Volterra Operators and Parabolicity

Anisotropic Pseudo–Differential Operators

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Abstract

Parabolic equations on manifolds with singularities require a new calculus of anisotropic pseudo–differential operators with operator–valued symbols. The paper develops this theory along the lines of an abstract wedge calculus with strongly continuous groups of isomorphisms on the involved Banach spaces. The corresponding pseudo–differential operators are continuous in anisotropic wedge Sobolev spaces, and they form an algebra. There is then introduced the concept of anisotropic parameter–dependent ellipticity, based on an order reduction variant of the pseudo–differential calculus. The theory is applied to a class of parabolic differential operators, and it is proved the invertibility in Sobolev spaces with exponential weights at infinity in time direction.

Contents

Introduction

The theory of parabolic partial differential equations is a classical field of mathematical analysis with a large variety of models in physics, engineering and applied sciences. A simplest example is the heat equation. Typical aspects are the construction of fundamental solutions, of inverse operators to initial–boundary value problems, or the regularity of solutions and asymptotic properties, cf., in particular, AGRANOVIČ, VIŠIK, [AV], SOLONNIKOV [So1], [So2], EIDEL'MAN [Ei1], FELLER [Fe], FRIEDMAN [Fi] and AMANN [Am].

Heat operators in several variants also occur in other areas of mathematics (spectral theory, index theory, geometry and topology), cf. ATHIYAH, PATODI, SINGER [AS] and GILKEY [Gi].

Similarly to the theory of elliptic equations, where pseudo–differential operators are a standard tool to express parametrices, to prove elliptic regularity and the Fredholm property in Sobolev spaces, etc., it is natural to ask for a pseudo–differential approach also for parabolic equations.

Another motivation to study parabolicity under the aspect of pseudo–differential operators is that non–smooth spatial configurations are very difficult to handle by using direct traditional methods only.

The present paper is the first one of a series of systematic investigations on parabolic pseudo–differential operators. We will obtain a general calculus aimed at solving initial boundary value problems on spatial manifolds that may have singularities of conical or edge type. Solutions will be given in terms of inverse operators expressed by means of various symbolic levels. The singularities require a special approach with pseudo–differential operators with operator–valued symbols. Moreover, parabolic operators have a typical anisotropic behaviour; the time covariable is involved in the symbols with another homogeneity than the spatial covariables. In the operator–valued set–up the homogeneity is connected with natural one–parameter groups of isomorphisms on the Banach spaces in which the symbols act as operator functions. Such groups are also contained in the corresponding Sobolev spaces of vector–valued distributions. This material is the main subject of the present paper.

In recent years the program to establish a pseudo–differential calculus on manifolds with singularities to solve elliptic problems in terms of parametrix constructions made considerable progress. This concerns, in particular, manifolds with singularities of cone, edge or corner type as well as with singular boundaries or noncompact exits to infinity, cf. the papers and monographs of SCHULZE [Su1], [Su3], [Su10], EGOROV, SCHULZE [ES], and papers jointly with DORSCHFELDT [DS], SCHROHE [SS2], FEDOSOV, TARKHANOV [FST], STERNIN, SHATALOV [SSS1], [SSS2]. Moreover, pseudo–differential methods for the singular set–up were developed by CORDES [Co1], [Co2], PLAMENEVSKIJ [Pl], ESKIN [Es1], [Es2], MELROSE, MENDOZA [MM], MELROSE [Me], UNTERBERGER, UPMEIER [UU].

Ellipticity and parabolicity may be understood in close connection to each other, cf. AGRANOVIČ, VIŠIK, [AV], SOLONNIKOV [So1], [So2], [So3]. Nevertheless, the pseudo–differential calculus for the parabolic theory was not so intensely developed as the elliptic one. This seems to be in fact a real gap in the literature, though there are some crucial papers, in particular, of PIRIOU [Pi1], [Pi2], that consider parabolic pseudo–differential operators under the aspect of algebras of anisotropic pseudo–differential operators and a subalgebra of Volterra operators.

Anisotropic pseudo–differential operators were studied under different aspects by HUNT, PIRIOU [HP], BOGGIATTO, BUZANO, RODINO [BBR] and many other authors. Parabolicity of boundary value problems in cylindrical domains were investigated by CHAN ŽUI CHO, ESKIN [CE]; the structure of fundamental solutions in terms of pseudo–differential analysis was studied in a series of papers of IWASAKI [Iw1], [Iw2]. Let us also note that ESKIN [Es3] considered particular parabolic problems for differential operators in non–smooth domains.

Parabolic theory for pseudo–differential operators is by no means a straightforward modification of the elliptic case, although some ideas, such as generalities on anisotropic pseudo–differential operators or anisotropic ellipticity are not so far from the isotropic case. There are certain problems and new difficulties, some of them really serious and unexpected at first glance, even for smooth spatial configurations. For instance, if we want to embed operators of the form $\frac{\partial}{\partial t} + A$ for elliptic pseudo–differential operators A (say on a C^∞ manifold) into a nice calculus, there appears the problem of handling symbols like $i\tau + a(x,\xi)$ for pseudo–differential symbols $a(x,\xi)$. This requires specific structure investigations, cf. the book of Grubb [Gr2].

Another new aspect is that parabolicity implies the Volterra property. This is roughly speaking holomorphy on the level of symbols in a complex τ half–plane, including the symbol estimates there. The standard operations with symbols that employ excisions in the covariables destroy this holomorphy. So, in particular, the asymptotic sums within such symbol spaces have to be carried out by other methods. We shall do this here on the level of kernels by means of kernel cut–offs, which preserve the required class.

The notion of parabolicity itself is another source of new discussion, in particular, when the spatial configurations have a singular geometry. There appear, like in the elliptic theory, additional (operator–valued) symbolic levels, caused by the singularities which have to be included in the parabolicity, in order to ensure unique solvability. Also the analysis of initial–boundary value problems with compatibility of initial and boundary conditions on the edge of the time–space cylinder is a rich program, in particular, when the given operator is pseudo–differential and the transmission property with respect to the initial surface is required.

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1 Anisotropic operator–valued symbols

In this section we define anisotropic symbols of pseudo–differential operators with values in the space of linear continuous operators between Banach or Fréchet spaces E and $\widetilde{E}$. Let us first assume that E and $\widetilde{E}$ are Banach spaces. Later on we will extend the theory to the case of Fréchet spaces.

1.1 Operator–valued symbols

Anisotropy refers to any fixed $1 \le l \in \mathbb{N}$. The isotropic case is contained in the anisotropic theory for $l = 1$. For $(\tau,\eta) \in \mathbb{R}_\tau \times \mathbb{R}^q_\eta = \mathbb{R}^{1+q}$ we define the anisotropic norm function

$$|\tau,\eta|_l = (|\tau|^2 + |\eta|^{2l})^{\frac{1}{2l}}. \tag{1}$$

and an anisotropic smoothed norm function

$$[\tau,\eta]_l := \omega(|\tau,\eta|_l) + (1 - \omega(|\tau,\eta|_l))|\tau,\eta|_l, \tag{2}$$

where $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+, [0,1])$ is supposed to be a cut–off function, i.e.,

$$\omega(r) = \begin{cases} 1 & \text{for} \quad 0 \le r \le 1, \\ 0 & \text{for} \quad r \ge 2. \end{cases} \tag{3}$$

Later on we need extensions of symbols with respect to the time covariable τ in the lower complex half space $\mathbb{C}_- = \{\zeta = \tau + i\vartheta \ : \ \vartheta < 0\}$. In this case we use also $|\cdot,\cdot|_l$ and $[\cdot,\cdot]_l$, where the first variable may be a complex number.

Let us now introduce anisotropic homogeneity:

Definition 1 *A function $f(\tau,\eta) \in C^\infty(\mathbb{R}^{1+q}_{\tau,\eta} \setminus \{0\})$ will be called anisotropic homogeneous of order $\nu \in \mathbb{R}$ (anisotropic ν–homogeneous), if for all $\lambda \in \mathbb{R}_+$ the relation*

$$f(\lambda^l\tau, \lambda\eta) = \lambda^\nu f(\tau,\eta) \tag{4}$$

holds. We call $f(\tau,\eta) \in C^\infty(\mathbb{R}^{1+q}_{\tau,\eta})$ anisotropic homogeneous of order $\nu \in \mathbb{R}$ for large $|\tau,\eta|_l$, if the homogeneity relation (4) is satisfied for all $\lambda \geq 1$ and $|\tau,\eta|_l \geq c$ for a constant $c > 0$. The smallest possible constant is called the homogeneity constant of the given function.

Note that $|\tau,\eta|_l$ is obviously anisotropic homogeneous of order 1 and $[\tau,\eta]_l$ anisotropic homogeneous of order 1 for large $|\tau,\eta|_l$. From now on we will shortly denote anisotropic homogeneity of order ν by ν–homogeneity. Set $D^\beta_{\tau,\eta} = (-i\partial_\tau)^{\beta_0}(-i\partial\eta)^{\beta'}$ and $|\beta|_l = l\beta_0 + |\beta'|$ for $\beta = (\beta_0,\beta') \in \mathbb{N}^{1+q}$.

Remark 2 Differentiation of both sides of (4) shows that the derivatives $D^\beta_{\tau,\eta} f(\tau,\eta)$ are $(\nu - |\beta|_l)$–homogeneous for any ν–homogeneous function $f(\tau,\eta)$ and arbitrary multi-indices $\beta \in \mathbb{N}^{1+q}$. A similar assertion holds for functions that are ν–homogeneous for large $|\tau,\eta|_l$.

Proposition 3 *A function $f(\tau,\eta) \in C^\infty(\mathbb{R}^{1+q}\backslash\{0\})$ is ν–homogeneneous iff it satisfies the modified Euler relation*

$$E_l f(\tau,\eta) := \left(l\tau\frac{\partial}{\partial\tau} + \sum_{j=1}^{q} \eta_j \frac{\partial}{\partial\eta_j} \right) f(\tau,\eta) = \nu f(\tau,\eta) \tag{5}$$

for all $(\tau,\eta) \in \mathbb{R}^{1+q}\backslash\{0\}$.

Proof: Using anisotropic polar coordinates $\rho = |\tau,\eta|_l$ and $\sigma_j = \frac{\eta_j}{\rho}$ the equation (5) takes the form

$$\left(\rho\frac{\partial}{\partial\rho}\right) f(\rho,\sigma) = \nu f(\rho,\sigma)$$

which is solved by all functions $f(\rho,\sigma) \in C^\infty(\mathbb{R}^{1+q} \setminus \{0\})$ of the form $f(\rho,\sigma) = \rho^\nu f(1,\sigma)$. In Cartesian coordinates this means $f(\tau,\eta) = |\tau,\eta|_l^\nu f\left(\frac{\tau}{|\tau,\eta|_l^l}, \frac{\eta}{|\tau,\eta|_l}\right)$ and hence $f(\lambda^l\tau,\lambda\eta) = \lambda^\nu f(\tau,\eta)$. □

Lemma 4 *There are constants $c, C > 0$, such that the inequalities*

$$c[\zeta,\eta]_l \leq (1 + |\zeta|^2 + |\eta|^{2l})^{\frac{1}{2l}} \leq C[\zeta,\eta]_l \tag{6}$$

and

$$c[\zeta,\eta]_l \leq [\zeta,\eta]_1 \leq C[\zeta,\eta]_l^l \tag{7}$$

hold for all $s \in \mathbb{R}$ and arbitrary $(\zeta,\eta) \in \mathbb{C}\times\mathbb{R}^q$. Moreover, the anisotropic version of Peetre's inequality is satisfied, i.e., there exists a constant $c > 0$ such that

$$[\zeta,\eta]_l^s \leq c^{|s|}[\zeta-\zeta',\eta-\eta']_l^{|s|}[\zeta',\eta']_l^s \tag{8}$$

holds for all $s \in \mathbb{R}$ and arbitrary $(\zeta,\eta), (\zeta',\eta') \in \mathbb{C}\times\mathbb{R}^q$.

Proof: To (6): For $(\zeta,\eta) \in \overline{B^l_2}$ it follows (6) from the compactness of the anisotropic ball $\overline{B^l_2} = \{|\zeta,\eta|_l \leq 2\}$, since $[\zeta,\eta]_l \geq 1$ and $(1 + |\zeta|^2 + |\eta|^{2l})^{\frac{1}{2l}} \geq 1$ for all $(\zeta,\eta) \in \mathbb{C}\times\mathbb{R}^q$. But for $|\zeta,\eta|_l > 2$ we have $[\zeta,\eta]_l = (|\zeta|^2 + |\eta|^{2l})^{\frac{1}{2l}}$ such that the estimate

$$[\zeta,\eta]_l < (1 + |\zeta|^2 + |\eta|^{2l})^{\frac{1}{2l}} < (2(|\zeta|^2 + |\eta|^{2l}))^{\frac{1}{2l}} = 2^{\frac{1}{2l}}[\zeta,\eta]_l$$

implies the desired inequality.

To (7): For (ζ,η) in the compact set $\overline{B^l_2}$ it follows (7) because of $[\zeta,\eta]_l > 1$ for arbitrary $l \in \mathbb{N}\setminus\{0\}$. By definition we have $\{|\zeta| \leq 1\} \cap \{|\eta| \leq 1\} \subset \overline{B^l_2}$ such that we only have to check the cases $|\zeta| > 1$ and $|\eta| > 1$. Moreover, for $(\zeta,\eta) \in (\mathbb{C}\times\mathbb{R}^q) \setminus \overline{B^l_2}$ we have $[\zeta,\eta]_l = |\zeta,\eta|_l$ and hence the estimate

$$\left(|\zeta|^2 + |\eta|^2\right)^l = \sum_{j=0}^{l} \binom{l}{j} |\zeta|^{2j}|\eta|^{2(l-j)} > \begin{cases} |\zeta|^{2l} + |\eta|^{2l} > |\zeta|^2 + |\eta|^{2l} & , |\zeta| > 1, \\ l|\zeta|^2|\eta|^{2(l-1)} + |\eta|^{2l} > |\zeta|^2 + |\eta|^{2l} & , |\eta| > 1, \end{cases}$$

which gives the first part of (7). The second part is clear for $|\eta| > 1$, whereas for $|\zeta| > 1$ and $|\eta| \le 1$ it follows from $|\zeta|^2 + |\eta|^2 \le |\zeta|^2 + |\zeta|^2 \le 2(|\zeta|^2 + |\eta|^{2l})$.

To (8): In view of inequality (6) it suffices to prove the anisotropic version of Peetre's inequality (8) in the form

$$(1 + |\zeta|^2 + |\eta|^{2l})^{\frac{s}{2l}} \le c^{|s|}(1 + |\zeta - \zeta'|^2 + |\eta - \eta'|^{2l})^{\frac{|s|}{2l}}(1 + |\zeta'|^2 + |\eta'|^{2l})^{\frac{s}{2l}}. \tag{9}$$

For arbitrary $a, \tilde{a}, b, \tilde{b} \in \mathbb{C}$ we have

$$\begin{aligned} 1 + |a + \tilde{a}|^2 + |b + \tilde{b}|^{2l} &\le 1 + (|a| + |\tilde{a}|)^2 + (|b| + |\tilde{b}|)^{2l} \\ &\le c(1 + |a|^2 + |b|^{2l})(1 + |\tilde{a}|^2 + |\tilde{b}|^{2l}) \end{aligned}$$

which implies (9) for $s \ge 0$ with $\zeta = a + \tilde{a},\ \eta = b + \tilde{b}$ and $\zeta' = \tilde{a},\ \eta' = \tilde{b}$. For $s < 0$ we get

$$(1 + |a + \tilde{a}|^2 + |b + \tilde{b}|^{2l})^{\frac{s}{2l}} c^{\frac{-s}{2l}} (1 + |a|^2 + |b|^{2l})^{\frac{-s}{2l}} \ge (1 + |\tilde{a}|^2 + |\tilde{b}|^{2l})^{\frac{s}{2l}}$$

which gives us (9) with $\zeta = \tilde{a},\ \eta = \tilde{b}$ and $\zeta' = -a,\ \eta' = -b$. □

The estimates (6) and (8) are also essential for the isotropic pseudo–differential calculus where $l = 1$. Sometimes the inequality (7) is used to reduce arguments to the isotropic case.

After these preparations we turn to the operator–valued theory. With the Banach spaces E and $\widetilde{E}$ we associate group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively, as follows: A family of automorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ is called a group action associated with E if the map $\lambda \mapsto \kappa_\lambda$ belongs to $C(\mathbb{R}_+, \mathcal{L}_\sigma(E))$ and if it satisfies $\kappa_\lambda \circ \kappa_\mu = \kappa_{\lambda\mu}$, $\kappa_\lambda^{-1} = \kappa_{\lambda^{-1}}$ for all $\lambda, \mu \in \mathbb{R}_+$. Here $\mathcal{L}_\sigma(E)$ is the space of all linear continuous operators $a : E \to E$ equipped with the strong operator topology, i.e., with the system of semi–norms $\mathcal{L}(E) \ni a \mapsto \|ae\|_E$ for $e \in E$. Note that for every group action there are constants $M, c > 0$ such that

$$\|\kappa_\lambda\|_{\mathcal{L}(E)} \le \begin{cases} c\lambda^{-M} & \text{for} \quad \lambda \le 1, \\ c\lambda^{M} & \text{for} \quad \lambda \ge 1. \end{cases} \tag{10}$$

Let us simply set $\kappa(\tau, \eta) := \kappa_{[\tau,\eta]_l}$ and $\tilde{\kappa}(\tau, \eta) := \tilde{\kappa}_{[\tau,\eta]_l}$. In view of (10) and (8) for $s = 1$ we obtain constants $c, M > 0$ such that

$$\left\|\kappa(\zeta, \eta) \circ \kappa^{-1}(\zeta', \eta')\right\|_{\mathcal{L}(E)} \le c[\zeta - \zeta', \eta - \eta']_l^M \tag{11}$$

is satisfied for arbitrary $(\zeta, \eta), (\zeta', \eta') \in \mathbb{C} \times \mathbb{R}^q$.

Definition 5 *Let $\nu \in \mathbb{R}$ and $\Omega = U_0 \times U$ with open sets $U_0 \subseteq \mathbb{R}^{p_0}, U \subseteq \mathbb{R}^p$; then the space*

$$S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E}) \tag{12}$$

of anisotropic operator–valued symbols is defined as the set of all $a(t, y, \tau, \eta) \in C^\infty(\Omega \times \mathbb{R}^{1+q}, \mathcal{L}(E, \widetilde{E}))$ such that

$$\|\tilde{\kappa}^{-1}(\tau, \eta)\{D_{t,y}^\alpha D_{\tau,\eta}^\beta a(t, y, \tau, \eta)\}\kappa(\tau, \eta)\|_{\mathcal{L}(E,\widetilde{E})} \le c[\tau, \eta]_l^{\nu - |\beta|_l} \tag{13}$$

holds for all $\alpha = (\alpha_0, \alpha') \in \mathbb{N}^{p_0} \times \mathbb{N}^p$, $\beta = (\beta_0, \beta') \in \mathbb{N} \times \mathbb{N}^q$ and all $(t, y) \in K$ for arbitrary compact $K \subset\subset \Omega$ and all $(\tau, \eta) \in \mathbb{R}^{1+q}$ with constants $c = c(\alpha, \beta, K) \ge 0$; $|\beta|_l = l\beta_0 + |\beta'|$ for $\beta \in \mathbb{N}^{1+q}$.

The best constants $c = c(\alpha, \beta, K)$ in (13) form a system of semi–norms that define a Fréchet topology on (12), cf. [Bu]. With this definition the space (12) depends on the specific choice of the $\kappa_\lambda, \tilde{\kappa}_\lambda$. They are always fixed in our applications; so for abbreviation we omit them in the notation. We only need the cases $\Omega = U_0 \times U$ with open $U_0 \subseteq \mathbb{R}$ and $U \subseteq \mathbb{R}^q$ or $\Omega = (U_0 \times U) \times (U_0 \times U)$, where in the latter case we call the elements of (12) amplitude functions with variables t, y, t', y', τ, η.

Of course, we also allow the special case $E = \mathbb{C}$ or $\widetilde{E} = \mathbb{C}$. Then we set $\kappa_\lambda = \mathrm{id}$ and $\tilde{\kappa}_\lambda = \mathrm{id}$ for all $\lambda \in \mathbb{R}_+$. With $E = \widetilde{E} = \mathbb{C}$ we get the scalar anisotropic symbols and write for short $S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}) = S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; \mathbb{C}, \mathbb{C})$ (see also [Pi1]).

Denote by $S^{\nu,l}(\mathbb{R}^{1+q};E,\widetilde{E})$ the closed subspace of anisotropic operator–valued symbols with constant coefficients, that means they do not depend on (t,y,t',y').

The following assertions are simple generalisations of analogous results of the isotropic theory and can be proved by the same techniques as in [Su1]. Therefore we will omit the proofs here. The following lemma is an immediate consequence of the definition of the symbol classes and of the nuclearity of $C^\infty(\Omega)$.

Lemma 6 *We have for an open set* $\Omega = U_0 \times U \subseteq \mathbb{R}^{1+q}$ *and* $\Omega^2 = \Omega \times \Omega$

$$S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q};E,\widetilde{E}) = C^\infty(\Omega^2, S^{\nu,l}(\mathbb{R}^{1+q};E,\widetilde{E})) = C^\infty(\Omega^2)\widehat{\otimes}_\pi S^{\nu,l}(\mathbb{R}^{1+q};E,\widetilde{E}).$$

Thus every $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q};E,\widetilde{E})$ *can be written as a convergent sum*

$$a(t,y,t',y',\tau,\eta) = \sum_{j=0}^{\infty} c_j b_j(t,y) a_j(\tau,\eta) d_j(t',y') \tag{14}$$

with sequences $(c_j) \in l_1$, $b_j, d_j \to 0$ *in* $C^\infty(\Omega)$ *and* $a_j \to 0$ *in* $S^{\nu,l}(\mathbb{R}^{1+q};E,\widetilde{E})$ *for* $j \to \infty$.

Lemma 7 *Let* $\Omega = U_0 \times U$ *with open sets* $U_0 \in \mathbb{R}^{p_0}$, $U \in \mathbb{R}^p$, *and let* $E, \widetilde{E}$ *and* $\widehat{E}$ *be Banach spaces and* $\{\kappa_\lambda\}$, $\{\tilde{\kappa}_\lambda\}$ *and* $\{\hat{\kappa}_\lambda\}$ *the associated group actions. Then for arbitrary* $\nu, \tilde{\nu} \in \mathbb{R}$ *we have the following relations:*

(i) *For all* $\nu \le \tilde{\nu}$ *there are continuous embeddings*

$$S^{\nu,l}(\Omega \times \mathbb{R}^{1+q};E,\widetilde{E}) \hookrightarrow S^{\tilde{\nu},l}(\Omega \times \mathbb{R}^{1+q};E,\widetilde{E}). \tag{15}$$

(ii) *It holds*

$$S^{\nu,l}(\Omega \times \mathbb{R}^{1+q};\widehat{E},\widetilde{E}) S^{\tilde{\nu},l}(\Omega \times \mathbb{R}^{1+q};E,\widehat{E}) \subseteq S^{\nu+\tilde{\nu},l}(\Omega \times \mathbb{R}^{1+q};E,\widetilde{E}), \tag{16}$$

with the point–wise composition of the operator–valued symbols.

(iii) *As a particular case of* (ii) *we have*

$$\left. \begin{array}{l} S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}) S^{\tilde{\nu},l}(\Omega \times \mathbb{R}^{1+q};E,\widetilde{E}) \\ S^{\nu,l}(\Omega \times \mathbb{R}^{1+q};E,\widetilde{E}) S^{\tilde{\nu},l}(\Omega \times \mathbb{R}^{1+q}) \end{array} \right\} \subseteq S^{\nu+\tilde{\nu},l}(\Omega \times \mathbb{R}^{1+q};E,\widetilde{E}), \tag{17}$$

with point–wise multiplication of the scalar symbols by the operator–valued ones.

(iv) *For all* $\alpha \in \mathbb{N}^{p_0+p}$, $\beta \in \mathbb{N}^{1+q}$ *and arbitrary* $\nu \in \mathbb{R}$ *we have*

$$D^\alpha_{t,y} D^\beta_{\tau,\eta} S^{\nu,l}(\Omega \times \mathbb{R}^{1+q};E,\widetilde{E}) \subseteq S^{\nu-|\beta|_l,l}(\Omega \times \mathbb{R}^{1+q};E,\widetilde{E}). \tag{18}$$

Equation (16) is the composition rule for operator–valued symbols. The proof of Lemma 7 is straightforward and left to the reader.

Example 8 Let $\phi(x,t,y) \in C^\infty(\mathbb{R}^n \times \Omega)$, $\Omega \subseteq \mathbb{R}^{1+q}$ open, be a function with $\phi(x,t,y) = 0$ for all $(x,t,y) \in \mathbb{R}^n \times \Omega$ with $|x| > c$ for some $c > 0$. Define the family $m_\phi(t,y)$ of multiplication operators by $m_\phi(t,y)u(x) = \phi(x,t,y)u(x)$ for u belonging to the standard Sobolev space $H^s(\mathbb{R}^n)$ on $\mathbb{R}^n$ of smoothness $s \in \mathbb{R}$. Then we have $m_\phi(t,y) \in S^{0,l}(\Omega \times \mathbb{R}^{1+q};H^s(\mathbb{R}^n),H^s(\mathbb{R}^n))$ with respect to the group action $(\kappa_\lambda u)(x) = \lambda^{\frac{n}{2}} u(\lambda x)$, $\lambda \in \mathbb{R}_+$.

Remark 9 The space $S^{-\infty}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$ has a representation as a projective limit of the form $S^{-\infty}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E}) = \varprojlim_{\nu \in \mathbb{R}} S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$, with the projective inclusion spectrum of the Fréchet spaces $\left(S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})\right)_{\nu \in \mathbb{R}}$, and it is independent of the particular choice of $\{\kappa_\lambda\}, \{\tilde{\kappa}_\lambda\}$ of the anisotropy l.

Moreover, the symbol estimates (13) imply $S^{-\infty}(\mathbb{R}^{1+q}; E, \widetilde{E}) = \mathcal{S}(\mathbb{R}^{1+q}, \mathcal{L}(E, \widetilde{E}))$, where the right hand side the Schwartz space of rapidely decreasing functions in $\mathbb{R}^{1+q}$ with values in $\mathcal{L}(E, \widetilde{E})$. Thus Lemma 6 and the stability of the projective tensor product under the projective limit allows us to write $S^{-\infty}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E}) = C^\infty(\Omega, \mathcal{S}(\mathbb{R}^{1+q}, \mathcal{L}(E, \widetilde{E})))$.

There is an important generalisation of the homogeneity of scalar functions to the operator–valued case. Let us start with a function

$$p(\tau, \eta, \xi) \in C^\infty(\mathbb{R}^{1+q+n} \setminus \{0\})$$

which is ν–homogeneous in the sense

$$p(\lambda^l \tau, \lambda\eta, \lambda\xi) = \lambda^\nu p(\tau, \eta, \xi)$$

for all $\lambda \in \mathbb{R}_+$ and $(\tau, \eta, \xi) \in \mathbb{R}^{1+q+n} \setminus \{0\}$ (here $(\tau, \eta) \in \mathbb{R}_\tau \times \mathbb{R}^q$, $\xi \in \mathbb{R}^n$). Then we can form a classical pseudo–differential operator in $\mathbb{R}^n$, defined by

$$f(\tau, \eta)u(x) = \iint e^{i(x-x')\xi} p(\tau, \eta, \xi) u(x') dx' đ\xi, \tag{19}$$

for every fixed $(\tau, \eta) \neq 0$. This can be regarded as an operator function

$$f(\tau, \eta) \in C^\infty(\mathbb{R}^{1+q} \setminus \{0\}, \mathcal{L}(H^s(\mathbb{R}^n), H^{s-\nu}(\mathbb{R}^n)))$$

for every $s \in \mathbb{R}$. Setting $(\kappa_\lambda u)(x) = \lambda^{\frac{n}{2}} u(\lambda x)$, $\lambda \in \mathbb{R}_+$, an elementary calculation shows that

$$f(\lambda^l \tau, \lambda\eta) = \lambda^\nu \kappa_\lambda f(\tau, \eta) \kappa_\lambda^{-1} \tag{20}$$

holds for every $\lambda \in \mathbb{R}_+$ and all $(\tau, \eta) \in \mathbb{R}^{1+q} \setminus \{0\}$. This motivates for general group actions the following definition.

Definition 10 *A function $f(\tau, \eta) \in C^\infty(\mathbb{R}^{1+q} \setminus \{0\}, \mathcal{L}(E, \widetilde{E}))$ is called anisotropic ν–homogeneous in the operator–valued sense if it satisfies*

$$f(\lambda^l \tau, \lambda\eta) = \lambda^\nu \tilde{\kappa}_\lambda f(\tau, \eta) \kappa_\lambda^{-1} \tag{21}$$

for all $\lambda \in \mathbb{R}_+$ and every $(\tau, \eta) \in \mathbb{R}^{1+q} \setminus \{0\}$.

We call the function $f(\tau, \eta) \in C^\infty(\mathbb{R}^{1+q}, \mathcal{L}(E, \widetilde{E}))$ anisotropic ν–homogeneous in the operator–valued sense for large $|\tau, \eta|_l$ if equation (21) holds for all $\lambda \geq 1$ and every $(\tau, \eta) \in \mathbb{R}^{1+q}$ with $|\tau, \eta|_l > c$ for some constant $c > 0$. The smallest constant c with this property is called homogeneity constant of f.

Lemma 11 *Every $a(\tau, \eta) \in C^\infty(\mathbb{R}^{1+q}, \mathcal{L}(E, \widetilde{E}))$ which is anisotropic ν–homogeneous in the operator–valued sense for large $|\tau, \eta|_l$ belongs to $S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$.*

Proof: Because of the (t, y)–independence of the function a we only have to check the estimates

$$\|\tilde{\kappa}^{-1}(\tau, \eta)\{D^\beta_{\tau,\eta} a(\tau, \eta)\}\kappa(\tau, \eta)\| \leq C[\tau, \eta]_l^{\nu - |\beta|_l} \tag{22}$$

for every $\beta \in \mathbb{N}^{1+q}$ and $(\tau,\eta) \in \mathbb{R}^{1+q}$. But for $|\tau,\eta|_l \le c$ we get (22) from the compactness of this anisotropic ball. For $|\tau,\eta|_l > c$, using the homogeneity, we obtain

$$\begin{aligned} & \|\tilde{\kappa}^{-1}(\tau,\eta)(D^\beta_{\tau,\eta}a)(\tau,\eta)\kappa(\tau,\eta)\| \\ = \ & \|\tilde{\kappa}^{-1}(\tau,\eta)[\tau,\eta]_l^{\nu-|\beta|_l}\tilde{\kappa}(\tau,\eta)(D^\beta_{\tau,\eta}a)\left(\frac{\tau}{[\tau,\eta]_l^l},\frac{\eta}{[\tau,\eta]_l}\right)\kappa^{-1}(\tau,\eta)\kappa(\tau,\eta)\| \\ = \ & [\tau,\eta]_l^{\nu-|\beta|_l}\left\|(D^\beta_{\tau,\eta}a)\left(\frac{\tau}{[\tau,\eta]_l^l},\frac{\eta}{[\tau,\eta]_l}\right)\right\| \\ \le \ & C[\tau,\eta]_l^{\nu-|\beta|_l}, \end{aligned}$$

with $C = \sup\{\|D^\beta_{\tau,\eta}a(\tau,\eta)\| \ : \ |\tau,\eta|_l = 1\}$.

In the above estimate we used that the derivatives of homogeneous functions are again homogeneous of the corresponding diminished order. This follows immediately by differentiating equation (21). □

Let us now observe an analogue of Euler's homogeneity relation, cf Proposition 3, in the operator–valued set–up. First we consider the particular case (19); then we pass to the general case.

Proposition 12 *Euler's homogeneity relation for $p(\tau,\eta,\xi)$, namely*

$$\left(l\tau\frac{\partial}{\partial\tau} + \sum_{j=1}^q \eta_j\frac{\partial}{\partial\eta_j} + \sum_{k=1}^n \xi_k\frac{\partial}{\partial\xi_k}\right)p(\tau,\eta,\xi) = \nu p(\tau,\eta,\xi), \tag{23}$$

implies a corresponding homogeneity relation for $f(\tau,\eta)$:

$$E_l f(\tau,\eta) = \left(l\tau\frac{\partial f}{\partial\tau} + \sum_{j=1}^q \eta_j\frac{\partial f}{\partial\eta_j}\right)(\tau,\eta) = (\nu I + H)\circ f(\tau,\eta) \tag{24}$$

with the operator H defined by $u(x) \mapsto Hu(x) = \sum\limits_{k=1}^n \frac{\partial}{\partial x_k}(x_k u)(x)$.
Conversely, (24) *implies* (23) *and hence* (20)*.*

Proof: Using relation (23) we obtain

$$\begin{aligned} \nu f(\tau,\eta)u(x) &= \int e^{ix\xi}\nu p(\tau,\eta,\xi)\hat{u}(\xi)đ\xi \\ &= \int e^{ix\xi}\left(l\tau\frac{\partial}{\partial\tau} + \sum_{j=1}^q\eta_j\frac{\partial}{\partial\eta_j} + \sum_{k=1}^n\xi_k\frac{\partial}{\partial\xi_k}\right)p(\tau,\eta,\xi)\hat{u}(\xi)đ\xi \\ &= \left(l\tau\frac{\partial}{\partial\tau} + \sum_{j=1}^q\eta_j\frac{\partial}{\partial\eta_j}\right)f(\tau,\eta)u(x) + \int e^{ix\xi}\sum_{k=1}^n\xi_k\frac{\partial}{\partial\xi_k}\left(p(\tau,\eta,\xi)\hat{u}(\xi)\right)đ\xi. \end{aligned}$$

Further we have

$$\begin{aligned} \int e^{ix\xi}\sum_{k=1}^n\xi_k\frac{\partial}{\partial\xi_k}\left(p(\tau,\eta,\xi)\hat{u}(\xi)\right)đ\xi &= -\sum_{k=1}^n\frac{\partial}{\partial x_k}\left(x_k\int e^{ix\xi}p(\tau,\eta,\xi)\hat{u}(\xi)đ\xi\right) \\ &= -(H\circ f(\tau,\eta))u(x) \end{aligned}$$

which gives us (24). For the converse direction we use that

$$\int e^{ix\xi}a(\xi)\hat{u}(\xi)đ\xi = \int e^{ix\xi}b(\xi)\hat{u}(\xi)đ\xi$$

for all $u(x) \in \mathcal{S}(\mathbb{R}^n)$ implies $a(\xi)\hat{u}(\xi) = b(\xi)\hat{u}(\xi)$ and hence $(a(\xi)-b(\xi))v(\xi)$ for $v \in \mathcal{S}(\mathbb{R}^n)$. In particular, for $v(\xi) = e^{-\frac{1}{1+|\xi|^2}} \ne 0$ we obtain $a(\xi) = b(\xi)$.

