A Remark on the Index of Symmetric Operators

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Abstract

We introduce a natural symmetry condition for a pseudodifferential operator on a manifold with cylindrical ends ensuring that the operator admits a doubling across the boundary. For such operators we prove an explicit index formula containing, apart from the Atiyah-Singer integral, a finite number of residues of the logarithmic derivative of the conormal symbol.

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Contents

1 Introduction

When studying the 'eta' invariant of an elliptic boundary value problem on a C^∞ compact manifold M, Gilkey and Smith [GS83a, GS83b] introduced a symmetry condition on a differential operator A ensuring that it admits a doubling across the boundary. In fact, the condition involves the symbol of A in a collar neighbourhood of ∂M and amounts to saying that two copies of A in the collar patch together smoothly by some bundle isomorphisms. This actually implies that in favourable cases the index of A subject to elliptic boundary conditions is one-half of the index of the operator on the double of M. Thus, the index formula on manifolds without boundary could be used to derive a corresponding cohomological formula for the index of the boundary value problem. Later in [SSS97] this observation was applied to elliptic differential operators on manifolds with a conical point treated as a cylindrical end, thus resulting in a simple index theorem. Unfortunately, this index theorem is valid for a narrow class of operators. When describing the contribution of a conical point in the index formula for singular surfaces, the authors [FST97b] observed that it reduces to a finite number of the residues of the logarithmic derivative of the function $f(\tau) = \det(M(\tau)+M^{-1}(\tau)-2)$ provided f is even with respect to a centre τ_0 in the complex plane. Here, $M(\tau)$ is the monodromy matrix for the ordinary differential equation defined by the conormal symbol. Thinking over the geometrical meaning of this property, we come to a generalisation of the symmetry condition encompassing, in particular, all first-order scalar differential operators. In the present paper we give an exposition of the symmetry condition in the general setting of pseudodifferential operators on higher-dimensional manifolds with conical singularities and show a simple index formula for the operators satisfying this condition.

2 Statement of the main result

Here we consider a manifold M of arbitrary dimension n with a conical point which is interpreted as a cylindrical end (see [FST97a]) and a pseudodifferential operator A between sections of smooth vector bundles E^0 and E^1 over M. We assume that A is *elliptic* with respect to a *weight line* $\Gamma = \{\Im\tau = \gamma\}$ in the complex τ-plane. This condition consists of two parts: *interior ellipticity* and *conormal ellipticity*. The first part means, similarly to the smooth case, that the interior principal symbol $\sigma_0(A) \in C^\infty(T^*M\setminus 0, \mathrm{Hom}\,(E^0, E^1))$ is invertible everywhere on $T^*M \setminus 0$. The second part we formulate for an operator A which on the cylindrical end is independent of the variable t along the cylinder axis. We also assume that all the structures (vector bundles, Riemannian metric, partition of unity and so on) are of product-type over the cylindrical end $\mathbb{R}_+ \times \partial M$ where ∂M means the cross-section of the cylinder. The operator A on the cylinder is treated as a pseudodifferential operator on $\mathbb{R}_+$ whose symbol $A_c = A_c(\tau)$ with values in pseudodifferential operators on ∂M satisfies the following conditions:

- it is holomorphic in some strip $S = \{|\Im\tau - \gamma| < \varepsilon\}$ around the weight line Γ;

- the operator $A_c(\tau) : C^\infty(\partial M, E^0) \to C^\infty(\partial M, E^1)$ is invertible for each $\tau \in S$ (*conormal ellipticity*).

Under these assumptions the operator $A\colon H^{s,\gamma}(M, E^0) \to H^{s-m,\gamma}(M, E^1)$ is Fredholm and the following index formula holds

$$
\begin{aligned}
\operatorname{ind} A \;=\; & \operatorname{Tr}\rho_0(1 - \overline{r} \circ \overline{a})|_N - \operatorname{Tr}\rho_0(1 - \overline{a} \circ \overline{r})|_N \\
& + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \operatorname{Tr}'\Big(A_c^{-1}(\tau + i\gamma - ih\gamma)A_c'(\tau + i\gamma - ih\gamma)|_{N-1} \\
& - \operatorname{Op}_{\partial M}(\overline{r}(\tau) \circ \overline{a}'(\tau)|_{N-1})\Big)\, d\tau
\end{aligned}
\tag{2.1}
$$

(notation and details see in [FST97a]).

