to a Green symbol $g(y,\eta) \in \mathcal{S}^m_G(\Omega' \times \mathbb{R}^q; w)$ extends to a continuous mapping

$$H^s_{\mathrm{comp}}(\Omega', \pi^* H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)) \to H^{s-m}_{\mathrm{loc}}(\Omega', \pi^* \mathcal{S}^\delta_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)),$$

for each $s \in \mathbb{R}$. Hence it follows that op(g) is smoothing in the interior of the wedge $\mathcal{W}$. We also note that any Green operator of order $-\infty$ is compact.

The following characterisation of Green edge symbols via their Schwartz kernels is the key to understanding the asymptotic expansions of solutions to elliptic edge problems.

Theorem 3.3.11 *If $k \in \mathcal{S}^m_{\mathrm{cl}}(\Omega' \times \mathbb{R}^q) \otimes_\pi \big(\mathcal{S}^\delta_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X) \otimes_\pi \mathcal{S}^{-\gamma}_{\tilde{\mathrm{as}}}(\bar{\mathbb{R}}_+ \times X)\big)$, then the operator family*

$$u \mapsto \int_0^\infty \int_X k(y,\eta; t\langle\eta\rangle, x, t'\langle\eta\rangle, x')\, u(t',x')\, \frac{dt'}{t'} dx', \quad u \in H^{0,\gamma}(\bar{\mathbb{R}}_+ \times X),$$

is a Green edge symbol of order m with asymptotic types as *and* $\tilde{\mathrm{as}}$. *Conversely, every Green edge symbol of order m with discrete asymptotic types* as *and* $\tilde{\mathrm{as}}$ *has such a representation.*

Proof. The proof of the first part of the theorem is immediate by using a familiar argument of topological tensor products. The proof of the second part is substantially subtler (cf. [ST98a]). □

To complete the local operator algebra close to edges, it remains to add the so-called smoothing Mellin operators in the wedge $\mathcal{W}$. On the other hand, they highlight a natural way in which Green operators with asymptotics appear.

A standard asymptotic summation allows one to invert, up to smoothing Mellin edge symbols, the Mellin edge symbols with invertible conormal symbol. Smoothing Mellin edge symbols are used in explicit form mainly for a finite weight interval $(-l, 0]$, with $l = 1, 2, \ldots$. By a *smoothing Mellin symbol* of order $m \in \mathbb{R}$, is meant a family

$$m(y,\eta) = \sum_{j=0}^{l-1} \sum_{|\alpha|\leq j} \frac{1}{t^m}\, \varphi_0(t\langle\eta\rangle) \left(t^j \operatorname{op}_{\mathcal{M},\gamma_{j,\alpha}}(\mathrm{h}_{j,\alpha}(\mathrm{y};\mathrm{z}))\, \eta^\alpha\right) \psi_0(t\langle\eta\rangle), \quad (3.3.4)$$

where φ_0, ψ_0 are cut-off functions close to $t = 0$, and $h_{j,\alpha}(y;z)$ are C^∞ functions of $y \in \Omega'$ with values in $\mathcal{A}(\mathbb{C} \setminus \sigma_{j,\alpha}, \Psi^{-\infty}(\mathrm{X}))$, every $\sigma_{j,\alpha}$ being a closed set in the complex plane. Moreover, for any $\sigma_{j,\alpha}$-excision function χ in the plane, the restriction of $\chi\, h_{j,\alpha}$ to each horizontal line $\Gamma_{-\gamma}$ is required to be a parameter-dependent pseudodifferential operator on X with parameter $\tau = \Re z$, uniformly for γ varying in finite segments of $\mathbb{R}$ (cf. Corollary 3.2.5).

Proposition 3.3.12 *Suppose that φ_0 and ψ_0 are arbitrary cut-off functions, $h(y;z) \in C^\infty_{\mathrm{loc}}(\Omega', \mathcal{A}(\mathbb{C} \setminus \sigma, \Psi^{-\infty}(\mathrm{X})))$ is pointwise a parameter-dependent pseudodifferential operator on X, and $\Gamma_{-\gamma} \cap \sigma = \emptyset$. Then the operator family*

$$\frac{1}{t^m}\, \varphi_0(t\langle\eta\rangle) \left(t^j \operatorname{op}_{\mathcal{M},\gamma}(\mathrm{h}(\mathrm{y};\mathrm{z}))\, \eta^\alpha\right) \psi_0(t\langle\eta\rangle)$$

belongs to $\mathcal{S}^{m-j+|\alpha|}_{\mathrm{cl}}(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{\infty,\gamma-m+j}(\bar{\mathbb{R}}_+ \times X)))$, for each $s \in \mathbb{R}$.

Proof. The proof is similar to that of Proposition 3.2.6. The symbol in question actually proves to be homogeneous of degree $m - j + |\alpha|$ for large η.

□

Returning to family (3.3.4) we assume that each set $\sigma_{j,\alpha}$ is a carrier of asymptotics. It is also possible to introduce the concept of asymptotic types for Mellin symbols, but we will not develop this point here. For details, the reader is referred to Subsection 3.3.2 in Schulze [Sch98].

Furthermore, we choose the weights $\gamma_{j,\alpha}$ in such a way that the line $\Gamma_{-\gamma_{j,\alpha}}$ is free from the singularities of $h_{j,\alpha}$, i.e., $\Gamma_{-\gamma_{j,\alpha}} \cap \sigma_{j,\alpha} = \emptyset$. The family $m(y,\eta)$ is thought of as acting from $H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)$ to $H^{\infty,\delta}(\bar{\mathbb{R}}_+ \times X)$, for any $s \in \mathbb{R}$. To ensure these mapping properties, by Proposition 3.3.12, the exponents m and $\gamma_{j,\alpha}$ involved must satisfy

$$\left\{ \begin{array}{rcl} \gamma_{j,\alpha} & \le & \gamma, \\ \gamma_{j,\alpha} - m + j & \ge & \delta \end{array} \right.$$

for all j and α. Under these conditions, (3.3.4) proves to be an operator-valued symbol in $\mathcal{S}^m_{\mathrm{cl}}(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{\infty,\delta}(\bar{\mathbb{R}}_+ \times X)))$, for every $s \in \mathbb{R}$. Note that

$$\sigma^m_{\mathrm{edge}}(m) = \sum_{j=0}^{l-1} \sum_{|\alpha|=j} \frac{1}{t^m}\, \varphi_0(t|\eta|) \left(t^j \, \mathrm{op}_{\mathcal{M},\gamma_{j,\alpha}}(\mathrm{h}_{j,\alpha}(\mathrm{y};\mathrm{z}))\, \eta^\alpha \right) \psi_0(t|\eta|).$$

Let us denote by $\mathcal{S}^m_M(\Omega' \times \mathbb{R}^q; w)$ the space of all smoothing Mellin edge symbols of order m and with respect to a double weight datum $w = (\gamma, \delta, (-l, 0])$.

It is easy to see that $\mathcal{S}^m_M(\Omega' \times \mathbb{R}^q; w) \subset \mathcal{S}^m_G(\Omega' \times \mathbb{R}^q; w)$ provided that $l \le \gamma - m - \delta$. As the operator of multiplication by t decreases the order m by 1, it follows that, for each smoothing Mellin edge symbol $m(y, \eta)$ of order m, the composition $t^N \, m(y, \eta)$ is a Green edge symbol of order $m - N$, provided N is large enough (precisely, $N \ge l - \gamma + m + \delta$). This is one of the motivations for introducing Green edge symbols. On the other hand, changing the cut-off functions φ_0 and ψ_0 in (3.3.4) results in perturbing $m(y, \eta)$ by a Green edge symbol of order m related to the same double weight datum w. For this reason smoothing Mellin symbols are not considered separately but along with Green edge symbols.

In order to define Mellin pseudodifferential operators we need gaps in the carriers of asymptotics of the Mellin symbols (so-called *quasi-discrete* asymptotic types). In general $\sigma_{j,\alpha}$ may consist of an infinite band without such gaps. This cannot happen in the case of discrete asymptotic types. For this reason it is convenient to represent an arbitrary carrier of asymptotics as the union of quasi-discrete ones, which results in considering the sums $m(y, \eta) = m_1(y, \eta) + m_2(y, \eta)$ of smoothing Mellin symbols of the form (3.3.4), each $m_\nu(y, \eta)$ being of quasi-discrete asymptotic type. Once again, the representation of an arbitrary $m(y, \eta)$ in such a form is independent of the particular partition of the carrier of asymptotics modulo Green operators.

Remark 3.3.13 *As defined above, the edge pseudodifferential operators with symbols $g(y, \eta) + m(y, \eta)$ form together an algebra, in which the subspace of Green operators is an ideal.*

3.4 *The operator algebra*

In this section we introduce an algebra of pseudodifferential operators on a C^∞ closed manifold with edges, (M, S). As above, there is no loss of generality in assuming that S is connected. Then, the corresponding stretched manifold $\mathcal{M}$ is a compact smooth manifold with boundary, the boundary being a C^∞ bundle over S. We fix a collar neighbourhood $O \cong \partial\mathcal{M} \times \mathbb{R}_+$ of the boundary in $\mathcal{M}$.

We begin with a local algebra of pseudodifferential operators in a wedge $\mathcal{W} = \Omega' \times \bar{\mathbb{R}}_+ \times X$, where Ω' is an open subset of $\mathbb{R}^q$. Let $m, \gamma \in \mathbb{R}$ and let $w = (\gamma, \gamma - m, (-l, 0])$, where $l > 0$. Denote by $\mathcal{S}^m(\Omega' \times \mathbb{R}^q; w)$ the set of all operator-valued functions

$$a(y,\eta) = a_0(y,\eta) + a_\infty(y,\eta) + m(y,\eta) + g(y,\eta) \tag{3.4.1}$$

on $\Omega' \times \mathbb{R}^q$, where

- a_0 and a_∞ originate from a typical symbol $a \in t^{-m}\mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{W})$ via (3.2.6), a vanishing for t large enough;
- $m(y,\eta)$ is a smoothing Mellin symbol of order m related to the weight datum w, as in (3.3.4); and
- $g(y,\eta)$ is a Green edge symbol of order m with asymptotics related to w.

In a recent paper by Gil, Schulze and Seiler [GSS98] a new representation of complete edge symbols (3.4.1) is shown. It has the advantage of reducing the use of cut-off aggregates like $\varphi_\nu(t\langle\eta\rangle)$, $\psi_\nu(t\langle\eta\rangle)$, etc., to cutting off by functions depending on t only, up to Green edge symbols.

For the same double weight data w, we may obviously consider the symbol spaces $\mathcal{S}^{m-j}(\Omega' \times \mathbb{R}^q; w)$, too, j being any non-negative integer.

Proposition 3.4.1 *Let $w = (\gamma, \gamma - m, (-l, 0])$, where $l > 0$. Then, for each $\mu = m - j$, $j \in \mathbb{Z}_+$, we have*

$$\mathcal{S}^\mu(\Omega' \times \mathbb{R}^q; w) \hookrightarrow \bigcap_{s\in\mathbb{R}} \mathcal{S}^\mu(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{s-\mu,\gamma-m}(\bar{\mathbb{R}}_+ \times X))).$$

Proof. This follows from (3.4.1), Propositions 3.2.6 and 3.3.12 and the definition of Green operators.

□

We next introduce the *principal symbols* of order μ for an element $a(y,\eta) \in \mathcal{S}^\mu(\Omega' \times \mathbb{R}^q; w)$. Write $a(y,\eta)$ in the form (3.4.1) with m replaced by μ. From (3.2.6) it follows that the sum $a_0(y,\eta) + a_\infty(y,\eta)$ possesses a principal homogeneous symbol of order μ in the interior of $\mathcal{W}$. In fact, it is given by the homogeneous component of order μ of the typical symbol $a(y,t,x;\eta,\tau,\xi)$ occurring in (3.2.6). Set

$$\sigma^\mu(a) = \sigma^\mu(a_0 + a_\infty) \tag{3.4.2}$$

thus obtaining a homogeneous function of degree μ defined away from the zero section of $T^*\mathring{\mathcal{W}}$. As a_0 and a_∞ originate from a typical symbol on $\mathcal{W}$, we deduce that

$$\begin{aligned} {}^b\sigma^\mu(a)(y,t,x;\tilde\eta,\tilde\tau,\xi) &= \sigma^\mu(a)(y,t,x;\tilde\eta,\tilde\tau,t\xi) \\ &= t^\mu\,\sigma^\mu(a)\left(y,t,x;\frac{\tilde\eta}{t},\frac{\tilde\tau}{t},\xi\right) \end{aligned} \tag{3.4.3}$$

is C^∞ up to $t=0$ (cf. (3.1.9)). By the above, ${}^b\sigma^\mu(a)$ is a function on the compressed cotangent bundle ${}^bT^*\mathcal{W}$. We thus obtain two symbol mappings which control the usual ellipticity of a in the interior of $\mathcal{W}$. When compared with $\sigma^\mu(a)$, the symbol ${}^b\sigma^\mu(a)$ has the advantage of being defined up to the boundary. However, the definition of ${}^b\sigma^\mu(a)$ relies on a global coordinate t measuring the distance to the boundary. We call either of the symbols $\sigma^\mu(a)$ and ${}^b\sigma^\mu(a)$ the *principal homogeneous interior symbols* of $a(y,\eta)$.

Yet another symbol mapping of great importance in the characterisation of elliptic edge problems is the principal *edge symbol* $\sigma^\mu_{\mathrm{edge}}(a)$. As mentioned, complete edge symbols (3.4.1) are not classical as operator-valued symbols. However, they possess principal edge symbols as is clear from the following lemma due to Behm [Beh95] (cf. Lemma 2.3.2.2 therein).

Lemma 3.4.2 *Each complete edge symbol $a(y,\eta)\in\mathcal{S}^\mu(\Omega'\times\mathbb{R}^q;w)$ possesses a principal symbol given by*

$$\sigma^\mu_{\mathrm{edge}}(a)=\sigma^\mu_{\mathrm{edge}}(a_0)+\sigma^\mu_{\mathrm{edge}}(a_\infty)+\sigma^\mu_{\mathrm{edge}}(m)+\sigma^\mu_{\mathrm{edge}}(g), \tag{3.4.4}$$

where

$$\begin{aligned} \sigma^\mu_{\mathrm{edge}}(a_0) &= t^{-\mu}\,\varphi_0(t|\eta|)\,\mathrm{op}_{\mathcal{M},\gamma}(\tilde{\mathrm{h}}(\mathrm{y},0;\mathrm{t}\eta,\mathrm{z}))\,\psi_0(\mathrm{t}|\eta|), \\ \sigma^\mu_{\mathrm{edge}}(a_\infty) &= t^{-\mu}\,\varphi_\infty(t|\eta|)\,\mathrm{op}_{\mathcal{F}_{\mathrm{t,x}}}(\tilde{\mathrm{a}}(\mathrm{y},0,\mathrm{x};\mathrm{t}\eta,\mathrm{t}\tau,\xi))\,\psi_\infty(\mathrm{t}|\eta|). \end{aligned}$$

Proof. By Definition 3.3.10 and Proposition 3.3.12 it suffices to show that each of the symbols $a_0(y,\eta)$ and $a_\infty(y,\eta)$ possesses a principal symbol of order μ. We give the proof only for the symbol $a_0(y,\eta)$; the arguments for $a_\infty(y,\eta)$ are similar. To this end, pick an excision function $\chi(\eta)$ in $\mathbb{R}^q$ such that $\langle\eta\rangle=|\eta|$ on the support of χ. We have

$$\begin{aligned} &\chi(\eta)\left(a_0-\sigma^\mu_{\mathrm{edge}}(a_0)\right) \\ &\quad=\chi(\eta)\,\frac{1}{t^\mu}\,\varphi_0(t\langle\eta\rangle)\,\mathrm{op}_{\mathcal{M},\gamma}(\tilde{\mathrm{h}}(\mathrm{y},\mathrm{t};\mathrm{t}\eta,\mathrm{z})-\tilde{\mathrm{h}}(\mathrm{y},0;\mathrm{t}\eta,\mathrm{z}))\,\psi_0(\mathrm{t}\langle\eta\rangle) \end{aligned}$$

for $(y,\eta)\in T^*\Omega'$. Choose $R>0$ such that both φ_0 and ψ_0 vanish away from the interval $(0,R]$ on the semiaxis. Then

$$\begin{aligned} \varphi_0(t\langle\eta\rangle) &\equiv 0, \\ \psi_0(t\langle\eta\rangle) &\equiv 0 \end{aligned}$$

for all $t\geq R'$, where $R'=R/\min\langle\eta\rangle$. On the other hand, since $\tilde h(y,t;\tilde\eta,z)$ is C^∞ in t up to $t=0$, we deduce that

$$\tilde h(y,t;\tilde\eta,z)-\tilde h(y,0;\tilde\eta,z)=t\,\tilde q(y,t;\tilde\eta,z)$$

for $t \in [0, R']$, where $\tilde{q}(y,t;\tilde{\eta},z)$ is a C^∞ function of $(y,t) \in \Omega' \times \bar{\mathbb{R}}_+$ with values in $\mathcal{M}(\mathbb{C}, \Psi^\mu(\mathrm{X}; \mathbb{R}^q_{\tilde{\eta}}))$ independent of t for t large enough. If moreover ω is a cut-off function on $\bar{\mathbb{R}}_+$ equal to 1 on the interval $[0, R']$, then we obtain

$$a_0 - \sigma^\mu_{\mathrm{edge}}(a_0) = (t\omega)\, \frac{1}{t^\mu}\, \varphi_0(t\langle\eta\rangle)\, \mathrm{op}_{\mathcal{M},\gamma}(\tilde{\mathrm{q}}(\mathrm{y},\mathrm{t};\mathrm{t}\eta,\mathrm{z}))\, \psi_0(\mathrm{t}\langle\eta\rangle)$$

for η in the support of χ. Summarising, we can assert that

$$\chi(\eta)\left(a_0 - \sigma^\mu_{\mathrm{edge}}(a_0)\right) = (t\omega)\, a_0^{(1)},$$

where $a_0^{(1)}$ lies in $\mathcal{S}^\mu\big(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{s-\mu,\gamma-\mu}(\bar{\mathbb{R}}_+ \times X))\big)$ for each $s, \gamma \in \mathbb{R}$. As the operator of multiplication by $t\omega$ belongs to

$$\bigcap_{s,\gamma} \mathcal{S}^{-1}\left(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X))\right),$$

we conclude that $\chi(\eta)\left(a_0 - \sigma^\mu_{\mathrm{edge}}(a_0)\right)$ is an edge symbol of order $\mu - 1$. From this the lemma follows.

□

This lemma ensures the existence of a *principal homogeneous edge symbol* $\sigma^\mu_{\mathrm{edge}}(a)$, for each symbol $a \in \mathcal{S}^\mu(\Omega' \times \mathbb{R}^q; w)$. This is a C^∞ function on $\Omega' \times (\mathbb{R}^q \setminus \{0\})$ taking its values in $\cap_s \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{s-\mu,\gamma-m}(\bar{\mathbb{R}}_+ \times X))$. As defined by (3.4.4), the function $\sigma^\mu_{\mathrm{edge}}(a)$ is homogeneous of degree μ in $\eta \neq 0$. It can be written (at least formally) as a limit

$$\sigma^\mu_{\mathrm{edge}}(a)\,(y,\eta) = \lim_{\theta\to\infty} \theta^{-m}\, \tilde{\kappa}_\theta^{-1} a(y,\theta\eta)\kappa_\theta$$

for $(y,\eta) \in T^*\Omega' \setminus \{0\}$. In fact, the limit is achieved in the strong operator topology of $\mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), H^{s-\mu,\gamma-m}(\bar{\mathbb{R}}_+ \times X))$, for each $s \in \mathbb{R}$.

Every $a(y,\eta) \in \mathcal{S}^\mu(\Omega' \times \mathbb{R}^q; w)$ gives rise to a pseudodifferential operator with respect to the variable y. As described in Section 3.3, we will write it simply $A = \mathrm{op(a)}$ when no confusion can arise. As but one application of Theorem 3.3.9 we show that the operator op(a) extends to a continuous mapping

$$\mathrm{op(a)}\colon\ \mathrm{H}^{\mathrm{s}}_{\mathrm{comp}}(\Omega', \pi^*\mathrm{H}^{\mathrm{s},\gamma}(\bar{\mathbb{R}}_+ \times \mathrm{X})) \to \mathrm{H}^{\mathrm{s}-\mu}_{\mathrm{loc}}(\Omega', \pi^*\mathrm{H}^{\mathrm{s}-\mu,\gamma-\mathrm{m}}(\bar{\mathbb{R}}_+ \times \mathrm{X})), \quad (3.4.5)$$

for each $s \in \mathbb{R}$.

For weight data $w = (\gamma, \gamma - m, (-l, 0])$ and $\mu = m - j$, $j \in \mathbb{Z}_+$, let $\Psi^\mu(\mathcal{W}; \mathrm{w})$ stand for the space of all operators op(a) corresponding to symbols $a \in \mathcal{S}^\mu(\Omega' \times \mathbb{R}^q; w)$.

Our next aim is to define pseudodifferential operators on a compact manifold $\mathcal{M}$ with fibred boundary. The idea is to glue together local operators from $\Psi^\mu(\mathcal{W}; \mathrm{w})$ near the boundary and the usual classical pseudodifferential operators in the interior of $\mathcal{M}$.

Fix a finite covering of the collar neighbourhood O of $\partial\mathcal{M}$ by charts with edges $(h_\iota, O_\iota)_{\iota\in I}$ on $\mathcal{M}$. Without loss of generality we can assume that each coordinate patch is of the form $O_\iota \cong O'_\iota \times \bar{\mathbb{R}}_+ \times X$, with O'_ι an open subset of S. Moreover, h_ι is a diffeomorphism of O_ι onto a stretched wedge $\mathcal{W}_\iota = \Omega'_\iota \times \bar{\mathbb{R}}_+ \times X$ where Ω'_ι is an open subset of $\mathbb{R}^q$. Let $(\varphi_\iota)_{\iota\in I}$ be a partition of unity on O subordinate to the covering $(O_\iota)_{\iota\in I}$, in other words, $\varphi_\iota \in C^\infty_{\mathrm{loc}}(O)$ satisfy $\operatorname{supp}\varphi_\iota \subset \mathrm{O}_\iota$ and $\sum_\iota \varphi_\iota \equiv 1$ on O. For each ι, we choose a function $\psi_\iota \in C^\infty_{\mathrm{loc}}(O)$ with a support in O_ι, such that $\psi_\iota = 1$ on the support of φ_ι. Then, to every system of operators $A_\iota \in \Psi^\mu(\mathcal{W}_\iota; \mathrm{w})$ we can assign a global operator

$$A_b = \sum_\iota \varphi_\iota \, h_\iota^\sharp A_\iota \, \psi_\iota \tag{3.4.6}$$

on O, where $h_\iota^\sharp A_\iota = h_\iota^* A_\iota h_{\iota *}$ stands for the operator pull-back under h_ι. The question of whether the definition of A_b is invariant under various choices of the atlas $(h_\iota, O_\iota)_{\iota\in I}$ on O lies beyond the range of the paper. Suffices it to note that if h_ι reduces to a diffeomorphism $O'_\iota \xrightarrow{\cong} \Omega'_\iota$ which does not touch the variables t and x, then the invariance is a standard fact from the calculus of pseudodifferential operators with operator-valued symbols along $\mathbb{R}^q$ (cf. Theorem 3.4.43 in [Sch98]). In the general case the invariance just amounts to saying that the space $\Psi^\mu(\mathcal{W}; \mathrm{w})$ is invariant, modulo reasonable "small" operators, under the diffeomorphisms $\delta\colon \mathcal{W} \to \mathcal{W}$ of the form (2.4.3). Such is easily verified to be the case for those diffeomorphisms which fulfil $\psi \equiv 0$. However, it is to be expected that $\Psi^\mu(\mathcal{W}; \mathrm{w})$ is invariant in the above sense under arbitrary diffeomorphisms (2.4.3) of $\mathcal{W}$.

Definition 3.4.3 *For weight data $w = (\gamma, \gamma - m, (-l, 0])$ and $\mu = m - j$, $j \in \mathbb{Z}_+$, denote by $\Psi^\mu(\mathcal{M}; \mathrm{w})$ the set of all operators*

$$A = \varphi_b A_b \psi_b + \varphi_i A_i \psi_i + S \tag{3.4.7}$$

on $\mathcal{M}$, where

- *A_b is of the form (3.4.6) close to $\partial\mathcal{M}$, $\varphi_b \in C^\infty_{\mathrm{comp}}(O)$ being a cut-off function near the boundary and $\psi_b \in C^\infty_{\mathrm{comp}}(O)$ covering φ_b;*
- *A_i is a classical pseudodifferential operator of order μ in the interior of $\mathcal{M}$, $\varphi_i = 1 - \varphi_b$ and $\psi_i \in C^\infty_{\mathrm{comp}}(\overset{\circ}{\mathcal{M}})$ covering φ_i; and*
- *S is a "small" global operator on $\mathcal{M}$ to be defined via its mapping properties.*

To describe more precisely "small" operators in (3.4.7) we note that the elements of $\Psi^\mu(\mathcal{M}; \mathrm{w})$ are supposed to act as $H^{s,\gamma}(\mathcal{M}) \to H^{s-\mu,\gamma-m}(\mathcal{M})$, for any $s \in \mathbb{R}$, possibly also in Sobolev spaces with asymptotics on $\mathcal{M}$. Thus, the "small" operators in question should be regularising relative to this scale of spaces. Apart from being *smoothing* this involves a *gain* in the weight exponent $\gamma - m$. The latter property can be achieved by requiring asymptotics in the image because the carrier of asymptotics has a gap away from the weight line. In this way we obtain what fits in the concept of a *smoothing Green operator* (cf. Definition 3.3.10). Namely, let $\Psi^{-\infty}(\mathcal{M};\mathrm{w})$ stand for the space of all operators S in weighted Sobolev spaces on M with the property that there are asymptotic types

$$\begin{array}{rcl} \mathrm{as} & \in & \mathrm{As}(\gamma - \mathrm{m}, (-1, 0]), \\ \tilde{\mathrm{as}} & \in & \mathrm{As}(-\gamma, (-1, 0]) \end{array}$$

such that

$$\begin{array}{rcl} S & \in & \cap_{s\in\mathbb{R}}\, \mathcal{L}(H^{s,\gamma}(\mathcal{M}), H^{\infty,\gamma-m}_{\mathrm{as}}(\mathcal{M})), \\ S^* & \in & \cap_{s\in\mathbb{R}}\, \mathcal{L}(H^{s,-\gamma+m}(\mathcal{M}), H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(\mathcal{M})), \end{array}$$

the 'asterisk' referring to the formal adjoint with respect to the conjugate linear pairing $H^{s,\gamma}(\mathcal{M}) \times H^{-s,-\gamma}(\mathcal{M}) \to \mathbb{C}$ induced by an inner product in $H^{0,0}(\mathcal{M})$. In this notation the condition on the operator S in (3.4.7) is just $S \in \Psi^{-\infty}(\mathcal{M};\mathrm{w})$.

The elements of $\Psi^{\mu}(\mathcal{M};\mathrm{w})$ are said to be pseudodifferential operators of order μ on $\mathcal{M}$ with respect to the weight data w. From what has already been proved it follows that the space $\Psi^{\mu}(\mathcal{M};\mathrm{w})$ is invariant under those diffeomorphisms of $\mathcal{M}$ which obey the fibration (2.2.1) of $\mathcal{M}$ close to the boundary.

The dual objects of functions are densities unless a Riemannian metric on $\mathcal{M}$ is tacitly fixed. Therefore, to discuss transposes of pseudodifferential operators on $\mathcal{M}$ we must make some comments on pseudodifferential operators between sections of vector bundles over $\mathcal{M}$. We first observe that a vector bundle $\pi\colon V \to \mathcal{M}$ over a manifold with fibred boundary is required to obey the boundary fibration $b:\partial\mathcal{M} \to S$ in the sense that there is a commutative diagram

$$\begin{array}{ccc} V|_{\partial\mathcal{M}} & \stackrel{h}{\longrightarrow} & V_S \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \partial\mathcal{M} & \stackrel{b}{\longrightarrow} & S \end{array} \qquad (3.4.8)$$

with V_S a vector bundle over S and h a bundle homomorphism. This diagram means simply that the restriction of V to the boundary of $\mathcal{M}$ allows a *push-forward* $b_* V|_{\partial\mathcal{M}} = V_S$ under the blow-down mapping b. By working over sets where a given pair of vector bundles, V and $\tilde{V}$, is trivial the space of operators $\Psi^{\mu}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$ from sections of V to sections of $\tilde{V}$ can be defined in a standard way. Since the space $\Psi^{\mu}(\mathcal{M};\mathrm{w})$ is a $C^{\infty}(\mathcal{M})$ module it is sufficient to assume any one covering of $\mathcal{M}$ by coordinate patches where both V and $\tilde{V}$ are trivial, and any one trivialisation of the bundles.

Theorem 3.4.4 *Let* $A \in \Psi^{\mu}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$. *For each* $s \in \mathbb{R}$ *and asymptotic types* $\mathrm{as} \in \mathrm{As}(\gamma, (-1, 0])$ *and* $\tilde{\mathrm{as}} \in \mathrm{As}(\gamma - \mathrm{m}, (-1, 0])$, *the operator* A

induces continuous mappings

$$\begin{array}{rcl} A & : & H^{s,\gamma}(\mathcal{M},V) \to H^{s-\mu,\gamma-m}(\mathcal{M},\tilde{V}), \\ A & : & H^{s,\gamma}_{\mathrm{as}}(\mathcal{M},V) \to H^{s-\mu,\gamma-m}_{\tilde{\mathrm{as}}}(\mathcal{M},\tilde{V}). \end{array}$$

Proof. This is an immediate consequence of the corresponding results from the local theory (cf. (3.4.5)). It merits mentioning here that the interior symbol of A does not contribute to asymptotics. On the other hand, the smoothing Mellin operators transform asymptotics occurring in the domain to other asymptotics in the range while the Green operators produce new asymptotics.

□

Every $A \in \Psi^{\mu}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$ possesses two principal symbols. These are bundle homomorphisms

$$\begin{array}{rcl} {}^b\sigma^{\mu}(A) & : & \pi^*V \to \pi^*\tilde{V}, \\ \sigma^{\mu}_{\mathrm{edge}}(A) & : & \pi^*\,H^{s,\gamma}(F^{-1}(\cdot)) \otimes V_S \to \pi^*\,H^{s-\mu,\gamma-m}(F^{-1}(\cdot)) \otimes \tilde{V}_S, \end{array} \tag{3.4.9}$$

for $s \in \mathbb{R}$. The first of the two is the principal homogeneous interior symbol of order μ, the homogeneity in the covariables being understood in the usual sense (cf. (3.4.3)). The second one is the principal homogeneous edge symbol of order μ, the homogeneity in the covariables referring to the group actions in the fibres $H^{s,\gamma}(F^{-1}(\cdot))$ over S (cf. (3.4.4)).

Theorem 3.4.5 *Suppose* $A \in \Psi^{\mu}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$. *If* ${}^b\sigma^{\mu}(A) = 0$ *and* $\sigma^{\mu}_{\mathrm{edge}}(A) = 0$, *then the mapping* $A : H^{s,\gamma}(\mathcal{M},\tilde{V}) \to H^{s-\mu,\gamma-m}(\mathcal{M},\tilde{V})$ *is compact for all* $s \in \mathbb{R}$.

Proof. Indeed, from ${}^b\sigma^{\mu}(A) = 0$ and $\sigma^{\mu}_{\mathrm{edge}}(A) = 0$ we conclude at once that $A \in \Psi^{\mu-1}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$. Hence it follows that A really maps $H^{s,\gamma}(\mathcal{M},\tilde{V})$ to $H^{s-\mu+1,\gamma-m}_{\mathrm{as}}(\mathcal{M},\tilde{V})$, for some as $\in \mathrm{As}(\gamma-\mathrm{m},(-\mathrm{l},0])$. This gives the desired conclusion when combined with the fact that the embedding $H^{s-\mu+1,\gamma-m}_{\mathrm{as}}(\mathcal{M},\tilde{V}) \hookrightarrow H^{s-\mu,\gamma-m}(\mathcal{M},\tilde{V})$ is compact, for each $s \in \mathbb{R}$.

□

This theorem ensures the existence of a parametrix construction on the symbol level in the algebra $\Psi^{\mu}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$. The ellipticity should refer to the pair of principal symbols $\big({}^b\sigma^{\mu}(A),\sigma^{\mu}_{\mathrm{edge}}(A)\big)$. These symbols behave in the usual way under compositions of operators and taking transposes and formal adjoints.

Theorem 3.4.6 *Let* $m_1-\mu_1 \in \mathbb{Z}_+$, $m_2-\mu_2 \in \mathbb{Z}_+$ *and let* l *be a positive integer. If*

$$\begin{array}{rclcrcl} A_1 & \in & \Psi^{\mu_1}(\mathcal{M};\mathrm{V}^1,\mathrm{V}^2;\mathrm{w}_1), & \qquad & w_1 & = & (\gamma,\gamma-m_1,(-l,0]), \\ A_2 & \in & \Psi^{\mu_2}(\mathcal{M};\mathrm{V}^2,\mathrm{V}^3;\mathrm{w}_2), & \qquad & w_2 & = & (\gamma-m_1,\gamma-m_1-m_2,(-l,0]), \end{array}$$

then $A_2A_1 \in \Psi^{\mu_1+\mu_2}(\mathcal{M};\mathrm{V}^1,\mathrm{V}^3;\mathrm{w}_2\circ\mathrm{w}_1)$ *with* $w_2\circ w_1 = (\gamma,\gamma-m_1-m_2,(-l,0])$ *and*

$$\begin{array}{rcl} {}^b\sigma^{\mu_1+\mu_2}(A_2A_1) & = & {}^b\sigma^{\mu_2}(A_2)\,{}^b\sigma^{\mu_1}(A_1), \\ \sigma^{\mu_1+\mu_2}_{\mathrm{edge}}(A_2A_1) & = & \sigma^{\mu_2}_{\mathrm{edge}}(A_2)\,\sigma^{\mu_1}_{\mathrm{edge}}(A_1). \end{array}$$

Proof. The proof is rather technical and it exceeds the scope of this paper. We refer the reader to Theorem 3.4.56 in [Sch98]. As mentioned, the new approach of [GSS98] allows one to avoid a number of voluminous calculations in the precise analysis of operator-valued edge symbols by a new quantisation of typical interior symbols in which a part of inconvenient combinations of the edge covariable and the distance to the boundary is dismissed.

□

Denote by $\Psi^{\mu}_{\mathrm{M+G}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$ the subspace of $\Psi^{\mu}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$ consisting of those operators for which the operator A_i in the representation (3.4.7) vanishes as well as the operators $a_0(y,\eta)$ and $a_\infty(y,\eta)$ in the local descriptions (3.4.1) of the edge symbol near the boundary. If in addition all ingradients $m(y,\eta)$ in (3.4.1) vanish, we obtain what will be referred to as $\Psi^{\mu}_{\mathrm{G}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$. By Remark 3.3.13, the spaces $\Psi^{\mu}_{\mathrm{M+G}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$ fit together to form an algebra in the sense of Theorem 3.4.6, wherein the subspaces $\Psi^{\mu}_{\mathrm{G}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$ form an ideal.

Theorem 3.4.7 *Each operator* $A \in \Psi^{\mu}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$ *allows a formal adjoint* $A^* \in \Psi^{\mu}(\mathcal{M};\tilde{\mathrm{V}},\mathrm{V};\mathrm{w}^*)$, *where* $w^* = (-\gamma+m,-\gamma,(-l,0])$. *Moreover,*

$$\begin{array}{rcl} {}^b\sigma^{\mu}(A^*) & = & ({}^b\sigma^{\mu}(A))^*, \\ \sigma^{\mu}_{\mathrm{edge}}(A^*) & = & (\sigma^{\mu}_{\mathrm{edge}}(A))^*. \end{array}$$

Proof. Cf. Theorem 3.4.55 in [Sch98]. We emphasise that the formal adjoint is understood with respect to the inner product of $H^{0,0}(\mathcal{M})$.

□

We omit discussion of the transpose A' just noting that it is related to the formal adjoint by the equality

$$A' = \star_V A^* \star_{\tilde{V}}^{-1},$$

where $\star_V : V \to V'$ and $\star_{\tilde{V}} : \tilde{V} \to \tilde{V}'$ are conjugate linear bundle isomorphisms induced by Hermitian metrics on V and $\tilde{V}$ and a positive volume form on $\mathcal{M}$ (cf. [Tar95, 2.1.3] for more details). For the purpose of this paper, the formal adjoint proves to be more important than the transpose.

Chapter 4

Elliptic Edge Problems

Similarly to boundary value problems where the Fredholm property depends on elliptic boundary conditions (in the pseudodifferential case of trace and potential type), the theory of edge problems requires elliptic conditions along the edges. These are also of trace and potential type in general, both occur even for differential operators when the cone fibres over the edge do not reduce to $\bar{\mathbb{R}}_+$, i.e., each fibre is a topological cone over a manifold of dimension at least 1. Moreover, the number of edge conditions depends on the chosen weights.

4.1 *The concept of ellipticity*

Let (M, S) be a C^∞ compact closed manifold with a connected edge S of dimension q, and let $\mathcal{M}$ be an associated stretched manifold. By definition, $\mathcal{M}$ is C^∞ compact manifold with fibred boundary close to which $\mathcal{M}$ is described by a stretched wedge $\mathcal{W} = \Omega' \times \bar{\mathbb{R}}_+ \times X$, with Ω' an open subset of $\mathbb{R}^q$ and X a compact closed manifold of dimension n.

Recall that, given any $s, \gamma \in \mathbb{R}$, we define the Sobolev space $H^{s,\gamma}(\mathcal{M})$ to consist of all $u \in H^s_{\mathrm{loc}}(\overset{\circ}{\mathcal{M}})$ such that $\varphi u \in H^{s,\gamma}(\mathcal{W})$ in the corresponding local coordinates near $\partial\mathcal{M}$, for every $\varphi \in C^\infty(\mathcal{M})$ supported close to the boundary (cf. Definition 2.2.2). Note that the transition diffeomorphisms for $\mathcal{M}$ near $\partial\mathcal{M}$ are assumed to satisfy (2.4.3).

On $\mathcal{M}$ live typical differential operators (3.1.2) which give rise to an "algebra" $\Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{w})$ of pseudodifferential operators on $\mathcal{M}$, equipped with a principal symbol structure $\left({}^b\sigma^m(A), \sigma^m_{\mathrm{edge}}(A)\right)$ (cf. Section 3.4). The algebra $\Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{w})$ is a subalgebra of $\Psi^{\mathrm{m}}_{\mathrm{cl}}(\overset{\circ}{\mathcal{M}})$ and the 'compressed' symbol ${}^b\sigma^m(A)$ substitutes the usual principal homogeneous symbol of order m, $\sigma^m(A)$, close to the boundary of $\mathcal{M}$. Every operator $A \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{w})$ extends to a continuous mapping $A\colon H^{s,\gamma}(\mathcal{M}) \to H^{s-m,\gamma-m}(\mathcal{M})$, for all $s \in \mathbb{R}$.

It is now a natural question whether this mapping is a Fredholm operator, for any one $s \in \mathbb{R}$, once A is *elliptic* with respect to the symbol ${}^b\sigma^m(A)$, i.e., ${}^b\sigma^m(A)$ is invertible away from the zero section of ${}^bT^*\mathcal{M}$. The answer is negative in general. A result of Schulze [Sch91] is that the Fredholm property requires the bijectivity of the operator-valued edge symbol, namely

$$\sigma^m_{\mathrm{edge}}(A)(y,\eta)\colon H^{s,\gamma}(\bar{\mathbb{R}}_+\times X)\to H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+\times X), \tag{4.1.1}$$

for each $y\in S$ and $\eta\in\mathbb{R}^q\setminus\{0\}$ (cf. (3.4.9)). However, the ellipticity with respect to ${}^b\sigma^m(A)$ implies only that (4.1.1) is a Fredholm operator for the weights $\gamma\in\mathbb{R}$ such that the line $\Gamma_{-\gamma}$ does not meet the spectrum of the conormal symbol of $\sigma^m_{\mathrm{edge}}(A)(y,\eta)$. This shows that we may expect exceptional weights where the Fredholm property of (4.1.1) is violated. Moreover, these weights s vary along with $y\in S$ and they may fill out the entire real axis. On the other hand, the Fredholm property of (4.1.1) itself is not sufficient.

The idea from Rempel and Schulze [RS82b], Schulze [Sch89b, Sch91], etc., is now to enlarge the class of operators by allowing matrices

$$\begin{pmatrix} A & P\\ T & B\end{pmatrix}\colon \begin{matrix} H^{s,\gamma}(\mathcal{M})\\ \oplus\\ H^s(S,W)\end{matrix} \to \begin{matrix} H^{s-m,\gamma-m}(\mathcal{M})\\ \oplus\\ H^{s-m}(S,\tilde{W})\end{matrix}, \tag{4.1.2}$$

where W and $\tilde{W}$ are smooth vector bundles over S. The meaning of the additional operators P, *corestriction* or *potential operator* with respect to S, T, *trace operator* with respect to S, and B, pseudodifferential operator on S, is analogous to that from pseudodifferential boundary problems (see Vishik and Eskin [VE65, VE67], Boutet de Monvel [BdM71], etc.). These operators will also be called *edge conditions*. They can be generated in local terms over the wedge $\mathcal{W}$ as pseudodifferential operators along Ω' with operator-valued symbols.

Our notion of ellipticity of edge problems $\mathcal{A}$ defined by (4.1.2) will refer (as it ought to be) to the leading symbols ${}^b\sigma^m(\mathcal{A})$ and $\sigma^m_{\mathrm{edge}}(\mathcal{A})$. As such it reflects a more general principle of establishing concepts of ellipticity in operator algebras with symbolic structures over manifolds with singularities. In those cases it will be natural to have more complicated hierarchies of leading symbolic levels, with interior compatibility conditions between the components. The ellipticity is the bijectivity of every component. Recall that for manifolds with conical singularities the hierarchy consists of two symbols ${}^b\sigma^m(A)$ and $\sigma_{\mathcal{M}}(A)$, the latter being a so-called *conormal symbol*. In edge problems we also have a leading conormal symbol $\sigma_{\mathcal{M}}(\mathcal{A}):=\sigma_{\mathcal{M}}(A)$. It is subordinate here to the other components of the hierarchy, namely to $\sigma^m_{\mathrm{edge}}(\mathcal{A})$. In fact, the bijectivity of $\sigma^m_{\mathrm{edge}}(\mathcal{A})$ implies, in particular, the Fredholm property of (4.1.1), which is in turn equivalent, by the cone theory, to the ellipticity of $\sigma^m_{\mathrm{edge}}(A)$ in the sense of the cone algebra. In other words, $\sigma_{\mathcal{M}}\left(\sigma^m_{\mathrm{edge}}(\mathcal{A})\right)(y,z)\neq 0$, for all $y\in\Omega'$ and $z\in\Gamma_{-\gamma}$, is necessarily satisfied. As well, the Fredholm property of (4.1.1) uses "exit nature" of the operators, when $t\to\infty$. Thus, $\sigma^m_{\mathrm{edge}}(\mathcal{A})(y,\eta)$, for $\eta\neq 0$, controls the subordinate cone conormal symbol as operators along X and exit symbols for $t\to\infty$.

