A Lefschetz Fixed Point Formula in the Relative Elliptic Theory

Bert-Wolfgang Schulze
Nikolai N. Tarkhanov*

Institut für Mathematik
Universität Potsdam
Postfach 60 15 53
14415 Potsdam
Germany

*Supported by the Max-Planck Gesellschaft.

Abstract

A version of the classical Lefschetz fixed point formula is proved for the cohomology of the cone of a cochain mapping of elliptic complexes. As a particular case we show a Lefschetz formula for the relative de Rham cohomology.

AMS subject classification: primary: 58G10; secondary: 58G05
Key words and phrases: elliptic complexes, relative cohomology, Lefschetz number.

Contents

1 A Brief Survey

Let M be a closed manifold and $f : M \to M$ a continuous mapping. The Lefschetz number of f is defined by $L(f) = \sum_i (-1)^i \mathrm{tr}\,(Hf)_i$, where $(Hf)_i$ denotes the induced endomorphism in the cohomology with real coefficients $H^i(M, \mathbb{R})$ and tr the trace. In 1926 Lefschetz proved that $L(f)$ is equal to the sum of the mapping degrees of $1 - f$ at fixed points of f, provided all these points are isolated (cf. [Lef26]). His argument is based on the intersection theory applied to the cycles Δ and Γ_f representing the diagonal and the graph of f in $M \times M$, respectively.

A few years later, considering simplicial mappings of finite simplicial complexes, Hopf [Hop29] proved an alternating sum formula which by simplicial approximation lead to an alternative proof of the Lefschetz formula.

The classical fixed point theorem of Lefschetz [Lef26] is easily formulated in terms of the de Rham complex. This complex, together with the well-known Dolbeault complex in complex analysis, gives rise to the important concept of an 'elliptic complex'. Elliptic complexes of pseudodifferential operators on manifolds arise in various problems of geometry and analysis rather than single elliptic operators.

In their paper [AB67], Atiyah and Bott established an analogue of the Lefschetz fixed point formula for geometric endomorphisms of elliptic complexes. The original proof of the formula in [AB67] can be considered as a generalisation of Hopf's argument. Its central point is again an alternating trace formula for endomorphisms of elliptic complexes given by pseudodifferential operators.

To state their result, let

$$\mathcal{E}(V^\cdot):\ 0 \longrightarrow \mathcal{E}(V^0) \xrightarrow{A_0} \mathcal{E}(V^1) \xrightarrow{A_1} \dots \xrightarrow{A_{N-1}} \mathcal{E}(V^N) \longrightarrow 0$$

be an elliptic complex, where V^i are complex vector bundles over M and A_i classical pseudodifferential operators of type $V^i \to V^{i+1}$ satisfying $A_{i+1}A_i = 0$. The ellipticity of $\mathcal{E}(V^\cdot)$ means that the corresponding sequence of principal symbols

$$0 \longrightarrow \pi^* V^0 \xrightarrow{\sigma(A_0)} \pi^* V^1 \xrightarrow{\sigma(A_1)} \dots \xrightarrow{\sigma(A_{N-1})} \pi^* V^N \longrightarrow 0$$

is exact in the complement of the zero section of T^*M. Here, $\pi^* V^i \to T^*M$ stands for the pull-back of the bundle V^i under the canonical mapping $\pi : T^*M \to M$. Just as in the case of the de Rham complex, the cohomology $H^i(\mathcal{E}(V^\cdot)) = \ker A_i / \mathrm{im}\, A_{i-1}$ of an elliptic complex is finite-dimensional at each step i. Suppose E is an endomorphism of the complex $\mathcal{E}(V^\cdot)$, i.e., a sequence $E_i : \mathcal{E}(V^i) \to \mathcal{E}(V^i)$ of linear mappings such that $A_i E_i = E_{i+1} A_i$. Then E preserves the spaces of cocycles and coboundaries of $\mathcal{E}(V^\cdot)$, hence after passing to quotient spaces it induces an endomorphism $(HE)_i$ of the cohomology $H^i(\mathcal{E}(V^\cdot))$, for every $i = 0, 1, \ldots, N$. As these are finite-dimensional, the traces $\mathrm{tr}\,(HE)_i$ are well-defined which yields the Lefschetz number of E by

$$L(E) = \sum_{i=0}^{N} (-1)^i \mathrm{tr}\,(HE)_i.$$

If $E = \mathrm{Id}$ is the identity endomorphism of $\mathcal{E}(V^\cdot)$, then $L(\mathrm{Id}) = \chi(\mathcal{E}(V^\cdot))$ is just the Euler characteristic of the complex $\mathcal{E}(V^\cdot)$. In particular, if $N = 1$, this becomes the index of the elliptic operator A_0. The question of how to compute $L(E)$ is therefore a generalisation of the index problem for elliptic operators. Atiyah and Bott [AB67] evaluated the Lefschetz number $L(E)$ in the case when E is a geometric endomorphism of $\mathcal{E}(V^\cdot)$. The latter is constructed via a smooth mapping f of the underlying manifold M and a family of smooth bundle homomorphisms $h_{V^i} \colon f^* V^i \to V^i$. For abbreviation, let us use the same letters h_{V^i} to designate the corresponding mappings $\mathcal{E}(f^* V^i) \to \mathcal{E}(V^i)$ of sections. An endomorphism E is said to be geometric if all E_i are of the form $E_i = h_{V^i} \circ f^*$. Then, the Atiyah-Bott formula reads

$$L(f) = \sum_{f(p)=p} \frac{\sum_{i=0}^{N} (-1)^i \mathrm{tr}\, h_{V^i}(p)}{|\det(\mathrm{Id} - df(p))|} \qquad (1.1)$$

provided f is of general position. Note that the bundle homomorphism $h_{V^i} \colon f^* V^i \to V^i$ is a family of linear mappings $h_{V^i}(p) \colon V^i_{f(p)} \to V^i_p$. Hence, at a fixed point p of f we have $V^i_{f(p)} = V^i_p$, and so $h_{V^i}(p)$ is an endomorphism of the vector space V^i_p. It follows that $\mathrm{tr}\, h_{V^i}(p)$ is well-defined.

Thus, the Atiyah-Bott formula expresses the Lefschetz number of a geometric endomorphism of an elliptic complex on a closed compact manifold via infinitesimal invariants of f and $h_{V^\cdot}$ at the fixed points of the mapping f. It is worth pointing out that formula (1.1) does not explicitly involve the pseudodifferential operators A_i. Thus it is much simpler than the Atiyah-Singer index formula. Of course the A_i are implicitly involved by the condition $A_i E_i = E_{i+1} A_i$.

New proofs of the Atiyah-Bott formula appeared in Kotake [Kot69], Toledo [Tol73], Nestke [Nes81], Bismut [Bis85] and Fedosov [Fed91].

A fixed point formula for higher-dimensional sets of fixed points was found by Gilkey in [Gil79] by means of heat equation methods.

A particular case of (1.1) is the Lefschetz fixed point formula for the Dolbeault complex which is referred to as the holomorphic Lefschetz formula. For direct constructions along more classical lines we refer the reader to Patodi [Pat73], Toledo and Tong [TT75], et al. Donnelly and Fefferman [DF86] found an analogue of the holomorphic Lefschetz formula for strictly pseudoconvex domains in $\mathbb{C}^n$ provided with the Bergman metric. This corresponds to the case of a non-compact manifold.

In the paper of Efremov [Efr88] the Atiyah-Bott fixed point formula is extended to universal coverings of a closed manifold. In the L^2-cohomology setting there are various further extensions of the Atiyah-Bott formula to non-compact manifolds by Shubin [Shu92] and by Shubin and Seifarth [SS90].

A new idea suggested by Fedosov [Fed93] is to consider endomorphisms of elliptic complexes which are induced by symplectic canonical transformations of T^*M rather than by a mapping f of the underlying manifold. Such endomorphisms can be realized on sections of vector bundles as Fourier integral operators obtained by quantising these symplectic canonical transformations. In the case of classical Hamiltonian flows $t : T^*M \to T^*M$, Fedosov showed an asymptotic expansion of the Lefschetz number as $\hbar \to 0$, in terms of fixed points of t. Later on, Sternin and Shatalov [SS94] gave an exposition of this result in the context of rather general symplectic canonical transformations of the cotangent bundle.

Brenner and Shubin [BS81] extended the Atiyah-Bott formula to elliptic complexes on a compact smooth manifold with boundary whose differentials are operators in Boutet de Monvel's algebra [BdM71]. We also mention an infinitesimal version of the classical Lefschetz formula for manifolds with boundary by Arnold [Arn79].

The aim of this paper is to extend the Atiyah-Bott formula to the case of relative cohomology. The motivation of this consists in the following. Let M and S be two C^∞ compact closed manifolds and $F\colon S \to M$ a differentiable mapping. The 'pull-back' $F^\sharp$ under F gives us a cochain mapping of the de Rham complex on M to that on S, namely, $F^\sharp : \mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}M) \to \mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}S)$. The cone $\mathcal{C}^\cdot$ of this cochain mapping is said to be the *cone* of F (cf. Dold [Dol72]). The complex $\mathcal{C}^\cdot$ is easily seen to be Fredholm, i.e., it bears a finite-dimensional cohomology $H^i(\mathcal{C}^\cdot)$, for each i. Moreover, $H^i(\mathcal{C}^\cdot)$ is naturally isomorphic to the relative cohomology of the pair $(M, F(S))$ with coefficients in $\mathbb{C}$, provided that F is an embedding. We now assume that f is a smooth mapping of the pair (M, S), i.e., $f = (f_M, f_S)$ where f_M and f_S are smooth mappings of M and S, respectively. If the diagram

$$\begin{array}{ccc} S & \xrightarrow{F} & M \\ \downarrow{\scriptstyle f_S} & & \downarrow{\scriptstyle f_M} \\ S & \xrightarrow{F} & M \end{array}$$

commutes, then f has a natural lift to the complex $\mathcal{C}^\cdot$, the lift being given by $f_M^\sharp \oplus f_S^\sharp$. Denote by $L(f)$ the Lefschetz number of this endomorphism of $\mathcal{C}^\cdot$. By the above, in case F is an embedding $L(f)$ coincides with the Lefschetz number in the relative cohomology of $(M, F(S))$ induced by f_M. This latter is well-defined because $F(S)$ is invariant under the mapping f_M. For the de Rham complex, our result reads

$$L(f) = L(f_M) - L(f_S) \tag{1.2}$$

which does not explicitly involve F. More generally, we consider two elliptic complexes $\mathcal{E}(V^\cdot)$ and $\mathcal{E}(W^\cdot)$ on M and S, respectively. To each cochain mapping $T\colon \mathcal{E}(V^\cdot) \to \mathcal{E}(W^\cdot)$ there corresponds a new complex $\mathcal{C}^\cdot$ called the *cone* of T (cf. *ibid.*). If T is of finite order relative to the scales of Sobolev spaces on M and S, then $\mathcal{C}^\cdot$ proves to be Fredholm. Such complexes of pseudodifferential operators seem to be first investigated by Dynin [Dyn72] in case S is a submanifold of M. In parallel to the relative de Rham cohomology, the corresponding theory in the case of general elliptic complexes on M and S is referred to as the 'relative elliptic theory'. Yet another example of great importance in geometry is the cone of a holomorphic mapping of two complex manifolds $F: S \to M$. In this latter case both $\mathcal{E}(V^\cdot)$ and $\mathcal{E}(W^\cdot)$ are the Dolbeault complexes on M and S, respectively. We prove formula (1.2) in the context of relative elliptic theory and show further refinements of this.

