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Preprint No. 97/23
*July 1997*

Potsdam 1997

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July 25, 1997

**Abstract**

The structure of symplectic (canonical) transformations on manifolds with conical singularities is established. The operators associated with these transformations are defined in the weight spaces and their properties investigated.

**Keywords**: manifolds with conical singularities, symplectic (canonical) transformations, quantization, Mellin transform

**1991 AMS classification**: primary 81S10, secondary 35J70

*Supported by Deutsche Forschungsgemeinschaft and RFBR grant No. 96-01-00720.

# Contents



# Introduction

The quantization of canonical transformations of smooth symplectic manifolds beginning from classical work by V. Fock [7] plays an important role in the theory of differential equations. The corresponding operators can be used, for instance, in the microlocal analysis (investigation of normal forms of Hamilton functions), for regularization of a wide class of partial differential equations, in different geometrical questions, e. g. in the Lefschetz theory (see, for example, [4] — [13], [15] – [17], [19], [23] – [26] and the citations therein). Therefore, it is natural to expect that quantization of canonical transforms for manifolds with singularities also will be an important tool for the theory of differential equations on singular manifolds. The simplest and rather well investigated model of singularities is a singularity of conical type, and in this paper we work out the theory of quantization of canonical transforms on manifolds with singularities of conical points[1]. In doing so, it is important to answer the following questions:

- What is the *cotangent bundle* of a manifold with conical singularities?

[1] The exact definition of objects in question will be given below.

- How to describe the structure of *canonical transformations* (that is, structure-preserving mappings of the cotangent bundle) near the singular points?
- What do the *quantized transformations* look like?
- How to define the notion of *conormal symbol* (which is a usual attribute of operators acting on manifolds with conical singularities) for these transformations?

All these issues are touched in the present paper. In a different context, some of these questions were studied by Melrose [13] who considered a manifold, $M$, with boundary $\partial M$ and the ring of pseudodifferential operators generated by the set of vector fields everywhere tangent to the boundary. Geometrically, this leads to the notion of the *compressed cotangent bundle* $\tilde{T}^*M$, which turns out to be adequate in the conical situation as well (essentially, the cotangent bundle of a manifold with conical singularities is the compressed cotangent bundle of the blow-up of this manifold). Melrose also considered (homogeneous) canonical transformations of the compressed cotangent bundle and the corresponding Fourier integral operators. However, the spaces in which his operators act are quite different from the spaces $H^{s,\gamma}$, natural in the theory of differential equations on manifolds with conical singularities. Melrose's spaces are spaces of distributions that behave regularly along the boundary in the sense that the operation of restriction to the boundary commutes with the action of the operators in question. That is why the notion of *conormal symbol*, which is crucial in dealing with operators on manifolds with conical singularities, is absent in Melrose's paper.

Our paper is organized as follows. It consists of two chapters. In the first chapter, the symplectic geometry of the cotangent bundle of a manifold with conical singularities is considered and general (not necessarily homogeneous) canonical transformations are studied. Their structure near the singularities is described, and generating functons are constructed. In the second chapter, the corresponding quantized objects (we shall call them Mellin–Maslov integral operators) are considered. The notion of conormal symbol is defined. The main properties of these operators are stated and their proof is outlined.

# 1 Symplectic Geometry for Manifolds with Conical Singularities

In this section we deal with some of the main notions of sympletic geometry (viz. the cotangent bundle, canonical transformations, and generating functions) as they appear in the theory of manifolds with conical singularities. We do not discuss Lagrangian manifolds other than graphs of canonical transformations (even though this might be an interesting topic), as they will not be needed in the sequel.

## 1.1 Manifolds with conical singularities and their cotangent bundles

What do we actually mean by saying that $M$ is a manifold with conical singularities? This notion is widely used in literature (e.g., see [3, 14, 20] and the literature), and various forms of the definition occur; to recall it, we proceed in the spirit of [22].

Let us first give some motivation. Recall that $\alpha \in M$ is a conical point if

*i)* $M\backslash\{\alpha\}$ is a smooth manifold in a neighborhood of $\alpha$ (which does not contain other conical points).

*ii)* The point $\alpha$ has a neighborhood $U$ homeomorphic to some cone

$$K = ([0,1) \times \Omega)/(\{0\} \times \Omega), \tag{1}$$

where $\Omega$ is a compact closed $C^\infty$ manifold, the factorization implies that all points of $\{0\} \times \Omega$ are identified into one point (referred to as the *cone vertex*), and the image of the point $\alpha$ under this homeomorphism is just the vertex; moreover, the restriction of this homeomorphism to $U\backslash\{\alpha\}$ is a diffeomorphism of $U\backslash\{\alpha\}$ onto $\overset{\circ}{K} = K\backslash\{\text{the vertex}\}$.

*iii)* $M$ is equipped with two "structure rings," one of functions and the other of differential operators. While outside the conical points these rings consist of smooth functions and differential operators with smooth coefficients, respectively, they can be described as follows in the neighborhood $U \cong K$. Let $(r, \omega)$ be local coordinates on $K$ such that $r \in [0,1)$ and $\omega = (\omega_1, \ldots, \omega_{n-1})$, $n = \dim M$, is a system of local coordinates[2] on $\Omega$. Then we deal with smooth functions $f(r,\omega) \in C^\infty([0,1) \times \Omega)$ (smoothness up to $r = 0$ is required!) and with differential operators[3]

$$\overset{\wedge}{H} = H\left(\overset{2}{r}, \overset{2}{\omega}, -i\overset{1}{\frac{\partial}{\partial\omega}}, ir\overset{1}{\frac{\partial}{\partial r}}\right) = \sum_{k+|\alpha|\le m} a_{k\alpha}(r,\omega)\left(ir\frac{\partial}{\partial r}\right)^k \left(-i\frac{\partial}{\partial\omega}\right)^\alpha \tag{2}$$

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[2]To simplify the notation, in the following we use the same notation for points $\omega \in \Omega$ and their coordinate representations $\omega = (\omega_1, \ldots, \omega_{n-1})$ in some local chart on $\Omega$. This will not lead to a misunderstanding.

[3]Throughout the following, we use the conventional notation of noncommutative analysis ([12]; see also [18]): the numbers over operators (Feynman indices) denote the order of action of these operators.

with smooth coefficients $a_{k\alpha}(r,\omega) \in C^\infty([0,1]\times\Omega)$.

While the requirement of necessarily having the point $\alpha$ attached is here only to conform with geometric intuition (actually, we always deal either with $\overset{\circ}{K}$ or with the blow-up $K^\wedge = [0,1)\times\Omega$ of $K$), it is condition *iii)* that is the most essential: it exactly describes the classes of smooth functions and differential operators we are going to deal with. Note that our requirment on the operators (2) is in fact a requirement on their symbols

$$H(r,\omega,p,q), \quad r\in[0,1), \quad p\in\mathbf{R}^1, \quad (\omega,q)\in T^*\Omega :$$

they must be polynomials in $(p,q)$ with coefficients in $C^\infty([0,1]\times\Omega)$. Moreover, if we intend to consider pseudodifferential operators, then the requirement that $H$ be a polynomial must be replaced by appropriate growth conditions as $|p|+|q|\to\infty$.