Starting with (24) and using the above computation in opposite direction we get the second part of the proposition. □

Remark 13 Notice that the operator $\hat{H} := \sum_{k=1}^{m} \xi_k \frac{\partial}{\partial \xi_k}$ acting on scalar symbols of the class $S^\nu(V \times \mathbb{R}^m)$, $V \subseteq \mathbb{R}^m$ open, also plays a role in the symbol estimates that are traditionally written

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq c(1+|\xi|)^{\nu-|\beta|}$$

for all $x \in K$, $\xi \in \mathbb{R}$, $K \subset\subset V$ arbitrary, for all multi–indices $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^m$, with constants $c = c(\alpha, \beta, K) > 0$. An alternative description is to define $S^\nu(V \times \mathbb{R}^n)$ as the subspace of all $p(x,\xi) \in C^\infty(V \times \mathbb{R}^n)$ satisfying

$$|D_x^\alpha \hat{H}^k p(x,\xi)| \leq c(1+|\xi|)^\nu$$

for all x, ξ as above and all $\alpha \in \mathbb{N}^m$ and $k \in \mathbb{N}$, with constants $c = c(\alpha, k, K) > 0$.

Remark 14 For the Fourier transform $\mathcal{F} \in \mathbb{R}^n$ acting, e.g., on $\mathcal{S}(\mathbb{R}^n)$ and the group action $(\kappa_\lambda u)(x) = \lambda^{\frac{n}{2}} u(\lambda x)$ we have

$$\begin{aligned}(\mathcal{F}\kappa_\lambda u)(\xi) &= \int e^{-ix\xi} \lambda^{\frac{n}{2}} u(\lambda x) dx \\ &= \int e^{-i\frac{x}{\lambda}\xi} \lambda^{\frac{n}{2}} u(x) \lambda^{-n} dx \\ &= \kappa_\lambda^{-1} (\mathcal{F}u)(\xi),\end{aligned}$$

i.e., $\kappa_\lambda = \mathcal{F}^{-1}\kappa_\lambda^{-1}\mathcal{F}$. This relation explains, in particular, the role of the exponent λ in front of the dilations.

To illustrate the nature of the homogeneity of operator–valued functions we want to consider a further example. Let us start with a function

$$p_0(\tilde{\rho}, \tilde{\tau}, \tilde{\eta}, \tilde{\tau}', \tilde{\eta}') \in C^\infty\left(\mathbb{R}_{\tilde{\rho}} \times \mathbb{R}^{2(1+q)}_{\tilde{\tau},\tilde{\eta},\tilde{\tau}',\tilde{\eta}'}\right).$$

Set $p(r, \tilde{\rho}, \tilde{\tau}, \tilde{\eta}, \tilde{\tau}', \tilde{\eta}') = r^{-\mu} p_0(\tilde{\rho}, \tilde{\tau}, \tilde{\eta}, \tilde{\tau}', \tilde{\eta}')$ and form

$$a(\tau,\eta)u := \iint e^{i(r-r')\rho} p(r, r\rho, r^l\tau, r\eta, r'^l\tau, r'\eta) u(r') dr' đ\rho,$$

regarded as an operator family

$$a(\tau,\eta) : C_0^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R}_+),$$

parametrised by $(\tau,\eta) \in \mathbb{R}^{1+q}$. Setting $(\kappa_\lambda u)(r) = \lambda^{\frac{1}{2}} u(\lambda r)$ for $\lambda \in \mathbb{R}_+$, it follows that

$$a(\lambda^l\tau, \lambda\eta) = \lambda^\nu \kappa_\lambda a(\tau,\eta) \kappa_\lambda^{-1}$$

for all $\lambda \in \mathbb{R}_+$, $(\tau,\eta) \in \mathbb{R}^{1+q}$. Writing

$$f(s, s', \rho, \tau, \eta) := p(s^{-1}, s^{-1}\rho, s^{-l}\tau, s^{-1}\eta, s'^{-l}\tau, s'^{-1}\eta)$$

we have $f(\lambda s, \lambda s', \lambda\rho, \lambda^l\tau, \lambda\eta) = \lambda^\mu f(s, s', \rho, \tau, \eta)$ for all $\lambda \in \mathbb{R}_+$, $s, s' \in \mathbb{R}_+$, $(\rho, \tau, \eta) \in \mathbb{R}^{1+1+q}$. Thus we get, cf. Proposition 3,

$$s\frac{\partial f}{\partial s} + s'\frac{\partial f}{\partial s'} + \rho\frac{\partial f}{\partial \rho} + l\tau\frac{\partial f}{\partial \tau} + \sum_{j=1}^{q} \eta_j \frac{\partial f}{\partial \eta_j} = \mu f, \tag{25}$$

and we can obtain a homogeneity relation for the operator family $a(\tau,\eta)$:

Proposition 15 *The operator family $a(\tau,\eta)$ satisfies the homogeneity relation*

$$E_l a(\tau,\eta) := \left(l\tau\frac{\partial a}{\partial \tau} + \sum_{j=1}^{q} \eta_j \frac{\partial a}{\partial \eta_j}\right)(\tau,\eta) = \mu a(\tau,\eta) + h a(\tau,\eta) - a(\tau,\eta) h \tag{26}$$

for the constant operator $h = r\frac{\partial}{\partial r}$.

Proof: Setting $g(s,s',\rho,\tau,\eta) := f(s^{-1},s'^{-1},\rho,\tau,\eta)$ we get from (25)

$$-s\frac{\partial g}{\partial s} - s'\frac{\partial g}{\partial s'} + \rho\frac{\partial g}{\partial \rho} + l\tau\frac{\partial g}{\partial \tau} + \sum_{j=1}^{q}\eta_j\frac{\partial g}{\partial \eta_j} = \mu g. \tag{27}$$

This yields

$$\{\mu a(\tau,\eta) - E_l a(\tau,\eta)\}u(t) = \iint e^{i(s-s')\rho}\left(-s\frac{\partial}{\partial s} - s'\frac{\partial}{\partial s'} + \rho\frac{\partial}{\partial \rho}\right) g(s,s',\rho,\tau,\eta)u(s')ds'đ\rho. \tag{28}$$

Using $\frac{\partial}{\partial s}\left(e^{is\rho}g\right) - i\rho e^{is\rho}g = e^{is\rho}\frac{\partial g}{\partial s}$ and $\frac{\partial}{\partial s'}\left(e^{-is'\rho}g\right) + i\rho e^{-is'\rho}g = e^{-is'\rho}\frac{\partial g}{\partial s'}$ the right hand side of (28) may be written

$$-s\frac{\partial}{\partial s}a(\tau,\eta)u(s) + I_1 + I_2 + I_3$$

for

$$\begin{aligned} I_1 &= \iint e^{i(s-s')\rho} i(s-s')\rho g u(s')ds'đ\rho, \\ I_2 &= \iint e^{i(s-s')\rho} g\frac{\partial}{\partial s'}(s'u(s'))ds'đ\rho, \\ I_3 &= \iint e^{i(s-s')\rho}\rho\frac{\partial g}{\partial \rho}u(s')ds'đ\rho. \end{aligned}$$

Applying $\rho\frac{\partial}{\partial\rho}\left(e^{i(s-s')\rho}g\right) = i(s-s')\rho e^{i(s-s')\rho}g + e^{i(s-s')\rho}\rho\frac{\partial g}{\partial \rho}$ it follows that $I_3 = I_0 - I_1$ for

$$I_0 = \iint \rho\frac{\partial}{\partial\rho}\left(e^{i(s-s')\rho}g\right)u(s')ds'đ\rho.$$

Moreover, for I_2 we obtain $I_2 = a(\tau,\eta)u(s) + a(\tau,\eta)s\frac{\partial}{\partial s}u(s)$. In I_0 we may replace $\rho\frac{\partial}{\partial\rho}$ by the operator $\frac{\partial}{\partial\rho}\rho - \mathrm{id}$. Thus

$$I_0 = -a(\tau,\eta)u(s)\iint \frac{\partial}{\partial\rho}\rho\left(e^{i(s-s')\rho}g\right)u(s')ds'đ\rho.$$

The second term of the right of the latter equation vanishes. So $I_3 = -a(\tau,\eta)u(s) - I_1$ and hence

$$\begin{aligned} I_1 + I_2 + I_3 &= I_1 + a(\tau,\eta)u(s) + a(\tau,\eta)hu(s) - a(\tau,\eta)u(s) - I_1 \\ &= a(\tau,\eta)hu(s) \end{aligned}$$

We thus obtained $\mu a(\tau,\eta) - E_l a(\tau,\eta) = -ha(\tau,\eta) + a(\tau,\eta)h$ which is the assertion. □

The following interesting generalisation of Euler's homogeneity relation for operator–valued functions is due to Th. Krainer. Recall that we are interested in $\mathcal{L}(E,\widetilde{E})$–valued functions, where E and $\widetilde{E}$ are Banach spaces endowed with group actions κ_λ and $\tilde{\kappa}_\lambda$. By A and $\widetilde{A}$ we denote the infinitesimal generators of the C_0 groups $Q:\mathbb{R}\to\mathcal{L}(E)$ with $Q(t) = \kappa_{\exp(t)}$ and $\widetilde{Q}:\mathbb{R}\to\mathcal{L}(\widetilde{E})$ with $\widetilde{Q}(t) = \tilde{\kappa}_{\exp(t)}$, respectively. Further let $D(A)$ and $D(\widetilde{A})$ be the domains of A and $\widetilde{A}$. Then we obtain the following result.

Proposition 16 *A function $f(\tau,\eta) \in C^\infty(\mathbb{R}^{1+q}\setminus\{0\},\mathcal{L}(E,\widetilde{E}))$ with $f(\tau,\eta)D(A) \subseteq D(\widetilde{A})$ for all $(\tau,\eta)\in\mathbb{R}^{1+q}$ is anisotropic ν–homogeneous in the operator–valued sense if and only if for every $u \in D(A)$ the differential equation*

$$E_l f(\tau,\eta)u = (\nu f(\tau,\eta) + \widetilde{A}f(\tau,\eta) - f(\tau,\eta)A)u$$

is satisfied.

Proof: Suppose first f to be anisotropic ν–homogeneous in the operator–valued sense. Then for all $(\tau,\eta)\in\mathbb{R}^{1+q}\setminus\{0\}$ we have

$$f(\tau,\eta) = |\tau,\eta|_l^\nu\,\tilde{\kappa}_{|\tau,\eta|_l}f\left(\frac{\tau}{|\tau,\eta|_l^l},\frac{\eta}{|\tau,\eta|_l}\right)\kappa^{-1}_{|\tau,\eta|_l}.$$

In anisotropic polar coordinates (ρ,σ), cf. Proposition 3, this means

$$f(\rho,\sigma)=\rho^{\nu}\tilde{\kappa}_{\rho}f(1,\sigma)\kappa_{\rho}^{-1}=\rho^{\nu}\widetilde{Q}(\ln\rho)f(1,\sigma)Q(-\ln\rho).$$

Then for $u\in D(A)$ we can form $\frac{\partial}{\partial\rho}f(\rho,\sigma)u$ and it holds

$$\begin{aligned}\left(\rho\frac{\partial}{\partial\rho}\right)f(\rho,\sigma)u \;&=\; \rho\Big(\nu\rho^{\nu-1}\widetilde{Q}(\ln\rho)f(1,\sigma)Q(-\ln\rho)\\ &\quad +\rho^{\nu-1}\widetilde{A}\widetilde{Q}(\ln\rho)f(1,\sigma)Q(-\ln\rho)\\ &\quad -\rho^{\nu-1}\widetilde{Q}(\ln\rho)f(1,\sigma)Q(-\ln\rho)A\Big)u\\ &=\; \Big(\nu\rho^{\nu}\tilde{\kappa}_{\rho}f(1,\sigma)\kappa_{\rho}^{-1}+\rho^{\nu}\widetilde{A}\tilde{\kappa}_{\rho}f(1,\sigma)\kappa_{\rho}^{-1}-\rho^{\nu}\tilde{\kappa}_{\rho}f(1,\sigma)\kappa_{\rho}^{-1}A\Big)u\\ &=\; (\nu f(\rho,\sigma)+\widetilde{A}f(\rho,\sigma)-f(\rho,\sigma)A)u\end{aligned}$$

which is the desired differential equation in polar coordinates.

On the other hand we suppose that f satisfies

$$\left(\rho\frac{\partial}{\partial\rho}\right)f(\rho,\sigma)u=(\nu f(\rho,\sigma)+\widetilde{A}f(\rho,\sigma)-f(\rho,\sigma)A)u \tag{29}$$

for all $u\in D(A)$. Then setting $F(\rho,\sigma)=\tilde{\kappa}_{\rho}^{-1}f(\rho,\sigma)\kappa_{\rho}=\widetilde{Q}(-\ln\rho)f(\rho,\sigma)Q(\ln\rho)$ we get

$$\left(\rho\frac{\partial}{\partial\rho}\right)F(\rho,\sigma)u=\left(-\widetilde{A}\tilde{\kappa}_{\rho}^{-1}f(\rho,\sigma)\kappa_{\rho}+\tilde{\kappa}_{\rho}^{-1}\left(\left(\rho\frac{\partial}{\partial\rho}\right)f(\rho,\sigma)\right)\kappa_{\rho}+\tilde{\kappa}_{\rho}^{-1}f(\rho,\sigma)\kappa_{\rho}\right)u.$$

Therefore inserting (29) we have $\left(\rho\frac{\partial}{\partial\rho}\right)F(\rho,\sigma)u=\rho^{\nu}F(1,\sigma)u$ or $f(\rho,\sigma)=\rho^{\nu}\tilde{\kappa}_{\rho}f(1,\sigma)\kappa_{\rho}^{-1}$, and that is equivalent to the anisotropic ν–homogeneity in the operator–valued sense. □

1.2 Classical symbols

We now turn the subspaces of classical anisotropic operator–valued symbols. The scalar versions of these spaces can be found in [Pi1]. Denote by $S^{(\nu),l}(\Omega\times(\mathbb{R}^{1+q}\backslash\{0\});E,\widetilde{E})$ the subspace of all operator–valued functions $f_{(\nu)}(t,y,\tau,\eta)\in C^{\infty}(\Omega\times(\mathbb{R}^{1+q}\backslash\{0\}),\mathcal{L}(E,\widetilde{E}))$ that are ν–homogeneous in (τ,η) for all $(t,y)\in\Omega$.

By definition we have

$$S^{(\nu),l}(\Omega\times(\mathbb{R}^{1+q}\backslash\{0\});E,\widetilde{E})\hookrightarrow C^{\infty}(\Omega\times(\mathbb{R}^{1+q}\backslash\{0\}),\mathcal{L}(E,\widetilde{E})) \tag{1}$$

and $S^{(\nu),l}(\Omega\times(\mathbb{R}^{1+q}\backslash\{0\});E,\widetilde{E})$ is a closed subspace of $C^{\infty}(\Omega\times(\mathbb{R}^{1+q}\backslash\{0\}),\mathcal{L}(E,\widetilde{E}))$ in the induced topology.

Furthermore, denote by $S^{[\nu],l}(\Omega\times\mathbb{R}^{1+q};E,\widetilde{E})$ the subspace of all operator–valued functions $f(t,y,\tau,\eta)\in C^{\infty}(\Omega\times\mathbb{R}^{1+q},\mathcal{L}(E,\widetilde{E}))$ that are ν–homogeneous in (τ,η) for large $|\tau,\eta|_l$. For an arbitrary excision function $\chi(\tau,\eta)$ we have

$$\chi(\tau,\eta)S^{(\nu),l}(\Omega\times(\mathbb{R}^{1+q}\backslash\{0\});E,\widetilde{E})\subset S^{[\nu],l}(\Omega\times\mathbb{R}^{1+q};E,\widetilde{E}).$$

From Lemma 11 it follows that $S^{[\nu],l}(\Omega\times\mathbb{R}^{1+q};E,\widetilde{E})\subset S^{\nu,l}(\Omega\times\mathbb{R}^{1+q};E,\widetilde{E})$.

Definition 1 *Let $\nu\in\mathbb{R}$ and $\Omega=U_0\times U$ with open $U_0\times U\in\mathbb{R}^{p_0+p}$ as in Definition 5. Then the space of classical anisotropic operator–valued symbols of order ν*

$$S_{cl}^{\nu,l}(\Omega\times\mathbb{R}^{1+q};E,\widetilde{E}) \tag{2}$$

is defined as the set of all $a(t,y,\tau,\eta) \in S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$ *such that there is a sequence* $(a_{(\nu-j)}(t,y,\tau,\eta))_{j=0}^{\infty}$ *with* $a_{(\nu-j)}(t,y,\tau,\eta) \in S^{(\nu-j),l}(\Omega \times (\mathbb{R}^{1+q}\setminus\{0\}); E, \widetilde{E})$ *for every* j*, satisfying the relation*

$$a(t,y,\tau,\eta) - \sum_{j=0}^{N} \chi(\tau,\eta) a_{(\nu-j)}(t,y,\tau,\eta) \in S^{\nu-N-1,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$$

for every $N \in \mathbb{N}$ *and for any excision function* $\chi(\tau,\eta)$*. Analogously to the isotropic case the function* $\sigma_{\wedge}^{\nu}(a)(t,y,\tau,\eta) := a_{(\nu)}(t,y,\tau,\eta)$ *is called the* ν*–homogeneous principal symbol of* a*.*

Remark 2 The $(\nu - j)$–homogeneous components $a_{(\nu-j)}(t,y,\tau,\eta)$ of an $a(t,y,\tau,\eta) \in S_{cl}^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$ are uniquely determined.

The notation $\sigma_{\wedge}^{\nu}$ for the homogeneous principal symbol is motivated by one of the main applications of the theory namely to equations on manifolds with edges. There is then an operator–valued homogeneous principal edge symbol (which is an analogue of the classical boundary symbol in boundary value problems) together with a homogeneous principal interior symbol, denoted by σ_{ψ}^{ν}. For higher singularities there are additional principal symbolic levels associated with lower–dimentional skeletons of the given configuration.

Remark 3 Every function $a(t,y,\tau,\eta) \in S^{[\nu],l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$ belongs to $S_{cl}^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$. Taking the uniquely determined ν–homogeneous extension $a_{(\nu)}(t,y,\tau,\eta) \in S^{(\nu),l}(\Omega \times (\mathbb{R}^{1+q}\setminus\{0\}); E, \widetilde{E})$ of $a(t,y,\tau,\eta)|_{\{[\tau,\eta]_l = K\}}$ with some constant $K > 0$ larger than the homogeneity constant of a, we have

$$a(t,y,\tau,\eta) - \chi(\tau,\eta) a_{(\nu)}(t,y,\tau,\eta) \in C^{\infty}(\Omega, \mathcal{S}(\mathbb{R}^{1+q}, \mathcal{L}(E, \widetilde{E})))$$

for every excision function $\chi(\tau,\eta)$. In other words $a(t,y,\tau,\eta) - \chi(\tau,\eta)a_{(\nu)}(t,y,\tau,\eta) \in S^{-\infty}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$.

Example 4 Setting $E = H^s(\mathbb{R}^n)$ and $\widetilde{E} = H^{s-\mu}(\mathbb{R}^n)$ endowed with the standard group actions $(\kappa_\lambda u)(x) = \lambda^{\frac{n}{2}} u(\lambda x),\ \lambda \in \mathbb{R}_+$, we form

$$a(\tau,\eta) = \mathrm{op}_x\left((|\xi|^l + i\tau + |\eta|^l)^{\frac{\mu}{l}} \right) = \int e^{i(x-x')\xi} (|\xi|^l + i\tau + |\eta|^l)^{\frac{\mu}{l}} u(x') dx' đ\xi,$$

where the powers are defined by the branch of logarithm that is real for real arguments. Then an elementary calculation shows $\kappa_\lambda^{-1} a(\lambda^l \tau, \lambda\eta) \kappa_\lambda u = \lambda^\mu a(\tau,\eta)u$, i.e., $a(\tau,\eta)$ is anisotropic μ–homogeneous in the operator–valued sense. Thus we have $\chi(\tau,\eta)a(\tau,\eta) \in S_{cl}^{\mu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$ for every excision function $\chi(\tau,\eta)$.

Next we want to define a Fréchet topology in the space of classical anisotropic operator–valued symbols. By definition we have a canonical embedding

$$S_{cl}^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E}) \hookrightarrow S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E}) \tag{3}$$

for every $\nu \in \mathbb{R}$. Moreover, the relation (1) induces a Fréchet topology in $S^{(\nu),l}(\Omega \times (\mathbb{R}^{1+q}\setminus\{0\}); E, \widetilde{E})$.

By $a \mapsto a_{(\nu-j)}$ we get a sequence of linear mappings

$$\alpha_j \ :\ S_{cl}^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E}) \longrightarrow S^{(\nu-j),l}(\Omega \times (\mathbb{R}^{1+q}\setminus\{0\}); E, \widetilde{E}),\ j \in \mathbb{N}. \tag{4}$$

Furthermore, by $a(t,y,\tau,\eta) \mapsto \chi(\tau,\eta) \sum_{j=0}^{k} a_{(\nu-j)}(t,y,\tau,\eta)$ we get a sequence of linear mappings

$$\beta_k \ :\ S_{cl}^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E}) \longrightarrow S^{\nu-k-1,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E}),\ k \in \mathbb{N} \tag{5}$$

that is well–defined by Remark 2. Now we topologise (2) with the weakest locally convex topology such that all the mappings (3), (4) and (5) are continuous. The completeness then follows analogously to the isotropic case.

Remark 5 The assertions of 1.1 Lemma 6 and 1.1 Lemma 7 also hold in analogous form for classical symbols.

1.3 Examples and remarks

In the applications the spaces E often run over scales of Hilbert spaces $\{E^s\}_{s\in\mathbb{R}}$ analogously to the standard Sobolev spaces. In particular, we have continuous embeddings $E^{s'} \hookrightarrow E^s$, when $s' \geq s$ and $E^\infty := \bigcap_{s\in\mathbb{R}} E^s$ is dense in every E^s, $s \in \mathbb{R}$. Further, there is a reference space E^0, where the scalar product

$$(\cdot,\cdot)_{E^0} : E^\infty \times E^\infty \to \mathbb{C}$$

extends to a non–degenerate sesquilinear pairing

$$(\cdot,\cdot)_{E^0} : E^s \times E^{-s} \to \mathbb{C}$$

for every $s \in \mathbb{R}$. Moreover, on E^0 we have a unitary group action (i.e. κ_λ is an unitary operator for all $\lambda \in \mathbb{R}_+$) that restricts (or extends) to a group action on E^s for all $s \in \mathbb{R}$. Setting $E = E^s$ and $E^{-s} = E'$ for fixed $s \geq 0$ we get Gelfand triples

$$\{E, E^0, E'; \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}\} \tag{1}$$

with unitary group actions. The simplest example is $E = H^s(\mathbb{R}^n)$, $E^0 = L^2(\mathbb{R}^n)$, $E' = H^{-s}(\mathbb{R}^n)$ for $s \geq 0$ and $(\kappa_\lambda u)(x) = \lambda^{\frac{n}{2}} u(\lambda x)$, $\lambda \in \mathbb{R}_+$.

More general Gelfand triples appear when we insert for E the weighted cone Sobolev spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$, $s, \gamma \in \mathbb{R}$. Here X is a closed compact C^∞ manifold of dimension n and $X^\wedge = \mathbb{R}_+ \times X$. First $\mathcal{H}^{0,0}(X^\wedge)$ is defined as $r^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times X)$, where the L^2 space refers to the measure $drdx$ with dx being associated with a Riemannian metric on X. Then $\mathcal{H}^{s,0}(X^\wedge)$ for $s \in \mathbb{N}$ is the subspace of all $u \in \mathcal{H}^{0,0}(X^\wedge)$ such that $(rD_r)^k v_1 \cdot \ldots \cdot v_N u \in \mathcal{H}^{0,0}(X^\wedge)$ for every choice of vector fields v_j on X and arbitrary k, N with $k + N \leq s$. Then by duality with respect to $(\cdot,\cdot)_{\mathcal{H}^{0,0}(X^\wedge)}$ we can pass to $\mathcal{H}^{s,0}(X^\wedge)$ for $-s \in \mathbb{N}$ and finally obtain $\mathcal{H}^{s,0}(X^\wedge)$ for arbitrary $s \in \mathbb{R}$ by interpolation.

The spaces $\mathcal{H}^{s,0}(X^\wedge)$ now form a scale in the above sense. More generally we set $\mathcal{H}^{s,\gamma}(X^\wedge) = r^\gamma \mathcal{H}^{s,0}(X^\wedge)$ for arbitrary $s, \gamma \in \mathbb{R}$. Here s is the smoothness as before and γ is a weight as it plays a role for the conical singularities $r \to 0$.

The applications require spaces that are modelled on the usual Sobolev spaces for $r \to \infty$ that is why we pass to the $\mathcal{K}^{s,\gamma}(X^\wedge)$ spaces defined by setting

$$\mathcal{K}^{s,\gamma}(X^\wedge) := [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]H^s(X^\wedge)$$

with some cut–off function $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$; here $H^s(X^\wedge)$ is the subspace of all $u \in H^s_{loc}(X^\wedge)$ with $(\chi_{\widetilde{V}})_*((1-\omega(r))\phi u) \in H^s(\mathbb{R}^{1+n})$ for every chart $(\widetilde{V}, \phi)$ on X.

Here we used the (non–direct) sum $G = E + F$ of Banach spaces E, F that are contained in a Hausdorff vector space. It consists of all $g = e + f$ with $e \in E$ and $f \in F$. If $\Delta = \{(g, -g) : g \in E \cap F\}$, then $G \cong (E \oplus F)/\Delta$ leads to a Banach structure in G. Further, if a Banach space E is a module over an algebra A, then $[a]E$ for $a \in A$ will denote the closure of $\{ae : e \in E\}$ in E. In particular, the spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}(X^\wedge)$ are modules over $C_0^\infty(\overline{\mathbb{R}}_+ \times X)$ as well as over the algebra of all $\phi(r,x) \in C^\infty(X^\wedge)$ which vanish near $r = 0$ and for which $1 - \phi \in C_0^\infty(\overline{\mathbb{R}}_+ \times X)$.

The cone Sobolev spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ are (Hilbertisable) Banach spaces and equipped with the group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ with $(\kappa_\lambda u)(r,x) = (\tilde{\kappa}_\lambda u)(r,x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x)$ for all $u(r,x) \in \mathcal{K}^{s,\gamma}(X^\wedge)$, $n = \dim X$. Then we can define the symbol spaces $S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge))$.

An example is the operator M_ϕ of multiplication by $\phi(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ which belongs to $S^{0,l}(\mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge))$ for arbitrary $s, \gamma \in \mathbb{R}$. Moreover, $M : \phi \mapsto M_\phi$ induces a continuous map $M : C_0^\infty(\overline{\mathbb{R}}_+) \to S^{0,l}(\mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge))$ for all $s \in \mathbb{R}$.

Let us now consider the case of Fréchet spaces E and $\widetilde{E}$. Assume that there are representations

$$E = \varprojlim_{j\in\mathbb{N}} E_j \text{ and } \widetilde{E} = \varprojlim_{k\in\mathbb{N}} \widetilde{E}_k$$

with projective inclusion spectra of Banach spaces $E_0 \hookleftarrow E_1 \hookleftarrow \cdots$ and $\widetilde{E}_0 \hookleftarrow \widetilde{E}_1 \hookleftarrow \cdots$ such that the associated sets $\{\kappa_\lambda^{(j)}\}_{j=0}^\infty$ and $\{\tilde{\kappa}_\lambda^{(k)}\}_{k=0}^\infty$ satisfy the compatibility conditions $\kappa_\lambda^{(j)}|_{E_{j+1}} = \kappa_\lambda^{(j+1)}$ and $\tilde{\kappa}_\lambda^{(k)}|_{\widetilde{E}_{k+1}} = \tilde{\kappa}_\lambda^{(k+1)}$ for all $j, k \in \mathbb{N}$. Without loss of generality we also assume that the corresponding norms q_j in E_j and $\tilde{q}_k$ in $\widetilde{E}_k$ are ordered norm systems in E and $\widetilde{E}$, respectively.

Definition 1 *Under the above assumptions we define for $\nu \in \mathbb{R}$ and an open set $\Omega = U_0 \times U \subseteq \mathbb{R}^{p_0+p}$ the space $S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$ of anisotropic symbols of order ν as the set of all functions $a(t, y, \tau, \eta)$ with values in $\mathcal{L}(E, \widetilde{E})$ such that for every $k \in \mathbb{N}$ there is some $j = j(k) \in \mathbb{N}$ with $a(t, y, \tau, \eta) \in S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E_j, \widetilde{E}_k)$.*

The subspaces of classical anisotropic symbols $S^{\nu,l}_{cl}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$ are defined by the condition $S^{\nu,l}_{cl}(\Omega \times \mathbb{R}^{1+q}; E_j, \widetilde{E}_k)$ for every $k \in \mathbb{N}$ with some $j = j(k) \in \mathbb{N}$.

Remark 2 The assertions of 1.1 Lemma 7 are also valid for Fréchet spaces E and $\widetilde{E}$.

Example 3 Let us describe a class of concrete Fréchet spaces which are necessary in the theory of parabolic operators on manifolds with conical and edge singularities, cf. [BS]. Fix a weight γ and a weight interval $\Theta = (\theta, 0]$, $-\infty \leq \theta < 0$, and set

$$\mathcal{K}^{s,\gamma}_\Theta(X^\wedge) = \varprojlim_{j\in\mathbb{N}} \mathcal{K}^{s,\gamma-\theta-\frac{1}{j+1}}(X^\wedge)$$

in the Fréchet topology of the projective limit. Denote by $\mathrm{As}(\gamma, \Theta)$ the set of all sequences, so–called asymptotic types, $P = \{(p_j, m_j, L_j)\}_{j=0,1\dots,N}$ for $N = N(P)$ when θ is finite, $P = \{(p_j, m_j, L_j)\}_{j\in\mathbb{N}}$ when θ is infinite, where $p_j \in \mathbb{C}$, $\frac{n+1}{2} - \gamma + \theta \leq \mathrm{Re}\, p_j \leq \frac{n+1}{2} - \gamma$ for all j, $\mathrm{Re}\, p_j \to \infty$ for $j \to \infty$ in the infinite case, $m_j \in \mathbb{N}$ and the L_j are finite–dimensional subspaces of $C^\infty(X)$ for all j.

Then $\mathcal{K}^{s,\gamma}_P(X^\wedge)$ for $s \in \mathbb{R}$ and $P \in \mathrm{As}(\gamma, \Theta)$ is defined as the subspace of all $u \in \mathcal{K}^{s,\gamma}(X^\wedge)$ for which there are coefficients $c_{jk} = c_{jk}(u) \in L - j$, $0 \leq k \leq m_j$, such that

$$u_N(r, x) = u(r, x) + \omega(r) \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk} r^{-p_j} \log^k r \in \mathcal{K}^{s,\gamma}_\Theta(X^\wedge)$$

holds for $N = N(P)$, when θ is finite, $u_N \in \mathcal{K}^{s,\gamma}_{(\beta,0]}(X^\wedge)$ for each $\beta < 0$ and all $N \geq N(\beta)$ for some $N(\beta) \in \mathbb{N}$ in the infinite case. Here $\omega(r)$ is an arbitrary cut–off function. The spaces $\mathcal{K}^{s,\gamma}_P(X^\wedge)$ are Fréchet in a natural way, and we then have for arbitrary $P \in \mathrm{As}(\gamma, \Theta)$ and $Q \in \mathrm{As}(\delta, \Theta)$ the symbol spaces $S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}_P(X^\wedge), \mathcal{K}^{r,\delta}_Q(X^\wedge))$ for $s, r \in \mathbb{R}$.

2 Pseudo–differential operators

We now pass to anisotropic pseudo–differential operators with operator–valued symbols. As in the previous section we first fix Banach spaces E and $\widetilde{E}$ and associated strongly continuous groups κ_λ and $\tilde{\kappa}_\lambda$, respectively.

2.1 Local operators and distributional kernels

Let $\mathcal{S}(\mathbb{R}^{1+q}, E) = \mathcal{S}(\mathbb{R}^{1+q})\widehat{\otimes}_\pi E$ be the Schwartz space of rapidely decreasing E–valued functions. We then define the Fourier transform

$$\mathcal{F} \ : \ \mathcal{S}(\mathbb{R}^{1+q}, E) \to \mathcal{S}(\mathbb{R}^{1+q}, E)$$

by $\mathcal{F} = F \otimes \mathrm{id}_E$, where F denotes the standard scalar Fourier transform. Setting

$$\mathcal{S}'(\mathbb{R}^{1+q}, E) := \mathcal{L}(\mathcal{S}(\mathbb{R}^{1+q}), E)$$

we define for any $T \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{1+q}_{t,y}), E)$ the Fourier transform $\mathcal{F}T = T \circ F \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{1+q}_{\tau,\eta}), E)$. Then we get the inverse Fourier transform by $\mathcal{F}^{-1}S = S \circ F^{-1}$ for $S \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{1+q}_{\tau,\eta}), E)$. We then obviously obtain $\mathcal{F}^{-1}\mathcal{F}T = T$. Since $\mathcal{F} = F$ and $\mathcal{F}^{-1} = F^{-1}$ for $E = \mathbb{C}$ we use the same letter for the Fourier transform of vector–valued distributions and scalar ones.

Definition 1 *Let $a(t, y, t', y', \tau, \eta) \in S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$ with a given open set $\Omega = U_0 \times U \subseteq \mathbb{R}^{1+q}$. Then we define*

$$\begin{aligned} \mathrm{Op}(a)u(t, y) &= \mathcal{F}^{-1}_{(\tau,\eta)\to(t,y)}\mathcal{F}_{(t',y')\to(\tau,\eta)}\{a(t, y, t', y', \tau, \eta)u(t', y')\} \\ &= \iint e^{i(t-t')\tau + i(y-y')\eta} a(t, y, t', y', \tau, \eta)u(t', y')dt'\, dy'\, đ\tau\, đ\eta \end{aligned} \tag{1}$$

for $u(t', y') \in C_0^\infty(\Omega, E)$.

We denote by $\Psi^{\nu,l}(\Omega; E, \widetilde{E})$ the Fréchet space of all operators $\mathrm{Op}(a)$ with $a(t, y, t', y', \tau, \eta) \in S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$. Furthermore, we write $\Psi^{\nu,l}_{cl}(\Omega; E, \widetilde{E})$ for the subspace of all $\mathrm{Op}(a)$ with classical symbols and call the elements of $\Psi^{\nu,l}(\Omega; E, \widetilde{E})$ anisotropic pseudo–differential operators, those in $\Psi^{\nu,l}_{cl}(\Omega; E, \widetilde{E})$ classical anisotropic pseudo–differential operators of order ν.

Remark 2 Similarly to the scalar theory each $A \in \Psi^{\nu,l}(\Omega; E, \widetilde{E})$ is continuous as operator

$$\mathrm{Op}(a) : \quad C_0^\infty(\Omega, E) \to C^\infty(\Omega, \widetilde{E}).$$

Analogously to the isotropic theory we use the notation

$$\Psi^{\infty,l}(\Omega; E, \widetilde{E}) = \bigcup_{\nu\in\mathbb{R}} \Psi^{\nu,l}(\Omega; E, \widetilde{E}), \tag{2}$$

$$\Psi^{-\infty}(\Omega; E, \widetilde{E}) = \bigcap_{\nu\in\mathbb{R}} \Psi^{\nu,l}(\Omega; E, \widetilde{E}). \tag{3}$$

The space $\Psi^{-\infty}(\Omega; E, \widetilde{E})$ is isomorphic to the space of all integral operators with kernel in $C^\infty(\Omega \times \Omega), \mathcal{L}(E, \widetilde{E}))$.