2 An Operator Algebra

We begin by introducing an operator algebra which contains the differentials of the cones of the cochain mappings in question.

Let M and S be differentiable compact closed manifolds and V and W be differentiable $\mathbb{C}$-vector bundles over M and S, respectively. Denote by $W \boxtimes V'$ the external tensor product of the bundles W and V', V' being the dual of V. This is the differentiable $\mathbb{C}$-vector bundle over $S \times M$ whose fibre over a point (y, x) is $W_y \otimes V'_x$, and whose transition matrices are tensor products of the transition matrices for W and V'. If $S = M$, the restriction of $W \boxtimes V'$ to the diagonal gives the internal tensor product $W \otimes V'$.

Sections $T(y,x) \in \mathcal{E}'(W \boxtimes V')$ are said to be *kernels* of type $V \to W$ on $S \times M$. Each kernel $T(y,x) \in \mathcal{E}'(W \boxtimes V')$ defines a continuous linear operator $T\colon \mathcal{E}(V) \to \mathcal{E}'(W)$ by the formula

$$\langle g, Tu \rangle_S = \langle T(y,x), g(y) \otimes u(x) \rangle_{S \times M}\,, \tag{2.1}$$

for $g \in \mathcal{E}(W')$ and $u \in \mathcal{E}(V)$. The Kernel Theorem of Schwartz in its global formulation asserts that this correspondence is a topological isomorphism.

We need a more delicate characterisation of kernels on $S \times M$. It is based on the observation that $\mathcal{E}'(W \boxtimes V') = \mathcal{E}'(W) \otimes_\pi \mathcal{E}'(V')$, where $\otimes_\pi$ stands for the projective tensor product. The kernels $T(y,x) \in \mathcal{E}(W) \otimes_\pi \mathcal{E}'(V')$ are said to be *regular*. They are characterised by the property that the corresponding operator T maps $\mathcal{E}(V)$ into $\mathcal{E}(W)$ continuously. The kernels $T(y,x) \in \mathcal{E}'(W) \otimes_\pi \mathcal{E}(V')$ are said to be *continuable*. Such kernels are characterised by the property that the corresponding operator T extends to a continuous mapping $\mathcal{E}'(V) \to \mathcal{E}'(W)$. A kernel $T(y,x)$ is said to be *biregular* if it is both regular and continuable.

For an operator $T: \mathcal{E}(V) \to \mathcal{E}'(W)$, let $T'\colon \mathcal{E}(W') \to \mathcal{E}'(V')$ stand for the transposed operator. By identifying the second dual of a bundle with the bundle itself it is easy to see that the kernel $T'(x,y)$ of the transposed operator is in $\mathcal{E}'(V' \boxtimes W)$. Moreover, a kernel $T(y,x)$ is regular if and only if the kernel $T'(x,y)$ is continuable, and inversely.

Later on we will use the same terminology both for operators and their kernels. This will not lead to misunderstandings in view of (2.1).

Example 2.1 Any pseudodifferential operator between sections of vector bundles over a differentiable compact closed manifold is a biregular operator.

□

A pseudodifferential operator followed by restriction to a proper submanifold fails to be a biregular operator while being regular. What is surviving under such a composition is the order of an operator with respect to the scale of Sobolev spaces on a manifold.

Definition 2.2 *An operator* $T : \mathcal{E}(V) \to \mathcal{E}'(W)$ *is said to be of finite order if there are real numbers* o_T *and* s_T *such that* T *extends to a continuous mapping* $H^s(V) \to H^{s-o_T}(W)$ *for all* $s \geq s_T$.

It follows from the Sobolev Embedding Theorem that each operator of finite order is regular.

Having disposed of this preliminary step, let us dwell upon a description of operators in our algebra. They are of the form

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ T & B \end{pmatrix}, \tag{2.2}$$

where A is a classical pseudodifferential operator of type $V \to \tilde{V}$ on M, T is an operator of type $V \to \tilde{W}$ and of finite order, and B is a classical pseudodifferential operator of type $W \to \tilde{W}$ on S.

Lemma 2.3 *Given any* $s, t \in \mathbb{R}$ *with* s *large enough, each operator of the form* (2.2) *induces a continuous linear mapping*

$$\mathcal{A} : \begin{matrix} H^s(V) \\ \oplus \\ H^t(W) \end{matrix} \to \begin{matrix} H^{s-o_A}(\tilde{V}) \\ \oplus \\ H^{t-o_B}(\tilde{W}) \end{matrix}. \tag{2.3}$$

Proof. Indeed, for $u_1 \in H^s(V)$ and $u_2 \in H^t(W)$, we have

$$\mathcal{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} Au_1 \\ Tu_1 + Bu_2 \end{pmatrix}.$$

From the continuity properties of pseudodifferential operators it follows that $Au_1 \in H^{s-o_A}(\tilde{V})$ and $Bu_2 \in H^{t-o_B}(\tilde{W})$. Since $Tu_1 \in H^{s-o_T}(\tilde{W})$ for $s \geq s_T$, we can assert that $Tu_1 + Bu_2 \in H^{t-o_B}(\tilde{W})$ provided $s - o_T \geq t - o_B$. This completes the proof. □

In particular, $\mathcal{A}$ restricts to a continuous mapping between the spaces of C^∞ sections of the corresponding bundles.

If S is a submanifold of M, then the entry T has the meaning of a *trace operator* in parallel to the operators in Boutet de Monvel's theory (cf. [BdM71]).

Lemma 2.4 *The composition of any two operators of the form* (2.2) *is again of the same form.*

Proof. Indeed,

$$\begin{pmatrix} \tilde{A} & 0 \\ \tilde{T} & \tilde{B} \end{pmatrix} \begin{pmatrix} A & 0 \\ T & B \end{pmatrix} = \begin{pmatrix} \tilde{A}A & 0 \\ \tilde{T}A + \tilde{B}T & \tilde{B}B \end{pmatrix}$$

provided the bundles in question allow compositions of the operators. Since both pseudodifferential operators and operators of finite order survive under the composition, the lemma follows.

□

In the sequel, we write $\mathrm{Alg}(V, \tilde{V}; W, \tilde{W})$ for the resulting "operator algebra."

Definition 2.5 *An operator* $\mathcal{A} \in \mathrm{Alg}(V, \tilde{V}; W, \tilde{W})$ *is said to be* elliptic *if its diagonal elements A and B are elliptic pseudodifferential operators on M and S, respectively.*

The main result of this section is that the ellipticity of $\mathcal{A}$ is equivalent to the Fredholm property of the mapping (2.3).

Lemma 2.6 *An operator* $\mathcal{A} \in \mathrm{Alg}(V, \tilde{V}; W, \tilde{W})$ *is elliptic if and only if the mapping* (2.3) *is Fredholm for each $s, t \in \mathbb{R}$ with $s \gg t$.*

Proof. It is immediately seen that for the mapping (2.3) to be Fredholm it is necessary and sufficient that both the mappings

$$\begin{array}{rl} A: & H^s(V) \to H^{s-o_A}(\tilde{V}), \\ B: & H^t(W) \to H^{t-o_B}(\tilde{W}) \end{array}$$

be Fredholm. As for pseudodifferential operators on a closed manifold the ellipticity is equivalent to the Fredholm property in Sobolev spaces, the lemma follows.

□

Under ellipticity it is easy to show an explicit parametrix construction for $\mathcal{A}$. To this end we fix pseudodifferential parametrices P and Q for the diagonal elements of the matrix $\mathcal{A}$, i.e.,

$$\begin{array}{rcl} P & \in & \Psi_{\mathrm{cl}}^{-o_A}(M; \tilde{V}, V), \\ Q & \in & \Psi_{\mathrm{cl}}^{-o_B}(S; \tilde{W}, W) \end{array}$$

such that

$$\begin{array}{rcl} PA &=& \mathrm{Id}, \\ AP &=& \mathrm{Id} \end{array} \quad \text{and} \quad \begin{array}{rcl} QB &=& \mathrm{Id}, \\ BQ &=& \mathrm{Id} \end{array}$$

modulo smoothing operators on M and S, respectively. Set

$$\mathcal{P} = \begin{pmatrix} P & 0 \\ -QTP & Q \end{pmatrix}. \tag{2.4}$$

Theorem 2.7 *As defined by* (2.4)*, the operator* $\mathcal{P}$ *belongs to the algebra* $\mathrm{Alg}(\tilde{V}, V; \tilde{W}, W)$ *and satisfies both* $\mathcal{P}\mathcal{A} = \mathrm{Id}$ *and* $\mathcal{A}\mathcal{P} = \mathrm{Id}$ *up to trace class operators.*

Proof. In fact,

$$\begin{array}{rcl} \mathcal{P}\mathcal{A} &=& \mathrm{Id} - \mathcal{S}_0, \\ \mathcal{A}\mathcal{P} &=& \mathrm{Id} - \mathcal{S}_1, \end{array}$$

where

$$\begin{array}{rcl} \mathcal{S}_0 &=& \begin{pmatrix} \mathrm{Id} - PA & 0 \\ -QT(\mathrm{Id} - PA) & \mathrm{Id} - QB \end{pmatrix}, \\ \mathcal{S}_1 &=& \begin{pmatrix} \mathrm{Id} - AP & 0 \\ -(\mathrm{Id} - BQ)TP & \mathrm{Id} - BQ \end{pmatrix}. \end{array}$$

All the entries of the matrices $\mathcal{S}_0$ and $\mathcal{S}_1$ are smoothing operators, with the exception of $O = -(\mathrm{Id} - BQ)TP$. On the other hand, this latter operator extends to a continuous linear mapping $H^{s-o_A}(\tilde{V}) \to H^{t-o_B}(\tilde{W})$, for each $s \geq s_T$. Since O actually maps $H^{s-o_A}(\tilde{V})$ to $\mathcal{E}(\tilde{W})$, the operator $O : H^{s-o_A}(\tilde{V}) \to H^{t-o_B}(\tilde{W})$ can be represented as the composition of the bounded operator $O : H^{s-o_A}(\tilde{V}) \to H^{t'}(\tilde{W})$ and the embedding $H^{t'}(\tilde{W}) \hookrightarrow H^{t-o_B}(\tilde{W})$, where $t' \geq t - o_B$. We next observe that the embedding $H^{t'}(\tilde{W}) \hookrightarrow H^{t-o_B}(\tilde{W})$ is of trace class for t' large enough (cf. for instance Maurin [Mau67]). Hence it follows that $O : H^{s-o_A}(\tilde{V}) \to H^{t-o_B}(\tilde{W})$ is a trace class operator if $s \geq s_T$. Thus, both

$$\mathcal{S}_0 : \begin{array}{c} H^s(V) \\ \oplus \\ H^t(W) \end{array} \to \begin{array}{c} H^s(V) \\ \oplus \\ H^t(W) \end{array},$$
$$\mathcal{S}_1 : \begin{array}{c} H^{s-o_A}(\tilde{V}) \\ \oplus \\ H^{t-o_B}(\tilde{W}) \end{array} \to \begin{array}{c} H^{s-o_A}(\tilde{V}) \\ \oplus \\ H^{t-o_B}(\tilde{W}) \end{array}$$

are trace class operators, provided s is large enough. This is our claim.