Although in the theory of manifolds with conical singularities one locally deals with Mellin (pseudo)differential operators $\overset{\wedge}{H} = H\left(\overset{2}{r},\overset{2}{\omega}, ir\overset{1}{\partial}/\partial r, -i\overset{1}{\partial}/\partial\omega\right)$, $r\in\mathbf{R}_+$, rather than with usual p.d.o. $f\left(\overset{2}{r},\overset{2}{\omega}, i\overset{1}{\partial}/\partial r, -i\overset{1}{\partial}/\partial\omega\right)$, the principal symbols $H(r,\omega,p,q)$ of these operators still can be interpreted as functions on the cotangent bundle $T^*\overset{\circ}{K}$. However, in this case $r$ and $p$ are no longer the standard (Darboux) canonical coordinates on $T^*\overset{\circ}{K}$. To preserve the characteristic property

$$\sigma([\overset{\wedge}{H}_1,\overset{\wedge}{H}_2]) = -i\{H_1,H_2\},$$

where

$$\sigma([\overset{\wedge}{H}_1,\overset{\wedge}{H}_2]) = -i\left\{-r\left(\frac{\partial H_1}{\partial p}\frac{\partial H_2}{\partial r} - \frac{\partial H_1}{\partial r}\frac{\partial H_2}{\partial p}\right) + \{H_1,H_2\}_{T^*\Omega}\right\}, \qquad (3)$$

is the principal symbol of the commutator of the operators $\overset{\wedge}{H}_1$ and $\overset{\wedge}{H}_2$ with principal symbols $H_1$ and $H_2$, $\{H_1,H_2\}$ is the Poisson bracket, and $\{H_1,H_2\}_{T^*\Omega}$ is the standard Poisson bracket on $T^*\Omega$, we must take the *symplectic structure* in the form

$$\omega^2 = -\frac{1}{r}dp\wedge dr + dq\wedge d\omega. \qquad (4)$$

This can be verified by straightforward computation, but it is perhaps easier to make the change of variables $r=e^{-t}$, which reduces $ir\partial/\partial r$ to $-i\partial/\partial t$, the symplectic structure to the standard form

$$\omega^2 = dp\wedge dt + dq\wedge d\omega, \qquad (5)$$

and the Poisson bracket to the conventional expression

$$\{H_1, H_2\} = \frac{\partial H_1}{\partial p}\frac{\partial H_2}{\partial t} - \frac{\partial H_1}{\partial t}\frac{\partial H_2}{\partial p} + \{H_1, H_2\}_{T^*\Omega}. \tag{6}$$

We see from (4) that the symbols of Mellin pseudodifferential operators are smooth functions on the *compressed cotangent bundle* $\tilde{T}^*K^\wedge$ (this notion was introduced by Melrose [13]). Recall that the compressed cotangent bundle $\tilde{T}^*N$ of a $C^\infty$ manifold $N$ with boundary $\partial N$ is invariantly defined as follows. Consider the $C^\infty(N)$-module $\mathrm{Vect}\,(N)$ of vector fields on $N$ everywhere tangent to $\partial N$. The dual module $\mathrm{Hom}_{C^\infty(N)}(\mathrm{Vect}_0(N), C^\infty(N))$ is a locally free $C^\infty(N)$-module and hence is the module of sections of a $C^\infty$ vector bundle over $N$, which is denoted by $\tilde{T}^*N$. We shall write $T^*K$ instead of $\tilde{T}^*K^\wedge$. Globalizing, we define the *compressed cotangent bundle* of a manifold $M$ with conical singular points $\alpha_1, \dots, \alpha_N$ by

$$T^*M \stackrel{\mathrm{def}}{=} \tilde{T}^*M^\wedge,$$

where $M^\wedge$ is the blow-up of $M$ over all the conical points $\alpha_1, \dots, \alpha_N$.

Thus, when dealing with a manifold with conical singularities, we deal with two important rings of functions, namely, the ring

$$C^\infty(M) \stackrel{\mathrm{def}}{=} C^\infty(M^\wedge)$$

and

$$C^\infty(T^*M) \stackrel{\mathrm{def}}{=} C^\infty(\tilde{T}^*M^\wedge).$$

The symbols of Mellin pseudodifferential operators belong to the latter ring.

Throughout the paper, we shall use two representations of a manifold with conical points in a neighborhood of each of these points. The first representation is the *conical* one. We represent the manifold in a neighborhood of the conical point as a cone

$$K = ([0,1] \times \Omega) \,/\, (\{0\} \times \Omega)$$

with base $\Omega$ a smooth compact manifold without boundary. Consequently, the coordinates near the corresponding point are $(r, \omega)$, where $r \in [0,1]$ and $\omega$ are local coordinates on $\Omega$. The second representation (which is in a lot of cases preferable) is the *cylindrical representation*

$$K = ([0,+\infty] \times \Omega) \,/\, (\{+\infty\} \times \Omega),$$

where the corresponding cone is replaced by a cylinder with the help of the variable change $r = e^{-t}$ (the coordinates on $\Omega$ remain unchanged).

The standard symplectic form on $T^*M$ in these coordinates is given by formulas (4) and (5), respectively.

**Remark 1** Note that in the conical coordinates, the symplectic form (4) is singular at $r = 0$, whereas the Poisson bracket (3) is not.

**Remark 2** In the cylindrical coordinates, the elements of $C^\infty(M)$ and $C^\infty(T^*M)$ can be described as functions exponentially stabilizing to functions independent of $t$ as $t \to \infty$.

**Lemma 1** *The rule*

$$(0, \omega, p, q) \mapsto p \tag{7}$$

*determines a well-defined function* $p : \partial T^*M \to \mathbf{R}^1$ *on the boundary of the compressed cotangent bundle (in other words, the value of* $p$ *is independent of the choice of the standard local coordinates near the conical point).*

**Proof.** Any admissible change of coordinates in a neighborhood of a conical point has the form

$$r' = rA(r, \omega), \quad \omega' = B(r, \omega), \tag{8}$$

where $A(r, \omega)$ and $B(r, \omega)$ are $C^\infty$ and $A(r, \omega) > 0$. To evaluate the effect of (8) on the variables $(p, q)$, we proceed to the cylindrical coordinates, where the symplectic structure has the standard form (4) and hence the momenta $(p, q)$ are trasformed by the well-known rule. We have

$$t' = t - \ln A(e^{-t}, \omega), \quad \omega' = B(e^{-t}, \omega).$$

The Jacobi matrix of this transformation at $t = \infty$ has the form

$$J = \left.\frac{\partial(t', \omega)}{\partial(t, \omega)}\right|_{t=\infty} = \begin{pmatrix} 1 & -\dfrac{\partial A}{\partial \omega} A^{-1} \\ 0 & \dfrac{\partial B}{\partial \omega} \end{pmatrix},$$

and consequently, for $t = \infty$ we have

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = {}^t(J^{-1}) \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & \left(\dfrac{\partial B}{\partial \omega}\right)^{-1} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ * \end{pmatrix},$$

where the asterisks stand for the entries whose value is inessential to us. Thus, $p' = p$, which proves the lemma.

## 1.2 Canonical transformations and their conormal families. Generating functions

In this subsection we consider structure-preserving mappings of $T^*M$, that is, canonical transformations. Homogeneous canonical transformations were considered earlier in [13], and we study their counterpart in the true (nonhomogeneous) symplectic situation. Furthermore, we introduce the notion of the *conormal family* of a canonical transformation and study generating functions of a special form; both issues were not touched in [13] (the notion of the boundary transformation $\partial\chi$ introduced there corresponds to the value of the conormal family at $p = 0$).

**Definition 1** Let $(M, \{\alpha_1, \dots, \alpha_N\})$ be a manifold with conical singularities. A *canonical transformation* of the cotangent bundle of $M$ is a smooth diffeomorphism

$$g : T^*M \to T^*M \tag{9}$$

of manifolds with boundary (in particular,

$$g(T^* \overset{\circ}{M}) = T^* \overset{\circ}{M} \quad \text{and} \quad g(\partial T^*M) = \partial T^*M,$$

where $\partial T^*M = T^*M - T^* \overset{\circ}{M}$ is the boundary of $T^*M$) such that

$$g^*\omega^2 = \omega^2, \tag{10}$$

where $\omega^2$ is the symplectic form on $T^*M$ (recall that it has the form (4) in the special coordinate systems near the conical points).