Note that in (1) we can also use operator–valued symbols depending only on $(t, y) \in \Omega$ or $(t', y') \in \Omega$. In this case we will call them left or right symbol, respectively. In the next section we will show, that it is possible to choose a left or right symbol as representative modulo smoothing operators for an arbitrary symbol, i.e., for every $a = a(t, y, t', y', \tau, \eta)$ belonging to $S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$ there exist $a_l = a_l(t, y, \tau, \eta) \in S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$ and $a_r = a_r(t', y', \tau, \eta) \in S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$ such that $\mathrm{Op}(a_l) + G_l = \mathrm{Op}(a) = \mathrm{Op}(a_r) + G_r$ with G_l, G_r belonging to $\Psi^{-\infty}(\Omega; E, \widetilde{E})$. Notice that a_l or a_r by not unique.

Next we discuss the distributional kernels of operators in $\Psi^{\nu,l}(\Omega; E, \widetilde{E})$. In view of Remark 2 every $A = \mathrm{Op}(a) \in \Psi^{\infty,l}(\Omega; E, \widetilde{E})$ has a kernel $K_A \in \mathcal{D}'(\Omega^2, \mathcal{L}(E, \widetilde{E}))$, i.e., we have $Au = \langle K_A, u\rangle$ for all $u \in C_0^\infty(\Omega, E)$.

Let us write

$$\mathrm{Op}(a)u(t, y) := \int K(a)(t, y, t', y', t - t', y - y')u(t', y')dt'\, dy'$$

with

$$K(a)(t,y,t',y',\rho,\sigma) := \int e^{i(\rho\tau+\sigma\eta)} a(t,y,t',y',\tau,\eta) đ\tau\, đ\eta. \tag{4}$$

These integrals exist in the classical sense for sufficiently negative order ν, otherwise in the distributional sense. Because of 1.1 Lemma 6 we can describe $K(a)$ first for symbols with constant coefficients. The case of variable coefficients is then an easy generalisation. For constant coefficients we have $K(a)(\rho,\sigma) = \int e^{i(\rho\tau+\sigma\eta)} a(\tau,\eta) đ\tau\, đ\eta$. Because of $S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E}) \subset \mathcal{S}'(\mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E}))$ we can write $K(a)(\rho,\sigma) = \mathcal{F}^{-1}_{(\tau,\eta)\to(\rho,\sigma)}\{a(\tau,\eta)\}$, and hence $K(a)$ belongs to $\mathcal{S}'(\mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E}))$. For $a(\tau,\eta) \in S^{-\infty}(\mathbb{R}^{1+q}; E, \widetilde{E}) = \mathcal{S}(\mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E}))$ the distribution $K(a)$ is even a regular distribution belonging to $\mathcal{S}(\mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E}))$. Using that the inverse Fourier transform induces an isomorphism $\mathcal{F}^{-1} : S^{-\infty}(\mathbb{R}^{1+q}; E, \widetilde{E}) \to \mathcal{S}(\mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E}))$ we define the space $T^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E}) := \mathcal{F}^{-1}_{(\tau,\eta)\to(\rho,\sigma)}\left(S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})\right)$ with constant coefficients; in the general case we set

$$T^{\nu,l}(\Omega^2\times\mathbb{R}^{1+q}; E, \widetilde{E}) := \mathcal{F}^{-1}_{(\tau,\eta)\to(\rho,\sigma)}\left(S^{\nu,l}(\Omega^2\times\mathbb{R}^{1+q}; E, \widetilde{E})\right).$$

We topologise these spaces with the Fréchet topology induced from the symbol spaces by the inverse Fourier transform. Analogously to 1.1 Lemma 6 we obtain

$$T^{\nu,l}(\Omega^2\times\mathbb{R}^{1+q}; E, \widetilde{E}) = C^\infty(\Omega^2, T^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})) = C^\infty(\Omega^2)\hat{\otimes}_\pi T^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})). \tag{5}$$

For any $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(\Omega^2\times\mathbb{R}^{1+q}; E, \widetilde{E})$ we then get the kernel $K_{\mathrm{Op}(a)}$ of the corresponding pseudo–differential operator by the formula

$$K_{\mathrm{Op}(a)}(t,y,t',y') = K(a)(t,y,t',y',\rho,\sigma)|_{(\rho,\sigma)=(t-t',y-y')}.$$

Denote by $\mathcal{S}'_0(\mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E}))$ the subspace of all $f(\rho,\sigma) \in \mathcal{S}'(\mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E}))$ with $\chi(\rho,\sigma)f(\rho,\sigma) \in \mathcal{S}(\mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E}))$ for every (ρ,σ)–excision function $\chi(\rho,\sigma)$.

Proposition 3 *For every $a(\tau,\eta) \in S^{\infty,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$ we have $K(a) \in \mathcal{S}'_0(\mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E}))$.*

Proof: By definition there is a $\nu \in \mathbb{R}$ with $a(\tau,\eta) \in S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$ for every $\nu \in \mathbb{R}$. We have to show

$$q_{\alpha,\beta}(\chi K(a)) := \sup_{(\rho,\sigma)\in\mathbb{R}^{1+q}} \left\|(\rho,\sigma)^\beta D^\alpha_{\rho,\sigma}\left(\chi(\rho,\sigma)K(a)(\rho,\sigma)\right)\right\|_{\mathcal{L}(E,\widetilde{E})} < \infty$$

for all $\alpha,\beta \in \mathbb{N}^{1+q}$, for an arbitrary excision function $\chi(\rho,\sigma)$. Without loss of generality we may assume that $|\rho| \geq 1$. Then it follows that

$$\left\|(\rho,\sigma)^\beta D^\alpha_{\rho,\sigma}\left(\chi(\rho,\sigma)K(a)(\rho,\sigma)\right)\right\| \leq \left\|\rho^F(\rho,\sigma)^\beta D^\alpha_{\rho,\sigma}\left(\chi(\rho,\sigma)K(a)(\rho,\sigma)\right)\right\|$$

for some $1 \leq F \in \mathbb{N}$. But then we obtain

$$\begin{aligned}
q_{\alpha,\beta}(\chi K(a)) &\leq \sup_{(\rho,\sigma)\in\mathbb{R}^{1+q}} \left\|\rho^F(\rho,\sigma)^\beta D^\alpha_{\rho,\sigma}\left(\chi(\rho,\sigma)\int e^{i(\rho\tau+\sigma\eta)}a(\tau,\eta)đ\tau đ\eta\right)\right\| \\
&= \sup_{(\rho,\sigma)\in\mathbb{R}^{1+q}} \left\| \int \rho^F(\rho,\sigma)^\beta \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\left(D^{\alpha-\gamma}_{\rho,\sigma}\chi(\rho,\sigma)\right)\right. \\
&\qquad\qquad \left.\left(D^\gamma_{\rho,\sigma}e^{i(\rho\tau+\sigma\eta)}\right)a(\tau,\eta)đ\tau đ\eta\right\| \\
&\leq \sup_{(\rho,\sigma)\in\mathbb{R}^{1+q}} \sum_{\gamma\leq\alpha} c_\gamma \left\|\int (\tau,\eta)^\gamma\left(D^F_\tau D^\beta_{\tau,\eta}e^{i(\rho\tau+\sigma\eta)}\right)a(\tau,\eta)đ\tau đ\eta\right\| \\
&\leq \sum_{\gamma\leq\alpha}\tilde{c}_\gamma \int |\tau,\eta|^{|\gamma|}\left\|D^F_\tau D^\beta_{\tau,\eta}a(\tau,\eta)\right\| đ\tau đ\eta \\
&\leq \sum_{\gamma\leq\alpha} C_\gamma \int [\tau,\eta]_l^{l|\gamma|+M+\widetilde{M}+\nu-|\beta|_l-lF} đ\tau đ\eta < \infty
\end{aligned}$$

with $lF + |\beta|_l > l|\alpha| + M + \widetilde{M} + \nu + 1 + q$ where M and $\widetilde{M}$ are the constants from (1.1.10) for κ_λ and $\tilde{\kappa}_\lambda$, respectively, which completes the proof. □

Corollary 4 *Anisotropic pseudo–differential operators with operator–valued symbols are pseudo-local, i.e.,* sing supp $Au \subseteq$ sing supp u *is satisfied for all* $u(t,y) \in \mathcal{D}'(\Omega, E)$ *and every* $A \in \Psi^{\infty,l}(\Omega; E, \widetilde{E})$.

In fact, equation (5) allows us to restrict the considerations to symbols with constant coefficients. The above Proposition 3 then gives sing supp $K(a)(\rho,\sigma) \subseteq \{0\}$ and hence sing supp $K_A(t,y,t',y') \subseteq \{(t,y) = (t',y')\}$, $A = \mathrm{Op}(a)$, $a(\tau,\eta) \in S^{\infty,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$. But then the assertion follows from

$$\text{sing supp } Au \subseteq \pi(\text{sing supp } K_A(t,y,t',y') \cap (\mathbb{R}^{1+q}_{t,y} \times \text{sing supp } u)),$$

where $\pi : \mathbb{R}^{1+q}_{t,y} \times \mathbb{R}^{1+q}_{t',y'} \to \mathbb{R}^{1+q}_{t,y}$ denotes the projection to the (t,y)-coordinates.

Proposition 5 *Let* $\phi(\rho,\sigma) \in \mathcal{S}(\mathbb{R}^{1+q})$ *and* $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$*, and set*

$$(h(\phi)a)(t,y,t',y',\tau,\eta) := \mathcal{F}_{(\rho,\sigma)\to(\tau,\eta)}\phi(\rho,\sigma)\mathcal{F}^{-1}_{(\tau,\eta)\to(\rho,\sigma)}a(t,y,t',y',\tau,\eta).$$

Then we get a continuous operator

$$h(\phi) \ : \ S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E}) \to S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E}).$$

Moreover, for fixed $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$ *the map* $\phi \mapsto h(\phi)a$ *is a continuous operator* $\mathcal{S}(\mathbb{R}^{1+q}) \to S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$.

Proof: It suffices to consider symbols with constant coefficients; the (t,y,t',y')–dependent case can be treated in an analogous manner. We shall prove the symbol estimates

$$\left\|\tilde{\kappa}^{-1}(\tau,\eta)D^{\beta}_{\tau,\eta}(h(\phi)a)(\tau,\eta)\kappa(\tau,\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} \le c_\beta(\phi)[\tau,\eta]_l^{\nu-|\beta|_l}$$

for arbitrary fixed group actions $\kappa_\lambda, \tilde{\kappa}_\lambda$, where $c_\beta(\phi)$ is some constant continuously depending on $\phi \in \mathcal{S}(\mathbb{R}^{1+q})$.

We have

$$\begin{aligned} D^{\beta}_{\tau,\eta}(h(\phi)a)(\tau,\eta) &= D^{\beta}_{\tau,\eta}(\hat{\phi} * a)(\tau,\eta) \\ &= D^{\beta}_{\tau,\eta}\int \hat{\phi}(\tau-\tau',\eta-\eta')a(\tau',\eta')\,đ\tau'đ\eta' \\ &= \int \hat{\phi}(\tau-\tau',\eta-\eta')D^{\beta}_{\tau',\eta'}a(\tau',\eta')\,đ\tau'đ\eta', \end{aligned}$$

and hence

$$\begin{aligned} &\left\|\tilde{\kappa}^{-1}(\tau,\eta)D^{\beta}_{\tau,\eta}(h(\phi)a)(\tau,\eta)\kappa(\tau,\eta)\right\| \\ &= \left\|\tilde{\kappa}^{-1}(\tau,\eta)\int \hat{\phi}(\tau-\tau',\eta-\eta')D^{\beta}_{\tau',\eta'}a(\tau',\eta')\,đ\tau'đ\eta'\kappa(\tau,\eta)\right\| \\ &\le c_1\int |\hat{\phi}(\tau-\tau',\eta-\eta')|\left\|\tilde{\kappa}^{-1}(\tau,\eta)D^{\beta}_{\tau',\eta'}a(\tau',\eta')\kappa(\tau,\eta)\right\|đ\tau'đ\eta' \\ &\le c_1c_\beta\int |\hat{\phi}(\tau-\tau',\eta-\eta')|\left\|\tilde{\kappa}_{\frac{[\tau',\eta']_l}{[\tau,\eta]_l}}\right\|[\tau',\eta']_l^{\nu-|\beta|_l}\left\|\kappa_{\frac{[\tau,\eta]_l}{[\tau',\eta']_l}}\right\|đ\tau'đ\eta' \\ &\le c_2c_\beta\int |\hat{\phi}(\tau-\tau',\eta-\eta')|[\tau-\tau',\eta-\eta']_l^{\widetilde{M}+M}[\tau',\eta']_l^{\nu-|\beta|_l}\,đ\tau'đ\eta' \qquad (6) \\ &\le c_2c_\beta c^{|\nu-|\beta|_l|}[\tau,\eta]_l^{\nu-|\beta|_l}\int |\hat{\phi}(\tau-\tau',\eta-\eta')|[\tau-\tau',\eta-\eta']_l^{\widetilde{M}+M+|\nu-|\beta|_l|}\,đ\tau'đ\eta' \qquad (7) \\ &\le c_\beta(\phi)[\tau,\eta]_l^{\nu-|\beta|_l}. \qquad (8) \end{aligned}$$

For the estimate (6) we used the inequality (1.1.11) and for (7) Peetre's inequality (1.1.8) with $s = \nu-|\beta|_l$. As desired the constant $c_\beta(\phi)$ depends continuously on ϕ. □

Remark 6 Proposition 5 extends to the case when ϕ depends as a rapidely decreasing function only on one variable or on a group of variables. Let, for instance, $\phi = \phi(\rho) \in \mathcal{S}(\mathbb{R})$ be given. Then we obtain $(h(\phi)a)(\tau,\eta) = \int_{-\infty}^{\infty} \hat{\phi}(\tau - \tau')a(\tau',\eta)\, đ\tau'$ and hence as above

$$\begin{aligned}
&\left\| \tilde{\kappa}^{-1}(\tau,\eta) D^{\beta}_{\tau,\eta}(h(\phi)a)(\tau,\eta)\kappa(\tau,\eta) \right\| \\
&= \left\| \tilde{\kappa}^{-1}(\tau,\eta) \int_{-\infty}^{\infty} \hat{\phi}(\tau-\tau') D^{\beta}_{\tau',\eta} a(\tau',\eta)\, đ\tau' \kappa(\tau,\eta) \right\| \\
&\le c_1 \int_{-\infty}^{\infty} |\hat{\phi}(\tau-\tau')| \left\| \tilde{\kappa}^{-1}(\tau,\eta) D^{\beta}_{\tau',\eta} a(\tau',\eta)\kappa(\tau,\eta) \right\| đ\tau' \\
&\le c_1 c_\beta \int_{-\infty}^{\infty} |\hat{\phi}(\tau-\tau')| \left\| \tilde{\kappa}_{\frac{[\tau',\eta]_l}{[\tau,\eta]_l}} \right\| [\tau',\eta]_l^{\nu-|\beta|_l} \left\| \kappa_{\frac{[\tau,\eta]_l}{[\tau',\eta]_l}} \right\| đ\tau' \\
&\le c_2 c_\beta \int_{-\infty}^{\infty} |\hat{\phi}(\tau-\tau')| [\tau-\tau',\eta-\eta]_l^{\widetilde{M}+M} [\tau',\eta]_l^{\nu-|\beta|_l}\, đ\tau' \\
&\le c_2 c_\beta c^{|\nu-|\beta|_l|} [\tau,\eta]_l^{\nu-|\beta|_l} \int_{-\infty}^{\infty} |\hat{\phi}(\tau-\tau')| [\tau-\tau',0]_l^{\widetilde{M}+M+|\nu-|\beta|_l|}\, đ\tau' \\
&\le c_\beta(\phi) [\tau,\eta]_l^{\nu-|\beta|_l}.
\end{aligned}$$

Thus $(h(\phi)a)(\tau,\eta)$ belongs to the space $S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$.

Example 7 Let $\psi \in C_0^\infty(\mathbb{R}^{1+q})$ be a (ρ,σ)–cut–off function with $\psi(\rho,\sigma) = 1$ in a neighbourhood of $(\rho,\sigma) = 0$. In this case the operator $h(\psi)$ is called a kernel cut–off operator. It satisfies the relation

$$h(\psi)a - a \in S^{-\infty}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E}). \tag{9}$$

In order to verify (9) we write $h(1)a = a$ which implies

$$\begin{aligned}
a - h(\psi)a &= h(1-\psi)a \\
&= \mathcal{F}_{(\rho,\sigma)\to(\tau,\eta)}(1-\psi)K(a)(t,y,t',y',\rho,\sigma).
\end{aligned}$$

The right hand side of this equation is the Fourier transform of a rapidely decreasing function and hence rapidely decreasing itself which yields the assertion.

Definition 8 *The operator $A = \mathrm{Op}(a) \in \Psi^{\nu,l}(\Omega; E, \widetilde{E})$ with operator–valued symbol is called properly supported if the corresponding distributional kernel $K_A(t,y,t',y')$ is properly supported in $\Omega \times \Omega$, i.e., if $\pi_j : \Omega \times \Omega \to \Omega$ denotes the projection to the j–th component, $\pi_j^{-1}(C) \cap \operatorname{supp} K_A$ is compact for every compact set $C \subset\subset \Omega$, $j = 1, 2$.*

Remark 9 From the definition it follows immediately that each properly supported pseudo–differential operator $A \in \Psi^{\nu,l}(\Omega; E, \widetilde{E})$ induces continuous operators

$$\begin{aligned}
&A : C_0^\infty(\Omega, E) \to C_0^\infty(\Omega, \widetilde{E}), \quad A : C^\infty(\Omega, E) \to C^\infty(\Omega, \widetilde{E}) \text{ as well as} \\
&A : C_0^\infty(\Omega) \to C_0^\infty(\Omega, \mathcal{L}(E, \widetilde{E})), \quad A : C^\infty(\Omega) \to C^\infty(\Omega, \mathcal{L}(E, \widetilde{E})).
\end{aligned}$$

Proposition 10 *Every pseudo–differential operator $A = \mathrm{Op}(a) \in \Psi^{\nu,l}(\Omega; E, \widetilde{E})$ possesses a decomposition $A = A_0 + A_1$ with a properly supported $A_0 \in \Psi^{\nu,l}(\Omega; E, \widetilde{E})$ and a smoothing operator $A_1 \in \Psi^{-\infty}(\Omega; E, \widetilde{E})$.*

In fact, it suffices to set $A_0 := \mathrm{Op}(\omega a)$ and $A_1 := \mathrm{Op}((1-\omega)a)$ where $\omega(t,y,t',y') \in C^\infty(\Omega \times \Omega)$ is a properly supported function which equals 1 in a neighbourhood of $\operatorname{diag} \Omega \times \Omega$.

2.2 The algebra

We call $a(t,y,t',y',\tau,\eta)$ asymptotic sum of a sequence $a_j(t,y,t',y',\tau,\eta) \in S^{\nu_j,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$, $j \in \mathbb{N}$, of anisotropic operator–valued symbols with $\nu_j \to -\infty$ for $j \to \infty$, if for every $\mu \in \mathbb{R}$ there is an $N_\mu \in \mathbb{N}$ such that

$$a(t,y,t',y',\tau,\eta) - \sum_{j=0}^{N} a_j(t,y,t',y',\tau,\eta) \in S^{\mu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E}), \tag{1}$$

holds for all $N \geq N_\mu$. In this case we write $a \sim \sum a_j$.

Proposition 1 *Let $a_j(t,y,t',y',\tau,\eta) \in S^{\nu_j,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$, $j \in \mathbb{N}$, be any sequence of anisotropic operator–valued symbols with $\nu_j \to -\infty$ for $j \to \infty$.*

Then $a(t,y,t',y',\tau,\eta) \sim \sum\limits_{j=0}^{\infty} a_j(t,y,t',y',\tau,\eta)$ and $\tilde{a}(t,y,t',y',\tau,\eta) \sim \sum\limits_{j=0}^{\infty} a_j(t,y,t',y',\tau,\eta)$ implies $a(t,y,t',y',\tau,\eta) = \tilde{a}(t,y,t',y',\tau,\eta) \bmod S^{-\infty}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$.

Proof: Since $a \sim \sum\limits_{j=0}^{\infty} a_j$ and $\tilde{a} \sim \sum\limits_{j=0}^{\infty} a_j$ for given $\mu \in \mathbb{R}$ there are constants N_0 and $\widetilde{N}_0$ such that $a(t,y,t',y',\tau,\eta) - \sum\limits_{j=0}^{N} a_j(t,y,t',y',\tau,\eta)$ as well as $\tilde{a}(t,y,t',y',\tau,\eta) - \sum\limits_{j=0}^{N} a_j(t,y,t',y',\tau,\eta)$ belongs to the space $S^{\mu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$ for all $N \geq \max\{N_0, \widetilde{N}_0\}$. Thus we obtain for all $\mu \in \mathbb{R}$

$$(a - \tilde{a})(t,y,t',y',\tau,\eta) = (a - \sum_{j=0}^{N} a_j - \tilde{a} + \sum_{j=0}^{N} a_j)(t,y,t',y',\tau,\eta) \in S^{\mu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$$

and hence $(a - \tilde{a})(t,y,t',y',\tau,\eta) \in S^{-\infty}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$. □

In the sequel the notation $\sum_{j=1}^{\infty} a_j$ will always be used in the sense of a choice of a representative in the class of all symbols that are equal modulo someone of order $-\infty$. All calculations below will be independent of the particular choice, modulo corresponding smoothing remainders.

Theorem 2 *Let $a_j(t,y,t',y',\tau,\eta) \in S^{\nu_j,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$, $j = 0, 1, \ldots$, be any sequence of anisotropic operator–valued symbols and suppose $\nu_j \to -\infty$ for $j \to \infty$. Then for every cut–off function $\omega(\rho,\sigma)$ there exists a sequence $\{c_j\}_{j=0}^{\infty}$ of constants such that the sum*

$$K(a)(t,y,t',y',\rho,\sigma) := \sum_{j=0}^{\infty} \omega(c_j\rho, c_j\sigma) K(a_j)(t,y,t',y',\rho,\sigma) \tag{2}$$

converges in $T^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$, $\nu = \max\{\nu_j : j \in \mathbb{N}\}$.

Setting $a := \mathcal{F}_{(\rho,\sigma)\to(\tau,\eta)} K(a)$ we have $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$ and $a \sim \sum\limits_{j=0}^{\infty} a_j$ holds.

Proof: Assuming the convergence of (2) within the space $T^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$ and fixing any order $\mu \in \mathbb{R}$ we may find $N_\mu \in \mathbb{N}$ such that

$$K\left(a - \sum_{j=0}^{N} a_j\right) = K(a) - \sum_{j=0}^{N} K(a_j) = \sum_{j=N+1}^{\infty} K(a_j)$$

belongs for all $N > N_\mu$ to $T^{\tilde{\nu}_{N_\mu},l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$. But this implies

$$a - \sum_{j=0}^{N} a_j \in S^{\tilde{\nu}_{N_\mu},l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E}) \subset S^{\mu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$$

for all $N > N_\mu$ such that we get $a \sim \sum\limits_{j=0}^{\infty} a_j$.

For any fixed semi–norm π in $T^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$ we have to find an index $j_0 = j_0(\pi)$ such that $\pi(\omega(c\rho, c\sigma)K(a_j)(t,y,t',y',\rho,\sigma)) \to 0$ for all $j \geq j_0$ as $c \to \infty$. Without loss of generality we choose an ordered system of semi-norms $\{\pi_k\}$ in the Fréchet space $T^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$. We will find constants $c_{jk} \in \mathbb{R}$ such that

$$\pi_k(\omega(c_{jk}\rho, c_{jk}\sigma)K(a_j)(t,x,\rho,\sigma)) < 2^{-j}$$

holds for all $j,k \in \mathbb{N}$. But then with $c_j := c_{jj}$ we obtain

$$\begin{aligned}\pi_k(\sum_{j=0}^{\infty}\omega(c_j\rho,c_j\sigma)K(a_j)) &\leq \pi_k(\sum_{j=0}^{k-1}\omega(c_j\rho,c_j\sigma)K(a_j)) + \pi_k(\sum_{j=k}^{\infty}\omega(c_j\rho,c_j\sigma)K(a_j))\\ &\leq \pi_k(\sum_{j=0}^{k-1}\omega(c_j\rho,c_j\sigma)K(a_j)) + \sum_{j=k}^{\infty}\pi_k(\omega(c_j\rho,c_j\sigma)K(a_j))\\ &\leq \pi_k(\sum_{j=0}^{k-1}\omega(c_j\rho,c_j\sigma)K(a_j)) + \sum_{j=k}^{\infty}2^j < \infty,\end{aligned}$$

and hence (2) converges .

Up to now we considered symbols with constant coefficients. The (t,y,t',y',τ,η)–dependent case can be treated in an analogous manner. For the case of constant coefficients the symbol estimates are of the form $\|\tilde{\kappa}^{-1}(\tau,\eta)D^{\beta}_{\tau,\eta}a(\tau,\eta)\kappa(\tau,\eta)\|_{\mathcal{L}(E,\widetilde{E})} \leq c_\beta[\tau,\eta]_l^{\nu-|\beta|_l}$. Because of $\sup_{(\tau,\eta)\in\mathbb{R}}[\tau,\eta]_l^{-\nu+|\beta|_l}\|\tilde{\kappa}(\tau,\eta)D^{\beta}_{\tau,\eta}a(\tau,\eta)\kappa(\tau,\eta)\| \leq \sup_{(\tau,\eta)\in\mathbb{R}}[\tau,\eta]_l^{-(\nu-\widetilde{M}-M)+|\beta|_l}\|D^{\beta}_{\tau,\eta}a(\tau,\eta)\|$, cf. (1.1.10), we can set $\kappa_\lambda = \mathrm{id}_E$ and $\tilde{\kappa}_\lambda = \mathrm{id}_{\widetilde{E}}$ for all $\lambda \in \mathbb{R}_+$ up to a fixed loss of order. Here and later on we write simply $\|\cdot\| := \|\cdot\|_{\mathcal{L}(E,\widetilde{E})}$.

In view of

$$\begin{aligned}\|(\tau,\eta)^\alpha D^{\beta}_{\tau,\eta}a(\tau,\eta)\| &\leq |(\tau,\eta)|^{|\alpha|}\|D^{\beta}_{\tau,\eta}a(\tau,\eta)\|\\ &\leq c[\tau,\eta]_l^{l|\alpha|}c_\beta[\tau,\eta]_l^{\nu-|\beta|_l}\\ &\leq \tilde{c}[\tau,\eta]_l^{\nu-|\beta|_l+l|\alpha|},\end{aligned}$$

i.e., $\|(\tau,\eta)^\alpha D^{\beta}_{\tau,\eta}a(\tau,\eta)\| \leq \tilde{c}[\tau,\eta]_l^{\nu+\mu}$ for $\alpha,\beta \in \mathbb{N}^{1+q}$ with $\mu + |\beta|_l \geq l|\alpha|$, we can pass to a system of semi–norms of the form

$$a \mapsto \sup_{(\tau,\eta)\in\mathbb{R}^{1+q}}\|(\tau,\eta)^\alpha D^{\beta}_{\tau,\eta}a(\tau,\eta)\| \tag{3}$$

for $\alpha,\beta \in \mathbb{N}^{1+q}$ with $|\beta|_l - \nu \geq l|\alpha|$. This may be replaced by the system of semi–norms

$$a \mapsto \left\{\int \|(\rho,\sigma)^{\tilde{\alpha}}D^{\tilde{\beta}}_{\rho,\sigma}K(a)(\rho,\sigma)\|^2 d\rho d\sigma\right\}^{\frac{1}{2}} \tag{4}$$

which is stronger than (3), up to a loss of order only depending on n.

Summing up we have to check

$$\lim_{c\to\infty}\int \|(\rho,\sigma)^{\tilde{\alpha}}D^{\tilde{\beta}}_{\rho,\sigma}\omega(c\rho,c\sigma)K(a_j)(\rho,\sigma)\|^2 d\rho d\sigma = 0.$$

For $|\tilde{\beta}| = 0$, $c > 1$ und arbitrary $F \in \mathbb{N}$ we obtain

$$\begin{aligned}(\rho,\sigma)^{\tilde{\alpha}}\omega(c\rho,c\sigma)K(a_j)(\rho,\sigma) &= \omega(\rho,\sigma)(\rho,\sigma)^{\tilde{\alpha}}\rho^{-F}\omega(c\rho,c\sigma)K(a_j)(\rho,\sigma)\rho^F\\ &= K_j(\rho,\sigma)\omega(c\rho,c\sigma)\rho^F\end{aligned}$$

with $K_j(\rho,\sigma) := \omega(\rho,\sigma)(\rho,\sigma)^{\tilde{\alpha}}\rho^{-F}K(a_j)(\rho,\sigma)$. Furthermore, since $\nu_j \to -\infty$ for every $m \in \mathbb{N}$, there is a j_0 such that $K(a_j)(\rho,\sigma) \in C^m(\mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E}))$ holds for all $j \geq j_0$. Subtracting a finite Taylor expansion at $\rho = 0$ we see that the remainder becomes flat of order m at $\rho = 0$, where the Taylor expansion together with the above cut–off factor is a Schwartz function which corresponds on symbol side to an error of order $-\infty$. Thus, after this change we may assume that $K(a_j)(\rho,\sigma)$ is flat of order m at $\rho = 0$.

Since $\nu_j \to -\infty$, we get an $j_0 \in \mathbb{N}$ so that the functions $K_j(\rho,\sigma)$ are continuous with compact support and hence bounded for all $j \geq j_0$. But then we have

$$\int \|K_j(\rho,\sigma)\omega(c\rho,c\sigma)\rho^F\|^2 d\rho d\sigma \leq \sup_{(\rho,\sigma)\in\mathbb{R}^{1+q}} \|K_j(\rho,\sigma)\|^2 \int |\omega(c\rho,c\sigma)\rho^F|^2\, d\rho d\sigma.$$

This implies

$$\begin{aligned}\int |\omega(c\rho,c\sigma)\rho^F|^2\, d\rho d\sigma &= \int |\omega(\rho,\sigma)\rho^F c^{-F}|^2 c^{-1-q}\, d\rho d\sigma \\ &= c^{-2F-1-q}\int |\omega(\rho,\sigma)\rho^F|^2\, d\rho d\sigma \to 0\end{aligned}$$

for $c \to \infty$ which is the desired behaviour. For $|\tilde{\beta}| \neq 0$ we can argue in an analogous manner, where the powers of c arising in the differentiation will be compensated by a suitable choice of F.

So it remains to prove that (4) gives a topology stronger than (3). For that reason we consider the operator–valued function $f(\tau,\eta) := (\tau,\eta)^\alpha D^\beta_{\tau,\eta} a(\tau,\eta)$. Note that by choosing $j_0 \in \mathbb{N}$ large enough we may assume f as fastly decreasing as we want. Then from $\|\mathcal{F}f\|_\infty \leq \|f\|_1$ it follows that

$$\begin{aligned}\|f\|_\infty &\leq \|\mathcal{F}^{-1}f\|_1 \\ &\leq \int \|\mathcal{F}^{-1}f(\rho,\sigma)\| d\rho d\sigma \\ &\leq \int \|\mathcal{F}^{-1}f(\rho,\sigma)\|(1+\rho^2+|\sigma|^{2l})^N(1+\rho^2+|\sigma|^{2l})^{-N} d\rho d\sigma \\ &\leq \left\{\int \|\mathcal{F}^{-1}f(\rho,\sigma)\|^2(1+\rho^2+|\sigma|^{2l})^{2N} d\rho d\sigma\right\}^{\frac{1}{2}} \left\{\int (1+\rho^2+|\sigma|^{2l})^{-2N} d\rho d\sigma\right\}^{\frac{1}{2}} \\ &\leq c\left\{\int \|(\rho,\sigma)^{\tilde{\alpha}} D^{\tilde{\beta}}_{\rho,\sigma} K(a)(\rho,\sigma)\|^2 d\rho d\sigma\right\}^{\frac{1}{2}},\end{aligned}$$

where the latter integral has the form (4) which completes the proof. □

Now we are able to check the standard algebra properties for anisotropic pseudo–differential operators with operator–valued symbols.

Proposition 3 *For every symbol $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(\Omega^2\times\mathbb{R}^{1+q};E,\widetilde{E})$ there exists a left symbol $\underline{a}(t,y,\tau,\eta) \in S^{\nu,l}(\Omega\times\mathbb{R}^{1+q};E,\widetilde{E})$ such that*

$$\mathrm{Op}(a) - \mathrm{Op}(\underline{a}) \in \Psi^{-\infty}(\Omega;E,\widetilde{E})$$

holds. Moreover, for $\underline{a}$ we have the asymptotic expansion

$$\underline{a}(t,y,\tau,\eta) \sim \sum_{\alpha\in\mathbb{N}^{1+q}} \frac{1}{\alpha!} D^\alpha_{t',y'}\partial^\alpha_{\tau,\eta} a(t,y,t',y',\tau,\eta)|_{(t',y')=(t,y)}.$$

For classical a the resulting $\underline{a}$ is again classical.

The proof is completely analogous to the isotropic case; so it will be left to the reader.

Corollary 4 *For every right symbol $a(t',y',\tau,\eta) \in S^{\nu,l}(\Omega\times\mathbb{R}^{1+q};E,\widetilde{E})$ exists a left symbol $\tilde{a}(t,y,\tau,\eta) \in S^{\nu,l}(\Omega\times\mathbb{R}^{1+q};E,\widetilde{E})$ such that*

$$\mathrm{Op}(a) - \mathrm{Op}(\tilde{a}) \in \Psi^{-\infty}(\Omega;E,\widetilde{E})$$

holds. Moreover, for $\tilde{a}$ we have the asymptotic expansion

$$\tilde{a}(t,y,\tau,\eta) \sim \sum_{\alpha\in\mathbb{N}^{1+q}} \frac{1}{\alpha!} D^\alpha_{t',y'}(-\partial_{\tau,\eta})^\alpha a(t',y',\tau,\eta)|_{(t',y')=(t,y)}.$$

For classical a the resulting $\tilde{a}$ is again classical.

Remark 5 The above left and right symbols satisfy the relation

$$(\mathcal{F}_{(t,y)\to(\tau,\eta)}\mathrm{Op}(a)u)(\tau,\eta) = \mathcal{F}_{(t',y')\to(\tau,\eta)}(\tilde{a}u)(\tau,\eta) + \mathcal{F}_{(t,y)\to(\tau,\eta)}Ru(\tau,\eta)$$

for all $u \in C_0^\infty(\Omega, E)$ with remainder $R \in \Psi^{-\infty}(\Omega; E, \widetilde{E})$.

Our next result is also analogous to the isotropic theory, namely that the symbol of the composition $\mathrm{Op}(a)\mathrm{Op}(b)$ of two pseudo–differential operator is the Leibniz product $a \# b$, given by the rule

$$(a \# b)(t,y,\tau,\eta) \sim \sum_{\alpha\in\mathbb{N}^{1+q}} \frac{1}{\alpha!}\partial^\alpha_{\tau,\eta}a(t,y,\tau,\eta)D^\alpha_{t,y}b(t,y,\tau,\eta). \tag{5}$$

Here $a(t,y,\tau,\eta) \in S^{\nu,l}(\Omega\times\mathbb{R}^{1+q}; F, \widetilde{E})$ and $b(t,y,\tau,\eta) \in S^{\nu,l}(\Omega\times\mathbb{R}^{1+q}; E, F)$ implies $(a \# b)(t,y,\tau,\eta) \in S^{\nu+\mu,l}(\Omega\times\mathbb{R}^{1+q}; E, \widetilde{E})$, cf. 1.1 Remark 7 and Theorem 2. If further a and b are classical then $a \# b$ is again classical.