□

3 Relative Elliptic Theory

Here we deal with elliptic complexes of operators in the algebra introduced in the preceding section.

Namely, let $\mathcal{E}(V^{\cdot})$ and $\mathcal{E}(W^{\cdot})$ be two complexes of pseudodifferential operators on the manifolds M and S and let A and B stand for the differentials of these complexes, respectively. Without loss of generality we may assume that the complexes are of the same length N, for if not, we complete one of them by zero bundles.

By a *cochain mapping* $T\colon \mathcal{E}(V^{\cdot}) \to \mathcal{E}(W^{\cdot})$ is understood any sequence of continuous linear mappings $T_i\colon \mathcal{E}(V^i) \to \mathcal{E}(W^i)$ satisfying $T_{i+1}A_i = B_i T_i$ for all $i = 0, 1, \ldots, N-1$. This just amounts to saying that the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{E}(V^0) & \xrightarrow{A_0} & \mathcal{E}(V^1) & \xrightarrow{A_1} & \ldots & \xrightarrow{A_{N-1}} & \mathcal{E}(V^N) & \longrightarrow & 0 \\ & & \Big\downarrow T_0 & & \Big\downarrow T_1 & & & & \Big\downarrow T_N & & \\ 0 & \longrightarrow & \mathcal{E}(W^0) & \xrightarrow{B_0} & \mathcal{E}(W^1) & \xrightarrow{B_1} & \ldots & \xrightarrow{B_{N-1}} & \mathcal{E}(W^N) & \longrightarrow & 0 \end{array}$$

commutes.

For a cochain mapping $T\colon \mathcal{E}(V^{\cdot}) \to \mathcal{E}(W^{\cdot})$, the *cone* of T is defined to be

$$\mathcal{C}^{\cdot}:\ 0 \longrightarrow \begin{array}{c} \mathcal{E}(V^0) \\ \oplus \\ \mathcal{E}(W^{-1}) \end{array} \xrightarrow{\mathcal{A}_0} \begin{array}{c} \mathcal{E}(V^1) \\ \oplus \\ \mathcal{E}(W^0) \end{array} \xrightarrow{\mathcal{A}_1} \ldots \xrightarrow{\mathcal{A}_N} \begin{array}{c} \mathcal{E}(V^{N+1}) \\ \oplus \\ \mathcal{E}(W^N) \end{array} \longrightarrow 0, \qquad (3.1)$$

where

$$\mathcal{A}_i = \begin{pmatrix} -A_i & 0 \\ T_i & B_{i-1} \end{pmatrix},$$

for $i = 0, 1, \ldots, N$ (cf. Dold [Dol72]). Recall that both V^i and W^i vanish unless $i = 0, 1, \ldots, N$.

In the case where M is a differentiable compact manifold with boundary and S the boundary of M, complexes (3.1) were first investigated by Dynin [Dyn72].

From now on we make a standing assumption on the cochain mappings $T : \mathcal{E}(V^{\cdot}) \to \mathcal{E}(W^{\cdot})$ under consideration. Namely, it is required that all the components T_i of T are operators of finite order. Hence it follows that the differentials $\mathcal{A}_i$ in (3.1) belong to the algebras $\mathrm{Alg}(V^i, V^{i+1}; W^{i-1}, W^i)$. We now invoke Lemma 2.3 to conclude that, for each $s, t \in \mathbb{R}$ with s large enough, complex (3.1) extends to a complex of continuous linear mappings in Hilbert spaces

$$0 \longrightarrow \begin{array}{c} H^{s_0}(V^0) \\ \oplus \\ H^{t_{-1}}(W^{-1}) \end{array} \xrightarrow{\mathcal{A}_0} \begin{array}{c} H^{s_1}(V^1) \\ \oplus \\ H^{t_0}(W^0) \end{array} \xrightarrow{\mathcal{A}_1} \ldots \xrightarrow{\mathcal{A}_N} \begin{array}{c} H^{s_{N+1}}(V^{N+1}) \\ \oplus \\ H^{t_N}(W^N) \end{array} \longrightarrow 0, \qquad (3.2)$$

where

$$\begin{aligned} s_i &= s - o_{A_0} - \ldots - o_{A_{i-1}}, \\ t_i &= t - o_{B_0} - \ldots - o_{B_{i-1}}. \end{aligned}$$

Definition 3.1 *Complex* (3.1) *is said to be* elliptic *if both* $\mathcal{E}(V^{\cdot})$ *and* $\mathcal{E}(W^{\cdot})$ *are elliptic complexes on* M *and* S*, respectively.*

The main result on an elliptic complex $\mathcal{C}^{\cdot}$ is that such a complex is Fredholm, i.e., has a finite-dimensional cohomology $H^i(\mathcal{C}^{\cdot})$. Moreover, this cohomology is independent on the Sobolev spaces at which it is evaluated. This is stated by our next result.

Lemma 3.2 *A complex* $\mathcal{C}^{\cdot}$ *is elliptic if and only if the complex* (3.2) *is Fredholm for each* $s, t \in \mathbb{R}$ *with* s *large enough.*

Proof. An easy computation shows that for the complex (3.2) to be Fredholm it is necessary and sufficient that both the complexes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^{s_0}(V^0) & \xrightarrow{A_0} & H^{s_1}(V^1) & \xrightarrow{A_1} & \ldots & \xrightarrow{A_{N-1}} & H^{s_N}(V^N) & \longrightarrow & 0 \\ 0 & \longrightarrow & H^{t_0}(W^0) & \xrightarrow{B_0} & H^{t_1}(W^1) & \xrightarrow{B_1} & \ldots & \xrightarrow{B_{N-1}} & H^{t_N}(W^N) & \longrightarrow & 0 \end{array}$$

be Fredholm. Since for complexes of pseudodifferential operators on a compact closed manifold the ellipticity is equivalent to the Fredholm property in Sobolev spaces, the proof is complete.

□

We complete Lemma 3.2 with an explicit parametrix construction for the complex (3.1). To this end we fix pseudodifferential parametrices P and Q for the complexes $\mathcal{E}(V^{\cdot})$ and $\mathcal{E}(W^{\cdot})$, i.e.,

$$\begin{aligned} P_i &\in \Psi_{\mathrm{cl}}^{-o_{A_{i-1}}}(M; V^i, V^{i-1}), \\ Q_i &\in \Psi_{\mathrm{cl}}^{-o_{B_{i-1}}}(S; W^i, W^{i-1}) \end{aligned}$$

such that

$$\begin{aligned} P_{i+1}A_i + A_{i-1}P_i &= \mathrm{Id} - R_i, \\ Q_{i+1}B_i + B_{i-1}Q_i &= \mathrm{Id} - S_i \end{aligned}$$

for all $i = 0, 1, \ldots, N$, where R_i and S_i are smoothing operators on M and S, respectively. Set

$$\mathcal{P}_i = \begin{pmatrix} -P_i & 0 \\ Q_{i-1}T_{i-1}P_i & Q_{i-1} \end{pmatrix}. \tag{3.3}$$

Theorem 3.3 *As defined by* (3.3)*, the operators* $\mathcal{P}_i$ *belong to the algebras* $\mathrm{Alg}(V^i, V^{i-1}; W^{i-1}, W^{i-2})$ *and satisfy*

$$\mathcal{P}_{i+1}\mathcal{A}_i + \mathcal{A}_{i-1}\mathcal{P}_i = \mathrm{Id} - \mathcal{S}_i \tag{3.4}$$

for each $i = 0, 1, \ldots, N+1$*. Moreover,* $\mathcal{S}_i$ *are trace class operators on the Sobolev spaces involved in* (3.2).

Proof. Indeed, for a fixed $i = 0, 1, \ldots, N+1$, equality (3.4) is fulfilled with

$$\mathcal{S}_i = \begin{pmatrix} R_i & 0 \\ Q_i T_i R_i - S_{i-1} T_{i-1} P_i & S_{i-1} \end{pmatrix},$$

as is easy to check. Analysis similar to that in the proof of Theorem 2.7 shows that $\mathcal{S}_i$ is a trace class operator on the space

$$\begin{array}{c} H^s(V^i) \\ \oplus \\ H^t(W^{i-1}) \end{array}$$

provided $s \geq s_{T_{i-1}} - o_{A_{i-1}}$. This is the desired conclusion.

□

As but one consequence of this theorem we conclude that the cohomology of complex (3.2) is isomorphic to that of complex (3.1) if s is sufficiently large. In particular, this cohomology is independent of s and t.

4 A Lefschetz Formula

In this section we introduce a Lefschetz fixed point theorem for the pair of manifolds (M, S).

