Exponential Function of Pseudo-Differential Operators

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May 26, 1997

Abstract

The paper is devoted to the construction of the exponential function of a matrix pseudo-differential operator which do not satisfy any of the known theorems (see, Sec.8 Ch.VIII and Sec.2 Ch.XI of [17]). The applications to the construction of the fundamental solution for the Cauchy problem for the hyperbolic operators with the characteristics of variable multiplicity are given, too.

AMS subject classification: primary: 35L80, 35S10 ; secondary: 45D05.
Key words and phrases: pseudodifferential operators, exponential function, Gevrey classes, hyperbolic operators, multiple characteristics.

Contents

Introduction

The remarkable progress made in the theory of linear partial differential over the last decades is essentially due to the extensive application of the microlocal analysis. In particular, the Gevrey pseudo-differential operators and related wave front sets were used starting from 1970 as a natural version of the corresponding C^∞ techniques. Nowadays they play a definite and important role, in connection with the study of the linear and nonlinear partial differential operators with multiple characteristics. In this order of ideas the pseudo-differential operators of infinite order are quite important.

Operators of infinite order appears in particular in the construction of the fundamental solution for the Cauchy problem for the operators with multiple characteristics when lower-oder term of operators satisfy so-called Gevrey-type Levi conditions. These conditions for the second order equation

$$D_t^2 u - t^{2l} D_x^2 u + at^k D_x u = f$$

implies $1 \leq \theta < (2l-k)/(l-k-1)$ where θ is Gevrey exponent, guarantee that the Cauchy problem is well-possed in Gevrey class $G^{(\theta)}(\Omega)$ [5], [6], [18]. On the other hand according to the well-known Hörmander theorem the wave front set of solutions is contained in the characteristic set of the principal symbol of the operator. This hints at the structure of the fundamental solution of the Cauchy problem, namely it can be written as a product of the Fourier integral operator with zero-order elliptic symbol and the pseudo-differential operator. If $k \geq l-1$ last pseudo-differential operators is of finite order [19] while for l, k, θ satisfying above inequalities, this is pseudo-differential operators of infinite order [15], [20]. This is one of the main steps of the construction of the fundamental solution carried out in [20] as well as of the proof of the uniqueness theorem for the degenerate elliptic operators [3]. Thus, the main ingredient of such construction is the exponential function of the pseudo-differential operator. The simplest case is described in Section 5 [20] where operators with x-independent coefficients are considered. In that article by means of Fourier transform the problem is reduced to the ordinary differential equation with the large parameter and turning point. Then the representation of any solution by amplitude functions and phase functions is constructed. Inverse Fourier transform gives the fundamental solution in the form of the sum of Fourier integral operators of infinite order.

1 Preliminaries

Let Ω be an open set in $\mathbb{R}^n$, K a compact subset of Ω and let $\theta > 1, A > 0$. We denote by $G^{(\theta),A}(K)$ the set of all complex valued functions $\varphi \in C^\infty(\Omega)$ such that

$$\|\varphi\|^{K,A} = \sum_{\alpha\in\mathbb{Z}^n_+} A^{-|\alpha|}\alpha!^{-\theta} \sup_{x\in K} |D^\alpha_x \varphi(x)| < +\infty.$$

We set

$$G^{(\theta)}(\Omega) = \lim_{\substack{\longleftarrow \\ K\to\Omega}} \lim_{\substack{\longrightarrow \\ A\to+\infty}} G^{(\theta),A}(K)\,,$$

$$G_0^{(\theta),A}(K) = G^{(\theta),A}(K) \cap C_0^\infty(K)\,,$$

and

$$G_0^{(\theta)}(\Omega) = \lim_{\substack{\longrightarrow \\ K\to\Omega}} \lim_{\substack{\longrightarrow \\ A\to+\infty}} G^{(\theta),A}(K) \cap C_0^\infty(K)\,.$$

The space of the ultradistributions of order θ on Ω, $G^{(\theta)'}(\Omega)$, $G_0^{(\theta)'}(\Omega)$, are defined as the duals of $G^{(\theta)}(\Omega)$, $G_0^{(\theta)}(\Omega)$, respectively. $G^{(\theta)'}(\Omega)$ can be identified with the subspace of ultradistributions of $G_0^{(\theta)'}(\Omega)$ with compact support. $G_0^{(\theta)}(\Omega)$, $G_0^{(\theta)}(\Omega)$, $G^{(\theta)'}(\Omega)$ and $G_0^{(\theta)'}(\Omega)$ are complete, Montel and Schwartz spaces.
The Fourier transform of $u \in G_0^{(\theta)'}(\mathbb{R}^n)$ is the function $\hat{u}(\xi) = u(\exp(-i < \cdot, \xi >))$, $\xi \in \mathbb{R}^n$. This function can be extended to an entire analytic function in $\mathbb{C}^n$, called the "Fourier-Laplace transform of u".
A theorem analogous to the Paley-Wiener theorem holds for the elements of $G_0^{(\theta)}(\mathbb{R}^n)$ and $G_0^{(\theta)'}(\mathbb{R}^n)$. For later use, we state it here. One can find that Theorem in [7], [8], [13]. In the exposition of the material of this section we follow the paper of Zanghirati [22] and the book of Rodino [13].

Theorem 1.1 ([7],[8], [13], [22]) *Let K be a compact convex set in $\mathbb{R}^n$ and let v be an entire function on $\mathbb{C}^n$. v is the Fourier-Laplace transform of a function $u \in G_0^{(\theta)}(\mathbb{R}^n)$ with support in K if and only if there exist constants c and $\varepsilon > 0$ such that*

$$|v(\zeta)| \leq c\exp(-\varepsilon|\xi|^{(\theta)'} + H_k(\zeta)), \quad \zeta \in \mathbb{C}^n,$$

where $H_k(\zeta) = \sup_{x\in K} < x, \mathrm{Im}\zeta >$.
v is the Fourier-Laplace transform of an ultradistribution $u \in G_0^{(\theta)}(\mathbb{R}^n)$ with support in K if and only if for every $\varepsilon > 0$ there exists a constant c_ε such that:

$$|v(\zeta)| \leq c\exp(-\varepsilon|\xi|^{1/\theta} + H_k(\zeta)), \quad \zeta \in \mathbb{C}^n,$$

Moreover a sequence $u_j \in G_0^{(\theta)}(\mathbb{R}^n)$ *(resp.* $G_0^{(\theta)'}(\mathbb{R}^n)$*) with support in* K *converges if and only if for some* $\varepsilon > 0$*)* $\exp(\varepsilon|\xi|^{1/\theta})|\hat{u}_j(\xi)|$ *(resp.* $\exp(\varepsilon|\xi|^{1/\theta})|\hat{u}_j(\xi)|$*) converges uniformly in* $\mathbb{R}^n$*.*

Finally we recall the Kernel theorem for ultradistributions:

Theorem 1.2 ([7],[8], [13], [22]) *Let* Ω *be an open set in* $\mathbb{R}^n$ *and denote by* $L(G_0^{(\theta)}(\Omega), G_0^{(\theta)'}(\Omega))$ *the space of all continuous linear maps* $T : G_0^{(\theta)}(\Omega) \longrightarrow G_0^{(\theta)'}(\Omega)$*. Then there exists an isomorphism between* $L(G_0^{(\theta)}(\Omega), G_0^{(\theta)'}(\Omega))$ *and* $G_0^{(\theta)'}(\Omega \times \Omega)$*.*

The ultradistributions in $\Omega \times \Omega$ corresponding to T is called the kernel of T.
Let Ω be an open set in $\mathbb{R}^n$ and let θ, ρ, δ be real numbers such that $\theta > 1$, $0 \leq \delta < \rho \leq 1$, $\theta\rho \geq 1$.