Example 4.1.1 Let Ω' be an open set in $\mathbb{R}^q$. Consider the Laplace operator $\Delta = D_t^2 + \left(D_{y_1}^2 + \ldots + D_{y_q}^2\right)$ in the cylinder $\Omega' \times \bar{\mathbb{R}}_+$. This corresponds to an one-point cone base X, i.e., $n = 0$. It is easy to see that $\Delta = \mathrm{op}(\mathrm{a}(\mathrm{y}, \eta))$, where

$$a(y, \eta) = \frac{1}{t^2}\left((tD_t)^2 + i(tD_t) + (t|\eta|)^2\right),$$

a being actually independent of y. Moreover, the edge symbol of a coincides with a itself, i.e., $\sigma^2_{\mathrm{edge}}(a)(y, \eta) = D_t^2 + |\eta|^2$. For any integer $s \geq 2$ and $\gamma \in \mathbb{R}$, this defines a family of continuous operators

$$\sigma^2_{\mathrm{edge}}(a)(y, \eta) : \; H^{s,\gamma}(\bar{\mathbb{R}}_+) \to H^{s-2,\gamma-2}(\bar{\mathbb{R}}_+)$$

parametrised by the point $(y, \eta) \in T^*\Omega'$. For $\eta \neq 0$, the ordinary differential equation $\sigma^2_{\mathrm{edge}}(a)u = 0$ has two solutions $u^{\pm}(t) = e^{\pm|\eta|t}$. Taking into account the behaviour of $u^{\pm}$ at $t = \infty$, we deduce that only u^- may belong to the domain of $\sigma^2_{\mathrm{edge}}(a)(y, \eta)$. Such is really the case for $\gamma < 0$, otherwise neither of the two lies in $H^{s,\gamma}(\bar{\mathbb{R}}_+)$. Hence it follows that

$$\dim\ker \sigma^2_{edge}(a)(y, \eta) = \begin{cases} 1 & \text{if } \gamma < 0; \\ 0 & \text{if } \gamma \geq 0. \end{cases}$$

On the other hand, we have

$$\begin{aligned} {}^b\sigma^2\left(\sigma^2_{\mathrm{edge}}(a)(y, \eta)\right)(t; \tilde{\tau}) &= \tilde{\tau}^2, \\ \sigma_{\mathcal{M}}\left(\sigma^2_{\mathrm{edge}}(a)(y, \eta)\right)(z) &= z^2 + iz, \\ \sigma^0_{\mathcal{F}_t,\mathrm{exit}}\left(\sigma^2_{\mathrm{edge}}(a)(y, \eta)\right)(t; \tau) &= \tau^2 + |\eta|^2, \\ \sigma^2\,\sigma^0_{\mathcal{F}_t,\mathrm{exit}}\left(\sigma^2_{\mathrm{edge}}(a)(y, \eta)\right)(t; \tau) &= \tau^2, \end{aligned}$$

hence $\sigma^2_{\mathrm{edge}}(a)(y, \eta)$ is elliptic in the sense of the cone algebra on $\bar{\mathbb{R}}_+ \times X$, for each $\eta \neq 0$, provided that $\gamma \neq 0, 1$. Note that the spectrum of the conormal symbol of $\sigma^2_{edge}(a)$ consists of two points, namely $z = 0$ and $z = -i$. Thus, $\gamma = 0$ and $\gamma = 1$ just correspond to those values γ for which the weight line $\Gamma_{-\gamma}$ meets the spectrum. Theorem 2.4.38 of [Sch98] now implies that $\sigma^2_{\mathrm{edge}}(a)(y, \eta)$ is a Fredholm mapping $H^{s,\gamma}(\bar{\mathbb{R}}_+) \to H^{s-2,\gamma-2}(\bar{\mathbb{R}}_+)$ for all $s \in \mathbb{R}$, unless $\eta = 0$ or $\gamma = 0, 1$. In particular, the range of $\sigma^2_{\mathrm{edge}}(a)(y, \eta)$ is isomorphic to the orthogonal complement of the null-space of the formal adjoint operator. Since the formal adjoint of $\sigma^2_{\mathrm{edge}}(a)(y, \eta)$ with respect to the inner product of $H^{0,-\frac{1}{2}}(\bar{\mathbb{R}}_+)$ is actually given by the same differential expression, we deduce from what has already been proved that

$$\dim\mathrm{coker}\, \sigma^2_{\mathrm{edge}}(\mathrm{a})(\mathrm{y}, \eta) = \begin{cases} 1 & \text{if } \gamma \in (1, \infty); \\ 0 & \text{if } \gamma \in (-\infty, 0) \cup (0, 1); \\ \infty & \text{if } \gamma = 0, 1. \end{cases}$$

For definiteness, consider the case $\gamma < 0$. Denote by $t(y, \eta)$ the family of linear functionals on $H^{s,\gamma}(\bar{\mathbb{R}}_+)$ given by

$$t(y, \eta)u = \langle\eta\rangle^2 \int_{\mathbb{R}_+} \left(\kappa_{\langle\eta\rangle} v\,(t)\right) u(t)\, dt, \quad u \in H^{s,\gamma}(\bar{\mathbb{R}}_+), \tag{4.1.3}$$

$v \in H^{\infty,-\gamma-1}(\bar{\mathbb{R}}_+)$ being a fixed function and $\kappa_\theta v\,(t) = \theta^{\frac{1}{2}} u(\theta t)$ for $\theta > 0$. A trivial verification shows that

$$\begin{aligned} t(y,\eta)\,\kappa_{\langle\eta\rangle} u &= \langle\eta\rangle^2 \int_{\mathbb{R}_+} \left(\kappa_{\langle\eta\rangle} v\,(t)\right) \left(\kappa_{\langle\eta\rangle} u\,(t)\right) dt \\ &= \langle\eta\rangle^2 \int_{\mathbb{R}_+} v(t)\, u(t)\, dt \end{aligned}$$

for each $\eta \in \mathbb{R}^q$, whence

$$\begin{aligned} \|(D_y^\alpha D_\eta^\beta t(y,\eta))\,\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+),\mathbb{C})} &\leq \|v\|_{H^{-s,-\gamma-1}(\bar{\mathbb{R}}_+)}\,|D_y^\alpha D_\eta^\beta\,\langle\eta\rangle^2| \\ &\leq c_\beta\,\langle\eta\rangle^{2-|\beta|} \end{aligned}$$

with c_β a constant independent of η (cf. (3.3.1)). We thus arrive at a family of isomorphisms

$$\begin{pmatrix} \sigma^2_{\mathrm{edge}}(a)(y,\eta) \\ t(y,\eta) \end{pmatrix} : \; H^{s,\gamma}(\bar{\mathbb{R}}_+) \to \begin{matrix} H^{s-2,\gamma-2}(\bar{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C} \end{matrix} ,$$

$(y,\eta) \in T^*\Omega' \setminus \{0\}$. Were the elements of $H^{s,\gamma}(\bar{\mathbb{R}}_+)$ continuous up to $t = 0$, one impose the local condition $t(y,\eta)u = u(0)$ and another way of stating these isomorphisms be to say that the *Dirichlet problem* for the Laplace equation is an *elliptic* or *coercive* boundary value problem.

□

This example demonstrates rather strikingly that the bijectivity of (4.1.1) is a generalisation of the classical *Lopatinskii condition* for boundary value problems. From this point of view, the following theorem shows that elliptic typical interior symbols are the best adapted to elliptic edge problems.

Theorem 4.1.2 *Suppose that* $a \in t^{-m}\,\mathcal{S}^m_{\mathrm{cl}}({}^bT^*\mathcal{W})$ *is polynomial in* τ. *If*

$${}^b\sigma^m(a)(y,0,x;\tilde{\eta},\tilde{\tau},\xi) \neq 0 \quad \text{for all} \quad (y,0,x;\tilde{\eta},\tilde{\tau},\xi) \in {}^bT^*\mathcal{W} \setminus \{0\}, \qquad (4.1.4)$$

then to every $y \in \Omega'$ *there corresponds a discrete set* $\sigma'(y)$ *on the real axis, such that* $\sigma^m_{\mathrm{edge}}(a)(y,\eta)\colon H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \to H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X)$ *is a Fredholm operator, for each* $\eta \in \mathbb{R}^q \setminus \{0\}$ *and for each* $s \in \mathbb{R}$, $\gamma \in \mathbb{R} \setminus \sigma'(y)$.

For the proof we need an auxiliary result. Recall that, for a *cone symbol* $h(t;z) \in C^\infty_{\mathrm{loc}}(\bar{\mathbb{R}}_+, \mathcal{M}(\mathbb{C}, \Psi^{\mathrm{m}}(\mathrm{X})))$, the conormal symbol of the Mellin operator $A = t^{-m}\,\mathrm{op}_{\mathcal{M},\gamma}(\mathrm{h})$ follows by putting $t = 0$ in h, i.e., $\sigma_{\mathcal{M}}(A)(z) = h(0;z)$. This behaves like a parameter-dependent pseudodifferential operator of order m on X.

Lemma 4.1.3 *Under condition* (4.1.4), *for every* $y \in \Omega'$ *there is a discrete set* $\sigma(y) \subset \mathbb{C}$ *such that every horizontal strip of finite width in* $\mathbb{C}$ *meets* $\sigma(y)$ *only at a finite number of points, and*

$$\sigma_{\mathcal{M}}\left(\sigma^m_{\text{edge}}(a)(y,\eta)\right)(z) : \ H^s(X) \to H^{s-m}(X) \tag{4.1.5}$$

is an isomorphism for all $z \in \mathbb{C} \setminus \sigma(y)$ *and* $s \in \mathbb{R}$.

Proof. Write

$$a(y,t,x;\tau,\xi,\eta) = \frac{1}{t^m}\,\tilde{a}(y,t,x;t\eta,t\tau,\xi),$$

where $\tilde{a}(y,t,x;\tilde{\eta},\tilde{\tau},\xi) \in \mathcal{S}^m_{\text{cl}}(\mathcal{W} \times \mathbb{R}^{q+1+n})$ is polynomial in $\tilde{\tau}$. By Lemma 3.2.2, we get

$$\sigma_{\mathcal{M}}\left(\sigma^m_{\text{edge}}(a)(y,\eta)\right)(z) = \text{op}_{\mathcal{F}_{\text{x}}}\left(\tilde{\text{a}}(\text{y},0,\text{x};0,\text{z},\xi)\right),$$

and so

$$\begin{aligned} \sigma^m\left(\sigma_{\mathcal{M}}\left(\sigma^m_{\text{edge}}(a)(y,\eta)\right)(z)\right)(x;\Re z,\xi) &= \tilde{a}_m(y,0,x;0,\Re z,\xi) \\ &= {}^b\sigma^m(a)(y,0,x;0,\Re z,\xi) \end{aligned}$$

whenever $y \in \Omega'$ and $z \in \mathbb{C}$. Combining this equality with (4.1.4) we see that, for any fixed $\gamma \in \mathbb{R}$, the operator $\sigma_{\mathcal{M}}\left(\sigma^m_{\text{edge}}(a)(y,\eta)\right)(z)$ is parameter-dependent elliptic of order m on X, with $z \in \Gamma_{-\gamma}$ and $\Re z$ as parameter. By a well-known property of parameter-dependent elliptic operators, to any finite segment $[a,b] \subset \mathbb{R}$ there corresponds a constant c such that (4.1.5) is an isomorphism for all $s \in \mathbb{R}$, provided $|\Re z| \geq c$ and $\Im z \in [a,b]$. To finish the proof it suffices to make use of the following abstract result on holomorphic Fredholm families (cf. Blekher [Ble69] and elsewhere). Let $a(z)\colon V \to \tilde{V}$ be a holomorphic family of Fredholm operators between Hilbert spaces defined for z in a domain $\Xi \subset \mathbb{C}$, and let $a(z)$ be invertible at some point $z_0 \in \Xi$. Then there is a discrete set $\sigma \subset \Xi$ such that $a(z)\colon V \to \tilde{V}$ is an isomorphism for all $z \in \Xi \setminus \sigma$.

□

Proof of Theorem 4.1.2. Pick an $y \in \Omega'$. Let $\sigma(y) \subset \mathbb{C}$ be the set given by Lemma 4.1.3, and let $\gamma \in \mathbb{R}$ satisfy $\Gamma_{-\gamma} \cap \sigma(y) = \emptyset$. As the operator $\sigma^m_{\text{edge}}(a)(y,\eta)$ belongs to the cone algebra on $\bar{\mathbb{R}}_+ \times X$, the Fredholm property of (4.1.1) will be established once we prove the ellipticity of $\sigma^m_{\text{edge}}(a)(y,\eta)$ with respect to the symbols ${}^b\sigma^m_{\mathcal{F}_{t,x}}$, $\sigma_{\mathcal{M}}$, $\sigma^0_{\mathcal{F}_t,\text{exit}}$ and $\sigma^m_{\mathcal{F}_t}\,\sigma^0_{\mathcal{F}_t,\text{exit}}$, the latter two being *exit* symbols. Lemma 4.1.3 just amounts to saying that $\sigma^m_{\text{edge}}(a)(y,\eta)$ is elliptic with respect to the conormal symbol $\sigma_{\mathcal{M}}$. As for the other three symbolic levels, an easy computation shows that

$$\begin{aligned} {}^b\sigma^m_{\mathcal{F}_{t,x}}\left(\sigma^m_{\text{edge}}(a)(y,\eta)\right)(t,x;\tilde{\tau},\xi) &= {}^b\sigma^m_{\mathcal{F}_{y,t,x}}(a)(y,0,x;0,\tilde{\tau},\xi), \\ \sigma^0_{\mathcal{F}_t,\text{exit}}\left(\sigma^m_{\text{edge}}(a)(y,\eta)\right)(t,x;\tau,\xi) &= {}^b\sigma^m_{\mathcal{F}_{y,t,x}}(a)(y,0,x;\eta,\tau,0), \\ \sigma^m_{\mathcal{F}_t}\,\sigma^0_{\mathcal{F}_t,\text{exit}}\left(\sigma^m_{\text{edge}}(a)(y,\eta)\right)(t,x;\tau,\xi) &= {}^b\sigma^m_{\mathcal{F}_{y,t,x}}(a)(y,0,x;0,\tau,0), \end{aligned}$$

hence they are subordinate to (4.1.4).

□

Note that Theorem 4.1.2 makes sense also in the context of arbitrary symbols $a \in \mathcal{S}^m(\Omega' \times \mathbb{R}^q; w)$ where $w = (\gamma, \gamma - m, (-l, 0])$ (cf. Theorem 3.5.1 in [Sch98]). This is a crucial step towards edge problems.

In all cases when

$$\sigma^m_{\text{edge}}(a)(y, \eta) \colon H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \to H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X)$$

is a Fredholm mapping, there are finite-dimensional subspaces

$$\begin{array}{rcl} \Sigma(y, \eta; \gamma) & \subset & \omega\, H^{\infty,\gamma}(\bar{\mathbb{R}}_+ \times X) + (1 - \omega)\, \mathcal{S}(\bar{\mathbb{R}}_+, C^\infty_{\text{loc}}(X)), \\ \tilde{\Sigma}(y, \eta; \gamma) & \subset & \mathcal{S}(\bar{\mathbb{R}}_+, C^\infty_{\text{loc}}(X)) \end{array}$$

such that

$$\begin{array}{rcl} \Sigma(y, \eta; \gamma) & = & \ker \sigma^m_{\text{edge}}(a)(y, \eta), \\ H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X) & = & \operatorname{im} \sigma^{\mathrm{m}}_{\text{edge}}(\mathrm{a})(\mathrm{y}, \eta) \oplus \tilde{\Sigma}(\mathrm{y}, \eta; \gamma) \end{array}$$

for each $y \in \Omega'$ and $\eta \neq 0$.

Setting

$$\begin{array}{rcl} l & = & \dim \tilde{\Sigma}(y, \eta; \gamma), \\ \tilde{l} & = & \dim \Sigma(y, \eta; \gamma), \end{array}$$

we find a matrix of operators

$$\begin{pmatrix} \sigma^m_{\text{edge}}(a) & p \\ t & b \end{pmatrix} \colon \begin{array}{c} H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^l \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^{\tilde{l}} \end{array} \tag{4.1.6}$$

which is an isomorphism for the given fixed (y, η).

For obtaining p and t it is sufficient to choose arbitrary isomorphisms $p \colon \mathbb{C}^l \xrightarrow{\cong} \tilde{\Sigma}(y, \eta; \gamma)$ and $t \colon \Sigma(y, \eta; \gamma) \xrightarrow{\cong} \mathbb{C}^{\tilde{l}}$, respectively. Moreover, we may set $b = 0$.

The edge calculus will require a choice of p, t and b such that (4.1.6) smoothly depends on $y \in \Omega'$ and $\eta \neq 0$, and that

$$\begin{pmatrix} \sigma^m_{\text{edge}}(a) & p \\ t & b \end{pmatrix} (y, \theta\eta) = \theta^m \begin{pmatrix} \kappa_\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma^m_{\text{edge}}(a) & p \\ t & b \end{pmatrix} (y, \eta) \begin{pmatrix} \kappa_\theta & 0 \\ 0 & 1 \end{pmatrix}^{-1} \tag{4.1.7}$$

for all $\theta > 0$. By the above, this homogeneity relation is satisfied for the upper left corner anyway. For the remaining entries it suffices to have an isomorphism (4.1.6) for arbitrary y and $|\eta| = 1$, and then to define the values of p, t and b at $\eta \neq 0$ via (4.1.7) by putting $\theta = |\eta|$ and replacing η by $\frac{\eta}{|\eta|}$.

It remains to choose p, t and b smoothly in y and η with $|\eta| = 1$. The existence of such a choice for y varying over a compact set $K \subset \Omega'$ is actually an easy consequence of generalities on families of Fredholm operators parametrised by a compact parameter set which is here $K \times \mathbb{S}^{q-1}$ (see for instance [Sch91, 2.2.5]). The restriction to compact K will be sufficient for our purposes, since Ω' below plays the role of a piece from a C^∞ compact manifold S (the edge). Since the dimensions l and $\tilde{l}$ may jump in general under varying y, we finally get operator families

$$\begin{pmatrix} \sigma^m_{\mathrm{edge}}(a)(y,\eta) & p(y,\eta) \\ t(y,\eta) & b(y,\eta) \end{pmatrix} : \begin{array}{c} H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ V_{(y,\eta)} \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \tilde{V}_{(y,\eta)} \end{array},$$

V and $\tilde{V}$ being vector bundles over $K \times \mathbb{S}^{q-1}$ and $\dim \tilde{V}_{(y,\eta)} - \dim V_{(y,\eta)} = \tilde{l} - l$. Here, the subscript (y,η) indicates the fibre over (y,η). Then, we have to allow $b \neq 0$ in general.

It is now a topological condition on the original symbol a that V and $\tilde{V}$ may be chosen as the local representatives of some vector bundles over S, i.e., the dependence on η disappears. Under this condition, we arrive at a family

$$\begin{pmatrix} \sigma^m_{\mathrm{edge}}(a) & p \\ t & b \end{pmatrix}(y,\eta) : \begin{array}{c} H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ V_y \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \tilde{V}_y \end{array} \tag{4.1.8}$$

which is then to be used below as the principal symbol of some *edge problem* associated with $A = \mathrm{op(a)}$.

We now proceed with the study of families (4.1.8) in the framework of "twisted" operator-valued symbols. Let us agree to consider the finite-dimensional spaces $\mathbb{C}^l$ and $\mathbb{C}^{\tilde{l}}$ with the identity group actions.

Definition 4.1.4 *An operator-valued function $p(y,\eta)$ on $\Omega' \times \mathbb{R}^q$ is said to be a potential edge symbol of order m with asymptotics if there is an asymptotic type*

$$\mathrm{as} \in \mathrm{As}(\delta, (-l, 0])$$

such that

$$\begin{array}{rcl} p(y,\eta) & \in & \mathcal{S}^m_{\mathrm{cl}}(\Omega' \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}, \mathcal{S}^\delta_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X))), \\ p^*(y,\eta) & \in & \cap_{s\in\mathbb{R}} \mathcal{S}^m_{\mathrm{cl}}(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,-\delta}(\bar{\mathbb{R}}_+ \times X), \mathbb{C})). \end{array}$$

For double weight data $w = (\gamma, \delta, (-l, 0])$, we denote by $\mathcal{S}^m_P(\Omega' \times \mathbb{R}^{''}; \precapprox)$ the space of all potential edge symbols of order m with asymptotics, as above.

Definition 4.1.5 *An operator-valued function $t(y,\eta)$ on $\Omega' \times \mathbb{R}^q$ is called a trace edge symbol of order m with asymptotics if there is an asymptotic type*

$$\tilde{\mathrm{as}} \in \mathrm{As}(-\gamma, (-l, 0])$$

such that

$$\begin{array}{rcl} t(y,\eta) & \in & \cap_{s\in\mathbb{R}} \mathcal{S}^m_{\mathrm{cl}}(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X), \mathbb{C})), \\ t^*(y,\eta) & \in & \mathcal{S}^m_{\mathrm{cl}}(\Omega' \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}, \mathcal{S}^{-\gamma}_{\tilde{\mathrm{as}}}(\bar{\mathbb{R}}_+ \times X))). \end{array}$$

We write $\mathcal{S}_T^m(\Omega' \times \mathbb{R}^q; w)$ for the space of all trace edge symbols of order m with asymptotics, as above.

Note that it is typical for the edge pseudodifferential calculus that the trace objects occur in integral form, in contrast to the case of standard boundary value problems. In other words, the traces which restrict the argument functions to the edges, possibly after differentiating them with respect to the t-variable (i.e., those of the form $t(y,\eta)u = D_t^j u\,|_{t=0}$, with $j \in \mathbb{Z}_+$), do not belong to the trace operators here. This would be impossible anyway, because in elliptic edge problems we cannot expect the solutions to have such traces on the edges. We will get in fact more general (e.g. discrete) asymptotics that are just the reason for our framework with arbitrary asymptotic types.

It follows directly from these definitions that $p(y,\eta)$ is a potential edge symbol if and only if the formal adjoint $p^*(y,\eta)$ is a trace edge symbol. For this reason, potential edge symbols are sometimes called corestriction edge symbols.

The composition of a potential edge symbol and a trace edge symbol is a Green edge symbol. Moreover, these concepts fit together to be treated from a uniform point of view. Namely, an operator-valued function $g(y,\eta)$ on $\Omega' \times \mathbb{R}^q$ is said to be a *generalised* Green edge symbol of order m with asymptotics if there are asymptotic types

$$\begin{array}{rcl} \mathrm{as} & \in & \mathrm{As}(\delta, (-l, 0]), \\ \tilde{\mathrm{as}} & \in & \mathrm{As}(-\gamma, (-l, 0]) \end{array}$$

such that

$$\begin{array}{rcl} g(y,\eta) & \in & \bigcap\limits_{s\in\mathbb{R}} \mathcal{S}_{\mathrm{cl}}^m(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^l, \mathcal{S}_{\mathrm{as}}^{\delta}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^{\tilde{l}})), \\ g^*(y,\eta) & \in & \bigcap\limits_{s\in\mathbb{R}} \mathcal{S}_{\mathrm{cl}}^m(\Omega' \times \mathbb{R}^q, \mathcal{L}(H^{s,-\delta}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^{\tilde{l}}, \mathcal{S}_{\tilde{\mathrm{as}}}^{-\gamma}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^l)). \end{array}$$

An important point to note here is the form of group actions in fibres which are $\kappa_\theta \oplus \mathrm{Id}$. Writing

$$g(y,\eta) = \begin{pmatrix} g(y,\eta) & p(y,\eta) \\ t(y,\eta) & b(y,\eta) \end{pmatrix} \tag{4.1.9}$$

we see at once that $g(y,\eta)$ is a Green edge symbol in the proper sense of Definition 3.3.10, $p(y,\eta)$ is an l-tuple of potential edge symbols, $t(y,\eta)$ is an $\tilde{l}$-tuple of trace edge symbols, and $b(y,\eta)$ is simply an $(\tilde{l} \times l)$-matrix of classical scalar symbols of order m along the edge.

For weight data $w = (\gamma, \delta, (-l, 0])$, let $\mathcal{S}_G^m(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\mathbb{C}^l, \mathbb{C}^{\tilde{l}}); \mathrm{w})$ stand for the space of all Green edge symbols of order m with asymptotics, as above.

We are able to describe the edge conditions which substitute the boundary conditions for the case of edge problems. When restricted to a collar neighbourhood of the boundary, these are simply Fourier pseudodifferential operators along the edge with the operator-valued symbols $g(y,\eta)$ in $\mathcal{S}^m_G(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\mathbb{C}^l, \mathbb{C}^{\tilde{l}}); \mathrm{w})$ that have 0 in the upper left corner. Using edge conditions (4.1.9), one corrects edge symbols originated with elliptic typical symbols in the wedge, thus attaining an isomorphism of the principal homogeneous edge symbol.

Our next objective is to introduce an algebra of edge problems on a manifold with fibred boundary. This follows by the same scheme as in Section 3.4. Fix weight data $w = (\gamma, \gamma - m, (-l, 0])$, where $m, \gamma \in \mathbb{R}$ and l is a positive integer. We first define a local algebra $\Psi^{\mathrm{m}}(\mathcal{W}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ on the wedge $\mathcal{W} = \Omega' \times \bar{\mathbb{R}}_+ \times X$, where

$$\begin{aligned} W &= S \times \mathbb{C}^l, \\ \tilde{W} &= S \times \mathbb{C}^{\tilde{l}} \end{aligned}$$

are trivial bundles over S. To do this, denote by $\mathcal{S}^m(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{W}, \tilde{\mathrm{W}}); \mathrm{w})$ the space of all operator-valued functions

$$a(y,\eta) = \begin{pmatrix} a(y,\eta) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & p(y,\eta) \\ t(y,\eta) & b(y,\eta) \end{pmatrix}$$

on $\Omega' \times \mathbb{R}^q$, where $a(y,\eta) \in \mathcal{S}^m(\Omega' \times \mathbb{R}^q; w)$ and the latter matrix belongs to $\mathcal{S}^m_G(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{W}, \tilde{\mathrm{W}}); \mathrm{w})$ (cf. (3.4.1)). Since the edge conditions are classical operator-valued symbols by the very definition, each symbol $a(y,\eta) \in \mathcal{S}^m(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{W}, \tilde{\mathrm{W}}); \mathrm{w})$ possesses two principal homogeneous symbols of order m, namely

$$\begin{aligned} {}^b\sigma^m(a) &= {}^b\sigma^m(a), \\ \sigma^m_{\mathrm{edge}}(a) &= \begin{pmatrix} \sigma^m_{\mathrm{edge}}(a) & \sigma^m_{\mathrm{edge}}(p) \\ \sigma^m_{\mathrm{edge}}(t) & \sigma^m_{\mathrm{edge}}(b) \end{pmatrix} \end{aligned} \tag{4.1.10}$$

(cf. (3.4.3) and (3.4.4)). Note that $\sigma^m_{\mathrm{edge}}(b) = \sigma^m(b)$, the latter being the usual principal homogeneous symbol of b. Just as in Section 3.4, every symbol $a(y,\eta) \in \mathcal{S}^m(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{W}, \tilde{\mathrm{W}}); \mathrm{w})$ gives rise a pseudodifferential operator $\mathcal{A} = \mathrm{op(a)}$ with respect to the variable y. For each $s \in \mathbb{R}$, it extends to a continuous mapping

$$\mathcal{A} : \begin{array}{c} H^s_{\mathrm{comp}}(\Omega', \pi^* H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)) \\ \oplus \\ H^s_{\mathrm{comp}}(\Omega', \mathbb{C}^l) \end{array} \to \begin{array}{c} H^{s-m}_{\mathrm{loc}}(\Omega', \pi^* H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X)) \\ \oplus \\ H^{s-m}_{\mathrm{loc}}(\Omega', \mathbb{C}^{\tilde{l}}) \end{array} \tag{4.1.11}$$

(cf. (3.4.5)). We write $\Psi^{\mathrm{m}}(\mathcal{W}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ for the set of all operators $\mathcal{A}$ with symbols in $\mathcal{S}^m(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{W}, \tilde{\mathrm{W}}); \mathrm{w})$.

To complete the algebra of edge problems on $\mathcal{M}$ it remains to glue together all the local algebras on $\mathcal{M}$. For this purpose we fix a covering of the collar neighbourhood O of $\partial\mathcal{M}$ by charts with edges $(h_\iota, O_\iota)_{\iota \in I}$ on $\mathcal{M}$.

We may take O_ι to be diffeomorphic to $O'_\iota \times \bar{\mathbb{R}}_+ \times X$, with O'_ι a coordinate patch on S, and h_ι to be a diffeomorphism of O_ι onto a stretched wedge $\mathcal{W}_\iota = \Omega'_\iota \times \bar{\mathbb{R}}_+ \times X$, with Ω'_ι an open set in $\mathbb{R}^q$. Pick a partition of unity $(\varphi_\iota)_{\iota\in I}$ on O subordinate to the covering $(O_\iota)_{\iota\in I}$. For each ι, choose a function $\psi_\iota \in C^\infty_{\mathrm{loc}}(O)$ such that $\operatorname{supp}\psi_\iota \subset \mathrm{O}_\iota$ and $\psi_\iota = 1$ on the support of φ_ι. Then, to every system of local edge problems $\mathcal{A}_\iota \in \Psi^\mu(\mathcal{W}_\iota; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ we can assign a global operator

$$\mathcal{A}_b = \sum_\iota \varphi_\iota \, h_\iota^\sharp \mathcal{A}_\iota \, \psi_\iota$$

on O, where $h_\iota^\sharp \mathcal{A}_\iota = h_\iota^* \mathcal{A}_\iota h_{\iota *}$ is the pull-back of $\mathcal{A}_\iota$ under h_ι. As each local diffeomorphism of $\mathcal{M}$ near the boundary acts trough a local diffeomorphism of S (cf. (2.4.1)), the invariance of $\mathcal{A}_b$ under local coordinates in O actually reduces to that for the operator in the upper left corner of $\mathcal{A}_b$. By the above, if h_ι does not touch the variables t and x, then $\mathcal{A}_b$ is independent of the various choices involved modulo smoothing Green operators in the generalised sense. These should be incorporated in the algebra anyway because they appear as remainder terms in the parametrix construction for elliptic edge problems. More precisely, denote by $\Psi^{-\infty}(\mathcal{M}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ the space of all edge problems $\mathcal{S}$ on M with the property that there are asymptotic types

$$\begin{aligned} \mathrm{as} &\in \mathrm{As}(\gamma - \mathrm{m}, (-\mathrm{l}, 0]), \\ \tilde{\mathrm{as}} &\in \mathrm{As}(-\gamma, (-\mathrm{l}, 0]) \end{aligned}$$

such that

$$\begin{aligned} \mathcal{S} &\in \cap_{s\in\mathbb{R}} \mathcal{L}(H^{s,\gamma}(\mathcal{M}) \oplus H^s(S, W), H^{\infty,\gamma-m}_{\mathrm{as}}(\mathcal{M}) \oplus H^\infty(S, \tilde{W})), \\ \mathcal{S}^* &\in \cap_{s\in\mathbb{R}} \mathcal{L}(H^{s,-\gamma+m}(\mathcal{M}) \oplus H^s(S, \tilde{W}), H^{\infty,-\gamma}_{\tilde{\mathrm{as}}}(\mathcal{M}) \oplus H^\infty(S, W)), \end{aligned}$$

the 'asterisk' meaning the formal adjoint with respect to the conjugate linear pairing induced by an inner product in $H^{0,0}(\mathcal{M}) \oplus H^0(S)$.

Definition 4.1.6 *Given weight data $w = (\gamma, \gamma - m, (-l, 0])$, denote by $\Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ the set of all operators*

$$\mathcal{A} = \varphi_b \mathcal{A}_b \psi_b + \varphi_i \begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix} \psi_i + \mathcal{S} \qquad (4.1.12)$$

on $\mathcal{M}$, where $\mathcal{A}_b$ is an operator near $\partial\mathcal{M}$ as above; A_i is a classical pseudodifferential operator of order m in $\mathring{\mathcal{M}}$; and $\mathcal{S} \in \Psi^{-\infty}(\mathcal{M}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$.

The elements of $\Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ are said to be *edge problems* of order m on $\mathcal{M}$ with respect to the weight data w. As mentioned, the space $\Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ is invariant under the diffeomorphisms of $\mathcal{M}$ which preserve the fibration (2.2.1) of $\mathcal{M}$ near the boundary.

An analogous definition makes sense in the case where the upper left corner of $\mathcal{A}$ is an operator between sections of vector bundles V and $\tilde{V}$ over

$\mathcal{M}$. Moreover, we may consider arbitrary vector bundles W and $\tilde{W}$ over S instead of the trivial bundles $W = S \times \mathbb{C}^l$ and $\tilde{W} = S \times \tilde{W}$, respectively. This yields the operator classes $\Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$. The details are left to the reader.

Theorem 4.1.7 *Let* $\mathcal{A} \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$. *For each* $s \in \mathbb{R}$ *and asymptotic types* $\mathrm{as} \in \mathrm{As}(\gamma, (-\mathrm{l}, 0])$ *and* $\tilde{\mathrm{as}} \in \mathrm{As}(\gamma - \mathrm{m}, (-\mathrm{l}, 0])$, *the operator* $\mathcal{A}$ *induces continuous mappings*

$$\begin{aligned} \mathcal{A} &: H^{s,\gamma}(\mathcal{M}, V) \oplus H^s(S, W) \to H^{s-m,\gamma-m}(\mathcal{M}, \tilde{V}) \oplus H^{s-m}(S, \tilde{W}), \\ \mathcal{A} &: H^{s,\gamma}_{\mathrm{as}}(\mathcal{M}, V) \oplus H^s(S, W) \to H^{s-m,\gamma-m}_{\tilde{\mathrm{as}}}(\mathcal{M}, \tilde{V}) \oplus H^{s-m}(S, \tilde{W}). \end{aligned}$$

Proof. This follows immediately from the corresponding results of the local theory (cf. (4.1.11)). We emphasise once again that it is generalised Green operators only that contribute to asymptotics.

□

Every edge problem $\mathcal{A} \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ bears two principal symbols. These are bundle homomorphisms

$$\begin{aligned} {}^b\sigma^m(\mathcal{A}) &: \pi^* V \to \pi^* \tilde{V}, \\ \sigma^m_{\mathrm{edge}}(\mathcal{A}) &: \pi^* \begin{matrix} H^{s,\gamma}(F^{-1}(\cdot)) \otimes V_S \\ \oplus \\ W \end{matrix} \to \pi^* \begin{matrix} H^{s-m,\gamma-m}(F^{-1}(\cdot)) \otimes \tilde{V}_S \\ \oplus \\ \tilde{W} \end{matrix}, \end{aligned} \tag{4.1.13}$$

over ${}^bT^*\mathcal{M}$ and T^*S, respectively, for $s \in \mathbb{R}$. We keep the terminology of the previous sections and call ${}^b\sigma^m(\mathcal{A})$ the *principal interior symbol* of $\mathcal{A}$, and $\sigma^m_{\mathrm{edge}}(\mathcal{A})$ the *principal edge symbol* of $\mathcal{A}$ (cf. (3.4.9)).

Theorem 4.1.8 *Suppose* $\mathcal{A} \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$. *If* ${}^b\sigma^m(\mathcal{A}) = 0$ *and* $\sigma^m_{\mathrm{edge}}(\mathcal{A}) = 0$, *then the mappings of Theorem 4.1.7 are compact for all* $s \in \mathbb{R}$.

Proof. Indeed, from ${}^b\sigma^m(\mathcal{A}) = 0$ and $\sigma^m_{\mathrm{edge}}(\mathcal{A}) = 0$ it follows easily that $A \in \Psi^{\mathrm{m}-1}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$. Thus, $\mathcal{A}$ acts trough the compact embedding

$$\begin{matrix} H^{s-m+1,\gamma-m+1}(\mathcal{M}, \tilde{V}) \\ \oplus \\ H^{s-m+1}(S, \tilde{W}) \end{matrix} \hookrightarrow \begin{matrix} H^{s-m,\gamma-m}(\mathcal{M}, \tilde{V}) \\ \oplus \\ H^{s-m}(S, \tilde{W}) \end{matrix},$$

and the proof is complete.

□

Theorem 4.1.8 makes it transparent that the pair of principal symbols $\left({}^b\sigma^\mu(A), \sigma^\mu_{\mathrm{edge}}(A)\right)$ allows one to construct a parametrix on the symbol level in the algebra $\Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$. We only need to show that this "algebra" is really closed under compositions of edge problems. By a *composition* of edge problems $\mathcal{A}_1$ and $\mathcal{A}_2$ on $\mathcal{M}$, we mean the composition of the corresponding mappings from Theorem 4.1.7. Of course, this tacitly assumes a compatibility of vector bundles and weight data.

Theorem 4.1.9 *As defined above, the composition $\mathcal{A}_2\mathcal{A}_1$ of any two operators*

$$\begin{array}{lll} \mathcal{A}_1 & \in & \Psi^{\mathrm{m}_1}(\mathcal{M};\mathrm{V}^1,\mathrm{V}^2;\mathrm{W}^1,\mathrm{W}^2;\mathrm{w}_1), \quad w_1=(\gamma,\gamma-m_1,(-l,0]),\\ \mathcal{A}_2 & \in & \Psi^{\mathrm{m}_2}(\mathcal{M};\mathrm{V}^2,\mathrm{V}^3;\mathrm{W}^1,\mathrm{W}^2;\mathrm{w}_2), \quad w_2=(\gamma-m_1,\gamma-m_1-m_2,(-l,0]),\end{array}$$

belongs to $\Psi^{\mathrm{m}_1+\mathrm{m}_2}(\mathcal{M};\mathrm{V}^1,\mathrm{V}^3;\mathrm{W}^1,\mathrm{W}^3;\mathrm{w}_2\circ\mathrm{w}_1)$. Moreover, it obeys the symbolic structure in the sense that

$$\begin{array}{rcl} {}^b\sigma^{m_1+m_2}(\mathcal{A}_2\mathcal{A}_1) & = & {}^b\sigma^{m_2}(\mathcal{A}_2)\,{}^b\sigma^{m_1}(\mathcal{A}_1),\\ \sigma^{m_1+m_2}_{\mathrm{edge}}(\mathcal{A}_2\mathcal{A}_1) & = & \sigma^{m_2}_{\mathrm{edge}}(\mathcal{A}_2)\,\sigma^{m_1}_{\mathrm{edge}}(\mathcal{A}_1).\end{array}$$

Proof. This is actually a consequence of Theorem 3.4.6. For a fuller treatment, see Theorem 3.4.56 in [Sch98].

□

We may also distinguish two "subalgebras" of great importance within $\Psi^{\mathrm{m}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{W},\tilde{\mathrm{W}};\mathrm{w})$, namely

$$\begin{array}{l} \Psi^{\mathrm{m}}_{\mathrm{M+G}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{W},\tilde{\mathrm{W}};\mathrm{w}),\\ \Psi^{\mathrm{m}}_{\mathrm{G}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{W},\tilde{\mathrm{W}};\mathrm{w}).\end{array} \tag{4.1.14}$$

Roughly speaking, they consists of those edge problems $\mathcal{A}$ on $\mathcal{M}$ for which the operator A in the upper left corner belongs to $\Psi^{\mathrm{m}}_{\mathrm{M+G}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$ or $\Psi^{\mathrm{m}}_{\mathrm{G}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$, respectively. Just as in the "scalar" case of operators in the upper left corner, the latter algebra in (4.1.14) is an ideal in the former.

Theorem 4.1.10 *Each edge problem $\mathcal{A}\in\Psi^{\mathrm{m}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{W},\tilde{\mathrm{W}};\mathrm{w})$ allows a formal adjoint $\mathcal{A}^*\in\Psi^{\mathrm{m}}(\mathcal{M};\tilde{\mathrm{V}},\mathrm{V};\tilde{\mathrm{W}},\mathrm{W};\mathrm{w}^*)$. Moreover,*

$$\begin{array}{rcl} {}^b\sigma^m(\mathcal{A}^*) & = & ({}^b\sigma^m(\mathcal{A}))^*,\\ \sigma^m_{\mathrm{edge}}(\mathcal{A}^*) & = & (\sigma^m_{\mathrm{edge}}(\mathcal{A}))^*.\end{array}$$

Proof. Cf. Theorem 3.4.55 in [Sch98]. Note that the formal adjoint is understood with respect to the inner product of $H^{0,0}(\mathcal{M})\oplus H^0(S)$.

□

We are now prepared to formulate a concept of ellipticity on a manifold with fibred boundary.

Definition 4.1.11 *An edge problem $\mathcal{A}\in\Psi^{\mathrm{m}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{W},\tilde{\mathrm{W}};\mathrm{w})$, for $w=(\gamma,\gamma-m,(-l,0])$, is said to be elliptic if*

1) *the principal interior symbol ${}^b\sigma^m(\mathcal{A})$ is invertible away from the zero section of ${}^bT^*\mathcal{M}$;*

2) *the principal edge symbol $\sigma^m_{\mathrm{edge}}(\mathcal{A})$ is invertible away from the zero section of T^*S, for any one $s\in\mathbb{R}$.*

The condition 1) of Definition 4.1.11 means nothing but the interior ellipticity of $\mathcal{A}$ up to the boundary. In fact, it is the ellipticity of the upper left corner of $\mathcal{A}$ on $\mathcal{M}$ in the sense of typical symbols, which is independent of the weight γ. The condition 2) is an analogue of the classical *Lopatinskii condition* in boundary value problems. It is known from the theory of pseudodifferential operators on the infinite stretched cone $\bar{\mathbb{R}}_+ \times X$ that the condition 2) is satisfied for all $s \in \mathbb{R}$ as soon as it holds for a particular $s_0 \in \mathbb{R}$. Moreover, this condition is known to depend essentially on $\gamma \in \mathbb{R}$. The index of $\sigma^m_{\mathrm{edge}}(A)(y, \eta)$, A being the upper left corner of $\mathcal{A}$, depends on γ, and it may happen that, for a particular $\gamma_0 \in \mathbb{R}$, the operator A cannot be completed to a matrix $\mathcal{A}$ satisfying 2) (cf. Lemma 4.1.3). In contrast to the case of boundary value problems, the ellipticity of an edge problem $\mathcal{A}$ for a typical differential operator A requires in general both trace and potential conditions. In other words, (4.1.1) will be a family of Fredholm operators with non-trivial kernels and cokernels whose dimensions depend on γ.

When applied in the interior of $\mathcal{M}$, the condition 1) means nothing but the usual ellipticity of the upper left corner A of $\mathcal{A}$ with respect to the principal interior symbol $\sigma^m(A)$. The *compressed* principal interior symbol ${}^b\sigma^m(\mathcal{A})$ is intended for substituting $\sigma^m(A)$ on the boundary of $\mathcal{M}$ where the latter symbol is not defined. On the other hand, we have some control of ${}^b\sigma^m(\mathcal{A})$ over $\partial\mathcal{M}$ via the condition 2). Indeed, this condition implies that $\sigma^m_{\mathrm{edge}}(A)(y, \eta)$ is a Fredholm mapping between weighted Sobolev spaces over an infinite stretched cone $\bar{\mathbb{R}}_+ \times X$, provided that $\eta \neq 0$. It follows that $\sigma^m_{\mathrm{edge}}(A)(y, \eta)$, $\eta \neq 0$, is an elliptic operator in the corresponding cone algebra, and so several relevant symbols of A at $t = 0$ must be invertible. One may ask whether or not, in the presence of condition 2), the condition 1) is fulfilled automatically over the boundary. Were such the case, the definition of an elliptic edge problem $\mathcal{A}$ on $\mathcal{M}$ be equivalent to the invertibility of the principal symbols $\sigma^m(\mathcal{A})$ over $T^*\mathring{\mathcal{M}} \setminus \{0\}$ and $\sigma^m_{\mathrm{edge}}(\mathcal{A})$ over $T^*S \setminus \{0\}$, i.e., the compressed principal interior symbol ${}^b\sigma^m(\mathcal{A})$ be not needed in the definition of elliptic edge problems on $\mathcal{M}$. The following example highlights this question.

Example 4.1.12 Consider the stretched wedge $\mathcal{W} = \mathbb{R}^1 \times \bar{\mathbb{R}}_+ \times \mathbb{R}^1$ and the typical differential operator

$$A = \frac{1}{t^2}\left((tD_t - tD_y)^2 + (tD_t - D_x)^2 + (tD_x)^2\right)$$

of second order on $\mathcal{W}$. As

$$\sigma^2(A)(t; \eta, \tau, \xi) = (\tau - \eta)^2 + (\tau - \frac{1}{t}\xi)^2 + \xi^2$$

for $t > 0$, we conclude at once that A is elliptic in the interior of $\mathcal{W}$. Further, we get

$$\sigma^2_{\mathrm{edge}}(A)(y, \eta) = \frac{1}{t^2}\left((tD_t - t\eta)^2 + (tD_t - D_x)^2\right)$$

whence

$$\begin{array}{rcl} {}^b\sigma^2\left(\sigma^2_{\mathrm{edge}}(A)\right)(\tilde{\tau},\xi) & = & \tilde{\tau}^2+(\tilde{\tau}-\xi)^2, \\ \sigma_{\mathcal{M}}\left(\sigma^2_{\mathrm{edge}}(A)\right)(z) & = & z^2+(z-D_x)^2, \\ \sigma^0_{\mathcal{F}_t,\mathrm{exit}}\left(\sigma^2_{\mathrm{edge}}(A)\right)(t;\tau) & = & (\tau-\eta)^2+\tau^2, \\ \sigma^2\,\sigma^0_{\mathcal{F}_t,\mathrm{exit}}\left(\sigma^2_{\mathrm{edge}}(A)\right)(t;\tau) & = & 2\tau^2. \end{array}$$

This shows that $\sigma^2_{\mathrm{edge}}(A)(y,\eta)$ is an elliptic operator in the cone algebra over $\bar{\mathbb{R}}_+\times\mathbb{R}^1$, provided that $\eta\neq 0$. However, the compressed principal interior symbol

$${}^b\sigma^2(A)(t;\tilde{\eta},\tilde{\tau},\xi)=(\tilde{\tau}-\tilde{\eta})^2+(\tilde{\tau}-\xi)^2+(t\xi)^2$$

vanishes on the entire line

$$\left\{\begin{array}{rcl} \tilde{\eta} & = & \vartheta, \\ \tilde{\tau} & = & \vartheta, \\ \xi & = & \vartheta \end{array}\right.$$

($\vartheta\in\mathbb{R}$) if $t=0$.