Next, we impose a *symmetry condition* on the conormal symbol $A_c(\tau)$. Let us denote by T_g an automorphism of the algebra of pseudodifferential operators on ∂M of the form

$$T_g\,\Psi = (g^{-1})^*\, v_1^{-1}\Psi\, v_0\, g^* \tag{2.2}$$

where g is a diffeomorphism of ∂M and

$$
\begin{aligned}
v_0 :&\;\; g^*E^0 \to E^0, \\
v_1 :&\;\; g^*E^1 \to E^1
\end{aligned}
$$

are isomorphisms of the vector bundles. The automorphism (2.2) is assumed to be an *involution*, that is $T_g^2\Psi = \Psi$ for any pseudodifferential operator Ψ on ∂M. Consequently, $g^2 = \mathrm{Id}$, so either $g = \mathrm{Id}$ or g is an involution itself.

Definition 2.1 *An operator A is called symmetric if its conormal symbol $A_c(\tau)$ has the following property: there exist a point $\tau_0 \in \mathbb{C}$ (the centre) and an automorphism T_g of the form* (2.2)*, such that*

- *$A_c(\tau)$ extends meromorphically to a strip $\widetilde{S} = \{|\Im(\tau - \tau_0)| < |\gamma - \Im\tau_0| + \varepsilon\}$, with a finite number of poles in each proper substrip of $\widetilde{S}$ and finite-dimensional principal parts of Laurent expansions at the poles* [1]*;*

- *for any $\tau \in \widetilde{S}$,*
$$A_c(2\tau_0 - \tau) = T_g\, A_c(\tau). \tag{2.3}$$

[1]Due to ellipticity the same is true for $A_c^{-1}(\tau)$.

In other words, (2.3) means that the symmetry transformation in the τ-plane acts on $A_c(\tau)$ as an automorphism generated by a change of variables g and changes of frames v_0, v_1. If g is identity, this condition was studied in [GS83a] in the context of boundary value problems for differential operators. Later in [SSS97] it was applied to cone differential operators resulting in a simple index theorem. We generalise the index theorem of [SSS97] in two directions, namely to both *pseudodifferential* operators A on M and *non-identical* involutions g of ∂M.

Theorem 2.2 *Let A be an elliptic pseudodifferential operator with respect to a weight line Γ and let the symmetry condition* (2.3) *be fulfilled. Then*

$$\operatorname{ind} A = \int_{\mathbb{S}^*M} \mathrm{AS}(A) + \frac{1}{2}(N-P)\,\operatorname{sgn}(\Im\tau_0 - \gamma) \tag{2.4}$$

where N and P denote the number of poles of $A_c^{-1}(\tau)$ and $A_c(\tau)$, respectively, in the strip between the weight line Γ and the line symmetric to Γ with respect to the centre τ_0.

3 Proof

We may assume that the symmetry centre τ_0 coincides with the origin. Indeed, otherwise we introduce an operator $\tilde{A} = f^{-1}Af$ with a function $f \neq 0$ such that $f = \exp(i\tau_0 t)$ over the cylindrical end. Then we get $\tilde{A}_c(\tau) = A_c(\tau+\tau_0)$, so $\tilde{A}$ is elliptic with respect to the weight line $\tilde{\Gamma}$ obtained by shifting Γ by $-\tau_0$, and clearly the index of A in $H^{s,\gamma}(M,E^0)$ is equal to the index of $\tilde{A}$ in $H^{s,\tilde{\gamma}}(M,E^0)$ with $\tilde{\gamma} = \gamma - \Im\tau_0$. On the other hand, the right-hand side of (2.4) for $\tilde{A}$ is the same as for A. Thus, in the sequel we take $\tau_0 = 0$ and the symmetry condition (2.3) becomes

$$A_c(-\tau) = T_g\, A_c(\tau). \tag{3.1}$$

The automorphism T_g acts on the algebra of formal symbols on ∂M because changes of variables and frames have sense for formal symbols as well (see [FST97a]). When choosing a family of formal symbols $\overline{a}(\tau)$ in (2.1) approximating $A_c(\tau)$, we would like to satisfy a symmetry condition similar to (3.1),