Definition 1.1 ([13],[22]) *We shall denote by* $S^{\infty,\theta}_{\rho,\delta}(\Omega)$ *the space of all functions* $p \in C^\infty(\Omega \times \mathbb{R}^n)$ *satisfying the following condition: for every compact subset* $K \subset \Omega$ *there exist constants* $C > 0$ *and* $B \geq 0$ *and for every* $\varepsilon > 0$ *there exists a constant* c_ε *such that*

$$\sup_{x\in K} \|D_\xi^\alpha D_x^\beta p(x,\xi)\| \leq C_\varepsilon C^{|\alpha+\beta|}\alpha!\beta!^{\theta(\rho-\delta)}(1+|\xi|)^{-\rho|\alpha|+\delta|\beta|}\exp(\varepsilon|\xi|^{1/\theta}) \qquad (1.1)$$

for every $\alpha, \beta \in \mathbb{Z}_+^n$ *and for every* $\xi \in \mathbb{R}^n$ *such that* $|\xi| \geq B|\alpha|^\theta$*.*

We remark that the product of elements of $S^{\infty,\theta}_{\rho,\delta}(\Omega)$ is in $S^{\infty,\theta}_{\rho,\delta}(\Omega)$ and that any θ-ultradifferential operator $P(D) = \sum_{|\alpha|=0}^\infty a_\alpha D^\alpha$, $a_\alpha \in C$, such that for every $\varepsilon > 0$ there is a constant c_ε such that $|a_\alpha| \leq c_\varepsilon \varepsilon^{|\alpha|}(\alpha!)^{-\theta}$, maps $S^{\infty,\theta}_{\rho,\delta}(\Omega)$ into itself.
If $p \in S^{\infty,\theta}_{\rho,\delta}(\Omega)$ and $u \in G_0^{(\theta)}$ the for the Paley-Wiener Theorem 1.1, the integral $\int e^{i<x,\xi>}p(x,\xi)\hat{u}(\xi)d\xi$ is convergent. Thus for every $p \in S^{\infty,\theta}_{\rho,\delta}(\Omega)$ we can define on $G_0^{(\theta)}(\Omega)$ the operator:

$$Pu(x) = (2\pi)^{-n}\int e^{i<x,\xi>}p(x,\xi)\hat{u}(\xi)d\xi, \quad u \in G_0^{(\theta)}(\Omega). \qquad (1.2)$$

Definition 1.2 ([13],[22]) $OPS^{\infty,\theta}_{\rho,\delta}(\Omega)$ *will denote the space of all operators* P *of the form (1.2) with* $p \in S^{\infty,\theta}_{\rho,\delta}(\Omega)$*. We shall write* $P = p(\cdot, D)$*.*

The following lemma plays a very important role in the study of operators in $OPS^{\infty,\theta}_{\rho,\delta}(\Omega)$.

Lemma 1.1 [13],[22] *Let $p \in S^{\infty,\theta}_{\rho,\delta}(\Omega)$ and let K be a compact subset of Ω and $A > 0$. There exist positive constants ε_0, C, B such that:*

$$\left| \int e^{i<x,\eta>} p(x,\xi)\hat{v}(x)dx \right| \leq C \exp(-2\varepsilon_0|\eta|^{1/\theta} + \varepsilon_0|\xi|^{1/\theta})\|v\|_{K,A} \tag{1.3}$$

for every $\xi \in \mathbb{R}^n, |\eta| > B, v \in G_0^{(\theta),A}(\Omega)$.

Theorem 1.3 [13],[22] *If $p \in S^{\infty,\theta}_{\rho,\delta}(\Omega)$, then P defined by (1.2) , is a continuous linear operator from $G_0^{(\theta)}(\Omega)$ to $G^{(\theta)}(\Omega)$ that can be extended as an operator from $G^{(\theta)'}(\Omega)$ to $G_0^{(\theta)'}(\Omega)$.*

Definition 1.3 ([13],[22]) *We denote by $FS^{\infty,\theta}_{\rho,\delta}(\Omega)$ the space of all formal sums $\sum_{j\geq 0} p_j(x,\xi)$, where $p_j \in C^\infty(\Omega \times \mathbb{R}^n)$ satisfies the following condition: for every compact set $K \subset \Omega$ there exist constants $C > 0$ and $B \geq 0$ and for every $\varepsilon > 0$ there exists a constant c_ε, such that for any $\alpha, \beta \in \mathbb{Z}^n_+$ and every $\xi \in R^n$ with $|\xi| \geq B(|\alpha|+j)^\theta$:*

$$\begin{aligned} \sup_{x\in K} |D^\alpha_\xi D^\beta_x p_j(x,\xi)| &\leq C_\varepsilon C^{|\alpha+\beta|+j} \alpha!(\beta!j!)^{\theta(\rho-\delta)} \\ &\quad \times (1+|\xi|)^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)j} \exp(\varepsilon|\xi|^{1/\theta}). \end{aligned} \tag{1.4}$$

Definition 1.4 ([13],[22]) *$\sum_{j\geq 0} p_j$ and $\sum_{j\geq 0} q_j$ from $FS^{\infty,\theta}_{\rho,\delta}(\Omega)$ are said to be equivalent*
$\left(\sum_{j\geq 0} p_j \sim \sum_{j\geq 0} q_j\right)$ if for any compact set $K \subset \Omega$ we can find $C > 0$ and $B \geq 0$ and for every $\varepsilon > 0$ we can find c_ε such that:

$$\begin{aligned} \sup_{x\in K} |D^\alpha_\xi D^\beta_x \sum_{j<s}(p_j(x,\xi) - q_j(x,\xi))| &\leq C_\varepsilon C^{|\alpha+\beta|+s} \alpha!\beta!s!^{\theta(\rho-\delta)} \\ &\quad \times (1+|\xi|)^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)s} \exp(\varepsilon|\xi|^{1/\theta}), \end{aligned} \tag{1.5}$$

for every $\alpha, \beta \in \mathbb{Z}^n_+$ and for $|\xi| \geq B(s+|\alpha|)^\theta$.

Theorem 1.4 ([13],[22]) *Let $\sum_{j\geq 0} p_j \in FS^{\infty,\theta}_{\rho,\delta}(\Omega)$. For every open relatively compact subset Ω' of Ω there exists $p \in S^{\infty,\theta}_{\rho,\delta}(\Omega)$ which equivalent to $\sum_{j\geq 0} p_j\Big|_{\Omega'}$ in the sense of the Definition 1.3.*

Proposition 1.1 ([13],[22]) *Let $p \sim 0$ in $FS^{\infty,\theta}_{\rho,\delta}(\Omega)$, then for every compact set $K \in \Omega$ there exist positive constants c, h, B such that:*

$$\sup_{x\in K} |D^\beta_x p(x,\xi)| \le c^{|\beta|+1}\beta!^{\theta(\rho-\delta)}(1+|\xi|)^{\delta|\beta|}\exp(-h|\xi|^{1/\theta}) \tag{1.6}$$

for every $\beta \in \mathbb{Z}^n_+$, $|\xi| \ge B$.