□

4.2 *Parametrix construction*

Assume that (M,S) is a compact closed C^∞ manifold with an edge S, and $\mathcal{M}$ is the corresponding stretched manifold.

Let $\mathcal{A}\in\Psi^{\mathrm{m}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{W},\tilde{\mathrm{W}};\mathrm{w})$ be an edge problem on the manifold $\mathcal{M}$, with $w=(\gamma,\gamma-m,(-l,0])$. By Theorem 4.1.7, $\mathcal{A}$ extends to a continuous mapping

$$\mathcal{A}:\begin{array}{c} H^{s,\gamma}(\mathcal{M},V) \\ \oplus \\ H^s(S,W) \end{array}\to\begin{array}{c} H^{s-m,\gamma-m}(\mathcal{M},\tilde{V}) \\ \oplus \\ H^{s-m}(S,\tilde{W}) \end{array}, \tag{4.2.1}$$

for each $s\in\mathbb{R}$. The Fredholm property of (4.2.1) is equivalent to the existence of a so-called regulariser to $\mathcal{A}$, i.e., an inverse up to compact operators. In the context of weighted Sobolev spaces compact operators are those which improve both the interior smoothness and the weight. We can attain a gain in the weight by requiring asymptotics in the range, for there is a gap between the carrier of asymptotics and the weight line. We are thus led to the following particular concept of a regulariser.

Definition 4.2.1 *An operator $\mathcal{P}\in\Psi^{-\mathrm{m}}(\mathcal{M};\tilde{\mathrm{V}};\mathrm{V};\tilde{\mathrm{W}},\mathrm{W};\mathrm{w}^{-1})$ where $w^{-1}=(\gamma-m,\gamma,(-l,0])$, is said to be a parametrix of $\mathcal{A}$ if*

$$\begin{array}{rcl} \mathcal{P}\mathcal{A}-1 & \in & \Psi^{-\infty}(\mathcal{M};\mathrm{V};\mathrm{W};\mathrm{w}^{-1}\circ\mathrm{w}), \\ \mathcal{A}\mathcal{P}-1 & \in & \Psi^{-\infty}(\mathcal{M};\tilde{\mathrm{V}};\tilde{\mathrm{W}};\mathrm{w}\circ\mathrm{w}^{-1}). \end{array} \tag{4.2.2}$$

Note that

$$\begin{array}{rcl} w^{-1}\circ w & = & (\gamma,\gamma,(-l,0]), \\ w\circ w^{-1} & = & (\gamma-m,\gamma-m,(-l,0]) \end{array}$$

and the operators on the right-hand side of (4.2.2) are compact in the corresponding spaces, which is clear from Theorem 4.1.8.

Theorem 4.1.9 shows that if $\mathcal{P} \in \Psi^{-\mathrm{m}}(\mathcal{M}; \tilde{\mathrm{V}}; \mathrm{V}; \tilde{\mathrm{W}}, \mathrm{W}; \mathrm{w}^{-1})$ is a parametrix of $\mathcal{A}$, then so is $\mathcal{P} + \mathcal{S}$ for any $\mathcal{S} \in \Psi^{-\infty}(\mathcal{M}; \tilde{\mathrm{V}}; \mathrm{V}; \tilde{\mathrm{W}}, \mathrm{W}; \mathrm{w}^{-1})$. Conversely, any two parametrices $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\mathcal{A}$ differ by an element of $\Psi^{-\infty}(\mathcal{M}; \tilde{\mathrm{V}}; \mathrm{V}; \tilde{\mathrm{W}}, \mathrm{W}; \mathrm{w}^{-1})$ because

$$\mathcal{P}_2 - \mathcal{P}_1 = (1 - \mathcal{P}_1\mathcal{A})\mathcal{P}_2 - \mathcal{P}_1(1 - \mathcal{A}\mathcal{P}_2).$$

Hence it follows that the parametrix of an edge problem $\mathcal{A}$ is defined uniquely modulo $\Psi^{-\infty}(\mathcal{M}; \tilde{\mathrm{V}}; \mathrm{V}; \tilde{\mathrm{W}}, \mathrm{W}; \mathrm{w}^{-1})$. Moreover, equating the principal symbols of the operators in (4.2.2) yields

$$\begin{array}{rcl} {}^b\sigma^{-m}(\mathcal{P}) & = & ({}^b\sigma^m(\mathcal{A}))^{-1}, \\ \sigma^{-m}_{\mathrm{edge}}(\mathcal{P}) & = & (\sigma^m_{\mathrm{edge}}(\mathcal{A}))^{-1}, \end{array} \tag{4.2.3}$$

and so for a parametrix to exist it is necessary and sufficient that $\mathcal{A}$ be elliptic in the sense of Definition 4.1.11. This condition proves to be also sufficient.

Theorem 4.2.2 *Each elliptic operator* $\mathcal{A} \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ *on* $\mathcal{M}$ *has a parametrix* $\mathcal{P} \in \Psi^{-\mathrm{m}}(\mathcal{M}; \tilde{\mathrm{V}}; \mathrm{V}; \tilde{\mathrm{W}}, \mathrm{W}; \mathrm{w}^{-1})$.

We have divided the proof into a sequence of lemmas. The reader who is more interested in the basic analytical ideas may confine himself to trivial bundles $V = \tilde{V} = \mathcal{M} \times \mathbb{C}$ and $W = S \times \mathbb{C}^l$, $\tilde{W} = S \times \mathbb{C}^{\tilde{l}}$. Moreover, the proof becomes considerably simpler if we impose the technical assumption that the spectrum of $\sigma_{\mathcal{M}}\left(\sigma^m_{\mathrm{edge}}(\mathcal{A})\right)(y, z)$, i.e., the set $\sigma(y)$ of Lemma 4.1.3, is independent of $y \in S$.

Lemma 4.2.3 *Let* $a(y, \eta) \in \mathcal{S}^m(\Omega' \times \mathbb{R}^q; w)$ *satisfy* (4.1.4). *Then, there exists a* $p(y, \eta) \in \mathcal{S}^{-m}(\Omega' \times \mathbb{R}^q; w^{-1})$ *such that*

$$\begin{array}{rcl} \sigma^{-m}_{\mathrm{edge}}(p)\, \sigma^m_{\mathrm{edge}}(a) & = & 1 \mod \sigma^0_{\mathrm{edge}}\, \mathcal{S}^0_{M+G}(\Omega' \times \mathbb{R}^q; w^{-1} \circ w), \\ \sigma^m_{\mathrm{edge}}(a)\, \sigma^{-m}_{\mathrm{edge}}(p) & = & 1 \mod \sigma^0_{\mathrm{edge}}\, \mathcal{S}^0_{M+G}(\Omega' \times \mathbb{R}^q; w \circ w^{-1}). \end{array} \tag{4.2.4}$$

Proof. Fix a finite covering (O''_ι) of X by coordinate patches and charts $h''_\iota \colon O''_\iota \to \Omega''_\iota$, with Ω''_ι an open set in $\mathbb{R}^n$. Then $\sigma^m(a)$ is represented locally by homogeneous components of order m of symbols

$$\frac{1}{t^m}\, \tilde{a}_\iota(y, t, x; t\eta, t\tau, \xi)$$

in $t^{-m}\, \mathcal{S}^m_{\mathrm{cl}}({}^bT^*(\Omega' \times \bar{\mathbb{R}}_+ \times \Omega''_\iota))$ which are compatible with respect to the symbol pull-backs under transition diffeomorphisms in the x-coordinates, modulo symbols of order $-\infty$. Condition (4.1.4) allows one to calculate the Leibniz inverse of every symbol $t^{-m}\, \tilde{a}_\iota(y, 0, x; t\eta, t\tau, \xi)$ with respect to the (t, x)-variables. Moreover, analysis similar to that in the proof of Proposition 3.1.6 shows they are of the form $t^m\, \tilde{p}_\iota(y, x; t\eta, t\tau, \xi)$ modulo symbols of order $-\infty$, for suitable $\tilde{p}_\iota \in \mathcal{S}^{-m}_{\mathrm{cl}}((\Omega' \times \Omega''_\iota) \times \mathbb{R}^{q+1+n}_{\tilde{\eta},\tilde{\tau},\xi})$. Let (φ_ι) be a partition of unity on X subordinate to the covering (O''_ι). Choose functions $\psi_\iota \in C^\infty_{\mathrm{comp}}(O''_\iota)$ satisfying $\varphi_\iota \psi_\iota = \varphi_\iota$ for all ι, and set

$$p_i(y,\eta) = t^m \sum_\iota \varphi_\iota \, (1 \times h''_\iota)^\sharp \, \mathrm{op}_{\mathcal{F}_{t,x}} \left(\tilde{p}_\iota(y,x;t\eta,t\tau,\xi)\right) \psi_\iota.$$

According to the results of Section 3.2, there exists a C^∞ function $\tilde{h}(y;\tilde{\eta},z)$ of $y \in \Omega'$ with values in $\mathcal{M}(\mathbb{C},\Psi^{-m}(X;\mathbb{R}^q_{\tilde{\eta}}))$, such that

$$p_i(y,\eta) = t^m \, \mathrm{op}_{\mathcal{M},\gamma-m}(h) \, (y,\eta) \quad \mathrm{mod} \ \ C^\infty_{\mathrm{loc}}(\Omega',\Psi^{-\infty}(\mathbb{R}_+ \times X;\mathbb{R}^q_\eta)),$$

where $h(y,t;\eta,z) = \tilde{h}(y;t\eta,z)$. Pick functions φ_b, φ_i and ψ_b, ψ_i, as in (3.4.1), and define

$$p(y,\eta) = \varphi_b \left(p_0(y,\eta) + p_\infty(y,\eta)\right) \psi_b + \varphi_i p_i(y,\eta) \psi_i \qquad (4.2.5)$$

for $(y,\eta) \in \Omega' \times \mathbb{R}^q$, where

$$\begin{array}{rcl} p_0(y,\eta) & = & t^m \, \varphi_0(t\langle\eta\rangle) \, \mathrm{op}_{\mathcal{M},\gamma-m}(\tilde{h}(y;t\eta,z)) \, \psi_0(t\langle\eta\rangle), \\ p_\infty(y,\eta) & = & \varphi_\infty(t\langle\eta\rangle) \, p_i(y,\eta) \, \psi_\infty(t\langle\eta\rangle) \end{array}$$

(cf. (3.2.6)). From (4.2.5) we deduce easily that $p(y,\eta) \in \mathcal{S}^{-m}(\Omega' \times \mathbb{R}^q; w^{-1})$ and

$$\sigma^{-m}_{\mathrm{edge}}(p) = t^m \varphi_0(t|\eta|) \, \mathrm{op}_{\mathcal{M},\gamma-m}(h)(y,\eta) \, \psi_0(t|\eta|) + \varphi_\infty(t|\eta|) \, p_i(y,\eta) \, \psi_\infty(t|\eta|).$$

It is now a simple matter to see that (4.2.4) holds. Indeed, because of the homogeneity in η it suffices to insert $|\eta| = 1$ and to carry out the compositions in the sense of cone pseudodifferential operators over $\bar{\mathbb{R}}_+ \times X$. These compositions are smoothing in the interior of $\bar{\mathbb{R}}_+ \times X$, i.e., of classes

$$\begin{array}{c} \Psi_{M+G}(\bar{\mathbb{R}}_+ \times X; w^{-1} \circ w), \\ \Psi_{M+G}(\bar{\mathbb{R}}_+ \times X; w \circ w^{-1}), \end{array}$$

respectively. For verifying (4.2.4) we can argue in a similar way, noting that the symbols $t^m \, \tilde{p}_\iota(y,x;t\eta,t\tau,\xi)$ can be Leibniz composed with $t^{-m} \, \tilde{a}_\iota(y,0,x;t\eta,t\tau,\xi)$ both from the left and from the right to yield 1 up to remainders of the desired type. This finishes the proof.

□

In the following lemma we assume that the weight interval $(-l,0]$ is finite, which is essential to the proof.

Lemma 4.2.4 *Let $r(y,\eta) \in \mathcal{S}^0_{M+G}(\Omega' \times \mathbb{R}^q; w)$, for $w = (\gamma,\gamma,(-l,0])$. Suppose that $\sigma_{\mathcal{M}}(r) = 0$. Then there is an $m(y,\eta) \in \mathcal{S}^0_{M+G}(\Omega' \times \mathbb{R}^q; w)$ such that*

$$\begin{array}{rcl} (1+m(y,\eta))(1+r(y,\eta)) & = & 1+g(y,\eta), \\ (1+r(y,\eta))(1+m(y,\eta)) & = & 1+g(y,\eta), \end{array}$$

for some $g(y,\eta) \in \mathcal{S}^0_G(\Omega' \times \mathbb{R}^q; w)$.

Proof. It suffices to take

$$1 + m(y, \eta) = \sum_{j=0}^{J} (-1)^j \, (r(y, \eta))^j$$

for J large enough. Indeed, we get

$$\begin{aligned} (1 + m(y, \eta))(1 + r(y, \eta)) &= (1 + r(y, \eta))(1 + m(y, \eta)) \\ &= 1 + (-1)^J \, (r(y, \eta))^{J+1}, \end{aligned}$$

and $g(y, \eta) = (-1)^J \, (r(y, \eta))^{J+1}$ is a Green edge symbol with asymptotics for sufficiently large J. This latter observation follows by invoking an additional symbolic structure on $\mathcal{S}^0_{M+G}(\Omega' \times \mathbb{R}^q; w)$ given by a sequence of conormal symbols (cf. Theorem 3.3.28 and Proposition 3.3.23 in [Sch98]), and the proof is complete.

□

Using Lemma 4.2.4, we can improve the soft regulariser $p(y, \eta)$ of Lemma 4.2.3.

Lemma 4.2.5 *Let $a(y, \eta) \in \mathcal{S}^m(\Omega' \times \mathbb{R}^q; w)$ with $w = (\gamma, \gamma - m, (-l, 0])$. Suppose $a(y, \eta)$ satisfies* (4.1.4)*, and let* (4.1.5) *be an isomorphism for each $y \in \Omega'$, $z \in \Gamma_{-\gamma}$ and some $s \in \mathbb{R}$. Then, there is a $p(y, \eta) \in \mathcal{S}^{-m}(\Omega' \times \mathbb{R}^q; w^{-1})$ such that*

$$\begin{aligned} \sigma^{-m}_{\mathrm{edge}}(p)\, \sigma^{m}_{\mathrm{edge}}(a) &= 1 \mod \sigma^0_{\mathrm{edge}}\, \mathcal{S}^0_G(\Omega' \times \mathbb{R}^q; w^{-1} \circ w), \\ \sigma^{m}_{\mathrm{edge}}(a)\, \sigma^{-m}_{\mathrm{edge}}(p) &= 1 \mod \sigma^0_{\mathrm{edge}}\, \mathcal{S}^0_G(\Omega' \times \mathbb{R}^q; w \circ w^{-1}) \end{aligned} \tag{4.2.6}$$

and

$$\begin{aligned} \varphi\, p(y, \eta)\, \psi\, a(y, \eta) &= \varphi \mod \mathcal{S}^0_G(\Omega' \times \mathbb{R}^q; w^{-1} \circ w), \\ \varphi\, a(y, \eta)\, \psi\, p(y, \eta) &= \varphi \mod \mathcal{S}^0_G(\Omega' \times \mathbb{R}^q; w \circ w^{-1}), \end{aligned} \tag{4.2.7}$$

for suitable cut-off functions $\varphi(t)$ and $\psi(t)$ satisfying $\psi\varphi = \varphi$.

Proof. Condition (4.1.4) implies that the symbol ${}^b\sigma^m(a)$ does not vanish for all $(y, t, x; \tilde{\eta}, \tilde{\tau}, \xi) \in {}^bT^*(\Omega' \times [0, \varepsilon) \times X) \setminus \{0\}$, with some $\varepsilon > 0$. In fact, we can achieve this by shrinking Ω' if necessary, which does not affect the conclusion. Thus, in the notation of Lemma 4.2.3 we can form the Leibniz inverses of the local symbols $t^{-m}\, \tilde{a}_\iota(y, t, x; t\eta, t\tau, \xi)$ with respect to the (t, x)-variables, for $t \in (0, \varepsilon)$. In this way we get a tuple of symbols $t^m\, \tilde{p}_\iota(y, t, x; t\eta, t\tau, \xi)$, for $t \in (0, \varepsilon)$, which are compatible with respect to the symbol pull-backs connected with the transition diffeomorphisms in the x-coordinates. Pick a cut-off function $\omega(t)$ supported by $[0, \varepsilon)$, and set

$$p^{(1)}_i(y, \eta) = t^m \sum_\iota \varphi_\iota\, (1 \times h''_\iota)^\sharp\, \omega(t)\, \mathrm{op}_{\mathcal{F}_{\mathrm{t,x}}}\, (\tilde{\mathrm{p}}_\iota(\mathrm{y}, \mathrm{x}; \mathrm{t}\eta, \mathrm{t}\tau, \xi))\, \psi_\iota \tag{4.2.8}$$

with the charts $h''_\iota : O''_\iota \to \Omega''_\iota$ and the functions φ_ι, ψ_ι from the proof of Lemma 4.2.3.

Applying Corollary 3.2.5 we now find a C^∞ function $\tilde{h}^{(1)}(y,t;\tilde{\eta},z)$ of $(y,t) \in \Omega' \times \bar{\mathbb{R}}_+$ with values in $\mathcal{M}(\mathbb{C}, \Psi^{-\mathrm{m}}(\mathrm{X};\mathbb{R}^{\mathrm{q}}_{\tilde{\eta}}))$, such that

$$p_i^{(1)}(y,\eta) = t^m \operatorname{op}_{\mathcal{M},\gamma-\mathrm{m}}(\mathrm{h}^{(1)})(\mathrm{y},\eta) \mod \mathrm{C}^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+ \times \mathrm{X};\mathbb{R}^{\mathrm{q}}_\eta)),$$

where $h^{(1)}(y,t;\eta,z) = \tilde{h}^{(1)}(y,t;t\eta,z)$. Moreover, this is compatible with the constructions of Lemma 4.2.3 because the Leibniz inversion of interior symbols followed by freezing at $t=0$ gives the symbol $p_i(y,\eta)$ modulo symbols of order $-\infty$. This implies in turn $\tilde{h}^{(1)}(y,0;\tilde{\eta},z) = \tilde{h}(y;\tilde{\eta},z)$ up to an element of $C^\infty_{\mathrm{loc}}(\Omega', \Psi^{-\infty}(\mathbb{R}_+ \times \mathrm{X};\mathbb{R}^{\mathrm{q}}_\eta))$. We now construct a symbol $p^{(1)}(y,\eta) \in \mathcal{S}^{-m}(\Omega' \times \mathbb{R}^q; w^{(-1)})$ analogously to (4.2.5), with $p_i(y,\eta)$ replaced by (4.2.8) and $\tilde{h}(y;\tilde{\eta},z)$ replaced by $\tilde{h}^{(1)}(y,t;\tilde{\eta},z)$.

Choosing cut-off functions $\varphi(t)$ and $\psi(t)$ in such a way that $\omega(t) \equiv 1$ on the supports of φ, ψ and $\psi\varphi = \varphi$, we obtain

$$\begin{aligned}\varphi\, p^{(1)}(y,\eta)\,\psi\, a(y,\eta) &= \varphi \mod \mathcal{S}^0_{M+G}(\Omega' \times \mathbb{R}^q; w^{-1} \circ w),\\ \varphi\, a(y,\eta)\,\psi\, p^{(1)}(y,\eta) &= \varphi \mod \mathcal{S}^0_{M+G}(\Omega' \times \mathbb{R}^q; w \circ w^{-1}).\end{aligned}$$

Set

$$f(y,z) = \left(\sigma_{\mathcal{M}}\left(\sigma^m_{\mathrm{edge}}(a)(y,\eta)\right)\right)^{-1}(z-m)$$

for $y \in \Omega'$ and $z \in \Gamma_{-\gamma+m}$. Since $\sigma_{\mathcal{M}}\left(\sigma^m_{\mathrm{edge}}(a)\right) = \sigma_{\mathcal{M}}(a)$ and

$$\sigma_{\mathcal{M}}(p^{(1)})(y,z) = (\sigma_{\mathcal{M}}(a)(y,z-m))^{-1} \mod \sigma_{\mathcal{M}}\, \mathcal{S}^0_{\mathrm{M+G}}(\Omega' \times \mathbb{R}^{\mathrm{q}}; \mathrm{w} \circ \mathrm{w}^{-1}),$$

the last equality being a consequence of the above ones, we deduce immediately that

$$\sigma_{\mathcal{M}}(p^{(1)})(y,z) = f(y,z) + s(y,z)$$

where $s(y,z)$ is a C^∞ function of $y \in \Omega'$ with values in the space of smoothing Mellin symbols with asymptotic information. More precisely, we have $s(y,z) \in \mathcal{M}_{\mathrm{as}}(\mathbb{C}, \Psi^{-\infty}(\mathrm{X}))$ for each fixed $y \in \Omega'$, the subscript 'as' referring to an asymptotic type of operator-valued Mellin symbols (cf. [Sch98, 2.3.4]).

Define

$$p^{(2)}(y,\eta) = p^{(1)}(y,\eta) - t^m\, \varphi_0(t\langle\eta\rangle)\operatorname{op}_{\mathcal{M},\gamma-\mathrm{m}}(\mathrm{s}(\mathrm{y},\mathrm{z}))\,\psi_0(\mathrm{t}\langle\eta\rangle)$$

for cut-off functions $\varphi_0(t)$ and $\psi_0(t)$. Then

$$\varphi\, p^{(2)}(y,\eta)\,\psi\, a(y,\eta) = 1 + r(y,\eta),$$

where $r(y,\eta) \in \mathcal{S}^0_{M+G}(\Omega' \times \mathbb{R}^q; w^{-1} \circ w)$ satisfies $\sigma_{\mathcal{M}}(r)(y,z) \equiv 0$. By Lemma 4.2.4 there is a symbol $m(y,\eta) \in \mathcal{S}^0_{M+G}(\Omega' \times \mathbb{R}^q; w^{-1} \circ w)$ with the property that

$$(1 + m(y,\eta))(1 + r(y,\eta)) = 1 + g(y,\eta)$$

for some $g(y,\eta) \in \mathcal{S}^0_G(\Omega' \times \mathbb{R}^q; w^{-1} \circ w)$. Setting

$$p(y,\eta) = (1 + m(y,\eta))p^{(2)}(y,\eta),$$

we comply the first equality of (4.2.7). The same construction applies to yield yet another symbol $p(y,\eta) \in \mathcal{S}^{-m}(\Omega' \times \mathbb{R}^q; w^{-1})$ satisfying the second equality of (4.2.7). Then, using the fact that the Green symbols form an ideal, one shows by standard algebraic manipulations that each of the symbols $p(y,\eta)$ so obtained actually satisfies both equalities in (4.2.7). To complete the proof, it suffices to use the observation that (4.2.7) implies (4.2.6).

□

We wish to arrange the remainder terms in (4.2.7) to be as smoothing as possible. For this purpose, we are going to make use of the second condition in Definition 4.1.11.

Lemma 4.2.6 *Let* $(g_{ij}(y,\eta)) \in \sigma^0_{\mathrm{edge}}\mathcal{S}^0_G(\Omega' \times \mathbb{R}^q; \mathrm{Hom(W)}; \mathrm{w})$, *where* $w = (\gamma, \gamma, (-l, 0])$. *Suppose*

$$\begin{pmatrix} 1+g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}(y,\eta) : \begin{array}{c} H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^l \end{array} \to \begin{array}{c} H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^l \end{array} \tag{4.2.9}$$

is invertible for all $(y,\eta) \in T^*\Omega' \setminus \{0\}$ *and any one* $s \in \mathbb{R}$. *Then* (4.2.9) *is invertible for all* s *and there is an* $(h_{ij}(y,\eta)) \in \sigma^0_{\mathrm{edge}}\mathcal{S}^0_G(\Omega' \times \mathbb{R}^q; \mathrm{Hom(W)}; \mathrm{w})$ *such that*

$$\begin{pmatrix} 1+g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} = \begin{pmatrix} 1+h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Proof. That the invertibility of (4.2.9) for any one $s \in \mathbb{R}$ implies that for all $s \in \mathbb{R}$, is a consequence of the cone theory (cf. Proposition 2.4.40 in [Sch98]). Thus, we can assume that $s = 0$. We next reduce the operator (4.2.9) to the case of weight $\gamma = 0$ by multiplying the block matrix from the left and from the right by suitable invertible matrices which preserve the class of Green edge symbols. Namely, when multiplied by an excision function in $\mathbb{R}^q$, the symbol

$$\begin{pmatrix} t^{-\gamma} & 0 \\ 0 & |\eta|^{-\gamma} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} t^{\gamma} & 0 \\ 0 & |\eta|^{\gamma} \end{pmatrix}$$

belongs to $\mathcal{S}^0_G(\Omega' \times \mathbb{R}^q; \mathrm{Hom(W)}; (0, 0, (-1, 0]))$, which is clear from the definition of Green edge symbols. In other words, without loss of generality we may assume that $\gamma = 0$. Since we are dealing with homogeneous operator-valued functions of degree 0 for $\eta \neq 0$, it suffices to invert (4.2.9) on the cosphere bundle $\mathbb{S}^*\Omega' = \{(y,\eta) \in T^*\Omega' : |\eta| = 1\}$ and then to extend the obtained block matrix by homogeneity to all of $T^*\Omega' \setminus \{0\}$. Set $V = H^{0,0}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^l$ and

$$1 + g(y,\eta) = \begin{pmatrix} 1+g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

by 1 being meant the identity operator in various spaces. Then the question is to specify the inverse of $1 + g(y,\eta) \in \mathcal{L}(V)$ in the form $1 + h(y,\eta)$, where

$$\begin{aligned} h(y,\eta) &\in C^\infty_{\mathrm{loc}}(\mathbb{S}^*\Omega', \mathcal{L}(V, \mathcal{S}^0_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^l), \\ h^*(y,\eta) &\in C^\infty_{\mathrm{loc}}(\mathbb{S}^*\Omega', \mathcal{L}(V, \mathcal{S}^0_{\tilde{\mathrm{as}}}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^l) \end{aligned}$$

for suitable asymptotic types $\mathrm{as}, \tilde{\mathrm{as}} \in \mathrm{As}(0, (-l, 0])$. To do this, let us observe, by assumption, that there is an operator $g(y,\eta) \in \mathcal{L}(V)$ such that $(1+g(y,\eta))^{-1} = 1 + h(y,\eta)$. This yields $(1+g(y,\eta))(1+h(y,\eta)) = 1$ whence

$$h(y,\eta) = -g(y,\eta)(1 + h(y,\eta)).$$

Thus, the continuity of $g(y,\eta)\colon V \to \mathcal{S}^0_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^l$ implies the continuity of $h(y,\eta)\colon V \to \mathcal{S}^0_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^l$. For the formal adjoint we can argue in an analogous manner. This gives us easily that both $h(y,\eta)$ and $h^*(y,\eta)$ are C^∞ functions on $\mathbb{S}^*\Omega'$ with values in the continuous operators of the required kind.

□

Until further notice we assume that Ω' is a relatively compact subset of an open set $U \subset \mathbb{R}^q$.

If $a(y,\eta) \in \mathcal{S}^m(U \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{W}, \tilde{\mathrm{W}}); \mathrm{w})$ has an invertible principal edge symbol over $T^*U \setminus \{0\}$, then

$$\sigma^m_{\mathrm{edge}}(a)(y,\eta)\colon\ H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \to H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X)$$

is a family of Fredholm operators parametrised by $(y,\eta) \in T^*U \setminus \{0\}$. The index of this mapping is the same for all $s \in \mathbb{R}$ and, by homogeneity, it depends only on $(y,\eta) \in \mathbb{S}^*U$. Thus, when restricted to the set $\mathbb{S}^*\Omega' \subset\subset T^*U$, the family $\sigma^m_{\mathrm{edge}}(a)(y,\eta)$ gives rise to an *index element* $\mathrm{ind}_{\mathbb{S}^*\Omega'}\, \sigma^{\mathrm{m}}_{\mathrm{edge}}(\mathrm{a}) \in \mathrm{K}(\mathbb{S}^*\Omega')$ (cf. Atiyah [Ati67]). Moreover,

$$\mathrm{ind}_{\mathbb{S}^*\Omega'}\, \sigma^{\mathrm{m}}_{\mathrm{edge}}(\mathrm{a}) = \pi^*[\Omega' \times \mathbb{C}^{\tilde{l}}, \Omega' \times \mathbb{C}^{l}]$$

where $\pi\colon \mathbb{S}^*\Omega' \to \Omega'$ stands for the canonical projection. Hence we deduce that $\mathrm{ind}_{\mathbb{S}^*\Omega'}\, \sigma^{\mathrm{m}}_{\mathrm{edge}}(\mathrm{a}) \in \pi^*\,\mathrm{K}(\Omega')$. [1]

Lemma 4.2.7 *Suppose* $a(y,\eta) \in \mathcal{S}^m(U \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{W}, \tilde{\mathrm{W}}); \mathrm{w})$ *is an elliptic edge symbol,* $w = (\gamma, \gamma - m, (-l, 0])$. *Then, for any cut-off function* $\omega(t)$, *there is a symbol* $p(y,\eta) \in \mathcal{S}^{-m}(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\tilde{\mathrm{W}}, \mathrm{W}); \mathrm{w}^{-1})$ *such that*

$$\begin{aligned} \omega\, {}^b\sigma^{-m}(p) &= \omega\, ({}^b\sigma^m(a))^{-1}, \\ \sigma^{-m}_{\mathrm{edge}}(p) &= (\sigma^m_{\mathrm{edge}}(a))^{-1}. \end{aligned}$$

[1] This condition is necessary to allow the interpretation of $a(y,\eta)$ as a upper left corner of a block matrix $a(y,\eta)$ with invertible principal edge symbol.

It is worth emphasising that the ellipticity of $a(y,\eta)$ merely means the invertibility of the principal symbols (4.1.10). We also mention that $p(y,\eta)$ may depend on ω.

Proof. The ellipticity of $a(y,\eta)$ implies that the upper left corner $a(y,\eta)$ of $a(y,\eta)$ meets the condition of Lemma 4.2.5 over U. Moreover, as the symbol ${}^b\sigma^m(a)$ is now invertible not only for $t=0$ but also over the whole semiaxis $\bar{\mathbb{R}}_+$, the cut-off functions $\varphi(t)$ and $\psi(t)$ with $\psi\varphi=\varphi$ may have arbitrary supports. Thus, there is a $p(y,\eta)\in\mathcal{S}^{-m}(U\times\mathbb{R}^q;w^{-1})$ such that

$$\sigma^{-m}_{\mathrm{edge}}(p)\,\sigma^{m}_{\mathrm{edge}}(a)=1+g \tag{4.2.10}$$

for some $g(y,\eta)\in\sigma^0_{\mathrm{edge}}\,\mathcal{S}^0_G(U\times\mathbb{R}^q;w^{-1}\circ w)$, and analogously for the composition in the reverse order.

This means that

$$\begin{aligned}\operatorname{ind}\sigma^{-\mathrm{m}}_{\mathrm{edge}}(\mathrm{p})(\mathrm{y},\eta) &= -\operatorname{ind}\sigma^{\mathrm{m}}_{\mathrm{edge}}(\mathrm{a})(\mathrm{y},\eta)\\ &= l-\tilde{l}\end{aligned}$$

for all $(y,\eta)\in T^*U\setminus\{0\}$, the last equality being due to the fact that $\sigma^m_{\mathrm{edge}}(a)$ is an isomorphism. The following constructions may be performed over $\mathbb{S}^*\Omega'$ which is a relatively compact set in T^*U. Let us fix $s=s_0$. We find a finite-dimensional subspace $\tilde{\Sigma}$ in $H^{s_0,\gamma}(\bar{\mathbb{R}}_+\times X)$ such that

$$H^{s_0,\gamma}(\bar{\mathbb{R}}_+\times X)=\operatorname{im}\sigma^{-\mathrm{m}}_{\mathrm{edge}}(\mathrm{p})(\mathrm{y},\eta)\oplus\tilde{\Sigma}$$

for all $(y,\eta)\in\mathbb{S}^*\Omega'$. Since $C^\infty_{\mathrm{comp}}(\mathbb{R}_+\times X)$ is dense in $H^{s_0,\gamma}(\bar{\mathbb{R}}_+\times X)$ we may actually choose $\tilde{\Sigma}$ within $C^\infty_{\mathrm{comp}}(\mathbb{R}_+\times X)$. Letting $\tilde{d}$ denote the dimension of $\tilde{\Sigma}$, we pick an isomorphism $c\colon\mathbb{C}^{\tilde{d}}\to\tilde{\Sigma}$. Then, the mapping

$$\left(\sigma^{-m}_{\mathrm{edge}}(p)(y,\eta),\ c\right):\quad\begin{array}{c}H^{s_0-m,\gamma-m}(\bar{\mathbb{R}}_+\times X)\\ \oplus\\ \mathbb{C}^{\tilde{d}}\end{array}\to H^{s_0,\gamma}(\bar{\mathbb{R}}_+\times X) \tag{4.2.11}$$

is surjective for all $(y,\eta)\in\mathbb{S}^*\Omega'$.

We have

$$\begin{aligned}\operatorname{ind}_{\mathbb{S}^*\Omega'}\sigma^{-\mathrm{m}}_{\mathrm{edge}}(\mathrm{p})(\mathrm{y},\eta) &= -\operatorname{ind}_{\mathbb{S}^*\Omega'}\sigma^{\mathrm{m}}_{\mathrm{edge}}(\mathrm{a})(\mathrm{y},\eta)\\ &\in \pi^*K(\Omega'),\end{aligned}$$

and so, by choosing $\tilde{d}$ large enough, we can assert that the kernel of (4.2.11) is a trivial subbundle of finite rank in $\mathbb{S}^*\Omega'\times\left(H^{s_0-m,\gamma-m}(\bar{\mathbb{R}}_+\times X)\oplus\mathbb{C}^{\tilde{d}}\right)$. We denote this bundle by Σ and the fibre dimension of Σ by d. Then, $\tilde{d}-d=\tilde{l}-l$. It is a consequence of the cone theory (cf. Proposition 2.4.40 in [Sch98]) that the fibres of Σ are subspaces of $\mathcal{S}^{\gamma-m}(\bar{\mathbb{R}}_+\times X)$.

Using the conjugate linear pairing

$$H^{s_0-m,\gamma-m}(\bar{\mathbb{R}}_+\times X)\times H^{-s_0+m,-\gamma+m}(\bar{\mathbb{R}}_+\times X)\to\mathbb{C}$$

induced by the scalar product of $H^{0,0}(\bar{\mathbb{R}}_+ \times X)$, we can also form a non-degenerate conjugate linear pairing

$$\left(H^{s_0-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^{\tilde{d}}\right) \times \left(H^{-s_0+m,-\gamma+m}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^{\tilde{d}}\right) \to \mathbb{C} \tag{4.2.12}$$

in an obvious manner by adding the scalar product of $\mathbb{C}^{\tilde{d}}$. This allows us to form a subbundle Σ^* of $\mathbb{S}^*\Omega' \times \left(H^{-s_0+m,-\gamma+m}(\bar{\mathbb{R}}_+ \times X) \oplus \mathbb{C}^{\tilde{d}}\right)$, such that the fibre $\Sigma^*_{(y,\eta)}$ of Σ^* over a point $(y,\eta) \in \mathbb{S}^*\Omega'$ is isomorphic to the corresponding fibre $\Sigma_{(y,\eta)}$ of Σ via (4.2.12). We can choose a basis $(e_1(y,\eta), \ldots, e_d(y,\eta))$ in $\Sigma^*_{(y,\eta)}$, which smoothly depends on $(y,\eta) \in \mathbb{S}^*\Omega'$. Then the pairing

$$\begin{array}{c} H^{s_0-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^{\tilde{d}} \end{array} \ni u \oplus v \mapsto \begin{pmatrix} (u \oplus v,\, e_1(y,\eta)) \\ \cdots \\ (u \oplus v,\, e_d(y,\eta)) \end{pmatrix} \tag{4.2.13}$$

may be interpreted as an (y,η)-dependent family of mappings

$$\begin{array}{c} H^{s_0-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^{\tilde{d}} \end{array} \to \mathbb{C}^d,$$

which induces a bundle isomorphism $\Sigma \overset{\cong}{\to} \mathbb{S}^*\Omega' \times \mathbb{C}^d$. Using the density of $C^\infty_{\text{comp}}(\mathbb{R}_+ \times X)$ in $H^{-s_0+m,-\gamma+m}(\bar{\mathbb{R}}_+ \times X)$ we may choose the vectors $e_j(y,\eta)$ by a suitable approximation as C^∞ functions of $(y,\eta) \in \mathbb{S}^*\Omega'$ with values in $C^\infty_{\text{comp}}(\mathbb{R}_+ \times X) \oplus \mathbb{C}^{\tilde{d}}$, such that the pairing (4.2.13) still induces a bundle isomorphism $\Sigma \overset{\cong}{\to} \mathbb{S}^*\Omega' \times \mathbb{C}^d$. According to the splittings in (4.2.12) we can write each $e_j(y,\eta)$ as a tuple $e_j(y,\eta) = (t_j(y,\eta),\, b_j(y,\eta))$ and form the block matrices

$$t(y,\eta) = \begin{pmatrix} t_1(y,\eta) \\ \cdots \\ t_d(y,\eta) \end{pmatrix}, \qquad b(y,\eta) = \begin{pmatrix} b_1(y,\eta) \\ \cdots \\ b_d(y,\eta) \end{pmatrix}.$$

When combined with (4.2.11), they yield a family of isomorphisms

$$\begin{pmatrix} \sigma^{-m}_{\text{edge}}(p) & c \\ t & b \end{pmatrix}(y,\eta) : \begin{array}{c} H^{s_0-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^{\tilde{d}} \end{array} \to \begin{array}{c} H^{s_0,\gamma}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^d \end{array} \tag{4.2.14}$$

parametrised by $(y,\eta) \in \mathbb{S}^*\Omega'$. We now extend (4.2.14) by homogeneity of degree $-m$ from $\mathbb{S}^*\Omega'$ to a symbol

$$\sigma^{-m}_{\text{edge}}(p^{(1)})(y,\eta) = \begin{pmatrix} \sigma^{-m}_{\text{edge}}(p)(y,\eta) & c(y,\eta) \\ t(y,\eta) & b(y,\eta) \end{pmatrix}$$

defined on all of $T^*\Omega' \setminus \{0\}$ via

$$c(y,\eta) = |\eta|^{-m}\,\kappa_{|\eta|}\,c,$$

$$t(y,\eta) = |\eta|^{-m}\,t\left(y,\frac{\eta}{|\eta|}\right)\kappa_{|\eta|}^{-1}, \qquad b(y,\eta) = |\eta|^{-m}\,b\left(y,\frac{\eta}{|\eta|}\right).$$

By abuse of notation, we use the same letters to designate the extensions of c, p and b.

Having disposed of this preliminary step, we can now turn to constructing a regularising symbol for $a(y,\eta)$. Our task is to construct the inverse of

$$\sigma^m_{\mathrm{edge}}(a) : \begin{array}{c} H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^l \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^{\tilde{l}} \end{array} \tag{4.2.15}$$

for $s = s_0$. This will then be the inverse of (4.2.15) for each $s \in \mathbb{R}$. To this end we recall that there is an integer N such that $l = d + N$ and $\tilde{l} = \tilde{d} + N$. If $N \geq 0$, we choose in the above construction for (4.2.11) the number $\tilde{d}$ equal to $\tilde{l}$ from the very beginning. Then we have automatically $d = l$. For $N < 0$, we enlarge both l and $\tilde{l}$ by $-N$ via replacing (4.2.15) by the direct sum

$$\sigma^m_{\mathrm{edge}}(a)(y,\eta) \oplus |\eta|^m\, I_{-N},$$

I_{-N} being the identity $(-N \times -N)$-matrix. If we construct the inverse of $\sigma^m_{\mathrm{edge}}(a)(y,\eta) \oplus |\eta|^m\, I_{-N}$, we get immediately also the inverse of $\sigma^m_{\mathrm{edge}}(a)(y,\eta)$ by omitting the superfluous entries. In other words, for simplicity of notation we may assume from now on that $d = l$ and $\tilde{d} = \tilde{l}$.

From (4.2.10) we get

$$\sigma^{-m}_{\mathrm{edge}}(p^{(1)})\,\sigma^m_{\mathrm{edge}}(a) = \begin{pmatrix} 1 + g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

where $(g_{ij}(y,\eta)) \in \sigma^0_{\mathrm{edge}}\,\mathcal{S}^0_G(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{W}); \mathrm{w}^{-1} \circ \mathrm{w})$. Since the operators on the left are invertible, we deduce that also

$$\begin{pmatrix} 1 + g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}(y,\eta) : \begin{array}{c} H^{s_0,\gamma}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^l \end{array} \to \begin{array}{c} H^{s_0,\gamma}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^l \end{array}$$

is invertible for all $(y,\eta) \in T^*\Omega' \setminus \{0\}$. Now Lemma 4.2.6 ensures the existence of $(h_{ij}(y,\eta)) \in \sigma^0_{\mathrm{edge}}\,\mathcal{S}^0_G(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{W}); \mathrm{w}^{-1} \circ \mathrm{w})$ with the property that

$$\left(\sigma^m_{\mathrm{edge}}(a)\right)^{-1} = \begin{pmatrix} 1 + h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \sigma^{-m}_{\mathrm{edge}}(p^{(1)})$$

over $T^*\Omega' \setminus \{0\}$.

Finally, we fix an excision function $\chi(\eta)$ on $\mathbb{R}^q$ and set

$$p^{(2)}(y,\eta) = \begin{pmatrix} 1 + \chi(\eta)h_{11}(y,\eta) & \chi(\eta)h_{12}(y,\eta) \\ \chi(\eta)h_{21}(y,\eta) & \chi(\eta)h_{22}(y,\eta) \end{pmatrix} p^{(1)}(y,\eta),$$

where
$$p^{(1)}(y,\eta) = \left(\begin{array}{cc} p(y,\eta) & \chi(\eta)c(y,\eta) \\ \chi(\eta)t(y,\eta) & \chi(\eta)b(y,\eta) \end{array} \right).$$
Then, for any cut-off functions $\varphi(t)$ and $\psi(t)$ satisfying $\psi\varphi = \varphi$, the symbol
$$p(y,\eta) = \left(\begin{array}{cc} \varphi(t) & 0 \\ 0 & 1 \end{array} \right) p^{(2)}(y,\eta) \left(\begin{array}{cc} \psi(t) & 0 \\ 0 & 1 \end{array} \right)$$
belongs to $\mathcal{S}^{-m}(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\tilde{\mathrm{W}}, \mathrm{W}); \mathrm{w}^{-1})$ and fulfils
$$\begin{array}{rcl} \varphi\, {}^b\sigma^{-m}(p) & = & \varphi\, ({}^b\sigma^m(a))^{-1}, \\ \sigma^{-m}_{\mathrm{edge}}(p) & = & (\sigma^m_{\mathrm{edge}}(a))^{-1}, \end{array}$$
as is easy to check. This proves the lemma.

□

Before formulating our next result let us mention that the "algebra" of residual operators on a stretched wedge $\mathcal{W} = \Omega' \times \bar{\mathbb{R}}_+ \times X$,
$$\Psi^{-\infty}(\mathcal{W}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w}),$$
is introduced analogously to that in the case of a compact manifold with fibred boundary, $\mathcal{M}$. A slight change we have to do consists in invoking the 'comp' and 'loc' versions of weighted Sobolev spaces with asymptotics on $\mathcal{W}$ (cf. Definition 3.4.1 in [Sch98]).