Clearly, away from the fibers over the conical points, $g$ may be an arbitrary canonical transformation in the usual sense. Let us consider the structure of $g$ near some conical point $\alpha \in M$. It is obvious from the definition[4] that $g$ sends the entire "fiber" of $T^*M$ over $\alpha$ to the "fiber" over some (possibly, the same) conical point $\beta \in M$. In the special coordinates, we have the transformation

$$g : T^*K \to T^*K, \tag{11}$$

where $K = [0, \infty) \times \Omega$ (since $g$ is a differomorphism, it follows that the bases of cones at $\alpha$ and $\beta$ are necessarily isomorphic). Furthermore, $g$ is given by smooth functions

$$\begin{aligned} \rho &= \rho(r, \omega, p, q) = r\rho_1(r, \omega, p, q), \\ \psi &= \psi(r, \omega, p, q), \\ \xi &= \xi(r, \omega, p, q), \\ \eta &= \eta(r, \omega, p, q), \end{aligned} \tag{12}$$

[4] We consider the case in which the bases of the cones are connected.

where $(\rho, \psi, \xi, \eta)$ are the standard conical coordinates on the second copy of $T^*K$, and has the property

$$g^* \left( -\frac{d\xi \wedge d\rho}{\rho} + d\eta \wedge d\psi \right) = -\frac{dp \wedge dr}{\rho} + dq \wedge d\omega. \tag{13}$$

We can restrict $g$ to the boundary

$$\partial T^*K \cong \mathbf{R}^1_p \times T^*\Omega \tag{14}$$

of $T^*K$; this restriction is given by the mappings

$$\begin{aligned} \xi &= \xi(0, \omega, p, q), \\ \psi &= \psi(0, \omega, p, q), \\ \eta &= \eta(0, \omega, p, q). \end{aligned} \tag{15}$$

The last two lines in (15) define a family of mappings

$$g(p) : \; T^*\Omega \to T^*\Omega \tag{16}$$

depending on the parameter $p \in \mathbf{R}^1$. (By Lemma 1, this definition is coordinate-independent).

**Definition 2** The family $g(p)$ given by (16) is called the *conormal family* of the canonical transformation (9) at the conical point $\alpha \in M$.

**Theorem 1** (a) *For each $p$, the mapping* (16) *is a symplectomorphism of $T^*\Omega$.*
(b) *The mapping $\xi = \xi(0, \omega, p, q)$ has the form*

$$\xi = p + c, \tag{17}$$

*where $c = c(g, \alpha)$ is a constant independent of $(p, q)$ and depending only on $g$. It will be referred to as the conormal shift of $g$ at the conical point $\alpha \in M$.*

*Proof*. We pass to the cylindrical coordinates and rewrite (12) in the form

$$\begin{aligned} \tau &= t + \chi(e^{-t}, \omega, p, q), \\ \psi &= \psi(e^{-t}, \omega, p, q), \\ \xi &= \xi(e^{-t}, \omega, p, q), \\ \eta &= \eta(e^{-t}, \omega, p, q), \end{aligned} \tag{18}$$

where

$$\chi = -\ln \rho_1. \tag{19}$$

(We have $\rho_1 > 0$, see Lemma 2 below).

To simplify the calculations, we introduce some notation. The independent variables are $(t, \omega, p, q)$. Let $d$ be the differential with respect to all the variables, $d'$ the differential with respect to $(\omega, p, q)$ with $t$ fixed, and $d''$ the differential with respect to $(\omega, q)$ with $t$ and $p$ fixed. Furthermore, the bar over a letter indicates the restriction to $t = +\infty$ $(r = 0)$. In the cylindrical coordinates, condition (13) has the familiar form

$$d\xi \wedge d\tau + d\eta \wedge d\psi = dp \wedge dt + dq \wedge d\omega. \tag{20}$$

Let us evaluate the left-hand side of (20) using (18). We have

$$\begin{aligned} d\tau &= (1 - e^{-t}\chi_r)\, dt + d'\chi, \\ d\xi &= -e^{-t}\xi_r dt + d'\xi, \\ d\psi &= -e^{-t}\psi_r dt + d'\psi, \\ d\eta &= -e^{-t}\eta_r dt + d'\eta, \end{aligned}$$

and consequently,

$$\begin{aligned} &d\xi \wedge d\tau + d\eta \wedge d\omega \\ &\quad = \{(1 - e^{-t}\chi_r)d'\xi + e^{-t}\xi_r d'\chi + e^{-t}\eta_r d'\xi - e^{-t}\xi_r d'\eta\} \wedge dt \\ &\quad\; + d'\xi \wedge d'\chi + d'\eta \wedge d'\xi. \end{aligned} \tag{21}$$

By equating this with $dp \wedge dt + dq \wedge d\omega$, we obtain

$$d'\xi + e^{-t}[\xi_r d'\chi + \eta_r d'\xi - \chi_r d'\xi - \psi_r d'\eta] = dp, \tag{22}$$

$$d'\xi \wedge d'\chi + d'\eta \wedge d'\psi = dq \wedge d\omega, \tag{23}$$

whence it follows that

$$d'\overline{\xi} = dp, \;\; \overline{\xi} = p + c \tag{24}$$

(we assume that $\Omega$ is connected), and furthermore,

$$d\overline{\xi} \wedge d\overline{\chi} + d\overline{\eta} \wedge d\overline{\psi} = dq \wedge d\omega, \tag{25}$$

or

$$dp \wedge d\overline{\chi} + d\overline{\eta} \wedge d\overline{\psi} = dq \wedge d\omega. \tag{26}$$

Equation (24) proves (b). Let us now further evaluate (26). We have (from now we omit the bars for brevity; all calculations pertain to $\{r = 0\}$)

$$d\chi = \chi_p dp + d''\chi,$$

and similarly for $\eta$ and $\psi$. Thus, by (26), we have

$$dp \wedge \{d''\chi + \eta_p d''\psi - \psi_p d''\eta\} + d''\eta \wedge d''\psi = dq \wedge d\omega,$$

or

$$d''\chi + \eta_p d''\psi - \psi_p d''\eta = 0$$

and

$$d''\eta \wedge d''\psi = dq \wedge d\omega. \tag{27}$$

Equation (27) just means that (a) is satisfied. The proof of Theorem 1 is complete.

Let us now proceed to a more informative description of the canonical transformation $g$.

In classical symplectic geometry, it is known that any canonical transformation can be locally described by a generating function [1]. Here the lemma about the local canonical coordinates is important. However, we unfortunately cannot directly apply this lemma here, because the symplectic form itself has a singularity as $r \to 0$. Hence, we must state and prove this lemma separately.

**Lemma 2 (canonical local coordinates on the graph of g)** *Let a canonical transformation* (11) *be given, and let*

$$(0, \psi_0, \xi_0, \eta_0) = g(0, \omega_0, p_0, q_0).$$

*Then there exists a subset $I \subset \{1, \ldots, n-1\}$ such that the functions $(\rho, \psi_I, \eta_{\overline{I}}, p, q)$, where $\overline{I} = \{1, \ldots, n-1\} \backslash I$, form a system of local coordinates on the graph of $g$ in a neighborhood of the point*

$$z_0 = (0, \omega_0, p_0, q_0; 0, \psi_0, \xi_0, \eta_0) \in \operatorname{graph} g \subset T^*K \times T^*K \tag{28}$$