Definition 1.5 ([13],[22]) *We shall denote by $V^\theta_R(\Omega)$ the space of all continuous linear operators from $G^{(\theta)}_0(\Omega)$ to $G^{(\theta)}(\Omega)$ which extend as continuous linear map of $G^{(\theta)'}(\Omega)$ into $G^{(\theta)'}_0(\Omega)$ and by $V^\theta_R(\Omega)$ the subspace of $V^\theta(\Omega)$ of the operators which extend as continuous linear maps of $G^{(\theta)'}$ into $G^{(\theta)}$. If $P \in V^\theta_R(\Omega)$, we shall say that P is θ-regularizing on Ω.*

Proposition 1.2 ([13],[22]) *Let $p \in C^\infty(\Omega \times \mathbb{R}^n)$ and suppose that for every compact set $K \subset \Omega$ there exist positive constants c, h, B for which:*

$$\sup_{x\in K} |D^\beta_x p(x,\xi)| \le c^{|\beta|+1}\beta!^{\theta}\exp(-h|\xi|^{1/\theta}), \quad |\xi| \ge B. \tag{1.7}$$

Then, the operator P defined on $G^{(\theta)}_0(\Omega)$ by:

$$Pu(x) = (2\pi)^{-n}\int e^{i<x,\xi>}p(x,\xi)\hat{u}(\xi)d\xi\,, \tag{1.8}$$

is θ-regularizing.
Moreover, if $p \in S^{\infty,\theta}_{\rho,\delta}(\Omega)$ then this is true even if (1.7) is satisfied only when $|\xi| \ge c_1|\xi|^\theta$, where the constant c_1 depends on K.

Corollary 1 ([13],[22]) *If $p \sim 0$ in $FS^{\infty,\theta}_{\rho,\delta}(\Omega)$ then $p(\cdot, D) \in V^\theta_R(\Omega)$.*

Definition 1.6 ([13],[22]) *If $p, q \in S^{\infty,\theta}_{\rho,\delta}(\Omega)$ we define Leibniz product $p \circ q$ to be the formal sum $\sum_{j\ge 0} r_j$, where:*

$$r_j(x,\xi) = \sum_{|\alpha|=j}(\alpha!)^{-1}\partial^\alpha_\xi p(x,\xi)D^\alpha_x q(x,\xi).$$

Proposition 1.3 ([13],[22]) *If* $p, q \in S^{\infty,\theta}_{\rho,\delta}(\Omega)$*, then* $p \circ q \in FS^{\infty,\theta}_{\rho,\delta}(\Omega)$.

Definition 1.7 ([13],[22]) *If* $\sum_{j\geq 0} p_j, \sum_{j\geq 0} q_j \in FS^{\infty,\theta}_{\rho,\delta}(\Omega)$ *let*

$$\left(\sum_{j\geq 0} p_j\right) \circ \left(\sum_{j\geq 0} q_j\right) = \sum_{j\geq 0} r_j,$$

where

$$r_j = \sum_{|\alpha|+h+k=j} (\alpha!)^{-1} \partial^\alpha_\xi p_h D^\alpha_x q_k.$$

Proposition 1.4 ([13],[22]) $\left(\sum_{j\geq 0} p_j\right) \circ \left(\sum_{j\geq 0} q_j\right) \in FS^{\infty,\theta}_{\rho,\delta}(\Omega)$.

Proposition 1.5 ([13],[22]) *If* $\sum_{j\geq 0} p'_j \sim \sum_{j\geq 0} p_j, \sum_{j\geq 0} q'_j \sim \sum_{j\geq 0} q_j$*, then*

$$\left(\sum_{j\geq 0} p'_j\right) \circ \left(\sum_{j\geq 0} q'_j\right) \sim \left(\sum_{j\geq 0} p_j\right) \circ \left(\sum_{j\geq 0} q_j\right).$$

In particular if $\sum_{j\geq 0} q_j \sim q$*, then* $\left(\sum_{j\geq 0} q_j\right) \circ p \sim p \circ p$.

Definition 1.8 ([13],[22]) *For* $p \in S^{\infty,\theta}_{\rho,\delta}(\Omega)$ *we define* $p^\#$ *to be the formal sum* $\sum_{j\geq 0} q_j$ *, where:*

$$q_j(x,\xi) = \sum_{|\alpha|=j} (\alpha!)^{-1} \partial^\alpha_\xi D^\alpha_x p(x,-\xi).$$

More generally, for $\sum_{j\geq 0} p_j \in FS^{\infty,\theta}_{\rho,\delta}(\Omega)$ *we define* $\left(\sum_{j\geq 0} p_j\right)^\#$ *to be formal* $\sum_{j\geq 0} q_j$*, where*

$$q_j(x,\xi) = \sum_{|\alpha|+h=j} (\alpha!)^{-1} \partial^\alpha_\xi D^\alpha_x p_h(x,-\xi).$$

Proposition 1.6 ([13],[22]) *If* $\sum_{j\geq 0} p_j \in FS^{\infty,\theta}_{\rho,\delta}(\Omega)$*, then also* $\left(\sum_{j\geq 0} p_j\right)^\# \in FS^{\infty,\theta}_{\rho,\delta}(\Omega)$ *and* $\left(\left(\sum_{j\geq 0} p_j\right)^\#\right)^\# \sim \sum_{j\geq 0} p_j$.

Now we enlarge our class of operators by considering operators of the form:

$$Au(x) = (2\pi)^{-n} \int \left(\int e^{i<x-y,\xi>} u(y) a(x,y,\xi) dy \right) d\xi \tag{1.9}$$

Definition 1.9 ([13],[22]) *Let θ, ρ, δ and Ω be as in Definition 1.1. We denote by $S^{\infty,\theta}_{\rho,\delta}(\Omega \times \Omega)$ the space of all functions $a(x,y,\xi) \in C^\infty(\Omega \times \Omega \times \mathbb{R}^n)$ satisfying the following condition:*

for every compact set $W \subset \Omega \times \Omega$ there exist constants $C \geq 0, B \geq 0$ and for every $\varepsilon > 0$ there exists a constant c_ε, such that:

$$\sup_{x \in K} \| D^\alpha_\xi D^\beta_x D^\gamma_y a(x,y,\xi) \| \leq C_\varepsilon C^{\alpha+\beta+\gamma} \alpha! (\beta! \gamma!)^{\theta(\rho-\delta)} (1 + |\xi|)^{-\rho|\alpha| + \delta|\beta+\gamma|} \exp(\varepsilon |\xi|^{1/\theta}) \tag{1.10}$$

for every $\alpha, \beta, \gamma \in \mathbb{Z}^\kappa_+$ and for every $\xi \in \mathbb{R}^n$ such that $|\xi| \geq B|\alpha|^\theta$.