Lemma 4.2.8 *Let* $a(y,\eta) \in \mathcal{S}^m(U \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{W}, \tilde{\mathrm{W}}); \mathrm{w})$ *be an elliptic symbol, with* $w = (\gamma, \gamma - m, (-l, 0])$*. Then, for any cut-off function* $\omega(t)$*, there exists an* $a^{(-1)}(y, y', \eta) \in \mathcal{S}^{-m}(\Omega' \times \Omega' \times \mathbb{R}^q; \mathrm{Hom}(\tilde{\mathrm{W}}, \mathrm{W}); \mathrm{w}^{-1})$ *such that*
$$\begin{array}{lcll} \left(\begin{array}{cc} \omega(t) & 0 \\ 0 & 1 \end{array} \right) \mathrm{op}(\mathrm{a}^{(-1)})\, \mathrm{op}(\mathrm{a}) & = & \left(\begin{array}{cc} \omega(t) & 0 \\ 0 & 1 \end{array} \right) & \mathrm{mod} \quad \Psi^{-\infty}(\mathcal{W}; \mathrm{W}; \mathrm{w}^{-1} \circ \mathrm{w}), \\ \left(\begin{array}{cc} \omega(t) & 0 \\ 0 & 1 \end{array} \right) \mathrm{op}(\mathrm{a})\, \mathrm{op}(\mathrm{a}^{(-1)}) & = & \left(\begin{array}{cc} \omega(t) & 0 \\ 0 & 1 \end{array} \right) & \mathrm{mod} \quad \Psi^{-\infty}(\mathcal{W}; \tilde{\mathrm{W}}; \mathrm{w} \circ \mathrm{w}^{-1}). \end{array}$$

For a double symbol $a^{(-1)}(y, y', \eta)$, we denote by $\mathrm{op}(\mathrm{a}^{(-1)})$ the operator $u(y') \mapsto \mathcal{F}^{-1}_{\eta \mapsto y} \mathcal{F}_{y' \mapsto \eta}\, a^{(-1)}(y, y', \eta) u(y')$.

Proof. Consider the symbol $p(y,\eta) \in \mathcal{S}^{-m}(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\tilde{\mathrm{W}}, \mathrm{W}); \mathrm{w}^{-1})$ that we constructed in Lemma 4.2.7. We have
$$p(y,\eta) a(y,\eta) = 1 + r^{(1)}(y,\eta)$$
where $r^{(1)}(y,\eta) \in \mathcal{S}^0(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{W}); \mathrm{w}^{-1} \circ \mathrm{w})$ satisfies both ${}^b\sigma^0(r^{(1)}) = 0$ for $t \in [0, T)$ and $\sigma^0_{\mathrm{edge}}(r^{(1)}) = 0$. From the construction of Lemma 4.2.7 we actually see that for every fixed $T > 0$ there is an appropriate $p(y,\eta)$ with these properties. Letting $\circ_y$ denote the Leibniz product with respect to the y-variables, we get
$$p(y,\eta) \circ_y a(y,\eta) = 1 + r(y,\eta)$$

for a symbol $r(y,\eta) \in \mathcal{S}^0(\Omega' \times \mathbb{R}^q; \mathrm{Hom(W)}; \mathrm{w}^{-1} \circ \mathrm{w})$ satisfying ${}^b\sigma^0(r) = 0$ for $t \in [0,T)$ and $\sigma^0_{\mathrm{edge}}(r) = 0$. Pick a cut-off function $\omega(t)$ which is supported in $[0,T)$ and equal to 1 in $[0,T/2)$. Then the asymptotic sum

$$s(y,\eta) = \sum_{\nu=0}^{\infty} (-1)^\nu \left(\begin{pmatrix} \omega(t) & 0 \\ 0 & 1 \end{pmatrix} r(y,\eta) \right)^{\circ \nu}$$

can be carried out in $\mathcal{S}^0(\Omega' \times \mathbb{R}^q; \mathrm{Hom(W)}; \mathrm{w}^{-1} \circ \mathrm{w})$, where $\circ\,\nu$ stands for the νth power with respect to $\circ_y$. We now denote by $a^{(-1)}(y,y',\eta)$ an element of $\mathcal{S}^{-m}(\Omega' \times \Omega' \times \mathbb{R}^q; \mathrm{Hom}(\tilde{\mathrm{W}}, \mathrm{W}); \mathrm{w}^{-1})$, such that

$$\mathrm{op}(\mathrm{a}^{(-1)}) = \mathrm{op}\,(\mathrm{s} \circ_{\mathrm{y}} \mathrm{p}) \quad \mathrm{mod} \quad \Psi^{-\infty}(\mathcal{W}; \tilde{\mathrm{W}}, \mathrm{W}; \mathrm{w}^{-1})$$

and $\mathrm{op}(\mathrm{a}^{(-1)})$ is properly supported with respect to the y-variables. Then we get

$$\begin{pmatrix} \omega(t) & 0 \\ 0 & 1 \end{pmatrix} \mathrm{op}(\mathrm{a}^{(-1)})\,\mathrm{op}(\mathrm{a}) = \begin{pmatrix} \omega(t) & 0 \\ 0 & 1 \end{pmatrix} \quad \mathrm{mod} \quad \Psi^{-\infty}(\mathcal{W}; \mathrm{W}; \mathrm{w}^{-1} \circ \mathrm{w})$$

for each cut-off function $\omega(t)$ with $\mathrm{supp}\,\omega \subset [0, \mathrm{T}/2)$. Similar arguments apply to the composition in the reverse direction, and the lemma follows.

□

We are now in a position to complete the proof of Theorem 4.2.2. The global invariant interpretations are straightforward, and we will tacitly use this here without further comments.

Proof of Theorem 4.2.2. Let us write our edge problem $\mathcal{A}$ in the form (4.1.12), where

$$\begin{aligned} \mathcal{A}_b &\in \Psi^{\mathrm{m}}(\mathrm{S} \times \bar{\mathbb{R}}_+ \times \mathrm{X}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w}), \\ A_i &\in \Psi^{\mathrm{m}}_{\mathrm{cl}}(\mathring{\mathcal{M}}; \mathrm{V}, \tilde{\mathrm{V}}), \\ \mathcal{S} &\in \Psi^{-\infty}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w}). \end{aligned}$$

For the parametrix construction we may neglect $\mathcal{S}$. The operator A_i is elliptic in the usual sense, hence it possesses a parametrix $P_i \in \Psi^{-\mathrm{m}}_{\mathrm{cl}}(\mathring{\mathcal{M}}; \tilde{\mathrm{V}}, \mathrm{V})$. Let we be moreover granted an operator

$$\mathcal{P}_b \in \Psi^{-\mathrm{m}}(\mathrm{S} \times \bar{\mathbb{R}}_+ \times \mathrm{X}; \tilde{\mathrm{V}}, \mathrm{V}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w}^{-1})$$

with the property that

$$\begin{aligned} \mathcal{P}_b\, \varphi_b\, \mathcal{A}_b\, \psi_b &= \varphi_b\, \mathrm{Id} \quad \mathrm{mod} \quad \Psi^{-\infty}(\mathcal{W}; \mathrm{V}; \mathrm{W}; \mathrm{w}^{-1} \circ \mathrm{w}), \\ \varphi_b\, \mathcal{A}_b\, \psi_b\, \mathcal{P}_b &= \varphi_b\, \mathrm{Id} \quad \mathrm{mod} \quad \Psi^{-\infty}(\mathcal{W}; \tilde{\mathrm{V}}; \tilde{\mathrm{W}}; \mathrm{w} \circ \mathrm{w}^{-1}), \end{aligned}$$

Id being the identity operator in various spaces. We can assume, by modifying P_i and $\mathcal{P}_b$ if necessary, that P_i is properly supported on $\mathring{\mathcal{M}}$ in the standard sense and $\mathcal{P}_b$ is properly supported with respect to the y-variables unless S itself is compact. Then,

$$\mathcal{A} = \varphi_b \mathcal{P}_b \psi_b + \varphi_i \begin{pmatrix} P_i & 0 \\ 0 & 0 \end{pmatrix} \psi_i \tag{4.2.16}$$

is a properly supported parametrix of $\mathcal{A}$. Thus, we are left with the task of constructing $\mathcal{P}_b$.

To this end, let O' be a coordinate neighbourhood on S and $h'\colon O' \to \Omega'$ be a chart, with Ω' an open set in $\mathbb{R}^q$. The restriction of $\mathcal{A}_b$ to $O' \times \bar{\mathbb{R}}_+ \times X$ may be written in the form

$$\mathcal{A}_b = \mathrm{op(a)} \quad \mathrm{mod} \quad \Psi^{-\infty}(\mathcal{W}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w}),$$

where $a(y,\eta) \in \mathcal{S}^m(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\mathrm{V}, \tilde{\mathrm{V}}); \mathrm{Hom}(\mathrm{W}, \tilde{\mathrm{W}}); \mathrm{w})$. We may shrink O' to a relatively compact subset of a larger open set where all the objects are still defined. Thus, we can assume without loss of generality that $\Omega' \subset\subset U$ and our objects are given on U. Then we may apply Lemma 4.2.8 appropriately extended to the case of general vector bundles. This gives as a symbol

$$a^{(-1)}(y,\eta) \in \mathcal{S}^{-m}(\Omega' \times \mathbb{R}^q; \mathrm{Hom}(\tilde{\mathrm{V}}, \mathrm{V}); \mathrm{Hom}(\tilde{\mathrm{W}}, \mathrm{W}); \mathrm{w}^{-1})$$

and $\mathrm{op}(\mathrm{a}^{(-1)})$ is a local representative over $O' \times \bar{\mathbb{R}}_+ \times X$ of a global operator in $\Psi^{-\mathrm{m}}(\mathrm{S} \times \bar{\mathbb{R}}_+ \times \mathrm{X}; \tilde{\mathrm{V}}, \mathrm{V}; \tilde{\mathrm{W}}, \mathrm{W}; \mathrm{w}^{-1})$. In the pull-back of $\mathrm{op}(\mathrm{a}^{(-1)})$ under $h' \otimes 1 \otimes 1 : O' \times \bar{\mathbb{R}}_+ \times X \to \Omega' \times \bar{\mathbb{R}}_+ \times X$ there are also involved local trivialisations of the corresponding vector bundles. We now proceed by a standard localisation procedure. Fix a locally finite open covering of S by coordinate patches (O'_ι). Let (φ_ι) be a partition of unity on S subordinate to this covering and let (ψ_ι) be another system of C^∞ functions on S, such that $\mathrm{supp}\, \psi_\iota \subset \mathrm{O}'_\iota$ and $\psi_\iota \varphi_\iota = \varphi_\iota$ for each ι. For every ι, denote by $\mathcal{P}_{b,\iota}$ the operator over $O'_\iota \times \bar{\mathbb{R}}_+ \times X$ constructed above. Then $\mathcal{P}_b = \sum_\iota \varphi_\iota \mathcal{P}_{b,\iota} \psi_\iota$ is easily verified to have the desired property, and the proof is complete.

□

We emphasise that the parametrix $\mathcal{P}$ guaranteed by Theorem 4.2.2 is modulo smoothing Green operators. These latter operators are known to be of trace class in the appropriate weighted Sobolev spaces on $\mathcal{M}$ provided $\mathcal{M}$ is compact (cf. [FST98a]). Hence it follows that each elliptic edge problem on a compact manifold with fibred boundary has a parametrix modulo trace class operators.

4.3 *Fredholm property*

It is just in the spirit of the classical elliptic theory that the existence of a regulariser modulo compact operators implies the Fredholm property of an edge problem as well as a specific regularity of solutions encoded in the structure of the regulariser.

Theorem 4.3.1 *Suppose* $\mathcal{A} \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ *is an elliptic edge problem on* $\mathcal{M}$, *with* $w = (\gamma, \gamma - m, (-l, 0])$. *Then, the mapping* (4.2.1) *is Fredholm for every* $s \in \mathbb{R}$.

Proof. Indeed, as defined by (4.2.2), the parametrix is a regulariser modulo trace class operators. Therefore, our result is a direct consequence of Theorem 4.2.2.

□

By Theorem 4.1.8, for the proof of the Fredholm property of an edge problem it suffices to have merely a regulariser modulo operators with vanishing principal interior and edge symbols. As mentioned, for the existence of such a regulariser it is necessary and sufficient that the edge problem be elliptic. Actually, for any one $s \in \mathbb{R}$, the Fredholm property itself is necessary for the ellipticity, but we will not develop this point here (cf. Corollary 4.2.2.8 in [Beh95] and [Dor98] for further elements).

Corollary 4.3.2 *If* $\mathcal{A}$ *is an elliptic edge problem on* $\mathcal{M}$, *then the equation* $\mathcal{A}u = f$ *is solvable if and only if* f *is orthogonal to the null-space of the formal adjoint edge problem* $\mathcal{A}^*$.

Proof. Denote by $\mathcal{A}^{\mathrm{adj}}$ the adjoint of $\mathcal{A}$ in the sense of Hilbert spaces, i.e.,

$$\mathcal{A}^{\mathrm{adj}} : \begin{array}{c} H^{s-m,\gamma-m}(\mathcal{M}, \tilde{V}) \\ \oplus \\ H^{s-m}(S, \tilde{W}) \end{array} \to \begin{array}{c} H^{s,\gamma}(\mathcal{M}, V) \\ \oplus \\ H^{s}(S, W) \end{array} .$$

Since $\mathcal{A}$ is a Fredholm operator, it follows that $\operatorname{im} \mathcal{A} = \left(\ker \mathcal{A}^{\mathrm{adj}}\right)^{\perp}$, the right-hind side being the orthogonal complement of the kernel of $\mathcal{A}^{\mathrm{adj}}$. On the other hand, the formal adjoint is related to the true adjoint by the equality

$$\mathcal{A}^* = \star \mathcal{A}^{\mathrm{adj}} \star^{-1},$$

where $\star$ stands for the topological isomorphism

$$\begin{array}{c} H^{s,\gamma}(\mathcal{M}, V) \\ \oplus \\ H^{s}(S, W) \end{array} \overset{\cong}{\to} \begin{array}{c} H^{-s,-\gamma}(\mathcal{M}, V) \\ \oplus \\ H^{-s}(S, W) \end{array}$$

defined via

$$(u, \star v)_{H^{0,0}(\mathcal{M},V) \oplus H^0(S,W)} = (u, v)_{H^{s,\gamma}(\mathcal{M},V) \oplus H^s(S,W)},$$

for $u, v \in H^{s,\gamma}(\mathcal{M}, V) \oplus H^s(S, W)$. Clearly this isomorphism depends on s, γ and the Riemannian metrics on V and W. It may be different in various applications but we write it $\star$ for short. Hence it follows that the orthogonal complement of $\ker \mathcal{A}^{\mathrm{adj}}$ in $H^{s-m,\gamma-m}(\mathcal{M}, \tilde{V}) \oplus H^{s-m}(S, \tilde{W})$ coincides with the annihilator of $\ker \mathcal{A}^*$ in $H^{-s+m,-\gamma+m}(\mathcal{M}, \tilde{V}) \oplus H^{-s+m}(S, \tilde{W})$. This completes the proof.

□

Note that the formal adjoint of an elliptic edge problem is elliptic, which is clear from Theorem 4.1.10.

Though Theorem 4.3.1 deals with elliptic edge problem, it highlights also important mapping properties of the operators in $\Psi^{\mathrm{m}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$ that are elliptic with respect to the mere principal interior symbol. As but one example of this we show the following statement.

Corollary 4.3.3 *Let $A \in \Psi^{\mathrm{m}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{w})$ be complemented to an elliptic edge problem on $\mathcal{M}$, with $w = (\gamma, \gamma - m, (-l, 0])$. Then the restriction of $A\colon H^{s,\gamma}(\mathcal{M}, V) \to H^{s-m,\gamma-m}(\mathcal{M}, \tilde{V})$ to the null-space of the corresponding trace operator has a closed range, for each $s \in \mathbb{R}$.*

We shall have established this corollary if we prove the following abstract result of functional analysis.

Lemma 4.3.4 *Let*

$$\mathcal{A} = \begin{pmatrix} A & P \\ T & B \end{pmatrix} : \begin{matrix} L_1 \\ \oplus \\ L_2 \end{matrix} \to \begin{matrix} \tilde{L}_1 \\ \oplus \\ \tilde{L}_2 \end{matrix}$$

be a Fredholm operator in Fréchet spaces. Then the restriction of the mapping $A\colon L_1 \to \tilde{L}_1$ to the null-space of T has a closed range in $\tilde{L}_1$.

Proof. This is the content of Theorem 3.5.19 in [Sch98], however, for the convenience of the reader we repeat the proof therefrom.

Let us first consider Fréchet spaces L and $\tilde{L}$ and continuous operators

$$\begin{aligned} O &: L \to \tilde{L}, \\ R &: \tilde{L} \to L \end{aligned}$$

satisfying $RO = 1$. Then $(OR)^2 = OR$, and so OR is a projection of $\tilde{L}$ onto the range of O. Hence it follows that $\operatorname{im} \mathrm{O}$ is a closed subspace of $\tilde{L}$ because it coincides with the null-space of the complementary projection $1 - OR$.

We now return to the original operator $\mathcal{A}$ and prove a reduced form of the lemma under the assumption that $\mathcal{A}$ is an isomorphism. Let

$$\mathcal{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} : \begin{matrix} \tilde{L}_1 \\ \oplus \\ \tilde{L}_2 \end{matrix} \to \begin{matrix} L_1 \\ \oplus \\ L_2 \end{matrix}$$

be the inverse of $\mathcal{A}$. Setting

$$O = \begin{pmatrix} A \\ T \end{pmatrix}, \quad R = \begin{pmatrix} P_{11} & P_{12} \end{pmatrix}, \quad L = L_1, \quad \tilde{L} = \begin{matrix} \tilde{L}_1 \\ \oplus \\ \tilde{L}_2 \end{matrix},$$

we get $RO = 1$. From what has already been proved it follows that the range of O is a closed subspace of $\tilde{L}$. Letting π denote the canonical projection of $\tilde{L}$ onto $\tilde{L}_2$, we deduce that the range of $A \colon \ker T \to \tilde{L}_1$ is naturally identified with $\mathrm{im}\, \mathrm{O} \cap \ker \pi$. Hence our assertion follows.

We now turn to the general case. As $\mathcal{A}$ is Fredholm, we can complete it to a surjective operator

$$\begin{pmatrix} A & P & O_{13} \\ T & B & O_{23} \end{pmatrix} : \begin{array}{c} L_1 \\ \oplus \\ L_2 \\ \oplus \\ \mathbb{C}^l \end{array} \to \begin{array}{c} \tilde{L}_1 \\ \oplus \\ \tilde{L}_2 \end{array},$$

where l is the codimension of the range of $\mathcal{A}$, and then to an isomorphism

$$\begin{pmatrix} A & P & O_{13} \\ T & B & O_{23} \\ O_{31} & O_{32} & O_{33} \end{pmatrix} : \begin{array}{c} L_1 \\ \oplus \\ L_2 \\ \oplus \\ \mathbb{C}^l \end{array} \to \begin{array}{c} \tilde{L}_1 \\ \oplus \\ \tilde{L}_2 \\ \oplus \\ \mathbb{C}^{\tilde{l}} \end{array},$$

where $\tilde{l}$ is the dimension of the null-space of $\mathcal{A}$. To reduce this to the case of (2×2)-block matrices, we set

$$\tilde{P} = \begin{pmatrix} P & O_{13} \end{pmatrix}, \quad \tilde{T} = \begin{pmatrix} T \\ O_{31} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B & O_{23} \\ O_{32} & O_{33} \end{pmatrix}$$

thus arriving at an isomorphism

$$\begin{pmatrix} A & \tilde{P} \\ \tilde{T} & \tilde{B} \end{pmatrix} : \begin{array}{c} L_1 \\ \oplus \\ L_2 \oplus \mathbb{C}^l \end{array} \to \begin{array}{c} \tilde{L}_1 \\ \oplus \\ \tilde{L}_2 \oplus \mathbb{C}^{\tilde{l}} \end{array}.$$

By the above, the restriction of $A \colon L_1 \to \tilde{L}_1$ to the null-space of the operator $\tilde{T}$ has a closed range in $\tilde{L}_1$. This is still true for the restriction of A to the null-space of T because $\tilde{T}$ and T differ by an operator of finite rank. The proof is complete.

□

Our next result substitutes the classical *elliptic regularity* for the analysis on manifolds with edges.

Theorem 4.3.5 *Suppose* $\mathcal{A} \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ *is an elliptic edge problem on* $\mathcal{M}$, *where* $w = (\gamma, \gamma - m, (-l, 0])$. *Then,*

$$u \in \begin{array}{c} H^{-\infty,\gamma}(\mathcal{M}, V) \\ \oplus \\ H^{-\infty}(S, W) \end{array}, \quad \mathcal{A}u \in \begin{array}{c} H^{s-m,\gamma-m}(\mathcal{M}, \tilde{V}) \\ \oplus \\ H^{s-m}(S, \tilde{W}) \end{array} \quad \textit{implies} \quad u \in \begin{array}{c} H^{s,\gamma}(\mathcal{M}, V) \\ \oplus \\ H^{s}(S, W) \end{array}.$$

Proof. Indeed, let $\mathcal{P} \in \Psi^{-\mathrm{m}}(\mathcal{M};\tilde{\mathrm{V}},\mathrm{V};\tilde{\mathrm{W}},\mathrm{W};\mathrm{w}^{-1})$ be a parametrix of $\mathcal{A}$. Then,

$$u = (1-\mathcal{P}\mathcal{A})u + \mathcal{P}\mathcal{A}u$$

for each $u \in H^{-\infty,\gamma}(\mathcal{M},V) \oplus H^{-\infty}(S,W)$. From Theorem 4.1.7 we conclude immediately that $(1-\mathcal{P}\mathcal{A})u \in H^{\infty,\gamma}(\mathcal{M},V) \oplus H^{\infty}(S,W)$. On the other hand, if $\mathcal{A}u \in H^{s-m,\gamma-m}(\mathcal{M},V) \oplus H^{s-m}(S,W)$ for some $s \in \mathbb{R}$, then $\mathcal{P}\mathcal{A}u \in H^{s,\gamma}(\mathcal{M},V) \oplus H^{s}(S,W)$, and the lemma follows.

□

The proof above gives more, namely the corresponding statement for weighted Sobolev spaces with asymptotics. Since asymptotics are of great importance in the analysis on manifolds with singularities, we formulate this result separately.

Corollary 4.3.6 *Under the assumptions of Theorem 4.3.5, for each asymptotic type* $\tilde{\mathrm{as}} \in \mathrm{As}\,(\gamma-\mathrm{m},(-\mathrm{l},0])$, $0 < l \le \infty$, *there is an asymptotic type* $\mathrm{as} \in \mathrm{As}\,(\gamma,(-\mathrm{l},0])$ *such that*

$$u \in \begin{array}{c} H^{-\infty,\gamma}(\mathcal{M},V) \\ \oplus \\ H^{-\infty}(S,W) \end{array}, \quad \mathcal{A}u \in \begin{array}{c} H^{s-m,\gamma-m}_{\tilde{\mathrm{as}}}(\mathcal{M},\tilde{V}) \\ \oplus \\ H^{s-m}(S,\tilde{W}) \end{array} \quad \textit{implies} \quad u \in \begin{array}{c} H^{s,\gamma}_{\mathrm{as}}(\mathcal{M},V) \\ \oplus \\ H^{s}(S,W) \end{array}.$$

Proof. This follows by the same method as in the proof of Theorem 4.3.5, if we make use of the part of Theorem 4.1.7 concerning spaces with asymptotics. The important point to note here is the form of the remainder $1-\mathcal{P}\mathcal{A}$ which is a smoothing Green operator with asymptotics. It contributes even to asymptotics of solutions of the homogeneous edge problem $\mathcal{A}u = 0$.

□

Let us have a look at how the asymptotics of solutions are generated in the case of elliptic edge problems $\mathcal{A}$ on a stretched wedge $\mathcal{W} = \Omega' \times \bar{\mathbb{R}}_+ \times X$ where Ω' is an open subset of $\mathbb{R}^q$. We restrict our attention to those $\mathcal{A}$ which bear a typical differential operator in the upper left corner, that is $A = t^{-m}\sum_{|\beta|+j\le m} A_{\beta,j}(tD_y)^{\beta}(tD_t)^{j}$ where $A_{\beta,j}(y,t)$ is a C^{∞} function on $\Omega' \times \bar{\mathbb{R}}_+$ with values in $\mathrm{Diff}^{\mathrm{m}-|\beta|-\mathrm{j}}(\mathrm{X})$). The main contribution comes from the conormal symbol of $\sigma^{m}_{\mathrm{edge}}(A)$, which is

$$\sigma_{\mathcal{M}}\left(\sigma^{m}_{\mathrm{edge}}(A)\right)(z) = \sum_{j=0}^{m} A_{0,j}(y,0)\,z^{j}.$$

This is a polynomial function of $z \in \mathbb{C}$ taking its values in elliptic differential operators on X. These are thought of as acting in the usual Sobolev spaces on X, and the non-bijectivity points of $\sum_{j=0}^{m} A_{0,j}(y,0)\,z^{j}$ in the complex plane are responsible for the exponents of the asymptotics. The set of all non-bijectivity points along with their multiplicities is known as the *spectrum* of the operator pencil $\sum_{j=0}^{m} A_{0,j}(y,0)\,z^{j}$. The simplest possible case is when the spectrum is independent of $y \in \Omega'$. This case can be handled in the framework of *discrete* asymptotic types as $= (\sigma, \Sigma)$ where $\sigma = (p_\mu)_{\mu=1,\dots,M}$ is a discrete set contained in the strip $-\gamma - l \le \Im z < -\gamma$, and Σ is a finite-dimensional space of analytic functionals of the form $u \mapsto \sum_{\mu=1}^{M} \sum_{j=0}^{j_\mu} f_{\mu j} (\partial/\partial z)^j u(p_\mu)$, with $f_{\mu j}(x)$ a C^∞ function on X (cf. Section 2.3). As defined above, the space $\mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ consists of the functions

$$u(t,x) = \omega(t) \sum_{\mu=1}^{M} \sum_{j=0}^{j_\mu} t^{ip_\mu} (i \log t)^j f_{\mu j}(x)$$

where $\omega(t)$ is a cut-off function on the semiaxis. A trivial verification shows that

$$\begin{aligned} \kappa_\theta u\,(t,x) &= \theta^{\frac{1+n}{2}} u(\theta t, x) \\ &= \omega(\theta t) \sum_{\mu=1}^{M} \sum_{j=0}^{j_\mu} t^{ip_\mu} (i \log t)^\iota f'_{\mu\iota}(x) \end{aligned}$$

with

$$f'_{\mu\iota}(x) = \theta^{\frac{1+n}{2} + ip_\mu} \sum_{j=\iota}^{j_\mu} \frac{j!}{\iota!(j-\iota)!} (i \log \theta)^{j-\iota} f_{\mu j}(x)$$

belonging to the same coefficient space as $f_{\mu j}$, for all $\theta > 0$. Hence it follows that $\kappa_\theta u$ leaves $\mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ for $\theta \ne 1$, although it behaves like an asymptotic of $\mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ up to a cut-off function. However, we may introduce the space $H^s_{\mathrm{loc}}(\Omega', \pi^* \mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X))$ by the following trick. Let V be a trivial Banach bundle over $\mathbb{R}^q$ and let $(\lambda(\eta))_{\eta \in \mathbb{R}^q}$ be a family of isomorphisms in the fibres of $\pi^* V$, as in Section 2.1. Then, the mapping

$$i: \ u \mapsto \mathcal{F}^{-1}_{\eta \mapsto y} \lambda(\eta)\, \mathcal{F}_{y \mapsto \eta} u$$

is easily seen to be an isometrical isomorphism $H^s(\mathbb{R}^q, \pi^* V) \overset{\cong}{\to} H^s(\mathbb{R}^q, V)$, the latter space being defined with respect to the identity action in the fibres of $\pi^* V$. This allows one to define the spaces $H^s(\mathbb{R}^q, \pi^* \Sigma)$ also for the bundles $\Sigma \subset V$ that are no longer invariant under $(\lambda(\eta))_{\eta \in \mathbb{R}^q}$. Suffices it to set $H^s(\mathbb{R}^q, \pi^* \Sigma) = i^{-1} H^s(\mathbb{R}^q, \Sigma)$. A typical example is $V = H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)$, with $\lambda(\eta) = \kappa^{-1}_{\langle \eta \rangle}$ and $\Sigma = \mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$. Then the space $H^s(\mathbb{R}^q, \pi^* \mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X))$ can be characterised as the linear span of all distributions of the form

$$\begin{aligned} &\mathcal{F}^{-1}_{\eta \mapsto y} \langle \eta \rangle^{\frac{1+n}{2}} \omega(t\langle \eta \rangle)\, (t\langle \eta \rangle)^{ip_\mu} (\log(t\langle \eta \rangle))^j f_{\mu j}(x)\, \mathcal{F}_{y \mapsto \eta} u_{\mu j} \\ &\quad = \sum_{|\alpha| = j} \frac{j!}{\alpha!} \left(\mathcal{F}^{-1}_{\eta \mapsto y} \omega(t\langle \eta \rangle) \langle \eta \rangle^{\frac{1+n}{2} + ip_\mu} (\log \langle \eta \rangle)^{\alpha_2} \mathcal{F}_{y \mapsto \eta} u_{\mu j} \right) t^{ip_\mu} (\log t)^{\alpha_1} f_{\mu j}(x) \end{aligned} \tag{4.3.1}$$

with arbitrary $u_{\mu j}(y) \in H^s(\mathbb{R}^q)$. Localising in $y \in \Omega'$ yields the spaces $H^s_{\mathrm{loc}}(\Omega', \pi^* \mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X))$, for $s \in \mathbb{R}$. Furthermore, both $H^{s,\gamma+l-0}(\bar{\mathbb{R}}_+ \times X)$

and $H^{s,\gamma}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X)$ are projective limits of Hilbert spaces which are invariant under (κ_θ). Hence we obtain the spaces $H^s_{\mathrm{loc}}(\Omega', \pi^* H^{s,\gamma+l-0}(\bar{\mathbb{R}}_+ \times X))$ and $H^s_{\mathrm{loc}}(\Omega', \pi^* H^{s,\gamma}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X))$ in a familiar way. The elements of the former may be interpreted as distributions of edge *flatness* $l-0$ with respect to the weight γ, whereas the elements of the latter are distributions with edge asymptotics of type 'as'. We next observe that if $V = V_1 + V_2$ in the sense of non-direct sums, then $H^s(\mathbb{R}^q, \pi^* V) = H^s(\mathbb{R}^q, \pi^* V_1) + H^s(\mathbb{R}^q, \pi^* V_2)$ for all $s \in \mathbb{R}$ (cf. Proposition 1.3.35 in [Sch98]). Combining this with (2.3.1) we deduce at once that

$$H^{s,\gamma}_{\mathrm{as,loc}}(\mathcal{W}) = H^s_{\mathrm{loc}}(\Omega', \pi^* \mathcal{A}_{\mathrm{as}}(\bar{\mathbb{R}}_+ \times X))$$

modulo distributions of edge flatness $l-0$. In this sense the edge asymptotics of asymptotic type as are of the form (4.3.1). It is worth pointing out that the smoothness of the coefficients of edge asymptotics (4.3.1) in y depends on $\Im p_\mu$. From (4.3.1) it is clear that the investigation of asymptotics becomes much more subtle when we drop the condition that the spectrum of $\sum_{j=0}^m A_{0,j}(y,0)\, z^j$ is independent of y. In general the non-bijectivity points of this operator pencil may vary under varying $y \in \Omega'$ and their multiplicities do so. This requires a generalisation of the concept of asymptotics in terms of analytic functionals in the plane, just as in Definition 2.3.1. It consists in replacing meromorphic functions on the complex plane by Mellin potentials of analytic functionals with values in $C^\infty(X)$, i.e., by $\langle f(x), t^{iz} \rangle$.

4.4 *Reductions of orders*

We begin by recalling a very popular now construction of an order reducing family on a C^∞ compact closed manifold X via parameter-dependent pseudodifferential operators. Namely, let $A(\lambda) \in \Psi^{\mathrm{m}}_{\mathrm{cl}}(\mathrm{X}; \mathrm{V}; \Lambda)$ be an elliptic pseudodifferential operator of order m with a parameter $\lambda \in \Lambda$, where V is a vector bundle over X and $\Lambda = \mathbb{R}^d$ for some d. Such an operator is known to possess a parameter-dependent parametrix, i.e., there exists a $P(\lambda) \in \Psi^{-\mathrm{m}}_{\mathrm{cl}}(\mathrm{X}; \mathrm{V}; \Lambda)$ with the property that

$$\begin{aligned} P(\lambda)A(\lambda) &= 1 - S_0(\lambda), \\ A(\lambda)P(\lambda) &= 1 - S_1(\lambda) \end{aligned}$$

with $S_0, S_1 \in \Psi^{-\infty}(\mathrm{X}; \mathrm{V}; \Lambda)$. Since $\Psi^{-\infty}(\mathrm{X}; \mathrm{V}; \Lambda) = \mathcal{S}(\Lambda, \Psi^{-\infty}(\mathrm{X}; \mathrm{V}))$, the right-hand side being the space of rapidly decreasing functions on Λ with values in $\Psi^{-\infty}(\mathrm{X}; \mathrm{V})$, it follows that the norm of the operator $S_0(\lambda)$ in each Sobolev space $H^s(V)$ is infinitesimal when $|\lambda| \to \infty$. Pick any one $s = s_0$. Then there is a number $R = R(s_0)$ such that the operator $1 - S_0(\lambda)$ is invertible in $H^{s_0}(V)$ for all $|\lambda| > R$. As the null-space of $1 - S_0(\lambda)$ in $H^s(V)$ consists of C^∞ sections and the same is true for the formal adjoint $1 - S_0^*(\lambda)$, we actually deduce that $1 - S_0(\lambda)$ is invertible in every space

$H^s(V)$ for $|\lambda| > R$, R being independent of s. Moreover, the inverse is given by the Neumann series

$$\begin{aligned}(1 - S_0(\lambda))^{-1} &= \sum_{j=0}^{\infty}(S_0(\lambda))^j \\ &= 1 + \tilde{S}_0(\lambda),\end{aligned}$$

for $|\lambda| > R$. In fact, we get $\tilde{S}_0(\lambda) \in \Psi^{-\infty}(\mathrm{X};\mathrm{V};\Lambda)$, which is clear from $\tilde{S}_0(\lambda) = S_0(\lambda)(1 - \tilde{S}_0(\lambda))$. Setting $A_l^{-1}(\lambda) = (1 + \tilde{S}_0(\lambda))P(\lambda)$, we see that $A_l^{-1}(\lambda) \in \Psi_{\mathrm{cl}}^{-\mathrm{m}}(\mathrm{X};\mathrm{V};\Lambda)$ and $A_l^{-1}(\lambda)A(\lambda) = 1$ for all $|\lambda| > R$. In the same manner we can conclude that $A_r^{-1}(\lambda) = P(\lambda)(1 + S_1(\lambda))^{-1}$ is a right inverse of $A(\lambda)$ for each $|\lambda| > R$, whence $A_l^{-1}(\lambda) = A_r^{-1}(\lambda)$ for $|\lambda| > R$. We can assume, be increasing the number of parameters is necessary, that $d \geq 2$. Write $\lambda = (\lambda', \lambda_d)$, where λ' varies over $\Lambda' = \mathbb{R}^{d-1}$. Then, $R_V^m(\lambda') = A(\lambda', R+1)$ belongs to $\Psi_{\mathrm{cl}}^{\mathrm{m}}(\mathrm{X};\mathrm{V};\Lambda')$ and extends to an isomorphism $H^s(V) \overset{\cong}{\to} H^{s-m}(V)$ for each $s \in \mathbb{R}$ and $\lambda' \in \Lambda'$. In this way we obtain what is referred to as an *order-reducing family* on X. This approach still works in the calculus of pseudodifferential operators on a manifold with edges.

The parameter-dependent calculus on a manifold with fibred boundary $\mathcal{M}$ is analogous to that of Chapter 3, while it has some new features. The interior symbols are parameter-dependent in the usual sense, with a parameter λ varying over the space of parameters $\Lambda = \mathbb{R}^d$. Near the boundary we restrict our discussion to typical interior symbols with parameter, i.e., those of the form $t^{-m}\,\tilde{a}(y,t,x;t\eta,t\lambda,t\tau,\xi)$ in the splitting of coordinates (y,t,x) over a stretched wedge $\mathcal{W} = \Omega' \times \bar{\mathbb{R}}_+ \times X$, with $\tilde{a} \in \mathcal{S}_{\mathrm{cl}}^m(\mathcal{W} \times \mathbb{R}^{q+d+1+n})$ smooth up to $t = 0$. For this reason, close to the boundary we replace the covariable $\eta \in \mathbb{R}^q$ in the operator-valued symbols along Ω' by (η, λ), where $\lambda \in \Lambda$. In this manner, given weight data $w = (\gamma, \gamma - m, (-l, 0])$ and smooth vector bundles V, $\tilde{V}$ over $\mathcal{M}$ and W, $\tilde{W}$ over S, we obtain symbol spaces

$$\mathcal{S}^m(\Omega' \times \mathbb{R}^q \times \Lambda; \mathrm{Hom}(\mathrm{V},\tilde{\mathrm{V}}); \mathrm{Hom}(\mathrm{W},\tilde{\mathrm{W}}); \mathrm{w})$$

as well as operator algebras

$$\Psi^{\mathrm{m}}(\mathcal{W}; \mathrm{V},\tilde{\mathrm{V}}; \mathrm{W},\tilde{\mathrm{W}}; \mathrm{w}; \Lambda),$$
$$\Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V},\tilde{\mathrm{V}}; \mathrm{W},\tilde{\mathrm{W}}; \mathrm{w}; \Lambda)$$

and those with subscripts $M + G$ and G. The details are left to the reader (cf. for instance Behm [Beh95]). The parameter-dependent theory on a smooth manifold suggests a proper choice of smoothing operators in the parameter-dependent calculus on $\mathcal{M}$. In fact, these are the elements of

$$\mathcal{S}(\Lambda, \Psi^{-\infty}(\mathcal{M}; \mathrm{V},\tilde{\mathrm{V}}; \mathrm{W},\tilde{\mathrm{W}}; \mathrm{w})) = \mathcal{S}(\Lambda) \otimes_\pi \Psi^{-\infty}(\mathcal{M}; \mathrm{V},\tilde{\mathrm{V}}; \mathrm{W},\tilde{\mathrm{W}}; \mathrm{w}),$$

the algebras $\Psi^{-\infty}(\mathcal{M}; \mathrm{V},\tilde{\mathrm{V}}; \mathrm{W},\tilde{\mathrm{W}}; \mathrm{w})$ being topologised via their symbol spaces.

Note that the edge problems with parameter may be handled in much the same way as those without parameter. Thus, Theorems 4.1.7–4.1.10 remain valid in this more general context. We next sharpen Theorem 4.1.7 and show explicit norm estimates of parameter-dependent operators in weighted Sobolev spaces on $\mathcal{M}$.

Theorem 4.4.1 *Let $\mathcal{M}$ be a compact manifold with fibred boundary and $\mathcal{A}(\lambda) \in \Psi^{\mathrm{m}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{W},\tilde{\mathrm{W}};\mathrm{w};\Lambda)$, where $w = (\gamma, \gamma - m, (-l, 0])$. Then, for each $\mu \geq m$, the norm of*

$$\mathcal{A}(\lambda) : \begin{array}{c} H^{s,\gamma}(M,V) \\ \oplus \\ H^{s}(S,W) \end{array} \to \begin{array}{c} H^{s-\mu,\gamma-m}(M,\tilde{V}) \\ \oplus \\ H^{s-\mu}(S,\tilde{W}) \end{array}$$

is estimated by

$$\|\mathcal{A}(\lambda)\| \leq \begin{cases} c\, \langle\lambda\rangle^{m+R+\tilde{R}} & \text{if} \quad \mu + R + \tilde{R} \geq 0; \\ c\, \langle\lambda\rangle^{m-\mu} & \text{if} \quad \mu + R + \tilde{R} \leq 0, \end{cases} \tag{4.4.1}$$

with c a constant independent of λ and s, γ, μ, where

$$\begin{array}{rcl} R &=& \max\left(|s|, |\gamma + \frac{1+n}{2}|\right), \\ \tilde{R} &=& \max\left(|s-\mu|, |\gamma - m + \frac{1+n}{2}|\right). \end{array}$$

Proof. Using a familiar localisation argument and the norm estimates of parameter-dependent pseudodifferential operators in the usual Sobolev spaces on a compact closed manifold (cf. [Shu87]), we reduce Theorem 4.4.1 easily to the case of operators $A(\lambda) = \mathrm{op}(\mathrm{a}(\mathrm{y},\eta,\lambda))$ over $\mathbb{R}^q$ with operator-valued symbols $a(y,\eta,\lambda) \in \mathcal{S}^m(T^*\mathbb{R}^q \times \Lambda, \mathcal{L}(V,\tilde{V}))$. Moreover, we can assume that $a(y,\eta,\lambda)$ is independent of y for $|y|$ large enough.

Let $(\kappa_\theta)_{\theta>0}$ and $(\tilde{\kappa}_\theta)_{\theta>0}$ stand for the group actions in the fibres of V and $\tilde{V}$, respectively. It is a general property of group actions that

$$\begin{array}{rcl} \|\kappa_\theta\|_{\mathcal{L}(V)} &\leq& c\, \max\left(\theta, \theta^{-1}\right)^{R}, \\ \|\tilde{\kappa}_\theta\|_{\mathcal{L}(\tilde{V})} &\leq& \tilde{c}\, \max\left(\theta, \theta^{-1}\right)^{\tilde{R}} \end{array}$$

for all $\theta > 0$, the constants R, $\tilde{R}$ and c, $\tilde{c}$ being independent of θ. In our application the spaces V and $\tilde{V}$ are

$$V = \begin{array}{c} H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^{l} \end{array}, \qquad \tilde{V} = \begin{array}{c} H^{s-\mu,\gamma-m}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^{\tilde{l}} \end{array},$$

and so the exponents R and $\tilde{R}$ are easily seen to coincide with those given by the theorem.

We first consider the case where $a(y,\eta,\lambda)$ is independent of $y \in \mathbb{R}^q$. An easy computation shows that

$$\begin{array}{l} \|\mathrm{op}(\mathrm{a}(\eta,\lambda))\mathrm{u}\|^2_{\mathrm{H}^{s-\mu}(\mathbb{R}^q,\pi^*\tilde{\mathrm{V}})} \leq \displaystyle\int_{\mathbb{R}^q} \langle\eta\rangle^{2(s-\mu)}\, \|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\tilde{\kappa}_{\langle\eta,\lambda\rangle}\|^2_{\mathcal{L}(\tilde{V})} \\ \quad \times\, \|\tilde{\kappa}^{-1}_{\langle\eta,\lambda\rangle} a(\eta,\lambda)\kappa_{\langle\eta,\lambda\rangle}\|^2_{\mathcal{L}(V,\tilde{V})}\, \|\kappa^{-1}_{\langle\eta,\lambda\rangle}\kappa_{\langle\eta\rangle}\|^2_{\mathcal{L}(V)}\, \|\kappa^{-1}_{\langle\eta\rangle}\mathcal{F}_{y\mapsto\eta}u\|^2_V\, d\eta \end{array}$$

for all $u \in H^s(\mathbb{R}^q, \pi^* V)$. The norm estimates for the group actions imply immediately that

$$\begin{aligned} \|\kappa^{-1}_{\langle \eta,\lambda \rangle} \kappa_{\langle \eta \rangle}\|_{\mathcal{L}(V)} &\leq c \left(\frac{\langle \eta,\lambda \rangle}{\langle \eta \rangle}\right)^R, \\ \|\tilde{\kappa}^{-1}_{\langle \eta \rangle} \tilde{\kappa}_{\langle \eta,\lambda \rangle}\|_{\mathcal{L}(\tilde{V})} &\leq \tilde{c} \left(\frac{\langle \eta,\lambda \rangle}{\langle \eta \rangle}\right)^{\tilde{R}}, \end{aligned}$$

whereas the symbol estimates give

$$\|\tilde{\kappa}^{-1}_{\langle \eta,\lambda \rangle} a(\eta,\lambda) \kappa_{\langle \eta,\lambda \rangle}\|_{\mathcal{L}(V,\tilde{V})} \leq C \, \langle \eta, \lambda \rangle^m$$

for all $\eta \in \mathbb{R}^q$ and $\lambda \in \Lambda$, where C depends only on a. Combining all these estimates we get

$$\|\mathrm{op}(\mathrm{a}(\eta,\lambda))\|_{\mathcal{L}(\mathrm{H}^s(\mathbb{R}^q,\pi^*\mathrm{V}),\mathrm{H}^{s-\mu}(\mathbb{R}^q,\pi^*\tilde{\mathrm{V}})} \leq \mathrm{const} \max_{\eta \in \mathbb{R}^q} \left(\frac{\langle \eta,\lambda \rangle}{\langle \eta \rangle}\right)^{\mathrm{R}+\tilde{\mathrm{R}}} \frac{\langle \eta,\lambda \rangle^{\mathrm{m}}}{\langle \eta \rangle^{\mu}},$$

the constant being independent of λ. This implies (4.4.1) when combined with the elementary inequality

$$\frac{\langle \eta,\lambda \rangle^m}{\langle \eta \rangle^{\mu}} \leq \begin{cases} c\,\lambda^m & \text{if} \quad \mu \geq 0; \\ c\,\lambda^{m-\mu} & \text{if} \quad \mu \leq 0, \end{cases}$$

which is a consequence of Peetre's inequality (cf. Lemma 2.1.1).