For $r(t,y,\tau,\eta) \in S^{\nu,l}(\Omega\times\mathbb{R}^{1+q}; E, E)$ we can also form powers with respect to the Leibniz product. In this case we write $r^{k\#} := \underbrace{r \# \cdots \# r}_{k-times}$ as well as $r^{0\#} := \mathit{1}$, where $\mathit{1}$ means the operator–valued function having the corresponding identity as constant value.

Proposition 6 *Let $A = \mathrm{Op}(a) \in \Psi^{\nu,l}(\Omega; F, \widetilde{E})$ and $B = \mathrm{Op}(b) \in \Psi^{\mu,l}(\Omega; E, F)$ with A or B properly supported be given. Then $AB \in \Psi^{\nu+\mu,l}(\Omega; E, \widetilde{E})$ and*

$$AB - \mathrm{Op}(a \# b) \in \Psi^{-\infty}(\Omega; E, \widetilde{E}).$$

For classical A and B is AB again classical.

Proof: Since A or B is properly supported the composition gives a map $AB : C_0^\infty(\Omega, E) \to C^\infty(\Omega, \widetilde{E})$. We prove the symbolic rule. In view of Remark 5 we have

$$(\mathcal{F}Bu)(\tau,\eta) = \int e^{-i(\tau t'+\eta y')}\tilde{b}(t',y',\tau,\eta)u(t',y')\,dt'dy' + (\mathcal{F}Ru)$$

with some $R \in \Psi^{-\infty}(\Omega; E, F)$. But then AB is given by

$$(ABu)(t,y) = \int\!\!\int e^{i(\tau(t-t')+\eta(y-y'))}a(t,y,\tau,\eta)\tilde{b}(t',y',\tau,\eta)u(t',y')\,dt'dy'đ\tau đ\eta$$

modulo smoothing operators and hence $AB = \mathrm{Op}(c)$ with $c(t,y,t',y',\tau,\eta) = a(t,y,\tau,\eta)\tilde{b}(t',y',\tau,\eta)$. Using Proposition 3 we get $AB = \mathrm{Op}(\underline{c}) \bmod \Psi^{-\infty}(\Omega; E, \widetilde{E})$, and in view of Corollary 4 we have

$$\begin{aligned}
\underline{c}(t,y,\tau,\eta) &\sim \sum_{\alpha\in\mathbb{N}^{1+q}} \frac{1}{\alpha!}\left(D^\alpha_{t',y'}\partial^\alpha_{\tau,\eta}\left(a(t,y,\tau,\eta)\tilde{b}(t',y',\tau,\eta)\right)\right)\Big|_{(t',y')=(t,y)} \\
&\sim \sum_{\alpha\in\mathbb{N}^{1+q}} \frac{1}{\alpha!}\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}\left(\partial^\beta_{\tau,\eta}a(t,y,\tau,\eta)\right)\left(D^\alpha_{t',y'}\partial^\gamma_{\tau,\eta}\tilde{b}(t',y',\tau,\eta)\right)\Big|_{(t',y')=(t,y)} \\
&\sim \sum_{\beta,\gamma\in\mathbb{N}^{1+q}} \frac{1}{\beta!\gamma!}\left(\partial^\beta_{\tau,\eta}a(t,y,\tau,\eta)\right)\sum_{\delta\in\mathbb{N}^{1+q}}(-1)^{|\delta|}\left(D^{\beta+\gamma+\delta}_{t,y}\partial^{\gamma+\delta}_{\tau,\eta}b(t,y,\tau,\eta)\right) \\
&\sim \sum_{\beta\in\mathbb{N}^{1+q}} \frac{1}{\beta!}\left(\partial^\beta_{\tau,\eta}a(t,y,\tau,\eta)\right)\sum_{\gamma+\delta=\lambda}\frac{(-1)^{|\delta|}}{\gamma!\delta!}\left(D^{\beta+\lambda}_{t,y}\partial^\lambda_{\tau,\eta}b(t,y,\tau,\eta)\right).
\end{aligned}$$

Set $m_0 = (1,\ldots,1) \in \mathbb{R}^{1+q}$, then for arbitrary multi–indices $\gamma,\delta,\lambda \in \mathbb{N}^{1+q}$ we have

$$\sum_{\gamma+\delta=\lambda}\frac{(-1)^{|\delta|}}{\gamma!\delta!} = \frac{1}{\lambda!}\sum_{\delta\le\lambda}(-1)^{|\delta|}\binom{\lambda}{\delta} = \frac{1}{\lambda!}(m_0-m_0)^\lambda = \begin{cases} 1 \text{ for } |\lambda| = 0 \\ 0 \text{ for } |\lambda| \ne 0 \end{cases}$$

and hence $\underline{c}(t,y,\tau,\eta) = (a \# b)(t,y,\tau,\eta) \in S^{\nu+\mu,l}(\Omega \times \mathbb{R}^{1+q}; E, E)$, i.e., $AB = \mathrm{Op}(a \# b) \bmod \Psi^{-\infty}(\Omega; E, \widetilde{E})$ as desired.

Finally, since the Leibniz product of classical symbols is classical we obtain the second part of the assertion. □

Let us now characterise the invertible elements with respect to the Leibniz product.

Proposition 7 *Let $a(t,y,\tau,\eta) \in S^{\nu,l}(\Omega \times \mathbb{R}^{1+q}; E, \widetilde{E})$ and assume that there is a $p_0(t,y,\tau,\eta) \in S^{-\nu,l}(\Omega \times \mathbb{R}^{1+q}; \widetilde{E}, E)$ satisfying*

$$(a \circ p_0 - 1)(t,y,\tau,\eta) \in S^{-1,l}(\Omega \times \mathbb{R}^{1+q}; \widetilde{E}, \widetilde{E}), \tag{6}$$

$$(p_0 \circ a - 1)(t,y,\tau,\eta) \in S^{-1,l}(\Omega \times \mathbb{R}^{1+q}; E, E). \tag{7}$$

Then there exists a Leibniz inverse $p(t,y,\tau,\eta) \in S^{-\nu,l}(\Omega \times \mathbb{R}^{1+q}; \widetilde{E}, E)$ in the sense

$$(a \# p - 1)(t,y,\tau,\eta) \in S^{-\infty}(\Omega \times \mathbb{R}^{1+q}; \widetilde{E}, \widetilde{E}), \tag{8}$$

$$(p \# a - 1)(t,y,\tau,\eta) \in S^{-\infty}(\Omega \times \mathbb{R}^{1+q}; E, E). \tag{9}$$

Proof: We will constuct p in terms of p_0. Using (6) we first get a symbol p_r satisfying (8). Analogously from (7) we get an element p_l satisfying (9). Then we obtain $p_r = p_r(ap_l) = (p_r a)p_l = p_l \bmod S^{-\infty}(\Omega \times \mathbb{R}^{1+q}; \widetilde{E}, E)$, such that we can omit the subscripts r and l modulo smoothing remainders.

Using the definition of the Leibniz product it follows immediately from (6)

$$-(a \# p_0 - 1)(t,y,\tau,\eta) \sim r(t,y,\tau,\eta) \in S^{-1,l}(\Omega \times \mathbb{R}^{1+q}; \widetilde{E}, \widetilde{E}).$$

We now form $(1-r) \# \sum_{k=0}^{\infty} r^{k\#}$ and check the relation $1 \sim (1-r) \# \sum_{k=0}^{\infty} r^{k\#}$ ($\sim$ indicates equality modulo symbols of order $-\infty$).

For every $M \in \mathbb{N}$ we have

$$1 - (1-r) \# \sum_{k=0}^{M} r^{k\#} \in S^{-M-1,l}(\Omega \times \mathbb{R}^{1+q}; \widetilde{E}, \widetilde{E})$$

which follows from

$$\begin{aligned} 1 - (1-r) \# \sum_{k=0}^{M} r^{k\#} &= 1 - \sum_{k=0}^{M} r^{k\#} + \sum_{k=0}^{M} r^{k+1\#} \\ &= -\sum_{k=1}^{M} r^{k\#} + \sum_{k=1}^{M} r^{k\#} + r^{M+1\#} \\ &= r^{M+1\#} \in S^{-M-1,l}(\Omega \times \mathbb{R}^{1+q}; \widetilde{E}, \widetilde{E}). \end{aligned}$$

But then we also get $1 \sim a \# p_0 \# \sum_{k=0}^{M} r^{k\#}$, and hence $p = p_0 \# \sum_{k=0}^{M} r^{k\#}$ belongs to $S^{-\nu,l}(\Omega \times \mathbb{R}^{1+q}; \widetilde{E}, E)$ which is the desired Leibniz inverse. □

2.3 Global operators

Next we are going to globalise the anisotropic pseudo–differential operators with operator–valued symbols with respect to the spatial variables. To this end we have to check the behaviour of the symbols under changing coordinates.

Let $\chi_1 : U \to \widetilde{U}$ be a diffeomorphism between open sets $U, \widetilde{U}$ in $\mathbb{R}^q$ and define

$$\chi : \mathbb{R} \times U \to \mathbb{R} \times \widetilde{U}. \tag{1}$$

by $\chi(t,y) := (t, \chi_1(y))$. Consider the pull–back

$$\chi^* \ : \ C^\infty(\mathbb{R}\times\widetilde{U},E) \to C^\infty(\mathbb{R}\times U,E),$$

$(\chi^* u)(t,y) := u(\chi(t,y)) = u(t,\chi_1(y))$ which restricts to a mapping

$$\chi^* \ : \ C_0^\infty(\mathbb{R}\times\widetilde{U},E) \to C_0^\infty(\mathbb{R}\times U,E).$$

For an operator

$$A \ : \ C_0^\infty(\mathbb{R}\times U,E) \to C^\infty(\mathbb{R}\times U,\widetilde{E})$$

we then get the operator push–forward

$$\chi_* A \ : \ C_0^\infty(\mathbb{R}\times\widetilde{U},E) \to C^\infty(\mathbb{R}\times\widetilde{U},\widetilde{E}),$$

defined by $(\chi_* A)\, u = (\chi^*)^{-1}\,(A(\chi^* u))$. As in the isotropic case, cf. [ES], we obtain the following result.

Proposition 1 *The operator push–forward $\widetilde{A} := \chi_* A$ of $A = \mathrm{Op}(a) \in \Psi^{\nu,l}(\mathbb{R}\times U;E,\widetilde{E})$ with respect to* (1) *belongs to $\Psi^{\nu,l}(\mathbb{R}\times\widetilde{U};E,\widetilde{E})$. Moreover, we have $\widetilde{A} = \mathrm{Op}(\tilde{a}) \mod \Psi^{-\infty}(\mathbb{R}\times\widetilde{U};E,\widetilde{E})$ where $\tilde{a}$ is given by the asymptotic expansion*

$$\tilde{a}(t,\chi_1(y),\tau,\tilde{\eta}) \sim \sum_{\alpha\in\mathbb{N}^q} \frac{1}{\alpha!}\varphi_\alpha(y,\tilde{\eta})\left(\partial_\eta^\alpha a\right)(t,y,\tau,{}^t d\chi_1(y)\tilde{\eta}), \tag{2}$$

where $\varphi_\alpha(y,\tilde{\eta})$ is polynomial in $\tilde{\eta}$ of order $\leq \frac{|\alpha|}{2}$ and $\varphi_0 \equiv 1$.

Remark 2 Note that (1) only concerns the spatial variables such that the anisotropic behaviour in the time direction does not play any role in the arguments. It would also be possible to treat diffeomorphisms in the time variable but it is our goal to investigate pseudo–differential operators on cylinders in space–time, where the base of the cylinder is a spatial manifold with singularities such that it suffices to take diffeomorphisms of the form (1).

The smoothing elements of the pseudo–differential algebra in question are given by the following definition.

Definition 3 *Let Y be a closed compact (Riemannian) C^∞ manifold of dimension q without boundary. Then the space $\Psi^{-\infty}(\mathbb{R}\times Y;E,\widetilde{E})$ is the set of all integral operators C on $\mathbb{R}\times Y$ with kernels $C(t,y,t',y') \in C^\infty((\mathbb{R}\times Y)\times(\mathbb{R}\times Y),\mathcal{L}(E,\widetilde{E}))$.*

We call the elements of $\Psi^{-\infty}(\mathbb{R}\times Y;E,\widetilde{E})$ smoothing operators, although they are not necessarily smoothing on the level of spaces E, $\widetilde{E}$, i.e., in case of scales $\left(E^s\right)$ and $\left(\widetilde{E}^t\right)$ (cf. Section 1.3).

For the definition of the global operators on Y we form N–tuples of triples $\{((\widetilde{U}_j,\tilde{\chi}_j),\phi_j,\psi_j)\}_{j=1}^N$. Such an N–tuple is a finite atlas $\{(\widetilde{U}_j,\tilde{\chi}_j)\}_{j=1}^N$ on Y endowed with a partition of unity $\{\phi_j\}_{j=1}^N$ and an N–tuple of functions $\{\psi_j \in C_0^\infty(\widetilde{U}_j)\}_{j=1}^N$ with $\phi_j\psi_j = \phi_j$ for all $j = 1,\dots,N$.

Definition 4 *We denote by $\Psi_{(cl)}^{\nu,l}(\mathbb{R}\times Y;E,\widetilde{E})$ the space of all operators $A : C_0^\infty(\mathbb{R}\times Y,E) \to C^\infty(\mathbb{R}\times Y,\widetilde{E})$, of the form*

$$A = \sum_{j=1}^N \phi_j\left\{\left(\chi_j^{-1}\right)_* \mathrm{Op}(a_j)\right\}\psi_j + C \tag{3}$$

with $\{a_j(t,y,\tau,\eta) \in S_{(cl)}^{\nu,l}(\mathbb{R}\times U_j\times\mathbb{R}^{1+q};E,\widetilde{E})\}_{j=1}^N$, $U_j = \tilde{\chi}_j(\widetilde{U}_j)$, and C belonging to $\Psi^{-\infty}(\mathbb{R}\times Y;E,\widetilde{E})$. Here $\left(\chi_j^{-1}\right)_*$ is the operator push–forward with respect to the inverse diffeomorphism χ_j^{-1} of $\chi_j : \mathbb{R}\times\widetilde{U}_j \to \mathbb{R}\times U_j$, $\chi_j(t,y) = (t,\tilde{\chi}_j(y))$.

The spaces $\Psi^{\nu,l}_{(cl)}(\mathbb{R}\times Y;E,\widetilde{E})$ are independent of the particular choice of $\{((\widetilde{U}_j,\tilde{\chi}_j),\phi_j,\psi_j)\}_{j=1}^N$.

Remark 5 In Definition 4 we may assume that the complete symbols are invariant modulo symbols of order $-\infty$ with respect to the symbol push–forward of Proposition 1 under $\chi_{jk}:\mathbb{R}\times\tilde{\chi}_j(\widetilde{U}_j\cap\widetilde{U}_k)\to\mathbb{R}\times\tilde{\chi}_k(\widetilde{U}_j\cap\widetilde{U}_k)$.

Then, when the atlas is fixed, the operator (3) is independent (up to a smoothing one) of the choice of the functions ϕ_j,ψ_j.

We shortly write

$$\Psi^{\infty,l}(\mathbb{R}\times Y;E,\widetilde{E}):=\bigcup_{\nu\in\mathbb{R}}\Psi^{\nu,l}(\mathbb{R}\times Y;E,\widetilde{E})$$

for the set of all global anisotropic pseudo–differential operators with $\mathcal{L}(E,\widetilde{E})$–valued symbols on Y. Analogously to the scalar theory every $A\in\Psi^{\infty,l}(\mathbb{R}\times Y;E,\widetilde{E})$ represents an operator

$$A:C_0^\infty(\mathbb{R}\times Y,E)\to C^\infty(\mathbb{R}\times Y,\widetilde{E}).$$

If A is properly supported we obtain continuous maps $A:C_0^\infty(\mathbb{R}\times Y,E)\to C_0^\infty(\mathbb{R}\times Y,\widetilde{E})$, $A:C^\infty(\mathbb{R}\times Y,E)\to C^\infty(\mathbb{R}\times Y,\widetilde{E})$. Furthermore, we have $AB\in\Psi^{-\infty}(\mathbb{R}\times Y;E,\widetilde{E})$ for $A\in\Psi^{-\infty}(\mathbb{R}\times Y;F,\widetilde{E})$ and $B\in\Psi^{\infty,l}(\mathbb{R}\times Y;E,F)$ or $A\in\Psi^{\infty,l}(\mathbb{R}\times Y;F,\widetilde{E})$ and $B\in\Psi^{-\infty}(\mathbb{R}\times Y;E,F)$, provided the composition exists.

Example 6 The operator M_f of multiplication by a function $f\in C^\infty(\mathbb{R}\times Y)$ represents an anisotropic pseudo–differential operator in $\Psi^{0,l}(\mathbb{R}\times Y;E,E)$ for each E.

Proposition 7 *For all $f,g\in C^\infty(\mathbb{R}\times Y)$ satisfying $fg\equiv 0$ we have*

$$M_f\left(\Psi^{\infty,l}(\mathbb{R}\times Y;E,\widetilde{E})\right)M_g\subseteq\Psi^{-\infty}(\mathbb{R}\times Y;E,\widetilde{E}).$$

Proof: Let us fix $\{((\widetilde{U}_j,\tilde{\chi}_j),\phi_j,\psi_j)\}_{j=1}^N$. Then the operator M_fAM_g for $A\in\Psi^{\nu,l}(\mathbb{R}\times Y;E,\widetilde{E})$ has a representation

$$M_fAM_g=M_f\left(\sum_{j=1}^N\phi_j\widetilde{A}_j\psi_j\right)M_g+M_fCM_g$$

with $\widetilde{A}_j=A|_{\widetilde{U}_j}=\left(\chi_j^{-1}\right)_*\mathrm{Op}(a_j)$. Here the second item is smoothing such that we only have to look at the first one. Moreover, it suffices to consider the action on functions u belonging to $C_0^\infty(\mathbb{R}\times\widetilde{U}_j,E)$ for some fixed j. Then with $\tilde{f}_j=f|_{\mathbb{R}\times\widetilde{U}_j}$ and $\tilde{g}_j=g|_{\mathbb{R}\times\widetilde{U}_j}$ we have locally

$$\phi_j\tilde{f}_j\widetilde{A}_j\tilde{g}_j\psi_j=\phi_j\left(\chi_j^{-1}\right)_*A_j\psi_j,$$

where $A_j:C_0^\infty(\mathbb{R}\times U_j;E)\to C^\infty(\mathbb{R}\times U_j,\widetilde{E})$, $U_j=\tilde{\chi}_j(\widetilde{U}_j)$ has the distributional kernel

$$K_j(t,y,t',y',t-t',y-y')=f_j(t,y)K(a_j)(t,y,t',y',t-t',y-y')g_j(t',y')$$

for $f_j=\left(\chi_j^{-1}\right)^*\tilde{f}_j$ and $g_j=\left(\chi_j^{-1}\right)^*\tilde{g}_j$ which implies $f_jg_j\equiv 0$.

Because of $f_j\equiv 0$ on supp g_j there exists an $\varepsilon>0$ such that $f_j(t,y)g_j(t',y')=0$ for all t,y,t',y' with $|(t,y)-(t',y')|<\varepsilon$. But then there also exists an excision function $\chi(\rho,\sigma)$ with $\chi\equiv 0$ near $(\rho,\sigma)=0\in\mathbb{R}^{1+q}$ and $\chi\equiv 1$ for $|\rho,\sigma|\geq\varepsilon$. Thus

$$\chi(t-t',y-y')K_j(t,y,t',y',t-t',y-y')=K_j(t,y,t',y',t-t',y-y').$$

But in view of 2.1 Proposition 3 the left hand side belongs to $C^\infty((\mathbb{R}\times U_j)\times(\mathbb{R}\times U_j),\mathcal{L}(E,\widetilde{E}))$ which yields after globalisation the assertion. □

Let us now turn to the composition of global operators. Remember, that the involved Banach spaces $E,\widetilde{E},F$ are assumed to be associated with corresponding group actions.

Theorem 8 *$A \in \Psi^{\nu,l}(\mathbb{R}\times Y; F, \widetilde{E})$ and $B \in \Psi^{\mu,l}(\mathbb{R}\times Y; E, F)$ and A or B properly supported implies $AB \in \Psi^{\nu+\mu,l}(\mathbb{R}\times Y; E, \widetilde{E})$. If A and B are classical pseudo–differential operators, then the composition AB is also classical.*

Moreover, for any fixed data as above

$$A = \sum_{j=1}^{N} \phi_j \left\{ \left(\chi_j^{-1}\right)_* \mathrm{Op}(a_j) \right\} \psi_j + C_A \quad \text{and} \quad B = \sum_{j=1}^{N} \phi_j \left\{ \left(\chi_j^{-1}\right)_* \mathrm{Op}(b_j) \right\} \psi_j + C_B$$

imply for AB a representation

$$AB = \sum_{j=1}^{N} \phi_j \left\{ \left(\chi_j^{-1}\right)_* \mathrm{Op}(a_j \# b_j) \right\} \psi_j + C, \tag{4}$$

$C \in \Psi^{-\infty}(\mathbb{R}\times Y; E, \widetilde{E})$.

Proof: Since A or B is properly supported, we have $AB : C_0^\infty(\mathbb{R}\times Y, E) \to C^\infty(\mathbb{R}\times Y, \widetilde{E})$ such that we only have to derive (4).

To this end we fix $\{((\widetilde{U}_j, \tilde{\chi}_j), \phi_j, \psi_j)\}_{j=1}^N$. Furthermore, we fix an N–tuple of functions $\left\{\tilde{\psi}_j \in C_0^\infty(\widetilde{U}_j)\right\}_{j=1}^N$ satisfying $\psi_j\tilde{\psi}_j = \tilde{\psi}_j$ and $\phi_j\tilde{\psi}_j = \phi_j$ for all $j = 1, \ldots, N$. Setting $\widetilde{A}_j = A|_{\widetilde{U}_j}$ and $\widetilde{B}_j = B|_{\widetilde{U}_j}$ we have

$$AB = \sum_{j=1}^{N} \Big(\phi_j \widetilde{A}_j \tilde{\psi}_j \widetilde{B}_j \psi_j + \phi_j \widetilde{A}_j (1-\tilde{\psi}_j) \widetilde{B}_j \psi_j + \phi_j \widetilde{A}_j \tilde{\psi}_j \widetilde{B}_j (1-\psi_j) + \phi_j \widetilde{A}_j (1-\tilde{\psi}_j) \widetilde{B}_j (1-\psi_j) \Big),$$

where $\sum_{j=1}^{N} \Big(\phi_j \widetilde{A}_j (1-\tilde{\psi}_j) \widetilde{B}_j \psi_j + \phi_j \widetilde{A}_j \tilde{\psi}_j \widetilde{B}_j (1-\psi_j) + \phi_j \widetilde{A}_j (1-\tilde{\psi}_j) \widetilde{B}_j (1-\psi_j) \Big)$ is smoothing since $\phi_j(1-\tilde{\psi}) = 0 = \tilde{\psi}_j(1-\psi_j)$. Moreover,

$$\sum_{j=1}^{N} \phi_j \widetilde{A}_j \tilde{\psi}_j \widetilde{B}_j \psi_j = \sum_{j=1}^{N} \Big(\phi_j \widetilde{A}_j \widetilde{B}_j \psi_j - \phi_j \widetilde{A}_j (1-\tilde{\psi}_j) \widetilde{B}_j \psi_j \Big),$$

where $\sum_{j=1}^{N} \phi_j \widetilde{A}_j (1-\tilde{\psi}_j) \widetilde{B}_j \psi_j$ is smoothing because of $\phi_j(1-\tilde{\psi}_j) = 0$. But locally in $\widetilde{U}_j$ we have

$$\widetilde{A}_j \widetilde{B}_j = \left(\chi_j^{-1}\right)_* \mathrm{Op}(b_j) \left(\chi_j^{-1}\right)_* \mathrm{Op}(a_j) = \left(\chi_j^{-1}\right)_* (\mathrm{Op}(a_j)\mathrm{Op}(b_j)) = \left(\chi_j^{-1}\right)_* \mathrm{Op}(a_j \# b_j)$$

which gives the desired relation (4). □

3 Abstract wedge Sobolev spaces

In this section we will introduce a scale of Sobolev spaces of E–valued distributions. It is our goal to prove the corresponding mapping property, i.e., the pseudo–differential operators of the last section should extend to continuous mappings between the corresponding Sobolev spaces, where the order of the operator describes the loss of smoothness of the distribution.

3.1 Definition and basic properties

Let a Banach space E and an associated group action κ_λ be given. Then we can introduce anisotropic abstract wedge Sobolev spaces of E–valued distributions.

Definition 1 *For every $s \in \mathbb{R}$, $l \in \mathbb{N} \setminus \{0\}$, we get by*

$$u \mapsto \|u\|_{s,l} := \left(\int_{\mathbb{R}^{1+q}} [\tau, \eta]_l^{2s} \|\kappa^{-1}(\tau,\eta) \mathcal{F}u(\tau,\eta)\|_E^2 đ\tau đ\eta \right)^{1/2} \tag{1}$$

a norm in $\mathcal{S}(\mathbb{R}^{1+q}, E)$. The abstract anisotropic wedge Sobolev space $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$ of smoothness s is defined as the completion of $\mathcal{S}(\mathbb{R}^{1+q}, E)$ with respect to the norm (1).

Recall that the smoothed anisotropic norm function $[\tau, \eta]_l$ was given by

$$[\tau, \eta]_l = \omega(|\tau, \eta|_l) + (1 - \omega(|\tau, \eta|_l))|\tau, \eta|_l$$

with the anisotropic norm function $|\tau, \eta|_l = (|\tau|^2 + |\eta|^{2l})^{1/2l}$, where $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ was supposed to be a cut–off function, cf. (1.1.3).

Replacing $[\tau, \eta]_l$ by equivalent functions we obtain equivalent norms on $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$, so, for instance, we will use the functions $(\tau, \eta) \mapsto \langle \tau, \eta \rangle_l := (1 + |\tau|^2 + |\eta|^{2l})^{1/2l}$ or $(\tau, \eta) \mapsto ([\tau]^2 + |\eta|^{2l})^{1/2l}$ with $[\tau] = \omega(|\tau|) + (1 - \omega(|\tau|))|\tau|$. Moreover, if $q = 0$, i.e., the spatial wedge has dimension 0, then we have obviously

$$\mathcal{W}^{s,l}(\mathbb{R}_t, E) = \mathcal{W}^{s/l}(\mathbb{R}_t, E).$$

Note that the anisotropic wedge Sobolev spaces are Banach spaces. As usually we write $H^{s,l}(\mathbb{R}^{1+q}, E)$ if we have $\kappa_\lambda = \mathrm{id}_E$ for all $\lambda \in \mathbb{R}_+$. With $E = \mathbb{C}$ we get the scalar version of anisotropic Sobolev spaces, which we denote by $H^{s,l}(\mathbb{R}^{1+q})$.

Remark 2 The isotropic case $l = 1$ is contained in our considerations. We will also need this special case here especially for distributions depending only on spatial or only on time variables. Then we will omit the index l in the notation, that means we write, for instance, $\mathcal{W}^s(\mathbb{R}_t, E) = \mathcal{W}^{s,1}(\mathbb{R}_t, E)$. This notation meets the original one introduced by Schulze [Su1].

Remark 3 Analogously to the isotropic case (cf. [Su1], Section 3.1), the operator $T = \mathcal{F}^{-1}_{(\tau,\eta)\to(t,y)} \kappa^{-1}(\tau, \eta) \mathcal{F}_{(t,y)\to(\tau,\eta)}$ extends by continuity to an isometry

$$T \; : \; \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \to H^{s,l}(\mathbb{R}^{1+q}, E). \tag{2}$$

This gives us the possibility to define $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{V})$ for subspaces $\mathcal{V} \subset E$ which are not necessary preserved under κ_λ by $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{V}) := T^{-1} H^{s,l}(\mathbb{R}^{1+q}, \mathcal{V})$.

Lemma 4 *For all $s \in \mathbb{R}$ the space $C_0^\infty(\mathbb{R}^{1+q}, E)$ is dense in $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$.*

Proof: By definition we only have to prove, that $C_0^\infty(\mathbb{R}^{1+q}, E)$ is dense with respect to the norm (1) in $\mathcal{S}(\mathbb{R}^{1+q}, E)$. The isotropic case $l = 1$ was treated, for instance, in [Su1], so that we have to show the assertion for $l > 1$.

Using the inequality $[\tau, \eta]_l \le c[\tau, \eta]_1$ we have $\|u\|_{s,l} \le \|u\|_{s,1}$ for all $s \in \mathbb{R}$ and every $u \in \mathcal{S}(\mathbb{R}^{1+q}, E)$. Thus the isotropic case implies the anisotropic case. □

Example 5 In our applications we are dealing with the case of $\mathcal{K}^{s,\gamma}(X^\wedge)$–valued distributions. As in Section 1.3 we take $\kappa_\lambda u(r) = \lambda^{\frac{n+1}{2}} u(\lambda r)$ as associated group action. Then we have from Definition 1 the Banach spaces $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s,\gamma}(X^\wedge))$. Setting $s = \gamma = 0$ we get the Hilbert space $\mathcal{W}^{0,l}(\mathbb{R}^{1+q}, \mathcal{K}^{0,0}(X^\wedge)) = \mathcal{W}^0(\mathbb{R}^{1+q}, \mathcal{K}^0(X^\wedge))$, which is independent of the anisotropy l. The corresponding scalar product is given by

$$(u, v)_0 = \int (\mathcal{F}u(\tau, \eta), \mathcal{F}v(\tau, \eta))_{\mathcal{K}^0(X^\wedge)} đ\tau đ\eta.$$

Of course, the space $C_0^\infty(\mathbb{R}^{1+q}, C_0^\infty(X^\wedge)) = C_0^\infty(\mathbb{R}^{1+q} \times X^\wedge)$ is dense in $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s,\gamma}(X^\wedge))$ for every $s, \gamma \in \mathbb{R}$, such that the form $(\cdot,\cdot)_0 \,:\, C_0^\infty(\mathbb{R}^{1+q} \times X^\wedge) \times C_0^\infty(\mathbb{R}^{1+q} \times X^\wedge) \to \mathbb{C}$ extends to a non–degenerate sesquilinear form

$$(\cdot,\cdot) \;:\; \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s,\gamma}(X^\wedge)) \times \mathcal{W}^{-s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{-s,-\gamma}(X^\wedge)) \to \mathbb{C}$$

for all $s, \gamma \in \mathbb{R}$ and all $l \in \mathbb{N} \setminus \{0\}$. This allows us to introduce formal adjoints A^* of operators

$$A \;:\; \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s,\gamma}(X^\wedge)) \to \mathcal{W}^{s-\nu,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge))$$

that are continuous operators

$$A^* \;:\; \mathcal{W}^{-s+\nu,l}(\mathbb{R}^{1+q}, \mathcal{K}^{-s+\nu,-\gamma+\mu}(X^\wedge)) \to \mathcal{W}^{-s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{-s,-\gamma}(X^\wedge))$$

for all $s \in \mathbb{R}$.

Remark 6 As in the isotropic case, cf. [Hi1], there exists a canonical embedding $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \hookrightarrow \mathcal{S}'(\mathbb{R}^{1+q}, E)$ given by $\langle u, \phi\rangle = \int u(t,y)\phi(t,y)\,dt\,dy$ with $u(t,y) \in \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$ and $\phi(t,y) \in \mathcal{S}(\mathbb{R}^{1+q})$.

Corollary 7 *An E–valued tempered distribution $u(t,y)$ belongs to the anisotropic wedge Sobolev space $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$ iff $\mathcal{F}u(\tau,\eta)$ is measurable and $\|u\|_{s,l} < \infty$.*

The next theorem is central for anisotropic wedge Sobolev spaces, because it shows, that an anisotropic space may be viewed as an isotropic space with values in another isotropic space. Therefore it allows us to generalise the properties of isotropic spaces to anisotropic ones.

Lemma 8 *Let E be a Banach space with group action κ_λ. Then for each $l \in \mathbb{N} \setminus \{0\}$ we get by*

$$\chi_\lambda u(y) = \kappa_{\lambda^{1/l}} \lambda^{q/2l} u(\lambda^{1/l} y), \quad \lambda \in \mathbb{R}_+,$$

a group action on $\mathcal{W}^s(\mathbb{R}^q, E)$ for all $s \in \mathbb{R}$. Moreover, if κ_λ is unitary on E^0 then the resulting χ_λ is unitary on $\mathcal{W}^0(\mathbb{R}^q, E^0)$.