We have

$$A : G^{(\theta)}_0(\Omega) \longrightarrow G^\theta(\Omega) \quad \text{is continuous} \tag{1.11}$$

Definition 1.10 ([13],[22]) *$OPS^{\infty,\theta}_{\rho,\delta}(\Omega \times \Omega)$ will denote the space of all operator of the form (1.9) with $a \in S^{\infty,\theta}_{\rho,\delta}(\Omega \times \Omega)$.*

By (1.4) we can define the kernel K of $A \in OPS^{\infty,\theta}_{\rho,\delta}(\Omega \times \Omega)$: K is the θ-ultradistribution in $G^\theta_0(\Omega)$ such that:

$$< K, v \otimes u > = < Au, v >, \quad u, v \in G^{(\theta)}_0(\Omega). \tag{1.12}$$

Formally

$$K(x,y) = (2\pi)^{-n} \int e^{<x-y,\xi>} a(x,y,\xi) d\xi\,. \tag{1.13}$$

Proposition 1.7 ([13],[22]) *The kernel of $A \in OPS^{\infty,\theta}_{\rho,\delta}(\Omega \times \Omega)$ is in $G^{(\theta)}(\Omega \times \Omega \setminus \Delta)$.*

Theorem 1.5 ([13],[22]) *Let $A \in OPS^{\infty,\theta}_{\rho,\delta}(\Omega \times \Omega)$. For every open relatively compact subset Ω' of Ω there exists $p(\cdot, D) \in OPS^{\infty,\theta}_{\rho,\delta}(\Omega')$ such that $A - p(\cdot, D) \in V^\theta_R(\Omega')$. Moreover, if $a(x,y,\xi)$ is an amplitude of A, then $p(x,\xi) \sim \sum_\alpha (\alpha!)^{-1} D^\alpha_\xi D^\alpha_y a(x,y,\xi)|_{y=x}$ in $FS^{\infty,\theta}_{\rho,\delta}(\Omega')$.*

Proposition 1.8 ([13],[22]) *If $P = p(\cdot, D) \in OPS^{\infty,\theta}_{\rho,\delta}(\Omega)$ then ${}^t p \in OPS^{\infty,\theta}_{\rho,\delta}(\Omega \times \Omega)$ and has amplitude $p(y,\xi)$. Moreover there exists Q in $OPS^{\infty,\theta}_{\rho,\delta}(\Omega)$ such that ${}^t P - Q \in V^{\theta}_R(\Omega)$ and $Q = q(\cdot, D)$ with $q(x,\xi) \sim p^{\#}(x,\xi)$.*

Proposition 1.9 ([13],[22]) *If $A \in OPS^{\infty,\theta}_{\rho,\delta}(\Omega\times\Omega)$ then there exists $B \in OPS^{\infty,\theta}_{\rho,\delta}(\Omega\times\Omega)$ with amplitude $b(y,\xi)$ depending only on y such that $A - B \in V^{\theta}_R(\Omega)$. In particular, if $A = q(\cdot, D) \in OPS^{\infty,\theta}_{\rho,\delta}(\Omega)$, then $b(y,\xi) \sim q^{\#}(y,-\xi)$ in $FS^{\infty,\theta}_{\rho,\delta}(\Omega)$.*

Definition 1.11 ([13],[22]) *A linear continuous operator A from $G^{(\theta)}_0(\Omega)$ to $G^{(\theta)'}_0(\Omega)$ is proper if its kernel K is a proper θ-ultradistribution on $\Omega \times \Omega$, i.e. if supp K has compact intersection with $H \times \Omega$ and with $\Omega \times H$ for every compact set $H \subset \Omega$. Alternatively A is proper if A and ${}^t A$ are continuous maps from $G^{(\theta)}_0(\Omega)$ to $G^{(\theta)'}_0(\Omega)$.*

Remark 1.1 ([13],[22]) *Let $A \in OPS^{\infty,\theta}_{\rho,\delta}(\Omega \times \Omega)$. If $\theta(\rho - \delta) > 1$ there is a properly supported operator $A_1 \in OPS^{\infty,\theta}_{\rho,\delta}(\Omega \times \Omega)$ such that $A - A_1 \in V^{\theta}_R(\Omega)$.*

Theorem 1.6 ([13],[22]) *Let $p(\cdot, D), q(\cdot, D) \in S^{\infty,\theta}_{\rho,\delta}(\Omega)$. Suppose that at least one of them is properly supported. Then $p(\cdot, D)q(\cdot, D) = t(\cdot, D) + R$ where $t(x,\xi) \sim p(x,\xi) \circ q(x,\xi)$ in $FS^{\infty,\theta}_{\rho,\delta}(\Omega)$ and $R \in V^{\theta}_R(\Omega)$.*

2 Result

We consider the Cauchy problem

$$\begin{cases} D_t Q(t,s) + \mathcal{R}(t,s)Q(t,s) + \mathcal{R}_0(t,s) \in C([0,T_1]; V^{\theta}_R(\mathbb{R}^n)), \\ Q(s,s) = 0 \quad (0 \le s \le t \le T_1), \end{cases} \tag{2.1}$$

where $R(t,s)$, $R_0(t,s)$ be matrix PDO of infinite order symbols $r(t,s,x,\xi)$, $r_0(t,s,x,\xi)$, respectively,belonging to the class $C([0,T_1] \times [0,T_1]; S^{\infty,\theta}_{\rho,\delta}(\mathbb{R}^n))$, where $\theta > 1, 0 \le \delta < \rho \le 1$, $\theta(\rho - \delta) > 1$. We have the following

Theorem 2.1 *We assume that for every compact subset $K \subset \mathbb{R}^n$ there exist constants $C > 0$ and $B \geq 0$ and for every $\varepsilon > 0$, there exists a constant c_ε, such that for any α, β for all $0 \leq s \leq t \leq T, \xi \in R^n, |\xi| \geq B|\alpha|^\theta$, $x \in \mathbb{R}^n$*

$$
\begin{aligned}
\sup_{x \in K} \|D_\xi^\alpha D_x^\beta r_0(t,s,x,\xi)\| &\leq c_\varepsilon C^{\alpha+\beta} \alpha! \beta!^{\theta(\rho-\delta)} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|} \\
&\quad \times \exp(\varepsilon \langle\xi\rangle^{1/\theta}) g_\varepsilon(t,\xi)\,, \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
\sup_{x \in K} \|D_\xi^\alpha D_x^\beta r(t,s,x,\xi)\| &\leq C^{\alpha+\beta} \alpha! \beta!^{\theta(\rho-\delta)} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|} g_\varepsilon(t,\xi)\,, \qquad (2.3)\\
\int_0^T g_\varepsilon(\tau,\xi) d\tau &\leq \varepsilon \langle\xi\rangle^{1/\theta} + C \ln\langle\xi\rangle\,, \quad \int_0^T g_\varepsilon(\tau,\xi) d\tau \leq C \langle\xi\rangle^{1/\theta}\,, \qquad (2.4)\\
g_\varepsilon(t,\xi) &\leq c_\varepsilon \exp(\varepsilon \langle\xi\rangle^{1/\theta})\,. \qquad (2.5)
\end{aligned}
$$

Then there exists a solution $Q(t,s)$ of the problem (2.1) with the symbol $q(t,s,x,\xi)$ satisfying all $0 \leq s \leq t \leq T, \xi \in R^n, |\xi| \geq B|\alpha|^\theta$, $x \in \Omega \subset \mathbb{R}^n$ with some constants C_1, C_2 and $0 \leq \delta_1 < \rho_1 \leq 1$ the inequalities

$$
\begin{aligned}
\sup_{x \in K} \|D_\xi^\alpha D_x^\beta q(t,s,x,\xi)\| &\leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^k \langle\xi\rangle^{-\rho_1|\alpha|+\delta_1|\beta|} \alpha! (\beta!)^{\theta(\rho-\delta)} \\
&\quad \times \exp(3\varepsilon \langle\xi\rangle^{1/\theta})(\varepsilon\langle\xi\rangle^{1/\theta} + C\ln\langle\xi\rangle) \qquad (2.6)
\end{aligned}
$$

for all $k = 0, 1, \ldots$, uniformly with respect to $(t,s) \in [0,T_1] \times [0,T_1]$. Thus,

$$
q(t,s,x,\xi) \in C([0,T_1] \times [0,T_1]; FS^{\infty,\theta}_{\rho_1,\delta_1}(\Omega))\,.
$$

The solution is unique modulo $V_R^{(\theta)}(\mathbb{R}^n)$.