The general case follows by a familiar trick with topological tensor products (cf. Proposition 3.3.2).

□

The crucial fact is that, for sufficiently small negative m, the norm tends to zero as $|\lambda| \to \infty$.

Definition 4.4.2 *An edge problem* $\mathcal{A}(\lambda) \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w}; \Lambda)$ *is said to be elliptic with parameter if*

1)

$${}^b\sigma^m(\mathcal{A})(p,c,\lambda): \ V_p \to \tilde{V}_p$$

is an isomorphism for each $(p,c) \in {}^bT^*\mathcal{M}$ *and* $\lambda \in \Lambda$ *with* $(c,\lambda) \neq 0$*;*

2)

$$\sigma^m_{\mathrm{edge}}(\mathcal{A})(y,\eta,\lambda): \begin{array}{c} H^{s,\gamma}(F^{-1}(y)) \otimes (V_S)_y \\ \oplus \\ W_y \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(F^{-1}(y)) \otimes (\tilde{V}_S)_y \\ \oplus \\ \tilde{W}_y \end{array}$$

is an isomorphism for each $(y,\eta) \in T^*S$ *and* $\lambda \in \Lambda$ *with* $(\eta,\lambda) \neq 0$*, and for any one* $s \in \mathbb{R}$*.*

Since compositions and formal adjoints of edge problems with parameter are available within the same parameter-dependent calculus on $\mathcal{M}$, the method of the proof of Theorem 4.2.2 carries over to elliptic edge problems with parameter.

Theorem 4.4.3 *Let* $\mathcal{A}(\lambda) \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w}; \Lambda)$ *be an elliptic edge problem with parameter on* $\mathcal{M}$*. Then* $\mathcal{A}(\lambda)$ *possesses a parameter-dependent parametrix, i.e., there is a* $\mathcal{P}(\lambda) \in \Psi^{-\mathrm{m}}(\mathcal{M}; \tilde{\mathrm{V}}; \mathrm{V}; \tilde{\mathrm{W}}, \mathrm{W}; \mathrm{w}^{-1}; \Lambda)$ *such that*

$$\begin{aligned} \mathcal{P}(\lambda)\mathcal{A}(\lambda) - 1 &\in \Psi^{-\infty}(\mathcal{M}; \mathrm{V}; \mathrm{W}; \mathrm{w}^{-1} \circ \mathrm{w}; \Lambda), \\ \mathcal{A}(\lambda)\mathcal{P}(\lambda) - 1 &\in \Psi^{-\infty}(\mathcal{M}; \tilde{\mathrm{V}}; \tilde{\mathrm{W}}; \mathrm{w} \circ \mathrm{w}^{-1}; \Lambda). \end{aligned}$$

Proof. This follows by analysis similar to that in Section 4.3. For details see [Beh95, 4.1.1] and [Dor98].

□

For reductions of orders we have to construct a parameter-dependent elliptic element in the algebra $\Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}; \mathrm{W}; \mathrm{w}; \Lambda)$ and to prove that it is invertible in the calculus for $\lambda \in \Lambda$ large enough. In fact, we show that the reductions of orders on $\mathcal{M}$ are available in the "diagonal" form

$$\mathcal{R}^m_{V,W}(\lambda) = \begin{pmatrix} R^m_V(\lambda) & 0 \\ 0 & R^m_W(\lambda) \end{pmatrix}, \tag{4.4.2}$$

where $R^m_V(\lambda) \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}; \mathrm{w}; \Lambda)$ is parameter-dependent elliptic on $\mathcal{M}$ without potential and trace conditions, and $R^m_W(\lambda) \in \Psi^{\mathrm{m}}(\mathrm{S}; \mathrm{W}; \Lambda)$ is parameter-dependent elliptic on S. As is described above, we can take as $R^m_W(\lambda)$ any one order-reducing family for the bundle W over S, for S is a C^∞ compact closed manifold. It remains to construct an $R^m_V(\lambda)$ with the desired properties.

Theorem 4.4.4 *For each weight data* $w = (\gamma, \gamma - m, (-l, 0])$ *and vector bundle* V *over* $\mathcal{M}$*, there exists a parameter-dependent elliptic operator* $A(\lambda) \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}; \mathrm{w}; \Lambda)$ *on* $\mathcal{M}$*.*

Proof. We restrict our attention to the case of a trivial vector bundle V of rank 1, i.e., $V = \mathcal{M} \times \mathbb{C}$, thus omitting V in the notation. The same proof still goes for general V.

Pick a collar neighbourhood $O \cong S \times \bar{\mathbb{R}}_+ \times X$ of $\partial\mathcal{M}$ in $\mathcal{M}$. We construct $A(\lambda)$ in the form

$$A(\lambda) = \varphi_b A_b(\lambda) \psi_b + \varphi_i A_i(\lambda) \psi_i,$$

where

$$\begin{aligned} A_b(\lambda) &\in \Psi^{\mathrm{m}}(\mathrm{O}; \mathrm{w}; \Lambda), \\ A_i(\lambda) &\in \Psi^{\mathrm{m}}_{\mathrm{cl}}(\mathring{\mathcal{M}}; \Lambda) \end{aligned}$$

and $\varphi_b \in C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+)$ is a cut-off function, $\psi_b \in C^\infty_{\mathrm{comp}}(\bar{\mathbb{R}}_+)$ is equal to 1 on the support of φ_b, $\varphi_i = 1 - \varphi_b$ is thought of as a C^∞ function on the whole manifold $\mathcal{M}$, and $\psi_i \in C^\infty_{\mathrm{comp}}(\mathring{\mathcal{M}})$ is equal to 1 on the support of φ_i.

We wish to arrange that $A_b(\lambda)$ and $A_i(\lambda)$ be compatible near the intersection of $\operatorname{supp}\varphi_{\mathrm{b}}$ and $\operatorname{supp}\varphi_{\mathrm{i}}$. In fact, it suffices that the principal interior symbols of $A_b(\lambda)$ and $A_i(\lambda)$ coincide close to $\operatorname{supp}\varphi_{\mathrm{b}} \cap \operatorname{supp}\varphi_{\mathrm{i}}$. If such is the case, then the parameter-dependent ellipticity of $A_b(\lambda)$ and $A_i(\lambda)$ with respect to the principal interior symbol implies that of $A(\lambda)$. Since the construction of $A_b(\lambda)$ requires much more efforts than that of $A_i(\lambda)$, we begin by discussing a suitable candidate for $A_b(\lambda)$.

To this end, let us fix a finite covering of O by charts with edges $(h_\iota, O_\iota)_{\iota\in I}$ on $\mathcal{M}$. We require each O_ι to be of the form $O'_\iota \times \bar{\mathbb{R}}_+ \times X$, with O'_ι a coordinate patch on S. Thus, h_ι is a diffeomorphism of O_ι onto a stretched wedge $\mathcal{W}_\iota = \Omega'_\iota \times \bar{\mathbb{R}}_+ \times X$, where Ω'_ι is an open set in $\mathbb{R}^q$, and h_ι restricts to a diffeomorphism $h'_\iota \colon O'_\iota \to \Omega'_\iota$. Let moreover $(h''_j, O''_j)_{j\in J}$ be a finite atlas on X, with h''_j a diffeomorphism of O''_j onto an open set Ω''_j in $\mathbb{R}^n$. Having fixed $\iota \in I$, we consider, for every $j \in J$, the typical symbol

$$a_{\iota,j}(t;\eta,\lambda,\tau,\xi,\vartheta) = \frac{1}{t^m}\left(1+|t\eta|^2+|t\lambda|^2+|t\tau|^2+|\xi|^2+\vartheta^2\right)^{\frac{m}{2}} \qquad (4.4.3)$$

over $\Omega'_\iota \times \bar{\mathbb{R}}_+ \times \Omega''_j$, where $\vartheta \in \mathbb{R}$ stands for an additional parameter to be chosen below. Choose a partition of unity $(\varphi''_j)_{j\in J}$ on X subordinate to the covering $(O''_j)_{j\in J}$, and functions $\psi''_j \in C^\infty_{\mathrm{comp}}(O''_j)$ with the property that $\psi''_j\varphi''_j = \varphi''_j$. Set

$$a_\iota(\eta,\lambda,\vartheta) = \sum_{j\in J} \varphi''_j\,(1\times h''_j)^\sharp\,\mathrm{op}_{\mathcal{F}_{t,x}}\left(a_{\iota,j}(t;\eta,\lambda,\tau,\xi,\vartheta)\right)\psi''_j$$

where $(1\times h''_j)^\sharp$ means the operator pull-back under the diffeomorphism $1\times h''_j$. As described at the end of Section 3.2, there exists a holomorphic function $\tilde{h}_\iota(\tilde{\eta},\tilde{\lambda},z,\vartheta)$ of $\mathcal{M}(\mathbb{C},\Psi^m(\Omega''_j;\mathbb{R}^{q+d+1}_{\tilde{\eta},\tilde{\lambda},\vartheta}))$, such that

$$a_\iota(\eta,\lambda,\vartheta) = \frac{1}{t^m}\,\mathrm{op}_{\mathcal{M},\gamma}(h_\iota(t;\eta,\lambda,z,\vartheta)) \mod \Psi^{-\infty}(\mathbb{R}_+\times X;\mathbb{R}^{q+d+1}_{\eta,\lambda,\vartheta})$$

for all $\gamma\in\mathbb{R}$, where $h_\iota(t;\eta,\lambda,z,\vartheta) = \tilde{h}_\iota(t\eta,t\lambda,z,\vartheta)$. Similarly to (3.2.6) we introduce two edge symbols

$$\begin{aligned} a_{0,\iota}(\eta,\lambda,\vartheta) &= t^{-m}\,\varphi_0(t\langle\eta,\lambda\rangle)\,\mathrm{op}_{\mathcal{M},\gamma}(\tilde{h}_\iota(t\eta,t\lambda,z,\vartheta))\,\psi_0(t\langle\eta,\lambda\rangle),\\ a_{\infty,\iota}(\eta,\lambda,\vartheta) &= \varphi_\infty(t\langle\eta,\lambda\rangle)\,a_\iota(\eta,\lambda,\vartheta)\,\psi_\infty(t\langle\eta,\lambda\rangle) \end{aligned}$$

belonging to $\mathcal{S}^m(\Omega'_\iota\times\mathbb{R}^{q+d+1}_{\eta,\lambda,\vartheta};w)$. It is a simple matter to see that

$$\begin{aligned} {}^b\sigma^m_{\mathcal{F}_{y,t,x}}(a_{0,\iota}+a_{\infty,\iota})(y,t,x;\tilde{\eta},\tilde{\lambda},\tilde{\tau},\xi,\vartheta) &= |(\tilde{\eta},\tilde{\lambda},\tilde{\tau},\xi,\vartheta)|^m,\\ \sigma^0_{\mathcal{F}_t,\mathrm{exit}}(a_{0,\iota}+a_{\infty,\iota})(t,\tau) &= |(\eta,\lambda,\tau)|^m, \end{aligned}$$

in local coordinates over $\mathcal{W}_\iota$, and so $a_{0,\iota}+a_{\infty,\iota}$ is elliptic with parameters $\lambda\in\Lambda$ and $\vartheta\in\mathbb{R}$ with respect to the principal interior symbol. Moreover, $a_{0,\iota}+a_{\infty,\iota}$ meets the condition of exit ellipticity for $t\to\infty$, provided that $(\eta,\lambda)\neq 0$.

Let us have a look at the principal edge symbol of $a_{0,\iota} + a_{\infty,\iota}$ which is a family of operators in weighted Sobolev spaces over the infinite stretched cone $\bar{\mathbb{R}}_+ \times X$. We check at once that

$$\begin{aligned} \sigma^m_{\mathrm{edge}}(a_{0,\iota}+a_{\infty,\iota})(y,\eta,\lambda) &= t^{-m}\varphi_0(t|\eta,\lambda|)\,\mathrm{op}_{\mathcal{M},\gamma}(\tilde{h}_\iota(t\eta,t\lambda,z,\vartheta))\,\psi_0(t|\eta,\lambda|) \\ &\quad + \varphi_\infty(t|\eta,\lambda|)\,a_\iota(\eta,\lambda,\vartheta)\,\psi_\infty(t|\eta,\lambda|) \end{aligned}$$

whence

$$\sigma_{\mathcal{M}}\left(\sigma^m_{\mathrm{edge}}(a_{0,\iota}+a_{\infty,\iota})(y,\eta,\lambda)\right)(z) = \tilde{h}_\iota(0,0,z,\vartheta), \tag{4.4.4}$$

the right side being regarded as a family of operators in $\mathcal{L}(H^s(X), H^{s-m}(X))$, for any one $s \in \mathbb{R}$. Were (4.4.4) an isomorphism for all $z \in \Gamma_{-\gamma}$, we would conclude from the cone theory that

$$\sigma^m_{\mathrm{edge}}(a_{0,\iota}+a_{\infty,\iota})(y,\eta,\lambda):\ H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \to H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X) \tag{4.4.5}$$

is a Fredholm operator for each $\eta \in \mathbb{R}^q$ and $\lambda \in \Lambda$ with $(\eta,\lambda) \neq 0$. However, this is not automatically the case. The conormal symbol $\tilde{h}_\iota(0,0,z,\vartheta)$ may have finitely many non-bijectivity points z on the weight line $\Gamma_{-\gamma}$. To avoid this, we make use of the additional parameter $\vartheta \in \mathbb{R}$. Namely, $\tilde{h}_\iota(0,0,z,\vartheta)$ is, by construction, a parameter-dependent elliptic operator of order m on X, with parameters $\tau = \Re z$ and ϑ. Hence there is a constant $R > 0$ such that $\tilde{h}_\iota(0,0,z,\vartheta)$ is an isomorphism for all $z \in \Gamma_{-\gamma}$ and $\vartheta \in \mathbb{R}$ satisfying $|(\Re z, \varepsilon)| > R$. Thus, by choosing $|\vartheta| > R$ we arrive at a family of isomorphisms

$$\tilde{h}_\iota(0,0,z,\vartheta):\ H^s(X) \to H^{s-m}(X)$$

for all $z \in \Gamma_{-\gamma}$ and for any $s \in \mathbb{R}$. From now on we make the assumption: $\vartheta > R$.

The advantage of using typical symbols (4.4.3) lies is in the fact that the principal edge symbol of $a_{0,\iota} + a_{\infty,\iota}$ is independent of $y \in \Omega'_\iota$ and (η,λ) lying on the unit sphere of $\mathbb{R}^q \times \Lambda$, i.e., $|\eta|^2 + |\lambda|^2 = 1$. Indeed, this is immediate for the symbols $a_\iota(\eta,\lambda,\vartheta)$, and the symbols $h_\iota(t;\eta,\lambda,z,\vartheta)$ inherit this property by the very construction of the Mellin quantisation (cf. Theorem 3.2.4). Hence it follows that

$$\mathrm{ind}_{\mathbb{S}^*S}\,\sigma^m_{\mathrm{edge}}(a_{0,\iota}+a_{\infty,\iota}) = \pi^*[S \times \mathbb{C}^{\tilde{l}}, S \times \mathbb{C}^{l}]$$

for some $l, \tilde{l} \in \mathbb{Z}_+$, where $\mathbb{S}^*S = \{(y,\eta,\lambda) \in T^*S \times \Lambda : |(\eta,\lambda)| = 1\}$ and $\pi : \mathbb{S}^*S \to S$ denotes the canonical projection.

It is now a consequence of the cone theory that the index of (4.4.5) does not depend on the particular choice of s, for $(\eta,\lambda) \neq 0$. Let us write i for this index. According to Theorem 2.4.43 of [Sch98], there exists a meromorphic Mellin symbol $s(z) \in \mathcal{M}_{\mathrm{as}}(\mathbb{C}, \Psi^{-\infty}(X))$ such that

$$1 + \varphi_0(t)\,\mathrm{op}_{\mathcal{M},\gamma}(s(z))\,\psi_0(t):\ H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \to H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)$$

is a Fredholm operator of index i, for all $s \in \mathbb{R}$. Set

$$m_\iota(\eta,\lambda) = \varphi_0(t\langle\eta,\lambda\rangle)\,\mathrm{op}_{\mathcal{M},\gamma}(\mathrm{s(z)})\,\psi_0(\mathrm{t}\langle\eta,\lambda\rangle), \tag{4.4.6}$$

for $(\eta,\lambda) \in \mathbb{R}^q \times \Lambda$. Then, $m_\iota(\eta,\lambda) \in \mathcal{S}^0_{M+G}(\Omega'_\iota \times \mathbb{R}^q \times \Lambda; w^{-1} \circ w)$ and

$$\sigma^0_{\mathrm{edge}}(1+m_\iota)(y,\eta,\lambda) = 1 + \varphi_0(t|\eta,\lambda|)\,\mathrm{op}_{\mathcal{M},\gamma}(\mathrm{s(z)})\,\psi_0(\mathrm{t}|\eta,\lambda|)$$

is a Fredholm operator of index i in $H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X)$, provided that $(\eta,\lambda) \neq 0$. Letting

$$a^{(1)}_\iota(\eta,\lambda,\vartheta) = (a_{0,\iota}(\eta,\lambda,\vartheta) + a_{\infty,\iota}(\eta,\lambda,\vartheta))\,(1 + m_\iota(\eta,\lambda))$$

we deduce at once that $a^{(1)}_\iota(\eta,\lambda,\vartheta) \in \mathcal{S}^m(\Omega'_\iota \times \mathbb{R}^{q+d+1}_{\eta,\lambda,\vartheta}; w)$ and

$$\begin{aligned} &\mathrm{ind}\,\sigma^{\mathrm{m}}_{\mathrm{edge}}(\mathrm{a}^{(1)}_\iota)(\mathrm{y},\eta,\lambda) \\ &\quad= \mathrm{ind}\,\sigma^{\mathrm{m}}_{\mathrm{edge}}(\mathrm{a}_{0,\iota} + \mathrm{a}_{\infty,\iota})(\mathrm{y},\eta,\lambda) + \mathrm{ind}\,\sigma^0_{\mathrm{edge}}(1+\mathrm{m}_\iota)(\mathrm{y},\eta,\lambda) \\ &\quad= 0 \end{aligned}$$

for all $(\eta,\lambda) \in \mathbb{R}^q \times \Lambda$ different from zero. As the principal edge symbol of $m_\iota(\eta,\lambda)$ is actually independent of $(y,\eta,\lambda) \in \mathbb{S}^*\Omega'_\iota$, it follows that

$$\mathrm{ind}_{\mathbb{S}^*\mathrm{S}}\,\sigma^{\mathrm{m}}_{\mathrm{edge}}(\mathrm{a}^{(1)}_\iota) = \pi^*[\mathrm{S}\times\mathbb{C}^\mathrm{l}, \mathrm{S}\times\mathbb{C}^\mathrm{l}]$$

for some $l \in \mathbb{Z}_+$.

Analysis similar to that in the proof of Lemma 4.2.7 shows that there is a Green symbol

$$g_\iota(\eta,\lambda) = \begin{pmatrix} 0 & p_\iota(\eta,\lambda) \\ t_\iota(\eta,\lambda) & b_\iota(\eta,\lambda) \end{pmatrix}$$

in $\mathcal{S}^m_G(\Omega'_\iota \times \mathbb{R}^q \times \Lambda; \mathrm{Hom}(\mathbb{C}^\mathrm{l}); \mathrm{w})$ with the property that

$$\sigma^m_{\mathrm{edge}}\begin{pmatrix} a^{(1)}_\iota & p_\iota \\ t_\iota & b_\iota \end{pmatrix}(y,\eta,\lambda) : \begin{matrix} H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^l \end{matrix} \to \begin{matrix} H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X) \\ \oplus \\ \mathbb{C}^l \end{matrix} \tag{4.4.7}$$

is an isomorphism for all non-zero $(\eta,\lambda) \in \mathbb{R}^q \times \Lambda$ and for each $s \in \mathbb{R}$. Recall that $g_\iota(\eta,\lambda)$ is first constructed on the sphere $|\eta,\lambda| = 1$, extended by the 'twisted' homogeneity of order m to all of $(\mathbb{R}^q \times \Lambda) \setminus \{0\}$, and finally smoothed at 0 by multiplying by an excision function $\chi(\eta,\lambda)$. Since $\mathrm{GL}(\mathrm{l},\mathbb{C})$ is dense in $\mathrm{Hom}(\mathbb{C}^\mathrm{l})$, we can always find $b_\iota(\eta,\lambda)$ with the property that $\sigma^m_{\mathrm{edge}}(b_\iota)(y,\eta,\lambda)$ is invertible away from $(\eta,\lambda) = 0$. The symbol

$$a^{(2)}_\iota(\eta,\lambda,\vartheta) = a^{(1)}_\iota(\eta,\lambda,\vartheta) - \chi(\eta,\lambda)\,p_\iota(\eta,\lambda)\left(\sigma^m_{\mathrm{edge}}(b_\iota)(y,\eta,\lambda)\right)^{-1} t_\iota(\eta,\lambda)$$

is easily verified to belong to $\mathcal{S}^m(\Omega'_\iota \times \mathbb{R}^q \times \Lambda; w)$. Moreover,

$$\sigma^m_{\mathrm{edge}}(a^{(2)}_\iota) = \sigma^m_{\mathrm{edge}}(a^{(1)}_\iota) - \sigma^m_{\mathrm{edge}}(p_\iota)\left(\sigma^m_{\mathrm{edge}}(b_\iota)\right)^{-1}\sigma^m_{\mathrm{edge}}(t_\iota)$$

is an isomorphism $H^{s,\gamma}(\bar{\mathbb{R}}_+ \times X) \overset{\cong}{\to} H^{s-m,\gamma-m}(\bar{\mathbb{R}}_+ \times X)$ for all $(\eta, \lambda) \in \mathbb{R}^q \times \Lambda$ with $(\eta, \lambda) \neq 0$, and for each $s \in \mathbb{R}$. To see this, it suffices to use Proposition 1.2.31 of [Sch98] along with the fact that (4.4.7) is an isomorphism.

We are now in a position to introduce the operator $A_b(\lambda)$. To do this, fix a partition of unity $(\varphi'_\iota)_{\iota \in I}$ on S subordinate to the covering $(O'_\iota)_{\iota \in I}$. For every ι, we find a function $\psi'_\iota \in C^\infty_{\mathrm{comp}}(O'_\iota)$ such that $\psi'_\iota = 1$ on the support of φ'_ι. Set

$$A_b(\lambda) = \sum_{\iota \in I} \varphi'_\iota \, (h'_\iota)^\sharp \operatorname{op}\left(\mathrm{a}^{(2)}_\iota(\eta, \lambda, \vartheta)\right) \psi'_\iota;$$

from what has already been proved it follows that $A_b \in \Psi^{\mathrm{m}}(\mathrm{O}; \mathrm{w}; \Lambda)$ is parameter-dependent elliptic with respect to the principal interior and edge symbols.

We are thus left with the task of determining the operator $A_i(\lambda)$ in the interior of $\mathcal{M}$ in a consistent way with $A_b(\lambda)$. Let us complete the atlas $(h_\iota, O_\iota)_{\iota \in I}$ on the collar O to an atlas on the whole manifold $\mathcal{M}$ by adding a finite number of charts $(h_\nu, O_\nu)_{\nu \in N}$, such that each O_ν lies along with the closure away from the support of φ_b and $\mathcal{M} \setminus O \subset \cup_{\nu \in N} O_\nu$. Here, h_ν is a diffeomorphism of O_ν onto an open set Ω_ν in $\mathbb{R}^{q+1+n}$. Fix a positive C^∞ function $\tilde{t}$ on $\mathring{\mathcal{M}}$ which coincides with the global coordinate t near the support of φ_b and is equal to 1 on every O_ν, $\nu \in N$. For $\iota \in I$, we consider the local symbols

$$a_{\iota,j}(\tilde{t}; \eta, \lambda, \tau, \xi, \vartheta) = \frac{1}{\tilde{t}^m} \left(1 + |\tilde{t}\eta|^2 + |\tilde{t}\lambda|^2 + |\tilde{t}\tau|^2 + |\xi|^2 + \vartheta^2\right)^{\frac{m}{2}}$$

on $\Omega'_\iota \times \bar{\mathbb{R}}_+ \times \Omega''_j$. These are slight corrections of (4.4.3) away from the support of φ_b, and we paste them together over the covering of X to get an operator-valued symbol $a_\iota(\eta, \lambda, \vartheta)$ with values in $\Psi^{\mathrm{m}}(\mathbb{R}_+ \times \mathrm{X}; \mathbb{R}^{\mathrm{q+d+1}}_{\eta,\lambda,\vartheta})$, as above. For short, we keep the same notation for the corrected symbols. In the intersections $O_\iota \cap O_\nu$, the factor $\tilde{t}$ reduces to 1, and so, for any $\nu \in N$, we consider the symbol

$$a_\nu(c, \lambda, \vartheta) = \left(1 + |c|^2 + |\lambda|^2 + \vartheta^2\right)^{\frac{m}{2}} \tag{4.4.8}$$

in the local coordinates of O_ν, where (p, c) stand for the coordinates in T^*O_ν. We see at once that (4.4.8) agrees with (4.4.3) after the coordinate t has been reduced to 1. We now proceed by a standard way. Choose a C^∞ partition of unity $(\varphi_\iota)_{\iota \in I}$, $(\varphi_\nu)_{\nu \in N}$ on $\mathcal{M}$, subordinate to the covering $(O_\iota)_{\iota \in I}$, $(O_\nu)_{\nu \in N}$, and functions $\psi_\iota \in C^\infty_{\mathrm{comp}}(O_\iota)$, $\psi_\nu \in C^\infty_{\mathrm{comp}}(O_\nu)$ such that $\psi_\iota \varphi_\iota = \varphi_\iota$, $\psi_\nu \varphi_\nu = \varphi_\nu$. Set

$$A_i(\lambda) = \sum_{\iota \in I} \varphi_\iota \, h^\sharp_\iota \operatorname{op}\left(\mathrm{a}_\iota(\eta, \lambda, \vartheta)\right) \psi_\iota + \sum_{\nu \in \mathrm{N}} \varphi_\nu \, \mathrm{h}^\sharp_\nu \operatorname{op}\left(\mathrm{a}_\nu(\mathrm{c}, \lambda, \vartheta)\right) \psi_\nu,$$

the operator 'op' in the first sum being with respect to $y \in \Omega'_\iota$ while that in the second sum being with respect to $p \in \Omega_\nu$. It is straightforward from the construction, that $A_i(\lambda) \in \Psi^{\mathrm{m}}_{\mathrm{cl}}(\mathring{\mathcal{M}}; \Lambda)$ is parameter-dependent elliptic.

We claim that $A(\lambda) = \varphi_b A_b(\lambda)\psi_b + \varphi_i A_i(\lambda)\psi_i$ is a parameter-dependent elliptic operator in $\Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{w}; \Lambda)$. Indeed, the principal interior symbol of $A(\lambda)$ is invertible away from zero section of ${}^bT^*\mathcal{M} \times \Lambda$ because the principal interior symbols of $A_b(\lambda)$ and $A_i(\lambda)$ agree close to $\operatorname{supp}\varphi_{\mathrm{b}} \cap \operatorname{supp}\varphi_{\mathrm{i}}$. On the other hand, the principal edge symbol of $A(\lambda)$ is invertible away from the zero section of $T^*S \times \Lambda$, for the principal edge symbol of $A_b(\lambda)$ does. This proves the theorem.

□

We can now return to order reductions in the algebra of pseudodifferential operators on a manifold with edges. They are of great importance in proving the necessity of ellipticity for the Fredholm property, and the so-called *spectral invariance*, i.e., invertibility within the algebra. The strong definition of an order reduction is as follows (cf. [Beh95, 4.2.1]).

Definition 4.4.5 *An operator $\mathcal{R}^m_{V,W}(\lambda) \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}; \mathrm{W}; \mathrm{w}; \Lambda)$ is said to be a parameter-dependent order reduction of order m and with respect to weight data $w = (\gamma, \gamma - m, (-l, 0])$ if*

$$\mathcal{R}^m_{V,W}(\lambda) : \begin{array}{c} H^{s,\gamma}(\mathcal{M}, V) \\ \oplus \\ H^s(S, W) \end{array} \rightarrow \begin{array}{c} H^{s-m,\gamma-m}(\mathcal{M}, V) \\ \oplus \\ H^{s-m}(S, W) \end{array}$$

is invertible for each $s \in \mathbb{R}$, and $\mathcal{R}^m_{V,W}(\lambda)^{-1} \in \Psi^{-\mathrm{m}}(\mathcal{M}; \mathrm{V}; \mathrm{W}; \mathrm{w}^{-1}; \Lambda)$.

As described above, the crucial step in constructing reductions of orders on $\mathcal{M}$ is the proof of the invertibility of parameter-dependent elliptic operators on $\mathcal{M}$ for large values of the parameter. Before formulating this result, we make more precise our requirement on the set of parameters. Namely, it should be open and contain, along with any point λ, the entire ray $\{\theta\lambda : \ \theta \geq 1\}$. In particular, given any $R > 0$, we may consider $\Lambda_R = \{\lambda \in \Lambda : \ |\lambda| > R\}$ instead of Λ.

Theorem 4.4.6 *Suppose $\mathcal{A}(\lambda) \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w}; \Lambda)$ is a parameter-dependent elliptic operator on $\mathcal{M}$, where $w = (\gamma, \gamma - m, (-l, 0])$. Then, there exists an $R > 0$ such that*

$$\mathcal{A}(\lambda) : \begin{array}{c} H^{s,\gamma}(\mathcal{M}, V) \\ \oplus \\ H^s(S, W) \end{array} \rightarrow \begin{array}{c} H^{s-m,\gamma-m}(\mathcal{M}, \tilde{V}) \\ \oplus \\ H^{s-m}(S, \tilde{W}) \end{array}$$

is invertible for all $\lambda \in \Lambda$ with $|\lambda| > R$, and for any $s \in \mathbb{R}$. Furthermore, the inverse belongs to $\Psi^{-\mathrm{m}}(\mathcal{M}; \tilde{\mathrm{V}}, \mathrm{V}; \tilde{\mathrm{W}}, \mathrm{W}; \mathrm{w}^{-1}; \Lambda_{\mathrm{R}})$.

Proof. Let $\mathcal{P}(\lambda) \in \Psi^{-\mathrm{m}}(\mathcal{M}; \tilde{\mathrm{V}}; \mathrm{V}; \tilde{\mathrm{W}}, \mathrm{W}; \mathrm{w}^{-1}; \Lambda)$ be a left parametrix of $\mathcal{A}(\lambda)$ guaranteed by Theorem 4.4.3. We thus have

$$\mathcal{P}(\lambda)\mathcal{A}(\lambda) = 1 - \mathcal{S}(\lambda)$$

for some $\mathcal{S}(\lambda) \in \Psi^{-\infty}(\mathcal{M};\mathrm{V};\mathrm{W};\mathrm{w}^{-1}\circ\mathrm{w};\Lambda)$. Fix $s = s_0$. By the Schwartz property of $\mathcal{S}(\lambda)$ and Theorem 4.4.1, there exists an $R > 0$ such that

$$\|\mathcal{S}(\lambda)\|_{\mathcal{L}(H^{s_0,\gamma}(\mathcal{M},V)\oplus H^{s_0}(S,W))} < \frac{1}{2}$$

for all $\lambda \in \Lambda$ satisfying $|\lambda| > R$. Hence it follows that $1 - \mathcal{S}(\lambda)$ is invertible on

$$\begin{array}{c} H^{s,\gamma}(\mathcal{M},V) \\ \oplus \\ H^{s}(S,W) \end{array}$$

for each λ with $|\lambda| > R$, provided $s = s_0$. On the other hand, the kernel and the cokernel of $1 - \mathcal{S}(\lambda)$ are independent on s, which is clear from the definition of $\Psi^{-\infty}(\mathcal{M};\mathrm{V};\mathrm{W};\mathrm{w}^{-1}\circ\mathrm{w};\Lambda)$. Therefore, $1 - \mathcal{S}(\lambda)$ is actually invertible for all $|\lambda| > R$ and for each $s \in \mathbb{R}$. Write $(1-\mathcal{S}(\lambda))^{-1} = 1+\mathcal{G}(\lambda)$, then $(1+\mathcal{G}(\lambda))(1-\mathcal{S}(\lambda)) = 1$ yields $\mathcal{G}(\lambda) = (1-\mathcal{S}(\lambda))^{-1}\mathcal{S}(\lambda)$. Consequently

$$\begin{aligned} \|\mathcal{G}(\lambda)\| &\le \|1-\mathcal{S}(\lambda)\|^{-1}\|\mathcal{S}(\lambda)\| \\ &\le (1-\|\mathcal{S}(\lambda)\|)^{-1}\|\mathcal{S}(\lambda)\| \\ &\le 2\,\|\mathcal{S}(\lambda)\| \end{aligned}$$

for $|\lambda| > R$, where $\|\cdot\|$ means the norm in $\mathcal{L}(H^{s_0,\gamma}(\mathcal{M},V)\oplus H^{s_0}(S,W))$. The same reasoning applies to the derivatives of $\mathcal{G}(\lambda)$ in $\lambda \in \Lambda_R$, showing $\mathcal{G}(\lambda) \in \Psi^{-\infty}(\mathcal{M};\mathrm{V};\mathrm{W};\mathrm{w}^{-1}\circ\mathrm{w};\Lambda_\mathrm{R})$ (cf. Lemma 1.2.3.2 in [Beh95]). Let

$$\mathcal{A}^{-1}(\lambda) = (1+\mathcal{G}(\lambda))\mathcal{P}(\lambda),$$

then $\mathcal{A}^{-1}(\lambda) \in \Psi^{-\mathrm{m}}(\mathcal{M};\tilde{\mathrm{V}},\mathrm{V};\tilde{\mathrm{W}},\mathrm{W};\mathrm{w}^{-1};\Lambda_\mathrm{R})$ is a left inverse of $\mathcal{A}(\lambda)$. In just the same way we get a right inverse of $\mathcal{A}(\lambda)$ from a right parametrix of $\mathcal{A}(\lambda)$, hence $\mathcal{A}^{-1}(\lambda)$ is the inverse of $\mathcal{A}(\lambda)$ for $|\lambda| > R$. This is the desired conclusion.

□

The main result of this section is now a straightforward consequence of Theorems 4.4.4 and 4.4.6.

Theorem 4.4.7 *Let (M,S) be a C^∞ compact closed manifold with edges and let V and W be vector bundles over $\mathcal{M}$ and S, respectively. Then, for each weight data $w = (\gamma,\gamma-m,(-l,0])$, there exists an order reduction $\mathcal{R}^m_{V,W}(\lambda) \in \Psi^\mathrm{m}(\mathcal{M};\mathrm{V};\mathrm{W};\mathrm{w};\Lambda)$. Moreover, it can be chosen without any potential and trace conditions.*

Proof. Indeed, Theorem 4.4.4 shows that there is a parameter-dependent elliptic operator $A(\lambda,\vartheta) \in \Psi^\mathrm{m}(\mathcal{M};\mathrm{V};\mathrm{w};\Lambda\times\mathbb{R})$. By Theorem 4.4.6, we can find an $R > 0$ such that $A(\lambda,\vartheta)$ is invertible within the calculus for all $(\lambda,\vartheta) \in \Lambda\times\mathbb{R}$ satisfying $|(\lambda,\vartheta)| > R$. Setting $R^m_V(\lambda) = A(\lambda,R+1)$ we get an invertible operator in $\Psi^\mathrm{m}(\mathcal{M};\mathrm{V};\mathrm{w};\Lambda)$ whose inverse belongs to

$\Psi^{-\mathrm{m}}(\mathcal{M};\mathrm{V};\mathrm{w}^{-1};\Lambda)$. The desired operator $\mathcal{R}^{m}_{V,W}(\lambda)$ is now given by (4.4.2), $R^m_W(\lambda)\in\Psi^{\mathrm{m}}_{\mathrm{cl}}(\mathrm{S};\mathrm{W};\Lambda)$ being an order reduction on S. The proof is complete. □

In the next result, $\mathcal{R}^{m}_{V,W}(\lambda)$ stands for an order reduction on $\mathcal{M}$ guaranteed by Theorem 4.4.7.

Corollary 4.4.8 *The mapping* $\mathcal{A}(\lambda)\mapsto\mathcal{R}^{\gamma-m}_{\tilde V,\tilde W}(\lambda)\mathcal{A}(\lambda)\mathcal{R}^{-\gamma}_{V,W}(\lambda)$ *induces an isomorphism*

$$\Psi^{\mathrm{m}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{W},\tilde{\mathrm{W}};\mathrm{w};\Lambda)\overset{\cong}{\to}\Psi^{0}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{W},\tilde{\mathrm{W}};(0,0,(-1,0]);\Lambda)$$

preserving the class of parameter-dependent elliptic operators.

Proof. This follows from Theorem 4.1.9, for the principal interior and edge symbols of $\mathcal{R}^{-\gamma}_{V,W}(\lambda)$ and $\mathcal{R}^{\gamma-m}_{\tilde V,\tilde W}(\lambda)$ are invertible. □

As an application of order reductions on $\mathcal{M}$ we extend to edge problems the well-known formula of Agranovich and Dynin [AD62] which compares the indices of two elliptic boundary value problems for the same differential operator in a domain.

Theorem 4.4.9 *Let*

$$\begin{array}{rcl}\mathcal{A}_1 & \in & \Psi^{\mathrm{m}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{W}^1,\tilde{\mathrm{W}}^1;\mathrm{w}),\\ \mathcal{A}_2 & \in & \Psi^{\mathrm{m}}(\mathcal{M};\mathrm{V},\tilde{\mathrm{V}};\mathrm{W}^2,\tilde{\mathrm{W}}^2;\mathrm{w})\end{array}$$

be two elliptic edge problems on $\mathcal{M}$, whose upper left corners coincide. Then, there is an elliptic operator $B\in\Psi^0(\mathrm{S};\tilde{\mathrm{W}}^1\oplus\mathrm{W}^2;\tilde{\mathrm{W}}^2\oplus\mathrm{W}^1)$ *such that* $\operatorname{ind}\mathcal{A}_2-\operatorname{ind}\mathcal{A}_1=\operatorname{ind}\mathrm{B}$.

The important point to note here is an explicit construction of the operator B through $\mathcal{A}_1$ and $\mathcal{A}_2$, given in the proof.

Proof. For $j=1,2$, pick order reductions

$$\begin{array}{rcl}\mathcal{R}^{\gamma}_{V,W^j} & \in & \Psi^{\gamma}(\mathcal{M};\mathrm{V};\mathrm{W}^{\mathrm{j}};(\gamma,0,(-1,0])),\\ \mathcal{R}^{\gamma-m}_{\tilde V,\tilde W^j} & \in & \Psi^{\gamma-\mathrm{m}}(\mathcal{M};\tilde{\mathrm{V}};\tilde{\mathrm{W}}^{\mathrm{j}};(\gamma-\mathrm{m},0,(-1,0])),\end{array}$$

as in (4.4.2), and define the operators $\tilde{\mathcal{A}}_j$ so as to make the following diagram commutative:

$$\begin{array}{ccc}
\begin{array}{c}H^{s,\gamma}(V)\\ \oplus\\ H^{s}(W^j)\end{array} & \xrightarrow{\mathcal{A}_j} & \begin{array}{c}H^{s-m,\gamma-m}(\tilde V)\\ \oplus\\ H^{s-m}(\tilde W^j)\end{array}\\
\Big\downarrow \mathcal{R}^{\gamma}_{V,W^j} & & \Big\downarrow \mathcal{R}^{\gamma-m}_{\tilde V,\tilde W^j}\\
\begin{array}{c}H^{s-\gamma,0}(V)\\ \oplus\\ H^{s-\gamma}(W^j)\end{array} & \xrightarrow{\tilde{\mathcal{A}}_j} & \begin{array}{c}H^{s-\gamma,0}(\tilde V)\\ \oplus\\ H^{s-\gamma}(\tilde W^j)\end{array}
\end{array}.$$

It follows that

$$\begin{array}{rcl} \tilde{\mathcal{A}}_1 & \in & \Psi^0(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}^1, \tilde{\mathrm{W}}^1; (0, 0, (-1, 0])), \\ \tilde{\mathcal{A}}_2 & \in & \Psi^0(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}^2, \tilde{\mathrm{W}}^2; (0, 0, (-1, 0])) \end{array}$$

are elliptic, the upper left corners of $\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_2$ coincide and $\operatorname{ind} \tilde{\mathcal{A}}_j = \operatorname{ind} \mathcal{A}_j$ for $j = 1, 2$. We are thus reduced to proving the theorem for operators of order $m = 0$.

Write

$$\mathcal{A}_j = \begin{pmatrix} A & P_j \\ T_j & B_j \end{pmatrix},$$

for $j = 1, 2$, and consider the operators

$$O_1 = \begin{pmatrix} A & P_2 & P_1 \\ T_1 & 0 & B_1 \\ 0 & 1 & 0 \end{pmatrix} : \begin{array}{c} H^{s,\gamma}(\mathcal{M}, V) \\ \oplus \\ H^s(S, W^2) \\ \oplus \\ H^s(S, W^1) \end{array} \to \begin{array}{c} H^{s,\gamma}(\mathcal{M}, \tilde{V}) \\ \oplus \\ H^s(S, \tilde{W}^1) \\ \oplus \\ H^s(S, W^2) \end{array},$$

$$O_2 = \begin{pmatrix} A & P_2 & P_1 \\ T_2 & B_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \begin{array}{c} H^{s,\gamma}(\mathcal{M}, V) \\ \oplus \\ H^s(S, W^2) \\ \oplus \\ H^s(S, W^1) \end{array} \to \begin{array}{c} H^{s,\gamma}(\mathcal{M}, \tilde{V}) \\ \oplus \\ H^s(S, \tilde{W}^2) \\ \oplus \\ H^s(S, W^1) \end{array}.$$

Obviously, both

$$\begin{array}{rcl} O_1 & \in & \Psi^0(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}^2 \oplus \mathrm{W}^1, \tilde{\mathrm{W}}^1 \oplus \mathrm{W}^2; \mathrm{w}), \\ O_2 & \in & \Psi^0(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}^2 \oplus \mathrm{W}^1, \tilde{\mathrm{W}}^2 \oplus \mathrm{W}^1; \mathrm{w}) \end{array}$$

are elliptic and $\operatorname{ind} \mathrm{O}_j = \operatorname{ind} \mathcal{A}_j$, for $j = 1, 2$.

Let

$$\mathcal{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

be a parametrix of $\mathcal{A}_1$. Then

$$\mathcal{R} = \begin{pmatrix} P_{11} & P_{12} & -P_{11}P_2 \\ 0 & 0 & 1 \\ P_{21} & P_{22} & -P_{21}P_2 \end{pmatrix}$$

is a parametrix of O_1, and

$$O_2\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 \\ T_2P_{11} & T_2P_{12} & -T_2P_{11}P_2 + B_2 \\ P_{21} & P_{22} & -P_{21}P_2 \end{pmatrix} \qquad (4.4.9)$$

modulo $\Psi^{-\infty}(\mathcal{M}; \tilde{\mathrm{V}}; \tilde{\mathrm{W}}^1 \oplus \mathrm{W}^2; \tilde{\mathrm{W}}^2 \oplus \mathrm{W}^1; \mathrm{w} \circ \mathrm{w}^{-1})$, which is due to the fact that $\mathcal{A}_1\mathcal{P} = 1$ modulo smoothing Green operators.

Setting

$$B = \begin{pmatrix} T_2 P_{12} & -T_2 P_{11} P_2 + B_2 \\ P_{22} & -P_{21} P_2 \end{pmatrix},$$

we deduce from (4.4.9) and the ellipticity of $O_2 \mathcal{R}$ that B is an elliptic operator in $\Psi^0(\mathrm{S}; \tilde{\mathrm{W}}^1 \oplus \mathrm{W}^2; \tilde{\mathrm{W}}^2 \oplus \mathrm{W}^1)$. Moreover,

$$\begin{aligned} \operatorname{ind} \mathrm{B} &= \operatorname{ind} \mathrm{O}_2 \mathcal{R} \\ &= \operatorname{ind} \mathrm{O}_2 - \operatorname{ind} \mathrm{O}_1 \\ &= \operatorname{ind} \mathcal{A}_2 - \operatorname{ind} \mathcal{A}_1, \end{aligned}$$

and the proof is complete.