Let $\mathcal{C}^\cdot$ be the cone of a cochain mapping of elliptic complexes on M and S as in (3.1). By an *endomorphism* of $\mathcal{C}^\cdot$ we mean a cochain mapping $\mathcal{E} : \mathcal{C}^\cdot \to \mathcal{C}^\cdot$, i.e., a family $\mathcal{E} = (\mathcal{E}_i)$ of linear mappings $\mathcal{E}_i : \mathcal{C}^i \to \mathcal{C}^i$ such that $\mathcal{E}_{i+1}\mathcal{A}_i = \mathcal{A}_i\mathcal{E}_i$ for all $i = 0, 1, \ldots, N$. Then $\mathcal{E}$ induces an endomorphism $(HE)_i$ of the cohomology $H^i(\mathcal{C}^\cdot)$, for every $i = 0, 1, \ldots, N+1$. As described above, these are finite-dimensional vector spaces, and so the traces $\mathrm{tr}\,(H\mathcal{E})_i$ are well-defined. We introduce the *Lefschetz number* of $\mathcal{E}$ by

$$L(\mathcal{E}) = \sum_{i=0}^{N+1} (-1)^i \mathrm{tr}\,(H\mathcal{E})_i.$$

We restrict our attention to those endomorphisms of $\mathcal{C}^\cdot$ which obey the shape of operators in our algebra. In other words, they are required to be of the form

$$\mathcal{E}_i = \begin{pmatrix} E_i & 0 \\ H_i & F_{i-1} \end{pmatrix} \tag{4.1}$$

for $i = 0, 1, \ldots, N+1$, where E_i is a regular operator of type $V^i \to V^i$ on M, H_i is a regular operator of type $V^i \to W^{i-1}$, and F_{i-1} is a regular operator of type $W^{i-1} \to W^{i-1}$ on S.

Lemma 4.1 *As defined by* (4.1)*, the family* $(\mathcal{E}_i)$ *is an endomorphism of* $\mathcal{C}^\cdot$ *if and only if the family* (E_i) *is an endomorphism of* $\mathcal{E}(V^\cdot)$*, the family* (F_i) *is an endomorphism of* $\mathcal{E}(W^\cdot)$*, and* $F_i T_i - T_i E_i = H_{i+1} A_i + B_{i-1} H_i$ *for each* $i = 0, 1, \ldots, N$.

Proof. Indeed,

$$\begin{aligned} \mathcal{E}_{i+1}\mathcal{A}_i &= \begin{pmatrix} -E_{i+1}A_i & 0 \\ -H_{i+1}A_i + F_iT_i & F_iB_{i-1} \end{pmatrix}, \\ \mathcal{A}_i\mathcal{E}_i &= \begin{pmatrix} -A_iE_i & 0 \\ T_iE_i + B_{i-1}H_i & B_{i-1}F_{i-1} \end{pmatrix} \end{aligned}$$

showing the lemma.

□

The lemma shows that each endomorphism of $\mathcal{C}^{\cdot}$ of the form (4.1) induces endomorphisms $E = (E_i)$ and $F = (f_i)$ of the complexes $\mathcal{E}(V^{\cdot})$ and $\mathcal{E}(W^{\cdot})$, respectively. Since these complexes are Fredholm, the Lefschetz numbers $L(E)$ of E and $L(F)$ of F are well-defined. Now our main result reads as follows.

Theorem 4.2 *Suppose that $\mathcal{E}$ is an endomorphism of $\mathcal{C}^{\cdot}$ of the form (4.1). Then,*

$$L(\mathcal{E}) = L(E) - L(F).$$

Proof. Consider the sequence

$$\begin{array}{cccccccc} 0 & \longrightarrow & H^0(\mathcal{C}^{\cdot}) & \xrightarrow{\pi} & H^0(\mathcal{E}(V^{\cdot})) & \xrightarrow{T} & H^0(\mathcal{E}(W^{\cdot})) & \xrightarrow{\delta} \\ & \longrightarrow & H^1(\mathcal{C}^{\cdot}) & \xrightarrow{\pi} & H^1(\mathcal{E}(V^{\cdot})) & \xrightarrow{T} & H^1(\mathcal{E}(W^{\cdot})) & \xrightarrow{\delta} & \ldots \\ & \longrightarrow & H^i(\mathcal{C}^{\cdot}) & \xrightarrow{\pi} & H^i(\mathcal{E}(V^{\cdot})) & \xrightarrow{T} & H^i(\mathcal{E}(W^{\cdot})) & \xrightarrow{\delta} & \ldots, \end{array} \tag{4.2}$$

where π, T and δ are defined by by

$$\begin{array}{llllll} \pi\colon & \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & \mathrm{mod}\ B^i(\mathcal{C}^{\cdot}) & \mapsto & u_1 & \mathrm{mod}\ B^i(\mathcal{E}(V^{\cdot})), \\ T\colon & u & \mathrm{mod}\ B^i(\mathcal{E}(V^{\cdot})) & \mapsto & T_iu & \mathrm{mod}\ B^i(\mathcal{E}(W^{\cdot})), \\ \delta\colon & f & \mathrm{mod}\ B^i(\mathcal{E}(W^{\cdot})) & \mapsto & \begin{pmatrix} 0 \\ f \end{pmatrix} & \mathrm{mod}\ B^{i+1}(\mathcal{C}^{\cdot}), \end{array}$$

for $\binom{u_1}{u_2} \in Z^i(\mathcal{C}^{\cdot})$, $u \in Z^i(\mathcal{E}(V^{\cdot}))$ and $f \in Z^i(\mathcal{E}(W^{\cdot}))$. Here, Z^i and B^i stand for the spaces of cocycles and coboundaries at the step i in the corresponding cochain complex. The sequence (4.2) is exact, as is a simple matter to see.

The endomorphism $\mathcal{E}$ induces an endomorphism of (4.2). Namely, the following diagram commutes

$$\begin{array}{ccccccc} H^i(\mathcal{C}^{\cdot}) & \xrightarrow{\pi} & H^i(\mathcal{E}(V^{\cdot})) & \xrightarrow{T} & H^i(\mathcal{E}(W^{\cdot})) & \xrightarrow{\delta} & H^{i+1}(\mathcal{C}^{\cdot}) \\ \big\downarrow \mathcal{E}_i & & \big\downarrow E_i & & \big\downarrow F_i & & \big\downarrow \mathcal{E}_{i+1} \\ H^i(\mathcal{C}^{\cdot}) & \xrightarrow{\pi} & H^i(\mathcal{E}(V^{\cdot})) & \xrightarrow{T} & H^i(\mathcal{E}(W^{\cdot})) & \xrightarrow{\delta} & H^{i+1}(\mathcal{C}^{\cdot}) \end{array} \tag{4.3}$$

for each $i = 0, 1, \ldots, N$, as is easy to check by Lemma 4.1.

From Lemma 3.2 and the remark after the proof of Theorem 3.3 it follows that all the vector spaces occurring in (4.2) are finite-dimensional.

Since the complex (4.2) is exact, we may invoke the algebraic *alternating sum formula* to conclude that the Lefschetz number of the endomorphism (4.3) of this complex is equal to zero. On the other hand, this number is easily seen to be $L(\mathcal{E}) - L(E) + L(F)$, whence the theorem follows.

□

The proof above is purely algebraic. We have used only the fact that the complex $\mathcal{C}^{\cdot}$ is Fredholm, i.e., bears a finite-dimensional cohomology.

In particular, setting $\mathcal{E}$ to be the identity endomorphism of $\mathcal{C}^{\cdot}$, we deduce that $\chi(\mathcal{C}^{\cdot}) = \chi(\mathcal{E}(V^{\cdot})) - \chi(\mathcal{E}(W^{\cdot}))$ (cf. Proposition 5 in [RS82, 3.2.3.1]).

5 Geometric Endomorphisms

In this section we are interested in evaluating the Lefschetz number for geometric endomorphisms of the complex $\mathcal{C}^{\cdot}$.

Assume that f is a differentiable mapping of the underlying pair (M, S), i.e., $f = (f_M, f_S)$ where f_M and f_S are differentiable mappings of the manifolds M and S, respectively. Let we be also given smooth bundle homomorphisms

$$\begin{array}{rcl} h_{V^i} : & f_M^* V^i & \to & V^i, \\ h_{W^i} : & f_S^* W^i & \to & W^i, \end{array}$$

for $i = 0, 1, \ldots, N$. Using the same notation for the induced mappings on sections, we can then define linear mappings $\mathcal{E}_i : \mathcal{C}^i \to \mathcal{C}^i$ as the compositions

$$\begin{array}{c} \mathcal{E}(V^i) \\ \oplus \\ \mathcal{E}(W^{i-1}) \end{array} \stackrel{f_M^* \oplus f_S^*}{\longrightarrow} \begin{array}{c} \mathcal{E}(f_M^* V^i) \\ \oplus \\ \mathcal{E}(f_S^* W^{i-1}) \end{array} \stackrel{h_{V^i} \oplus h_{W^{i-1}}}{\longrightarrow} \begin{array}{c} \mathcal{E}(V^i) \\ \oplus \\ \mathcal{E}(W^{i-1}) \end{array} . \qquad (5.1)$$

Thus,

$$\mathcal{E}_i = \begin{pmatrix} h_{V^i} f_M^* & 0 \\ 0 & h_{W^{i-1}} f_S^* \end{pmatrix}$$

meets condition (4.1), for each $i = 0, 1, \ldots, N+1$. It is worth pointing out that $f_M^* u_1(x) \oplus f_S^* u_2(y) \in V^i_{f_M(x)} \oplus W^{i-1}_{f_S(y)}$, but the bundle homomorphism $h_{V^i \oplus W^{i-1}}$ takes us back to $V^i_x \oplus W^{i-1}_y$.

If further $\mathcal{E}_{i+1} \mathcal{A}_i = \mathcal{A}_i \mathcal{E}_i$, then the family $(\mathcal{E}_i)$ defines an endomorphism of the complex $\mathcal{C}^{\cdot}$. In this case we say that the mapping f has a *lift* (i.e., $\mathcal{E}$) to the complex $\mathcal{C}^{\cdot}$. Endomorphisms of this type we call *geometric* endomorphisms of $\mathcal{C}^{\cdot}$.

The components of a geometric endomorphism are operators of finite order, as is easy to see.

If E is a geometric endomorphism of $\mathcal{C}^*$ defined by a differentiable mapping $f = (f_M, f_S)$ of the pair (M, S), we write $L(f)$ instead of $L(E)$.

Lemma 4.1 shows that if the mapping $f = (f_M, f_S)$ has a lift to the cone $\mathcal{C}^{\cdot}$ of a cochain mapping $T : \mathcal{E}(V^{\cdot}) \to \mathcal{E}(W^{\cdot})$, then the components f_M and f_S have lifts to the complexes $\mathcal{E}(V^{\cdot})$ and $\mathcal{E}(W^{\cdot})$, respectively. Let us denote by $L(f_M)$ and $L(f_S)$ the Lefschetz numbers of the corresponding geometric endomorphisms of $\mathcal{E}(V^\cdot)$ and $\mathcal{E}(W^\cdot)$.