*Proof.* First, note that the function $\rho_1(r, \omega, p, q)$ in (12) does not vanish for $r = 0$. Indeed, we have

$$0 \neq \frac{D(\rho, \psi, \xi, \eta)}{D(r, \omega, p, q)} = \left(\rho_1 \frac{D(\psi, \xi, \eta)}{D(\omega, p, q)}\right)\bigg|_{r=0} + O(r) \tag{29}$$

by virtue of (12). Let $L = \operatorname{graph} g$. We claim that the differentials $d\rho$, $dp$ and $dq$ are linearly independent on $L$ at $z_0$. Indeed, suppose that

$$\varepsilon\, d\rho - \gamma\, dp + \delta\, dq = 0. \tag{30}$$

By (12), we have

$$d\rho = \rho_1 dr + r\, d\rho_1, \tag{31}$$

and since $r = 0$ at $z_0$, we can combine (30) with (31) to obtain

$$\varepsilon\rho_1\, dr + \gamma\, dp + \delta\, dq = 0 \tag{32}$$

at $z_0$. But the differentials $dr$, $dp$, and $dq$ are linearly independent on $L$, since $L$ is the graph of a smooth mapping. Taking into account the fact that $\rho_1 \neq 0$, we obtain $\varepsilon = \gamma = \delta = 0$, as desired. Now let $I \subset \{1, \ldots, n-1\}$ be a maximal subset such that the system

$$(d\rho = \rho_1 dr, d\psi_I, dp, dq)$$

is linearly independent on $L$ at $z_0$. Then $(\rho, \psi_I, \eta_{\overline{I}}, p, q)$ is the desired coordinate system. Indeed, let us fix $\rho$ and $p$; then $(d''\psi_I, dq)$ is a maximal linearly independent system in $(d''\psi, dq)$. By the standard lemma on local coordinates [11, 15], applied to the canonical transformation $g(p_0)$, it follows that the system $(d''\psi_I, d''\eta_{\overline{I}}, dq)$ is linearly independent. Hence, so is $(d\rho, d\psi_I, d\eta_{\overline{I}}, dp, dq)$, and the proof of the lemma is complete.

We can now descibe generating functions. For simplicity, we first consider the "nonsingular" case ($\overline{I} = \emptyset$); the result in the general case is similar, except that the notation is more complicated. In the nonsingular case, the generating function, $S$, depends on the variables $(\rho, \psi, p, q)$, and the transformation itself is defined by the implicit equations

$$\begin{aligned} \xi &= -\rho\frac{\partial S}{\partial \rho}(\rho, \psi, p, q), \\ \eta &= \frac{\partial S}{\partial \psi}(\rho, \psi, p, q), \\ r &= \exp\left(-\frac{\partial S}{\partial p}(\rho, \psi, p, q)\right), \\ \omega &= \frac{\partial S}{\partial q}(\rho, \psi, p, q) \end{aligned} \tag{33}$$

(the easiest way to obtain (33) is to pass to the cylindrical coordinates and use the standard formulas known in symplectic geometry).

Let us try to find a function $S(\rho, \psi, p, q)$ so that the transformation defined by (33) coincides with $g$. To this end, let us solve the first pair of equation in (12) for $(r, \omega)$ as functions of $(\rho, \psi, p, q)$ (which is possible in the nonsingular chart) and substitute the result into the second pair of equations in (12). Then we obtain

$$\begin{aligned} r &= \rho F(\rho, \psi, p, q), \\ \omega &= G(\rho, \psi, p, q), \\ \xi &= p + c + \rho H(\rho, \psi, p, q), \\ \eta &= I(\rho, \psi, p, q) \end{aligned} \tag{34}$$

(in the equation for $\xi$, we have taken into account Theorem 1, (b)), where $F, G, H$ and $I$ are smooth functions, $F \neq 0$).

The generating function, by virtue of (33), must satisfy the Pfaff equation

$$\begin{aligned} dS &= -\frac{\xi}{\rho} d\rho + \eta\, d\psi + \omega\, dq - \ln r \cdot dp \\ &\equiv -\left(\frac{p+c}{\rho} + H\right) d\rho - (\ln\, \rho + \Phi)\, dp + I\, d\psi + G\, dq, \end{aligned} \tag{35}$$

where $\Phi = \ln F$ is a smooth function. Of course, the integrability of (35) is guaranteed, since (12) is a canonical transformation. It readily follows from (35) that

$$S(\rho, \psi, p, q) = -(p+c) \ln \rho + S_1(\rho, \psi, p, q), \tag{36}$$

where $S_1$ is a smooth function; moreover, for each fixed $p$ the function $S_1(0, \psi, p, q)$ is a generating function of $g(p)$, where $\{g(p)\}$ is the conormal family of $g$.

Likewise, for "singular" charts ($\overline{I} \neq \emptyset$), the canonical transformation is defined by a generating function of the form

$$S_I(\rho, \psi_I, \eta_{\overline{I}}, p, q) = -(p+c) \ln \rho + S_{1I}(\rho, \psi_I, \eta_{\overline{I}}, p, q). \tag{37}$$

Let us study the following question. Under what conditions does a function of the form (36) define a canonical transformation? The answer is obvious. It is necessary that the Jacobian $D(\xi, \eta)/D(p, q)$ be nonzero:

$$\frac{D(\xi, \eta)}{D(p, q)} = \det \begin{Vmatrix} 1 - \rho \dfrac{\partial^2 S_1}{\partial\rho\partial p} & -\rho \dfrac{\partial^2 S_1}{\partial\rho\partial q} \\ \dfrac{\partial^2 S_1}{\partial\psi\partial p} & \dfrac{\partial^2 S_1}{\partial\psi\partial q} \end{Vmatrix} \neq 0. \tag{38}$$

Note that for $\rho = 0$ we have

$$\left.\frac{D(\xi, \eta)}{D(p, q)}\right|_{\rho=0} = \det \frac{\partial^2 S_1(0, \psi, p, q)}{\partial\psi\partial q} \neq 0, \tag{39}$$

which agrees with the fact that the function $S_1(0, \psi, p, q)$ determines the conormal family of canonical transformations

$$g(p) : T^*\Omega \to T^*\Omega,$$

depending on the parameter $p$.

Using (36), let us rewrite (13) in the cylindrical coordinates

$$t = -\ln r, \omega, p, q; \quad \tau = -\ln \rho, \psi, \xi, \eta.$$

We obtain the following formulas:

$$\begin{aligned}
\xi &= p + c - e^{-\tau}\frac{\partial S_1}{\partial \rho}(e^{-\tau}, \psi, p, q),\\
\eta &= \frac{\partial S_1}{\partial \psi}(e^{-\tau}, \psi, p, q),\\
t &= \tau + \frac{\partial S_1}{\partial p}(e^{-\tau}, \psi, p, q),\\
\omega &= \frac{\partial S_1}{\partial q}(e^{-\tau}, \psi, p, q).
\end{aligned}$$

Similar formulas hold for singular charts ($\overline{I} \neq \emptyset$). Specifically, let $S_I(\rho, \psi_I, \eta_{\overline{I}}, p, q)$ be the generating function of $g$ in a singular chart of type $I$, $\overline{I} \neq \emptyset$. Then

$$S_I(\rho, \psi_I, \eta_{\overline{I}}, p, q) = -(p+c)\ln\rho + S_{1I}(\rho, \psi_I, \eta_{\overline{I}}, p, q),$$

where $S_{1I}$ is a smooth function. Moreover, the formulas describing the canonical transformation read

$$\begin{aligned}
&\xi = -\rho\frac{\partial S_I}{\partial \rho}(\rho, \psi_I, \eta_{\overline{I}}, p, q) \equiv p + c - \rho\frac{\partial S_{1I}}{\partial \rho}(\rho, \psi_I, \eta_{\overline{I}}, p, q),\\
&r = \exp\left(-\frac{\partial S_I}{\partial p}(\rho, \psi_I, \eta_{\overline{I}}, p, q)\right) \equiv \rho \cdot \exp\left(-\frac{\partial S_{1I}}{\partial \rho}(\rho, \psi_I, \eta_{\overline{I}}, p, q)\right),\\
&\eta_I = \frac{\partial S_{1I}}{\partial \psi_I}(\rho, \psi_I, \eta_{\overline{I}}, p, q),\\
&\psi_{\overline{I}} = -\frac{\partial S_{1I}}{\partial \eta_{\overline{I}}}(\rho, \psi_I, \eta_{\overline{I}}, p, q),\\
&\omega = \frac{\partial S_{1I}}{\partial q}(\rho, \psi_I, \eta_{\overline{I}}, p, q),
\end{aligned}$$

or, in the cylindrical coordinates,

$$\begin{aligned}
&\xi = p + c - e^{-\tau}\frac{\partial S_{1I}}{\partial \rho}(e^{-\tau}, \psi_I, \eta_{\overline{I}}, p, q),\\
&t = \tau + \frac{\partial S_{1I}}{\partial p}(e^{-\tau}, \psi_I, \eta_{\overline{I}}, p, q),\\
&\eta_I = \frac{\partial S_{1I}}{\partial \psi_I}(e^{-\tau}, \psi_I, \eta_{\overline{I}}, p, q),
\end{aligned}$$

$$\psi_{\overline{I}} = -\frac{\partial S_{1I}}{\partial \eta_{\overline{I}}}(e^{-\tau}, \psi_I, \eta_{\overline{I}}, p, q),$$

$$\omega = -\frac{\partial S_{1I}}{\partial q}(e^{-\tau}, \psi_I, \eta_{\overline{I}}, p, q).$$