□

The method of proof carries over to elliptic complexes of edge problems on $\mathcal{M}$ to be considered in the next chapter.

Chapter 5

Complexes over a Manifold with Edges

In the analysis on manifolds with singularities it is natural to consider also complexes of pseudodifferential operators. As but one reason of this we mention that elliptic complexes of differential operators arise in geometry rather than single elliptic operators. On the other hand, the parametrix to such a complex derived from the Hodge theory is an elliptic complex of pseudodifferential operators. Also in the construction of the tensor product of elliptic complexes of differential operators it is useful to allow complexes whose differentials are pseudodifferential operators. For obtaining parametrices and other useful information for elliptic complexes it is necessary to have an elliptic theory for single operators because essential questions lead to the sequence of Laplacians. In this chapter we give explicitly applications to the complexes on manifolds with edges. In order to arrange the theory of elliptic complexes in a way which looks like a canonical extension of the smooth case we emphasise those symbolic levels the bijectivity of which is necessary and sufficient for the Fredholm property. In other words we distinguish a system of adequate principal symbols with internal compatibility conditions, which determines the structure of the operators on a manifold with edges modulo regular ones. These are just Green operators. As described in the previous chapter, they are smoothing in the interior and map distributions to functions with asymptotics close to the edges. We prove an analogue of the Hodge decomposition and establish a general result on the asymptotics of harmonic sections. Then we discuss external products of elliptic complexes, which lead to higher singularities of the underlying manifolds. For instance, if X is a manifold with conical singularities, then the topological cone over X has a natural C^∞ structure with corners. It is important for the concept of ellipticity that external multiplication of Fredholm complexes leads again to Fredholm complexes over the direct product. A corresponding abstract result for Fredholm complexes of Fréchet spaces may be found in the literature, cf. Grothendieck [Gro54], Mantlik [Man95], etc. Here we reproduce another independent argument which applies also to more general cases. Complexes of differential operators on manifolds with singularities were also studied by other authors, see Teleman [Tel79, Tel83], Cheeger [Che80, Che83], et al. However, these papers deal with geometric complexes and the analysis of the operators remains obscure in the sense of parametrix construction on the symbol level and of the nature of associated classes of pseudodifferential operators. This gap was filled by Shaw [Sha83] and Schulze [Sch88a] for manifolds with conical singularities. Let us also mention that elliptic complexes of boundary value problems were treated by Dynin [Dyn72] and Pillat and Schulze [PS80]. In this chapter we develop the theory for manifolds with edges.

5.1 *Fredholm quasicomplexes*

In the category of Fréchet spaces, a *complex* (L, d) is given by a sequence of Fréchet spaces L^i, $i \in \mathbb{Z}$, and continuous linear mappings $d_i \colon L^i \to L^{i+1}$ such that $d_i \circ d_{i-1} = 0$. We will write it simply L when no confusion can arise. A complex L is said to be *finite* if all the L^i are of finite dimension; *elementary* if there exists an index i_0 such that $L^i = 0$ unless $i = i_0, i_0 + 1$, d_i being an isomorphism; and *bounded* if $L^i = 0$ for $|i|$ large enough. When considering bounded complexes we can certainly assume that $L^i = 0$ for i different from $0, 1, \ldots, N$, for if not, we shift the indexing. Let $Z^i(L)$ and $B^i(L)$ stand for the spaces of *cocycles* and *coboundaries* of a complex L at step $i \in \mathbb{Z}$. The quotient $H^i(L) = Z^i(L)/B^i(L)$ is called the *cohomology* of L at step i. It is separated if and only if $B^i(L)$ is a closed subspace of L^i. If such is not the case, one often considers the so-called *reduced* cohomology of L at step i, that is $\bar{H}^i(L) = Z^i(L)/\overline{B^i(L)}$. Finite complexes are objects of homological algebra. There is a more general class of complexes of Fréchet spaces, which are almost as easy to be handled as finite complexes. These are *Fredholm complexes*, i.e., those complexes L of Fréchet spaces which have a finite-dimensional (and hence separated) cohomology at every step $i \in \mathbb{Z}$. In particular, any elliptic complex of pseudodifferential operators on a C^∞ compact closed manifold is known to induce Fredholm complexes of Sobolev spaces or spaces of C^∞ sections of vector bundles. The ellipticity here is controlled by the sequence of principal symbols which inherits the property of being a complex.

We note that the complexes, so defined, are *cochain* complexes, i.e., their differentials have degree $+1$ with respect to the grading. We will also deal with *chain* complexes, i.e., those with differentials of degree -1 with respect to the grading.

Replacing the relations $d_i \circ d_{i-1} = 0$ by requiring $d_i \circ d_{i-1}$ to be "small" operators yields what will be referred to as a *quasicomplex*. As but one example of this we show the sequence of *covariant differentiations* related to a *connection* of a vector bundle V over a C^∞ manifold (cf. for instance [Wel73, Ch. 3]). In this case the compositions $d_i \circ d_{i-1}$ are identified with the *curvature* of the connection, which is a smooth differential form of degree 2 with values in the endomorphisms of V. From the point of view of analysis, quasicomplexes seem to be much more natural objects than complexes. Indeed, "small" perturbations of Fredholm operators do not affect the Fredholm property. In particular, perturbing a single Fredholm operator by compact operators leads to a Fredholm operator. It would be desirable to have the same property for Fredholm complexes but most of the perturbations lead out the class of 'complexes'. For example, perturbing an elliptic complex by lower order terms does not change the sequence of principal symbols which remains to be exact away from the zero section of the cotangent bundle. However, the operators no longer satisfy $d_i \circ d_{i-1} = 0$, and so the standard theory does not apply to the deformed complex. We are thus lead to a class of sequences $L = (L^i, d_i)_{i\in\mathbb{Z}}$ bearing the property that the compositions $d_i \circ d_{i-1}$ are small in some sense.

Definition 5.1.1 *By a (cochain) quasicomplex (L, d) is meant any sequence of Fréchet spaces L^i, $i \in \mathbb{Z}$, and operators $d_i \in \mathcal{L}(L^i, L^{i+1})$ satisfying $d_i \circ d_{i-1} = 0$ modulo compact operators.*

The composition $d_i \circ d_{i-1}$ is said to be the *curvature* of a quasicomplex (L, d) at step i.

Denote by $\mathcal{K}(L, \tilde{L})$ the subspace of $\mathcal{L}(L, \tilde{L})$ consisting of compact operators. For $m_1, m_2 \in \mathcal{L}(L, \tilde{L})$, we write $m_1 \sim m_2$ if $m_1 - m_2 \in \mathcal{K}(L, \tilde{L})$.

Let (L, d) and $(\tilde{L}, \tilde{d})$ be two quasicomplexes. By a *cochain mapping* of (L, d) into $(\tilde{L}, \tilde{d})$ is meant any collection of operators $m_i \in \mathcal{L}(L^i, \tilde{L}^i)$, $i \in \mathbb{Z}$, such that $\tilde{d}_i m_i \sim m_{i+1} d_i$ for all $i \in \mathbb{Z}$. In particular, the families $0 = (0_{L^i})_{i\in\mathbb{Z}}$ and $1 = (1_{L^i})_{i\in\mathbb{Z}}$ are cochain mappings of (L, d) into itself, and so are all their compact perturbations.

Cochain mappings $(m_i^{(1)})_{i\in\mathbb{Z}}$ and $(m_i^{(2)})_{i\in\mathbb{Z}}$ of (L, d) into $(\tilde{L}, \tilde{d})$ are said to be *homotopic* if there is a collection $h_i \in \mathcal{L}(L^i, \tilde{L}^{i-1})$, $i \in \mathbb{Z}$, such that $m_i^{(2)} - m_i^{(1)} \sim \tilde{d}_{i-1} h_i + h_{i+1} d_i$ for all $i \in \mathbb{Z}$.

The task is now to introduce the concept of a *Fredholm quasicomplex*. Recall that an operator $d \in \mathcal{L}(L, \tilde{L})$ in Fréchet spaces is Fredholm if and only if its image in the *Calkin algebra* $\mathcal{L}(L, \tilde{L})/\mathcal{K}(L, \tilde{L})$ is invertible. Thus, the idea is to pass in a given quasicomplex to quotients modulo spaces of compact operators and require exactness. To this end, we make use of a functor ϕ_Σ studied by Putinar [Put82].

For Fréchet spaces L and Σ, set $\phi_\Sigma(L) = \mathcal{L}(\Sigma, L)/\mathcal{K}(\Sigma, L)$. Moreover, given any $d \in \mathcal{L}(L, \tilde{L})$, we define $\phi_\Sigma(d) \in \mathcal{L}(\phi_\Sigma(L), \phi_\Sigma(\tilde{L}))$ by the formula

$$\phi_\Sigma(d)\,(m + \mathcal{K}(\Sigma, L)) = dm + \mathcal{K}(\Sigma, \tilde{L})$$

for $m \in \mathcal{L}(\Sigma, L)$. Clearly, this operator is well-defined. It is easily seen that $\phi_\Sigma(d_2 d_1) = \phi_\Sigma(d_2)\phi_\Sigma(d_1)$ for all $d_1 \in \mathcal{L}(L^1, L^2)$ and $d_2 \in \mathcal{L}(L^2, L^3)$. If 1_L is the identity operator on L, then $\phi_\Sigma(1_L)$ is the identity operator on $\phi_\Sigma(L)$. These remarks show that ϕ_Σ is a *covariant functor* in the category of

Fréchet spaces. The crucial fact is that ϕ_Σ vanishes on compact operators, for every Fréchet space Σ. Conversely, if $d \in \mathcal{L}(L, \tilde{L})$ and $\phi_\Sigma(d) = 0$ for any Fréchet space Σ, then $d \in \mathcal{K}(L, \tilde{L})$. Indeed, taking $\Sigma = L$, we deduce from the equality

$$\begin{aligned} \phi_L(d)\,(1_L + \mathcal{K}(L,L)) &= d + \mathcal{K}(L, \tilde{L}) \\ &= \mathcal{K}(L, \tilde{L}) \end{aligned}$$

that $d \in \mathcal{K}(L, \tilde{L})$.

Note that if (L, d) is an arbitrary quasicomplex, then $(\phi_\Sigma(L), \phi_\Sigma(d))$ is a complex, for each Fréchet space Σ. Thus, the functor ϕ_Σ transforms quasicomplexes into ordinary complexes. Furthermore, cochain mappings of quasicomplexes transform under ϕ_Σ into cochain mappings of complexes, and ϕ_Σ preserves the homotopy classes of cochain mappings.

Definition 5.1.2 *A quasicomplex (L, d) is called Fredholm if the complex $(\phi_\Sigma(L), \phi_\Sigma(d))$ is exact, for each Fréchet space Σ.*

Let (L, d) and $(L, \tilde{d})$ be two quasicomplexes, such that $d_i \sim \tilde{d}_i$ for all $i \in \mathbb{Z}$. Then the complexes $(\phi_\Sigma(L), \phi_\Sigma(d))$ and $(\phi_\Sigma(L), \phi_\Sigma(\tilde{d}))$ coincide, for every Fréchet space Σ. Therefore, (L, d) and $(L, \tilde{d})$ are simultaneously Fredholm. In other words, any compact perturbation of a Fredholm quasicomplex is a Fredholm quasicomplex.

Theorem 5.1.3 *A bounded above quasicomplex (L, d) is Fredholm if and only if the identity mapping of (L, d) is homotopic to the zero one.*

This theorem goes back at least as far as Putinar [Put82] wherein the designation 'essential complexes' is used for what we call 'quasicomplexes' here.

Proof. *Necessity.* Let (L, d) be Fredholm and bounded above, i.e., $L^i = 0$ for all but $i \le N$. Our goal is to show that there are operators $\pi_i \in \mathcal{L}(L^i, L^{i-1})$, $i \in \mathbb{Z}$, such that

$$d_{i-1}\pi_i + \pi_{i+1}d_i = 1_{L^i} - c_i \qquad (5.1.1)$$

for all $i \in \mathbb{Z}$, where $c_i \in \mathcal{K}(L^i)$.

Set $\pi_i = 0$ for all integers $i > N$. If $i = N$, then from the exactness of the complex $(\phi_\Sigma(L), \phi_\Sigma(d))$, $\Sigma = L^N$, at step N if follows that there is an operator $\pi_N \in \mathcal{L}(L^N, L^{N-1})$ such that $d_{N-1}\pi_N \sim 1_{L^N}$. Denoting by c_N the difference $1_{L^N} - d_{N-1}\pi_N$, we thus get $c_N \in \mathcal{K}(L^N)$.

We now proceed by induction. Suppose we have already found mappings

$$\begin{array}{llll} \pi_i, & \pi_{i+1}, & \ldots; \\ c_i, & c_{i+1}, & \ldots, \end{array}$$

such that the equality (5.1.1) is satisfied at steps $i, i+1, \ldots$, for some $i \le N$. Note that

$$\begin{array}{rcl} d_{i-1}\left(1_{L^{i-1}} - \pi_i d_{i-1}\right) & = & d_{i-1} - \left(1_{L^i} - c_i - \pi_{i+1} d_i\right) d_{i-1} \\ & = & c_i d_{i-1} + \pi_{i+1} d_i d_{i-1} \\ & \sim & 0 \end{array}$$

in virtue of (5.1.1). From the exactness of $(\phi_\Sigma(L), \phi_\Sigma(d))$, $\Sigma = L^{i-1}$, at step $i-1$ if follows that there exists an operator $\pi_{i-1} \in \mathcal{L}(L^{i-1}, L^{i-2})$ such that $d_{i-2}\pi_{i-1} \sim 1_{L^{i-1}} - \pi_i d_{i-1}$. Setting $c_{i-1} = 1_{L^{i-1}} - \pi_i d_{i-1} - d_{i-2}\pi_{i-1}$, we obtain $c_{i-1} \in \mathcal{K}(L^{i-1})$ and (5.1.1) fulfilled at step $i-1$. This establishes the existence of solutions π_i, c_i to (5.1.1) for each $i \in \mathbb{Z}$, i.e., the homotopy between the identity and zero cochain mappings of (L, d).

Sufficiency. If the identity mapping $1 = (1_{L^i})_{i\in\mathbb{Z}}$ is homotopic to the zero mapping $0 = (0_{L^i})_{i\in\mathbb{Z}}$ on (L, d), then the identity mapping on the cohomology $H^i(\phi_\Sigma(L), \phi_\Sigma(d))$ vanishes for all $i \in \mathbb{Z}$. In other words, the complex $(\phi_\Sigma(L), \phi_\Sigma(d))$ is exact for each Fréchet space Σ, as required.

□

Any solution $\pi_i \in \mathcal{L}(L^i, L^{i-1})$, $i \in \mathbb{Z}$, to (5.1.1) is called a *parametrix* of the quasicomplex (L, d). Thus, Theorem 5.1.3 just amounts to saying that a bounded above quasicomplex is Fredholm if and only if it possesses a parametrix. We will be mainly interested in those parametrices (L, π) which themselves are (chain) quasicomplexes. Obviously, a quasicomplex (L, π) is a parametrix of (L, d) if and only if (L, d) is a parametrix of (L, π).

A standard way of constructing parametrices for Fredholm quasicomplexes is to reduce the matter to a single Fredholm operator. Recall that an operator $d \colon L \to \tilde{L}$ between Hilbert spaces is Fredholm if and only if both Laplacians $d^*d \in \mathcal{L}(L)$, $dd^* \in \mathcal{L}(\tilde{L})$ are Fredholm operators, d^* being the Hilbert adjoint. In Fréchet spaces, the adjoint is no longer available, however, we can generalise this in the following way. In order that an operator $d \in \mathcal{L}(L, \tilde{L})$ be Fredholm it is necessary and sufficient that there be an operator $\delta \in \mathcal{L}(L, \tilde{L})$ such that both $\delta d \in \mathcal{L}(L)$ and $d\delta \in \mathcal{L}(\tilde{L})$ are Fredholm. Thus, δ substitutes the adjoint while δd and $d\delta$ substitute the Laplacians d^*d and dd^*, respectively. By choosing a suitable δ it is sometimes possible to attain either of δd and $d\delta$ to be of "good" structure. If g_0 is a left parametrix of δd, then $g_0\delta$ is a left parametrix of d, and if g_1 is a right parametrix of $d\delta$, then δg_1 is a right parametrix of d.

To extend this idea to quasicomplexes, the basic algebraic ingredient is the concept of an associated quasicomplex. More precisely, given two quasicomplexes

$$\begin{array}{lcccccccc} (L,d): & \ldots & \longrightarrow & L^{i-1} & \xrightarrow{d_{i-1}} & L^i & \xrightarrow{d_i} & \ldots, \\ (L,\delta): & \ldots & \longleftarrow & L^{i-1} & \xleftarrow{\delta_i} & L^i & \xleftarrow{\delta_{i+1}} & \ldots, \end{array} \qquad (5.1.2)$$

we say that (L, δ) is an *associated quasicomplex* for (L, d) if all the operators $\Delta_i = d_{i-1}\delta_i + \delta_{i+1}d_i$, $i \in \mathbb{Z}$, are Fredholm. The operators $\Delta_i \in \mathcal{L}(L^i)$,

$i \in \mathbb{Z}$, are obvious generalisations of the Laplace operators from Hodge theory. Vice versa, if (L, δ) is an associated quasicomplex for (L, d), then (L, d) is an associated quasicomplex for (L, δ). Since compact operators form an ideal in the algebra of all bounded operators and single Fredholm operators are stable under compact perturbations, we deduce at once that associated quasicomplexes survive under compact perturbations. Moreover, the availability of an associated quasicomplex is equivalent to the existence of a special parametrix.

Theorem 5.1.4 *In order that a quasicomplex (L, d) have a parametrix being a quasicomplex it is necessary and sufficient that (L, d) possess an associated complex.*

Proof. *Necessity.* Indeed, if a quasicomplex (L, π) is a parametrix of (L, d), then it is an associated quasicomplex for (L, d), too, which is clear from (5.1.1).

Sufficiency. This follows by the same method as in [AB67, Section 6]. Fix an associated quasicomplex (L, δ) for (L, d). By definition,

$$\Delta_i = d_{i-1}\delta_i + \delta_{i+1}d_i$$

is a Fredholm operator for each $i \in \mathbb{Z}$. Moreover, we have

$$\begin{array}{rcl} d_i\Delta_i & \sim & \Delta_{i+1}d_i, \\ \delta_i\Delta_i & \sim & \Delta_{i-1}\delta_i, \end{array} \tag{5.1.3}$$

so that the sequence of Δ_i defines an endomorphism of both quasicomplexes (L, d) and (L, δ) modulo compact operators. Now, we can find a parametrix $g_i \in \mathcal{L}(L^i)$ for Δ_i so that

$$\begin{array}{rcl} g_i\Delta_i & \sim & 1, \\ \Delta_i g_i & \sim & 1, \end{array} \tag{5.1.4}$$

for each $i \in \mathbb{Z}$. Multiplying (5.1.3) on the left by g_{i+1} and on the right by g_i, we then obtain

$$\begin{array}{rcl} g_{i+1}d_i & \sim & d_i g_i, \\ g_{i-1}\delta_i & \sim & \delta_i g_i \end{array} \tag{5.1.5}$$

for any $i \in \mathbb{Z}$. We now claim that $\pi_i = g_{i-1}\delta_i$ is the required parametrix for L. Indeed,

$$\begin{array}{rcl} \pi_i\pi_{i+1} & \sim & g_{i-1}g_{i-1}\delta_i\delta_{i+1} \\ & \sim & 0, \end{array}$$

the first relation being due to (5.1.5). On the other hand, invoking (5.1.5) and (5.1.4) we get

$$\begin{array}{rcl} d_{i-1}\pi_i + \pi_{i+1}d_i & = & d_{i-1}g_{i-1}\delta_i + g_i\delta_{i+1}d_i \\ & \sim & g_i d_{i-1}\delta_i + g_i\delta_{i+1}d_i \\ & = & g_i\Delta_i \\ & \sim & 1, \end{array}$$

and so $d_{i-1}\pi_i + \pi_{i+1}d_i = 1 - c_i$ with c_i compact. The proof is complete.

□

Theorem 5.1.3 shows, given any Fredholm quasicomplex (L, d), that if $f \in L^i$ satisfies $d_i f = 0$, then $f = c_i f + d_{i-1}\pi_i f$, where (L, π) is a parametrix for (L, d) as in (5.1.1). In other words the operator d_{i-1} has a right inverse π_i on $Z^i(L)$ modulo compact operators. However, since the compositions $d_i d_{i-1}$ need not vanish for a quasicomplex L, the range of d^{i-1} no longer lies in $Z^i(L)$. It follows that the usual cohomology does not make sense for L. The question on a proper substitute of the cohomology for quasicomplexes seems to be very subtle.

Example 5.1.5 Take an exact complex (L, d). Perturb the operator d_i in this complex by a compact operator $\Delta d_i \in \mathcal{L}(L^i, L^{i+1})$. What we obtain in this way, is a Fredholm quasicomplex, the operator at step i being $d_i + \Delta d_i$. Given $f \in L^i$, the equation $d_{i-1}u = f$ is solvable if and only if $d_i f = 0$. To achieve this equality for an f satisfying $(d_i + \Delta d_i)f = 0$, we must impose additional conditions on f guaranteeing that $\Delta d_i f = 0$. Hence it follows that the complement of $B^i(L)$ in the space of cocycles of the quasicomplex at step i is infinite-dimensional unless the operator Δd_i is of finite rank.

□

For quasicomplexes with curvatures of finite rank a reasonable substitute of the cohomology can be defined as follows. To have encompassed the ranges of d_{i-1}, let us look for a necessary condition of solvability of the inhomogeneous equation $d_{i-1}u = f$. If $f = d_{i-1}u$ for some $u \in L^{i-1}$, then $d_i f$ should belong to the image of L^{i-1} under the compact operator $d_i d_{i-1}$. Thus, for every $i \in \mathbb{Z}$, we introduce the space of *quasicocycles* $\tilde{Z}^i(L)$ of L at step i to consist of all $f \in L^i$ with the property that $d_i f \in d_i d_{i-1} L^{i-1}$. It is easy to see that $\tilde{Z}^i(L) = Z^i(L) + B^i(L)$, the sum being non-direct, and consequently $B^i(L) \subset \tilde{Z}^i(L)$ for each $i \in \mathbb{Z}$.

Lemma 5.1.6 *Let the curvature of L at step i is of finite rank. Then, $\tilde{Z}^i(L)$ is a closed subspace of L^i.*

Proof. As the operator $d_i d_{i-1}$ is of finite rank, there are elements $u_1, \ldots, u_J \in L^{i-1}$ such that $d_i d_{i-1} u_1, \ldots, d_i d_{i-1} u_J$ are linearly independent and span the range of $d_i d_{i-1}$. If $f \in \tilde{Z}^i(L)$, then $d_i f = \sum_{j=1}^{J} c_j\, d_i d_{i-1} u_j$, the coefficients $c_j \in \mathbb{C}$ depending continuously on f. Hence it follows that

$$f = f_0 + \sum_{j=1}^{J} c_j\, d_{i-1} u_j \qquad (5.1.6)$$

where $f_0 \in Z^i(L)$. The differentials $d_{i-1}u_1, \ldots, d_{i-1}u_J$ are linearly independent and $d\left(\sum_{j=1}^{J} c_j\, d_{i-1} u_j\right) \neq 0$ unless $c_1 = \ldots = c_J = 0$, since otherwise the system $d_i d_{i-1} u_1, \ldots, d_i d_{i-1} u_J$ be linearly dependent. Thus, the decomposition (5.1.6) induces a topological isomorphism

$$\tilde{Z}^i(L) \cong Z^i(L) \oplus \mathbb{C}^J$$

via $f \mapsto f_0 \oplus (c_1, \ldots, c_J)$, the space $Z^i(L) \oplus \mathbb{C}^J$ being equipped with the product topology. To complete the proof it remains to note that $Z^i(L)$ is a closed subspace of L^i, and consequently $Z^i(L) \oplus \mathbb{C}^J$ is complete.

□

We always regard $\tilde{Z}^i(L)$ and $B^i(L)$ as subspaces of L^i with the induced topology. The quotient

$$\begin{aligned} \tilde{H}^i(L) &= \frac{\tilde{Z}^i(L)}{B^i(L)} \\ &= \frac{Z^i(L) + B^i(L)}{B^i(L)} \end{aligned} \tag{5.1.7}$$

is said to be the *quasicohomology* of L at step i. It is endowed with the quotient topology. By Example 5.1.5, the quasicohomology of a Fredholm quasicomplex may be infinite-dimensional, if arbitrary compact curvature is allowed. However, any Fredholm quasicomplex whose curvature is of finite rank bears a finite-dimensional quasicohomology.

Theorem 5.1.7 *Suppose L is a Fredholm quasicomplex whose curvature at steps i and $i+1$ is of finite rank. Then,*

1) $\dim \tilde{H}^i(L) < \infty$*;*
2) $B^i(L)$ *is closed in* L^i*; and*
3) $\tilde{Z}^i(L)$ *is a topological direct factor of* L^i*.*

Proof. The proof is a slight modification of the proof of Proposition 6.5 in [AB67]. Let (L, π) be a parametrix for (L, π) so that (5.1.1) take place with $c_i \in \mathcal{L}(L^i)$ compact. For $u \in L^i$, we have

$$(1 - c_i - \pi_{i+1} d_i) u = d_{i-1} \pi_i u$$

whence $(1 - c_i - \pi_{i+1} d_i) L^i \subset B^i(L)$. In particular, $1 - c_i - \pi_{i+1} d_i$ restricts to a mapping $\tilde{Z}^i(L) \to \tilde{Z}^i(L)$.

We claim that the restriction of $c_i + \pi_{i+1} d_i$ is a compact operator in $\tilde{Z}^i(L)$. To prove this, pick a bounded set σ in $\tilde{Z}^i(L)$. By (5.1.6), there are bounded sets σ' in $Z^i(L)$ and σ'' in L^{i-1} such that $\sigma \subset \sigma' + d_{i-1}\sigma''$, the sum of two subsets in L^i being understood element-wise. Therefore, we have

$$\begin{aligned} (c_i + \pi_{i+1} d_i)\, \sigma &\subset c_i\, \sigma + \pi_{i+1} d_i\, (\sigma' + d_{i-1} \sigma'') \\ &\subset c_i\, \sigma + \pi_{i+1} d_i d_{i-1}\, \sigma'' \end{aligned}$$

because d_i vanishes on σ'. Since both c_i and $d_i d_{i-1}$ are compact operators, the image of σ under c_i and the image of σ'' under $\pi_{i+1} d_i d_{i-1}$ are relatively compact subsets of L^i, then so is their sum. We thus deduce that the image of σ under $c_i + \pi_{i+1} d_i$ lies in a relatively compact subset of L^i. It follows that this image is a relatively compact subset of $\tilde{Z}^i(L)$, for this subspace is closed by Lemma 5.1.6. Hence, $c_i + \pi_{i+1} d_i$ restricts to a compact operator in $\tilde{Z}^i(L)$.

We can now assert that $1 - c_i - \pi_{i+1} d_i$ is a Fredholm operator in $\tilde{Z}^i(L)$, and so $(1 - c_i - \pi_{i+1} d_i) \tilde{Z}^i(L)$ is closed and of finite codimension in $\tilde{Z}^i(L)$. The same is then true of $B^i(L)$ and, as $\tilde{Z}^i(L)$ is closed in L^i, we deduce 1) and 2).

To prove 3), we consider the mapping

$$T_i \colon \tilde{Z}^i(L) \oplus B^{i+1}(L) \to L^i$$

given by $T_i (f \oplus F) = f + \pi_{i+1} F$. Then $\mathrm{im}\, \mathrm{T_i} \supset \mathrm{im}\, (1 - \mathrm{c_i})$ and so $\mathrm{im}\, \mathrm{T_i}$ is closed and of finite codimension. Also we have

$$\begin{aligned} (1 - c_{i+1} - \pi_{i+2} d_{i+1}) F &= d_i \pi_{i+1} F \\ &= d_i T_i (f \oplus F) - d_i f \\ &= d_i T_i (f \oplus F) - \textstyle\sum_{j=1}^{J} c_j \, d_i d_{i-1} u_j, \end{aligned}$$

the last equality being a consequence of (5.1.6). Thus, $T_i (f \oplus F) = 0$ implies that the image of F under the mapping $1 - c_{i+1} - \pi_{i+2} d_{i+1}$ belongs to the finite-dimensional subspace $d_i d_{i-1} L^{i-1}$ of L^{i+1}. As the restriction of $1 - c_{i+1} - \pi_{i+2} d_{i+1}$ to $\tilde{Z}^{i+1}(L)$ is a Fredholm operator, we conclude that the space of all $F \in B^{i+1}(L)$ satisfying $T_i (f \oplus F) = 0$ is of finite dimension. However, $T_i (f \oplus F) = 0$ yields $f = -\pi_{i+1} F$ showing that the null-space of T_i is finite-dimensional. Thus, T_i is a Fredholm operator, and it induces an isomorphism (algebraic and hence topological because all spaces are Fréchet) of

$$\tilde{Z}^i(L) \oplus \frac{B^{i+1}(L)}{\pi_{i+1}^{-1} \left(\tilde{Z}^i(L) \right) \cap B^{i+1}(L)}$$

onto a closed subspace of finite codimension Σ^i in L^i. If ${\Sigma^i}^\perp$ denotes a complement of Σ^i, it follows that

$$\frac{\pi_{i+1} B^{i+1}(L)}{\tilde{Z}^i(L) \cap \pi_{i+1} B^{i+1}(L)} + {\Sigma^i}^\perp$$

is a complement of $\tilde{Z}^i(L)$ in L^i. This proves 3).

□

We now return to general Fredholm quasicomplexes. As mentioned above, any single Fredholm operator $d \in \mathcal{L}(L, \tilde{L})$ can be thought of as a Fredholm quasicomplex $0 \to L \stackrel{d}{\to} \tilde{L} \to 0$. Conversely, to each quasicomplex (L, d) possessing an associated quasicomplex there corresponds a sequence of Fredholm operators $\Delta_i = d_{i-1} \delta_i + \delta_{i+1} d_i$. They are not canonically defined by (L, d) itself and depend also on the choice of an associated quasicomplex

(L, δ). To avoid purely technical details related to infinite sums of Fréchet spaces, we restrict our attention to bounded quasicomplexes. Set $L = \oplus L^i$. The operator $\Delta = \oplus \Delta_i$ in L admits the factorisation $\Delta = (d + \delta)^2$ modulo compact operators, where

$$d = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & 0 & \dots \\ \dots & d_{i-1} & 0 & 0 & \dots \\ \dots & 0 & d_i & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \delta = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & \delta_i & 0 & \dots \\ \dots & 0 & 0 & \delta_{i+1} & \dots \\ \dots & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{5.1.8}$$

are continuous operators in L. Indeed, from the definition of a quasicomplex it follows that both d^2 and δ^2 vanish modulo compact operators, whence

$$\begin{aligned} (d + \delta)^2 &\sim d\delta + \delta d \\ &= \Delta \end{aligned}$$

as required. Since Δ is a Fredholm operator, so is $d + \delta$. Conversely, if $d + \delta$ is a Fredholm operators, then so is Δ. The index of $d + \delta$ is a half of the index of Δ, which is zero in many interesting cases, for instance, if δ is a parametrix of d. To get Fredholm operators of non-zero index, we split L into the sum

$$L = L^{\mathrm{even}} \oplus L^{\mathrm{odd}},$$

with $L^{\mathrm{even}} = \oplus L^{2i}$ and $L^{\mathrm{odd}} = \oplus L^{2i+1}$. Let now

$$\begin{aligned} (d + \delta)_e : \ & L^{\mathrm{even}} \to L^{\mathrm{odd}}, \\ (d + \delta)_o : \ & L^{\mathrm{odd}} \to L^{\mathrm{even}} \end{aligned}$$

denote the restrictions of $d + \delta$ to L^{even} and L^{odd}, respectively.

Lemma 5.1.8 *As defined above, the operators $(d + \delta)_e$ and $(d + \delta)_o$ are Fredholm, and*

$$\frac{1}{2} \operatorname{ind} \Delta = \operatorname{ind} (\mathrm{d} + \delta)_\mathrm{e} + \operatorname{ind} (\mathrm{d} + \delta)_\mathrm{o}.$$

Proof. Indeed, an easy computation shows that the operator $d + \delta$ is represented by the matrix

$$d + \delta = \begin{pmatrix} 0 & (d + \delta)_o \\ (d + \delta)_e & 0 \end{pmatrix}$$

with respect to the splitting $L = L^{\mathrm{even}} \oplus L^{\mathrm{odd}}$. According to the above remark,

$$(d + \delta)^2 = \begin{pmatrix} (d + \delta)_o (d + \delta)_e & 0 \\ 0 & (d + \delta)_e (d + \delta)_o \end{pmatrix}$$

coincides with Δ modulo compact operators. This make it obvious that both $(d+\delta)_e$ and $(d+\delta)_o$ are Fredholm operators. Furthermore, we deduce, by the logarithmic property of the index, that

$$\begin{aligned} \operatorname{ind}\Delta &= \operatorname{ind}(\mathrm{d}+\delta)_{\mathrm{o}}(\mathrm{d}+\delta)_{\mathrm{e}} + \operatorname{ind}(\mathrm{d}+\delta)_{\mathrm{e}}(\mathrm{d}+\delta)_{\mathrm{o}} \\ &= 2\left(\operatorname{ind}(\mathrm{d}+\delta)_{\mathrm{e}} + \operatorname{ind}(\mathrm{d}+\delta)_{\mathrm{o}}\right), \end{aligned}$$

and this is precisely the assertion of the lemma.

□

The operators $(d+\pi)_e$ and $(d+\pi)_o$ are actually Fredholm for *each* parametrix (L,π) of (L,d), not necessarily a quasicomplex. Indeed, we have $(d+\pi)^2 = 1+\delta^2$ modulo $\mathcal{K}(L)$. As (L,d) is bounded, we deduce easily that the operator δ is nilpotent, i.e., some power of δ vanishes. It follows that $1+\delta^2$ is invertible, and so

$$\begin{aligned} (1+\delta^2)^{-1}(d+\pi)^2 &= 1, \\ (d+\pi)^2(1+\delta^2)^{-1} &= 1 \end{aligned}$$

modulo $\mathcal{K}(L)$. Hence $d+\pi$ is a Fredholm operator on L, and analysis similar to that in the proof of Lemma 5.1.8 shows that both $(d+\pi)_e$ and $(d+\pi)_o$ are Fredholm.

Were (L,d) a Fredholm complex of Hilbert spaces with the associated complex (L,d^*), d_i^* being the adjoint of d_i, the Hodge theory would allow one to conclude that the index of $(d+d^*)_e$ is equal to the *Euler characteristic* of (L,d). This gives a hint on a reasonable definition of the Euler characteristic for Fredholm quasicomplexes, which might be $\chi(L,d) = \operatorname{ind}(\mathrm{d}+\delta)_{\mathrm{e}}$. What is still lacking is an explicit description of the associated complex (L,δ) to be used.

Theorem 5.1.9 *The index of $(d+\delta)_e$ does not depend on the particular choice of the associated quasicomplex (L,δ) with the property that $d_{i-1}\delta_i + \delta_{i+1}d_i$ is linearly homotopic to 1_{L^i} modulo compact operators, for all $i \in \mathbb{Z}$.*

Proof. We first prove a reduced form of the theorem that the index of $(d+\pi)_e$ is independent on the choice of a parametrix (L,π) for (L,d). Let $(L,\pi^{(0)})$ and $(L,\pi^{(1)})$ be any two parametrices of (L,d). For $t\in[0,1]$, set $\pi_i^{(t)} = (1-t)\pi_i^{(0)} + t\pi_i^{(1)}$. Then, $(L,\pi^{(t)})$ is easily seen to be a parametrix of (L,d), for any t. By the above, $(d+\pi^{(t)})_e$ is a continuous family of Fredholm operators on L, parametrised by $t\in[0,1]$. It follows that the index of $(d+\pi^{(t)})_e$ is independent of t. In particular, the indices of $(d+\pi^{(0)})_e$ and $(d+\pi^{(1)})_e$ coincide, which is our claim.

Let now (L,δ) be an arbitrary associated quasicomplex for (L,d), such that $(1-t)1_{L^i} + t\Delta_i$ is a Fredholm operator on L for all $t\in[0,1]$. Denote by $\pi_i = g_{i-1}\delta_i$, $i\in\mathbb{Z}$, a parametrix of (L,d), as is constructed in Theorem 5.1.4. We shall have established the theorem if we prove that the index of $(d+\delta)_e$ is equal to that of $(d+\pi)_e$. For this purpose, consider the family

$$\begin{aligned} \delta_i^{(t)} &= g_{i-1}\left((1-t)1_{L^i} + t\Delta_i\right)\delta_i \\ &\sim (1-t)\pi_i + t\delta_i \end{aligned}$$

of operators in $\mathcal{L}(L^i, L^{i-1})$, parametrised by $t \in [0,1]$. From (5.1.5) it follows that $\delta_i^{(t)}\delta_{i+1}^{(t)} \sim 0$ for each $i \in \mathbb{Z}$, i.e., $(L, \delta^{(t)})$ is a quasicomplex. Furthermore, all the operators

$$\begin{aligned} d_{i-1}\delta_i^{(t)} + \delta_{i+1}^{(t)} d_i &\sim (1-t)\,(d_{i-1}\pi_i + \pi_{i+1}d_i) + t\,(d_{i-1}\delta_i + \delta_{i+1}d_i) \\ &\sim (1-t)1_{L^i} + t\Delta_i \end{aligned}$$

are, by assumption, Fredholm, for any $t \in [0,1]$. This means that $(L, \delta^{(t)})$ is an associated quasicomplex for (L, d). According to Lemma 5.1.8, $(d + \delta^{(t)})_e$ is a continuous family of Fredholm operators on L, the parameter t varying over $[0,1]$. Hence the index of $(d + \delta^{(t)})_e$ is independent of $t \in [0,1]$, and the theorem follows.

□

As mentioned in the proof, any parametrix (L, π) of (L, d) meets the condition of Theorem 5.1.9. If (L, d) is a complex of Hilbert spaces, then the adjoint complex (L, d^*) satisfies the condition of Theorem 5.1.9. Indeed, in this case $\Delta_i = d_{i-1}d_{i-1}^* + d_i^* d_i$ is a non-negative Fredholm operator in L^i, hence $(1-t)1_{L^i} + t\Delta_i$ is Fredholm for all $t \in [0,1]$.

For a recent account of the theory of quasicomplexes of Banach spaces we refer the reader to Ambrozie and Vasilescu [AV95].

5.2 *Elliptic quasicomplexes*

We now want to study quasicomplexes of weighted Sobolev spaces over a compact manifold with edges (M, S), whose differentials belong to the algebra of edge problems on M we constructed in Chapter 4. Recall that the analysis always takes place on an associated stretched manifold $\mathcal{M}$ which is a compact C^∞ manifold with boundary. The boundary of $\mathcal{M}$ bears the structure of a fibre bundle over S whose typical fibre is a compact closed manifold X. Moreover the fibration of $\partial\mathcal{M}$ gives rise to a fibration of $\mathcal{M}$ in a collar neighbourhood of the boundary, whose typical fibre is the stretched cone $\bar{\mathbb{R}}_+ \times X$.

Let we be given sequences of vector bundles $(V^i)_{i\in\mathbb{Z}}$ and $(W^i)_{i\in\mathbb{Z}}$ over $\mathcal{M}$ and S, respectively. Assume that both V^i and W^i are zero for all but finitely many i. Suppose that there is a sequence of pseudodifferential operators $\mathcal{A}_i \in \Psi^{m_i}(\mathcal{M}; V^i, V^{i+1}; W^i, W^{i+1}; w_i)$ such that all compositions $\mathcal{A}_i \circ \mathcal{A}_{i-1}$ are well-defined. This means just the compatibility of weight data $w_i = (\gamma_i, \gamma_{i+1}, (-l, 0])$ in the sense that $\gamma_i = \gamma_{i-1} - m_{i-1}$ for all i, the exponent $\gamma = \gamma_0$ being arbitrary. Then $(\mathcal{A}_i)_{i\in\mathbb{Z}}$ can be gathered together to map as in the following sequence[1]:

$$(L, \mathcal{A}): \ \dots \longrightarrow \begin{array}{c} H^{s_{i-1},\gamma_{i-1}}(V^{i-1}) \\ \oplus \\ H^{s_{i-1}}(W^{i-1}) \end{array} \xrightarrow{\mathcal{A}_{i-1}} \begin{array}{c} H^{s_i,\gamma_i}(V^i) \\ \oplus \\ H^{s_i}(W^i) \end{array} \xrightarrow{\mathcal{A}_i} \dots, \qquad (5.2.1)$$

[1]For simplicity we denote in this chapter $H^{s,\gamma}(\mathcal{M}, V)$ by $H^{s,\gamma}(V)$, not to be confused with the weighted Sobolev space on V.

with $s_i = s_{i-1} - m_{i-1}$, the exponent $s_0 = s$ being chosen arbitrarily.

Recall that we have a distinguished class of "small" operators in the calculus on $\mathcal{M}$ that are smoothing Green operators with asymptotics. Thus, a sequence $(L, \mathcal{A})$ of edge problems on $\mathcal{M}$ is said to be a *quasicomplex* if

$$\mathcal{A}_i \mathcal{A}_{i-1} \in \Psi^{-\infty}(\mathcal{M}; \mathrm{V}^{i-1}, \mathrm{V}^{i+1}; \mathrm{W}^{i-1}, \mathrm{W}^{i+1}; \mathrm{w}_i \circ \mathrm{w}_{i-1}) \tag{5.2.2}$$

for all $i \in \mathbb{Z}$. By Theorem 4.1.8, the composition $\mathcal{A}_i \circ \mathcal{A}_{i-1}$ of any two operators in a quasicomplex is compact. Hence the definition of a quasicomplex of edge problems specifies that given in the preceding section.

Associated with the sequence (5.2.1) are the sequences of principal interior and edge symbols. The first of these is a sequence of bundle homomorphisms over the compressed cotangent bundle of $\mathcal{M}$, namely

$${}^b\sigma^m(L, \mathcal{A}) : \ \ldots \longrightarrow \pi^* V^{i-1} \stackrel{{}^b\sigma^{m_{i-1}}(\mathcal{A}_{i-1})}{\longrightarrow} \pi^* V^i \stackrel{{}^b\sigma^{m_i}(\mathcal{A}_i)}{\longrightarrow} \ldots \tag{5.2.3}$$

where $\pi : {}^bT^*\mathcal{M} \to \mathcal{M}$ stands for the canonical projection. The second sequence consists of homomorphisms of Hilbert bundles over the cotangent bundle of S,

$$\sigma^m_{\mathrm{edge}}(L, \mathcal{A}) : \ \ldots \longrightarrow \pi^* \begin{array}{c} H^{i-1} \otimes V_S^{i-1} \\ \oplus \\ W^{i-1} \end{array} \stackrel{\sigma^{m_{i-1}}_{\mathrm{edge}}(\mathcal{A}_{i-1})}{\longrightarrow} \pi^* \begin{array}{c} H^{i} \otimes V_S^{i} \\ \oplus \\ W^{i} \end{array} \stackrel{\sigma^{m_i}_{\mathrm{edge}}(\mathcal{A}_i)}{\longrightarrow} \ldots, \tag{5.2.4}$$

where $H^i = H^{s_i, \gamma_i}(F^{-1}(\cdot))$ and $\pi : T^*S \to S$ is the canonical projection.

If $(L, \mathcal{A})$ is a quasicomplex, then it follows from (5.2.2) and the multiplicative property of a symbol mapping that both (5.2.3) and (5.2.4) are complexes. Loosely speaking, the complex (5.2.3) controls the usual ellipticity on the "smooth" part of M up to the edges. On the other hand, the complex (5.2.4) controls the contribution of potential and trace conditions on S to compensate the lack of ellipticity in the directions normal to the edges.

Definition 5.2.1 *A quasicomplex* $(L, \mathcal{A})$ *is called elliptic if the associated symbol sequences* (5.2.3) *and* (5.2.4) *are exact apart from the zero sections of the corresponding cotangent bundles.*

A single elliptic edge problem $\mathcal{A} \in \Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{W}, \tilde{\mathrm{W}}; \mathrm{w})$ is a simple example of an elliptic quasicomplex. Further examples are given by geometric complexes on a manifold with edges to be discussed in a forthcoming paper.

The symbol sequence (5.2.3) is irrelevant to the particular choice of s and γ in (5.2.1). This is not the case for the symbol sequence (5.2.4) including both s and γ. The analysis of elliptic complexes on a manifold with conical singularities makes it obvious that the exactness of (5.2.4) depends essentially on γ (cf. Schulze [Sch88a], Melrose [Mel93], etc.) However, it is independent of the choice of s.