$$\overline{a}(-\tau) = T_g\,\overline{a}(\tau). \tag{3.2}$$

Such a choice is always possible. Indeed, using averaging, we may pass to a new symbol

$$\tilde{\overline{a}}(\tau) = \frac{1}{2}\left(\overline{a}(\tau) + T_g\overline{a}(-\tau)\right)$$

which clearly satisfies (3.2) since $T_g^2 = 1$. On the other hand,

$$\begin{aligned} \tilde{A}_c(\tau) &= \frac{1}{2}\left(A_c(\tau) + T_g A_c(-\tau)\right) \\ &\equiv A_c(\tau) \end{aligned}$$

since $A_c(\tau)$ does satisfy (3.1). Thus, the new formal symbol $\tilde{\overline{a}}(\tau)$ also defines a suitable approximation of $A_c(\tau)$.

Lemma 3.1 *Suppose* (3.2) *holds. Then the boundary term in* (2.1) *is equal to*

$$\frac{1}{2}\,(N-P)\,\mathrm{sgn}\,(\Im\tau_0-\gamma).$$

Proof. For definiteness, consider the case $\gamma<\Im\tau_0$. Let Γ' be the line symmetric to the weight line Γ with respect to the centre τ_0. By Remark 3.4 in [FST97a] we may replace Γ by Γ' adding the sum of residues of $\mathrm{Tr}'\,\mathrm{res}\,A_c^{-1}A_c'$ in the strip between Γ and Γ'. Thus,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\mathrm{Tr}'\Big(A_c^{-1}(\tau+i\gamma-ih\gamma)A_c'(\tau+i\gamma-ih\gamma)|_{N-1}-\mathrm{Op}_{\partial M}(\overline{r}(\tau)\circ\overline{a}'(\tau)|_{N-1})\Big)\,d\tau\\
&=2\pi i\;(N-P)\\
&+\int_{-\infty}^{\infty}\mathrm{Tr}'\Big(A_c^{-1}(\tau-i\gamma+ih\gamma)A_c'(\tau-i\gamma+ih\gamma)|_{N-1}-\mathrm{Op}_{\partial M}(\overline{r}(\tau)\circ\overline{a}'(\tau)|_{N-1})\Big)\,d\tau.
\end{aligned}\tag{3.3}$$

Now, by (3.1)

$$\begin{aligned}
A_c^{-1}(-z)A_c'(-z) &= -A_c^{-1}(-z)\,\frac{d}{dz}A_c(-z)\\
&= -T_g\,A_c^{-1}(z)A_c'(z)
\end{aligned}$$

with

$$T_g\,\Psi=(g^{-1})^*\,v_1^{-1}\Psi\,v_0\,g^*,$$

and

$$\begin{aligned}
\overline{r}(-\tau)\circ\overline{a}'(-\tau) &= -\,\overline{r}(-\tau)\circ\frac{d}{d\tau}\overline{a}(-\tau)\\
&= -\,T_g\,\overline{r}(\tau)\circ\overline{a}'(\tau).
\end{aligned}$$

the latter equality being a consequence of (3.2). Thus, the last integral in (3.3) may be rewritten as

$$-\int_{-\infty}^{\infty}\mathrm{Tr}'\,T_g\Big(A_c^{-1}(\tau+i\gamma-ih\gamma)A_c'(\tau+i\gamma-ih\gamma)|_{N-1}-\mathrm{Op}_{\partial M}(\overline{r}(\tau)\circ\overline{a}'(\tau)|_{N-1})\Big)\,d\tau.$$

For the automorphism T_g we have

$$\mathrm{Tr}'\,T_g\Psi=\mathrm{Tr}'\,\Psi$$

if Ψ is a trace class operator. Thus, the integral on the right-hand side of (3.3) coincides with the integral on the left-hand side with the opposite sign, proving the lemma.

$\square$

It remains to show that

$$\mathrm{Tr}\,\rho_0(1-\overline{r}\circ\overline{a})|_N-\mathrm{Tr}\,\rho_0(1-\overline{a}\circ\overline{r})|_N=\int_{\mathbb{S}^*M}\mathrm{AS}(A).\tag{3.4}$$

To this end we construct a manifold $\widehat{M}$ (a double of M) by gluing together two copies of M with the help of the diffeomorphism g. The symmetry condition allows one to construct an operator $\widehat{A}$ on $\widehat{M}$ whose restriction to each copy of M coincides with A.