3 The Proof of Theorem 2.1

Existence. We select proper representatives of the equivalence classes of operators $R(t,s)$ and $R_0(t,s)$ and we construct proper operator $Q(t,s)$.
We shall seek the solution in the form

$$
q \sim q_0 + q_1 + q_2 + \cdots \qquad (3.1)
$$

where

$$
D_t q_k + r q_k + r_k = 0\,, \quad q_k(s,s) = 0\,, \quad k = 0,1,2,\ldots, \qquad (3.2)
$$

$$
r_k = \sum_{l=0}^{k-1} \sum_{|\alpha|=k-l} \frac{1}{\alpha!} \partial_\xi^\alpha r D_x^\alpha q_l\,, \quad k = 1,2,\ldots, \qquad (3.3)
$$

$$q_k = -i\int_s^t r_k(s_1)ds_1 + \sum_{l=2}^{\infty}(-i)^l \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{l-1}} ds_l r(s_1)\cdots r(s_{l-2}) r_k(s_{l-1})\,. \tag{3.4}$$

We introduce the operator $(Ir)(t) = \int_s^t r(s_1)ds_1$. If g is a scalar function, then $\underbrace{IgIg\cdots Ig}_{l} = (Ig)^l/l!$. We rewrite (3.4) in the form

$$q_k = -iIr_k + \sum_{l=2}^{\infty}(-i)^l \underbrace{Ir\cdots IrIr_k}\,. \tag{3.5}$$

Lemma 3.1 *There exist constants C_1, C_2 and $B \geq 0$ and for every $\varepsilon > 0$, there exists a constant c_ε, such that for any α, β, k $(k = 1, 2, \ldots)$ for all $0 \leq s \leq t \leq T$, $\xi \in \mathbb{R}^n$, $|\xi| \geq B|\alpha|^\theta$, $x \in \mathbb{R}^n$*

$$\sup_{x\in K} \|D_\xi^\alpha D_x^\beta r_k(t,s,x,\xi)\| \leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^k \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)k} \frac{(|\alpha|+k)!}{k!}((|\beta|+k)!)^{\theta(\rho-\delta)} \times \exp(\varepsilon\langle\xi\rangle^{1/\theta}) g_\varepsilon(t,\xi)(1+Ig)^{k-1}\sum_{l=0}^{\infty}\frac{(Ig)^l}{l!} \tag{3.6}$$

$$\sup_{x\in K} \|D_\xi^\alpha D_x^\beta q_k(t,s,x,\xi)\| \leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^k \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)k} \frac{(|\alpha|+k)!}{k!}((|\beta|+k)!)^{\theta(\rho-\delta)} \times \exp(\varepsilon\langle\xi\rangle^{1/\theta}) Ig(1+Ig)^{k}\sum_{l=1}^{\infty}\frac{(Ig)^l}{l!}\,. \tag{3.7}$$

Proof. We prove the lemma by induction on k. Since from (3.5) we have

$$q_0 = \sum_{l=1}^{\infty}(-i)^l \underbrace{Ir\cdots IrIr_0}_{l},$$

it follows that

$$\sup_{x\in K}\|D_\xi^\alpha D_x^\beta q_0\| \leq \sum_{l=1}^{\infty} \sum_{\substack{\alpha_1,\ldots,\alpha_{l-1} \\ \alpha_1\leq\alpha,\ldots,\alpha_{l-1}\leq\alpha}} \sum_{\substack{\beta_1,\ldots,\beta_{l-1} \\ \beta_1\leq\beta,\ldots,\beta_{l-1}\leq\beta}} \frac{\alpha!}{\alpha_1!\cdots\alpha_{l-1}!(\alpha-\alpha_1-\cdots-\alpha_{l-1})!} \times\frac{\beta!}{\beta_1!\cdots\beta_{l-1}!(\beta-\beta_1-\cdots-\beta_{l-1})!}$$

$$
\begin{aligned}
&\times\|D_\xi^{\alpha_1}D_x^{\beta_1}Ir\|\cdots\|D_\xi^{\alpha_{l-1}}D_x^{\beta_{l-1}}Ir\|\;\|D_\xi^{\alpha-\sum_{j=1}^{l-1}\alpha_j}D_x^{\beta-\sum_{j=1}^{l-1}\beta_j}Ir_0\|\\
\le\;& c_\varepsilon C_1^{|\alpha+\beta|}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}\alpha!(\beta!)^{\theta(\rho-\delta)}\exp(\varepsilon\langle\xi\rangle^{1/\theta})\sum_{l=1}^{\infty}\sum_{\substack{\alpha_1,\cdots,\alpha_{l-1}\\ \alpha_1\le\alpha,..,\alpha_{l-1}\le\alpha}}\\
&\times\sum_{\substack{\beta_1,\cdots,\beta_{l-1}\\ \beta_1\le\beta,..,\beta_{l-1}\le\beta}}\left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots+\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}\\
&\times\left(\frac{\beta_1!\cdots\beta_{l-1}!(\beta-\beta_1-\cdots-\beta_{l-1})}{\beta!}\right)^{\theta(\rho-\delta)-1}\underbrace{Ig\cdots Ig}_{l}\\
\le\;& c_\varepsilon C_1^{|\alpha+\beta|}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}|\alpha|!(|\beta|!)^{\theta(\rho-\delta)}\exp(\varepsilon\langle\xi\rangle^{1/\theta})\\
&\times\sum_{l=1}^{\infty}\sum_{\substack{\alpha_1,\cdots,\alpha_{l-1}\\ \alpha_1\le\alpha,..,\alpha_{l-1}\le\alpha}}\sum_{\substack{\beta_1,\cdots,\beta_{l-1}\\ \beta_1\le\beta,..,\beta_{l-1}\le\beta}}\left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots+\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}\frac{(Ig)^l}{l!}\\
\le\;& c_\varepsilon C_1^{|\alpha+\beta|}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}|\alpha|!|\beta|!^{\theta(\rho-\delta)}\exp(\varepsilon\langle\xi\rangle^{1/\theta})\sum_{l=1}^{\infty}\frac{(Ig)^l}{l!}\,.
\end{aligned}
$$