Proof: Let us first note $\mathcal{F}_{y\to\eta}\{\chi_\lambda u(y)\} = \mathcal{F}_{y\to\eta}\{\kappa_{\lambda^{1/l}}\lambda^{q/2l}u(\lambda^{1/l}y)\} = \kappa_{\lambda^{1/l}}\lambda^{-q/2l}\mathcal{F}_{y\to\eta}u(\lambda^{-1/l}\eta)$. This gives

$$\begin{aligned}
\|\chi_\lambda u\|^2_{\mathcal{W}^s(\mathbb{R}^q,E)} &= \int [\eta]^{2s} \|\kappa^{-1}_{[\eta]} \mathcal{F}_{y\to\eta}\{\chi_\lambda u(y)\}\|^2_E \,đ\eta \\
&= \int [\eta]^{2s} \|\kappa^{-1}_{[\eta]} \kappa_{\lambda^{1/l}} \lambda^{-q/2l} \mathcal{F}_{y\to\eta} u(\lambda^{-1/l}\eta)\|^2_E \,đ\eta \\
&= \int [\lambda^{1/l}\eta]^{2s} \|\kappa^{-1}_{[\lambda^{1/l}\eta]} \kappa_{\lambda^{1/l}} \mathcal{F}_{y\to\eta} u(\eta)\|^2_E \,đ\eta.
\end{aligned}$$

This shows that χ_λ is unitary on $\mathcal{W}^0(\mathbb{R}^q, E^0)$ if κ_λ is unitary on E^0. In order to prove that χ_λ is a group action on $\mathcal{W}^s(\mathbb{R}^q, E)$ we show $\|u - \chi_\lambda u\|_{\mathcal{W}^s(\mathbb{R}^q,E)} \to 0$ for $\lambda \to 1$. To this end we use Lebesgue's dominated convergence theorem. Of course,

$$[\eta]^{2s}\|\kappa^{-1}_{[\eta]}\mathcal{F}_{y\to\eta}(u - \chi_\lambda u)(\eta)\|^2_E = [\eta]^{2s}\|\kappa^{-1}_{[\eta]}(\mathcal{F}_{y\to\eta}u(\eta) - \kappa_{\lambda^{1/l}}\lambda^{-q/2l}\mathcal{F}_{y\to\eta}u(\lambda^{-1/l}\eta))\|^2_E$$

tends to zero for $\lambda \to 1$, since κ_λ is a group action. Moreover,we have

$$\|\kappa^{-1}_{[\eta]}(\mathcal{F}u(\eta) - \kappa_{\lambda^{1/l}}\lambda^{-q/2l}\mathcal{F}u(\lambda^{-1/l}\eta))\|_E \le \|\kappa^{-1}_{[\eta]}\mathcal{F}u(\eta)\|_E + \|\kappa^{-1}_{[\eta]}\kappa_{\lambda^{1/l}}\lambda^{-q/2l}\mathcal{F}u(\lambda^{-1/l}\eta))\|_E,$$

such that it remains to check that $[\eta]^{2s}\|\kappa^{-1}_{[\eta]}\kappa_{\lambda^{1/l}}\lambda^{-q/2l}\mathcal{F}u(\lambda^{-1/l}\eta))\|^2_E$ is L_1 for $\lambda \in [\frac{1}{2}, 2]$. But this holds, since $\frac{[\eta]\lambda^{1/l}}{[\lambda^{1/l}\eta]}$ is bounded and bounded away from zero. □

Theorem 9 *Let us associate to $\mathcal{W}^s(\mathbb{R}^q, E)$ the group action χ_λ given in Lemma 8. Then we have $\mathcal{W}^{s/l}(\mathbb{R}, \mathcal{W}^s(\mathbb{R}^q, E)) = \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$.*

Proof: Since $\mathcal{S}(\mathbb{R}\times\mathbb{R}^q, E) = \mathcal{S}(\mathbb{R}, \mathcal{S}(\mathbb{R}^q, E))$ we only have to check the equivalence of the corresponding norms. Let $U(t) \in \mathcal{S}(\mathbb{R}, \mathcal{S}(\mathbb{R}^q, E))$ be given as $(U(t))(y) = u(t, y)$ with $u(t, y) \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^q, E)$. Then we have

$$\|u\|_{s,l} \sim \left\{ \int\int ([\tau]^2 + |\eta|^{2l})^{s/l} \|\kappa^{-1}_{([\tau]^2+|\eta|^{2l})^{1/2l}} \mathcal{F}_{(t,y)\to(\tau,\eta)} u(\tau, \eta)\|^2_E đ\tau đ\eta \right\}^{1/2},$$

where $\sim$ means equivalence of norms. Moreover, for the corresponding U we have

$$\|U\|_{s/l} \sim \left\{ \int [\tau]^{2s/l} \|\chi^{-1}_{[\tau]} \mathcal{F}_{t\to\tau} U(\tau)\|^2_{\mathcal{W}^s(\mathbb{R}^q,E)} đ\tau \right\}^{1/2}$$

Now with $\langle\eta\rangle := (1 + |\eta|^{2l})^{1/2l}$ we compute

$$\begin{aligned}
\|U\|^2_{s/l} &\sim \int [\tau]^{2s/l} \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\to\eta}\{\chi^{-1}_{[\tau]} \mathcal{F}_{t\to\tau} u(\tau, y)\}\|^2_E đ\eta đ\tau \\
&= \int\int ([\tau]^2 + [\tau]^2|\eta|^{2l})^{s/l} \|\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\to\eta}\{\kappa^{-1}_{[\tau]^{1/l}} [\tau]^{-q/2l} \mathcal{F}_{t\to\tau} u(\tau, [\tau]^{-1/l} y)\}\|^2_E đ\eta đ\tau \\
&= \int\int ([\tau]^2 + ([\tau]^{1/l}|\eta|)^{2l})^{s/l} \|\kappa^{-1}_{\langle\eta\rangle[\tau]^{1/l}} [\tau]^{q/2l} \mathcal{F}_{(t,y)\to(\tau,\eta)} u(\tau, [\tau]^{1/l}\eta)\|^2_E đ\eta đ\tau \\
&= \int\int ([\tau]^2 + |\eta|^{2l})^{s/l} \|\kappa^{-1}_{([\tau]^2+|\eta|^{2l})^{1/2l}} \mathcal{F}_{(t,y)\to(\tau,\eta)} u(\tau, \eta)\|^2_E đ\eta đ\tau \sim \|u\|^2_{s,l},
\end{aligned}$$

which is the desired equivalence. □

Corollary 10 *Let E be a Banach space with group action κ_λ and assumme that E' is a continuously embedded subspace of E connected with the group action $\kappa'_\lambda = \kappa_\lambda|_{E'}$. Then there are continuous embeddings*

$$\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \hookrightarrow \mathcal{W}^{s',l}(\mathbb{R}^{1+q}, E')$$

for all $s' \geq s$.

Proof: The isotropic case is proven, for instance, in [Su11] Proposition 1.3.30. Using this twice we get first continuous emmbeddings

$$\mathcal{W}^s(\mathbb{R}^q, E) \hookrightarrow \mathcal{W}^{s'}(\mathbb{R}^q, E')$$

and then

$$\mathcal{W}^{s/l}(\mathbb{R}, \mathcal{W}^s(\mathbb{R}^q, E)) \hookrightarrow \mathcal{W}^{s'/l}(\mathbb{R}, \mathcal{W}^{s'}(\mathbb{R}^q, E'))$$

for all $s' \geq s$. But then the assertion follows from Theorem 9. □

Definition 11 *Let $\Omega \subseteq \mathbb{R}^{1+q}$ be open and $K \subset\subset \Omega$ be compact. Then we define*

$$\begin{aligned}
\mathcal{W}^{s,l}_K(\Omega, E) &:= \{u \in \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \ : \ \text{supp}\, u \subseteq K\}, \\
\mathcal{W}^{s,l}_{comp}(\Omega, E) &:= \bigcup_{K\subset\subset\Omega} \mathcal{W}^{s,l}_K(\Omega, E), \\
\mathcal{W}^{s,l}_{loc}(\Omega, E) &:= \{u \in \mathcal{S}'(\mathbb{R}^{1+q}, E) \ : \ \omega u \in \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \textit{ for all } \omega \in C^\infty_0(\Omega)\}.
\end{aligned}$$

3.2 Mapping properties

Theorem 1 *Let $\Omega \subset \mathbb{R}^{1+q}$ be an open set and $a(t, y, t', y', \tau, \eta) \in S^{\nu,l}(\Omega^2 \times \mathbb{R}^{q+1}; E, \widetilde{E})$ be an anisotropic operator–valued symbol of order $\nu \in \mathbb{R}$ and $\mathrm{Op}(a) \in \Psi^{\nu,l}(\Omega; E, \widetilde{E})$ the associated pseudo–differential operator. Then $\mathrm{Op}(a) : \mathcal{W}^{s,l}_{comp}(\Omega, E) \to \mathcal{W}^{s-\nu,l}_{loc}(\Omega, \widetilde{E})$ is continuous for all $s \in \mathbb{R}$.*

The proof will be given in terms of a tensor product argument using the two following lemmata.

Lemma 2 *For every $v \in \mathcal{S}(\mathbb{R}^{1+q})$ the operator $M : C_0^\infty(\mathbb{R}^{1+q}, E) \ni \phi \mapsto v\phi \in C_0^\infty(\mathbb{R}^{1+q}, E)$ has a unique continuous extension to $M : \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \to \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$. Furthermore, the map $M : \mathcal{S}(\mathbb{R}^{1+q}) \ni v \mapsto M \in \mathcal{L}(\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E))$ is continuous for all $s \in \mathbb{R}$.*

Proof: Let $v(t,y) \in \mathcal{S}(\mathbb{R}^{1+q})$ and $\phi \in C_0^\infty(\mathbb{R}^{1+q}, E)$, then (1.1.8) and (1.1.11) imply

$$\begin{aligned}
\|M_v\phi\|_{s,l}^2 &= \int [\tau,\eta]_l^{2s} \left\|\kappa^{-1}_{[\tau,\eta]_l}\mathcal{F}(v\phi)(\tau,\eta)\right\|_E^2 \,đ\tau đ\eta \\
&= \int [\tau,\eta]_l^{2s} \left\|\kappa^{-1}_{[\tau,\eta]_l}\int \mathcal{F}v(\tau-\tau',\eta-\eta')\mathcal{F}\phi(\tau',\eta')\,đ\tau' đ\eta'\right\|_E^2 \,đ\tau đ\eta \\
&\le c_1 \int\!\!\int [\tau,\eta]_l^{2s} \left\|\kappa_{\frac{[\tau',\eta']_l}{[\tau,\eta]_l}}\right\|_{\mathcal{L}(E)}^2 |\mathcal{F}v(\tau-\tau',\eta-\eta')|^2 \left\|\kappa^{-1}_{[\tau',\eta']_l}\mathcal{F}\phi(\tau',\eta')\right\|_E^2 \,đ\tau' đ\eta' đ\tau đ\eta \\
&\le c^{2|s|}c_2 \int\!\!\int [\tau-\tau',\eta-\eta']_l^{2(M+|s|)}|\mathcal{F}v(\tau-\tau',\eta-\eta')|^2\,đ\tau đ\eta [\tau',\eta']_l^{2s}\left\|\kappa^{-1}_{[\tau',\eta']_l}\mathcal{F}\phi(\tau',\eta')\right\|_E^2 \,đ\tau' đ\eta' \\
&\le C_v c^{|2s|}c_2 \int [\tau',\eta']_l^{2s}\left\|\kappa^{-1}_{[\tau',\eta']_l}\mathcal{F}\phi(\tau',\eta')\right\|_E^2 \,đ\tau' đ\eta' = C\|\phi\|_{s,l}^2,
\end{aligned}$$

and hence also the extension $M_v : \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \to \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$ is continuous. Moreover, $C_v \to 0$ for $v \to 0$ within $\mathcal{S}(\mathbb{R}^{1+q})$ which gives the second part of the lemma. □

Lemma 3 *Let $a(\tau,\eta) \in S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$ be an anisotropic operator–valued symbol with constant coefficients of order $\nu \in \mathbb{R}$ and $\mathrm{Op}(a) \in \Psi^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$ the associated pseudo–differential operator. Then $\mathrm{Op}(a) : \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \to \mathcal{W}^{s-\nu,l}(\mathbb{R}^{1+q}, \widetilde{E})$ is a continuous map for all $s \in \mathbb{R}$, and we have $\|\mathrm{Op}(a)\| \le \sup_{(\tau,\eta)\in\mathbb{R}^{1+q}}[\tau,\eta]_l^{-\nu}\|\tilde\kappa^{-1}(\tau,\eta)a(\tau,\eta)\kappa(\tau,\eta)\|_{\mathcal{L}(E,\widetilde{E})} =: p_{0,0}^{(\nu)}(a)$.*

Proof: Since $(\mathrm{Op}(a)u)(t,y)$ belongs to $\mathcal{S}'(\mathbb{R}^{1+q}, \widetilde{E})$ for all $u \in \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$ we only have to check $\|\mathrm{Op}(a)u\|_{s-\nu,l} \le p_{0,0}^{(\nu)}(a)$. But the symbol estimates yields

$$\begin{aligned}
\|\mathrm{Op}(a)u\|_{s-\nu,l}^2 &= \int [\tau,\eta]_l^{2(s-\nu)}\|\tilde\kappa^{-1}_{[\tau,\eta]_l}a(\tau,\eta)\mathcal{F}u(\tau,\eta)\|_{\widetilde{E}}^2\,đ\tau\,đ\eta \\
&= \int [\tau,\eta]_l^{2(s-\nu)}\|\tilde\kappa^{-1}_{[\tau,\eta]_l}a(\tau,\eta)\kappa_{[\tau,\eta]_l}\kappa^{-1}_{[\tau,\eta]_l}\mathcal{F}u(\tau,\eta)\|_{\widetilde{E}}^2\,đ\tau\,đ\eta \\
&\le \int (p_{0,0}^{(\nu)}(a))^2[\tau,\eta]_l^{2s}\|\kappa^{-1}_{[\tau,\eta]_l}\mathcal{F}u(\tau,\eta)\|_E^2\,đ\tau\,đ\eta \\
&= (p_{0,0}^{(\nu)}(a))^2\|u\|_{s,l}^2
\end{aligned}$$

as desired. □

Proof: (of Theorem 1) For every $K \subset\subset \Omega$ there exists a function $\phi \in C_0^\infty(\Omega)$ such that $\phi(t,y) \equiv 1$ in a neighbourhood of K. We want to prove that for every fixed $K \subset\subset \Omega$ with such a $\phi \in C_0^\infty(\Omega)$ and arbitrary $\psi \in C_0^\infty(\Omega)$ the operator

$$M_\psi \mathrm{Op}(a) M_\phi \;:\; \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \to \mathcal{W}^{s-\nu,l}(\mathbb{R}^{1+q}, \widetilde{E}) \tag{1}$$

is continuous for all $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(\Omega^2 \times \mathbb{R}^{q+1}; E, \widetilde{E})$.

Because of $S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E}) = C^\infty(\Omega)\widehat{\otimes}_\pi S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})\widehat{\otimes}_\pi C^\infty(\Omega)$ we have the representation $a(t,y,t',y',\tau,\eta) = \sum_{j=0}^\infty \lambda_j b_j(t,y)a_j(\tau,\eta)d_j(t',y')$, where $b_j \to 0$ and $d_j \to 0$ in $C^\infty(\Omega)$, $a_j \to 0$ in $S^{\nu,l}(\mathbb{R}^{q+1}; E, \widetilde{E})$ and $\{\lambda_j\}_{j=1}^\infty \in l_1$.

Thus we obtain

$$M_\psi \mathrm{Op}(a) M_\phi = \sum_{j=1}^\infty \lambda_j M_{\psi b_j}\mathrm{Op}(a_j)M_{\phi d_j},$$

where $\psi b_j \to 0$ and $\phi d_j \to 0$ in $C_0^\infty(\Omega)$. Therefore, we get by Lemma 2 and Lemma 3

$$\begin{aligned}\|M_\psi \mathrm{Op}(a) M_\phi u\|_{s-\nu,l} &= \|\sum_{j=1}^{\infty} \lambda_j M_{\psi b_j} \mathrm{Op}(a_j) M_{\phi d_j} u\|_{s-\nu,l} \\ &\le \sum_{j=1}^{\infty} |\lambda_j| \, \|M_{\psi b_j}\| \, \|\mathrm{Op}(a_j)\| \, \|M_{\phi d_j}\| \, \|u\|_{s,l} \\ &\le \sum_{j=1}^{\infty} |\lambda_j| c_{\psi b_j} p_{0,0}^{(\nu)}(a_j) c_{\phi d_j} \|u\|_{s,l}.\end{aligned}$$

The convergence of $\{c_{\psi b_j}\}$, $\{c_{\phi d_j}\}$ and $\{p_{0,0}^{(\nu)}(a_j)\}$ implies, in particular, boundedness, which gives the continuity of (1). □

Like in Section 1.3 we deal with the case of Fréchet spaces E and $\widetilde{E}$, which have representations as projective limits of Banach spaces, where the associated group actions satisfy the corresponding compatibility conditions.

In this case we also set $\Psi^{\nu,l}(\Omega; E, \widetilde{E}) := \mathrm{Op}(S^{\nu,l}(\Omega^2 \times \mathbb{R}^{1+q}; E, \widetilde{E}))$ for arbitrary open $\Omega \subset \mathbb{R}^{1+q}$ and define $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) = \varprojlim_{k\in\mathbb{N}} \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E_k)$ as well as

$$\mathcal{W}^{s,l}_{comp}(\Omega, E) = \varprojlim_{k\in\mathbb{N}} \mathcal{W}^{s,l}_{comp}(\Omega, E_k) \text{ and } \mathcal{W}^{s,l}_{loc}(\Omega, E) = \varprojlim_{k\in\mathbb{N}} \mathcal{W}^{s,l}_{loc}(\Omega, E_k).$$

We then obtain the following mapping property for pseudo–differential operators with symbols that have values in $\mathcal{L}(E, \widetilde{E})$.

Theorem 4 *If E and $\widetilde{E}$ are Fréchet spaces with above properties, then every $A \in \Psi^{\nu,l}(\Omega; E, \widetilde{E})$ has a unique extension to a linear continuous operator $A : \mathcal{W}^{s,l}_{comp}(\Omega, E) \to \mathcal{W}^{s-\nu,l}_{loc}(\Omega, \widetilde{E})$ for all $s \in \mathbb{R}$.*

3.3 Global spaces

Let us now form the global abstract wedge Sobolev spaces in the anisotropic set–up. To this end we assume that we have a Hilbert space E and a Hilbert space E^0 equipped with a unitary group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$. Moreover, we suppose $E \subseteq E^0$ or $E^0 \subseteq E$ and κ_λ restricts or extends to a group action on E. For this pair of spaces we further assume the existence of an isomorphism $a : E \to E^0$ such that the map $\lambda \mapsto \kappa_\lambda a \kappa_\lambda^{-1}$ belongs to $C^\infty(\mathbb{R}_+, \mathcal{L}(E, E^0))$.

As before (cf. 2.3) we start with a diffeomorphism $\chi_1 : U \to \widetilde{U}$ that induces by $\chi(t, y) = (t, \chi_1(y))$ a diffeomorphism $\chi : \mathbb{R} \times U \to \mathbb{R} \times \widetilde{U}$. Then the pull–back of E–valued distributions

$$\chi^* \ : \ \mathcal{D}'(\mathbb{R} \times \widetilde{U}, E) \to \mathcal{D}'(\mathbb{R} \times U, E) \tag{1}$$

is given by $\langle \chi^* u, \varphi\rangle = \langle u, (\chi*)^{-1}\varphi\rangle$ for $u \in \mathcal{D}'(\mathbb{R} \times \widetilde{U}, E)$ and $\varphi \in C_0^\infty(\mathbb{R} \times U)$.

These relations allow us to state the following proposition that can be proven completely analogous io the isotropic case (cf. [Su10]).

Proposition 1 *The pull–back* (1) *of E–valued distributions* (1) *restricts to continuous mappings*

$$\begin{aligned}\chi^* &: \mathcal{W}^{s,l}_{comp}(\mathbb{R} \times \widetilde{U}, E) \to \mathcal{W}^{s,l}_{comp}(\mathbb{R} \times U, E) \\ \chi^* &: \mathcal{W}^{s,l}_{loc}(\mathbb{R} \times \widetilde{U}, E) \to \mathcal{W}^{s,l}_{loc}(\mathbb{R} \times U, E)\end{aligned}$$

for all $s \in \mathbb{R}$.

Next, we fix on the smooth compact manifold Y a finite atlas $\{\widetilde{U}_j, \tilde{\chi}_j\}_{j=1}^N$ with charts $\tilde{\chi}_j : \widetilde{U}_j \to U_j \subset \mathbb{R}^q$ and a subordinate partition of unity $\{\phi_j\}_{j=1}^N$. With

$$u \mapsto \|u\|_{\mathcal{W}^{s,l}(\mathbb{R}\times Y,E)} := \sum_{j=1}^N \left\| \left(\chi_j^*\right)^{-1} (\phi_j u) \right\|_{s,l} \tag{2}$$

we then get a norm in $C_0^\infty(\mathbb{R}\times Y, E)$. Here χ_j^* is the pull–back with respect to $\chi_j : \mathbb{R}\times\widetilde{U} \to \mathbb{R}\times U$ defined by $\chi_j(t,y) = (t, \tilde{\chi}_j(y))$. In the sequel we will also write $\|\cdot\|_{s,l}$ instead of $\|\cdot\|_{\mathcal{W}^{s,l}(\mathbb{R}\times Y,E)}$.

Definition 2 *The global abstract anisotropic wedge Sobolev space $\mathcal{W}^{s,l}(\mathbb{R}\times Y, E)$ of smoothness $s \in \mathbb{R}$ is the completion of $C_0^\infty(\mathbb{R}\times Y, E)$ with respect to* (2).

From Proposition 1 it follows that another choice of data $\widetilde{U}_j, \chi_j, \phi_j$ yields an equivalent norm on $\mathcal{W}^{s,l}(\mathbb{R}\times Y, E)$.

Of course, we have $\mathcal{W}^{s,l}(\mathbb{R}\times Y, E) \subset \mathcal{D}'(\mathbb{R}\times Y, E)$ for all $s \in \mathbb{R}$. The space $\mathcal{W}^{s,l}_{comp}(\mathbb{R}\times Y, E)$ is then the subspace of all distributios $u \in \mathcal{W}^{s,l}(\mathbb{R}\times Y, E)$ with compact support in $\mathbb{R}\times Y$; and $\mathcal{W}^{s,l}_{loc}(\mathbb{R}\times Y, E)$ is the space of all $u \in \mathcal{D}'(\mathbb{R}\times Y, E)$ with $\varphi u \in \mathcal{W}^{s,l}_{comp}(\mathbb{R}\times Y, E)$ for all $\varphi \in C_0^\infty(\mathbb{R}\times Y)$.

Proposition 3 *Every $A \in \Psi^{\nu,l}(\mathbb{R}\times Y; E, \widetilde{E})$ extends for every $s \in \mathbb{R}$ to a continuous map $A : \mathcal{W}^{s,l}_{comp}(\mathbb{R}\times Y, E) \to \mathcal{W}^{s-\nu,l}_{loc}(\mathbb{R}\times Y, \widetilde{E})$.*

Proof: By definition we have a representation

$$A = \sum_{j=1}^N \phi_j \left((\chi_j)_*^{-1} A_j\right) \psi_j + C$$

with $A_j \in \Psi^{\nu,l}(\mathbb{R}\times U_j; E, \widetilde{E})$ for all $j = 1,\dots,N$ and $C \in \Psi^{-\infty}(\mathbb{R}\times Y; E, \widetilde{E})$ (cf. 2.3 Definition 4). Since C is smoothing, we can restrict the consideration to the first part of the sum. But for this part we can use Proposition 1 and the local mapping property 3.2 Theorem 1 to check the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{W}^{s,l}_{comp}(\mathbb{R}\times\widetilde{U}_j, E) & \overset{(\chi_j)_*^{-1} A_j}{\longrightarrow} & \mathcal{W}^{s-\nu,l}_{loc}(\mathbb{R}\times\widetilde{U}_j, \widetilde{E}) \\ (\chi_j^*)^{-1} \Big\downarrow & & \chi_j^* \Big\uparrow \\ \mathcal{W}^{s,l}_{comp}(\mathbb{R}\times U_j, E) & \overset{A_j}{\longrightarrow} & \mathcal{W}^{s-\nu,l}_{loc}(\mathbb{R}\times U_j, \widetilde{E}) \end{array}$$

for every $j = 1,\dots,N$, and this gives the assertion. □

4 Parameter–dependent ellipticity

4.1 Parameter–dependent pseudo–differential operators

The definitions and results from the preceding sections have a parameter–dependent variant, where the covariables $(\tau,\eta) \in \mathbb{R}^{1+q}$ in the operator–valued symbols are to be replaced by (ζ,η) with $\zeta = \tau + i\vartheta \in \mathbb{C}$, and $\vartheta = \mathrm{Im}\ \zeta$ plays the role of a further anisotropic parameter.

Recall that for $(\zeta,\eta) \in \mathbb{C}\times\mathbb{R}^{1+q}$ we also have an anisotropic norm function

$$|\zeta,\eta|_l = (|\zeta|^2 + |\eta|^{2l})^{\frac{1}{2l}},$$

and the corresponding smoothed version is given by

$$[\zeta,\eta]_l := \omega(|\zeta,\eta|_l) + (1-\omega(|\zeta,\eta|_l))|\zeta,\eta|_l.$$

Here $\omega(r)$ is a cut–off function, cf. (1.1.3). As above we set $\kappa(\zeta,\eta) = \kappa_{[\zeta,\eta]_l}$ for a group κ_λ, $\lambda \in \mathbb{R}_+$, acting in some Banach space E.

Definition 1 *Let $\nu \in \mathbb{R}$, $U \subseteq \mathbb{R}^p$ be open and $\overline{\mathbb{C}}_- := \{\zeta = \tau + i\vartheta \in \mathbb{C} : \vartheta \le 0\}$. Then the space*

$$S^{\nu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E}) \tag{1}$$

is defined as the set of all $a(y,\zeta,\eta) \in C^\infty(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q, \mathcal{L}(E,\widetilde{E}))$ satisfying

$$\|\tilde{\kappa}^{-1}(\zeta,\eta)\{D_y^\alpha D_\eta^{\beta'} D_\tau^{\beta_{01}} D_\vartheta^{\beta_{02}} a(y,\zeta,\eta)\}\kappa(\zeta,\eta)\|_{\mathcal{L}(E,\widetilde{E})} \le c[\zeta,\eta]_l^{\nu-|\beta|_l} \tag{2}$$

for all $\alpha \in \mathbb{N}^p$, $\beta = (\beta_{01},\beta_{02},\beta') \in \mathbb{N}^2 \times \mathbb{N}^q$ and all $y \in K$ for arbitrary compact $K \subset\subset U$ and all $(\zeta,\eta) \in \overline{\mathbb{C}}_- \times \mathbb{R}^q$ with constants $c = c(\alpha,\beta,K) \ge 0$; here $|\beta|_l = l(\beta_{01}+\beta_{02}) + |\beta'|$ for $\beta \in \mathbb{N}^{2+q}$.

The elements of $S^{\nu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ are called parameter–dependent symbols (or amplidude functions) with parameter $\zeta \in \overline{\mathbb{C}}_-$. There is a trivial modification of Definition 1 leading to time–dependent amplitude functions denoted by $S^{\nu,l}(\Omega \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ where $\Omega \subset \mathbb{R}^{p_0} \times \mathbb{R}^p$ is an open set, cf. 1.1 Definition 5.

We will not repeat here the standard properties of these symbol spaces, that are analogous to those of Section 1.1. In particular, we have the corresponding versions of 1.1 Lemma 6, 1.1 Lemma 7 and 1.1 Remark 9. Moreover, the symbol estimates (2) give

$$S^{-\infty}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E}) := S^{-\infty,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E}) = C^\infty(U, \mathcal{S}(\overline{\mathbb{C}}_- \times \mathbb{R}^q, \mathcal{L}(E,\widetilde{E}))),$$

where $\mathcal{S}(\overline{\mathbb{C}}_- \times \mathbb{R}^q, \mathcal{L}(E,\widetilde{E}))$ denotes the space of restrictions of elements of $\mathcal{S}(\mathbb{C} \times \mathbb{R}^q, \mathcal{L}(E,\widetilde{E}))$ to $\overline{\mathbb{C}}_- \times \mathbb{R}^q$.

Note that the substitution $\varsigma = \zeta^l$ allows us to replace the parameter space $\overline{\mathbb{C}}_-$ by $\Sigma = \{\varsigma = \varrho e^{i\varphi} \in \mathbb{C} : 0 \le \varphi \le \pi/l\}$ as it was done in [AV]. For instance, the symbol estimates (2) then take the form

$$\|\tilde{\kappa}^{-1}(\varsigma,\eta)\{D_y^\alpha D^\beta_{\mathrm{Re}\ \varsigma, \mathrm{Im}\ \varsigma, \eta} a(y,\varsigma,\eta)\}\kappa(\varsigma,\eta)\|_{\mathcal{L}(E,\widetilde{E})} \le c[\varsigma,\eta]^{\nu-|\beta|}$$

for all $\alpha \in \mathbb{N}^p$, $\beta = (\beta_0,\beta') \in \mathbb{N}^2 \times \mathbb{N}^q$ and all $y \in K$ for arbitrary compact $K \subset\subset U$ and all $(\varsigma,\eta) \in \Sigma \times \mathbb{R}^q$ with constants $c = c(\alpha,\beta,K) \ge 0$; here $[\varsigma,\eta] := [\varsigma,\eta]_1$ which is also involved in $\kappa(\varsigma,\eta)$ and $\tilde{\kappa}(\varsigma,\eta)$, respectively.

In the anisotropic parameter–dependent setting we also have an analogue of homogeneity in the operator–valued sense namely

$$f(\lambda^l\zeta, \lambda\eta) = \lambda^\nu \tilde{\kappa}_\lambda f(\zeta,\eta)\kappa_\lambda^{-1}$$

for all $\lambda \in \mathbb{R}_+$ and all $(\zeta,\eta) \in (\overline{\mathbb{C}}_- \times \mathbb{R}^q) \setminus \{0\}$. We then get a natural notion of classical symbols in (ζ,η). An element $a(y,y',\zeta,\eta) \in S^{\nu,l}(U \times U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ is called classical if there exists a sequence $a_{(\nu-j)}(y,y',\zeta,\eta) \in C^\infty(U \times U \times (\overline{\mathbb{C}}_- \times \mathbb{R}^q) \setminus \{0\}; \mathcal{L}(E,\widetilde{E}))$, $j \in \mathbb{N}$, satisfying

$$a_{(\nu-j)}(y,y',\lambda^l\zeta,\lambda\eta) = \lambda^{\nu-j}\tilde{\kappa}_\lambda a_{(\nu-j)}(y,y',\zeta,\eta)\kappa_\lambda^{-1}$$

for all $\lambda \in \mathbb{R}_+$, all $(y,y') \in U \times U$ and all $(\zeta,\eta) \in (\overline{\mathbb{C}}_- \times \mathbb{R}^q) \setminus \{0\}$, such that

$$a(y,y',\zeta,\eta) - \sum_{j=0}^N \chi(\zeta,\eta) a_{(\nu-j)}(y,y',\zeta,\eta) \in S^{\nu-(N+1),l}(U \times U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$$

for each $N \in \mathbb{N}$, where $\chi(\zeta,\eta)$ is an arbitrary excision function. The $a_{(\nu-j)}(y,y',\zeta,\eta)$, called anisotropic $(\nu-j)$–homogeneous components, are uniquely determined by the symbol $a(y,y',\zeta,\eta)$. In particular, we set $\sigma_\wedge^\nu(a)(y,y',\zeta,\eta) := a_{(\nu)}(y,y',\zeta,\eta)$ for the operator–valued principal symbol. The subspace of all classical symbols will be denoted by $S_{cl}^{\nu,l}(U \times U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$. Analogously to 1.2 Remark 3 we obtain, that smooth operator–valued functions that are anisotropic homogeneous in the operator–valued sense for large $[\zeta,\eta]_l$ are classical parameter–dependent symbols.

Let us now introduce the associated parameter–dependent pseudo–differential operators, first in local form and with time–independent symbols. For open $U \subseteq \mathbb{R}^q$ every $a(y,y',\zeta,\eta) \in S^{\nu,l}(U \times U \times \overline{\mathbb{C}}_- \times$

$\mathbb{R}^q; E, \widetilde{E})$ gives a ζ–dependent family of symbols in $S^\nu(U \times U \times \mathbb{R}^q; E, \widetilde{E})$. This allows us to form the corresponding parameter–dependent pseudo–differential operators $\mathrm{Op}_y(a)(\zeta)$

$$\left(\mathrm{Op}_y(a)(\zeta)\right) u(y) = \int\int e^{i(y-y')\eta} a(y, y', \zeta, \eta) u(y') dy' đ\eta \tag{3}$$

for $u(y) \in C_0^\infty(U, E)$ which defines a family of mappings

$$\mathrm{Op}_y(a)(\zeta) : C_0^\infty(U, E) \to C^\infty(U, \widetilde{E})$$

that can be extended in the standard way to a family of operators between suitable vector–valued Sobolev spaces.

Denote by $\Psi^{\nu,l}_{(cl)}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ the space of parameter–dependent pseudo–differential operators which is the set of all $\mathrm{Op}_y(a)(\zeta)$, cf. formula (3), for arbitrary $a(y, y', \zeta, \eta) \in S^{\nu,l}_{(cl)}(U \times U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$. Furthermore, we set

$$\begin{aligned} \Psi^{-\infty}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-) &:= \bigcap_{\nu \in \mathbb{R}} \Psi^{\nu,l}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-) \\ &= \mathcal{S}(\overline{\mathbb{C}}_-; \Psi^{-\infty}(U; E, \widetilde{E})) \\ &= \mathcal{S}(\mathbb{C}; \Psi^{-\infty}(U; E, \widetilde{E}))|_{\overline{\mathbb{C}}_-}, \end{aligned}$$

with $\mathbb{C}$ identified with $\mathbb{R}^2 \ni (\tau, \vartheta)$.

Similarly as above we can introduce the spaces of operator–valued distributions

$$T^{\nu,l}(U \times U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E}) := \mathcal{F}^{-1}_{\eta \to \sigma} \left(S^{\nu,l}(U \times U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E}) \right).$$

In other words, $T^{\nu,l}(U \times U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ consists of all elements

$$K(a)(y, y', \zeta, \sigma) := \int e^{i\sigma\eta} a(y, y', \zeta, \eta) đ\eta, \tag{4}$$

for arbitrary $a(y, y', \zeta, \eta) \in S^{\nu,l}(U \times U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$, and

$$K_{\mathrm{Op}_y(a)(\zeta)}(y, y', \zeta) = K(a)(y, y', \zeta, \sigma)|_{\sigma = y - y'}$$

is then the ζ–dependent family of (operator–valued) distributional kernels of $\mathrm{Op}_y(a)(\zeta)$. The singular support of $K_{\mathrm{Op}_y(a)(\zeta)}$ is contained in $\mathrm{diag}(U \times U)$ for all ζ, cf. 2.1 Proposition 3.

Definition 2 *A family $A(\zeta) \in \Psi^{\nu,l}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ of pseudo–differential operators is called properly supported if it is properly supported for every $\zeta \in \overline{\mathbb{C}}_-$, and for every compact set $C \subset\subset U$ there exists a compact set $M \subset\subset U \times U$ with*

$$(\pi_j^{-1}(C) \times \overline{\mathbb{C}}_-) \cap \mathrm{supp}\ K_{\mathrm{Op}_y(a)(\zeta)} \subseteq M \times \overline{\mathbb{C}}_-$$

for all $\zeta \in \mathbb{C}$. Here $K(a)(\zeta)$ denotes the distributional kernel of $A(\zeta)$ and $\pi_j : U \times U \to U$ means the projection to the j–th component, $j = 1, 2$.

Analogously to the case of anisotropic pseudo–differential operators with operator–valued symbols we obtain the following result.

Proposition 3 *Every family $A(\zeta) \in \Psi^{\nu,l}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ has a decomposition $A(\zeta) = A_0(\zeta) + A_1(\zeta)$ with a properly supported $A_0(\zeta) \in \Psi^{\nu,l}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ and an $A_1(\zeta) \in \Psi^{-\infty}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$.*

Remark 4 Let $\phi(y), \psi(y) \in C^\infty(U)$ and suppose $\mathrm{supp}\ \phi \cap \mathrm{supp}\ \psi = \emptyset$. Then $\phi A(\zeta)\psi \in \Psi^{-\infty}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ for every $A(\zeta) \in \Psi^{\nu,l}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$.

Remark 5 Notice that it also makes sense to consider a real anisotropic parameter $\tau \in \mathbb{R}$ instead of $\zeta \in \overline{\mathbb{C}}_-$. Also in this case we are interested in symbols $a(y,y',\tau,\eta) \in S^{\nu,l}(\mathbb{R} \times U \times \mathbb{R} \times U \times \mathbb{R}^{1+q}; E, \widetilde{E})$, and we get

$$\mathrm{Op}(a)u(t,y) = \mathrm{Op}_t(\mathrm{Op}_y(a)(\tau))u(t,y) := \mathcal{F}^{-1}_{\tau\to t}\mathrm{Op}_y(a)(\tau)\mathcal{F}_{t'\to\tau}u(t,y),$$

first for $u(t,y) \in C_0^\infty(\mathbb{R} \times U, E)$ and then extended to the corresponding wedge Sobolev spaces.

Regarding the imargenary part of the parameter ζ as a further covariable we can repeat the arguments of Section 2.2 to see the algebra properties also for the parameter–dependent calculus, i.e., we have the following results.