In the intersections of the charts, the generating functions of different types are related by the Legendre transformation [1]. For example, in the intersection of the nonsingular chart and the chart of type $I$ we have

$$S_I = S - \eta_{\overline{I}}\psi_{\overline{I}}, \quad \eta_{\overline{I}}\psi_{\overline{I}} \stackrel{\text{def}}{=} \sum_{j\in\overline{I}} \eta_i\psi_i,$$

where both sides are understood as functions on $L = \operatorname{graph} g$.

Let $H \in C^\infty(T^*M)$. Then corresponding Poissonian vector field

$$V(H) = \{H, \cdot\}$$

is defined, where $\{\cdot, \cdot\}$ is the Poisson bracket. In the standard canonical coordinates $(x, \xi)$ on $T^* \overset{\circ}{M}$, this is juct the Hamiltonian vector field

$$V(H) = H_\xi \partial_x - H_x \partial_\xi,$$

whereas in the conical coordinates $(r, \omega, p, q)$ near a singular point we have

$$V(H) = -r\frac{\partial H}{\partial p}\frac{\partial}{\partial r} + r\frac{\partial H}{\partial r}\frac{\partial}{\partial p} + \frac{\partial H}{\partial q}\frac{\partial}{\partial \omega} - \frac{\partial H}{\partial \omega}\frac{\partial}{\partial q}. \tag{40}$$

Of course, in the cylindrical coordinates we have the usual Hamiltonian expression for this field, but with regard to the special behavior of the coefficients of $H$ as $t \to \infty$:

$$V(H) = \frac{\partial H}{\partial p}\frac{\partial}{\partial t} + e^{-t}\frac{\partial H}{\partial r}\frac{\partial}{\partial p} + \frac{\partial H}{\partial q}\frac{\partial}{\partial \omega} - \frac{\partial H}{\partial \omega}\frac{\partial}{\partial q}. \tag{41}$$

Consider the phase flow $\{g_H^\tau\}$ of the field $V(H)$. This is obviously a one-parameter family of (generally, local) diffeomorphisms of $T^* \overset{\circ}{M}$ preserving the form $\omega^2$, i.e., a one-parameter family of canonical transformations of $T^* \overset{\circ}{M}$. But what is the boundary behavior of these transformations? The following important theorem shows that the class of transformations we consider in natural.

**Theorem 2** *The phase flow $\{g_H^\tau\}$ consists of canonical transformations of the compressed cotangent bundle $T^*M$.*

*Proof.* Using the conical representation (40), we see that the field $V(H)$ is smooth up to the boundary on $T^*M$ and that the $\partial/\partial r$ component of this field vanishes on $\partial T^*M$, whence the desired assertion follows readily.

# 2 Quantization

In this section, we deal with *asymptotic quantization on* $M$ (e.g., see [10, 19]). We introduce appropriate Sobolev spaces, rewrite the definition of a pseudodifferential operator so as to involve the small parameter $h$, and finally introduce Mellin–Maslov integral operators on $M$ and state the boundedness and composition theorems. The construction of these operators is very much parallel to the construction of Fourier integral operators in [13]; however, the central point in our exposition is the notion of the *conormal symbol* of a Mellin–Maslov integral operator, which is lacking in [13].

## 2.1 Weighted Sobolev spaces with small parameter $h$

Let $M$ be a compact manifold with conical singularities $\{\alpha_1, \dots, \alpha_N\}$. By $\overset{\circ}{M}$ we denote the smooth part of the manifold $M$, that is, the complement in $M$ of the set $\{\alpha_1, \dots, \alpha_N\}$.

The operators considered in this paper will act in weighted Sobolev spaces $H_h^{s,\gamma}(M)$, where $h \to 0$ is a small parameter. These are defined as follows. Let

$$\gamma : \{\alpha_1, \dots, \alpha_N\} \to \mathbf{R}$$

be a given mapping (the *weight exponent vector*). Let $s \in \mathbf{R}$. For a function $u$ on $M$, the norm $||u||_{s,\gamma}$ is defined with the help of a partition of unity

$$\sum_{j=0}^{N} e_j(x) = 1, \quad x \in M,$$

where $e_0(x) \equiv 0$ in a neighborhood of the set $\{\alpha_1, \dots, \alpha_N\}$ and each function $e_j(x)$, $j = 1, \dots, N$, is supported in a neighborhood $U_j$ of $\alpha_j$ where the local conical representation

$$U_j \backslash \{\alpha_j\} = (0, 1) \times \Omega_j$$

is valid. We set

$$||u||_{s,\gamma}^2 = \sum_{j=0}^{N} ||e_j u||_{s,\gamma}^2,$$

where $||e_0 u||_{s,\gamma} = ||e_0 u||_s$ is one of the (equivalent) norms on the usual Sobolev space $H_h^s\left(M \backslash \cup_{j=1}^N U_j\right)$ (say,

$$||e_0 u||_s^2 = \int \left| \left(1 - h^2 \frac{\partial^2}{\partial x^2}\right)^{s/2} u \right|^2 dx$$

in local coordinates, see [15]), and $||e_j u||_{s,\gamma}$, $j = 1, \dots, N$, is defined as follows in the conical coordinates $(r, \omega) \in (0,1) \times \Omega_j$:

$$||e_j u||^2_{s,\gamma} = ||e_j u||_{s,\gamma(\alpha_j)} = \int r^{-2\gamma(\alpha_j)} \left\| \left( 1 + \left( ihr\frac{\partial}{\partial r} \right)^2 + h^2 \Delta_{\Omega_j} \right)^{s/2} e_j u \right\|^2_{L^2(\Omega_j)} \frac{dr}{r}.$$

Here $\Delta_{\Omega_j}$ is the positive Beltrami–Laplace on $\Omega_j$ with respect to some Riemannian metric. For sections of a vector bundle[5] $E$ over $M$, the spaces $H^{s,\gamma}_h(M, E)$ are now defined in an obvious way.

## 2.2 Symbol spaces and $1/h$-pseudodifferential operators. The $1/h$-Mellin transform

We shall consider operators that are usual $1/h$-pseudodifferential operators outside a neighborhood of a conical points and that can be represented in the form

$$\widehat{A} = A \left( \overset{2}{r}, \overset{2}{\omega}, ihr \overset{1}{\frac{\partial}{\partial r}}, -ih \overset{1}{\frac{\partial}{\partial \omega}}, h \right)$$

in the conical coordinate system $(r, \omega)$ near each conical point. We consider the following symbol spaces $\Sigma^{m,\mu} = \Sigma^{m,\mu}(T^*M \times [0,1])$. Outside a neighborhood of the conical points, any symbol $H(x,p,h)$, where $(x,p)$ are canonical coordinates on $T^*M$, must satisfy the estimates

$$|D^\alpha_x D^\beta_p \varphi(x,p,h)| \le C_{\alpha\beta} h^\mu (1 + |p|^2)^{(m-|\beta|)/2} \tag{42}$$

and be representable in the form

$$\varphi(x,p,h) = \sum_{k=0}^{N} h^{\mu+k} \varphi_k(x,p) + R_N(x,p,h)$$

for any integer $N$. Here the function $R_N(x,p,h)$ must satisfy the estimates (42) with $m$ replaced by $m + N + 1$. Near each conical point, in the conical coordinates $(r, \omega, p, q)$, where

$$p \leftrightarrow ihr\frac{\partial}{\partial r}, \quad q \leftrightarrow -ih\frac{\partial}{\partial \omega},$$