Since by means of the appropriate choice of the constant C_1 we can get an inequality

$$
\sum_{\substack{\alpha_1,\cdots,\alpha_{l-1}\\ \alpha_1\le\alpha,..,\alpha_{l-1}\le\alpha}}\sum_{\substack{\beta_1,\cdots,\beta_{l-1}\\ \beta_1\le\beta,..,\beta_{l-1}\le\beta}}\left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots+\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}\le 1\,,
$$

if $|\alpha+\beta|\neq 0$. (For $\alpha=\beta=0$, the estimate is easily seen from (2.2), (2.3)).
We perform the transfer to $k=1$. Since

$$
r_1=\sum_{|\gamma|=1}\frac{1}{\gamma!}(\partial_\xi^\gamma r)(D_x^\gamma q_0)\,,
$$

we have

$$
\begin{aligned}
&\sup_{x\in K}\|D_\xi^\alpha D_x^\beta r_1\|\\
\le\;& \sum_{|\gamma|=1}\sum_{\alpha_1\le\alpha}\sum_{\beta_1\le\beta}\frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!}\frac{\beta!}{\beta_1!(\beta-\beta_1)!}\frac{1}{\gamma!}\|D_\xi^{\alpha_1+\gamma}D_x^{\beta_1}r\|\;\|D_\xi^{\alpha-\alpha_1}D_x^{\beta-\beta_1+\gamma}q_0\|\\
\le\;& c_\varepsilon\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)}g_\varepsilon(t,\xi)\exp(\varepsilon\langle\xi\rangle^{1/\theta})\\
&\times\sum_{|\gamma|=1}\sum_{\alpha_1\le\alpha}\sum_{\beta_1\le\beta}\frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!}\frac{\beta!}{\beta_1!(\beta-\beta_1)!}\frac{1}{\gamma!}C^{|\alpha_1+\beta_1+\gamma|}C_1^{|\alpha-\alpha_1+\gamma+\beta-\beta_1|}
\end{aligned}
$$

$$
\begin{aligned}
&\times(\alpha_1+\gamma)!(\beta_1!)^{\theta(\rho-\delta)}|\alpha-\alpha_1|!(|\gamma+\beta-\beta_1|!)^{\theta(\rho-\delta)}\sum_{l=1}^{\infty}\frac{(Ig)^l}{l!}\\
\leq\ & c_\varepsilon C_1^{|\alpha+\beta|}C_2\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)}g_\varepsilon(t,\xi)\exp(\varepsilon\langle\xi\rangle^{1/\theta})\\
&\times(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)}\sum_{l=1}^{\infty}\frac{(Ig)^l}{l!}\sum_{|\gamma|=1}\sum_{\alpha_1\leq\alpha}\sum_{\beta_1\leq\beta}\left(\frac{C}{C_1}\right)^{|\alpha_1+\beta_1|}\left(\frac{CC_1}{C_2}\right)^{|\gamma|}\\
&\times\frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!}\frac{\beta!}{\beta_1!(\beta-\beta_1)!}\frac{1}{\gamma!}\frac{(\alpha_1+\gamma)!|\alpha-\alpha_1|!}{(|\alpha|+1)!}\left(\frac{\beta_1!\,|\gamma+\beta-\beta_1|!}{(|\beta|+1)!}\right)^{\theta(\rho-\delta)}\\
\leq\ & c_\varepsilon C_1^{|\alpha+\beta|}C_2\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)}g_\varepsilon(t,\xi)\exp(\varepsilon\langle\xi\rangle^{1/\theta})(|\alpha|+1)!\\
&\times((|\beta|+1)!)^{\theta(\rho-\delta)}\sum_{l=1}^{\infty}\frac{(Ig)^l}{l!}\sum_{|\gamma|=1}\sum_{\alpha_1\leq\alpha}\sum_{\beta_1\leq\beta}\left(\frac{C}{C_1}\right)^{|\alpha_1+\beta_1|}\left(\frac{CC_1}{C_2}\right)^{|\gamma|}\\
&\times\frac{\alpha!}{(\alpha-\alpha_1)!}\frac{(|\alpha-\alpha_1|+1)!}{(|\alpha|+1)!}\frac{(\alpha_1+\gamma)!}{\alpha_1!\gamma!}\frac{|\beta|!}{|\beta_1|!|\beta-\beta_1|!}\left(\frac{|\beta_1|!\,|\gamma+\beta-\beta_1|!}{(|\beta|+1)!}\right)^{\theta(\rho-\delta)}\\
\leq\ & c_\varepsilon C_1^{|\alpha+\beta|}C_2\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)}g_\varepsilon(t,\xi)\exp(\varepsilon\langle\xi\rangle^{1/\theta})(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)}\\
&\times\sum_{l=1}^{\infty}\frac{(Ig)^l}{l!}\sum_{|\gamma|=1}\sum_{\alpha_1\leq\alpha}\sum_{\beta_1\leq\beta}\left(\frac{C}{C_1}\right)^{|\alpha_1+\beta_1|}\left(\frac{CC_1}{C_2}\right)^{|\gamma|}n^{|\alpha_1+\gamma|}\\
&\times\frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!}\frac{\alpha_1!}{|\alpha_1|!}\frac{|\alpha_1|!(|\alpha-\alpha_1|+1)!}{(|\alpha|+1)!}\\
&\times\frac{|\beta|!(|\beta-\beta_1|+1)!}{(|\beta|+1)!|\beta-\beta_1|!}\left(\frac{|\beta_1|!\,|\gamma+\beta-\beta_1|!}{(|\beta|+1)!}\right)^{\theta(\rho-\delta)-1}\\
\leq\ & c_\varepsilon C_1^{|\alpha+\beta|}C_2\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)}g_\varepsilon(t,\xi)\exp(\varepsilon\langle\xi\rangle^{1/\theta})(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)}\\
&\times\sum_{l=1}^{\infty}\frac{(Ig)^l}{l!}\sum_{|\gamma|=1}\sum_{\alpha_1\leq\alpha}\sum_{\beta_1\leq\beta}\left(\frac{C}{C_1}\right)^{|\alpha_1+\beta_1|}\left(\frac{CC_1}{C_2}\right)^{|\gamma|}n^{|\alpha_1+\gamma|}\\
&\times\frac{|\alpha|!}{|\alpha_1|!|\alpha-\alpha_1|!}\frac{|\alpha_1|!(|\alpha-\alpha_1|+1)!}{(|\alpha|+1)!}\frac{(|\beta-\beta_1|+1)}{(|\beta|+1)}\left(\frac{|\beta_1|!\,|\gamma+\beta-\beta_1|!}{(|\beta|+1)!}\right)^{\theta(\rho-\delta)-1}\\
\leq\ & c_\varepsilon C_1^{|\alpha+\beta|}C_2\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)}g_\varepsilon(t,\xi)\exp(\varepsilon\langle\xi\rangle^{1/\theta})(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)}\\
&\times\sum_{l=1}^{\infty}\frac{(Ig)^l}{l!}\sum_{|\gamma|=1}\sum_{\alpha_1\leq\alpha}\sum_{\beta_1\leq\beta}\left(\frac{C}{C_1}\right)^{|\alpha_1+\beta_1|}\left(\frac{CC_1}{C_2}\right)^{|\gamma|}n^{|\alpha_1+\gamma|}\frac{|\alpha|!(|\alpha-\alpha_1|+1)!}{|\alpha-\alpha_1|!(|\alpha|+1)!}\\
\leq\ & c_\varepsilon C_1^{|\alpha+\beta|}C_2\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)}g_\varepsilon(t,\xi)\exp(\varepsilon\langle\xi\rangle^{1/\theta})(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)}
\end{aligned}
$$

$$\times\sum_{l=1}^{\infty}\frac{(Ig)^l}{l!}\sum_{|\gamma|=1}\sum_{\alpha_1\le\alpha}\sum_{\beta_1\le\beta}\left(\frac{Cn}{C_1}\right)^{|\alpha_1+\beta_1|}\left(\frac{nCC_1}{C_2}\right)^{|\gamma|}\frac{(|\alpha_1|+1)}{(|\alpha|+1)}$$
$$\le c_\varepsilon C_1^{|\alpha+\beta|}C_2\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)}g_\varepsilon(t,\xi)\exp(\varepsilon\langle\xi\rangle^{1/\theta})$$
$$\times(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)}\sum_{l=1}^{\infty}\frac{(Ig)^l}{l!}$$