Theorem 6 *Let $U \subseteq \mathbb{R}^q$ be any open set and set $U^2 = U \times U$. Then we have the following properties.*

(i) Let $a_j(y,y',\zeta,\eta) \in S^{\nu_j,l}(U^2 \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$, $j = 0,1,\ldots$, be any sequence of parameter–dependent symbols and suppose $\nu_j \to -\infty$ for $j \to \infty$. Then there exists an symbol $a(y,y',\zeta,\eta) \in S^{\nu,l}(U^2 \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$, $\nu := \max \nu_j$, such that for every $\mu \in \mathbb{R}$ there is an $N(\mu) \in \mathbb{N}$ with

$$a(y,y',\zeta,\eta) - \sum_{j=0}^{N} a_j(y,y',\zeta,\eta) \in S^{\mu,l}(U^2 \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$$

for all $N \geq N(\mu)$, and a is unique modulo $S^{-\infty}(U^2 \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$,

We then write $a \sim \sum a_j$.

(ii) For every $a(y,y',\zeta,\eta) \in S^{\nu,l}(U^2 \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ There exists an $\underline{a}(y,\zeta,\eta) \in S^{\nu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ such that

$$\mathrm{Op}(a) - \mathrm{Op}(\underline{a}) \in \Psi^{-\infty}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$$

holds. For $\underline{a}$ we have the asymptotic expansion

$$\underline{a}(y,\zeta,\eta) \sim \sum_{\alpha\in\mathbb{N}^q} \frac{1}{\alpha!} D_{y'}^\alpha \partial_\eta^\alpha a(y,y',\zeta,\eta)|_{y'=y}.$$

(iii) For every $a(y',\zeta,\eta) \in S^{\nu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ there exists an $\tilde{a}(y,\zeta,\eta) \in S^{\nu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ such that

$$\mathrm{Op}(a) - \mathrm{Op}(\tilde{a}) \in \Psi^{-\infty}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$$

holds. For $\tilde{a}$ we have the asymptotic expansion

$$\tilde{a}(y,\zeta,\eta) \sim \sum_{\alpha\in\mathbb{N}^q} \frac{1}{\alpha!} D_{y'}^\alpha (-\partial_\eta)^\alpha a(y',\zeta,\eta)|_{y'=y}.$$

(iv) Let parameter–dependent operators $A(\zeta) = \mathrm{Op}_y(a)(\zeta) \in \Psi^{\nu,l}(U; F, \widetilde{E}; \overline{\mathbb{C}}_-)$ and $B(\zeta) = \mathrm{Op}_y(b)(\zeta) \in \Psi^{\mu,l}(U; E, F; \overline{\mathbb{C}}_-)$ with $A(\zeta)$ or $B(\zeta)$ properly supported be given. Then the point-wise (in ζ) composition $AB(\zeta) = A(\zeta)B(\zeta)$ belongs to $\Psi^{\nu+\mu,l}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ and

$$AB(\zeta) - \mathrm{Op}_y(a \#_y b)(\zeta) \in \Psi^{-\infty}(\Omega; E, \widetilde{E}; \overline{\mathbb{C}}_-).$$

Here and in the sequel we set

$$(a \#_y b)(y,\zeta,\eta) \sim \sum_{\alpha\in\mathbb{N}^q} \frac{1}{\alpha!} \partial_\eta^\alpha a(y,\zeta,\eta) D_y^\alpha b(y,\zeta,\eta).$$

(*v*) *An element* $a(y,\zeta,\eta) \in S^{\nu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ *has a Leibniz inverse* $p(y,\zeta,\eta) \in S^{-\nu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; \widetilde{E}, E)$*, i.e.,*

$$(a \#_y p - 1)(y,\zeta,\eta) \in S^{-\infty}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; \widetilde{E}, \widetilde{E}),$$
$$(p \#_y a - 1)(y,\zeta,\eta) \in S^{-\infty}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, E),$$

exactly when there exists a $b(y,\zeta,\eta) \in S^{-\nu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; \widetilde{E}, E)$ *satisfying*

$$(a \circ b - 1)(y,\zeta,\eta) \in S^{-1,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; \widetilde{E}, \widetilde{E}),$$
$$(b \circ a - 1)(y,\zeta,\eta) \in S^{-1,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, E).$$

(*vi*) *Let* $U, \widetilde{U} \subseteq \mathbb{R}^q$ *be open sets and let* $\chi : U \to \widetilde{U}$ *be a diffeomorphism. Then the operator push–forward* $\widetilde{A}(\zeta) := \chi_* A(\zeta)$ *of* $A(\zeta) = \mathrm{Op}_y(a)(\zeta) \in \Psi^{\nu,l}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ *belongs to* $\Psi^{\nu,l}(\widetilde{U}; E, \widetilde{E}; \overline{\mathbb{C}}_-)$*, and we have* $\widetilde{A}(\zeta) = \mathrm{Op}(\tilde{a})(\zeta) \bmod \Psi^{-\infty}(\widetilde{U}; E, \widetilde{E}; \overline{\mathbb{C}}_-)$*, where* $\tilde{a}$ *has the asymptotic expansion*

$$\tilde{a}(\chi(y), \zeta, \tilde{\eta}) \sim \sum_{\alpha \in \mathbb{N}^q} \frac{1}{\alpha!} \varphi_\alpha(y, \tilde{\eta}) \left(\partial_\eta^\alpha a\right)(y, \zeta, {}^t d\chi(y)\tilde{\eta})$$

where $\varphi_\alpha(y,\tilde{\eta})$ *is polynomial in* $\tilde{\eta}$ *of order* $\leq \frac{|\alpha|}{2}$ *and* $\varphi_0 \equiv 1$.

Remark 7 Note that the diffeomorphism χ in Theorem 6,(*vi*) does not concern the parameter such that the anisotropic behaviour plays no role in the arguments.

We will employ the parameter–dependent pseudo–differential operators in global form along the (isotropic) edge Y, where Y is a closed compact C^∞ manifold. Notice that here was assumed that E is a Hilbert space.

First we have the global (isotropic) Sobolev spaces $\mathcal{W}^s(Y,E)$, $s \in \mathbb{R}$. The details of the precise definition can easily be red off from the above anisotropic construction by setting $l = 1$ and replacing the involved variables by y. The time–variable varying along $\mathbb{R}$ can be omitted. The spaces $\mathcal{W}^s(Y,E)$ may be found in the monograph [Su10]. We also have the spaces of (isotropic) pseudo–differential operators $\Psi^\nu_{(cl)}(Y; E, \widetilde{E})$, $\nu \in \mathbb{R}$, with respect to pairs $\{E, \kappa_\lambda\}$, $\{\widetilde{E}, \tilde{\kappa}_\lambda\}$ of Hilbert spaces with fixed strongly continuous groups of isomorphisms. Also this can easily be derived from the above definitions by omitting the time direction such that the anisotropy disappears. Note, in particular, that

$$\Psi^{-\infty}(Y; E, \widetilde{E}) \cong C^\infty(Y \times Y; \mathcal{L}(E, \widetilde{E}))$$

which is independent of the particular group actions.

Theorem 8 *Every* $A \in \Psi^\nu(Y; E, \widetilde{E})$*, first regarded as a continuous operator*

$$A : C^\infty(Y, E) \to C^\infty(Y, \widetilde{E}),$$

extends to a continuous operator

$$\mathcal{W}^s(Y,E) \to \mathcal{W}^{s-\nu}(Y, \widetilde{E})$$

for each $s \in \mathbb{R}$.

This result can be proved by analogous methods as above in the more complicated anisotropic case.

To introduce the global parameter–dependent pseudo–differential operators with anisotropic dependence on the parameters (τ, ϑ) we set

$$\Psi^{-\infty}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-) = \mathcal{S}(\overline{\mathbb{C}}_-; \Psi^{-\infty}(Y; E, \widetilde{E})) = \mathcal{S}(\mathbb{C}; \Psi^{-\infty}(Y; E, \widetilde{E}))|_{\overline{\mathbb{C}}_-},$$

where $\Psi^{-\infty}(Y;E,\widetilde{E})$ is endowed with its natural Fréchet topology.

Recall (cf. Section 2.3) that in order to formulate global operators on a closed compact C^∞ manifold Y we fix an N–tuple of triples $\{((\widetilde{U}_j,\chi_j),\phi_j,\psi_j)\}_{j=1}^N$, where $\{(\widetilde{U}_j,\chi_j)\}_{j=1}^N$ is a fixed atlas on Y, $\{\phi_j\}_{j=1}^N$ is a subordinate partition of unity and $\{\psi_j\in C_0^\infty(\widetilde{U}_j)\}_{j=1}^N$ N–tuple of functions satisfying $\phi_j\psi_j=\phi_j$ for all j.

Definition 9 *The space $\Psi^{\nu,l}_{(cl)}(Y;E,\widetilde{E};\overline{\mathbb{C}}_-)$ is defined as the set of all ζ–dependent operator families that take the form*

$$A(\zeta)=\sum_{j=1}^N\phi_j\left\{\left(\chi_j^{-1}\right)_*\operatorname{Op}_y(a_j)(\zeta)\right\}\psi_j+C(\zeta),\tag{5}$$

where $a_j(y,\zeta,\eta)\in S^{\nu,l}_{(cl)}(U_j\times\overline{\mathbb{C}}_-\times\mathbb{R}^q;E,\widetilde{E})$, $U_j=\chi_j(\widetilde{U}_j)$, $j=1,\dots,N$, and $C(\zeta)$ belongs to $\Psi^{-\infty}(Y;E,\widetilde{E};\overline{\mathbb{C}}_-)$.

Remark 10 Again, without loss of generality we may assume the following compatibility conditions of the tuple of symbols $a_j(y,\zeta,\eta)$ in Definition 9. If $\chi_{jk}:U_k\cap\chi_k(\widetilde{U}_j\cap\widetilde{U}_k)\to U_j\cap\chi_j(\widetilde{U}_j\cap\widetilde{U}_k)$ is the system of transition diffeomorphisms to the above charts, then for the symbol push–forward of the complete symbols we have

$$a_j|_{U_j\cap\chi_j(\widetilde{U}_j\cap\widetilde{U}_k)}=(\chi_{jk})_*\,a_k|_{U_k\cap\chi_k(\widetilde{U}_j\cap\widetilde{U}_k)}$$

modulo $S^{-\infty}(U_j\cap\chi_j(\widetilde{U}_j\cap\widetilde{U}_k)\times\overline{\mathbb{C}}_-\times\mathbb{R}^q;E,\widetilde{E})$ for all $j,k\in 1,\dots,N$.

Due to Theorem 6 (ii) and (iii) we get the same classes of operators if the local amplitude functions depend on (y,y',ζ,η) or (y',ζ,η), respectively. Moreover, it is clear that $A(\zeta_0)$ belongs to $\Psi^{\nu}_{(cl)}(Y;E,\widetilde{E})$ for every $\zeta_0\in\overline{\mathbb{C}}_-$.

Theorem 11 *Let $A(\zeta)\in\Psi^{\nu,l}(Y;E,\widetilde{E};\overline{\mathbb{C}}_-)$ and $\nu,\mu\in\mathbb{R}$ with $\mu\ge\nu$ be given and set $\rho=\max\{\nu,\nu-\mu\}$. Then the inequality*

$$\|A(\zeta)\|_{\mathcal{L}(\mathcal{W}^s(Y,E),\mathcal{W}^{s-\mu}(Y,\widetilde{E}))}\le c(s,\nu,A)(1+|\zeta|^2)^{(M+\widetilde{M}+\rho)/2l}\tag{6}$$

holds for all $\zeta\in\overline{\mathbb{C}}_-$. Here M and $\widetilde{M}$ are the constants associated to κ_λ and $\tilde{\kappa}_\lambda$, respectively, according to (1.1.10).

Proof: Because of the compactness of Y it suffices to check the norm estimate (6) locally. Moreover, by construction every local operator family $A_j(\zeta)$ is uniformly (in ζ) compactly supported with respect to (y,y'), i.e., there is a compact set $K\subset\mathbb{R}^{2q}$ such that the support of the kernel of $A_j(\zeta)$ lies in K. Thus we only need to look at the case of (y,y')–independent symbols and the general case then follows by standard tensor product arguments, cf. [Su10] Theorem 1.2.19.

Now let $A(\zeta)=\operatorname{Op}-y(a)(\zeta)\in\Psi^{\nu,l}(\mathbb{R}^q;E,\widetilde{E};\overline{\mathbb{C}}_-)$, $a(\zeta,\eta)\in S^{\nu,l}(\overline{\mathbb{C}}_-\times\mathbb{R}^q;E,\widetilde{E})$. Then with $\langle\eta\rangle=(1+|\eta|^2)^{1/2}$ we have

$$\begin{aligned}\|A(\zeta)u\|^2_{s-\mu}&=\int\langle\eta\rangle^{2(s-\mu)}\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}(\mathcal{F}A(\zeta)u)(\eta)\|^2_{\widetilde{E}}đ\eta\\&\le\int\langle\eta\rangle^{2(s-\mu)}\left\|\tilde{\kappa}_{\frac{[\zeta,\eta]_l}{\langle\eta\rangle}}\right\|^2\left\|\tilde{\kappa}^{-1}(\zeta,\eta)a(\zeta,\eta)\kappa(\zeta,\eta)\right\|^2\left\|\kappa_{\frac{\langle\eta\rangle}{[\zeta,\eta]_l}}\right\|^2\|\kappa^{-1}_{\langle\eta\rangle}(\mathcal{F}u)(\eta)\|^2_E đ\eta.\end{aligned}$$

Using $\left\|\tilde{\kappa}_{[\zeta,\eta]_l/\langle\eta\rangle}\right\|\le c\left([\zeta,\eta]_l/\langle\eta\rangle\right)^{\widetilde{M}}$ and $\left\|\kappa_{\langle\eta\rangle/[\zeta,\eta]_l}\right\|\le c\left([\zeta,\eta]_l/\langle\eta\rangle\right)^{M}$ and the symbol estimate $\left\|\tilde{\kappa}^{-1}(\zeta,\eta)a(\zeta,\eta)\kappa(\zeta,\eta)\right\|\le c(a)[\zeta,\eta]_l^\nu$ we obtain

$$\|A(\zeta)u\|^2_{s-\mu}\le\int\frac{[\zeta,\eta]_l^{2(M+\widetilde{M}+\nu)}}{\langle\eta\rangle^{2(M+\widetilde{M}+\mu)}}\langle\eta\rangle^{2s}\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}(\mathcal{F}u)(\eta)\|^2_E đ\eta.$$

But then we get the desired norm estimate, since the assumptions concerning μ, ν and ρ imply the inequality

$$[\zeta,\eta]_l^{M+\widetilde{M}+\nu} \leq c\langle\eta\rangle^{M+\widetilde{M}+\mu}(1+|\zeta|^2)^{(M+\widetilde{M}+\rho)/2l}. \tag{7}$$

In order to see (7) we consider the inequality

$$\langle\zeta,\eta\rangle_l^{\tilde{\nu}} \leq c\langle\eta\rangle^{\tilde{\mu}}\langle\zeta\rangle^{\frac{\tilde{\rho}}{l}} \tag{8}$$

where $\tilde{\mu},\tilde{\nu} \in \mathbb{R}$, $\tilde{\mu} \geq \tilde{\nu}$, $\tilde{\rho} = \max\{\tilde{\nu}, \tilde{\nu}-\tilde{\mu}\}$, and $\langle\zeta,\eta\rangle_l = (1+|\zeta|^2+|\eta|^{2l})^{\frac{1}{2l}}$. Since $\langle\zeta,\eta\rangle_l \sim [\zeta,\eta]_l$ the inequality (8) is equivalent to (7). But for $\tilde{\mu} \geq 0$, where $\tilde{\rho} = \tilde{\nu}$, we have

$$\langle\eta\rangle^{l\tilde{\mu}}\langle\zeta\rangle^{\tilde{\nu}} \geq \langle\eta\rangle^{l\tilde{\nu}}\langle\zeta\rangle^{\tilde{\nu}} \geq c\langle\zeta,\eta\rangle_l^{l\tilde{\nu}},$$

and for $\tilde{\mu} < 0$, where $\tilde{\rho} = \tilde{\nu}-\tilde{\mu} \leq 0$, we have

$$\langle\eta\rangle^{-l\tilde{\mu}}\langle\zeta\rangle^{-\tilde{\rho}} = \langle\eta\rangle^{-l\tilde{\mu}}\langle\zeta\rangle^{-(\tilde{\nu}-\tilde{\mu})} \leq c\langle\zeta,\eta\rangle_l^{-l\tilde{\mu}}\langle\zeta,\eta\rangle_l^{-l(\tilde{\nu}-\tilde{\mu})} = c\langle\zeta,\eta\rangle_l^{-l\tilde{\nu}}$$

which yields (8). □

Corollary 12 *Let $A(\zeta) \in \Psi^{0,l}(Y;E,\widetilde{E};\overline{\mathbb{C}}_-)$ be given and assume that the corresponding group actions κ_λ and $\tilde{\kappa}_\lambda$ associated to E and $\widetilde{E}$, respectively, are unitary. Then inserting $M = \widetilde{M} = 0$ in the above estimate* (6) *it follows that for all $s \in \mathbb{R}$ there is a constant $c = c(s) > 0$ such that we have*

$$\|A(\zeta)\|_{\mathcal{L}(\mathcal{W}^s(Y,E),\mathcal{W}^s(Y,\widetilde{E}))} \leq c$$

for all $\zeta \in \overline{\mathbb{C}}_-$.

Theorem 11 allows a further essential application. In the process of inverting a pseudo–differential operator under certain conditions we arrive at families of smoothing operators. Then it remains to invert $I + C(\zeta)$ for some $C(\zeta) \in \Psi^{-\infty}(Y;E,E;\overline{\mathbb{C}}_-)$. Here and in the sequel we denote by I the corresponding identity. We will invert $I+C(\zeta)$ using a Neumann series argument, such that we have to ensure that the norm of $C(\zeta)$ is sufficiently small. This will be reached by enlarging $|\zeta|$, since for $\nu < -(M+\widetilde{M}+|\mu|)$ the right hand side of (6) decreases for increasing $|\zeta|$.

Proposition 13 *For every $C(\zeta) \in \Psi^{-\infty}(Y;E,E;\overline{\mathbb{C}}_-)$ and every $s \in \mathbb{R}$ there exists a constant $c = c(s) > 0$ such that the mapping*

$$I + C(\zeta) : \mathcal{W}^s(Y,E) \to \mathcal{W}^s(Y,E)$$

is invertible for all $\zeta \in \overline{\mathbb{C}}_-$ with $|\zeta| \geq c(s)$. Moreover, there exists an operator family $G(\zeta) \in \Psi^{-\infty}(Y;E,E;\overline{\mathbb{C}}_-)$ with $(I+C(\zeta))(I+G(\zeta)) = I + H(\zeta)$ with $H(\zeta) = 0$ for $|\zeta| \geq c(s)$.

Proof: We first construct $I + G_0(\zeta)$ as point–wise inverse of $I + C(\zeta)$ for sufficiently large $|\zeta|$ as a Neumann series. This is possible, since Theorem 11 yields $\|C(\zeta)\|_{s,s} < 1$ if $|\zeta| \geq c_0$ for some constant $c_0 = c_0(s) > 0$. Then we set $G(\zeta) = \left(1-\omega\left(\frac{|\zeta|}{c_0}\right)\right)G_0(\zeta)$ with a cut–off function $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+,[0,1])$ with

$$\omega(r) = \begin{cases} 1 & \text{for} \quad 0 \leq r \leq 1, \\ 0 & \text{for} \quad r \geq 2 \end{cases}. \tag{9}$$

It is obvious that $G(\zeta) \in \Psi^{-\infty}(Y;E,E)$ for $\zeta \in \overline{\mathbb{C}}_-$ and $(I+C(\zeta))(I+G(\zeta)) = I+H(\zeta)$ with $H(\zeta) = 0$ for $|\zeta| \geq c(s) = 2c_0(s)$. Moreover, we have to verify that $G(\zeta)$ is rapidely decreasing. This follows by induction, since every derivative $G^{(k)}(\zeta) = \frac{d^k}{d\zeta^k}G(\zeta)$ may be written as a sum of products of bounded functions, where at least one factor is rapidely decreasing.

Indeed, for $k=0$ we have $G=-(I+C)^{-1}C$, where $C(\zeta)$ is rapidely decreasing and $(I+C(\zeta))^{-1} = I+G_0(\zeta)$ is uniformly bounded for large $|\zeta|$. This gives us

$$G^{(1)}(\zeta) = -(I+C(\zeta))^{-1}C^{(1)}(\zeta)(I+C(\zeta))^{-1} = -(I+G_0(\zeta))C^{(1)}(\zeta)(I+G_0(\zeta)).$$

By induction we then obtain that $G^{(k)}(\zeta)$ is a linear combination of compositions of $I+G_0(\zeta)$, $I+C(\zeta)$ and $C^{(n)}(\zeta)$, $n\le k$, where every summand contains at least one factor $C^{(n)}(\zeta)$ which is rapidely decreasing. □

Proposition 14 *Let $C(\zeta)\in\Psi^{-\infty}(Y;E,E;\overline{\mathbb{C}}_-)$ be an arbitrary element and assume that*

$$I+C(\zeta):\mathcal{W}^s(Y,E)\to\mathcal{W}^s(Y,E) \tag{10}$$

is invertible for some $s=s_0\in\mathbb{R}$ and all $\zeta\in\overline{\mathbb{C}}_-$. Then (10) *is invertible for all $s\in\mathbb{R}$ and all $\zeta\in\overline{\mathbb{C}}_-$, and the inverse has the form $I+G(\zeta)$ with $G(\zeta)\in\Psi^{-\infty}(Y;E,E;\overline{\mathbb{C}}_-)$.*

An analogous result holds if (10) *is invertible for all $\zeta\in\overline{\mathbb{C}}_-$ with $|\zeta|\ge c$ with some constant $c>0$.*

Proof: We only prove the first statement. The second one is easier, since we only have to check that the constant $c(s)$ in Proposition 13 may be chosen uniformly for all $s\in\mathbb{R}$.

In order to prove the first part we set

$$G(\zeta) := -C(\zeta)+C(\zeta)(I+C(\zeta))^{-1}C(\zeta).$$

Then an easy calculation shows $(I+G(\zeta))(I+C(\zeta)) = (I+G(\zeta))(I+C(\zeta)) = I$.

Moreover, for each ζ we have to show that $G(\zeta):\mathcal{W}^r(Y,E)\to\mathcal{W}^\infty(Y,E)$ is continuous for every $r\in\mathbb{R}$. Since this is known for $C(\zeta)$ it suffices to look at $G(\zeta)+C(\zeta)$ which is the composition of $C(\zeta):\mathcal{W}^r(Y,E)\to\mathcal{W}^\infty(Y,E)\hookrightarrow\mathcal{W}^{s_0}(Y,E)$, $(I+C(\zeta))^{-1}:\mathcal{W}^{s_0}(Y,E)\to\mathcal{W}^{s_0}(Y,E)$ and $C(\zeta):\mathcal{W}^{s_0}(Y,E)\to\mathcal{W}^\infty(Y,E)$. This gives point–wise $G(\zeta)\in\Psi^{-\infty}(Y;E,E)$. The simple arguments for $G(\zeta)\in C^\infty(\overline{\mathbb{C}}_-,\Psi^{-\infty}(Y;E,E))$ will be omitted.

In a last step we have to check $G(\zeta)\in\mathcal{S}(\overline{\mathbb{C}}_-,\Psi^{-\infty}(Y;E,E))$. But this follows from Proposition 13, since the uniqueness of the inverse implies that for large $|\zeta|$ the family $G(\zeta)$ equals that of Proposition 13, and we proved there that it is rapidely decreasing. □

4.2 Ellipticity, parametrix construction and invertibility

We now turn to the parameter–dependent ellipticity in the class $\Psi^{\mu,l}(U;E,\widetilde{E};\overline{\mathbb{C}}_-)$, $\mu\in\mathbb{R}$, for open $U\subseteq\mathbb{R}^q$.

Definition 1 *An element $A(\zeta)\in\Psi^{\mu,l}(U;E,\widetilde{E};\overline{\mathbb{C}}_-)$ is called anisotropic parameter–dependent elliptic of order $\mu\in\mathbb{R}$ if for any representation of $A(\zeta)$ in the form $A(\zeta)=\mathrm{Op}_y(a)(\zeta)+C(\zeta)$, cf.* (4.1.3), *for some $a(y,\zeta,\eta)\in S^{\mu,l}(U\times\overline{\mathbb{C}}_-\times\mathbb{R}^q;E,\widetilde{E})$ and $C(\zeta)\in\Psi^{-\infty}(U;E,\widetilde{E};\overline{\mathbb{C}}_-)$ the symbol a satisfies the following*

Condition E: *For every $K\subset\subset U$ there are constants $R,c>0$ such that*

$$a(y,\zeta,\eta)|_{U\times\{|\zeta,\eta|_l>R\}}:E\to\widetilde{E}$$

is a family of isomorphisms satisfying the inequality

$$\|\tilde{\kappa}^{-1}(\zeta,\eta)a(y,\zeta,\eta)\kappa(\zeta,\eta)u\|_{\widetilde{E}}\ge c[\zeta,\eta]_l^\mu\|u\|_E \tag{1}$$

for all $u\in E$, all $y\in K$ and all $(\zeta,\eta)\in\overline{\mathbb{C}}_-\times\mathbb{R}^q$ with $|\zeta,\eta|_l\ge R$.

Proposition 2 *Let* $a(y,\zeta,\eta) \in S^{\mu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ *be given. Then Condition* E *is equivalent to the existence of an element* $b(y,\zeta,\eta) \in S^{-\mu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; \widetilde{E}, E)$ *with*

$$a(y,\zeta,\eta)b(y,\zeta,\eta) - 1 \in S^{-1,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; \widetilde{E}, \widetilde{E}), \tag{2}$$

$$b(y,\zeta,\eta)a(y,\zeta,\eta) - 1 \in S^{-1,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, E). \tag{3}$$

Here 1 is the identity as constant operator–valued function and $a(y,\zeta,\eta)b(y,\zeta,\eta)$ *means the point–wise composition of the operator–valued functions.*

Proof: Condition E gives a point–wise inverse $(a(y,\zeta,\eta))^{-1}$ for $|\zeta,\eta|_l > R$ and we set

$$b(y,\zeta,\eta) := \begin{cases} 0 & \text{for} \quad |\zeta,\eta|_l \leq R, \\ \chi(|\zeta,\eta|_l)(a(y,\zeta,\eta))^{-1} & \text{for} \quad |\zeta,\eta|_l > R, \end{cases}$$

where $\chi(r) \in C^\infty(\overline{\mathbb{R}}_+, [0,1])$ is an excision function satisfying

$$\chi(r) = \begin{cases} 0 & \text{for} \quad r \leq R, \\ 1 & \text{for} \quad r \geq 2R. \end{cases}$$

Then $b(y,\zeta,\eta)$ belongs to $C^\infty(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q, \mathcal{L}(\widetilde{E}, E))$. It only remains to check the symbol estimates. Outside the compact set $B^l_{2R} := \{|\zeta,\eta|_l \leq 2R\}$ we have

$$\kappa^{-1}(\zeta,\eta)b(y,\zeta,\eta)\tilde{\kappa}(\zeta,\eta) = \kappa^{-1}(\zeta,\eta)(a(y,\zeta,\eta))^{-1}\tilde{\kappa}(\zeta,\eta) = (\tilde{\kappa}^{-1}(\zeta,\eta)a(y,\zeta,\eta)\kappa(\zeta,\eta))^{-1}.$$

Let $T := \tilde{\kappa}^{-1}(\zeta,\eta)a(y,\zeta,\eta)\kappa(\zeta,\eta)$; then (1) corresponds to

$$\|Tu\|_{\widetilde{E}} \leq c[\zeta,\eta]_l^\mu \|u\|_E$$

for all $u \in E$. But this implies

$$\|T^{-1}v\|_E \leq c^{-1}[\zeta,\eta]_l^{-\mu}\|TT^{-1}v\|_{\widetilde{E}} = c^{-1}[\zeta,\eta]_l^{-\mu}\|v\|_{\widetilde{E}}$$

for any $v \in \widetilde{E}$ and hence

$$\|\kappa^{-1}(\zeta,\eta)b(y,\zeta,\eta)\tilde{\kappa}(\zeta,\eta)\|_{\mathcal{L}(E,\widetilde{E})} \leq c^{-1}[\zeta,\eta]_l^{-\mu}. \tag{4}$$

The symbol estimates for $\alpha, \beta \neq 0$ then follow by induction from (4) and the symbol estimates for $a(y,\zeta,\eta)$.

This implies $b(y,\zeta,\eta) \in S^{-\mu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; \widetilde{E}, E)$. Note that differentiation with respect to τ or ϑ reduces the order of the symbol by l instead of 1. We obtain

$$ba - 1 = (\chi - 1)1 \in C^\infty(U, C_0^\infty(\overline{\mathbb{C}}_- \times \mathbb{R}^q, \mathcal{L}(E,E))) \subset S^{-1,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, E)$$

and

$$ab - 1 = (\chi - 1)1 \in C^\infty(U, C_0^\infty(\overline{\mathbb{C}}_- \times \mathbb{R}^q, \mathcal{L}(\widetilde{E},\widetilde{E}))) \subset S^{-1,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; \widetilde{E}, \widetilde{E}),$$

as desired.

Conversely, if there exists for the given symbol $a(y,\zeta,\eta)$ an element $b(y,\zeta,\eta) \in S^{-\mu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; \widetilde{E}, E)$ with (3) we get

$$r(y,\zeta,\eta) := b(y,\zeta,\eta)a(y,\zeta,\eta) - 1 \in S^{-1,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, E)$$

such that there is a constant R_1 with $\|r(y,\zeta,\eta)\|_{\mathcal{L}(E,E)} < 1$ for $|\zeta,\eta|_l \geq R_1$. But then there exists $(1 + r(y,\zeta,\eta))^{-1}$ point–wise as a Neumann series and $1 = (1+r)^{-1}(1+r) = (1+r)^{-1}ba$ yields a left inverse of $a(y,\zeta,\eta)$ for $|\zeta,\eta|_l \geq R_1$. Analogously (2) gives the existence of a point–wise right inverse of $a(y,\zeta,\eta)$ for $|\zeta,\eta|_l \geq R_2$ with a constant R_2. Thus $a(y,\zeta,\eta) : E \to \widetilde{E}$ is an isomorphism for $|\zeta,\eta|_l \geq \max\{R_1,R_2\}$.

Moreover, for every $K \subset\subset U$ we obtain an $R = R(K) > 0$ such that $\|\kappa^{-1}(\zeta,\eta)(1+r(y,\zeta,\eta))\kappa(\zeta,\eta)\|_{\mathcal{L}(E,E)} > c > 0$ holds for all $(y,\zeta,\eta) \in K \times \overline{\mathbb{C}}_- \times \mathbb{R}^q$ with $|\zeta,\eta|_l \geq R$. This implies for every $u \in E$

$$\begin{aligned} c\|u\|_E &\leq \|\kappa^{-1}(\zeta,\eta)(1+r(y,\zeta,\eta))\kappa(\zeta,\eta)u\|_E \\ &= \|\kappa^{-1}(\zeta,\eta)[\zeta,\eta]_l^{\mu} b(y,\zeta,\eta)\tilde{\kappa}(\zeta,\eta)\tilde{\kappa}^{-1}(\zeta,\eta)[\zeta,\eta]_l^{-\mu} a(y,\zeta,\eta))\kappa(\zeta,\eta)u\|_E \\ &\leq \|\kappa^{-1}(\zeta,\eta)[\zeta,\eta]_l^{\mu} b(y,\zeta,\eta)\tilde{\kappa}(\zeta,\eta)\|_{\mathcal{L}(\widetilde{E},E)}\|\tilde{\kappa}^{-1}(\zeta,\eta)[\zeta,\eta]_l^{-\mu} a(y,\zeta,\eta))\kappa(\zeta,\eta)u\|_{\widetilde{E}} \\ &\leq C\|\tilde{\kappa}^{-1}(\zeta,\eta)[\zeta,\eta]_l^{-\mu} a(y,\zeta,\eta))\kappa(\zeta,\eta)u\|_{\widetilde{E}}, \end{aligned}$$

and hence $\|\tilde{\kappa}^{-1}(\zeta,\eta)[\zeta,\eta]_l^{-\mu} a(y,\zeta,\eta))\kappa(\zeta,\eta)u\|_{\widetilde{E}} \geq c/C\|u\|_E$. Thus Condition E is verified for $a(y,\zeta,\eta)$.
□

Corollary 3 *It follows immediately from Proposition 2 that if $a = a_\mu + a_{\mu-1}$ is a decomposition of the given symbol a into a symbol a_μ of order μ and a symbol $a_{\mu-1}$ of order $\mu-1$ then a is parameter–dependent anisotropic elliptic if and only if a_μ has this property. Especially, the notion of parameter–dependent anisotropic ellipticity of $A(\zeta)$ is independent of the representation $A(\zeta) = \mathrm{Op}_y(a)(\zeta) + C(\zeta)$.*

Corollary 3 allows us to characterise parameter–dependent ellipticity of classical families of pseudo–differential operators by means of the anisotropic homogeneous principal symbol.

Proposition 4 *A classical family of pseudo–differential operators $A(\zeta) = \mathrm{Op}_y(a)(\zeta) + C(\zeta) \in \Psi_{cl}^{\mu,l}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ is parameter–dependent elliptic if and only if the corresponding μ–homogeneous principal symbol $\sigma_\wedge^\mu(a)(y,\zeta,\eta) : E \to \widetilde{E}$ is invertible for all $y \in U$ and all $(\zeta,\eta) \in \mathbb{R}^{1+q} \setminus \{0\}$.*

Proof: From the definition of classical families we have $a(y,\zeta,\eta) = \chi(\zeta,\eta)\sigma_\wedge^\mu(A)(y,\zeta,\eta) + a_{\mu-1}(y,\zeta,\eta)$ for an arbitrary excision function $\chi(\zeta,\eta)$ and a remainder $a_{\mu-1}(y,\zeta,\eta) \in S_{cl}^{\mu-1,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$. Hence in view of Corollary 3 the ellipticity of $a(y,\zeta,\eta)$ is equivalent to the ellipticity of $a_\mu(y,\zeta,\eta) := \chi(\zeta,\eta)\sigma_\wedge^\mu(A)(y,\zeta,\eta)$.

On one hand the invertibility of $\sigma_\wedge^\mu(A)(y,\zeta,\eta)$ for $(\zeta,\eta) \neq 0$ implies that the operator function $a_\mu(y,\zeta,\eta)$ takes values in isomorphisms for $[\zeta,\eta]_l > R$ and all $y \in U$ with some constant $R > 0$ depending only upon the excision function χ. Moreover, the homogeneity of $a_\mu(y,\zeta,\eta)$ in the operator–valued sense for large $[\zeta,\eta]_l$ yields (1) for all $u \in E$, all y belonging to some compact set K and all $(\zeta,\eta) \in \overline{\mathbb{C}}_-$ with $|\zeta,\eta|_l \geq R$. Hence the highest order part a_μ satisfies Condition E.