[5]A vector bundle over $M$ is defined as a vector bundle over $M^\wedge$.

the symbol $H(r,\omega,p,q,h)$ must satisfy the estimates

$$\left|\frac{\partial^\alpha}{\partial r^\alpha}\frac{\partial^\beta}{\partial \omega^\beta}\frac{\partial^\gamma}{\partial p^\gamma}\frac{\partial^\delta}{\partial q^\delta}H(r,\omega,p,q,h)\right| \le C_{\alpha\beta\gamma\delta}h^\mu(1+|p|^2+|q|^2)^{(m-|\gamma|-|\delta|)/2} \tag{43}$$

and possess the expansion

$$H(r,\omega,p,q,h) = \sum_{k=0}^{N} h^{\mu+k}H_k(r,\omega,p,q,h) + R_N(r,\omega,p,q,h), \tag{44}$$

where $R_N$ satisfies the same estimates with $\mu$ replaced by $\mu+N+1$, for any integer $N$.

**Lemma 3** *If $H \in \Sigma^{m,\mu}$, then*

$$\widehat{H} = H\left(\overset{2}{r},\overset{2}{\omega}, ihr\,\overset{1}{\frac{\partial}{\partial r}}, -ih\,\overset{1}{\frac{\partial}{\partial \omega}}, h\right) \,:\, H_h^{s,0} \to H_h^{s-m,0} \tag{45}$$

*is a continuous operator for any $s \in \mathbf{R}$, and*[6]

$$\|\widehat{H}\|_{s,0\to s-m,0} \le Ch^\mu.$$

The proof does not differ very much from the proof of a similar statement (see e.g. [20]).

To ensure boundedness of $\widehat{H}$ in weighted Sobolev spaces with $\gamma \ne 0$, we must require the symbol $H$ to be defined and satisfy (43) and (44) on the line

$$\{\operatorname{Im} p = h\gamma\}.$$

This is obvious in view of the fact that the multiplication by $r^\gamma$ is an isomorphism between $H_h^{s,0}$ and $H_h^{s,\gamma}$ and the fact that

$$r^{-\gamma}\left(ihr\frac{\partial}{\partial r}\right)r^\gamma = ihr\frac{\partial}{\partial r} + ih\gamma, \tag{46}$$

so that the argument $p$ of the symbol undergoes a shift by $ih\gamma$. Note also the following obvious formula for the operator

$$\widehat{H} \,:\, H_h^{s,\gamma} \to H_h^{s-m,\gamma}.$$

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[6]Unless otherwise specified, all constants in the estimates are independent of $h$.

One has

$$[\widehat{H}u](r,\omega) = \mathcal{M}_{h,\gamma}^{-1} H\Big(r, \overset{2}{\omega}, p, -ih\,\overset{1}{\frac{\partial}{\partial\omega}}\Big)\mathcal{M}_{h,\gamma}u, \tag{47}$$

where

$$[\mathcal{M}_{h,\gamma}f](p) = \int_0^\infty r^{ip/h} f(r)\frac{dr}{r}, \qquad p \in \mathcal{L}_{h\gamma} = \{\,\mathrm{Im}\,p = h\gamma\}, \tag{48}$$

is the $(h^{-1},\gamma)$-*Mellin transform* of a function $f$, and $\mathcal{M}_{h,\gamma}^{-1}$ is the inverse transform, given by

$$[\mathcal{M}_{h,\gamma}^{-1}\,\tilde{f}](r) = \frac{1}{2\pi h}\int_{\mathcal{L}_{\gamma h}} r^{-ip/h}\,\tilde{f}\,(p)\,dp. \tag{49}$$

By virtue of the isomorphism (46), we can always reduce the case $\gamma \neq 0$ to the case $\gamma = 0$, which is assumed in the sequel.

**Lemma 4 (the composition formula)** *If $H_i \in \Sigma^{m_i,\mu_i}$, $i = 1,2$, then $\widehat{H}_1 \circ \widehat{H}_2 = \widehat{H}$, where*

$$H(r,\omega,p,q,h) = H_1\Big(\overset{2}{r}, \overset{2}{\omega}, p + ihr\,\overset{1}{\frac{\partial}{\partial r}}, q - ih\,\overset{1}{\frac{\partial}{\partial\omega}}, h\Big)(H_2) \in \Sigma^{m_1+m_2,\mu_1+\mu_2}.$$

The proof is a standard exercise in noncommutative analysis (cf. [21]).

The notion of the conormal symbol [20] is generalized to $h^{-1}$-pseudodifferential operator in the standard way. Let $\hat{D}$ be a pseudodifferential operator on $M$. Thus, in the conical coordinates we have

$$\widehat{D} = \widehat{D}\Big(\overset{2}{r}, ihr\overset{1}{\frac{d}{dr}}\Big), \tag{50}$$

where $\widehat{D}(r,p)$ is the operator-valued symbol of $\widehat{D}$ (with values in the algebra of $h^{-1}$-pseudodifferential operators on $\Omega$). The operator family

$$\widehat{D}_0(p) = \widehat{D}(0,p) \tag{51}$$

is called the *conormal symbol* of the operator $\widehat{D}$ given by (50).

The conormal symbol can be obtained in the following special way.

Let $U_\lambda$, $\lambda > 0$, be the operator acting on functions of $r$ by the rule

$$U_\lambda f(r) = f(\lambda r); \tag{52}$$

then $U_\lambda^{-1} = U_{\lambda^{-1}}$ and

$$U_\lambda \circ \left(-ir\frac{d}{dr}\right) \circ U_\lambda^{-1} = -ir\frac{d}{dr}, \quad U_\lambda \circ r \circ U_\lambda^{-1} = \lambda r. \tag{53}$$

(Note that in the cylindrical coordinates $U_\lambda$ is represented by the translation by $\ln\lambda$). Then

$$s\text{-}\lim_{\lambda\to 0} U_\lambda \overset{\wedge}{D}\left(\overset{2}{r}, ih\overset{1}{\frac{\partial}{\partial r}}\right) U_\lambda^{-1} = s\text{-}\lim_{\lambda\to 0} \widehat{D}\left(\lambda r, ihr\frac{d}{dr}\right) = \widehat{D}\left(0, ihr\frac{d}{dr}\right)$$
$$= \widehat{D}_0\left(ihr\frac{d}{dr}\right) = \mathcal{M}_h^{-1} \circ \widehat{D}_0(p) \circ \mathcal{M}_h, \tag{54}$$

where $\widehat{D}_0(p)$ is the conormal symbol, $\mathcal{M}_h$ is the $1/h$-Mellin transform, and s-lim stands for the strong limit in $\operatorname{Hom}(H^{s,\gamma}(K) \to H^{s-m,\gamma}(K))$ (here $m = \operatorname{ord}\hat{D}$ and $K$ is the model cone).

## 2.3 Quantizations of canonical transformations

Let

$$g : T^*M \to T^*M$$

be a conical transformation of the compressed cotangent bundle $T^*M$, and let $a \in C_0^\infty(T^*M)$ be a smooth compactly supported function (note that this condition requires neither that $\operatorname{supp} a \cap \partial T^*M = \emptyset$ nor even that $a \equiv 0$ everywhere on $T^*M$). Under an additional condition, we intend to define an operator $T(g,a)$, associated with the pair $(g,a)$, in the Mellin–Sobolev scale $H^{s,0}(M)$.