First we chose constant C_1 then constant C_2 such that $(|\gamma|>0)$

$$\sum_{|\gamma|=1}\sum_{\alpha_1\le\alpha}\sum_{\beta_1\le\beta}\left(\frac{Cn}{C_1}\right)^{|\alpha_1+\beta_1|}\left(\frac{nCC_1}{C_2}\right)^{|\gamma|}<1\,.$$

We consider now q_1

$$q_1=\sum_{l=1}^{\infty}\underbrace{Ir\dots IrIr_1}_{l} \tag{3.8}$$

from where

$$\sup_{x\in K}\|D_\xi^\alpha D_x^\beta q_1\|\le\sum_{l=1}^{\infty}\sum_{\substack{\alpha_1,\dots,\alpha_{l-1}\\ \alpha_1\le\alpha,\dots,\alpha_{l-1}\le\alpha}}\sum_{\substack{\beta_1,\dots,\beta_{l-1}\\ \beta_1\le\beta,\dots,\beta_{l-1}\le\beta}}\frac{\alpha!}{\alpha_1!\cdots\alpha_{l-1}!(\alpha-\alpha_1-\cdots-\alpha_{l-1})!}$$
$$\times\frac{\beta!}{\beta_1!\cdots\beta_{l-1}(\beta-\beta_1-\cdots-\beta_{l-1})!}$$
$$\times\|D_\xi^{\alpha_1}D_x^{\beta_1}Ir\|\cdots\|D_\xi^{\alpha_{l-1}}D_x^{\beta_{l-1}}Ir\|\|D_\xi^{\alpha-\alpha_1-\dots-\alpha_{l-1}}D_x^{\beta-\beta_1-\dots-\beta_{l-1}}Ir_1\|$$
$$\le c_\varepsilon\langle\xi\rangle^{\rho|\alpha|+\delta|\beta|-(\rho-\delta)}\sum_{l=1}^{\infty}\sum_{\substack{\alpha_1,\dots,\alpha_{l-1}\\ \alpha_1\le\alpha,\dots,\alpha_{l-1}\le\alpha}}\sum_{\substack{\beta_1,\dots,\beta_{l-1}\\ \beta_1\le\beta,\dots,\beta_{l-1}\le\beta}}C^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}$$
$$\times C_1^{|\alpha-\alpha_1-\dots-\alpha_{l-1}+\beta-\beta_1-\dots-\beta_{l-1}|}C_2$$
$$\times\frac{\alpha!}{\alpha_1!\cdots\alpha_{l-1}!(\alpha-\alpha_1-\cdots-\alpha_{l-1})!}\alpha_1!\cdots\alpha_{l-1}!(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+1)!$$
$$\times\frac{\beta!}{\beta_1!\cdots\beta_{l-1}!(\beta-\beta_1-\cdots-\beta_{l-1})!}$$
$$\times(\beta_1!\cdots\beta_{l-1}!(|\beta-\beta_1-\cdots-\beta_{l-1}|+1)!)^{\theta(\rho-\delta)}\underbrace{Ig\cdots Ig}_{l}\sum_{m=1}^{\infty}\frac{(Ig)^m}{m!}$$
$$\le c_\varepsilon C_1^{|\alpha+\beta|}C_2\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)}\exp(\varepsilon\langle\xi\rangle^{1/\theta})(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)}$$

$$
\begin{aligned}
&\times \sum_{l=1}^{\infty} \sum_{\substack{\alpha_1,\dots,\alpha_{l-1}\\ \alpha_1\le\alpha,\dots,\alpha_{l-1}\le\alpha}} \sum_{\substack{\beta_1,\dots,\beta_{l-1}\\ \beta_1\le\beta,\dots,\beta_{l-1}\le\beta}} \left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}\\
&\times \frac{\alpha!}{\alpha_1!\cdots\alpha_{l-1}!(\alpha-\alpha_1-\cdots-\alpha_{l-1})!} \frac{\alpha_1!\cdots\alpha_{l-1}!(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+1)!}{(|\alpha|+1)!}\\
&\times \frac{\beta!}{\beta_1!\cdots\beta_{l-1}!(\beta-\beta_1-\cdots-\beta_{l-1})!}\\
&\times \left(\frac{\beta_1!\cdots\beta_{l-1}!(|\beta-\beta_1-\cdots-\beta_{l-1}|+1)!}{(|\beta|+1)!}\right)^{\theta(\rho-\delta)} \underbrace{Ig\cdots Ig}_{l} \sum_{m=1}^{\infty}\frac{(Ig)^m}{m!}\\
\le\ & c_\varepsilon C_1^{|\alpha+\beta|} C_2 \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta})(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)}\\
&\times \sum_{l=1}^{\infty} \sum_{\substack{\alpha_1,\dots,\alpha_{l-1}\\ \alpha_1\le\alpha,\dots,\alpha_{l-1}\le\alpha}} \sum_{\substack{\beta_1,\dots,\beta_{l-1}\\ \beta_1\le\beta,\dots,\beta_{l-1}\le\beta}} \left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|} \sum_{m=1}^{\infty}\frac{(Ig)^{m+l}}{(m+l)!}\\
&\times \frac{|\alpha|!}{|\alpha_1|!\cdots|\alpha-\alpha_1-\cdots-\alpha_{l-1}|!} \frac{|\alpha_1|!\cdots|\alpha_{l-1}|!(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+1)!}{(|\alpha|+1)!}\\
&\times \frac{|\beta|!}{|\beta_1|!\cdots|\beta-\beta_1-\cdots-\beta_{l-1}|!} \left(\frac{|\beta_1|!\cdots|\beta_{l-1}|!(|\beta-\beta_1-\cdots-\beta_{l-1}|+1)!}{(|\beta|+1)!}\right)^{\theta(\rho-\delta)}\\
\le\ & c_\varepsilon C_1^{|\alpha+\beta|} C_2 \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta})(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)}\\
&\times \sum_{l=1}^{\infty} \sum_{\substack{\alpha_1,\dots,\alpha_{l-1}\\ \alpha_1\le\alpha,\dots,\alpha_{l-1}\le\alpha}} \sum_{\substack{\beta_1,\dots,\beta_{l-1}\\ \beta_1\le\beta,\dots,\beta_{l-1}\le\beta}} \left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|} \sum_{k=l}^{\infty}\frac{(Ig)^{k+1}}{(k+1)!}\\
&\times \frac{|\alpha|!(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+1)!}{|\alpha-\alpha_1-\cdots-\alpha_{l-1}|!(|\alpha|+1)!} \frac{|\beta|!(|\beta-\beta_1-\cdots-\beta_{l-1}|+1)!}{(|\beta|+1)!|\beta-\beta_1-\cdots-\beta_{l-1}|!}\\
&\times \left(\frac{|\beta_1|!\cdots|\beta_{l-1}|!(|\beta-\beta_1-\cdots-\beta_{l-1}|+1)!}{(|\beta|+1)!}\right)^{\theta(\rho-\delta)-1}\\
\le\ & c_\varepsilon C_1^{|\alpha+\beta|} C_2 \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta})(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)}\\
&\times \sum_{k=1}^{\infty}\frac{(Ig)^{k+1}}{(k+1)!} \sum_{l=1}^{k} \sum_{\substack{\alpha_1,\dots,\alpha_{l-1}\\ \alpha_1\le\alpha,\dots,\alpha_{l-1}\le\alpha}} \sum_{\substack{\beta_1,\dots,\beta_{l-1}\\ \beta_1\le\beta,\dots,\beta_{l-1}\le\beta}} \left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}\\
&\times \frac{(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+1)}{(|\alpha|+1)} \frac{(|\beta-\beta_1-\cdots-\beta_{l-1}|+1)}{(|\beta|+1)}\\
&\times \left(\frac{|\beta_1|!\cdots|\beta_{l-1}|!(|\beta-\beta_1-\cdots-\beta_{l-1}|+1)!}{(|\beta|+1)!}\right)^{\theta(\rho-\delta)-1}
\end{aligned}
$$