Lemma 5.2.2 *If the sequence of principal edge symbols* (5.2.4) *is exact for any one* $s = s_0$*, then it is exact for all* $s \in \mathbb{R}$.

Proof. We give the proof only for those quasicomplexes which are elliptic in the interior of $\mathcal{M}$ up to the boundary. This setting will be sufficient for our applications.

The proof is embedded in the context of elliptic complexes on the infinite stretched cone $\bar{\mathbb{R}}_+ \times X$ over a smooth compact closed manifold X. Indeed, if localised at a point $(y, \eta) \in T^*S$, $\eta \neq 0$, the sequence (5.2.4) becomes

$$\ldots \longrightarrow \begin{array}{c} H_y^{i-1} \otimes V_{S,y}^{i-1} \\ \oplus \\ W_y^{i-1} \end{array} \stackrel{\sigma_{\mathrm{edge}}^{m_{i-1}}(\mathcal{A}_{i-1})(y,\eta)}{\longrightarrow} \begin{array}{c} H_y^{i} \otimes V_{S,y}^{i} \\ \oplus \\ W_y^{i} \end{array} \stackrel{\sigma_{\mathrm{edge}}^{m_{i}}(\mathcal{A}_{i})(y,\eta)}{\longrightarrow} \ldots, \qquad (5.2.5)$$

where

$$\begin{array}{rcl} H_y^i & = & H^{s_i,\gamma_i}(F^{-1}(y)) \\ & = & H^{s_i,\gamma_i}(\bar{\mathbb{R}}_+ \times X). \end{array}$$

The fibres $V_{S,y}^i$ and W_y^i are finite-dimensional vector spaces whose dimensions k_i and l_i are the ranks of the bundles V_S^i and W^i, respectively. Choosing local frames for these bundles near y, we identify $V_{S,y}^i$ and W_y^i with $\mathbb{C}^{k_i}$ and $\mathbb{C}^{l_i}$, respectively.

For fixed $(y, \eta) \in T^*S \setminus \{0\}$, we denote by a_i the upper left corner of $\sigma_{\mathrm{edge}}^{m_i}(\mathcal{A}_i)(y, \eta)$. Under the above identification, this is an element of the algebra $\Psi^{m_i}(\bar{\mathbb{R}}_+ \times X; \mathbb{C}^{k_i}, \mathbb{C}^{k_{i+1}}; w_i)$ over the infinite stretched cone $\bar{\mathbb{R}}_+ \times X$, where we write $\mathbb{C}^{k_i}$ instead of $(\bar{\mathbb{R}}_+ \times X) \times \mathbb{C}^{k_i}$ for short. Since (5.2.5) is a complex, we conclude that

$$\ldots \longrightarrow H^{s_{i-1},\gamma_{i-1}}(\bar{\mathbb{R}}_+ \times X)^{k_{i-1}} \stackrel{a_{i-1}}{\longrightarrow} H^{s_i,\gamma_i}(\bar{\mathbb{R}}_+ \times X)^{k_i} \stackrel{a_i}{\longrightarrow} \ldots \qquad (5.2.6)$$

is a quasicomplex of Hilbert spaces, whose curvature is a Green operator of finite rank at each step $i \in \mathbb{Z}$.

It is easy to see that (5.2.5) is a Fredholm complex if and only if (5.2.6) is a Fredholm quasicomplex. In fact, if $(\pi_i)_{i\in\mathbb{Z}}$ is a parametrix for (5.2.6), then

$$\begin{pmatrix} \pi_i & 0 \\ 0 & 0 \end{pmatrix}_{i\in\mathbb{Z}} \qquad (5.2.7)$$

is a parametrix for (5.2.5).

The schema of the proof of the lemma is as follows. Suppose (5.2.5) is exact for some $s = s_0$. Then (5.2.6) is a Fredholm quasicomplex, for $s = s_0$. From this we deduce that (5.2.6) is actually an elliptic quasicomplex, the ellipticity referring to the algebra of pseudodifferential operators over $\bar{\mathbb{R}}_+ \times X$. It follows that (5.2.6) has a parametrix within the algebra. We can then transfer the parametrix to the complex (5.2.5), as explained above, and deduce from the mapping properties of this parametrix, by a standard procedure, that the cohomology of (5.2.5) is independent of the particular choice of s. Consequently, (5.2.5) is exact for all values of s.

Recall that elliptic quasicomplexes (5.2.6) in the algebra over $\bar{\mathbb{R}}_+ \times X$ are characterised by the following four symbol sequences

$$\begin{array}{ccccccc} \ldots & \longrightarrow & \mathbb{C}^{k_{i-1}} & \overset{{}^b\sigma^{m_{i-1}}_{\mathcal{F}_{t,x}}(a_{i-1})}{\longrightarrow} & \mathbb{C}^{k_i} & \overset{{}^b\sigma^{m_i}_{\mathcal{F}_{t,x}}(a_i)}{\longrightarrow} & \ldots, \\ \ldots & \longrightarrow & H^{s_{i-1}}(X)^{k_{i-1}} & \overset{\sigma_{\mathcal{M}}(a_{i-1})(z+i(\gamma-\gamma_{i-1}))}{\longrightarrow} & H^{s_i}(X)^{k_i} & \overset{\sigma_{\mathcal{M}}(a_i)(z+i(\gamma-\gamma_i))}{\longrightarrow} & \ldots, \\ \ldots & \longrightarrow & \mathbb{C}^{k_{i-1}} & \overset{\sigma^0_{\mathcal{F}_t,\mathrm{exit}}(a_{i-1})}{\longrightarrow} & \mathbb{C}^{k_i} & \overset{\sigma^0_{\mathcal{F}_t,\mathrm{exit}}(a_i)}{\longrightarrow} & \ldots, \\ \ldots & \longrightarrow & \mathbb{C}^{k_{i-1}} & \overset{\sigma^{m_{i-1}}_{\mathcal{F}_t}\sigma^0_{\mathcal{F}_t,\mathrm{exit}}(a_{i-1})}{\longrightarrow} & \mathbb{C}^{k_i} & \overset{\sigma^{m_i}_{\mathcal{F}_t}\sigma^0_{\mathcal{F}_t,\mathrm{exit}}(a_i)}{\longrightarrow} & \ldots, \end{array} \tag{5.2.8}$$

the first of these being exact over ${}^bT^*(\bar{\mathbb{R}}_+ \times X) \setminus \{0\}$, the second one being exact for all $z \in \Gamma_{-\gamma}$, the third one being exact over all of $T^*(\mathbb{R}_+ \times X)$, and the fourth one being exact away from the zero section of $T^*(\mathbb{R}_+ \times X)$. We thus see that merely the complex of conormal symbols in (5.2.8) includes s via $s_i = s - m_0 - \ldots - m_{i-1}$. On the other hand, the exactness of the other three complexes in (5.2.8) follows automatically from the exactness of (5.2.3) in much the same way as in the proof of Theorem 4.1.2.

Note that the differentials of the second complex in (5.2.8) are matrices of classical pseudodifferential operators on X, a C^∞ compact closed manifold. From the compatibility of the principal interior and conormal symbols we see that this complex is elliptic whenever the first complex (5.2.8) is exact for all $(0, x; \tau, \xi) \in {}^bT^*(\bar{\mathbb{R}}_+ \times X) \setminus \{0\}$. Hence it follows that the cohomology of the second complex (5.2.8) is independent on the Sobolev spaces where it is evaluated, i.e., on s.

For a fuller treatment of elliptic complexes on a manifold with conical points we refer the reader to Section 3.1 in Schulze [Sch88a]. The only fact from this theory we need here is that the exactness of the complex of conormal symbols is necessary for the Fredholm property of (5.2.6) for any one $s = s_0$.

We are now in a position to finish the proof. Let (5.2.5) be exact for some $s \in \mathbb{R}$. Then, (5.2.6) is an elliptic quasicomplex. Consequently, (5.2.6) possesses a parametrix within the calculus, i.e., there is a sequence

$$\pi_i \in \Psi^{-m_{i-1}}(\bar{\mathbb{R}}_+ \times X; \mathbb{C}^{k_i}, \mathbb{C}^{k_{i-1}}; w_{i-1}^{-1}), \quad i \in \mathbb{Z},$$

such that

$$a_{i-1}\pi_i + \pi_{i+1}a_i = 1 \mod \Psi^{-\infty}(\bar{\mathbb{R}}_+ \times X; \mathbb{C}^{k_i}; w_i^{-1} \circ w_i),$$

for all i. Now (5.2.7) yields a parametrix of (5.2.5) which behaves properly in the weighted Sobolev spaces on $\bar{\mathbb{R}}_+ \times X$. This completes the proof.

□

Let us apply cochain mappings of quasicomplexes to reduce the weights in the weighted Sobolev spaces entered into (5.2.1). To this end, fix, for each $i \in \mathbb{Z}$, a pair of elliptic edge problems on $\mathcal{M}$,

$$\begin{array}{rcl} \mathcal{R}_i & \in & \Psi^{\gamma_i}(\mathcal{M}; \mathrm{V}^i, \tilde{\mathrm{V}}^i; \mathrm{W}^i, \tilde{\mathrm{W}}^i; (\gamma_i, 0, (-1, 0])), \\ \mathcal{R}_i^{(-1)} & \in & \Psi^{-\gamma_i}(\mathcal{M}; \tilde{\mathrm{V}}^i, \mathrm{V}^i; \tilde{\mathrm{W}}^i, \mathrm{W}^i; (0, \gamma_i, (-1, 0])), \end{array}$$

such that $\mathcal{R}_i^{(-1)}$ is a parametrix of $\mathcal{R}_i$. The existence of a large number of such pairs is guaranteed by Theorems 4.4.4 and 4.2.2. Set

$$\tilde{\mathcal{A}}_i = \mathcal{R}_{i+1} \mathcal{A}_i \mathcal{R}_i^{(-1)} \tag{5.2.9}$$

and consider the diagram

$$\begin{array}{ccccccc} (L, \mathcal{A}): \ \ldots \longrightarrow & \begin{array}{c} H^{s_{i-1},\gamma_{i-1}}(V^{i-1}) \\ \oplus \\ H^{s_{i-1}}(W^{i-1}) \end{array} & \xrightarrow{\mathcal{A}_{i-1}} & \begin{array}{c} H^{s_i,\gamma_i}(V^i) \\ \oplus \\ H^{s_i}(W^i) \end{array} & \xrightarrow{\mathcal{A}_i} & \ldots \\ \mathcal{R}^{(-1)} \Big\uparrow \Big\downarrow \mathcal{R} & \mathcal{R}_{i-1}^{(-1)} \Big\uparrow \Big\downarrow \mathcal{R}_{i-1} & & \mathcal{R}_i^{(-1)} \Big\uparrow \Big\downarrow \mathcal{R}_i & & \\ (\tilde{L}, \tilde{\mathcal{A}}): \ \ldots \longrightarrow & \begin{array}{c} H^{s-\gamma,0}(\tilde{V}^{i-1}) \\ \oplus \\ H^{s-\gamma}(\tilde{W}^{i-1}) \end{array} & \xrightarrow{\tilde{\mathcal{A}}_{i-1}} & \begin{array}{c} H^{s-\gamma,0}(\tilde{V}^i) \\ \oplus \\ H^{s-\gamma}(\tilde{W}^i) \end{array} & \xrightarrow{\tilde{\mathcal{A}}_i} & \ldots . \end{array} \tag{5.2.10}$$

The operators $\tilde{\mathcal{A}}_i$ are defined so as to make this diagram commutative modulo smoothing Green operators on $\mathcal{M}$.

Theorem 5.2.3 *As defined above, the sequence* $(\tilde{L}, \tilde{\mathcal{A}})$ *is an elliptic quasicomplex. Moreover, the diagram* (5.2.10) *establishes an isomorphism of quasicomplexes* $(L, \mathcal{A})$ *and* $(\tilde{L}, \tilde{\mathcal{A}})$ *modulo smoothing Green operators.*

Proof. From (5.2.9) we conclude that

$$\tilde{\mathcal{A}}_i \in \Psi^0(\mathcal{M}; \tilde{\mathrm{V}}^i, \tilde{\mathrm{V}}^{i+1}; \tilde{\mathrm{W}}^i, \tilde{\mathrm{W}}^{i+1}; (0, 0, (-1, 0]))$$

and

$$\tilde{\mathcal{A}}_i \tilde{\mathcal{A}}_{i-1} = \mathcal{R}_{i+1} \mathcal{A}_i \mathcal{A}_{i-1} \mathcal{R}_{i-1}^{(-1)} + \mathcal{R}_{i+1} \mathcal{A}_i \left(\mathcal{R}_i^{(-1)} \mathcal{R}_i - 1 \right) \mathcal{A}_{i-1} \mathcal{R}_{i-1}^{(-1)},$$

which makes it obvious that $(\tilde{L}, \tilde{\mathcal{A}})$ is a quasicomplex with entries in the pseudodifferential calculus on $\mathcal{M}$. Moreover, combining Theorem 4.1.9 with equalities (4.2.3) yields

$$\begin{array}{rcl} {}^b\sigma^0(\tilde{\mathcal{A}}_i) & = & {}^b\sigma^{\gamma_{i+1}}(\mathcal{R}_{i+1}) {}^b\sigma^{m_i}(\mathcal{A}_i) \left({}^b\sigma^{\gamma_i}(\mathcal{R}_i) \right)^{-1}, \\ \sigma^0_{\mathrm{edge}}(\tilde{\mathcal{A}}_i) & = & \sigma^{\gamma_{i+1}}_{\mathrm{edge}}(\mathcal{R}_{i+1}) \sigma^{m_i}_{\mathrm{edge}}(\mathcal{A}_i) \left(\sigma^{\gamma_i}_{\mathrm{edge}}(\mathcal{R}_i) \right)^{-1}, \end{array}$$

and so a trivial verification shows that the quasicomplex $(\tilde{L}, \tilde{\mathcal{A}})$ is elliptic.

The quasicomplexes $(L, \mathcal{A})$ and $(\tilde{L}, \tilde{\mathcal{A}})$ are actually isomorphic modulo smoothing Green operators. Indeed, multiplying (5.2.9) on the right by $\mathcal{R}_i$, we obtain $\tilde{\mathcal{A}}_i \mathcal{R}_i = \mathcal{R}_{i+1} \mathcal{A}_i$ modulo smoothing Green operators, for each $i \in \mathbb{Z}$, showing that $\mathcal{R} = (\mathcal{R}_i)_{i \in \mathbb{Z}}$ is a cochain mapping of $(L, \mathcal{A})$ into $(\tilde{L}, \tilde{\mathcal{A}})$. The inverse of this mapping modulo smoothing Green operators is given by $\mathcal{R}^{(-1)} = \left(\mathcal{R}_i^{(-1)} \right)_{i \in \mathbb{Z}}$, and the proof is complete.

□

Our next concern will be the Fredholm property of elliptic quasicomplexes and the parametrix construction within the pseudodifferential calculus on $\mathcal{M}$. By a *parametrix* of a quasicomplex $(L, \mathcal{A})$ we mean any sequence of edge problems $\mathcal{P}_i \in \Psi^{-\mathrm{m}_{\mathrm{i}-1}}(\mathcal{M}; \mathrm{V}^{\mathrm{i}}, \mathrm{V}^{\mathrm{i}-1}; \mathrm{W}^{\mathrm{i}}, \mathrm{W}^{\mathrm{i}-1}; \mathrm{w}_{\mathrm{i}-1}^{-1})$, $i \in \mathbb{Z}$, such that

$$\mathcal{A}_{i-1} \mathcal{P}_i + \mathcal{P}_{i+1} \mathcal{A}_i = 1 - \mathcal{S}_i \tag{5.2.11}$$

for all $i \in \mathbb{Z}$, where $\mathcal{S}_i \in \Psi^{-\infty}(\mathcal{M}; \mathrm{V}^{\mathrm{i}}; \mathrm{W}^{\mathrm{i}}; \mathrm{w}_{\mathrm{i}}^{-1} \circ \mathrm{w}_{\mathrm{i}})$. In other words, $(\mathcal{P}_i)_{i \in \mathbb{Z}}$ is a *cochain homotopy* between the identity and a smoothing Green endomorphisms of $(L, \mathcal{A})$.

Corollary 5.2.4 *Let $\tilde{\mathcal{P}}_i = \mathcal{R}_{i-1} \mathcal{P}_i \mathcal{R}_i^{(-1)}$, for $i \in \mathbb{Z}$. Then, $(\mathcal{P}_i)_{i \in \mathbb{Z}}$ is a parametrix of $(L, \mathcal{A})$ if and only if $\left(\tilde{\mathcal{P}}_i \right)_{i \in \mathbb{Z}}$ is a parametrix of $(\tilde{L}, \tilde{\mathcal{A}})$.*

Proof. Indeed,

$$\begin{aligned} \tilde{\mathcal{A}}_{i-1} \tilde{\mathcal{P}}_i + \tilde{\mathcal{P}}_{i+1} \tilde{\mathcal{A}}_i &= \mathcal{R}_i \left(\mathcal{A}_{i-1} \mathcal{R}_{i-1}^{(-1)} \mathcal{R}_{i-1} \mathcal{P}_i + \mathcal{P}_{i+1} \mathcal{R}_{i+1}^{(-1)} \mathcal{R}_{i+1} \mathcal{A}_i \right) \mathcal{R}_i^{(-1)} \\ &= \mathcal{R}_i \left(\mathcal{A}_{i-1} \mathcal{P}_i + \mathcal{P}_{i+1} \mathcal{A}_i \right) \mathcal{R}_i^{(-1)}, \end{aligned}$$

the last equality being modulo smoothing Green operators. As $\mathcal{R}_i$ is invertible modulo smoothing Green operators, the inverse being $\mathcal{R}_i^{(-1)}$, this proves the corollary.

□

We are left with the task of constructing parametrices for those elliptic quasicomplexes (5.2.1) whose differentials are given by operators of order zero on $\mathcal{M}$. The advantage of using such a reduction lies in the fact that we can invoke the sequence of Laplacians for the study of $(\tilde{L}, \tilde{\mathcal{A}})$. For general quasicomplexes on $\mathcal{M}$, the Laplacians are not available within the calculus because the formal adjoint acts in the weighted Sobolev spaces of opposite weights. Thus, it can be identified with the adjoint in the sense of Hilbert spaces only for operators of order zero acting in Sobolev spaces of weight zero.

Lemma 5.2.5 *A quasicomplex $(\tilde{L}, \tilde{\mathcal{A}})$ of edge problems of order zero is elliptic if and only if the Laplacian*

$$\tilde{\Delta}_i = \tilde{\mathcal{A}}_{i-1} \tilde{\mathcal{A}}_{i-1}^* + \tilde{\mathcal{A}}_i^* \tilde{\mathcal{A}}_i \tag{5.2.12}$$

is an elliptic operator in $\Psi^0(\mathcal{M}; \tilde{\mathrm{V}}^{\mathrm{i}}; \tilde{\mathrm{W}}^{\mathrm{i}}; (0, 0, (-1, 0]))$ at each step $i \in \mathbb{Z}$.

Proof. Indeed, the ellipticity of $\tilde{\Delta}_i$ means that every of the principal symbols

$$\begin{array}{rcl} {}^b\sigma^0(\tilde{\Delta}_i) & : & \pi^*\tilde{V}^i \to \pi^*\tilde{V}^i, \\ \sigma^0_{\mathrm{edge}}(\tilde{\Delta}_i) & : & \pi^* \begin{array}{c} H^{s,0}(F^{-1}(\cdot)) \otimes \tilde{V}^i_S \\ \oplus \\ \tilde{W}^i \end{array} \to \pi^* \begin{array}{c} H^{s,0}(F^{-1}(\cdot)) \otimes \tilde{V}^i_S \\ \oplus \\ \tilde{W}^i \end{array} \end{array}$$

is an isomorphism away from the zero section of the corresponding cotangent bundle. Moreover, being an isomorphism for the edge symbol is independent on the particular choice of s; we take $s = 0$. By Theorem 4.1.10, we get

$$\begin{array}{rcl} {}^b\sigma^0(\tilde{\Delta}_i) & = & {}^b\sigma^0(\tilde{\mathcal{A}}_{i-1}) \left({}^b\sigma^0(\tilde{\mathcal{A}}_{i-1})\right)^* + \left({}^b\sigma^0(\tilde{\mathcal{A}}_i)\right)^* {}^b\sigma^0(\tilde{\mathcal{A}}_i), \\ \sigma^0_{\mathrm{edge}}(\tilde{\Delta}_i) & = & \sigma^0_{\mathrm{edge}}(\tilde{\mathcal{A}}_{i-1}) \left(\sigma^0_{\mathrm{edge}}(\tilde{\mathcal{A}}_{i-1})\right)^* + \left(\sigma^0_{\mathrm{edge}}(\tilde{\mathcal{A}}_i)\right)^* \sigma^0_{\mathrm{edge}}(\tilde{\mathcal{A}}_i), \end{array} \quad (5.2.13)$$

the 'asterisk' indicating the formal adjoint in the cone algebra. Further, Lemma 5.2.2 shows that the complex of principal edge symbols $\sigma^0_{\mathrm{edge}}(\tilde{L}, \tilde{\mathcal{A}})$ can be equally evaluated in the spaces

$$\pi^* \begin{array}{c} H^{0,0}(F^{-1}(\cdot)) \otimes \tilde{V}^i_S \\ \oplus \\ \tilde{W}^i \end{array}.$$

Under this choice, the formal adjoints $\left(\sigma^0_{\mathrm{edge}}(\tilde{\mathcal{A}}_i)\right)^*$ in (5.2.13) coincide with the adjoints in the sense of Hilbert spaces. Now the required conclusion is a consequence of the well-known algebraic fact that for a complex of Hilbert spaces

$$\ldots \to L^{i-1} \stackrel{d_{i-1}}{\to} L^i \stackrel{d_i}{\to} \ldots$$

to be exact at step i it is necessary and sufficient that $\Delta_i = d_{i-1}d^*_{i-1} + d^*_i d_i$ be an isomorphism of L^i.

□

The principal significance of the lemma is in the assertion that the formal adjoint of an elliptic quasicomplex of edge problems of order zero on $\mathcal{M}$ is an associated quasicomplex. Thus, we can adopt the schema of Atiyah and Bott [AB67] to construct a parametrix for $(\tilde{L}, \tilde{\mathcal{A}})$.

Theorem 5.2.6 *Each elliptic quasicomplex of edge problems on $\mathcal{M}$ has a parametrix being a quasicomplex.*

Proof. Combining Theorem 5.2.3 and Lemma 5.2.5 we conclude that all the Laplacians (5.2.12) are elliptic operators. Since $\tilde{\mathcal{A}}_i\tilde{\mathcal{A}}_{i-1}$ is a smoothing Green operator, so is $\tilde{\mathcal{A}}^*_{i-1}\tilde{\mathcal{A}}^*_i = (\tilde{\mathcal{A}}_i\tilde{\mathcal{A}}_{i-1})^*$, for each $i \in \mathbb{Z}$. Therefore, we get

$$\begin{array}{rcl} \tilde{\mathcal{A}}_i\tilde{\Delta}_i & = & \tilde{\Delta}_{i+1}\tilde{\mathcal{A}}_i, \\ \tilde{\mathcal{A}}^*_{i-1}\tilde{\Delta}_i & = & \tilde{\Delta}_{i-1}\tilde{\mathcal{A}}^*_{i-1} \end{array} \quad (5.2.14)$$

modulo smoothing Green operators. In other words, the sequence of $\tilde{\Delta}_i$ defines an endomorphism of both $(\tilde{L}, \tilde{\mathcal{A}})$ and $(\tilde{L}, \tilde{\mathcal{A}}^*)$ modulo smoothing Green operators. By Theorem 4.2.2, we can find a parametrix

$$\tilde{\mathcal{G}}_i \in \Psi^0(\mathcal{M}; \tilde{\mathrm{V}}^{\mathrm{i}}; \tilde{\mathrm{W}}^{\mathrm{i}}; (0, 0, (-1, 0]))$$

for $\tilde{\Delta}_i$, so that

$$\begin{array}{rcl} \tilde{\mathcal{G}}_i \tilde{\Delta}_i - 1 & \in & \Psi^{-\infty}(\mathcal{M}; \tilde{\mathrm{V}}^{\mathrm{i}}; \tilde{\mathrm{W}}^{\mathrm{i}}; (0, 0, (-1, 0])), \\ \tilde{\Delta}_i \tilde{\mathcal{G}}_i - 1 & \in & \Psi^{-\infty}(\mathcal{M}; \tilde{\mathrm{V}}^{\mathrm{i}}; \tilde{\mathrm{W}}^{\mathrm{i}}; (0, 0, (-1, 0])), \end{array} \tag{5.2.15}$$

for each $i \in \mathbb{Z}$. Multiplying (5.2.14) on the left by $\tilde{\mathcal{G}}_{i+1}$ and on the right by $\tilde{\mathcal{G}}_i$, we obtain

$$\begin{array}{rcl} \tilde{\mathcal{G}}_{i+1} \tilde{\mathcal{A}}_i & = & \tilde{\mathcal{A}}_i \tilde{\mathcal{G}}_i, \\ \tilde{\mathcal{G}}_{i-1} \tilde{\mathcal{A}}^*_{i-1} & = & \tilde{\mathcal{A}}^*_{i-1} \tilde{\mathcal{G}}_i \end{array} \tag{5.2.16}$$

modulo smoothing Green operators, for any $i \in \mathbb{Z}$. We now claim that $\tilde{\mathcal{P}}_i = \tilde{\mathcal{G}}_{i-1} \tilde{\mathcal{A}}^*_{i-1}$ is a parametrix for $(\tilde{L}, \tilde{\mathcal{A}})$ which is actually a quasicomplex. Indeed,

$$\begin{array}{rcl} \tilde{\mathcal{P}}_i \tilde{\mathcal{P}}_{i+1} & = & \tilde{\mathcal{G}}_{i-1} \tilde{\mathcal{G}}_{i-1} \tilde{\mathcal{A}}^*_{i-1} \tilde{\mathcal{A}}^*_i \\ & = & 0 \end{array}$$

modulo smoothing Green operators, the first relation being due to (5.2.16). On the other hand, invoking (5.2.16) and (5.2.15) we get

$$\begin{array}{rcl} \tilde{\mathcal{A}}_{i-1} \tilde{\mathcal{P}}_i + \tilde{\mathcal{P}}_{i+1} \tilde{\mathcal{A}}_i & = & \tilde{\mathcal{A}}_{i-1} \tilde{\mathcal{G}}_{i-1} \tilde{\mathcal{A}}^*_{i-1} + \tilde{\mathcal{G}}_i \tilde{\mathcal{A}}^*_i \tilde{\mathcal{A}}_i \\ & = & \tilde{\mathcal{G}}_i \tilde{\mathcal{A}}_{i-1} \tilde{\mathcal{A}}^*_{i-1} + \tilde{\mathcal{G}}_i \tilde{\mathcal{A}}^*_i \tilde{\mathcal{A}}_i \\ & = & \tilde{\mathcal{G}}_i \tilde{\Delta}_i \\ & = & 1 \end{array}$$

modulo smoothing Green operators, as required.

To complete the proof it suffices to use Corollary 5.2.4 according to which $\mathcal{P}_i = \mathcal{R}^{(-1)}_{i-1} \tilde{\mathcal{P}}_i \mathcal{R}_i$, $i \in \mathbb{Z}$, is exactly what we need.

□

Theorems 5.2.6 and 5.1.3 may be summarised by saying that every elliptic quasicomplex is Fredholm. The method of proof carries over to those sequences (5.2.1) which meet a weaker condition than (5.2.2), namely

$$\begin{array}{rcl} {}^b\sigma^{m_i + m_{i-1}}(\mathcal{A}_i \mathcal{A}_{i-1}) & = & 0, \\ \sigma^{m_i + m_{i-1}}_{\mathrm{edge}}(\mathcal{A}_i \mathcal{A}_{i-1}) & = & 0 \end{array}$$

for all i. By Theorem 4.1.8, these are still quasicomplexes in the sense of Definition 5.1.1. The concept of ellipticity still applies to such sequences, and an analogue of Theorem 4.1.8 gives a parametrix within the calculus, now merely modulo operators of order -1.

The question arises of whether the ellipticity is necessary for the Fredholm property of a quasicomplex of edge problems on $\mathcal{M}$.

Theorem 5.2.7 *For $s = \gamma + \frac{1+n}{2}$, the Fredholm property of* (5.2.1) *implies the ellipticity of this quasicomplex.*

Proof. The proof of the theorem lies beyond the range of this paper. We only give the main ideas.

Let $(L, \mathcal{A})$ be a Fredholm quasicomplex of edge problems on $\mathcal{M}$ as in (5.2.1), with $s = \gamma + \frac{1+n}{2}$. Starting with complexes of principal symbols (5.2.3) and (5.2.4), we construct a complex $(L, \tilde{\mathcal{A}})$ of edge problems on $\mathcal{M}$, such that

$$\begin{array}{rcl} {}^b\sigma^{m_i}(\tilde{\mathcal{A}}_i) & = & {}^b\sigma^{m_i}(\mathcal{A}_i), \\ \sigma^{m_i}_{\text{edge}}(\tilde{\mathcal{A}}_i) & = & \sigma^{m_i}_{\text{edge}}(\mathcal{A}_i) \end{array}$$

for each $i \in \mathbb{Z}$. As $\tilde{\mathcal{A}}_i$ and $\mathcal{A}_i$ differ by compact operators, we deduce at once that the complex $(L, \tilde{\mathcal{A}})$ is Fredholm.

To reduce $(L, \tilde{\mathcal{A}})$ to a complex of edge problems of order zero, we make use of a diagram similar to (5.2.10). Namely, we take as $\mathcal{R}_i$ the order reductions ensured by Theorem 4.4.7, $\mathcal{R}_i$ being of order $\gamma + \frac{1+n}{2}$. Then the complex at the bottom of (5.2.10) corresponds to the smoothness 0 and the weight $-\frac{1+n}{2}$.

We now set $H^{0,-\frac{1+n}{2}}(\bar{\mathbb{R}}_+ \times X)$ to be the reference space, so that the dual spaces and formal adjoints will be defined with respect to the scalar product in $H^{0,-\frac{1+n}{2}}(\bar{\mathbb{R}}_+ \times X)$. Under this shift of weights, we can introduce the Laplacians (5.2.12) to act in the spaces of weight $-\frac{1+n}{2}$. For the reduced complex, the Fredholm property proves to be equivalent to the Fredholm property of all the Laplacians.

For single edge problems of order 0 in $H^{0,-\frac{1+n}{2}}(\mathcal{M}, V^i) \oplus H^0(S, W^i)$, the Fredholm property just amounts to the ellipticity. By Lemma 5.2.5 and Theorem 5.2.3 we can then assert that $(L, \tilde{\mathcal{A}})$ is an elliptic complex.

Finally, since the sequences of principal symbols are the same for $(L, \mathcal{A})$ and $(L, \tilde{\mathcal{A}})$, it follows that the quasicomplex $(L, \mathcal{A})$ is elliptic, too. This is the desired conclusion.

□

We finish this section by deriving an interesting formula for solutions of the homogeneous edge problems $\mathcal{A}_i u = 0$ on $\mathcal{M}$. Namely, if (5.2.1) is an elliptic quasicomplex and $u \in H^{s,\gamma_i}(V^i) \oplus H^s(W^i)$ satisfies $\mathcal{A}_i u = 0$, then

$$u = \mathcal{S}_i u + \mathcal{A}_{i-1}\mathcal{P}_i u$$

by (5.2.11), where $\mathcal{S}_i u \in H^{\infty,\gamma_i}_{\text{as}}(V^i) \oplus H^\infty(W^i)$ for some asymptotic type 'as' subordinate to $(\gamma_i, (-l, 0])$, and $\mathcal{P}_i u \in H^{s+m_{i-1},\gamma_{i-1}}(V^{i-1}) \oplus H^{s+m_{i-1}}(W^{i-1})$.

5.3 *Hodge theory*

Let us now establish an analogue of the Hodge decomposition for the de Rham complex (cf. [Hod41]) in the case of elliptic complexes of edge problems on $\mathcal{M}$.

We begin with some generalities on complexes of Hilbert spaces, refining the material of Section 5.1. For such a complex (L, d), there is always a natural choice of an associated complex, namely, the adjoint complex:

$$\begin{array}{llllllll} (L,d): & \ldots & \longrightarrow & L^{i-1} & \xrightarrow{d_{i-1}} & L^i & \xrightarrow{d_i} & \ldots, \\ (L,d^*): & \ldots & \longleftarrow & L^{i-1} & \xleftarrow{d^*_{i-1}} & L^i & \xleftarrow{d^*_i} & \ldots . \end{array} \tag{5.3.1}$$

Lemma 5.3.1 *For a complex (L, d) of Hilbert spaces, the following conditions are equivalent:*

1) (L, d) is a Fredholm complex;

2) (L, d^*) is a Fredholm complex;

3) the Laplacians $\Delta_i = d_{i-1}d^*_{i-1} + d^*_i d_i$ are Fredholm operators in L^i, for $i \in \mathbb{Z}$.

Proof. See Theorem 4 in Rempel and Schulze [RS82a, 3.2.3.1], and elsewhere.

□

An important point to note here is the selfadjointness of the Laplacians Δ_i in L^i, $i \in \mathbb{Z}$. Thus, either they are both injective and surjective or possess neither of these properties. It is easy to see that the null-space of Δ_i consists of all $u \in L^i$ such that $d_i u = 0$ and $d^*_{i-1}u = 0$.

Theorem 5.3.2 *Suppose that (L, d) is a Fredholm complex of Hilbert spaces. Then, for each $i \in \mathbb{Z}$, the null-space of Δ_i is finite-dimensional and we have an orthogonal decomposition*

$$L^i = \ker \Delta_i \oplus d_{i-1}d^*_{i-1}g_i L^i \oplus d^*_i d_i g_i L^i. \tag{5.3.2}$$

Proof. Fix $i \in \mathbb{Z}$. According to Lemma 5.3.1, Δ_i is a Fredholm operator in L^i, hence $\ker \Delta_i$ is finite-dimensional. Denote by $(\ker \Delta_i)^\perp$ the orthogonal complement of $\ker \Delta_i$ in L^i. As Δ_i is a selfadjoint Fredholm operator in L^i, it follows that Δ_i restricts to a topological isomorphism of $(\ker \Delta_i)^\perp$. Letting h_i stand for the orthogonal projection of L^i onto $\ker \Delta_i$, we introduce a bounded operator

$$g_i = \left(\Delta_i \,|_{(\ker \Delta_i)^\perp}\right)^{-1} (1_{L^i} - h_i)$$

in L^i. Then we get $1_{L^i} - h_i = \Delta_i g_i$, which is precisely the decomposition (5.3.2).

□

The operators $g_i \in \mathcal{L}(L^i)$ in (5.3.2) have the property that $d_i g_i = g_{i+1} d_i$ for all $i \in \mathbb{Z}$. Indeed, applying (5.3.2) at steps i and $i+1$, we obtain easily $d_i = d_i d^*_i d_i g_i$ and $d_i = d_i d^*_i g_{i+1} d_i$, whence $\Delta_{i+1}(d_i g_i - g_{i+1} d_i) = 0$. However, the operator $d_i g_i - g_{i+1} d_i$ maps to $(\ker \Delta_{i+1})^\perp$, hence $d_i g_i - g_{i+1} d_i = 0$, as required. Thus, the equalities (5.3.2) can be written equivalently in the form

$$d_{i-1}\left(d^*_{i-1} g_i\right) + \left(d^*_i g_{i+1}\right) d_i = 1_{L^i} - h_i,$$

for $i \in \mathbb{Z}$, showing that $\pi_i = d^*_{i-1} g_i$, $i \in \mathbb{Z}$, is a very special parametrix for (L, d).

Corollary 5.3.3 *For any Fredholm complex of Hilbert spaces (L, d), we have topological isomorphisms*

$$H^i(L, d) \cong \ker \Delta_i, \quad i \in \mathbb{Z}. \tag{5.3.3}$$

Proof. Indeed, the desired isomorphisms are given by $[u] \mapsto h_i u$ where $[u]$ means the class in $H^i(L, d)$ of a cocycle $u \in Z^i(L, d)$.

□

When compared with the abstract Hodge decomposition (5.3.2), such a decomposition for elliptic complexes of edge problems bears more information on the operators g_i.

Let $(L, \mathcal{A})$ be an elliptic complex within the calculus on $\mathcal{M}$, as in (5.2.1) but $\mathcal{A}_i \mathcal{A}_{i-1} = 0$ for all $i \in \mathbb{Z}$. Pick, for each $i \in \mathbb{Z}$, an order reduction $\mathcal{R}_i = \mathcal{R}^{\gamma_i}_{V^i, W^i}$ of $\Psi^{\gamma_i}(\mathcal{M}; \mathrm{V}^i; \mathrm{W}^i; (\gamma_i, 0, (-1, 0]))$, as is guaranteed by Theorem 4.4.7. Consider the diagram

$$\begin{array}{ccccccc}
(L, \mathcal{A}) : \ \ldots \longrightarrow & \begin{array}{c} H^{s_{i-1}, \gamma_{i-1}}(V^{i-1}) \\ \oplus \\ H^{s_{i-1}}(W^{i-1}) \end{array} & \stackrel{\mathcal{A}_{i-1}}{\longrightarrow} & \begin{array}{c} H^{s_i, \gamma_i}(V^i) \\ \oplus \\ H^{s_i}(W^i) \end{array} & \stackrel{\mathcal{A}_i}{\longrightarrow} & \ldots \\
\mathcal{R}^{-1} \Big\uparrow \Big\downarrow \mathcal{R} & \mathcal{R}^{-1}_{i-1} \Big\uparrow \Big\downarrow \mathcal{R}_{i-1} & & \mathcal{R}^{-1}_i \Big\uparrow \Big\downarrow \mathcal{R}_i & & \\
(L, \tilde{\mathcal{A}}) : \ \ldots \longrightarrow & \begin{array}{c} H^{s-\gamma, 0}(V^{i-1}) \\ \oplus \\ H^{s-\gamma}(W^{i-1}) \end{array} & \stackrel{\tilde{\mathcal{A}}_{i-1}}{\longrightarrow} & \begin{array}{c} H^{s-\gamma, 0}(V^i) \\ \oplus \\ H^{s-\gamma}(W^i) \end{array} & \stackrel{\tilde{\mathcal{A}}_i}{\longrightarrow} & \ldots .
\end{array} \tag{5.3.4}$$

the operators $\tilde{\mathcal{A}}_i$ being defined so as to make this diagram commutative, i.e., $\tilde{\mathcal{A}}_i = \mathcal{R}_{i+1} \mathcal{A}_i \mathcal{R}_i^{-1}$.

When compared with (5.2.10), the commutativity relations in (5.3.4) hold precisely, not only modulo smoothing Green operators. It follows that $(L, \tilde{\mathcal{A}})$ is an elliptic complex of edge problems of order zero on $\mathcal{M}$. This allows one to make use of the Laplacians $\tilde{\Delta}_i$, $i \in \mathbb{Z}$, defined by (5.2.12).

Theorem 5.3.4 *Suppose $(L, \mathcal{A})$ is an elliptic complex of edge problems on $\mathcal{M}$. Then, to every $i \in \mathbb{Z}$ there corresponds operators*

$$\begin{array}{rcl}
\mathcal{H}_i & \in & \Psi^{-\infty}(\mathcal{M}; \mathrm{V}^i; \mathrm{W}^i; \mathrm{w}_i^{-1} \circ \mathrm{w}_i), \\
\mathcal{P}_i & \in & \Psi^{-\mathrm{m}_{i-1}}(\mathcal{M}; \mathrm{V}^i, \mathrm{V}^{i-1}; \mathrm{W}^i, \mathrm{W}^{i-1}; \mathrm{w}_{i-1}^{-1})
\end{array}$$

such that

1) *$\mathcal{H}_i$ is a projection of finite rank, satisfying $\mathcal{A}_i \mathcal{H}_i = 0$, $\mathcal{H}_i \mathcal{A}_{i-1} = 0$;*

2) *for each $u \in H^{s, \gamma_i}(V^i) \oplus H^s(W^i)$ with $s \geq \gamma_i$, we have*

$$u = \mathcal{H}_i u + \mathcal{A}_{i-1} \mathcal{P}_i u + \mathcal{P}_{i+1} \mathcal{A}_i u, \tag{5.3.5}$$

the decomposition being orthogonal with respect to the scalar product $(u, v) \mapsto (\mathcal{R}_i u, \mathcal{R}_i v)_{H^{0,0}(V^i) \oplus H^0(W^i)}$.

Proof. Since $\tilde{\mathcal{A}}_i \in \Psi^0(\mathcal{M}; \mathrm{V^i}, \mathrm{V^{i+1}}; \mathrm{W^i}, \mathrm{W^{i+1}}; (0, 0, (-\mathrm{l}, 0]))$, they map as in the following sequence:

$$(\tilde{L}, \tilde{\mathcal{A}}) : \ \ldots \longrightarrow \begin{array}{c} H^{0,0}(V^{i-1}) \\ \oplus \\ H^0(W^{i-1}) \end{array} \xrightarrow{\tilde{\mathcal{A}}_{i-1}} \begin{array}{c} H^{0,0}(V^{i}) \\ \oplus \\ H^0(W^{i}) \end{array} \xrightarrow{\tilde{\mathcal{A}}_{i}} \ldots . \tag{5.3.6}$$

From Lemma 5.2.2 and the ellipticity of the complex $(L, \tilde{\mathcal{A}})$ in (5.3.4) we conclude that (5.3.6) is an elliptic complex. Lemma 5.2.5 now shows that the Laplacians $\tilde{\Delta}_i = \tilde{\mathcal{A}}_{i-1}\tilde{\mathcal{A}}_{i-1}^* + \tilde{\mathcal{A}}_i^*\tilde{\mathcal{A}}_i$ restrict to elliptic operators in $H^{0,0}(V^i) \oplus H^0(W^i)$. Moreover, when mapping $H^{0,0}(V^i) \oplus H^0(W^i)$ to $H^{0,0}(V^{i-1}) \oplus H^0(W^{i-1})$, the formal adjoint $\tilde{\mathcal{A}}_{i-1}^*$ coincides with the adjoint of $\tilde{\mathcal{A}}_{i-1}$ in the sense of Hilbert spaces. Hence it follows that $\tilde{\Delta}_i$ is a selfadjoint operator in $H^{0,0}(V^i) \oplus H^0(W^i)$, for each $i \in \mathbb{Z}$.

By the above, the null-space of $\tilde{\Delta}_i$ consists of all $u \in H^{0,0}(V^i) \oplus H^0(W^i)$ satisfying both $\tilde{\mathcal{A}}_i u = 0$ and $\tilde{\mathcal{A}}_{i-1}^* u = 0$. Moreover, since $\tilde{\Delta}_i$ is elliptic, $\ker \tilde{\Delta}_i$ is a finite-dimensional subspace of $H^{0,0}_{\mathrm{as}}(V^i) \oplus H^0(W^i)$, for some asymptotic type 'as' subordinate to the weight data $(0, (-l, 0])$. Denote by $\tilde{\mathcal{H}}_i$ the orthogonal projection of $H^{0,0}(V^i) \oplus H^0(W^i)$ onto the null-space of $\tilde{\Delta}_i$. If $\left(u_\nu^{(i)}\right)$ is an orthonormal basis for $\ker \tilde{\Delta}_i$, ν varying over a finite set of indices, then $\tilde{\mathcal{H}}_i$ is an integral operator with the kernel $\sum_\nu u_\nu^{(i)} \otimes \star u_\nu^{(i)}$, where '$\star$' is a Hodge star operator associated to the scalar product of $H^{0,0}(V^i) \oplus H^0(W^i)$. From this we deduce immediately that $\tilde{\mathcal{H}}_i$ is a smoothing Green operator, i.e.,

$$\tilde{\mathcal{H}}_i \in \Psi^{-\infty}(\mathcal{M}; \mathrm{V^i}; \mathrm{W^i}; (0, 0, (-\mathrm{l}, 0])). \tag{5.3.7}$$

Let us write $(\ker \tilde{\Delta}_i)^\perp$ for the orthogonal complement of $\ker \tilde{\Delta}_i$ in $H^{0,0}(V^i) \oplus H^0(W^i)$. Then, the operator $\tilde{\Delta}_i$ restricts to a topological isomorphism of $(\ker \tilde{\Delta}_i)^\perp$, hence

$$\tilde{G}_i = \left(\tilde{\Delta}_i \left|_{(\ker \tilde{\Delta}_i)^\perp}\right.\right)^{-1} (1 - \tilde{\mathcal{H}}_i)$$

is a bounded operator in $H^{0,0}(V^i) \oplus H^0(W^i)$. It is clear from the definition that

$$\begin{aligned} 1 - \tilde{\mathcal{H}}_i &= \tilde{\Delta}_i \tilde{G}_i \\ &= \tilde{G}_i \tilde{\Delta}_i \end{aligned} \tag{5.3.8}$$

on $H^{0,0}(V^i) \oplus H^0(W^i)$.