Corollary 5.1 *Suppose that E is a geometric endomorphism of $\mathcal{C}^\cdot$ defined by a mapping $f = (f_M, f_S)$ of the pair (M, S). Then,*

$$L(f) = L(f_M) - L(f_S).$$

Proof. This is a very particular case of Theorem 4.2.

□

For a mapping $m \colon \sigma \to \sigma$, we denote by $\mathrm{Fix}(m, \sigma)$ the set of all fixed points of m on σ.

In particular, let $f = (f_M, f_S)$ be a mapping of the pair (M, S) such that both f_M and f_S have only simple fixed points. If f has a lift to $\mathcal{C}^\cdot$, then

$$L(f) = \sum_{p \in \mathrm{Fix}(f_M, M)} \frac{\sum\limits_{i=0}^{N} (-1)^i \mathrm{tr}\, h_{V^i}(p)}{|\det(\mathrm{Id} - df_M(p))|} - \sum_{p \in \mathrm{Fix}(f_S, S)} \frac{\sum\limits_{i=0}^{N} (-1)^i \mathrm{tr}\, h_{W^i}(p)}{|\det(\mathrm{Id} - df_S(p))|}, \tag{5.2}$$

as follows from Corollary 5.1 and formula (1.1).

6 Relative de Rham Cohomology

In this section we indicate how formula (5.2) may be used to derive an explicit formula for the Lefschetz number in relative de Rham cohomology.

Let M be a smooth compact closed manifold of dimension n and S be a submanifold of M of dimension q. For simplicity we assume that M is orientable.

For $i \in \mathbb{Z}$, we denote by $\Lambda^i T^*_{\mathbb{C}} M$ the complexified bundle of exterior forms of degree i over M. These bundles are non-zero only for $i = 0, 1, \ldots, n$. They fit together to form a complex $\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} M)$ on M whose differential is given by the exterior derivative on differential forms. This complex is referred to as the *de Rham complex* on M and is known to be elliptic.

Similarly, we have the de Rham complex $\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} S)$ on S. The length of this latter is actually equal to $q < n$. However, we may complete it by the zero bundles $\Lambda^{q+1} T^*_{\mathbb{C}} S, \ldots, \Lambda^n T^*_{\mathbb{C}} S$ thus arriving at a complex of length n.

Let ι stand for the embedding $S \hookrightarrow M$. Thus, ι is a differentiable mapping and we denote by $\iota^\sharp$ the corresponding 'pull-back' operator on differential forms. Then $\iota^\sharp$ is well known to be a cochain mapping of the complexes $\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} M) \to \mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} S)$. The cone of this mapping is

$$\mathcal{C}^\cdot : \; 0 \longrightarrow \begin{array}{c} \mathcal{E}(M) \\ \oplus \\ 0 \end{array} \stackrel{\mathcal{A}_0}{\longrightarrow} \begin{array}{c} \mathcal{E}(\Lambda^1 T^*_{\mathbb{C}} M) \\ \oplus \\ \mathcal{E}(S) \end{array} \stackrel{\mathcal{A}_1}{\longrightarrow} \ldots \stackrel{\mathcal{A}_{n-1}}{\longrightarrow} \begin{array}{c} \mathcal{E}(\Lambda^n T^*_{\mathbb{C}} M) \\ \oplus \\ \mathcal{E}(\Lambda^{n-1} T^*_{\mathbb{C}} S) \end{array} \longrightarrow 0, \tag{6.1}$$

where

$$A_i = \begin{pmatrix} -d_i & 0 \\ \iota^\sharp & d_{i-1} \end{pmatrix},$$

d_i meaning the exterior derivative restricted to differential forms of degree i (cf. (3.1)).

The key result on the complex (6.1) is that it bears an information on the relative singular cohomology of the pair (M, S). The following result can be certainly attributed to the mathematical *folk lore* (cf. for instance Proposition 2.10 in Brenner and Shubin [BS81]).

Lemma 6.1 *There are natural isomorphisms*

$$H^i(\mathcal{C}^\cdot) \cong H^i((M, S), \mathbb{C}), \quad i = 0, 1, \ldots, n,$$

$H^i((M, S), \mathbb{C})$ *being the relative cohomology of the pair* (M, S) *with complex coefficients.*

Proof. By the de Rham Theorem, we have natural isomorphisms

$$\begin{array}{rcl} H^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} M)) & \cong & H^i(M, \mathbb{C}), \\ H^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} S)) & \cong & H^i(S, \mathbb{C}) \end{array} \tag{6.2}$$

for each i. We are going to make use of these to derive the desired isomorphisms in the relative cohomology.

To this end we invoke a standard exact long sequence of singular homology with coefficients in $\mathbb{C}$,

$$\begin{array}{lllllll} 0 & \longleftarrow & H_0((M, S), \mathbb{C}) & \stackrel{i}{\longleftarrow} & H_0(M, \mathbb{C}) & \stackrel{i}{\longleftarrow} & H_0(S, \mathbb{C}) & \stackrel{\partial}{\longleftarrow} & \\ & \longleftarrow & H_1((M, S), \mathbb{C}) & \stackrel{i}{\longleftarrow} & H_1(M, \mathbb{C}) & \stackrel{i}{\longleftarrow} & H_1(S, \mathbb{C}) & \stackrel{\partial}{\longleftarrow} & \ldots \\ & \longleftarrow & H_i((M, S), \mathbb{C}) & \stackrel{i}{\longleftarrow} & H_i(M, \mathbb{C}) & \stackrel{i}{\longleftarrow} & H_i(S, \mathbb{C}) & \stackrel{\partial}{\longleftarrow} & \ldots, \end{array} \tag{6.3}$$

i being induced by the inclusion of cycles and ∂ being induced by the boundary operator. Dual to this we have a standard exact long cohomological sequence

$$\begin{array}{lllllll} 0 & \longrightarrow & H^0((M, S), \mathbb{C}) & \stackrel{i'}{\longrightarrow} & H^0(M, \mathbb{C}) & \stackrel{i'}{\longrightarrow} & H^0(S, \mathbb{C}) & \stackrel{\delta}{\longrightarrow} & \\ & \longrightarrow & H^1((M, S), \mathbb{C}) & \stackrel{i'}{\longrightarrow} & H^1(M, \mathbb{C}) & \stackrel{i'}{\longrightarrow} & H^1(S, \mathbb{C}) & \stackrel{\delta}{\longrightarrow} & \ldots \\ & \longrightarrow & H^i((M, S), \mathbb{C}) & \stackrel{i'}{\longrightarrow} & H^i(M, \mathbb{C}) & \stackrel{i'}{\longrightarrow} & H^i(S, \mathbb{C}) & \stackrel{\delta}{\longrightarrow} & \ldots, \end{array} \tag{6.4}$$

δ being known as the *coboundary* operator. Recall that this latter sequence is obtained from (6.3) by applying the functor $\mathrm{Hom}_{\mathbb{C}}(\cdot, \mathbb{C})$.

The task is now to construct a sequence of the de Rham cohomology analogous to (6.4), i.e.,

$$\begin{array}{lllllll} 0 & \longrightarrow & H^0(\mathcal{C}^\cdot) & \stackrel{\pi}{\longrightarrow} & H^0(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} M)) & \stackrel{\iota^\sharp}{\longrightarrow} & H^0(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} S)) & \stackrel{\delta}{\longrightarrow} & \\ & \longrightarrow & H^1(\mathcal{C}^\cdot) & \stackrel{\pi}{\longrightarrow} & H^1(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} M)) & \stackrel{\iota^\sharp}{\longrightarrow} & H^1(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} S)) & \stackrel{\delta}{\longrightarrow} & \ldots \\ & \longrightarrow & H^i(\mathcal{C}^\cdot) & \stackrel{\pi}{\longrightarrow} & H^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} M)) & \stackrel{\iota^\sharp}{\longrightarrow} & H^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}} S)) & \stackrel{\delta}{\longrightarrow} & \ldots . \end{array} \tag{6.5}$$

Namely, we define π, $\iota^\sharp$ and δ by

$$\begin{array}{llll} \pi\colon & \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \bmod B^i(\mathcal{C}^\cdot) & \mapsto & u_1 \bmod B^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}M)), \\ \iota^\sharp\colon & u \bmod B^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}M)) & \mapsto & \iota^\sharp u \bmod B^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}S)), \\ \delta\colon & f \bmod B^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}S)) & \mapsto & \begin{pmatrix} 0 \\ f \end{pmatrix} \bmod B^{i+1}(\mathcal{C}^\cdot), \end{array}$$

for $\binom{u_1}{u_2} \in Z^i(\mathcal{C}^\cdot)$, $u \in Z^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}M))$ and $f \in Z^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}S))$ (cf. (4.2)). As mentioned in the proof of Theorem 4.2, the sequence (6.5) is exact.

From (6.2), (6.4) and (6.5) it already follows that the spaces $H^i(\mathcal{C}^\cdot)$ and $H^i((M,S),\mathbb{C})$ are of the same dimension, for each i. However, the lemma states more, namely there is a natural isomorphism of these spaces. The existence of such an isomorphism is a consequence of the fact that there is a duality between sequences (6.5) and (6.3). This duality is given on

$$\begin{array}{rcl} H^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}M)) & \times & H_i(M,\mathbb{C}), \\ H^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}S)) & \times & H_i(S,\mathbb{C}) \end{array}$$

by integrating differential forms over singular cycles, just as in the classical de Rham Theorem (cf. [dR55]). On $H^i(\mathcal{C}^\cdot) \times H_i((M,S),\mathbb{C})$ the duality is defined by

$$\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \bmod B^i(\mathcal{C}^\cdot),\ \sum_{\nu=1}^N c_\nu \Delta_\nu \bmod B_i((M,S),\mathbb{C})\right) \mapsto \sum_{\nu=1}^N c_\nu \left(\int_{\Delta_\nu} u_1 + \int_{\partial\Delta_\nu} u_2\right) \tag{6.6}$$

for $\binom{u_1}{u_2} \in Z^i(\mathcal{C}^\cdot)$ and $\sum_{\nu=1}^N c_\nu \Delta_\nu \in Z_i((M,S),\mathbb{C})$, where Δ_ν are singular simplexes and $c_\nu \in \mathbb{C}$. It is immediate that (6.6) is well-defined.