On the other hand if Condition E holds for $a_\mu(y,\zeta,\eta)$, then the operator–valued function $\sigma_\wedge^\mu(A)(y,\zeta,\eta)$ consists of isomorphisms for all $(y,\zeta,\eta) \in U \times S_R$, where $S_R = (\overline{\mathbb{C}}_- \times \mathbb{R}^q) \cap \{|\zeta,\eta|_l = R\}$ denotes the anisotropic half–sphere. Choosing R large enough, we get $a_\mu(y,\zeta,\eta)|_{S_R^l} \equiv \sigma_\wedge^\mu(A)(y,\zeta,\eta)|_{S_R^l}$ such that the corresponding extension by homogeneity produces the μ–homogeneous principal symbol $\sigma_\wedge^\mu(A)(y,\zeta,\eta)$, and it is an isomorphism for all $(\zeta,\eta) \neq 0$ and arbitrary $y \in U$. □

Definition 5 *Let $A(\zeta) \in \Psi^{\mu,l}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ and $P(\zeta) \in \Psi^{-\mu,l}(U; \widetilde{E}, E; \overline{\mathbb{C}}_-)$ be given and assume that $A(\zeta)$ or $P(\zeta)$ is properly supported in the sense of 4.1 Definition 2. Then $P(\zeta)$ is called a parameter–dependent parametrix of $A(\zeta)$ if*

$$A(\zeta)P(\zeta) - I \in \Psi^{-\infty}(U; \widetilde{E}, \widetilde{E}; \overline{\mathbb{C}}_-), \quad P(\zeta)A(\zeta) - I \in \Psi^{-\infty}(U; E, E; \overline{\mathbb{C}}_-).$$

Proposition 6 *For any $A(\zeta) \in \Psi^{\mu,l}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ which is anisotropic parameter–dependent elliptic of order μ there exists a (properly supported) parametrix $P(\zeta) \in \Psi^{-\mu,l}(U; \widetilde{E}, E; \overline{\mathbb{C}}_-)$; it is classical when $A(\zeta)$ is classical.*

Proof: Without loss of generality we may assume $A(\zeta) = \mathrm{Op}_y(a)(\zeta)$ for some $a(y,\zeta,\eta) \in S^{\mu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ satisfying Condition E. Let $b(y,\zeta,\eta) \in S^{-\mu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; \widetilde{E}, E)$ be the symbol from Proposition 2. Then we can apply 4.1 Theorem 6,(v) and pass to a Leibniz inverse $p(y,\zeta,\eta)$ of $a(y,\zeta,\eta)$ with parameter ζ. Now applying 4.1 Proposition 3 to $\mathrm{Op}_y(p)(\zeta)$ we find the required properly supported parametrix $P(\zeta)$. □

We now formulate ellipticity and parametrices for a closed compact C^∞ manifold Y of dimension $q \in \mathbb{N}$.

Definition 7 *An $A(\zeta) \in \Psi^{\mu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ is called anisotropic parameter–dependent elliptic of order $\mu \in \mathbb{R}$ if for any representation of $A(\zeta)$ in the form* (4.1.5) *the local symbols a_j, $j = 1, \dots, N$, satisfy Condition* E.

Moreover, an operator family $P(\zeta) \in \Psi^{-\mu,l}(Y; \widetilde{E}, E; \overline{\mathbb{C}}_-)$ is called a parameter–dependent parametrix of $A(\zeta)$ if

$$A(\zeta)P(\zeta) - I \in \Psi^{-\infty}(Y; \widetilde{E}, \widetilde{E}; \overline{\mathbb{C}}_-), \quad P(\zeta)A(\zeta) - I \in \Psi^{-\infty}(Y; E, E; \overline{\mathbb{C}}_-).$$

Theorem 8 *For any $A(\zeta) \in \Psi^{\mu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ which is anisotropic parameter–dependent elliptic of order μ there exists a parametrix $P(\zeta) \in \Psi^{-\mu,l}(Y; \widetilde{E}, E; \overline{\mathbb{C}}_-)$.*

Proof: By definition $A(\zeta)$ can be written in the form

$$A(\zeta) = \sum_{j=1}^{N} \phi_j \left\{ \left(\chi_j^{-1}\right)_* \mathrm{Op}_y(a_j)(\zeta) \right\} \psi_j + C(\zeta),$$

cf. 4.1 Definition 9, where we may assume $C(\zeta) = 0$. In addition we may assume the compatibility of the local symbols a_j in the sense of 4.1 Remark 10. Let us form the Leibniz inverse p_j of a_j for every j in the sense of Proposition 6. Then also the symbols p_j satisfy the corresponding compatibility condition, cf. 4.1 Theorem 6,(vi). Thus defining

$$P(\zeta) = \sum_{j=1}^{N} \phi_j \left\{ \left(\chi_j^{-1}\right)_* \mathrm{Op}_y(p_j)(\zeta) \right\} \psi_j,$$

we just obtain a parameter–dependent parametrix of $A(\zeta)$, which belongs to the space $\Psi^{-\mu,l}(Y; \widetilde{E}, E; \overline{\mathbb{C}}_-)$, cf. analogously 2.2 Theorem 8, where we employ 4.1 Remark 4. □

Let us set $T_{-i\gamma}A(\zeta) = A(\zeta - i\gamma)$ and define

$$\Psi^{\nu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_{-\gamma}) := \{T_{-i\gamma}A(\zeta) : A(\zeta) \in \Psi^{\nu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)\}.$$

Then $\Psi^{\nu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_{-(\gamma+\rho)}) = \{T_{-i\rho}B(\zeta) : B(\zeta) \in \Psi^{\nu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_{-\gamma})\}$, and moreover, we will shortly write $\Psi^{\nu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_{-\gamma}) = T_{-i\gamma}\Psi^{\nu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$.

Proposition 9 *$A(\zeta) \in \Psi^{\nu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ implies $A(\zeta)|_{\mathrm{Im}\ \zeta \le -\gamma} \in T_{-i\gamma}\Psi^{\nu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$.*

Proof: Since every translation maps smoothing operators to smoothing operators it suffices to check the symbol estimates for the symbols of the local operators. Let an arbitrary local operator family $\mathrm{Op}_y(a)(\zeta) \in \Psi^{\nu,l}(U; E, \widetilde{E}; \overline{\mathbb{C}}_-)$, $U \subseteq \mathbb{R}^q$, with symbol $a(y,\zeta,\eta) \in S^{\nu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$ be given. Then we have to show $T_{-i\gamma}a(y,\zeta,\eta) = a(y,\zeta - i\gamma,\eta) \in S^{\nu,l}(U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$.

It is clear that $a(y,\zeta - i\gamma,\eta)$ is a smooth $\mathcal{L}(E, \widetilde{E})$–valued function on $U \times \overline{\mathbb{C}}_- \times \mathbb{R}^q$, such that it remains to check the symbol estimates. For lucidity we will only use (y,η)–independent symbols, the general case then follows analogously.

But then for $\zeta = \tau + i\vartheta \in \overline{\mathbb{C}}_-$ and arbitrary $\beta_{01}, \beta_{02} \in \mathbb{N}$ we have

$$\begin{aligned}
\|\tilde{\kappa}^{-1}(\zeta,\eta) D_\tau^{\beta_{01}} D_\vartheta^{\beta_{02}} T_{-i\gamma} a(\zeta)\kappa(\zeta,\eta)\| &= \|\tilde{\kappa}^{-1}(\zeta,\eta) D_\tau^{\beta_{01}} D_\vartheta^{\beta_{02}} a(\zeta - i\gamma)\kappa(\zeta,\eta)\| \\
&= \|\tilde{\kappa}_{\frac{[\zeta-i\gamma,\eta]_l}{[\zeta,\eta]_l}} \tilde{\kappa}^{-1}_{[\zeta-i\gamma,\eta]_l} D_\tau^{\beta_{01}} D_\vartheta^{\beta_{02}} a(\zeta - i\gamma)\kappa_{[\zeta-i\gamma,\eta]_l} \kappa_{\frac{[\zeta,\eta]_l}{[\zeta-i\gamma,\eta]_l}}\| \\
&\le c[\gamma,0]_l^{M+\widetilde{M}} [\zeta - i\gamma, \eta]_l^{\nu - l(\beta_{01}+\beta_{02})} \\
&\le C\gamma^{(M+\widetilde{M}+|\nu-l(\beta_{01}+\beta_{02})|)/l} [\zeta,\eta]_l^{\nu - l(\beta_{01}+\beta_{02})}
\end{aligned}$$

by (1.1.11) and Peetre's inequality (1.1.8). □

Theorem 10 *Let $A(\zeta) \in \Psi^{\mu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ be anisotropic parameter–dependent elliptic of order μ. Then there exists a constant $\gamma > 0$ such that*

$$A(\zeta) : \mathcal{W}^s(Y, E) \to \mathcal{W}^{s-\mu}(Y, \widetilde{E}) \tag{5}$$

is invertible for all $|\zeta| \ge \gamma$ and all $s \in \mathbb{R}$.

Proof: Applying Theorem 8 we find a parametrix $P(\zeta) \in \Psi^{-\mu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ to $A(\zeta)$. This gives us

$$P(\zeta)A(\zeta) = I + C(\zeta) \tag{6}$$

for $C(\zeta) \in \Psi^{-\infty}(Y; E, E; \overline{\mathbb{C}}_-)$. From 4.1 Proposition 13 we conclude that $I + C(\zeta)$ is invertible as operator in $\mathcal{W}^s(Y, E)$ for a fixed $s \in \mathbb{R}$ and $|\zeta| \ge \gamma(s)$. Then by 4.1 Proposition 14 we get invertibility for all $s \in \mathbb{R}$ and γ independent upon s, and there exists a $G(\zeta) \in \Psi^{-\infty}(Y; E, E; \overline{\mathbb{C}}_-)$ such that $(I + G(\zeta))(I + C(\zeta)) = I$ for all $|\zeta| \ge \gamma$. Then (6) implies $(I + G(\zeta))P(\zeta)A(\zeta) = I$ for all $|\zeta| \ge \gamma$, in other words $A(\zeta)$ is invertible in $\mathcal{W}^s(Y, E)$ for all $s \in \mathbb{R}$ and $B(\zeta) = (I + G(\zeta))P(\zeta)$ belongs to $\Psi^{-\mu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ and equals $A^{-1}(\zeta)$ for $|\zeta| \ge \gamma$. □

Remark 11 Let $A(\zeta) \in \Psi^{\mu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ satisfy the conditions of Theorem 10 and γ be the corresponding constant. Then $(A(\zeta)|_{\mathrm{Im}\ \zeta \le -\gamma})^{-1} \in T_{-i\gamma}\Psi^{-\mu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$.

In the sequel we assume, that we have a Hilbert space E and a Hilbert space E^0 associated with an unitary group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$. Moreover, we assume $E \subseteq E^0$ or $E^0 \subseteq E$ and κ_λ restricts or extends to a group action on E. For such a pair of spaces we define the following condition.
Condition R: There exists an isomorphism $a : E \to E^0$ with the property $\kappa_\lambda a \kappa_\lambda^{-1} \in C^\infty(\mathbb{R}_+, \mathcal{L}(E, E^0))$.

Remark 12 Set $a(\lambda) = \kappa_\lambda a \kappa_\lambda^{-1}$ for some $a : E \to E^0$ from Condition R. Then the family of the (point–wise) inverses $a^{-1}(\lambda) = \kappa_\lambda a^{-1} \kappa_\lambda^{-1}$ belongs to $C^\infty(\mathbb{R}_+, \mathcal{L}(E^0, E))$.

In fact, to see this we use that $a(\lambda)a^{-1}(\lambda) = \mathrm{id}_{E^0}$ implies

$$\frac{d^n}{d\lambda^n} a^{-1}(\lambda) = -a^{-1}(\lambda) \left(\sum_{k=1}^{n-1} \binom{n}{k} \left(\frac{d^k}{d\lambda^k} a(\lambda) \right) \left(\frac{d^{n-k}}{d\lambda^{n-k}} a^{-1}(\lambda) \right) + \left(\frac{d^n}{d\lambda^n} a(\lambda) \right) a^{-1}(\lambda) \right).$$

Thus the smoothness of $a^{-1}(\lambda)$ follows by induction from the smoothness of $a(\lambda)$ and the continuity of $a^{-1}(\lambda)$. In order to show the continuity of $a^{-1}(\lambda)$ we write

$$a(\lambda) = a(\lambda_0)(1 - a^{-1}(\lambda_0)(a(\lambda_0) - a(\lambda)))$$

which is possible for all $\lambda, \lambda_0 \in \mathbb{R}_+$ since $a(\lambda)$ is a family of isomorphisms. Moreover,

$$\|a^{-1}(\lambda_0)(a(\lambda_0) - a(\lambda))\| \le \|a^{-1}(\lambda_0)\| \, \|a(\lambda_0) - a(\lambda)\|$$

is small for small $|\lambda_0 - \lambda|$ by the continuity of $a(\lambda)$. Hence we can invert $a(\lambda)$ using the corresponding Neumann series for $1 - a^{-1}(\lambda_0)(a(\lambda_0) - a(\lambda))$ and obtain

$$\begin{aligned} a^{-1}(\lambda) &= \left(\sum_{j=0}^{\infty} \left(a^{-1}(\lambda_0)(a(\lambda_0) - a(\lambda))\right)^j \right) a^{-1}(\lambda_0) \\ &= a^{-1}(\lambda_0) + \left(\sum_{j=1}^{\infty} \left(a^{-1}(\lambda_0)(a(\lambda_0) - a(\lambda))\right)^j \right) a^{-1}(\lambda_0). \end{aligned}$$

But then we have

$$\begin{aligned} \|a^{-1}(\lambda) - a^{-1}(\lambda_0)\| &\leq \left\| \sum_{j=1}^{\infty} \left(a^{-1}(\lambda_0)(a(\lambda_0) - a(\lambda))\right)^j \right\| \|a^{-1}(\lambda_0)\| \\ &\leq \sum_{j=1}^{\infty} \left\|a^{-1}(\lambda_0)(a(\lambda_0) - a(\lambda))\right\|^j \|a^{-1}(\lambda_0)\| \\ &= \frac{\|a^{-1}(\lambda_0)(a(\lambda_0) - a(\lambda))\| \, \|a^{-1}(\lambda_0)\|}{1 - \|a^{-1}(\lambda_0)(a(\lambda_0) - a(\lambda))\|} \\ &\leq C(\lambda_0)\|a(\lambda) - a(\lambda_0)\| \end{aligned}$$

with some constant $C(\lambda_0) > 0$ which gives the continuity of a^{-1}.

Example 13 Let us set $E^0 = L^2(\mathbb{R}^m)$, $E = H^s(\mathbb{R}^m)$ for some fixed $s \in \mathbb{R}$ and $(\kappa_\lambda u)(x) = \lambda^{m/2}u(\lambda x)$. Then Condition R is satisfied, since we can take $a = \mathrm{op}_x(p)$ for the symbol $p(\xi) = (1 + |\xi|^2)^{s/2}$. In fact, an easy calculation shows $\kappa_\lambda \mathrm{op}_x(p)\kappa_\lambda^{-1} = \mathrm{op}_x(p_\lambda)$ with $p_\lambda(\xi) = p\left(\frac{\xi}{\lambda}\right)$. It is clear that thean $\frac{\partial^k}{\partial \lambda^k} p_\lambda$ is also a symbol of order s.

Lemma 14 *Assume E, E^0 are Hilbert spaces with a group action κ_λ satisfying Condition* R. *Then for every $\mu \in \mathbb{R}$ there exist an parameter–dependent operator–valued symbol $a^\mu(\zeta, \eta) \in S^{\mu,l}_{cl}(\overline{\mathbb{C}}_- \times \mathbb{R}^q; E, E^0)$ that gives isomorphisms $a^\mu(\zeta, \eta) : E \to E^0$ for all $(\zeta, \eta) \in \overline{\mathbb{C}}_- \times \mathbb{R}^q$.*

Proof: Set $a^\mu(\zeta, \eta) = [\zeta, \eta]_l^\mu \kappa(\zeta, \eta) a \kappa^{-1}(\zeta, \eta)$ which belongs to $C^\infty(\overline{\mathbb{C}}_- \times \mathbb{R}^q; \mathcal{L}(E, E^0))$ in view of Condition R . Of course, $a^\mu(\zeta, \eta)$ is an isomorphism for every fixed $(\zeta, \eta) \in \overline{\mathbb{C}}_- \times \mathbb{R}^q$ with inverse $[\zeta, \eta]_l^{-\mu} \kappa(\zeta, \eta) a^{-1} \kappa^{-1}(\zeta, \eta)$. Moreover, an easy calculation shows, that it is anisotropic μ–homogeneous in the operator–valued sense for large $|\zeta, \eta|_l$, which yields $a^\mu(\zeta, \eta) \in S^{\mu,l}_{cl}(\overline{\mathbb{C}}_- \times \mathbb{R}^q; E, E^0)$. □

Corollary 15 *Under the conditions of Lemma 14 for every order $\mu \in \mathbb{R}$ there exists an anisotropic parameter–dependent elliptic operator $A(\zeta) \in \Psi^{\mu,l}(Y; E, E^0; \overline{\mathbb{C}}_-)$ such that*

$$A(\zeta) : \mathcal{W}^s(Y, E) \to \mathcal{W}^{s-\mu}(Y, E^0)$$

is an isomorphism for all $s \in \mathbb{R}$ and all $\zeta \in \overline{\mathbb{C}}_-$.

Moreover, the (point–wise for all ζ) inverse $A^{-1}(\zeta)$ is of analogous structure of order $-\mu$.

Indeed, defining $A_0(\zeta)$ locally as $\mathrm{Op}_y(a)$ with $a(\zeta, \eta)$ given as in Lemma 14 and globalising with respect to Y we obtain an operator family that satisfies the conditions of Theorem 10. Hence $A(\zeta) = A_0(\zeta - i\gamma)$ is an operator family with the desired properties.

Let us fix an order $s \in \mathbb{R}$ and choose a parameter–dependent elliptic element $R^s(\zeta) \in \Psi^{s,l}(Y; E, E^0; \overline{\mathbb{C}}_-)$ with the properties of Corollary 15. Then we can define a parameter–dependent norm in the space $\mathcal{W}^s(Y, E)$ of the form

$$\|u\|_{\mathcal{W}^s(Y,E)_\zeta} := \|R^s(\zeta)u\|_{\mathcal{W}^0(Y,E^0)}.$$

$\mathcal{W}^s(Y,E)$ endowed with this parameter–dependent norm will be denoted by $\mathcal{W}^s(Y,E)_\zeta$. It is clear that for every fixed $\zeta = \zeta_0$ we have

$$\|u\|_{\mathcal{W}^s(Y,E)_{\zeta_0}} \sim \|u\|_{\mathcal{W}^s(Y,E)}$$

(where $\sim$ means equivalence of norms). Moreover, the parameter–dependent norms are independent of the particular choice of $R^s(\zeta)$. In other words, if $\widetilde{R}^s(\zeta)$ is another order reducing family with the above properties, we have for suitable constants $c_1, c_2 > 0$

$$c_1\|R^s(\zeta)u\|_{\mathcal{W}^0(Y,E^0)} \le \|\widetilde{R}^s(\zeta)u\|_{\mathcal{W}^0(Y,E^0)} \le c_2\|R^s(\zeta)u\|_{\mathcal{W}^0(Y,E^0)}$$

for all $\zeta \in \overline{\mathbb{C}}_-$. In fact,

$$\begin{aligned} \|\widetilde{R}^s(\zeta)u\|_{\mathcal{W}^0(Y,E^0)} &= \|\widetilde{R}^s(\zeta)R^{-s}(\zeta)R^s(\zeta)u\|_{\mathcal{W}^0(Y,E^0)} \\ &\le \|\widetilde{R}^s(\zeta)R^{-s}(\zeta)\|_{\mathcal{L}(\mathcal{W}^0(Y,E^0))}\|R^s(\zeta)u\|_{\mathcal{W}^0(Y,E^0)}. \end{aligned}$$

But now we can apply 4.1 Corollary 12 which implies $\|\widetilde{R}^s(\zeta)R^{-s}(\zeta)\|_{\mathcal{L}(\mathcal{W}^0(Y,E^0))} \le c$ for all $\zeta \in \overline{\mathbb{C}}_-$ with some constant $c > 0$.

Theorem 16 *Let E, E^0 and $\widetilde{E}, \widetilde{E}^0$ be Hilbert spaces with group actions κ_λ and $\tilde{\kappa}_\lambda$, respectively, and suppose that Condition* R *is satisfied. Then every $A(\zeta) \in \Psi^{\nu,l}(Y;E,\widetilde{E};\overline{\mathbb{C}}_-)$ induces a family of continuous operators*

$$A(\zeta) : \mathcal{W}^s(Y,E) \to \mathcal{W}^{s-\nu}(Y,\widetilde{E}),$$

$s \in \mathbb{R}$, satisfying

$$\|A(\zeta)u\|_{\mathcal{W}^{s-\nu}(Y,\widetilde{E})_\zeta} \le C\|u\|_{\mathcal{W}^s(Y,E)_\zeta}$$

for all $u \in \mathcal{W}^s(Y,E)$ with a constant C independent of ζ.

Proof: We only have to check the boundedness of $A(\zeta)$ with respect to the parameter–dependent norm. But this follows from

$$\begin{aligned} \|A(\zeta)u\|_{\mathcal{W}^{s-\nu}(Y,\widetilde{E})_\zeta} &= \|\widetilde{R}^{s-\nu}(\zeta)A(\zeta)u\|_{\mathcal{W}^0(Y,\widetilde{E}^0)} \\ &= \|\widetilde{R}(\zeta)^{s-\nu}A(\zeta)R^{-s}(\zeta)R^s(\zeta)u\|_{\mathcal{W}^0(Y,\widetilde{E}^0)} \\ &\le \|\widetilde{R}(\zeta)^{s-\nu}A(\zeta)R^{-s}(\zeta)\|_{\mathcal{L}(\mathcal{W}^0(Y,E^0),\mathcal{W}^0(Y,\widetilde{E}^0))}\|R^s(\zeta)u\|_{\mathcal{W}^0(Y,E^0)} \\ &\le C\|u\|_{\mathcal{W}^s(Y,E)_\zeta}. \end{aligned}$$

Here we used that in view of 4.1 Theorem 6,(iv) $\widetilde{R}(\zeta)^{s-\nu}A(\zeta)R^{-s}(\zeta)$ belongs to $\Psi^{0,l}(Y;E^0,\widetilde{E}^0;\overline{\mathbb{C}}_-)$ and is uniformly bounded with respect to ζ, cf. 4.1 Corollary 12. □

Proposition 17 *Let $A(\zeta) \in \Psi^{\mu,l}(Y;E,\widetilde{E};\overline{\mathbb{C}}_-)$ be parameter–dependent anisotropic elliptic. Then for arbitrary $s \in \mathbb{R}$ there are constants $c, \gamma > 0$ such that the estimate*

$$\|u\|_{\mathcal{W}^{s+\mu}(Y,E)_\zeta} \le c\|A(\zeta)u\|_{\mathcal{W}^s(Y,\widetilde{E})_\zeta}$$

holds for all $|\zeta| \ge \gamma$.

Proof: Applying Theorem 10 the anisotropic parameter–dependent ellipticity implies the existence of an element $B(\zeta) \in \Psi^{-\mu,l}(\widetilde{E},E;\overline{\mathbb{C}}_-)$ which equals $A^{-1}(\zeta)$ for $|\zeta| > \gamma$ for some $\gamma > 0$. But then for any $u \in \mathcal{W}^{s+\mu}(Y,E)_\zeta$, $|\zeta| > \gamma$, the assertion follows from $\|u\|_{\mathcal{W}^{s+\mu}(Y,E)_\zeta} = \|B(\zeta)A(\zeta)u\|_{\mathcal{W}^{s+\mu}(Y,E)_\zeta}$. □

For the scalar case, i.e., when $E = \widetilde{E} = \mathbb{C}$, our results generalise corresponding observations in the paper [AV]. Consider a closed compact C^∞ manifold X and let $\Psi^{\mu,l}(X,\overline{\mathbb{C}}_-)$ be the anisotropic parameter–dependent class of pseudo–differential operators on X of order μ. The Sobolev spaces are here the standard ones, namely $H^s(X)$, $s \in \mathbb{R}$. It is then well known that for every $\mu \in \mathbb{R}$ there exists an element $R(\zeta) \in \Psi^{\mu,l}(X, \overline{\mathbb{C}}_-)$ such that

$$R(\zeta) : H^s(X) \to H^{s-\mu}(X)$$

is an isomorphism for all $s \in \mathbb{R}$. This allows us to introduce parameter dependent norms within the Sobolev spaces, namely

$$\|u\|_{H^s(X)_\zeta} = \|R^s(\zeta)u\|_{H^0(X)}.$$

Again we denote the corresponding spaces by $H^s(X)_\zeta$ and it is clear that the norm is independent of the choice of $R^s(\zeta)$ up to equivalence.

Agranovič, Višik [AV] used on $H^s(X)$ for $s \in \mathbb{N}$ the parameter–dependent norm $u \mapsto \|u\|_{H^s(X)} + |\zeta|^s \|u\|_{H^0(X)}$. We shall show that this norm is equivalent to that defined in this section.

Lemma 18 *For all $s \in \overline{\mathbb{R}}_+$ ther are constants $c, C > 0$ such that we have*

$$c\|u\|_{H^s(X)_\zeta} \leq \|u\|_{H^s(X)} + |\zeta|^s \|u\|_{H^0(X)} \leq C\|u\|_{H^s(X)_\zeta} \tag{7}$$

Proof: The first part is nothing else than the mapping property of the corresponding order reduction. To see the second we obtain for $0 \leq s' \leq s$

$$\begin{aligned} \|u\|_{H^{s'}(X)_\zeta} = \|R^{s'}(\zeta)u\|_{H^0(X)} &= \|R^{s'}(\zeta)R^{-s}(\zeta)R^s(\zeta)u\|_{H^0(X)} \\ &\leq \|R^{s'-s}(\zeta)\|_{\mathcal{L}(H^0(X))}\|R^s(\zeta)u\|_{H^0(X)} \\ &\leq C_{s',s}(1+|\zeta|)^{s'-s}\|R^s(\zeta)u\|_{H^0(X)} \\ &\leq C_{s',s}(1+|\zeta|)^{s'-s}\|u\|_{H^s(X)_\zeta}, \end{aligned} \tag{8}$$

which follows from 4.1 Theorem 11 for $E = \widetilde{E} = \mathbb{C}$ and $l = 1$, since $R^{s'-s}(\zeta)$ gives by definition a continuous mapping

$$R^{s'-s}(\zeta) : H^0(X) \to H^{s-s'}(X) \hookrightarrow H^0(X).$$

Using $\|u\|_{H^s(X)} \leq c_s \|u\|_{H^s(X)_\zeta}$ and (8) for $s' = 0$ we get

$$\begin{aligned} \|u\|_{H^s(X)} + (1+|\zeta|)^s\|u\|_{H^0(X)} &\leq c_s\|u\|_{H^s(X)_\zeta} + C_{0,s}\|u\|_{H^s(X)_\zeta} \\ &\leq C\|u\|_{H^s(X)_\zeta}, \end{aligned}$$

which is the right hand side of (7). □

Note that (8) implies the often used interpolation inequality

$$|\zeta|^{s-s'}\|u\|_{H^{s'}(X)} \leq C_{s',s}(\|u\|_{H^s(X)} + |\zeta|^s\|u\|_{H^0(X)})$$

for $0 \leq s' \leq s$; but here s and s' need not to be natural numbers.

4.3 Application to parabolic differential operators

In this section we shall apply the parameter–dependent calculus of the previous sections to treat parabolic differential operators on infinite cylinders $\overline{\mathbb{R}}_+ \times Y$, $\overline{\mathbb{R}}_+ = [0, \infty)$, where Y is a closed compact q–dimensional C^∞ manifold without boundary. In local coordinates the differential operator A in consideration will have the form

$$A(y, D_t, D_y) = a_0 D_t - \sum_{|\alpha| \leq l} a_\alpha(y) D_y^\alpha \tag{1}$$

where $a_0 \in \mathcal{L}(E, \widetilde{E})$ and $a_\alpha(y) \in C^\infty(Y, \mathcal{L}(E, \widetilde{E}))$ are given such that the corresponding symbol

$$a(y, \tau, \eta) = a_0 \tau - \sum_{|\alpha| \leq l} a_\alpha(y) \eta^\alpha$$

belongs to $S^{l,l}(\mathbb{R}^q \times \mathbb{R}^{1+q}; E, \widetilde{E})$ (even classical) with Hilbert spaces E, $\widetilde{E}$ belonging to Gelfand triples $(E, E^0, E'; \kappa_\lambda)$ and $(\widetilde{E}, \widetilde{E}^0, \widetilde{E}'; \tilde{\kappa}_\lambda)$ with unitary group actions, respectively, given as above, cf. 1.3.

Then the operator A belongs to the space $\Psi^{l,l}_{cl}(\mathbb{R} \times Y; E, \widetilde{E})$ but its symbol does not depend on the time variable $t \in \mathbb{R}$, and in view of Section 3.3 A extends to a continuous operator

$$A : \mathcal{W}^{s,l}_{comp}(\mathbb{R} \times Y, E) \to \mathcal{W}^{s-l,l}_{loc}(\mathbb{R} \times Y, \widetilde{E})$$

for arbitrary $s \in \mathbb{R}$. Note that because of the compactness of Y the subscripts *comp* and *loc* only refer to the time variable. Moreover, since the coefficients of the operator do not depend on t we can extend A to an operator

$$A : \mathcal{W}^{s,l}(\mathbb{R} \times Y, E) \to \mathcal{W}^{s-l,l}(\mathbb{R} \times Y, \widetilde{E}).$$

In order to specify the mapping properties with respect to the infinite time cylinder $\overline{\mathbb{R}}_+ \times Y$ we introduce anisotropic Sobolev spaces with some weight $\gamma > 0$ describing the behaviour of the corresponding distribution for t tending to infinity.

To this end, we form $\mathcal{W}^{s,l}_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$ as the subspace of all distributions in $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$ supported in $\overline{\mathbb{R}}_+ \times \mathbb{R}^q$ and topologise this space with the induced Banach topology.

Lemma 1 *For all $s \in \mathbb{R}$ we have:*

(i) The space $C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^q, E)$ is dense in $\mathcal{W}^{s,l}_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$.

(ii) $\mathcal{W}^{s,l}_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$ is a closed subspace of $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$.

Proof: To (i): For given $u(t,y) \in \mathcal{W}^{s,l}_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$ and any $\delta > 0$ we set $u_\delta(t,y) := u(t-\delta, y)$. Then we have supp $u_\delta \subseteq [\delta, \infty) \times \mathbb{R}^q$ and Lebesgue's dominated convergence theorem yields $\|u - u_\delta\|_{s,l} \to 0$ for $\delta \to 0$. Next, in view of 3.1 Lemma 4 we approximate u_δ as element of $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$ with a sequence of smooth functions with compact support. Since $u_\delta \equiv 0$ for $t < \delta$ we can choose a subsequence supported in $\mathbb{R}_+$ that approximates u_δ and this gives the assertion.

To (ii): Let $\{u_n\} \subset \mathcal{W}^{s,l}_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$ and $u \in \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$ with $\|u - u_n\|_{s,l} \to 0$ for $n \to \infty$ be given. We have to check supp $u \subseteq \overline{\mathbb{R}}_+ \times \mathbb{R}^q$.

In view of (i) we have sequences $\{\phi^n_k\}^\infty_{k=1} \subset C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^q, E)$ with $\phi^n_k \to u_n$ for $k \to \infty$. Then an easy diagonal argument gives a sequence $\{\phi^n_{k(n)}\}^\infty_{n=1}$ tending to u within $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$. Remember that the elements of $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$ belong to $\mathcal{S}'(\mathbb{R}^{1+q}, E) = \mathcal{L}(\mathcal{S}(\mathbb{R}^{1+q}), E)$. Then our convergence means

$$u(\psi) = \lim_{n\to\infty} \int \phi^n_{k(n)}(t,y)\psi(t,y)dtdy$$

for every $\psi \in \mathcal{S}(\mathbb{R}^{1+q})$.

But then choosing some arbitrary $\psi \in \mathcal{S}(\mathbb{R}^{1+q})$ with supp $\psi \subset (-\infty, 0) \times \mathbb{R}^q$ we get

$$\int \phi^n_{k(n)}(t,y)\psi(t,y)dtdy = 0$$

for every n and hence $u(\psi) = 0$ that is the desired property. □

In order to use the results of Sections 4.1 and 4.2 we convert the time–dependent operator via Fourier–Laplace transformation $\mathcal{L}$ into a parameter–dependent pseudo–differential operator. This transformation is given by the formula

$$\mathcal{L}_{t\to\zeta}u(\zeta, y) := \int_{-\infty}^{\infty} e^{-it\zeta}u(t,y)dt.$$

Proposition 2 *For any Banach space F let $\mathcal{S}_0(\overline{\mathbb{R}}_+, F)$ be the subspace of all elements in $\mathcal{S}(\mathbb{R}, F)$ supported in $\overline{\mathbb{R}}_+$. Then $\hat{u}(\zeta)$ is the Fourier–Laplace transform of $u(t) \in \mathcal{S}_0(\overline{\mathbb{R}}_+, F)$ if and only if $\hat{u}(\zeta)$ is holomorphic in $\mathbb{C}_-$, smooth in $\overline{\mathbb{C}}_-$, and for all $m, k \in \mathbb{N}$ there exists a constant $C = C(m,k) > 0$ such that the inequality*

$$(1 + |\zeta|)^m \|D^k_\zeta u(\zeta)\|_F \le C$$

holds for all $\zeta \in \overline{\mathbb{C}}_-$.

The proof is completely analogous to the scalar case treated in [Es1] and will be omitted here.

Using the Fourier–Laplace transformation we can form pseudo–differential operators with symbols where the covariable corresponding to the time runs over subsets of the complex plane. Remember that in view of the Paley–Wiener theorem the Fourier transform of a smooth function with compact support may be extended analytically into the complex plane and for functions supported in $\overline{\mathbb{R}}_+$ this is nothing else than its Fourier–Laplace transform. Multiplying the resulting function by some convenient symbol and transforming back gives a pseudo–differential operation. In order to do that we need a further condition concerning the order reduction.

Condition H: For every $s \in \mathbb{R}$ there exists an element $r(\zeta,\eta) \in S^{s,l}_{cl}(\overline{\mathbb{C}}_- \times \mathbb{R}^q; E, E^0)$ which is holomorphic for all $\zeta \in \mathbb{C}_-$ and parameter–dependent anisotropic elliptic.

Corollary 3 *Condition* H *implies the existence of an order reduction* $R^s(\zeta) \in \Psi^{s,l}(Y; E, E^0; \overline{\mathbb{C}}_-)$ *that is holomorphic for all* $\zeta \in \mathbb{C}_-$.

In fact, according to the notation of (4.1.5) we can form

$$R^s(\zeta) = \sum_{j=1}^{N} \phi_j \left\{ \left(\chi_j^{-1}\right)_* \mathrm{Op}_y(r_j)(\zeta + i\gamma_0) \right\} \psi_j + C(\zeta)$$

for $\gamma_0 \geq 0$ which is parameter–dependent elliptic with parameter $\zeta \in \overline{\mathbb{C}}_-$ and holomorphic in $\mathbb{C}_-$. Then applying 4.2 Corollary 15 the operator

$$R^s(\zeta) : \mathcal{W}^s(Y, E) \to \mathcal{W}^0(Y, E^0)$$

is an isomorphism for all $\zeta \in \overline{\mathbb{C}}_-$ as soon as γ_0 is sufficiently large.

In the sequel we allways assume that we have Gelfand triples $(E, E^0, E', \kappa_\lambda)$ and $(\widetilde{E}, \widetilde{E}^0, \widetilde{E}', \tilde{\kappa}_\lambda)$, respectively, satisfying the Condition H with resulting order reductions $R^s(\zeta)$ and $\widetilde{R}^s(\zeta)$.