Our next concern will be the regularity property of $\tilde{G}_i$. To this end, we first observe that if $u \in H^{0,0}(V^i) \oplus H^0(W^i)$, then $\tilde{\Delta}_i \tilde{G}_i u = u - \tilde{\mathcal{H}}_i u$, where $\tilde{\mathcal{H}}_i u$ lies, by (5.3.7), in $H^{\infty,0}_{\mathrm{as}}(V^i) \oplus H^\infty(W^i)$. We now invoke Theorem 4.3.5 and Corollary 4.3.6 on elliptic regularity, for $\tilde{\Delta}_i$, to conclude that $\tilde{G}_i$ preserves $H^{\infty,0}(V^i) \oplus H^\infty(W^i)$ and the corresponding spaces with asymptotics.

We next fix any one parametrix $\tilde{\mathcal{G}}_i \in \Psi^0(\mathcal{M}; \mathrm{V}^{\mathrm{i}}; \mathrm{W}^{\mathrm{i}}; (0,0,(-1,0]))$ for $\tilde{\Delta}_i$, i.e.,

$$\begin{array}{rcl} \tilde{\mathcal{G}}_i\tilde{\Delta}_i - 1 & \in & \Psi^{-\infty}(\mathcal{M}; \mathrm{V}^{\mathrm{i}}; \mathrm{W}^{\mathrm{i}}; (0,0,(-1,0])), \\ \tilde{\Delta}_i\tilde{\mathcal{G}}_i - 1 & \in & \Psi^{-\infty}(\mathcal{M}; \mathrm{V}^{\mathrm{i}}; \mathrm{W}^{\mathrm{i}}; (0,0,(-1,0])) \end{array}$$

(cf. (5.2.15)). By (5.3.8), we have

$$\left(\tilde{G}_i - \tilde{\mathcal{G}}_i\right)\tilde{\Delta}_i = \left(1 - \tilde{\mathcal{G}}_i\tilde{\Delta}_i\right) - \tilde{\mathcal{H}}_i$$

on $H^{0,0}(V^i) \oplus H^0(W^i)$. Applying the operator $\tilde{\mathcal{G}}_i$ to this equality from the right, we get

$$\left(\tilde{G}_i - \tilde{\mathcal{G}}_i\right) = \left(\tilde{G}_i - \tilde{\mathcal{G}}_i\right)\left(1 - \tilde{\Delta}_i\tilde{\mathcal{G}}_i\right) + \left(1 - \tilde{\mathcal{G}}_i\tilde{\Delta}_i\right)\tilde{\mathcal{G}}_i - \tilde{\mathcal{H}}_i\tilde{\mathcal{G}}_i$$

showing that $\tilde{G}_i$ and $\tilde{\mathcal{G}}_i$ differ by a smoothing Green operator. We can thus assert that

$$\tilde{G}_i \in \Psi^0(\mathcal{M}; \mathrm{V}^{\mathrm{i}}; \mathrm{W}^{\mathrm{i}}; (0,0,(-1,0])). \tag{5.3.9}$$

From what has been proved we see that each $u \in H^{0,0}(V^i) \oplus H^0(W^i)$ splits into

$$u = \tilde{\mathcal{H}}_i u + \tilde{\mathcal{A}}_{i-1}\tilde{\mathcal{A}}_{i-1}^*\tilde{G}_i u + \tilde{\mathcal{A}}_i^*\tilde{\mathcal{A}}_i\tilde{G}_i u, \tag{5.3.10}$$

the decomposition being orthogonal with respect to the scalar product in $H^{0,0}(V^i) \oplus H^0(W^i)$.

We claim that $\tilde{\mathcal{A}}_i\tilde{G}_i = \tilde{G}_{i+1}\tilde{\mathcal{A}}_i$. Indeed, if $u \in H^{0,0}(V^i) \oplus H^0(W^i)$ satisfies $\tilde{\mathcal{A}}_i u = 0$, then $\tilde{\mathcal{A}}_i\tilde{\mathcal{A}}_i^*\tilde{\mathcal{A}}_i\tilde{G}_i u = 0$. Hence it easily follows that $\tilde{\mathcal{A}}_i\tilde{G}_i u = 0$. Thus, for arbitrary $u \in H^{0,0}(V^i) \oplus H^0(W^i)$, we have $\tilde{\mathcal{A}}_i u = \tilde{\mathcal{A}}_i\tilde{\mathcal{A}}_i^*\tilde{\mathcal{A}}_i\tilde{G}_i u$ on the one hand, and $\tilde{\mathcal{A}}_i u = \tilde{\mathcal{A}}_i\tilde{\mathcal{A}}_i^*\tilde{G}_{i+1}\tilde{\mathcal{A}}_i u$ on the other. Hence it follows that $\tilde{\Delta}_i(\tilde{\mathcal{A}}_i\tilde{G}_i u - \tilde{G}_{i+1}\tilde{\mathcal{A}}_i u) = 0$, and since $\tilde{\mathcal{A}}_i\tilde{G}_i u - \tilde{G}_{i+1}\tilde{\mathcal{A}}_i u$ is orthogonal to $\ker\tilde{\Delta}_{i+1}$ then $\tilde{\mathcal{A}}_i\tilde{G}_i u - \tilde{G}_{i+1}\tilde{\mathcal{A}}_i u = 0$, as required. Thus, (5.3.10) results in

$$u = \tilde{\mathcal{H}}_i u + \tilde{\mathcal{A}}_{i-1}\tilde{\mathcal{P}}_i u + \tilde{\mathcal{P}}_{i+1}\tilde{\mathcal{A}}_i u$$

for all $u \in H^{0,0}(V^i) \oplus H^0(W^i)$, where $\tilde{\mathcal{P}}_i = \tilde{\mathcal{A}}_{i-1}^*\tilde{G}_i$. Note that the summands on the right are still orthogonal with respect to the scalar product in $H^{0,0}(V^i) \oplus H^0(W^i)$.

We are now able to finish the proof. Let $u \in H^{s,\gamma_i}(V^i) \oplus H^s(W^i)$ where $s \geq \gamma_i$. Then $\mathcal{R}_i u \in H^{0,0}(V^i) \oplus H^0(W^i)$ whence

$$\mathcal{R}_i u = \tilde{\mathcal{H}}_i\mathcal{R}_i u + \tilde{\mathcal{A}}_{i-1}\tilde{\mathcal{P}}_i\mathcal{R}_i u + \tilde{\mathcal{P}}_{i+1}\tilde{\mathcal{A}}_i\mathcal{R}_i u.$$

Applying the inverse $\mathcal{R}_i^{-1}$ from the left and substituting $\tilde{\mathcal{A}}_i = \mathcal{R}_{i+1}\mathcal{A}_i\mathcal{R}_i^{-1}$, we arrive at the decomposition (5.3.5), with

$$\begin{array}{rcl} \mathcal{H}_i & = & \mathcal{R}_i^{-1}\tilde{\mathcal{H}}_i\mathcal{R}_i, \\ \mathcal{P}_i & = & (\mathcal{R}_{i-1}^*\mathcal{R}_{i-1})^{-1}\mathcal{A}_{i-1}^*\mathcal{R}_i^*\tilde{G}_i\mathcal{R}_i \end{array}$$

having the desired properties because of (5.3.7) and (5.3.9). The proof is complete. □

The principal interest of the theorem is that it allows one to represent the cohomology classes at step i for the complex $(L, \mathcal{A})$ by elements of the invariant subspace of the projection $\mathcal{H}_i$.

Corollary 5.3.5 *For any elliptic complex $(L, \mathcal{A})$ on $\mathcal{M}$, the correspondence $[u] \mapsto \mathcal{H}_i u$ induces a topological isomorphism of $H^i(L, \mathcal{A})$ onto the invariant subspace of the projection $\mathcal{H}_i$.*

Proof. Indeed, from item 1) of Theorem 5.3.4 it follows that the correspondence $[u] \mapsto \mathcal{H}_i u$ is a well-defined continuous mapping of $H^i(L, \mathcal{A})$ onto the invariant subspace of the projection $\mathcal{H}_i$. If $\mathcal{H}_i u = 0$ for some $u \in Z^i(L, \mathcal{A})$, then $u = \mathcal{A}_{i-1} \mathcal{P}_i u$, and so $[u] = 0$. This proves the injectivity of the mapping $[u] \mapsto \mathcal{H}_i u$. On the other hand, the surjectivity of this mapping is obvious because the invariant subspace of $\mathcal{H}_i$ belongs to $Z^i(L, \mathcal{A})$. To complete the proof, it suffices to observe that the inverse of $[u] \mapsto \mathcal{H}_i u$ is given by the canonical mapping of the invariant subspace of $\mathcal{H}_i$ to $H^i(L, \mathcal{A})$. □

In particular, the cohomology of an elliptic complex of edge problems on $\mathcal{M}$ is independent of the *smoothness* of the weighted Sobolev spaces where it is evaluated (but does depend on the *weights*).

5.4 *External multiplication*

Given any two Hilbert spaces H and L, we denote by $H \otimes L$ the *Hilbert tensor product* of H and L, i.e., the Hilbert space generated by the formal products $v \otimes u$, where $v \in H$ and $u \in L$, satisfying the bilinearity relations and equipped with the scalar product $(v_1 \otimes u_1, v_2 \otimes u_2) = (v_1, v_2)(u_1, u_2)$. For operators $\partial \in \mathcal{L}(H, \tilde{H})$ and $d \in \mathcal{L}(L, \tilde{L})$, the Hilbert tensor product $\partial \otimes d \in \mathcal{L}(H \otimes L, \tilde{H} \otimes \tilde{L})$ is uniquely determined via its restriction to elements of the form $v \otimes u$, where $v \in H$ and $u \in L$, the restriction being $(\partial \otimes d)(v \otimes u) = \partial v \otimes du$. If both ∂ and d are of trace class, then so is $\partial \otimes d$ and $\operatorname{tr} \partial \otimes \mathrm{d} = \operatorname{tr} \partial \operatorname{tr} \mathrm{d}$. However, the Fredholm property does not survive under tensor multiplication of operators, as is easy to see. To cope with this drawback and gain the multiplicative property of the index, one uses the following construction. Let us identify the operators ∂ and d with the complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \stackrel{\partial}{\longrightarrow} & \tilde{H} & \longrightarrow & 0, \\ 0 & \longrightarrow & L & \stackrel{d}{\longrightarrow} & \tilde{L} & \longrightarrow & 0 \end{array}$$

and define the *Hilbert tensor product* of these by

$$0 \longrightarrow H \otimes L \stackrel{D_0}{\longrightarrow} (\tilde{H} \otimes L) \oplus (H \otimes \tilde{L}) \stackrel{D_1}{\longrightarrow} \tilde{H} \otimes \tilde{L} \longrightarrow 0, \qquad (5.4.1)$$

where

$$D_0 = \begin{pmatrix} \partial \otimes 1 \\ 1 \otimes d \end{pmatrix}, \qquad D_1 = \begin{pmatrix} -1 \otimes d & \partial \otimes 1 \end{pmatrix}.$$

It is a simple matter to see that $D_1 D_0 = 0$, i.e., (5.4.1) is a complex of Hilbert spaces.

Lemma 5.4.1 *If both* $\partial \in \mathcal{L}(H, \tilde{H})$ *and* $d \in \mathcal{L}(L, \tilde{L})$ *are Fredholm, then so is* (5.4.1). *Moreover, the Euler characteristic of this complex is equal to* $\operatorname{ind}\partial \operatorname{ind} \mathrm{d}$.

Proof. Indeed, the Laplacians of complex (5.4.1) are easily verified to be

$$\begin{aligned} D_0^* D_0 &= \partial^*\partial \otimes 1 + 1 \otimes d^* d, \\ D_0 D_0^* + D_1^* D_1 &= \begin{pmatrix} \partial\partial^* \otimes 1 + 1 \otimes d^* d & 0 \\ 0 & \partial^*\partial \otimes 1 + 1 \otimes dd^* \end{pmatrix}, \\ D_1 D_1^* &= \partial\partial^* \otimes 1 + 1 \otimes dd^*. \end{aligned}$$

The null-space of $\partial^*\partial \otimes 1 + 1 \otimes d^* d$ is spanned by elements of the form $v \otimes u$, where $v \in \ker \partial$ and $u \in \ker d$, and similarly for the other diagonal elements of the Laplacians. Hence it follows that

$$\begin{aligned} \dim\ker D_0^* D_0 &= \dim\ker\partial \, \dim\ker d, \\ \dim\ker(D_0 D_0^* + D_1^* D_1) &= \dim\ker\partial^* \, \dim\ker d + \dim\ker\partial \, \dim\ker d^*, \\ \dim\ker D_1 D_1^* &= \dim\ker\partial^* \, \dim\ker d^*, \end{aligned}$$

and so the Euler characteristic of (5.4.1) amounts to

$$\dim\ker D_0^* D_0 - \dim\ker(D_0 D_0^* + D_1^* D_1) + \dim\ker D_1 D_1^* = \operatorname{ind}\partial \operatorname{ind} \mathrm{d},$$

as required.

□

By Lemma 5.1.8, the Fredholm property of complex (5.4.1) is equivalent to the Fredholm property of the operator $(D + D^*)_e$. A trivial verification yields

$$(D + D^*)_e = \begin{pmatrix} \partial \otimes 1 & -1 \otimes d^* \\ 1 \otimes d & \partial^* \otimes 1 \end{pmatrix} \qquad (5.4.2)$$

mapping as $(H \otimes L) \oplus (\tilde{H} \otimes \tilde{L}) \to (\tilde{H} \otimes L) \oplus (H \otimes \tilde{L})$. The operator on the right-hand side of (5.4.2) is sometimes referred to as the #-product of ∂ and d. Thus, if ∂ and d are Fredholm operators, then so is $\partial \# d$ and $\operatorname{ind}\partial\#\mathrm{d} = \operatorname{ind}\partial \operatorname{ind}\mathrm{d}$. Note that the product $\partial \# d$ is commutative and distributive with respect to the direct sum of factors modulo homotopies. Moreover, if any one of the factors is invertible, then so is the product.

Lemma 5.4.1 is a very particular case of a Künneth formula for topological tensor products of complexes of Fréchet spaces. This formula goes back to the work of Grothendieck [Gro54] who treated the case of nuclear Fréchet spaces. In the context of Hilbert tensor products a Künneth formula was proved by Grosu and Vasilescu [GV82]. Recently Mantlik [Man95] developed a product theory including a Künneth formula for complexes of so-called *hilbertisable* Fréchet spaces, a notion adapted from Jarchow [Jar81]. For this purpose the topology τ used on the tensor product $H \otimes L$ of two locally convex (l.c.) spaces should fulfil at least two requirements. Namely, it has to be

- "*injective*", i.e., for any subspace Σ of H , the topology of $\Sigma \otimes_\tau L$ coincides with that induced by $H \otimes_\tau L$;

- "*projective*", i.e., for any subspace Σ of H, the canonical quotient mapping $H \otimes_\tau L \to (H/\Sigma) \otimes_\tau L$ is open.

It is known, e.g., that the π-tensor topology is projective but not injective (cf. Floret [Flo73], Stegall and Retherford [SR72]) whereas the ε-tensor topology is injective but not projective (cf. Kaballo [Kab77]). Recall that the π- and ε-topologies coincide in the presence of nuclearity, which explains the result of Grothendieck [Gro54].

Definition 5.4.2 *Let H be a Hausdorff l.c. space. By $\chi(H)$ we denote the system of all continuous Hermitean forms on H. For $h \in \chi(H)$, let $|u|_h = \sqrt{h(u,u)}$ stand for the corresponding seminorm. We call H hilbertisable if the topology of H is generated by the system $\{|\cdot|_h : \ h \in \chi(H)\}$.*

In other words, H is hilbertisable if and only if the completion $\hat{H}$ is representable as a reduced projective limit of Hilbert spaces. This class of spaces includes, e.g., the distribution spaces like $H^s_{\mathrm{loc}}(V)$ and $H^s_{\mathrm{comp}}(V)$, V being a vector bundle over a smooth manifold M, further all nuclear Fréchet spaces, and, of course, the Hilbert spaces. The property of being hilbertisable is inherited by subspaces, various products, Hausdorff quotients, completions and countable inductive limits (cf. [Jar81, 11.9, 21.1]). Under suitable conditions also the strong dual H'_β of a hilbertisable space H is hilbertisable; this is the case, e.g., if H is metrisable (cf. Hollstein [Hol85]).

Suppose H and L are fixed hilbertisable spaces. Our goal is to introduce a natural hilbertisable topology σ on the algebraic tensor product $H \otimes L$. For any $h \in \chi(H)$ and $g \in \chi(L)$, we define a Hermitean form $h \otimes g$ on $H \otimes L$ by

$$h \otimes g \left(\sum_j v'_j \otimes u'_j, \sum_k v''_k \otimes u''_k \right) = \sum_{j,k} h\left(v'_j, v''_k\right) g\left(u'_j, v''_k\right),$$

the summations being over $j = 1, \ldots, J$ and $k = 1, \ldots, K$. It follows that $h \otimes g \in \chi(H \otimes L)$, and we denote by $H \otimes_\sigma L$ the space $H \otimes L$ with the topology generated by the system of Hermitean forms $\{h \otimes g : \ h \in \chi(H), g \in \chi(L)\}$. One can check that the corresponding system of seminorms $|\cdot|_{h\otimes g}$, where $h \in \chi(H)$ and $g \in \chi(L)$, is fundamental, i.e., each continuous seminorm on $H \otimes_\sigma L$ is majorised by some $|\cdot|_{h\otimes g}$.

By Proposition 2.3 of Mantlik [Man95], there exist continuous inclusions $H \otimes_\pi L \hookrightarrow H \otimes_\sigma L \hookrightarrow H \otimes_\varepsilon L$. Thus, if H or L is nuclear, then these three topologies coincide. For Hilbert spaces, $H \otimes_\sigma L$ is the usual Hilbert tensor product.

Since $H \otimes_\varepsilon L$ is Hausdorff, so is $H \otimes_\sigma L$. In particular, $H \otimes_\sigma L$ is a hilbertisable space and so is the completion $H \hat{\otimes}_\sigma L$.

Let we be given two complexes of hilbertisable Fréchet spaces and closed operators

$$\begin{array}{llllllll} (H,\partial): & \ldots & \longrightarrow & H^{i-1} & \stackrel{\partial_{i-1}}{\longrightarrow} & H^i & \stackrel{\partial_i}{\longrightarrow} & \ldots, \\ (L,d): & \ldots & \longrightarrow & L^{i-1} & \stackrel{d_{i-1}}{\longrightarrow} & L^i & \stackrel{d_i}{\longrightarrow} & \ldots, \end{array}$$

the domains of the operators not being necessarily dense. We require both the complexes to be bounded below. Fix isomorphisms $\eta_i \in \mathcal{L}(H^i)$, $i \in \mathbb{Z}$, preserving the domains of ∂_i and satisfying $\partial_i \eta_i = -\eta_{i+1}\partial_i$ on $\mathrm{Dom}\,\partial_i$, for each $i \in \mathbb{Z}$. A usual choice of η_i is $\eta_i v = (-1)^i v$, if $v \in H^i$. For any $i \in \mathbb{Z}$, set

$$(H \otimes_\sigma L)^i = \bigoplus_{j+k=i} H^j \otimes_\sigma L^k,$$

the sum being finite because the complexes (H,∂) and (L,d) are bounded below. We endow each $(H \otimes_\sigma L)^i$ with the product topology. Define an operator $D_i\colon (H \otimes_\sigma L)^i \to (H \otimes_\sigma L)^{i+1}$ with domain $\oplus_{j+k=i} \mathrm{Dom}\,\partial_j \otimes_\sigma \mathrm{Dom}\,d_k$ by

$$D_i = \partial_j \otimes 1_{L^k} + \eta_j \otimes d_k$$

on $\mathrm{Dom}\,\partial_j \otimes_\sigma \mathrm{Dom}\,d_k$, for $j+k=i$. It is easy to see that $\mathrm{im}\,D_{i-1} \subset \ker D_i$ for all i. Thus, we obtain a complex $(H \otimes_\sigma L, D)$ called the *algebraic σ-tensor product* of (H,∂) and (L,d) [2]. Furthermore, each operator D_i, $i \in \mathbb{Z}$, proves to be closable in $(H \hat{\otimes}_\sigma L)^i \times (H \hat{\otimes}_\sigma L)^{i+1}$, where $(H \hat{\otimes}_\sigma L)^i = \oplus_{j+k=i} H^j \hat{\otimes}_\sigma L^k$. Letting $\bar{D}_i$ denote the closure of D_i, we deduce at once that $\bar{D}_i \bar{D}_{i-1} = 0$ for all $i \in \mathbb{Z}$. This yields a complex of hilbertisable Fréchet spaces and closed operators

$$(H \hat{\otimes}_\sigma L, \bar{D}): \quad \ldots \longrightarrow (H \hat{\otimes}_\sigma L)^{i-1} \stackrel{\bar{D}_{i-1}}{\longrightarrow} (H \hat{\otimes}_\sigma L)^i \stackrel{\bar{D}_i}{\longrightarrow} \ldots \qquad (5.4.3)$$

which will be referred to as the *completed σ-tensor product* of the complexes (H,∂) and (L,d). We emphasise that from the point of view of analysis the completed σ-tensor product is much more interesting than the algebraic one.

The Künneth formulas link the cohomology of the complex $(H \hat{\otimes}_\sigma L, \bar{D})$ with those of (H,∂) and (L,d).

Theorem 5.4.3 *Suppose (H,∂) and (L,d) are topological complexes of hilbertisable Fréchet spaces. Then,*

$$\begin{array}{rcl} Z^i(H \hat{\otimes}_\sigma L) & = & \overline{Z^i(H \otimes_\sigma L)}, \\ B^i(H \hat{\otimes}_\sigma L) & = & \overline{B^i(H \otimes_\sigma L)} \end{array}$$

[2]Note that quasicomplexes do not survive under tensor multiplication.

for any $i \in \mathbb{Z}$*, the closures being in* $(H \hat{\otimes}_\sigma L)^i$*. In particular,* $(H \hat{\otimes}_\sigma L, \bar{D})$ *is a topological complex, and the natural mapping* $H(H) \otimes H(L) \to H(H \otimes L)$ *extends to a topological isomorphism*

$$H(H \hat{\otimes}_\sigma L) \cong H(H) \hat{\otimes}_\sigma H(L). \tag{5.4.4}$$

Recall that a complex (L, d) is said to be *topological* if the range of d_{i-1} is closed in L^i for each $i \in \mathbb{Z}$. This is the case if and only if every mapping $d_{i-1} : \mathrm{Dom}\, \mathrm{d_{i-1}} \to \mathrm{L^i}$ is open, $\mathrm{Dom}\, \mathrm{d_{i-1}}$ being equipped with the induced topology or with the graph topology. In particular, every Fredholm complex is topological.

Proof. For a thorough proof of the theorem we refer the reader to the original paper of Mantlik [Man95].

□

Here is another way of stating (5.4.4): for each $i \in \mathbb{Z}$, we have a topological isomorphism

$$H^i(H \hat{\otimes}_\sigma L) \cong \bigoplus_{j+k=i} H^j(H) \hat{\otimes}_\sigma H^k(L).$$

In particular, if both (H, ∂) and (L, d) are Fredholm complexes, then so is $(H \hat{\otimes}_\sigma L, \bar{D})$ and

$$\dim H^i(H \hat{\otimes}_\sigma L) \cong \sum_{j+k=i} \dim H^j(H)\, \dim H^k(L), \tag{5.4.5}$$

for any $i \in \mathbb{Z}$.

Corollary 5.4.4 *Let* (H, ∂) *and* (L, d) *be Fredholm complexes of hilbertisable Fréchet spaces. Then, the complex* $(H \hat{\otimes}_\sigma L, \bar{D})$ *is Fredholm and we have* $\chi(H \hat{\otimes}_\sigma L, \bar{D}) = \chi(H, \partial)\, \chi(L, d)$.

Proof. Indeed, using (5.4.5) gives

$$\begin{aligned}
\chi(H \hat{\otimes}_\sigma L, \bar{D}) &= \sum_i (-1)^i \sum_{j+k=i} \dim H^j(H)\, \dim H^k(L) \\
&= \sum_{j,k} (-1)^j \dim H^j(H)\, (-1)^k \dim H^k(L) \\
&= \chi(H, \partial)\, \chi(L, d),
\end{aligned}$$

as desired.

□

Suppose (H, ∂) and (L, d) are Fredholm complexes of hilbertisable Fréchet spaces. We restrict ourselves to the case where ∂_j and d_k are continuous. Consider the completed σ-tensor product $(H \hat{\otimes}_\sigma L, \bar{D})$, where $\eta_i = (-1)^i 1_{H^i}$ for $i \in \mathbb{Z}$. By Corollary 5.4.4, it is a Fredholm complex.

Let $(H\hat{\otimes}_\sigma L, \bar{\delta})$ be an associated complex for $(H\hat{\otimes}_\sigma L, \bar{D})$, the operators $\bar{\delta}_i$ mapping as in the sequence

$$(H\hat{\otimes}_\sigma L, \delta): \quad \ldots \longleftarrow (H\hat{\otimes}_\sigma L)^{i-1} \xleftarrow{\bar{\delta}_i} (H\hat{\otimes}_\sigma L)^i \xleftarrow{\bar{\delta}_{i+1}} \ldots$$

and the Laplacians $\Delta_i = \bar{D}_{i-1}\bar{\delta}_i + \bar{\delta}_{i+1}\bar{D}_i$, $i \in \mathbb{Z}$, being Fredholm operators. We can construct $\bar{\delta}_i$, $i \in \mathbb{Z}$, e.g., as follows. Choose associated complexes $(H, \delta^{(H)})$ and $(L, \delta^{(L)})$ for (H, ∂) and (L, d), respectively. Consider the mapping $\delta_i : (H \otimes_\sigma L)^i \to (H \otimes_\sigma L)^{i-1}$ whose restriction to $H^j \otimes_\sigma L^k$, $j + k = i$, is given by

$$\delta_i = \delta_j^{(H)} \otimes 1_{L^k} + (-1)^j 1_{H^j} \otimes \delta_k^{(L)}.$$

By the above, we have $\delta_i \delta_{i+1} = 0$ for each $i \in \mathbb{Z}$. Moreover, we check at once that

$$D_{i-1}\delta_i + \delta_{i+1}D_i = \left(\partial_{j-1}\delta_j^{(H)} + \delta_{j+1}^{(H)}\partial_j\right) \otimes 1_{L^k} + 1_{H^j} \otimes \left(d_{k-1}\delta_k^{(L)} + \delta_{k+1}^{(L)} d_k\right)$$

if evaluated on $H^j \otimes_\sigma L^k$ with $j + k = i$. Denote by $\bar{\delta}_i$ the closure of δ_i in $(H\hat{\otimes}_\sigma L)^i \times (H\hat{\otimes}_\sigma L)^{i-1}$. Then, the operators $\bar{D}_{i-1}\bar{\delta}_i + \bar{\delta}_{i+1}\bar{D}_i$, $i \in \mathbb{Z}$, are Fredholm, i.e., $(H\hat{\otimes}_\sigma L, \bar{\delta})$ is an associated complex for $(H\hat{\otimes}_\sigma L, \bar{D})$, as desired. We can now apply Lemma 5.1.8 to see that

$$(\bar{D} + \bar{\delta})_e : \ (H\hat{\otimes}_\sigma L)^{\text{even}} \to (H\hat{\otimes}_\sigma L)^{\text{odd}} \tag{5.4.6}$$

is a Fredholm operator. Assume moreover that the associated complexes $(H, \delta^{(H)})$ and $(L, \delta^{(L)})$ meet the condition that

$$\begin{array}{rcl} \partial_{i-1}\delta_i^{(H)} & + & \delta_{i+1}^{(H)}\partial_i, \\ d_{i-1}\delta_i^{(L)} & + & \delta_{i+1}^{(L)} d_i \end{array}$$

are linearly homotopic to $(1/2)1_{H^i}$ and $(1/2)1_{L^i}$, respectively, for all $i \in \mathbb{Z}$. Then $\bar{D}_{i-1}\bar{\delta}_i + \bar{\delta}_{i+1}\bar{D}_i$ is linearly homotopic to $1_{(H\hat{\otimes}_\sigma L)^i}$, for any $i \in \mathbb{Z}$. By Theorem 5.1.9, the index of $(\bar{D} + \bar{\delta})_e$ is equal to the Euler characteristic of $(H\hat{\otimes}_\sigma L, \bar{D})$, i.e., $\chi(H, \partial)\chi(L, d)$. This generalises the construction (5.4.2) to arbitrary complexes of hilbertisable Fréchet spaces.

Example 5.4.5 Consider two single Fredholm operators $\partial \in \mathcal{L}(H, \tilde{H})$ and $d \in \mathcal{L}(L, \tilde{L})$ in hilbertisable Fréchet spaces. Specifying them within complexes, we fix associated complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \xrightarrow{\partial} & \tilde{H} & \longrightarrow & 0, \\ 0 & \longleftarrow & H & \xleftarrow{\delta^{(H)}} & \tilde{H} & \longleftarrow & 0; \end{array} \qquad \begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{d} & \tilde{L} & \longrightarrow & 0, \\ 0 & \longleftarrow & L & \xleftarrow{\delta^{(L)}} & \tilde{L} & \longleftarrow & 0. \end{array}$$

Define the #-product of ∂ and d by

$$\partial \# d = \begin{pmatrix} \partial \otimes 1 & -1 \otimes \delta^{(L)} \\ 1 \otimes d & \delta^{(H)} \otimes 1 \end{pmatrix}$$

mapping as $(H\hat{\otimes}_\sigma L) \oplus (\tilde{H}\hat{\otimes}_\sigma \tilde{L}) \to (\tilde{H}\hat{\otimes}_\sigma L) \oplus (H\hat{\otimes}_\sigma \tilde{L})$ (cf. (5.4.2)). It follows that $\partial \# d$ is a Fredholm operator. If moreover

$$\begin{array}{ll} t\,\delta^{(H)}\partial + (1-t)\,(1/2)\,1_H, & t\,\delta^{(L)}d + (1-t)\,(1/2)\,1_L, \\ t\,\partial\delta^{(H)} + (1-t)\,(1/2)\,1_{\tilde{H}}, & t\,d\delta^{(L)} + (1-t)\,(1/2)\,1_{\tilde{L}} \end{array}$$

are Fredholm operators for any $t \in (0,1)$, then $\operatorname{ind}\partial\#\mathrm{d} = \operatorname{ind}\partial\operatorname{ind}\mathrm{d}$ independently of the choice of associated complexes.

□

We now turn to the case where (H,∂) and (L,d) are complexes of pseudodifferential operators on different manifolds $\mathcal{S}$ and $\mathcal{V}$, both manifolds bearing singularities. The completed σ-tensor product of (H,∂) and (L,d) is of great importance because it lives on the Cartesian product of the manifolds S and V, which bears higher order singularities. If both (H,∂) and (L,d) are elliptic, then (5.4.6) yields an example of an elliptic operator on $S \times V$, thus giving rise to a concept of ellipticity on manifolds with higher order singularities. For instance, if S is a C^∞ closed manifold and V is a C^∞ closed manifold with conical points, then $S \times V$ is a C^∞ manifold with edges. On the other hand, manifolds with edges survive under multiplication by a C^∞ closed manifold S, which contributes merely to the edges. If S and V are C^∞ closed manifolds with conical points and edges, then $S \times V$ bears a natural C^∞ structure of a manifold with corners. This class of manifolds seems to be most important because it survives under Cartesian multiplication.

When discussing analysis on manifolds with edges, we confine ourselves to the case where S is a C^∞ compact closed manifold and V is a C^∞ compact closed manifold with conical points which are $v_1, \ldots, v_I$. Set $M = S \times V$. This is a C^∞ compact closed manifold with edges $S \times \{v_i\}$, $i = 1, \ldots, I$, and the corresponding stretched manifold is $\mathcal{M} = S \times \mathcal{V}$, where $\mathcal{V}$ is obtained by blowing up $\mathcal{V}$ at every conical point. Thus, $\mathcal{V}$ is a smooth compact manifold with a boundary bearing the structure of a fibre bundle over the set of conical points. Consider two complexes of pseudodifferential operators on S and V,

$$\begin{array}{lcccccc} (H,B): & \ldots & \longrightarrow & H^{t_{i-1}}(W^{i-1}) & \xrightarrow{B_{i-1}} & H^{t_i}(W^i) & \xrightarrow{B_i} & \ldots, \\ (L,A): & \ldots & \longrightarrow & H^{s_{i-1},\gamma_{i-1}}(V^{i-1}) & \xrightarrow{A_{i-1}} & H^{s_i,\gamma_i}(V^i) & \xrightarrow{A_i} & \ldots, \end{array} \tag{5.4.7}$$

both complexes being bounded above. Here,

$$\begin{array}{lcl} B_i & \in & \Psi^{\mathrm{o_i}}_{\mathrm{cl}}(\mathrm{S}; \mathrm{W^i}, \mathrm{W^{i+1}}), \\ A_i & \in & \Psi^{\mathrm{m_i}}(\mathcal{V}; \mathrm{V^i}, \mathrm{V^{i+1}}; \mathrm{w_i}) \end{array}$$

belong to "algebras" of pseudodifferential operators on S and V, respectively. Thus,

$$\begin{array}{lcl} t_i & = & t_{i-1} - o_{i-1}, \\ s_i & = & s_{i-1} - m_{i-1} \end{array}$$

for all $i \in \mathbb{Z}$. However, the weight data $w_i = (\gamma_i, \gamma_{i+1}, (-l, 0])$ may be arbitrary, not necessarily meeting $\gamma_i = \gamma_{i-1} - m_{i-1}$, as is allowed in the cone theory (cf. Schulze [Sch98]).

As mentioned, all the spaces of sections entering into (5.4.7) are hilbertisable Fréchet spaces. Consequently, the product theory sketched above is applicable. Setting

$$\begin{array}{rcl} (H \hat{\otimes}_\sigma L)^i & = & \bigoplus_{j+k=i} H^{t_j}(W^j) \hat{\otimes}_\sigma H^{s_k,\gamma_k}(V^k), \\ D_i \,|_{H^{t_j}(W^j) \otimes_\sigma H^{s_k,\gamma_k}(V^k)} & = & B_j \otimes 1 + (-1)^j \, 1 \otimes A_k \end{array} \tag{5.4.8}$$

for $j + k = i$, we arrive at the completed σ-tensor product of complexes (5.4.7)

$$(H \hat{\otimes}_\sigma L, \bar{D}) : \quad \ldots \longrightarrow (H \hat{\otimes}_\sigma L)^{i-1} \stackrel{\bar{D}_{i-1}}{\longrightarrow} (H \hat{\otimes}_\sigma L)^i \stackrel{\bar{D}_i}{\longrightarrow} \ldots,$$

where $\bar{D}_i$ stands for the closure of D_i in $(H \hat{\otimes}_\sigma L)^i \times (H \hat{\otimes}_\sigma L)^{i+1}$.

Denote by $W^j \boxtimes V^k$ the external tensor product of the bundles W^j and V^k. This is a vector bundle over $S \times \mathcal{V}$ whose fibre over a point $(y, w) \in S \times \mathcal{V}$ is $W^j_y \otimes V^k_w$. The completed σ-tensor product $H^{t_j}(W^j) \hat{\otimes}_\sigma H^{s_k,\gamma_k}(V^k)$ can be identified within distribution sections of $W^j \boxtimes V^k$ over $\mathcal{M}$. We write $H^{t_j,s_k,\gamma_k}(W^j \boxtimes V^k)$ for it. By the above, $H^{t_j,s_k,\gamma_k}(W^j \boxtimes V^k)$ is a hilbertisable Fréchet space.

Lemma 5.4.6 *Let W and V be vector bundles over S and $\mathcal{V}$, respectively, and let $\gamma \in \mathbb{R}$. Then*

$$H^\infty(W) \hat{\otimes}_\sigma H^{\infty,\gamma}(V) \stackrel{\text{top.}}{\cong} H^{\infty,\gamma}(W \boxtimes V).$$

Proof. Recall that the space $H^{s,\gamma}(W \boxtimes V)$ is obtained by gluing together the spaces $H^s(S, \pi^*(W \otimes H^{s,\gamma}(V)))$ close to $\partial\mathcal{M} = S \times \partial\mathcal{V}$ and $H^s(W \boxtimes V)$, away from $\partial\mathcal{M}$, the former being defined with respect to a group action $(\kappa_\theta)_{\theta>0}$ in the fibres $W_y \otimes H^{s,\gamma}(V)$, $y \in S$ (cf. (2.2.3)). As is mentioned in the proof of Theorem 4.4.1, the group action fulfils an estimate

$$\|\kappa_\theta\|_{\mathcal{L}(W_f \otimes H^{s,\gamma}(V))} \leq c \ \max\left(\theta, \theta^{-1}\right)^R$$

for all $\theta > 0$, with $R = \max(|s|, |\gamma + \frac{1+n}{2}|)$ and c independent of θ. Hence it follows that

$$\begin{array}{rcl} \|\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta} u\|_{W_f \otimes H^{s,\gamma}(V)} & \leq & c \, \langle\eta\rangle^R \, \|\mathcal{F}_{y\mapsto\eta} u\|_{W_f \otimes H^{s,\gamma}(V)}, \\ \|\mathcal{F}_{y\mapsto\eta} u\|_{W_f \otimes H^{s,\gamma}(V)} & \leq & c \, \langle\eta\rangle^R \, \|\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta} u\|_{W_f \otimes H^{s,\gamma}(V)} \end{array}$$

for all $\eta \in \mathbb{R}^q$. Integrating these estimates over $\eta \in \mathbb{R}^q$ leads to continuous embeddings

$$H^{t+R}(S, W \otimes H^{s,\gamma}(V)) \hookrightarrow H^t(S, \pi^*(W \otimes H^{s,\gamma}(V))) \hookrightarrow H^{t-R}(S, W \otimes H^{s,\gamma}(V)),$$

for any $t \in \mathbb{R}$, where the spaces on the very left and right are defined with respect to the identity action in the fibres $W_y \otimes H^{s,\gamma}(V)$, $y \in S$. Since the completed σ-tensor product and the Hilbert tensor product coincide for Hilbert spaces, we see that $H^t(S, W \otimes H^{s,\gamma}(V)) = H^t(W) \hat{\otimes}_\sigma H^{s,\gamma}(V)$, and, in consequence,

$$H^{2s}(W) \hat{\otimes}_\sigma H^{2s,\gamma}(V) \hookrightarrow H^{s,\gamma}(W \boxtimes V) \hookrightarrow H^{s/2}(W) \hat{\otimes}_\sigma H^{s/2,\gamma}(V) \qquad (5.4.9)$$

for all $s \in \mathbb{R}$ large enough. This proves the lemma.

□

The lemma shows that, for $t_j = \infty$ and $s_k = \infty$, the spaces involved in the completed σ-tensor product $(H \hat{\otimes}_\sigma L, \bar{D})$ are weighted Sobolev spaces on $\mathcal{M}$, i.e., those serving as domains for operators in the calculus on $\mathcal{M}$ developed in Chapter 3. However, the operators $\bar{D}_i$ of (5.4.8) do not belong to the algebras $\Psi^{\mathrm{m}}(\mathcal{M}; \mathrm{V}, \tilde{\mathrm{V}}; \mathrm{w})$ described therein, unless $\gamma_i = \gamma_{i-1} - m_{i-1}$ for each i. If the weight data w_i meet this condition, then $\bar{D}_i$, $i \in \mathbb{Z}$, belong rather to the norm closures of the algebras in question than to the calculus itself, as is the case provided that all the B_j and A_k are differential operators.

Loosely speaking, the action of $B_j \otimes 1$ on a section $u(y, w)$ of $W^j \boxtimes V^k$ is given by the operator B_j applied in $y \in S$ whereas the action of $1 \otimes A_k$ on $u(y, w)$ amounts to A_k applied in $w \in V$. This is why one uses the specification "*external multiplication*" for a tensor product of operators on different manifolds.

Theorem 5.4.7 *If (H, B) and (L, A) are elliptic complexes on S and V, respectively, then the complex $(H \hat{\otimes}_\sigma L, \bar{D})$ given by (5.4.8) is Fredholm and its cohomology does not depend on the particular choice of t_j and s_k.*

Proof. Indeed, both (H, B) and (L, A) are Fredholm complexes, which is clear from the Hodge theory on S and V (cf. Theorem 5.3.4). By Theorem 5.4.3, the complex $(H \hat{\otimes}_\sigma L, \bar{D})$ is Fredholm and its cohomology at step i amounts to $\oplus_{j+k=i} H^j(H) \hat{\otimes}_\sigma H^k(L)$. This gives the desired conclusion when combined with the fact that the cohomologies of (H, B) and (L, A) are independent of the choice of t_j and s_k, respectively.

□

Theorem 5.4.7 gains in interest if we realise that, when combined with construction (5.4.6), it yields interesting elliptic operators on $\mathcal{M}$ without potential and trace conditions, the ellipticity being in the Douglis-Nirenberg sense. Moreover, starting with an elliptic complex (L, A) of Euler characteristic 1 on V and varying (H, B) over elliptic complexes on S, we arrive at elliptic operators without potential and trace conditions and of a given index on $\mathcal{M}$ [3].

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[3]This result was presented by F. Mantlik in the Workshop "Partial Differential Equations" at the University of Potsdam (July 13 – July 19, 1997).

The point of Theorem 5.4.7 is in the assertion that completed σ-tensor products of elliptic complexes bear all essential features of elliptic complexes, namely, a finite-dimensional cohomology independent of the Sobolev spaces where it is evaluated. Therefore, they are natural candidates for the class of elliptic complexes in a calculus of pseudodifferential operators on $\mathcal{M}$ to be specified. As mentioned, this calculus is different from ours; in fact, it is much simpler because it relies on the identity action in the fibres of $\mathcal{M}$ over S. For more details we refer the reader to Luke [Luk72]. Of course, it is not the problem that the components $B_j \otimes 1$ and $1 \otimes A_k$ of D_i are allowed to have different orders o_j and m_k. Such operators can be handled within the concept of weighted homogeneity and ellipticity in the Douglis-Nirenberg sense. Let us have look at how the ellipticity of (H, B) and (L, A) is inherited by the product complex. The interior ellipticity of $(H \hat{\otimes}_\sigma L, \bar{D})$ is controlled by the complex of *principal interior symbols* ${}^b\sigma^{o,m}(H \hat{\otimes}_\sigma L, \bar{D})$, where ${}^b\sigma^{o,m}(\bar{D}_i) = \sigma^{o_j}(B_j) \otimes 1 + (-1)^j 1 \otimes {}^b\sigma^{m_k}(A_k)$ on $\pi^*(W^j \boxtimes V^k)$. We check at once that

$$ {}^b\sigma^{o,m}(H \hat{\otimes}_\sigma L, \bar{D}) = \sigma^o(H, B) \otimes {}^b\sigma^m(L, A), $$

and so the exactness of ${}^b\sigma^{o,m}(H \hat{\otimes}_\sigma L, \bar{D})$ over $(T^*S \setminus \{0\}) \times ({}^bT^*\mathcal{V} \setminus \{0\})$ follows from that of $\sigma^o(H, B)$ and ${}^b\sigma^m(L, A)$ by Künneth formula (5.4.4). On the other hand, the ellipticity of $(H \hat{\otimes}_\sigma L, \bar{D})$ in the normal directions to the edge S relies on the complex of *principal edge symbols* $\sigma^o_{\mathrm{edge}}(H \hat{\otimes}_\sigma L, \bar{D})$, where $\sigma^o_{\mathrm{edge}}(\bar{D}_i) = \sigma^{o_j}(B_j) \otimes 1 + (-1)^j 1 \otimes A_k$ on $\pi^*(W^j \otimes H^{s_k, \gamma_k}(V^k))$. Once again we see that

$$ \sigma^o_{\mathrm{edge}}(H \hat{\otimes}_\sigma L, \bar{D}) = \sigma^o(H, B) \otimes (L, A), $$

and so the exactness of $\sigma^o_{\mathrm{edge}}(H \hat{\otimes}_\sigma L, \bar{D})$ over $(T^*S \setminus \{0\})$ follows from the exactness of $\sigma^o(H, B)$ and the Fredholm property of (L, A) by Künneth formula (5.4.4).

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