Take $t_0 \in \mathbb{R}_+$ and a closed neighbourhood $[t_0 - \varepsilon, t_0 + \varepsilon] \subset \mathbb{R}$, with $\varepsilon > 0$. Let $U \subset M$ stand for the annular neighbourhood $[t_0 - \varepsilon, t_0 + \varepsilon] \times \partial M$ on M. We throw out the points with $t > t_0 + \varepsilon$ and obtain a manifold (still denoted by M) with boundary $\{t_0 + \varepsilon\} \times \partial M$. Consider two copies M_1, M_2 of M and the corresponding annular neighbourhoods $U_1 \subset M_1$, $U_2 \subset M_2$. For $y_j = (t_j, x_j) \in U_j$ with $t_j \in [t_0 - \varepsilon, t_0 + \varepsilon]$ and $x_j \in \partial M_j$, $j = 1, 2$, we write $y_1 \sim y_2$ or $y_2 = f(y_1)$ if

$$(t_2, x_2) = (-t_1, g(x_1)). \tag{3.5}$$

Since g is an involution, (3.5) may be rewritten also in the form

$$(t_1, x_1) = (-t_2, g(x_2))$$

which means that the relation $\sim$ is symmetric. We define $\widehat{M}$ to be a disjoint union $M_1 \sqcup M_2$ with identification of the points $y_1 \in U_1$ and $y_2 \in U_2$ for which $y_1 \sim y_2$. We write

$$\widehat{M} = M_1 \sqcup M_2 / \sim .$$

Any point $y \in \widehat{M}$ is either a point $y_1 \in M_1$ or $y_2 \in M_2$, at least one of them being an interior point. This gives a C^∞ structure on $\widehat{M}$. If both y_1 and y_2 are interior points, we may use coordinate charts either in M_1 or in M_2. The transitions between them are given by the diffeomorphism (3.5).

We have embeddings $\iota_1 : M_1 \to \widehat{M}$ and $\iota_2 : M_2 \to \widehat{M}$. The annular neighbourhoods U_1, U_2 go to the same neighbourhood $\widehat{U} \subset \widehat{M}$. We don't care of orientations, the manifold M itself may be non-orientable. It is also possible to obtain a non-orientable manifold $\widehat{M}$ starting with an orientable manifold M.

Define a vector bundle $\widehat{E}^0$ over $\widehat{M}$ taking two copies of E^0 over M_1 and M_2 with the transition isomorphism $v_0 g^* \colon E^0 \to E^0$ over $\widehat{U}$. In more detail, a section $\widehat{u}(y) \in C^\infty(M_1, \widehat{E}^0)$ is defined by two sections

$$\begin{array}{rcl} u_1(y_1) & \in & C^\infty(M_1, E^0), \\ u_2(y_2) & \in & C^\infty(M_2, E^0) \end{array}$$

such that we have

$$u_2(y_2) = v_0(y_1) u_1(f(y_1)).$$

for $y_2 = f(y_1)$. The same construction gives us a bundle $\widehat{E}^1$.

Now, define an operator

$$\widehat{A} \colon C^\infty(\widehat{M}, \widehat{E}^0) \to C^\infty(\widehat{M}, \widehat{E}^1)$$

taking two copies A_1, A_2 of the operator A on M_1, M_2. The symmetry condition implies that on U_1, U_2 we have

$$\begin{aligned} A_1\left(\frac{\partial}{\partial t}\right) &= (g^{-1})^* \, v_1^{-1} A_2 \left(-\frac{\partial}{\partial t}\right) v_0 \, g^* \\ &= (f^{-1})^* \, v_1^{-1} A_2 \left(\frac{\partial}{\partial t}\right) v_0 \, f^* \\ &:= T_f \, A_2 \end{aligned} \tag{3.6}$$

which means that A_1 and A_2 obey the transition rule under the diffeomorphism $f \colon (t,x) \mapsto (-t, g(x))$ and the bundle isomorphisms $v_0 f^*$ and $v_1 f^*$ over $\widehat{U}$.