Let

$$L_g = \Big\{(\rho,\psi,\xi,\eta,r,\omega,p,q) \in T^*([0,1]\times\Omega)\times T^*([0,1]\times\Omega) \mid (\rho,\psi,\xi,\eta) = g(r,\omega,p,q)\Big\}$$

be the graph of $g$, and let

$$\pi_i : L_g \to T^*([0,1]\times\Omega)$$

be the projection onto the $i$th factor, $i = 1, 2$. We set

$$\tilde{a} = \pi_2^* a. \tag{55}$$

Thus, $\widetilde{a}$ is a function on $L_g$.

The manifold $L_g$ is Lagrangian in $T^*M \times T^*M$ with respect to the 2-form

$$\tilde{\omega}^2 = \pi_1^*\omega^2 - \pi_2^*\omega^2,$$

where $\omega^2$ is the standard symplectic structure on the compressed cotangent bundle $T^*M$. We impose the following condition.

**Condition 1** *The manifold $L_g$ is quantized* ([11], [15]).

Condition 1 is necessarily satisfied if, for example, $g = g_H^\tau$ for some $\tau$ and some Hamiltonian $H \in C^\infty(T^*M)$.

If Condition 1 is satisfied, then Maslov's canonical operator $K_{L_g}$ ([11], [15]) is well defined on $L_g$; for any $\varphi \in C_0^\infty(L)$,

$$(K_{L_g}\varphi)(x,y), \ \ (x,y) \in M \times M$$

is the kernel of some integral operator acting on $M$. We define $T(g,a)$ as the integral operator with kernel $(K_{L_g}\widetilde{a})(x,y)$. Outside a neighborhood of the conical points, this definition is quite standard; let us write out the expression for $T(g,a)$ for the case in which supp $\widetilde{a}$ is contained in a chart of type $I$ near the conical point (as is usual in the theory of Maslov's canonical operator, the global definition is readily patched from local ones with the help of the partition of unity technique). Let $S_I(\rho, \psi_I, \eta_{\overline{I}}, p, q)$ be the generating function of $g$ in that chart.

We express $\widetilde{a}$ via the nonsingular canonical coordinates $(\rho, \psi_I, \eta_{\overline{I}}, p, q)$ on the graph of $g$ and write $\widetilde{a} = \widetilde{a}\,(\rho, \psi_I, \eta_{\overline{I}}, p, q)$.

Then the *operator* $T(g,a)$ acts on functions $u(r,\omega)$, $(r,\omega) \in [0,1] \times \Omega$ as follows:

$$\begin{aligned}[][T(g,a)u](\rho,\psi) &= \frac{1}{2\pi h}\left(\frac{i}{2\pi h}\right)^{(n+|\overline{I}|-1)/2} \iint e^{(i/h)[S_I(\rho,\psi_I,\eta_{\overline{I}},p,q)+\psi_{\overline{I}}\eta_{\overline{I}}]}\,\widetilde{a}\,(\rho,\psi,p,q) \\ &\quad \times \left(\frac{D(\xi,\eta_I,\psi_{\overline{I}})}{D(p,q)}\right)^{1/2}(\xi,\psi_I,\eta_{\overline{I}},p,q)\widetilde{u}(p,q)\,dp\,dq\,d\eta_{\overline{I}}, \qquad (56)\end{aligned}$$

where

$$\widetilde{u}(p,q) = -\left(\frac{i}{2\pi h}\right)^{(n-1)/2} \int_0^\infty \frac{dr}{r} \int_{\mathbf{R}^n} d\omega \left\{ r^{ip/h} e^{-(i/h)q\omega} u(q,\omega) \right\} \qquad (57)$$

is the $1/h$-Fourier–Mellin transform of $u(r,\omega)$ and the argument of the Jacobian is chosen in the way prescribed by the construction of the canonical operator. Let us write

$$\widetilde{a}\left(\frac{D(\xi,\eta_I,\psi_{\overline{I}})}{D(p,q)}\right)^{1/2} = b$$

for brevity. We can rewrite $T(g,a)$ as the integral operator with the kernel

$$\begin{aligned} K_{(g,a)}(\rho,\psi,r,\omega) &\equiv K(\rho,\psi,r,\omega) \\ &= \left(\frac{1}{2\pi h}\right)^{n+|\overline{I}|/2} \iint r^{ip/h} e^{(i/h)[S_I(\rho,\psi_I,\eta_{\overline{I}},p,q)-q\omega+\psi_{\overline{I}}\eta_{\overline{I}}]} \\ &\quad \cdot b(\rho,\psi_I,\eta_{\overline{I}},p,q)\,dp\,dq\,d\eta_{\overline{I}}, \end{aligned} \tag{58}$$

where the integration measure for this kernel is, of course, $d\omega\,dr/r$ rather than $d\omega\,dr$.

Using the explicit form of the generating function $S_I$, we can rewrite (58) as follows:

$$\begin{aligned} K(\rho,\psi,r,\omega) &= \left(\frac{1}{2\pi h}\right)^{n} + |\overline{I}|/2 \iint \left(\frac{r}{\rho}\right)^{ip/h} \rho^{-ic/h} e^{(i/h)[S_{1I}(\rho,\psi_I,\eta_{\overline{I}},p,q)-q\omega+\psi_{\overline{I}}\eta_{\overline{I}}]} \\ &\quad \cdot b(\rho,\psi_I,\eta_{\overline{I}},p,q)\,dp\,dq\,d\eta_{\overline{I}}. \end{aligned} \tag{59}$$

Let us now state the main properties of the operators $T(g,a)$. In the following theorem, $g$, $g_1$ and $g_2$ are arbitrary canonical transformations of $T^*M$, $a, a_1, a_2 \in C_0^\infty(T^*M)$ and $H \in \Sigma^{m,0}(T^*M)$.

**Theorem 3** (a) *The operator*

$$T(g,a):\ H^{s,0}(M) \to H^{k,0}(M)$$

*is continuous uniformly with respect to* $h \in (0,1]$ *for any* $s$ *and* $k$.

(b) *For any* $g_1$, $g_2$ $a_1$, *and* $a_2$, *there exists a function* $a \in C_0^\infty(T^*M \times [0,1])$ *such that*

$$T(g_1,a_1) \circ T(g_2,a_2) = T(g_1,g_2,a) + O(h^N)$$

*for any* $N > 0$, *where* $O(h^N)$ *stands for an operator whose norm in the pair of spaces* $(H^{s,0}(M), H^{k,0}(M))$ *is bounded for any* $s$ *and* $k$. *Moreover,*

$$a = (g_2^* a_1) \cdot a_2 + O(h).$$

(c) *If* $\hat{H}$ *is a* $1/h$*-pseudodifferential operator on* $M$ *with principal symbol* $H$, *then*

$$\hat{H} \circ T(g,a) = T(g,a_1) + O(h^N)$$

*for any* $N$, *where*

$$a_1 = g^*(H) \cdot a + O(h).$$

(d) *We have*

$$T(g,a_1)^* T(g,a_2) = \overset{\wedge}{a} + O(h^N) \tag{60}$$

*for any $N$, where the symbol $a$ of the pseudodifferential operator $\overset{\wedge}{a}$ is compactly supported and satisfies*

$$\tilde{a} = \overline{a}_1 a_2.$$

*In (60), the asterisk denotes the adjoint operator in $L^2(M) \equiv H^{0,0}(M)$.*