Thus, both (6.4) and (6.5) are dual to (6.3). This gives natural homomorphisms of the spaces in (6.5) to the corresponding spaces in (6.4). Hence we arrive at the commutative diagram

$$\begin{array}{ccccccc} H^{i-1}(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}S)) & \xrightarrow{\delta} & H^i(\mathcal{C}^\cdot) & \xrightarrow{\pi} & H^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}M)) & \xrightarrow{\iota^\sharp} & H^i(\mathcal{E}(\Lambda^\cdot T^*_{\mathbb{C}}S)) \\ \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ H^{i-1}(S,\mathbb{C}) & \xrightarrow{\delta} & H^i((M,S),\mathbb{C}) & \xrightarrow{i'} & H^i(M,\mathbb{C}) & \xrightarrow{i'} & H^i(S,\mathbb{C}) \end{array}$$

with exact rows, for each $i = 0, 1, \ldots, n$. The homomorphisms marked by the vertical arrows in the diagram are actually isomorphisms, with the exception of $H^i(\mathcal{C}^\cdot) \to H^i((M,S),\mathbb{C})$. Applying the 'Lemma on Five Isomorphisms' we can therefore assert that this latter homomorphism is also an isomorphism. This is our assertion.

□

Having disposed of this preliminary step, we can now return to the Lefschetz fixed point formula.

Let f be a differentiable mapping of the manifold M with the property that $f(S) \subset S$. Then f induces a mapping $f = (f_M, f_S)$ of the pair (M, S) via $f_M = f|_M$, $f_S = f|_S$. The 'pull-back' operator $f^\sharp$ under f commutes with the exterior derivative on both M and S. Moreover, we have

$$\begin{aligned} f_S^\sharp \iota^\sharp &= (\iota \circ f_S)^\sharp \\ &= (f_M \circ \iota)^\sharp \\ &= \iota^\sharp f_M^\sharp \end{aligned}$$

the second equality being due to the fact that $f(S) \subset S$. Hence it follows, by Lemma 4.1, that f has a lift to the complex $\mathcal{C}^\cdot$, the lift being given by $f^\sharp$. We write $L(f, (M, S))$ for the corresponding Lefschetz number. Lemma 6.1 allows one to conclude that $L(f, (M, S))$ is just the classical Lefschetz number of f with respect to the relative cohomology of the pair (M, S).

Suppose $p \in S$ is a fixed point of f. Then, the tangent mappings to f_M and f_S induce linear transformations

$$\begin{aligned} df_M(p) &: T_pM \to T_pM, \\ df_S(p) &: T_pS \to T_pS, \end{aligned}$$

of the tangent spaces to M and S at the point p, respectively. Letting $d_M f = df_M$ and $d_S f = df_S$, we thus arrive at a linear transformation of the quotient space T_pM/T_pS, namely

$$d_{M/S} f(p) \ : \ \frac{T_pM}{T_pS} \to \frac{T_pM}{T_pS}.$$

Moreover,

$$\det(\mathrm{Id} - d_M f(p)) = \det(\mathrm{Id} - d_S f(p)) \det\left(\mathrm{Id} - d_{M/S} f(p)\right),$$

as is easy to see by using local coordinates at p.

In particular, we deduce that if $p \in S$ is a simple fixed point of f_M, then the determinant of $\mathrm{Id} - d_{M/S} f(p)$ is different from zero. Denote by $\mathrm{Fix}^{(\pm)}(f, S)$ the set of all simple fixed points of f on M with the property that $\pm \det\left(\mathrm{Id} - d_{M/S} f(p)\right) > 0$.

Corollary 6.2 *Let f be a differentiable mapping of the pair (M, S) with simple fixed points. Then*

$$L(f) = \sum_{p \in \mathrm{Fix}(f, M \setminus S)} \mathrm{sgn}\ \det(\mathrm{Id} - df(p)) + 2 \sum_{p \in \mathrm{Fix}^{(-)}(f,S)} \mathrm{sgn}\ \det(\mathrm{Id} - df(p)).$$

Proof. Indeed, if $p \in S$ is a simple fixed point of f, then

$$\mathrm{sgn}\ \det(\mathrm{Id} - d_M f(p)) = \pm\ \mathrm{sgn}\ \det(\mathrm{Id} - d_S f(p))$$

where '+' is taken for $p \in \mathrm{Fix}^{(+)}(f, S)$ and '$-$' for $p \in \mathrm{Fix}^{(-)}(f, S)$. Thus, the contributions of the points $p \in \mathrm{Fix}^{(+)}(f, S)$ in (5.2) cancel while the

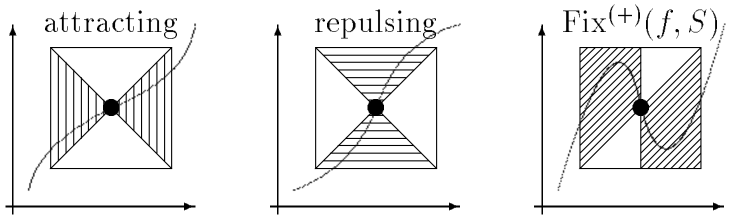

Fig. 1: Specification of the graphs of functions close to a fixed point.

contributions of the points $p \in \mathrm{Fix}^{(-)}(f,S)$ duplicate. This establishes the formula.

□

In contrast to Brenner and Shubin [BS81], the specification of simple fixed points of f on S by those in $\mathrm{Fix}^{(+)}(f,S)$ and $\mathrm{Fix}^{(-)}(f,S)$ is much more complicated than the specification by being *attracting* or *repulsing*. As is shown in Fig. 1, $\mathrm{Fix}^{(+)}(f,S)$ contains all attracting fixed points of f on S along with a part of repulsing fixed points.

Note that Corollary 6.2 can be also obtained from the algebraic *alternating sum formula* applied to sequence (6.4). Indeed, this sequence is exact and f induces an endomorphism $f^\sharp$ of the sequence, which results in the equality $L(f,(M,S)) = L(f_M, M) - L(f_S, S)$. This agrees with Theorem 4.2.

7 Extension to More General Operators

Formula (5.2) extends to complexes of operators more general than those in (3.1). Before describing these operators, we recall that a regular operator $T\colon \mathcal{E}(V) \to \mathcal{E}(W)$ extends to a continuous mapping $H^{s_T}(V) \to \mathcal{E}(W)$, for some $s_T \in \mathbb{R}$, if and only if it has a kernel in $\mathcal{E}(W) \otimes_\pi H^{-s_T}(V')$. If such is the case, we say that T is an operator *of order* $-\infty$.

The operators in question are of the form

$$\mathcal{A} = \begin{pmatrix} A+G & C \\ T & B \end{pmatrix}, \tag{7.1}$$

where A is a classical pseudodifferential operator of type $V \to \tilde{V}$ on M, G is an operator of type $V \to \tilde{V}$ and of order $-\infty$, C is a smoothing operator of type $W \to \tilde{V}$, T is an operator of type $V \to \tilde{W}$ and of finite order, and B is a classical pseudodifferential operator of type $W \to \tilde{W}$ on S. Every operator of the form (7.1) extends to a continuous mapping of Sobolev spaces as in (2.3). Moreover, the composition of any two operators of the form (7.1) is of the same form, as is easy to check. For this reason we say that operators (7.1) form an "operator algebra" and write $\mathrm{Alg}(V,\tilde{V};W,\tilde{W})$

for it. This "operator algebra" bears a symbolic structure given by the 'principal symbol'

$$\sigma(\mathcal{A}) = \begin{pmatrix} \sigma(A) & 0 \\ 0 & \sigma(B) \end{pmatrix}$$

for $\mathcal{A}$ of the form (7.1), the symbol $\sigma(\mathcal{A})$ being thought of as a bundle homomorphism of $\pi_M^* V \oplus \pi_S^* W$ to $\pi_M^* \tilde{V} \oplus \pi_S^* \tilde{W}$. Thus, an operator $\mathcal{A} \in \mathrm{Alg}(V, \tilde{V}; W, \tilde{W})$ is said to be *elliptic* if both A and B are elliptic pseudodifferential operators on M and S, respectively. Note that Lemma 2.6 and Theorem 2.7 are still true for operators (7.1) because (7.1) differs from (2.2) by a compact operator.

We next consider complexes whose differentials take form (7.1), more precisely,

$$\mathcal{C}^{\cdot} : \; 0 \longrightarrow \begin{matrix} \mathcal{E}(V^0) \\ \oplus \\ \mathcal{E}(W^{-1}) \end{matrix} \xrightarrow{\mathcal{A}_0} \begin{matrix} \mathcal{E}(V^1) \\ \oplus \\ \mathcal{E}(W^0) \end{matrix} \xrightarrow{\mathcal{A}_1} \ldots \xrightarrow{\mathcal{A}_N} \begin{matrix} \mathcal{E}(V^{N+1}) \\ \oplus \\ \mathcal{E}(W^N) \end{matrix} \longrightarrow 0, \qquad (7.2)$$

where

$$\mathcal{A}_i = \begin{pmatrix} -A_i + G_i & C_{i-1} \\ T_i & B_{i-1} \end{pmatrix}$$

belongs to $\mathrm{Alg}(V^i, V^{i+1}; W^{i-1}, W^i)$, for each $i = 0, 1, \ldots, N$. Given any $s, t \in \mathbb{R}$ with s large enough, complex (7.2) induces a complex of continuous linear mappings of Sobolev spaces as in (3.2). Complex (7.2) is said to be *elliptic* if the corresponding sequence of principal symbols $\sigma(\mathcal{A}_i)$ is exact away from the zero sections of T^*M and T^*S. This amounts to saying that the sequences of principal symbols $\sigma(A_i)$ and $\sigma(B_i)$ are exact in the complements of the zero sections of T^*M and T^*S, respectively. We emphasise that the sequences $\mathcal{E}(V^{\cdot})$ and $\mathcal{E}(W^{\cdot})$ themselves need not be complexes. Every elliptic complex (7.2) is Fredholm. Moreover, its cohomology is independent on the Sobolev spaces at which it is evaluated. The standard way to prove this is to construct a proper parametrix of the complex, which is our next task.

To this end we observe that from $\mathcal{A}_{i+1}\mathcal{A}_i = 0$ it follows that $A_{i+1}A_i = 0$ up to an operator of type $V^i \to V^{i+2}$ and of order $-\infty$ and $B_i B_{i-1} = 0$ up to a smoothing operator of type $W^{i-1} \to W^{i+1}$. Since the Laplacians of $\mathcal{E}(V^{\cdot})$ and $\mathcal{E}(W^{\cdot})$ are elliptic operators, the usual parametrix construction for elliptic complexes still applies to the sequences $\mathcal{E}(V^{\cdot})$ and $\mathcal{E}(W^{\cdot})$. This results in pseudodifferential parametrices P and Q for the sequences $\mathcal{E}(V^{\cdot})$ and $\mathcal{E}(W^{\cdot})$, respectively. More precisely, there exist

$$\begin{aligned} P_i &\in \Psi_{\mathrm{cl}}^{-o_{A_{i-1}}}(M; V^i, V^{i-1}), \\ Q_i &\in \Psi_{\mathrm{cl}}^{-o_{B_{i-1}}}(S; W^i, W^{i-1}) \end{aligned}$$

such that

$$\begin{aligned} P_{i+1}A_i + A_{i-1}P_i &= \mathrm{Id} - R_i, \\ Q_{i+1}B_i + B_{i-1}Q_i &= \mathrm{Id} - S_i \end{aligned} \qquad (7.3)$$

for all $i = 0, 1, \dots, N$, where R_i is an operator of type $V^i \to V^i$ and of order $-\infty$ and S_i is a smoothing operator of type $W^i \to W^i$. If defined by (3.3), the operators $\mathcal{P}_i$ meet the assertion of Theorem 3.3, thus providing us with a "good" parametrix to every complex (3.2).