Lemma 4 *Let* $R^s(\zeta) \in \Psi^{s,l}(Y; E, E^0; \overline{\mathbb{C}}_-)$ *be an order reduction following from Condition* H. *Then* $D^j_\zeta R^s(\zeta) \in \Psi^{s-jl,l}(Y; E, E^0; \overline{\mathbb{C}}_-)$, *and there are constants* $c, M > 0$ *such that for* $\rho = \max\{s - jl, 0\}$

$$\|D^j_\zeta R^s(\zeta)\|_{\mathcal{L}(\mathcal{W}^{s-jl,l}(Y,E), \mathcal{W}^0(Y,E^0))} \leq c(1 + |\zeta|^2)^{(\rho+M)/2l} \tag{2}$$

holds for all $j \in \mathbb{N}$.

Proof: Since $R^s(\zeta)$ is holomorphic in $\mathbb{C}_-$ we can form $D^j_\zeta R^s(\zeta)$ and it belongs to $\Psi^{s-jl,l}(Y; E, E^0; \overline{\mathbb{C}}_-)$. This follows immediately from the symbol estimates. The norm estimate (2) follows from 4.1 Theorem 11. Here we set $\mu = s - jl$ and $\widetilde{M} = 0$ since κ_λ is supposed to be unitary on E^0. □

Example 5 Setting $E = H^s(\mathbb{R}^n)$ and $\widetilde{E} = H^{s-\mu}(\mathbb{R}^n)$ endowed with the standard group actions $(\kappa_\lambda u)(x) = \lambda^{\frac{n}{2}} u(\lambda x)$, $\lambda \in \mathbb{R}_+$, we form $R^\mu(\zeta)$ according to the notation of 4.1 Definition 9 as

$$R^\mu(\zeta) = \sum_{j=1}^{N} \phi_j \left\{ (\chi_j^{-1})_* \mathrm{Op}_y(r_j)(\zeta) \right\} \psi_j$$

with

$$r_j(\zeta, \eta) = \mathrm{op}_x \left((1 + i\zeta + |\xi|^l + |\eta|^l + |\beta|)^{\frac{\mu}{l}} \right)$$

where $\beta \in \mathbb{R}^b$ is an extra parameter that will be chosen below. The powers are defined by the branch of the logarithm in $\zeta \in \overline{\mathbb{C}}_-$ which is real for real arguments.

We then obtain an operator family

$$R^\mu(\zeta) : \mathcal{W}^s(Y, H^s(\mathbb{R}^n)) \to \mathcal{W}^{s-\mu}(Y, H^{s-\mu}(\mathbb{R}^n))$$

that becomes an isomorphism for sufficiently large $|\beta|$, for all $\zeta \in \overline{\mathbb{C}}_-$ and all $s \in \mathbb{R}$. After fixing this β we get $R^\mu(\zeta) \in \Psi^{\mu,l}(Y, H^s(\mathbb{R}^n), H^{s-\mu}(\mathbb{R}^n); \overline{\mathbb{C}}_-)$ for all s. In particular, inserting $\mu = s$ we obtain an order reduction in the sense of Condition H.

Proposition 6 *The operator* $\mathrm{Op}_t(R^s) := \mathcal{F}^{-1}_{\tau\to t} R^s(\tau)\mathcal{F}_{t\to\tau}$ *gives an isomorphism*

$$\mathrm{Op}_t(R^s) : \mathcal{W}_0^{s,l}(\overline{\mathbb{R}}_+ \times Y, E) \to \mathcal{W}_0^0(\overline{\mathbb{R}}_+ \times Y, E^0)$$

for all $s \in \mathbb{R}$.

Proof: It suffices to invert the order reduction locally since then it remains to invert $I + C(\zeta)$ with some $C(\zeta) \in \Psi^{-\infty}(Y; E, E^0; \overline{\mathbb{C}}_-)$ which can be done with the help of 4.1 Proposition 14. For simplicity we will denote for the moment the local order reduction also by $R^s(\zeta)$.

We have $\mathcal{W}^{s,l}(\mathbb{R}\times\mathbb{R}^q, E) = \mathcal{W}^{s/l}(\mathbb{R}, \mathcal{W}^s(\mathbb{R}^q, E))$ such that $\mathrm{Op}_t(R^s)$ at least extends to a continuous map $\mathrm{Op}_t(R^s) : \mathcal{W}^{s/l}(\mathbb{R}, \mathcal{W}^s(\mathbb{R}^q, E)) \to \mathcal{W}^{0/l}(\mathbb{R}, \mathcal{W}^0(\mathbb{R}^q, E^0)) = \mathcal{W}^0(\mathbb{R}\times\mathbb{R}^q, E^0)$ since it is an anisotropic pseudo–differential operator of order s with coefficients independent of $t \in \mathbb{R}$. Therefore it remains to prove that $\mathrm{Op}_t(R^s)$ maps distributions supported in $\overline{\mathbb{R}}_+$ to distributions supported in $\overline{\mathbb{R}}_+$. Since in view of Lemma 1 the space $\mathcal{S}_0(\overline{\mathbb{R}}_+, \mathcal{W}^s(\mathbb{R}^q, E))$ is dense in $\mathcal{W}_0^{s/l}(\overline{\mathbb{R}}_+, \mathcal{W}^s(\mathbb{R}^q, E))$ we can restrict our considerations to $\mathcal{W}^s(\mathbb{R}^q, E)$–valued rapidely decreasing functions supported in $\overline{\mathbb{R}}_+$.

Using Proposition 2 we obtain that the Fourier transform $\hat{u}(\tau)$ of an element $u(t) \in \mathcal{S}_0(\overline{\mathbb{R}}_+, \mathcal{W}^s(\mathbb{R}^q, E))$ possesses an analytic extension $\hat{u}(\zeta)$ into the lower complex half–plane that satisfies

$$\|D_\zeta^{\tilde{k}} u(\zeta)\|_F \le C_{\tilde{k},\tilde{m}}(1+|\zeta|)^{-\tilde{m}} \tag{3}$$

for all $\tilde{k}, \tilde{m} \in \mathbb{N}$ with some constant $C_{\tilde{k},\tilde{m}} > 0$.

Because of Condition H and the (ζ–wise) mapping property of R^s the function $\hat{v}(\zeta) := R^s(\zeta)\hat{u}(\zeta)$ is analytic in the lower complex half–plane and takes values in $\mathcal{W}^0(\mathbb{R}\times\mathbb{R}^q, E^0)$. So by Proposition 2 $\mathcal{F}^{-1}_{\tau\to t}\hat{v}|_{\mathrm{Im}\ \zeta=0} = \mathrm{Op}_t(R^s)u$ belongs to $\mathcal{S}_0(\overline{\mathbb{R}}_+, \mathcal{W}^0(\mathbb{R}^q, E^0))$ if the inequality

$$(1+|\zeta|)^m \|D_\zeta^k \hat{v}(\zeta)\|_F \le C \tag{4}$$

holds for all $\zeta \in \overline{\mathbb{C}}_-$ and arbitrary $m, k \in \mathbb{N}$.

To check this inequality we only have to remark that $D_\zeta^k \hat{v}(\zeta) = D_\zeta^k(R^s(\zeta)\hat{u}(\zeta))$ is a finite linear combination of terms of the form $D_\zeta^j R^s(\zeta) D_\zeta^{k-j}\hat{u}(\zeta)$. But in view of (2) and (3) and choosing $\tilde{m}$ sufficiently large we get (4).

Finally recall that an order reduction is a family of isomorphisms. Hence we can form $(R^s(\zeta))^{-1}$ (for every $\zeta \in \overline{\mathbb{C}}_-$) which is also an analytic family of isomorphisms. Therefore $\mathrm{Op}_t((R^s)^{-1})$ is the inverse operator of $\mathrm{Op}_t(R^s)$ which completes the proof. □

Definition 7 *For any weight* $\gamma > 0$ *we define the abstract weighted Sobolev space*

$$\mathcal{T}_\gamma^{s,l}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$$

on the half–space $\overline{\mathbb{R}}_+ \times \mathbb{R}^q$ *as the set of all* $u(t,y) \in \mathcal{S}'(\mathbb{R}^{1+q}, E)$ *such that* $e^{-\gamma t}u(t,y)$ *belongs to* $\mathcal{W}_0^{s,l}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$. *We endow* $\mathcal{T}_\gamma^{s,l}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$ *with the norm* $\|\cdot\|_{s,l;\gamma}$ *given by*

$$\|u\|_{s,l;\gamma} := \|e^{-\gamma t}u\|_{s,l} = \left\{ \int [\tau,\eta]_l^{2s} \|\kappa^{-1}(\tau,\eta)\mathcal{F}\{e^{-\gamma t}u\}(\tau,\eta)\|_E^2 đ\tau đ\eta \right\}^{\frac{1}{2}}. \tag{5}$$

Lemma 8 *For all* $s, \gamma \in \mathbb{R}$, $\gamma > 0$, *we have:*

(i) *The space* $\mathcal{T}_\gamma^{s,l}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$ *is a Banach space, that is Hilbertisable.*

(ii) *The space* $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^q, E)$ *is dense in* $\mathcal{T}_\gamma^{s,l}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$.

Proof: To (i): The completeness of $\mathcal{T}^{s,l}_\gamma(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$ follows immediately from the completeness of $\mathcal{W}^{s,l}_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$. In order to obtain a scalar product we set

$$(u,v)_{s,l;\gamma} := (e^{-\gamma t}u, e^{-\gamma t}v)_{s,l}$$

where $(\cdot,\cdot)_{s,l}$ denotes the scalar product in $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$.

To (ii): For arbitrary $u \in \mathcal{T}^{s,l}_\gamma(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$ we set $v := e^{-\gamma t}u$. Then v belongs to $\mathcal{W}^{s,l}_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$ and $\|u\|_{s,l;\gamma} = \|v\|_{s,l}$. Using (i) of Lemma 1, namely the density of $C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^q, E)$ in $\mathcal{W}^{s,l}_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, E)$, there is e sequence $(\phi_n) \subset C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^q, E)$ such that $\|\phi_n - v\|_{s,l} \to 0$ for $n \to \infty$. But then setting $\psi_n = e^{\gamma t}\phi_n$ we obtain $\psi_n \in C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^q, E)$ and $\|\psi_n - u\|_{s,l;\gamma} = \|e^{-\gamma t}(\psi_n - u)\|_{s,l} = \|\phi_n - v\|_{s,l} \to 0$ for $n \to \infty$. □

Furthermore, we have for the smooth compact C^∞ manifold Y the spaces $\mathcal{W}^{s,l}(\mathbb{R} \times Y, E)$ as well as $\mathcal{W}^{s,l}_0(\overline{\mathbb{R}}_+ \times Y, E)$ such that we can form the anisotropic abstract weighted Sobolev spaces

$$\mathcal{T}^{s,l}_\gamma(\overline{\mathbb{R}}_+ \times Y, E) := e^{\gamma t}\mathcal{W}^{s,l}_0(\overline{\mathbb{R}}_+ \times Y, E)$$

on the infinite time cylinder $\overline{\mathbb{R}}_+ \times Y$ with base Y.

Notice that supposing E^0 as Hilbert space the above definitions give

$$\mathcal{T}^{0,l}_0(\overline{\mathbb{R}}_+ \times Y, E^0) = \mathcal{W}^{0,l}_0(\overline{\mathbb{R}}_+ \times Y, E^0) = L^2(\mathbb{R}_+ \times Y, E^0),$$

since the correspondig group action is unitary, and the Fourier–Laplace transform of some function belonging to this space is nothing else than the analytic extension of its Fourier transform into the lower complex half–plane. So there should be an analogue of the Paley–Wiener theorem, which is our next goal.

Definition 9 *For $s, \gamma \in \mathbb{R}$ with $\gamma \geq 0$ the space $\mathcal{Z}^{s,l}(\overline{\mathbb{C}}_{-\gamma} \times Y, E)$ consists of all $\hat{u}(\zeta) : \overline{\mathbb{C}}_{-\gamma} \to \mathcal{W}^s(Y, E)$ holomorphic for all $\zeta \in \mathbb{C}_{-\gamma}$ such that*

$$\|u\|_{\mathcal{Z}^{s,l}(\overline{\mathbb{C}}_{-\gamma} \times Y, E)} := \sup_{\vartheta \leq -\gamma} \left\{ \int \|\hat{u}(\tau + i\vartheta)\|^2_{\mathcal{W}^s(Y,E)_\zeta} đ\tau \right\}^{\frac{1}{2}} < \infty.$$

holds. Here we set $\mathbb{C}_{-\gamma} = \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta < -\gamma\}$ and $\overline{\mathbb{C}}_{-\gamma}$ is the corresponding closure.

With this definition we obtain the following theorem.

Theorem 10 *For all $s, \gamma \in \mathbb{R}$, $\gamma \geq 0$, the space $\mathcal{Z}^{s,l}(\overline{\mathbb{C}}_{-\gamma} \times Y, E)$ is isometric to $\mathcal{T}^{s,l}_\gamma(\overline{\mathbb{R}}_+ \times Y, E)$ and the isomorphism is given by the Fourier–Laplace transformation.*

Corollary 11 *Theorem 10 implies that $\mathcal{Z}^{s,l}(\overline{\mathbb{C}}_{-\gamma} \times Y, E)$ is Banach and since E is a Hilbert space, even Hilbertsable. Moreover, the space $\mathcal{L}(C^\infty_0(\mathbb{R}_+ \times Y), E)$ is dense in $\mathcal{Z}^{s,l}(\overline{\mathbb{C}}_{-\gamma} \times Y, E)$.*

In order to prove Theorem 10 we need as a proposition the following vector–valued version of the Paley–Wiener theorem. It characterises the set of Fourier transforms of all L^2–functions supported in $\overline{\mathbb{R}}_+$. We follow here the proof of Rudin in [Ru] Theorem 19.2, where it was shown for scalar L^2–functions.

Proposition 12 *Let H be an arbitrary Hilbert space. Then $\mathcal{F}(L^2(\mathbb{R}_+, H))$ is the space of all H–valued functions $\hat{u}(\tau)$ with the following conditions:*

(i) $\hat{u}$ possesses an analytic extension into $\mathbb{C}_-$.

(ii) There is a constant $C > 0$ such that

$$\sup_{\vartheta < 0} \int_{-\infty}^{\infty} \|\hat{u}(\tau + i\vartheta)\|^2_H đ\tau \leq C \tag{6}$$

holds.

Proof: Remember that $L^2(\mathbb{R}_+, H)$ denotes the subspace of all functions $u(t) \in L^2(\mathbb{R}, H)$ supported in $\overline{\mathbb{R}}_+$. For all such functions and $\zeta = \tau + i\vartheta \in \overline{\mathbb{C}}_-$ the analytic extension of the Fourier transform (if it exists) is given by the formula

$$\hat{u}(\zeta) = \int_{-\infty}^{\infty} e^{-it\zeta} u(t) dt = \int_0^{\infty} e^{-it\zeta} u(t) dt = \mathcal{L}u(\zeta). \tag{7}$$

Since $|e^{-it\zeta}| = e^{t\vartheta}$ and $t\vartheta \le 0$ for $t \ge 0$ and $\vartheta \le 0$ the second integral in (7) exists for all $\zeta \in \overline{\mathbb{C}}_-$. Let now $\zeta \in \mathbb{C}_\delta := \{\text{Im } \zeta < -\delta\}$ for some $\delta > 0$ be given. Then for any sequence $\zeta_n \subset \mathbb{C}_\delta$ with $\zeta_n \to \zeta$ we get

$$\begin{aligned}\lim_{n\to\infty} \|\hat{u}(\zeta_n) - \hat{u}(\zeta)\|_H &= \lim_{n\to\infty} \Big\| \int_0^{\infty} (e^{-it\zeta_n} - e^{-it\zeta}) u(t) dt \Big\|_H \\ &\le \lim_{n\to\infty} \Big(\int_0^{\infty} |e^{-it\zeta_n} - e^{-it\zeta}|^2 dt \Big)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R},H)} = 0\end{aligned}$$

since the integral $\int_0^\infty |e^{-it\zeta_n} - e^{-it\zeta}|^2 dt$ tends to 0 for $\zeta_n \to \zeta$ because the integrand is bounded by $4e^{-2\delta t}$ and tends to zero for every $t > 0$. Thus $\hat{u}$ is continuous in $\mathbb{C}_-$. But then Fubini's and Cauchy's theorems imply

$$\int_\Pi \hat{u}(\zeta) d\zeta = \int_\Pi \int_0^{\infty} e^{-it\zeta} u(t) dt d\zeta = 0$$

for every closed path Π in $\mathbb{C}_-$. Hence $\hat{u}$ is holomorphic in $\mathbb{C}_-$ by Morera's theorem. Further, using Plancherel's formula for an arbitrary fixed $\vartheta = \text{Im } \zeta < 0$ we get

$$\begin{aligned}\int_{-\infty}^{\infty} \|\hat{u}(\tau + i\vartheta)\|_H^2 đ\tau &= \int_{-\infty}^{\infty} \|\mathcal{F}_{t\to\tau}\{e^{t\vartheta} u(t)\}\|_H^2 đ\tau \\ &= \int_0^{\infty} e^{2t\vartheta} \|u(t)\|_H^2 dt \\ &\le \int_{-\infty}^{\infty} \|u(t)\|_H^2 dt = \|u\|_{L^2(\mathbb{R},H)}^2\end{aligned}$$

which gives the second claim.

On the other hand for given $\hat{u}$ with (i) and (ii) we are looking for $u(t) \in L^2(\mathbb{R}_+, H)$ such that $\hat{u}(\zeta) = \mathcal{L}u(\zeta)$. To this end we consider the integral

$$\int_{-\infty}^{\infty} e^{it(\tau + i\vartheta)} \hat{u}(\tau + i\vartheta) đ\tau$$

and prove that it does not depend on ϑ for $\vartheta < 0$. First for fixed $\vartheta_0 < 0$ Cauchy's theorem gives

$$\int_{\Gamma_\alpha} e^{it\zeta} \hat{u}(\zeta) đ\zeta = 0 \tag{8}$$

for the rectangular path Γ_α with vertices $\pm\alpha + i$ and $\pm\alpha + i\vartheta_0$. Setting

$$\Phi(\beta) := \int_{\beta+i}^{\beta+i\vartheta_0} e^{it\zeta} \hat{u}(\zeta) đ\zeta$$

and $I = [\vartheta_0, -1]$ for $\vartheta_0 < -1$ or $I = [-1, \vartheta_0]$ for $\vartheta_0 > -1$ we get

$$\|\Phi(\beta)\|_H^2 = \Big\| \int_I e^{it(\beta+i\vartheta)} \hat{u}(\beta + i\vartheta) đ\vartheta \Big\|_H^2 \le \int_I \|\hat{u}(\beta + i\vartheta)\|_H^2 đ\vartheta \int_I e^{2t\vartheta} d\vartheta.$$

For $\Lambda(\beta) := \int_I \|\hat{u}(\beta + i\vartheta)\|_H^2 đ\vartheta$ we obtain

$$\int_{-\infty}^{\infty} \Lambda(\beta) đ\beta \le C \frac{|\vartheta_0 + 1|}{2\pi}.$$

Here we used (ii) and Fubini's theorem. Hence there is a sequence α_j with $\alpha_j \to \infty$ and $\Lambda(\alpha_j) + \Lambda(-\alpha_j) \to 0$ for $j \to \infty$. This gives us altogether

$$\Phi(\alpha_j) \to 0 \text{ and } \Phi(-\alpha_j) \to 0 \tag{9}$$

for $j \to \infty$ and this holds for all $\vartheta_0 < 0$ and the α_j may be chosen independently of ϑ_0.

Now setting

$$v_j(\vartheta, t) := \int_{-\alpha_j}^{\alpha_j} e^{it\tau} \hat{u}(\tau + i\vartheta) đ\tau$$

from (8) and (9) it follows that

$$\lim_{j\to\infty} \left(e^{-t\vartheta} v_j(\vartheta, t) - e^t v_j(-1, t)\right) = 0. \tag{10}$$

Next we write $\hat{u}_\vartheta(\tau)$ for $\hat{u}(\tau + i\vartheta)$ and by hypothesis we have $\hat{u}_\vartheta \in L^2(\mathbb{R}, H)$. Then Plancherel's formula gives us

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} \|\mathcal{F}^{-1}_{\tau\to t} \hat{u}_\vartheta(t) - v_j(\vartheta, t)\|^2_H dt = 0.$$

But this implies that there is a subsequence of $\{v_j(\vartheta, t)\}$ converging pointwise a.e. to $\mathcal{F}^{-1}_{\tau\to t}\hat{u}_\vartheta(t)$.

Now we define $u(t) := e^t \mathcal{F}^{-1}_{\tau\to t} \hat{u}_{-1}(t)$ and (10) yields

$$u(t) = e^{-t\vartheta} \mathcal{F}^{-1}_{\tau\to t} \hat{u}_\vartheta(t). \tag{11}$$

Note that by definition $u(t)$ does not depend on ϑ. Moreover, (11) holds for every $\vartheta \leq 0$. Again by Plancherel's formula we have

$$\|e^{t\vartheta} u\|_{L^2(\mathbb{R}_t, H)} = \|\mathcal{F}^{-1}_{\tau\to t} \hat{u}_\vartheta\|_{L^2(\mathbb{R}_t, H)} = \|\hat{u}_\vartheta\|_{L^2(\mathbb{R}_\tau, H)} < c.$$

This shows for $\vartheta \to -\infty$ that u vanishes for $t < 0$ and for $\vartheta \to 0$ that u belongs to the space $L^2(\mathbb{R}, H)$. Finally we compute

$$\begin{aligned}
\mathcal{L}u(\zeta) &= \int_0^{\infty} e^{it\zeta} u(t) dt \\
&= \int_{-\infty}^{\infty} e^{it\zeta} u(t) dt \\
&= \int_{-\infty}^{\infty} e^{it\zeta} e^{-t\vartheta} \mathcal{F}^{-1}_{\tau'\to t} \hat{u}_\vartheta(t) dt \\
&= \int_{-\infty}^{\infty} e^{it\tau} \mathcal{F}^{-1}_{\tau'\to t} \hat{u}_\vartheta(t) dt \\
&= \hat{u}_\vartheta(\tau) = \hat{u}(\zeta)
\end{aligned}$$

which comletes the proof. □

Proof: (of Theorem 10) Let us consider in a first step the case $s = \gamma = 0$ and values in E^0. Since $\mathcal{T}_0^{0,l}(\overline{\mathbb{R}}_+ \times Y, E^0) = L^2(\mathbb{R}_+, L^2(Y, E^0))$ and $L^2(Y, E^0)$ is a Hilbert space, we can use Proposition 12 for this case. Note that the Fourier–Laplace transform is nothing else than the analytic extension of the Fourier transform, such that it gives in view of Proposition 12 an isomorphism between $\mathcal{T}_0^{0,l}(\overline{\mathbb{R}}_+ \times Y, E^0)$ and $\mathcal{Z}^{0,l}(\overline{\mathbb{C}}_- \times Y, E^0)$. Moreover, since the group action on E^0 is unitary, it turns out that $\|\cdot\|_{0,l;0}$ is the L^2–norm such that Plancherel's formula shows that the Fourier–Laplace transformation preserves the norm.

Next we consider the case $s \neq 0$. But then, since

$$\mathrm{Op}_t(R^s) : \mathcal{W}_0^{s,l}(\overline{\mathbb{R}}_+ \times Y, E) \to \mathcal{W}_0^{0,l}(\overline{\mathbb{R}}_+ \times Y, E^0)$$

and

$$M_{R^s} : \mathcal{Z}_0^{s,l}(\overline{\mathbb{C}}_- \times Y, E) \ni u(\zeta) \mapsto M_{R^s} u(\zeta) := R^s(\zeta) u(\zeta) \in \mathcal{Z}_0^{0,l}(\overline{\mathbb{C}}_- \times Y, E^0)$$

are isomorphisms, the theorem follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{W}_0^{s,l}(\overline{\mathbb{R}}_+ \times Y, E) & \xrightarrow{\mathcal{L}} & \mathcal{Z}^{s,l}(\overline{\mathbb{C}}_- \times Y, E) \\ \downarrow \operatorname{Op}_t(R^s) & & \downarrow M_{R^s} \\ \mathcal{W}_0^{0,l}(\overline{\mathbb{R}}_+ \times Y, E^0) & \xrightarrow{\mathcal{L}} & \mathcal{Z}^{0,l}(\overline{\mathbb{C}}_- \times Y, E^0). \end{array}$$

For an arbitrary weight γ we have to check

$$\mathcal{L}\left(\mathcal{T}_\gamma^{s,l}(\overline{\mathbb{R}}_+ \times Y, E)\right) = \mathcal{L}\left(e^{\gamma t}\mathcal{W}_0^{s,l}(\overline{\mathbb{R}}_+ \times Y, E)\right) = \mathcal{Z}^{s,l}(\overline{\mathbb{C}}_{-\gamma} \times Y, E),$$

but this follows immediately from $\mathcal{L}\{e^{\gamma t}u\}(\zeta) = \mathcal{L}u(\zeta + i\gamma)$. □

Proposition 13 *Let* $A(\zeta) \in \Psi^{0,l}(Y; E^0, \widetilde{E}^0; \overline{\mathbb{C}}_-)$ *be holomorphic for all* $\zeta \in \mathbb{C}_-$. *Then* $\operatorname{Op}_t(A) := \mathcal{F}^{-1}_{\tau\to t}A(\zeta)|_{\operatorname{Im}\,\zeta=0}\mathcal{F}_{t'\to\tau}$ *gives a continuous map*

$$\operatorname{Op}_t(A) : \mathcal{W}_0^0(\overline{\mathbb{R}}_+ \times Y, E^0) \to \mathcal{W}_0^0(\overline{\mathbb{R}}_+ \times Y, \widetilde{E}^0).$$

Proof: Since $\mathcal{W}_0^0(\overline{\mathbb{R}}_+ \times Y, E^0) = L^2(\mathbb{R}_+, L^2(Y, E^0))$ and $\mathcal{W}_0^0(\overline{\mathbb{R}}_+ \times Y, \widetilde{E}^0) = L^2(\mathbb{R}_+, L^2(Y, \widetilde{E}^0))$ we can use Proposition 12 to check $A(\zeta)\mathcal{L}u(\zeta) \in \mathcal{L}L^2(\mathbb{R}_+, L^2(Y, \widetilde{E}^0))$. Of course, with $A(\zeta)$ and $\mathcal{L}u(\zeta)$ also $A(\zeta)\mathcal{L}u(\zeta)$ is holomorphic in $\mathbb{C}_-$. So it only remains to note

$$\sup_{\vartheta<0}\int_{-\infty}^{\infty} \|A(\tau+i\vartheta)\mathcal{L}u(\tau+i\vartheta)\|^2_{L^2(Y,\widetilde{E}^0)} d\!\!{}^-\tau \le c\sup_{\vartheta<0}\int_{-\infty}^{\infty} \|\mathcal{L}u(\tau+i\vartheta)\|^2_{L^2(Y,E^0)} d\!\!{}^-\tau \le cC,$$

which used 4.1 Corollary 12 and Proposition 12. □

Corollary 14 *Every* $A(\zeta) \in \Psi^{\nu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ *which is holomorphic for all* $\zeta \in \mathbb{C}_-$ *induces a continuous map*

$$\operatorname{Op}_t(A) : \mathcal{W}_0^{s,l}(\overline{\mathbb{R}}_+ \times Y, E) \to \mathcal{W}_0^{s-\nu,l}(\overline{\mathbb{R}}_+ \times Y, \widetilde{E})$$

for all $s \in \mathbb{R}$.

In fact, we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{W}_0^{s,l}(\overline{\mathbb{R}}_+ \times Y, E) & \xrightarrow{\operatorname{Op}_t(A)} & \mathcal{W}_0^{s-\nu,l}(\overline{\mathbb{R}}_+ \times Y, \widetilde{E}) \\ \downarrow \operatorname{Op}_t(R^s) & & \downarrow \operatorname{Op}_t(\widetilde{R}^{s-\nu}), \\ \mathcal{W}_0^0(\overline{\mathbb{R}}_+ \times Y, E^0) & \xrightarrow{\operatorname{Op}_t(A_0)} & \mathcal{W}_0^0(\overline{\mathbb{R}}_+ \times Y, \widetilde{E}^0) \end{array}$$

where $A_0(\zeta) = \widetilde{R}^{s-\nu}(\zeta)A(\zeta)\left(R^s(\zeta)\right)^{-1} \in \Psi^{0,l}(Y; E^0, \widetilde{E}^0; \overline{\mathbb{C}}_-)$. But then the desired continuity follows from Proposition 13 and Proposition 6.

Definition 15 *An operator* $A = \operatorname{Op}_t(A(\zeta))$ *for* $A(\zeta) \in \Psi^{\mu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ *holomorphic in* $\mathbb{C}_-$ *is called parabolic of order* μ *if* $A(\zeta)$ *is parameter–dependent anisotropic elliptic.*

Theorem 16 *Let* $A = \operatorname{Op}_t(A(\zeta))$ *for* $A(\zeta) \in \Psi^{\mu,l}(Y; E, \widetilde{E}; \overline{\mathbb{C}}_-)$ *be parabolic of order* μ, *then there exists a weight* $\gamma_0 > 0$ *such that*

$$A : \mathcal{T}_\gamma^{s,l}(\overline{\mathbb{R}}_+ \times Y, E) \to \mathcal{T}_\gamma^{s-\mu,l}(\overline{\mathbb{R}}_+ \times Y, E)$$

is invertible for all $\gamma \ge \gamma_0$ *and all* $s \in \mathbb{R}$.

Proof: In view of the diagram

$$\begin{array}{ccc} \mathcal{T}_\gamma^{s,l}(\overline{\mathbb{R}}_+ \times Y, E) & \xrightarrow{A} & \mathcal{T}_\gamma^{s-\mu,l}(\overline{\mathbb{R}}_+ \times Y, \widetilde{E}) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ \mathcal{Z}^{s,l}(\overline{\mathbb{C}}_{-\gamma} \times Y, E) & \xrightarrow{M_{A(\zeta)}} & \mathcal{Z}^{s-\mu,l}(\overline{\mathbb{C}}_{-\gamma} \times Y, \widetilde{E}) \end{array}$$

we only have to check the invertibility of the multiplication

$$M_{A(\zeta)} : \mathcal{Z}^{s,l}(\overline{\mathbb{C}}_{-\gamma} \times Y, E) \to \mathcal{Z}^{s-\mu,l}(\overline{\mathbb{C}}_{-\gamma} \times Y, \widetilde{E})$$

for all $\gamma > \gamma_0$ with some $\gamma_0 > 0$. But in view of 4.2 Theorem 10 the anisotropic parameter–dependent ellipticity of $A(\zeta)$ implies the existence of a weight $\gamma_0 > 0$ such that $A(\zeta)$ is (ζ–wise) invertible as map $\mathcal{W}^s(Y,E) \to \mathcal{W}^{s-\mu}(Y,\widetilde{E})$. Denoting this inverse by $(A(\zeta))^{-1}$ (it belongs to $T_{-i\gamma_0}\Psi^{-\mu,l}(Y;\widetilde{E},E;\overline{\mathbb{C}}_-)$) we obtain $M_{(A(\zeta))^{-1}} = M^{-1}_{A(\zeta)}$ and hence $M_{A(\zeta)}$ is invertible for sufficiently large γ, which gives the desired invertibility of A for $\gamma \geq \gamma_0$. □

Remark 17 Note that $B(\zeta) = (A(\zeta))^{-1}$ itself is a family of pseudo–differential operators that is holomorphic for all $\zeta \in \mathbb{C}_{-\gamma_0}$. Therefore the translated family $B(\zeta + i\gamma_0)$ belongs to $\Psi^{-\mu,l}(Y;\widetilde{E},E;\overline{\mathbb{C}}_-)$ and is holomorphic in $\mathbb{C}_-$. Further considerations will show that if we set $B_{\gamma_0} = \mathrm{Op}_t(B(\zeta + i\gamma_0))$ then $AB_{\gamma_0} - I$ and $B_{\gamma_0}A - I$ turn out to be pseudo–differential operators of order $-l$. But this will be the starting point for the construction of a parametrix and the inverse operator within a subspace of operators in $\Psi^{\nu,l}(\mathbb{R} \times Y; E, \widetilde{E})$.

Remember that the differential operator A has in local coordinates the form (1), i.e.,

$$A(y, D_t, D_y) = a_0 D_t - \sum_{|\alpha| \leq l} a_\alpha(y) D_y^\alpha$$

where $a_0 \in \mathcal{L}(E,\widetilde{E})$ and $a_\alpha(y) \in C^\infty(\mathbb{R}^q, \mathcal{L}(E,\widetilde{E}))$ are given such that the corresponding symbol

$$a(y,\tau,\eta) = a_0\tau - \sum_{|\alpha| \leq l} a_\alpha(y)\eta^\alpha$$

belongs to $S^{l,l}(\mathbb{R}^q \times \mathbb{R}^{1+q}; E, \widetilde{E})$ (even classical) with Hilbert spaces E, $\widetilde{E}$. Thus A extends to a continuous map

$$A : \mathcal{T}^{s,l}_\gamma(\overline{\mathbb{R}}_+ \times Y, E) \to \mathcal{T}^{s-l,l}_\gamma(\overline{\mathbb{R}}_+ \times Y, \widetilde{E}) \tag{12}$$

for all $s, \gamma \in \mathbb{R}$. Moreover, if A is parabolic, then there exists an weight $\gamma_0 \in \mathbb{R}$ such that (12) is invertible for all $\gamma \geq \gamma_0$.

As an example consider, in particular, the operator

$$A = \frac{\partial}{\partial t} - \Delta_Y - \Delta_{\mathbb{R}^m},$$

where Δ_Y is the Laplace–Beltrami operator on Y with respect to a Riemannian metric, and $\Delta_{\mathbb{R}^m}$ is the Laplacian in $\mathbb{R}^m \ni x$ with the symbol $-|\xi|^2$.

Let $E = H^s(\mathbb{R}^m)$, $\widetilde{E} = H^{s-2}(\mathbb{R}^m)$ both be endowed with the group action $(\kappa_\lambda u)(x) = \lambda^{m/2}u(\lambda x)$, $\lambda \in \mathbb{R}_+$. Then in local coordinates

$$a(\tau,\eta) = i\tau + |\eta|^2 - \Delta_{\mathbb{R}^m} : E \to \widetilde{E}$$

belongs to $S^{2,2}_{cl}(\mathbb{R}^{1+q}; E, \widetilde{E})$. There is an extension $a(\zeta,\eta) = i\zeta + |\eta|^2 - \Delta_{\mathbb{R}^m} \in S^{2,2}_{cl}(\overline{\mathbb{C}}_- \times \mathbb{R}^q; E, \widetilde{E})$. The operator–valued anisotropic homogeneous principal symbol equals $i\zeta + |\eta|^2 - \Delta_{\mathbb{R}^m}$ which is an isomorphism $E \to \widetilde{E}$ for all $|\zeta,\eta| \neq 0$.

We have $A(\zeta) = i\zeta - \Delta_Y - \Delta_{\mathbb{R}^m}$ which belongs to $\Psi^{2,2}(Y, E, \widetilde{E}; \overline{\mathbb{C}}_-)$, and $A(\zeta)$ is parameter–dependent anisotropic elliptic in the sense of 4.2 Definition 7. Thus $A = \mathrm{Op}_t(A(\zeta))$ is parabolic and Theorem 16 is valid for $\mu = 2$.

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