(e) *If $H$ is a real-valued function for which the Hamiltonian flow is globally well defined, then there exists a function $a_t \in C_0^\infty(T^*M \times \mathbf{R}_t)$ such that the operator*

$$U_t = T(g_H^t g, a_t)$$

*satisfies the Cauchy problem*

$$-ih\frac{\partial U_t}{\partial t} + \overset{\wedge}{H}\, U_t = O(h^N) \ \textit{ for any } N,$$

$$U_t|_{t=0} = T(g,a).$$

**Proof.** All these assertions are well known for the case in which $M$ is a $C^\infty$ manifold, and so they follow automatically once the support of the amplitude $a$ of the operator $T(g,a)$ in question has an empty intersection with $\partial T^*M$. Thus, it suffices to carry out the proof for the case in which $\operatorname{supp} a$ lies in a small neighborhood of $\partial T^*M$. Moreover, we can assume that $\operatorname{supp} a$ is small enough to ensure that only one canonical chart is involved in the definition of the canonical operator. To simplify the exposition, we assume that this is a *nonsingular* chart, that is, $\overline{I} = \emptyset$. Thus, $T(g,a)$ is an integral operator with the kernel

$$K(\rho,\psi,r,\omega) = \left(\frac{1}{2\pi h}\right)^n \iint \left(\frac{r}{\rho}\right)^{ip/h} \rho^{-ic/h} e^{(i/h)S_1(\rho,\psi,p,q)-q\omega} b(\rho,\psi,p,q)\,dp\,dq, \tag{61}$$

where $b(\rho,\psi,p,q)$ is a compactly supported function. Let us proceed to the cylindrical coordinates by setting $r = e^{-t}$, $\rho = e^{-\tau}$. Then we arrive at the kernel (which we denote by the same letter)

$$K(\tau,\psi,t,\omega) = \left(\frac{1}{2\pi h}\right)^n \iint e^{(i/h)p(\tau-t)} e^{(i/h)c\tau} e^{(i/h)S_1(e^{-\tau},\psi,p,q)-q\omega} b(e^{-\tau},\psi,p,q)\,dp\,dq. \tag{62}$$

Set

$$\Phi(\rho,\psi,p,q) = S_1(\rho,\psi,p,q) - q\psi. \tag{63}$$

Then it is obvious from (62) that $T(g,a)$ can be rewritten in the form

$$T(g,a) = e^{(i/h)[\Phi(\overset{2}{e^{-t}},\overset{2}{\omega},-ih\overset{1}{\partial/\partial t},-ih\overset{1}{\partial/\partial\omega})+c\overset{2}{t}]} b\left(\overset{2}{e^{-t}},\overset{2}{\omega},-ih\overset{1}{\frac{\partial}{\partial t}},-ih\overset{1}{\frac{\partial}{\partial\omega}}\right). \qquad (64)$$

Let us prove (c). Let

$$\hat{H} = H\left(\overset{2}{t},\overset{2}{\omega},-ih\overset{1}{\frac{\partial}{\partial t}},-ih\overset{1}{\frac{\partial}{\partial\omega}}\right)$$

be a $1/h$-pseudodifferential operator; then, by virtue of the composition formulas for functions of noncommuting operators [12, 18]

$$\begin{aligned}
&\mathrm{Smbl}\,(\hat{H}\circ T(g,a)) \\
&= H\left(\overset{2}{t},\overset{2}{\omega},p-ih\overset{1}{\frac{\partial}{\partial t}},q-ih\overset{1}{\frac{\partial}{\partial\omega}}\right) e^{(i/h)[\Phi(e^{-t},\omega,p,q)+ct]}b(e^{-t},\omega,p,q) \\
&= e^{(i/h)[\Phi(e^{-t},\omega,p,q)+ct]} \\
&\times H\left(\overset{2}{t},\overset{2}{\omega},\overset{1}{\overline{p+c-ih\frac{\partial}{\partial t}+e^{-t}\frac{\partial\Phi}{\partial r}}},\overset{1}{\overline{q-ih\frac{\partial}{\partial\omega}+\frac{\partial\Phi}{\partial\omega}}}\right) b(e^{-t},\omega,p,q).
\end{aligned}$$

By expanding this in power of $h$ in the standard manner, we obtain (c); moreover, we can readily obtain the subsequent terms of the expansion, i.e., continue the expansion to an arbitrary high power of $h$.

To prove (e), one solves the Cauchy problem asymptotically in the standard way [11].

Now items (a), (b), and (d) can be proved by the Cauchy problem method as follows. Any canonical transformation $g$ can be locally included in a family $\{g^t\}$ of canonical transformations such that $g^1 = g$ and $g^0 = id$ (see [16, 15]). Let $H$ be a Hamiltonian such that $g_H^t = g^t$ in a sufficiently large ball and $H \equiv 0$ outside a larger ball (the size of the interior ball is chosen so that it includes $\mathrm{supp}\,((g^t)^* a)$ for all $t\in[0,1]$). Then it is easy to construct an amplitude $a_t \in C_0^\infty(M)$, $t\in[0,1]$, so that

$$U_t = T(g^t, a^t)$$

satisfies the Cauchy problem

$$\begin{cases} -ih\frac{\partial \hat{U}_t}{\partial t} + \frac{1}{2}(\hat{H}+\hat{H}^*)\hat{U}_t = O(h^N), \\ \hat{U}_t\big|_{t=0} = \hat{a}_0 \end{cases} \qquad (65)$$

for arbitrary large $N$.

Now (a) follow by the method of energy inequalities: it follows from (62) that $\|T(g,a)\|_{L^2}$ can be estimated by const $h^{-n}$, and from (65) we obtain

$$-ih\frac{\partial}{\partial t}(\hat{U}_t^*\hat{U}_t) = O(h^N). \qquad (66)$$

Hence,

$$U_t^*U_t = \hat{a}_0^*\hat{a}_0 + O(h^N),$$

and the boundedness of $T(g,a) = \hat{U}_1$ in $L^2$ follows from the boundedness theorem for Mellin pseudodifferential operators. The proof of (b) and (d) can readily be conducted by the same method.

## 2.4 The conormal symbol of a quantized canonical transformation

Let $T(g,a)$ be an operator on $M$. We intend to define the conormal symbol of $T(g,a)$ at a conical point $\alpha \in M$.

Consider the function family $a(p)$, $p \in \mathbf{R}$, on $T^*\Omega$ given by

$$a(p)(\omega,q) = a(r,p,\omega,q)\big|_{r=0}, \quad (\omega,q) \in T^*\Omega. \qquad (67)$$

Furthermore, let $g(p)$ be the conormal family of canonical transformations associated with $g$. Then the family

$$T(p) = T(g(p),a(p)) \qquad (68)$$

of operators on $\Omega$ will be called the *conormal symbol of the operator* $T(g,a)$.

**Proposition 1** *Suppose that the conormal shift of $g$ is zero* [7]. *Then*

$$s\text{-}\lim_{\lambda\to 0} U_\lambda T(g,a) U_\lambda^{-1} = \mathcal{M}_h^{-1} \circ T(p) \circ \mathcal{M}_h, \qquad (69)$$

*where $T(p)$ is the conormal symbol of $T(g,a)$ and $\mathcal{M}_h$ is the $1/h$-Mellin transform.*

*Proof.* The proof is by straightforward computation. We have

$$[T(g,a)u](\rho,\psi) = \int K(\rho,\psi,r,\omega)u(r,\omega)\frac{d\omega\,dr}{r}, \qquad (70)$$

[7] The conormal shift was defined in Theorem 1.

where the kernel $K(\rho,\psi,r,\omega)$ is given by (59) with $c=0$. Then for the kernel $K_\lambda(\rho,\psi,r,\omega)$ of the operator $U_\lambda T(g,a)U_\lambda^{-1}$ we have (in the nonsingular chart)

$$\begin{aligned} K_\lambda(\rho,\psi,r,\omega) &= K(\lambda\rho,\psi,\lambda r,\omega) \qquad (71)\\ &= \left(\frac{1}{2\pi h}\right)^n \iint \left(\frac{\lambda r}{\lambda\rho}\right)^{ip/h} e^{(i/h)[S_1(\lambda\rho,\psi,p,q)-\omega q]} b(\lambda\rho,\psi,p,q)\,dp\,dq\\ &\xrightarrow{\lambda\to 0} \left(\frac{1}{2\pi h}\right)^n \iint \left(\frac{r}{\rho}\right)^{ip/h} e^{(i/h)[S_1(0,\psi,p,q)-\omega q]} b(0,\psi,p,q)\,dp\,dq, \end{aligned}$$

which readily proves (69). In the charts of general type ($\overline{I}\neq\emptyset$), the computation is similar.

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