When considering endomorphisms $\mathcal{E} = (\mathcal{E}_i)$ of complex (7.2), we will restrict our attention to those obeying the shape of operators in our algebra. They are of the form

$$\mathcal{E}_i = \begin{pmatrix} E_i & K_{i-1} \\ H_i & F_{i-1} \end{pmatrix} \tag{7.4}$$

for $i = 0, 1, \dots, N+1$, where E_i is a regular operator of type $V^i \to V^i$ on M, K_{i-1} is a smoothing operator of type $W^{i-1} \to V^i$, H_i is a regular operator of type $V^i \to W^{i-1}$, and F_{i-1} is a regular operator of type $W^{i-1} \to W^{i-1}$ on S. Lemma 4.1 still holds in this more general setting, provided that all the commutativity relations involved therein are understood up to operators of order $-\infty$ (or smoothing operators). In particular, (E_i) is an endomorphism of $\mathcal{E}(V^\cdot)$ modulo operators of order $-\infty$, and (F_i) is an endomorphism of $\mathcal{E}(W^\cdot)$ modulo smoothing operators.

If $\mathcal{P} = (\mathcal{P}_i)$ is a parametrix of complex (7.2) satisfying (3.4), then the family $\mathcal{S} = (\mathcal{S}_i)$ is easily verified to be an endomorphism of (7.2). It is homotopic to the identity endomorphism of (7.2), hence the Lefschetz number of $\mathcal{S}$ is equal to the Euler characteristic of this complex. Moreover, the composition $\mathcal{E}\mathcal{S} = (\mathcal{E}_i\mathcal{S}_i)$ is still an endomorphism of (7.2), for each endomorphism $\mathcal{E}$ of this complex. The advantage of using a parametrix of $\mathcal{C}^\cdot$ lies in the fact that $\mathcal{E}\mathcal{S}$ is homotopic to $\mathcal{E}$ while the mapping properties of this new endomorphism are much better than those of $\mathcal{E}$.

Lemma 7.1 *Let $\mathcal{P}$ be the parametrix of the complex $\mathcal{C}^\cdot$ given by* (3.3). *Then, for each endomorphism $\mathcal{E}$ of $\mathcal{C}^\cdot$, we have $L(\mathcal{E}) = L(\mathcal{E}\mathcal{S})$.*

Proof. By the homotopy formula (3.4), we get

$$\mathcal{P}_{i+1}\mathcal{A}_i u + \mathcal{A}_{i-1}\mathcal{P}_i u = u - \mathcal{S}_i u$$

for all $u \in \mathcal{E}(\mathcal{C}^i)$. Applying the endomorphism $\mathcal{E}$ to both sides of this equality yields

$$(\mathcal{E}_i\mathcal{P}_{i+1})\mathcal{A}_i u + \mathcal{A}_{i-1}(\mathcal{E}_{i-1}\mathcal{P}_i)u = \mathcal{E}_i u - (\mathcal{E}_i\mathcal{S}_i)u$$

for each $u \in \mathcal{E}(\mathcal{C}^i)$. We next observe that the composition $\mathcal{E}_{i-1}\mathcal{P}_i$ is a regular operator of type $V^i \oplus W^{i-1} \to V^{i-1} \oplus W^{i-2}$, for each $i = 1, \dots, N+1$. Hence it follows that $(H\,\mathcal{E})_i = (H\,\mathcal{E}\mathcal{S})_i$ for all i, and so $L(\mathcal{E}) = L(\mathcal{E}\mathcal{S})$, as desired. □

Let us emphasise that every component $\mathcal{E}_i\mathcal{S}_i$ is an operator of type $V^i \oplus W^{i-1} \to V^i \oplus W^{i-1}$ and of order $-\infty$. In particular, it is a trace class operator on any space $H^s(V^i) \oplus H^t(W^{i-1})$ with s large enough, for it factors trough a trace class embedding of Sobolev spaces.

Lemma 7.2 *For any endomorphism $\mathcal{E} = (\mathcal{E}_i)$ of $\mathcal{C}^\cdot$ of the form (7.4), it follows that*

$$L(\mathcal{E}) = \sum_{i=0}^{N}(-1)^i \ (\ \mathrm{tr}\,(E_i - E_i P_{i+1} A_i - A_{i-1} E_{i-1} P_i) \\ - \ \mathrm{tr}\,(F_i - F_i Q_{i+1} B_i - B_{i-1} F_{i-1} Q_i)\,).$$

Proof. For $s, t \in \mathbb{R}$, set

$$\mathcal{H}^{s,t} = \begin{array}{c} H^s(V^i) \\ \oplus \\ H^t(W^{i-1}) \end{array},$$

the space being Hilbert in a canonical way. As mentioned, the composition $O = \mathcal{E}_i \mathcal{S}_i$ extends to a trace class operator $O|_{s,t} : \mathcal{H}^{s,t} \to \mathcal{H}^{s,t}$ provided s is sufficiently large. Moreover, both the eigenfunctions and the associated functions of $O|_{s,t}$ subject to a non-zero eigenvalue belong to $\mathcal{H}^{\infty,\infty}$. Therefore, the non-zero eigenvalues of the operators $O|_{s,t}$ and $O|_{s',t'}$, if counted along with their multiplicities, coincide for all s, t and s', t' with s and s' sufficiently large. Now the *Lidskii Theorem* shows that the trace of $O|_{s,t}$ is independent of s and t provided s is large enough. In the sequel, by $\mathrm{tr}\, O$ we mean the trace of $O|_{s,t}$ evaluated at a pair $s, t \in \mathbb{R}$ with large s. From what has already been proved it follows that $\mathrm{tr}\, O$ is well-defined.

The endomorphism $\mathcal{E}\mathcal{S}$ of complex (7.2) clearly extends to an endomorphism $\mathcal{E}\mathcal{S}|_{s,t}$ of complex (3.2), for large s. Since the cohomology of (7.2) is isomorphic to that of (3.2), it follows that $L(\mathcal{E}\mathcal{S}) = L(\mathcal{E}\mathcal{S}|_{s,t})$ for s large enough. Thus, combining Lemma 7.1 with Theorem 19.1.15 of [Hör85] we deduce that

$$L(\mathcal{E}) = \sum_{i=0}^{N+1} (-1)^i \,\mathrm{tr}\, \mathcal{E}_i \mathcal{S}_i. \tag{7.5}$$

We now express $\mathcal{S}_i$ from equality (3.4) and substitute these expressions into (7.5). Rearranging the summands, we thus obtain

$$\begin{aligned} L(\mathcal{E}) &= \sum_{i=0}^{N+1} (-1)^i \,\mathrm{tr}\,(\mathcal{E}_i - \mathcal{E}_i \mathcal{P}_{i+1} \mathcal{A}_i - \mathcal{E}_i \mathcal{A}_{i-1} \mathcal{P}_i) \\ &= \sum_{i=0}^{N+1} (-1)^i \,\mathrm{tr}\,(\mathcal{E}_i - \mathcal{E}_i \mathcal{P}_{i+1} \mathcal{A}_i - \mathcal{A}_{i-1} \mathcal{E}_{i-1} \mathcal{P}_i) \\ &= \mathrm{tr} \sum_{i=0}^{N} (-1)^i \,(\mathcal{E}_i + \mathcal{A}_i \mathcal{E}_i \mathcal{P}_{i+1} - \mathcal{E}_i \mathcal{P}_{i+1} \mathcal{A}_i), \end{aligned}$$

the second equality being due to the fact that $\mathcal{E}$ is an endomorphism of (7.2). Write

$$\begin{aligned} \mathcal{A}_i &= \mathcal{A}'_i + \mathcal{A}''_i, \\ \mathcal{E}_i &= \mathcal{E}'_i + \mathcal{E}''_i, \end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}'_i &= \begin{pmatrix} -A_i & 0 \\ T_i & B_{i-1} \end{pmatrix}, & \mathcal{A}''_i &= \begin{pmatrix} G_i & C_{i-1} \\ 0 & 0 \end{pmatrix}; \\
\mathcal{E}'_i &= \begin{pmatrix} E_i & 0 \\ H_i & F_{i-1} \end{pmatrix}, & \mathcal{E}''_i &= \begin{pmatrix} 0 & K_{i-1} \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

By the above, $\mathcal{A}''_i$ and $\mathcal{E}''_i$ are operators of order $-\infty$, and hence trace class operators in Sobolev spaces of smoothness large enough. Hence it follows, by the Lidskii Theorem, that

$$\begin{aligned}
\operatorname{tr} \mathcal{A}'_i \mathcal{E}''_i \mathcal{P}_{i+1} &= \operatorname{tr} \mathcal{E}''_i \mathcal{P}_{i+1} \mathcal{A}'_i, \\
\operatorname{tr} \mathcal{A}''_i \mathcal{E}'_i \mathcal{P}_{i+1} &= \operatorname{tr} \mathcal{E}'_i \mathcal{P}_{i+1} \mathcal{A}''_i, \\
\operatorname{tr} \mathcal{A}''_i \mathcal{E}''_i \mathcal{P}_{i+1} &= \operatorname{tr} \mathcal{E}''_i \mathcal{P}_{i+1} \mathcal{A}''_i
\end{aligned}$$

for all $i = 0, 1, \ldots, N$. This implies

$$\begin{aligned}
L(\mathcal{E}) &= \operatorname{tr} \sum_{i=0}^{N} (-1)^i \left(\mathcal{E}_i + \mathcal{A}'_i \mathcal{E}'_i \mathcal{P}_{i+1} - \mathcal{E}'_i \mathcal{P}_{i+1} \mathcal{A}'_i \right) \\
&= \sum_{i=0}^{N+1} (-1)^i \operatorname{tr} \left(\mathcal{E}_i - \mathcal{E}'_i \mathcal{P}_{i+1} \mathcal{A}'_i - \mathcal{A}'_{i-1} \mathcal{E}'_{i-1} \mathcal{P}_i \right).
\end{aligned}$$

We are left with the task of determining the traces of the summands on the right-hand side. In fact,

$$\begin{aligned}
\operatorname{tr} \left(\mathcal{E}_i - \mathcal{E}'_i \mathcal{P}_{i+1} \mathcal{A}'_i - \mathcal{A}'_{i-1} \mathcal{E}'_{i-1} \mathcal{P}_i \right) &= \operatorname{tr} \left(E_i - E_i P_{i+1} A_i - A_{i-1} E_{i-1} P_i \right) \\
&\quad + \operatorname{tr} \left(F_{i-1} - F_{i-1} Q_i B_{i-1} - B_{i-2} F_{i-2} Q_{i-1} \right),
\end{aligned}$$

as is easy to check. Hence the lemma follows.