Finally, the formal symbol $\overline{a}$ approximating A and satisfying the symmetry condition (3.2) defines a pair $\overline{a_1}$, $\overline{a_2}$ of identical formal symbols on M_1, M_2 obeying the transition rule $\overline{a_1} = T_f \, \overline{a_2}$ similar to (3.6). Thus, they define a formal symbol $\widehat{\overline{a}}$ on $\widehat{M}$ approximating $\widehat{A}$.

The operator $\widehat{A}$ is elliptic and its index is equal to

$$\operatorname{ind} \widehat{A} = \widehat{\operatorname{Tr}}\,(1 - \widehat{\overline{r}} \circ \widehat{\overline{a}})|_N - \widehat{\operatorname{Tr}}\,(1 - \widehat{\overline{a}} \circ \widehat{\overline{r}})|_N,$$

where $\widehat{\operatorname{Tr}}$ means a trace on $\widehat{M}$ (see [FST97a, (1.6)]). We choose a partition of unity ρ_j, $j = 1,2$, and $\rho = \rho(t)$ with $\operatorname{supp} \rho_j \subset M_j$ and $\operatorname{supp} \rho \subset \widehat{U}$. Observe that

$$\begin{aligned} &\widehat{\operatorname{Tr}}\,\rho(1 - \widehat{\overline{r}} \circ \widehat{\overline{a}})|_N - \widehat{\operatorname{Tr}}\,\rho(1 - \widehat{\overline{a}} \circ \widehat{\overline{r}})|_N \\ &\quad = \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \rho(t) dt \int_{-\infty}^{\infty} \left(\operatorname{Tr}'\,(1 - \overline{r}(\tau) \circ \overline{a}(\tau))|_N - \operatorname{Tr}'\,(1 - \overline{a}(\tau) \circ \overline{r}(\tau))|_N\right) d\tau \\ &\quad = 0 \end{aligned}$$

since the expression in parenthesis is equal to the index of $A_c(\tau)$ which is zero for $A_c(\tau)$ is invertible. The interior terms in (2.1) may thus be written in the form

$$\operatorname{Tr} \rho_1 (1 - \overline{r} \circ \overline{a})|_N - \operatorname{Tr} \rho_1 (1 - \overline{a} \circ \overline{r})|_N = \frac{1}{2} \operatorname{ind} \widehat{A}.$$

On the other hand, by the Atiyah-Singer theorem

$$\operatorname{ind} \widehat{A} = \int_{\mathbb{S}^* \widehat{M}} \operatorname{AS}\left(\sigma_0(\widehat{A})\right).$$

The integrand vanishes over $\widehat{U}$ since the principal symbol is independent of t. The integrals over $M_1 \setminus U_1$ and $M_2 \setminus U_2$ are equal to each other, so we may write

$$\operatorname{ind} \widehat{A} = 2 \int_{\mathbb{S}^* M} \operatorname{AS}\left(\sigma_0(A)\right)$$

proving the theorem.

□

4 Some examples

As an example we first mention the Cauchy-Riemann operator in the plane with a conical point at infinity. Its conormal symbol has the form

$$A_c(\tau) = \frac{i}{2}\left(\tau + \frac{\partial}{\partial x}\right), \quad x \in \mathbb{R} \pmod{2\pi},$$

the bundles E^0 and E^1 over U being trivial one-dimensional bundles. The symmetry condition is fulfilled with

$$\begin{aligned} \tau_0 &= 0; \\ g &: x \mapsto -x; \\ v_0 &= 1; \\ v_1 &= -1. \end{aligned}$$

More general symmetric operators on two-dimensional manifolds M are considered in Example 3.5 of [FST98].

Now, let M be an even-dimensional oriented manifold with a cylindrical end. As usual, we assume that there is a Riemannian metric on M having the form $dt^2 + ds^2$ over the cylinder where ds^2 is a metric on ∂M. Suppose also that there exists an orientation-reversing involution $g : \partial M \to \partial M$ which is an isometry. The Hirzebruch operator on M is symmetric with $\tau_0 = 0$ and with isomorphisms v_0, v_1 given on the bundles of exterior forms by the differential df where $f : (t, x) \mapsto (-t, g(x))$. Since f preserves the orientation, the operators d, d^*, $*$ on two copies M_1, M_2 agree on $\widehat{U}$, thus defining the signature operator on $\widehat{M}$.

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