□

As is noted by Fedosov [Fed91, p. 203], the advantage of using this formula lies in the fact that the operators E_i and F_i therein need not satisfy the commutativity relations $E_{i+1} A_i = A_i E_i$ and $F_{i+1} B_i = B_i F_i$ precisely, but only up to trace class operators. Moreover, perturbations of A, P and B, Q by nuclear terms do not affect the alternating sum of the traces.

Theorem 7.3 *Suppose $f = (f_M, f_S)$ is a mapping of the pair (M, S), such that both f_M and f_S have only simple fixed points. If f has a lift to* (7.2)*, then the Lefschetz number of f can be evaluated by formula* (5.2).

Proof. The theorem follows from Lemma 7.2 by the scheme suggested by Fedosov [Fed91, p. 204]. Namely, we shall have established (5.2) if we prove that

$$\sum_{i=0}^{N} (-1)^i \operatorname{tr} \left(E_i - E_i P_{i+1} A_i - A_{i-1} E_{i-1} P_i \right) = \sum_{p \in \operatorname{Fix}(f_M, M)} \frac{\sum_{i=0}^{N} (-1)^i \operatorname{tr} h_{V^i}(p)}{|\det(\mathrm{Id} - df_M(p))|}$$

and similarly for the quasicomplex $\mathcal{E}(W^\cdot)$ on S.

To do this, pick a partition of unity (ϕ_ν) on M with the property that each ϕ_ν either vanishes or is equal to 1 in a neighbourhood of any fixed point of f_M. Let further ψ_0 be a function of compact support on T^*M such that $\psi_0(\xi) \equiv 1$ near $\xi = 0$, and let $\psi_\infty = 1 - \psi_0$. In local coordinates on M, we introduce operators $\Psi_{0,\nu}$ and $\Psi_{\infty,\nu}$ by

$$\begin{aligned}\Psi_{0,\nu} u &= F^{-1}_{\xi \mapsto x} \psi_0(h\xi) F_{x\mapsto\xi} (\phi_\nu u),\\ \Psi_{\infty,\nu} u &= F^{-1}_{\xi \mapsto x} \psi_\infty(h\xi) F_{x\mapsto\xi} (\phi_\nu u),\end{aligned}$$

F being the Fourier transform and h a positive number. These operators decompose the identity operator; moreover, the operators $\Psi_{0,\nu}$ are smoothing and hence of trace class on each Sobolev space. We can assert, by the Lidskii Theorem, that

$$\operatorname{tr} A_i E_i P_{i+1} \Psi_{0,\nu} = \operatorname{tr} E_i P_{i+1} \Psi_{0,\nu} A_i$$

whence

$$\begin{aligned}&\sum_{i=0}^{N} (-1)^i \operatorname{tr} (E_i - E_i P_{i+1} A_i - A_{i-1} E_{i-1} P_i)\\ &= \sum_\nu \sum_{i=0}^{N} (-1)^i \operatorname{tr} E_i \Psi_{0,\nu}\\ &+ \sum_\nu \sum_{i=0}^{N} (-1)^i \operatorname{tr} (E_i - E_i P_{i+1} A_i - A_{i-1} E_{i-1} P_i) \Psi_{\infty,\nu}\\ &- \sum_\nu \sum_{i=0}^{N-1} (-1)^i \operatorname{tr} E_i P_{i+1} [A_i, \Psi_{0,\nu}],\end{aligned} \tag{7.6}$$

$[A_i, \Psi_{0,\nu}]$ being the commutator of A_i and $\Psi_{0,\nu}$.

In a local chart close to a fixed point of f_M, the operator $E_i \Psi_{0,\nu}$ is given by the iterated integral

$$E_i \Psi_{0,\nu} u\,(x) = \frac{1}{(2\pi h)^n} \iint e^{i\frac{\xi}{h}(f_M(x) - y)}\, h_{V^i}(x) \psi_0(\xi)\, \phi_\nu(y) u(y)\, dy d\xi,$$

and consequently

$$\operatorname{tr} E_i \Psi_{0,\nu} = \frac{1}{(2\pi h)^n} \iint e^{i\frac{\xi}{h}(f_M(x) - x)} \operatorname{tr} h_{V^i}(x)\, \psi_0(\xi) \phi_\nu(x)\, d\xi dx.$$

For $h \to 0$, the limit of the integral on the right-hand side of this equality can be evaluated by the method of stationary phase. Moreover, the stationary points are just the points where $\xi = 0$ and $f_M(x) - x = 0$. In the principal part independent of h the contribution of a fixed point p is equal to

$$\frac{\operatorname{tr} h_{V^i}(p)}{|\det(\mathrm{Id} - df_M(p))|},$$

and so the alternating sum of these contributions gives us the right-hand side of (5.2). On the other hand, the remaining terms on the right side of (7.6) are oscillatory integrals whose exponent has no critical points. Indeed,

$$\begin{aligned} [A_i, \Psi_{0,\nu}] &= [A_i, \Psi_{0,\nu} - \mathrm{Id}] \\ &= -[A_i, \Psi_{\infty,\nu}] \end{aligned}$$

close to each fixed point and the function ψ_∞ vanishes in a neighbourhood of $\xi = 0$. Hence it follows that the remaining summands in (7.6) are rapidly decreasing as $h \to 0$. Since the left-hand side of (7.6) is actually independent of h, we arrive at the desired formula.

□

References

[Arn79] V. I. Arnold, *Indices of singular points of 1-forms, swinging of invariants of groups generated by mappings, and singular projections of smooth surfaces*, Uspekhi Mat. Nauk **34** (1979), no. 2, 3–38.

[AB67] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes. 1:*, Ann. Math. **86** (1967), no. 2, 374–407.

[Bis85] M. Bismut, *The infinitesimal Lefschetz formulas: A heat equation proof*, J. Funct. Anal. **62** (1985), 437–457.

[BdM71] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), no. 1–2, 11–51.

[BS81] A. V. Brenner and M. A. Shubin, *The Atiyah-Bott-Lefschetz theorem for manifolds with boundary*, Funkts. Analiz **15** (1981), no. 4, 67–68.

[Dol72] A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, Berlin-Heildelberg-New York, 1972.

[DF86] H. Donnelly and C. Fefferman, *Fixed point formula for the Bergman kernel*, Amer. J. Math. **108** (1986), no. 5, 1241–1258.

[dR55] Georges de Rham, *Variétés différentiables. Formes, courants, formes harmoniques*, Actualites Sci. Indust., vol. 1222, Hermann, Paris, 1955.

[Dyn72] A. S. Dynin, *Elliptic boundary problems for pseudo-differential complexes*, Funkts. Analiz **6** (1972), no. 1, 75–76.

[Efr88] A. V. Efremov, *The Atiyah-Bott-Lefschetz formula for Hilbert bundles*, Vestnik Mosk. Univ., Mat., Meh. (1988), no. 4, 89–92.

[Fed91] B. V. Fedosov, *Index Theorems*, Modern Problems of Mathematics. Fundamental Directions. Vol. 65, VINITI, Moscow, 1991, 165–268.

[Fed93] B. V. Fedosov, *Trace formula for Schrödinger operator*, Zh. Mat. Phisiki **1** (1993), no. 4, 447–463.

[Gil79] P. B. Gilkey, *Lefschetz fixed point formulas and the heat equation*, Comm. Pure Appl. Math. **48** (1979), 91–147.

[Hop29] H. Hopf, *Über die algebraische Anzahl von Fixpunkten*, Math. Z. **29** (1929), 493–524.

[Hör85] L. Hörmander, *The Analysis of Linear Partial Differential Operators. Vol. 3: Pseudo-Differential Operators*, Springer-Verlag, Berlin et al., 1985.

[Kot69] T. Kotake, *The fixed point theorem of Atiyah-Bott via parabolic operators*, Comm. Pure Appl. Math. **22** (1969), no. 6, 789–806.

[Lef26] S. Lefschetz, *Intersections and transformations of complexes and manifolds*, Trans. Amer. Math. Soc. **28** (1926), 1–49.

[Mau67] K. Maurin, *Methods of Hilbert Spaces*, Polish Scientific Publishers, Warszawa, 1967.

[Nes81] A. Nestke, *Cohomology of elliptic complexes and the fixed point formula of Atiyah and Bott*, Math. Nachr. **94** (1981), 289–313.

[Pat73] V. K. Patodi, *Holomorphic Lefschetz fixed point formula*, Bull. Amer. Math. Soc. **79** (1973), 825–828.

[RS82] St. Rempel and B.-W. Schulze, *Index Theory of Elliptic Boundary Problems*, Akademie-Verlag, Berlin, 1982.

[Shu92] M. A. Shubin, *A Lefschetz fixed point formula in reduced L^2-cohomologies on manifolds with cylinders*, Symposium "Analysis on Manifolds with Singularities," Breitenbrunn, 1990. Teubner-Texte zur Mathematik 131, Teubner-Verlag, Leipzig, 1992, pp. 290–297.

[SS90] M. A. Shubin and S. Seifarth, *A Lefschetz fixed point formula on manifolds with cylinder ends*, C. R. Acad. Sci. Paris, Sér. 1, Math. **310** (1990), no. 12, 849–853.

[SS94] B. Yu. Sternin and V. E. Shatalov, *Quantization of Symplectic Transformations and the Lefschetz Fixed Point Theorem*, Preprint MPI/94-92, Max-Planck-Inst. für Math., Bonn, 1994, 22 pp.

[Tar95] N. N. Tarkhanov, *Complexes of Differential Operators*, Kluwer Academic Publishers, Dordrecht, NL, 1995.

[Tol73] D. Toledo, *On the Atiyah-Bott formula for isolated fixed points*, J. Differential Geom. **8** (1973), no. 3, 401–436.

[TT75] D. Toledo and Y. L. Tong, *Duality and intersection theory in complex manifolds. II: The holomorphic Lefschetz formula*, Ann. of Math. **108** (1975), 519–538.