$$
\begin{aligned}
&\le c_\varepsilon C_1^{|\alpha+\beta|} C_2 \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta})(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)} \\
&\quad\times \sum_{k=1}^{\infty} \frac{(Ig)^{k+1}}{(k+1)!} \sum_{l=1}^{k} \sum_{\substack{\alpha_1,\dots,\alpha_{l-1} \\ \alpha_1\le\alpha,\dots,\alpha_{l-1}\le\alpha}} \sum_{\substack{\beta_1,\dots,\beta_{l-1} \\ \beta_1\le\beta,\dots,\beta_{l-1}\le\beta}} \left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|} \\
&\le c_\varepsilon C_1^{|\alpha+\beta|} C_2 \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta})(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)} \sum_{k=1}^{\infty} \frac{k(Ig)^{k+1}}{(k+1)!}.
\end{aligned}
$$

Since by means of the choice of constant C_1 we can achieve

$$
\sum_{l=1}^{k} \sum_{\substack{\alpha_1,\dots,\alpha_{l-1} \\ \alpha_1\le\alpha,\dots,\alpha_{l-1}\le\alpha}} \sum_{\substack{\beta_1,\dots,\beta_{l-1} \\ \beta_1\le\beta,\dots,\beta_{l-1}\le\beta}} \left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|} \le k
$$

if $|\alpha+\beta| \ne 0$. Consequently, we obtain

$$
\begin{aligned}
\sup_{x\in K} \|D_\xi^\alpha D_x^\beta q_1\| &\le c_\varepsilon C_1^{|\alpha+\beta|} C_2 \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta}) \\
&\quad\times(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)} \sum_{k=1}^{\infty} \frac{(k+1)(Ig)^{k+1}}{(k+1)!} \\
&\le c_\varepsilon C_1^{|\alpha+\beta|} C_2 \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta}) \\
&\quad\times(|\alpha|+1)!((|\beta|+1)!)^{\theta(\rho-\delta)} Ig \sum_{k=1}^{\infty} \frac{(Ig)^{k}}{k!}.
\end{aligned}
$$

For $\alpha=\beta=0$, we have

$$
\begin{aligned}
\sup_{x\in K} \|q_1\| &\le \sum_{l=1}^{\infty} \|Ir\|\cdots\|Ir\|\|Ir_1\| \\
&\le c_\varepsilon \langle\xi\rangle^{-(\rho-\delta)} C_2 \exp(\varepsilon\langle\xi\rangle^{1/\theta}) \sum_{l=1}^{\infty} \underbrace{Ig\cdots Ig}_{l} \sum_{m=1}^{\infty} \frac{(Ig)^m}{m!} \\
&\le c_\varepsilon \langle\xi\rangle^{-(\rho-\delta)} C_2 \exp(\varepsilon\langle\xi\rangle^{1/\theta}) \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{(Ig)^{m+l}}{(m+l)!} \\
&\le c_\varepsilon \langle\xi\rangle^{-(\rho-\delta)} C_2 \exp(\varepsilon\langle\xi\rangle^{1/\theta}) \sum_{l=1}^{\infty} \sum_{k=l}^{\infty} \frac{(Ig)^{k+1}}{(k+1)!} \\
&\le c_\varepsilon \langle\xi\rangle^{-(\rho-\delta)} C_2 \exp(\varepsilon\langle\xi\rangle^{1/\theta}) \sum_{k=1}^{\infty} \sum_{l=1}^{k} \frac{(Ig)^{k+1}}{(k+1)!}
\end{aligned}
$$

$$\leq \quad c_\varepsilon \langle\xi\rangle^{-(\rho-\delta)} C_2 \exp(\varepsilon\langle\xi\rangle^{1/\theta}) \sum_{k=1}^{\infty} \frac{k(Ig)^{k+1}}{(k+1)!}$$
$$\leq \quad c_\varepsilon \langle\xi\rangle^{-(\rho-\delta)} C_2 \exp(\varepsilon\langle\xi\rangle^{1/\theta}) Ig \sum_{k=1}^{\infty} \frac{(Ig)^{k}}{k!}\,.$$

Thus, (3.7) is proved for $k = 1$. We assume now that (3.6), (3.7) have been proved for k and we prove them $k+1$. We have

$$r_{k+1} = \sum_{l=0}^{k} \sum_{|\gamma|=k+1-l} \frac{1}{\gamma!} (\partial_\xi^\gamma r)(D_x^\gamma q_l)\,.$$

It follows

$$\sup_{x\in K} \|D_\xi^\alpha D_x^\beta r_{k+1}\|$$
$$= \left\| \sum_{l=0}^{k} \sum_{|\gamma|=k+1-l} \frac{1}{\gamma!} \sum_{\alpha_1\leq\alpha} \sum_{\beta_1\leq\beta} \frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!} \frac{\beta!}{\beta_1!(\beta-\beta_1)!} (\partial_\xi^{\gamma+\alpha_1} D_x^{\beta_1} r)(\partial_\xi^{\alpha-\alpha_1} D_x^{\gamma+\beta-\beta_1} q_l) \right\|$$
$$\leq \sum_{l=0}^{k} \sum_{|\gamma|=k+1-l} \sum_{\alpha_1\leq\alpha} \sum_{\beta_1\leq\beta} \frac{1}{\gamma!} \frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!} \frac{\beta!}{\beta_1!(\beta-\beta_1)!} c_\varepsilon C^{|\gamma+\alpha_1+\beta_1|} g_\varepsilon(t,\xi)(\gamma+\alpha_1)!$$
$$\times (\beta_1!)^{\theta(\rho-\delta)} \langle\xi\rangle^{-\rho|\alpha_1+\gamma|+\delta|\beta_1|-\rho|\alpha-\alpha_1|+\delta|\gamma+\beta-\beta_1|-(\rho-\delta)l} C_1^{|\alpha-\alpha_1+\beta-\beta_1+\gamma|} C_2^l$$
$$\times \frac{(|\alpha-\alpha_1|+l)!}{l!} ((|\beta-\beta_1+\gamma|+l)!)^{\theta(\rho-\delta)} Ig(1+Ig)^{l-1} \sum_{m=1}^{\infty} \frac{(Ig)^m}{m!}$$
$$\leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^{k+1} \frac{(|\alpha|+k+1)!}{(k+1)!} ((|\beta|+k+1)!)^{\theta(\rho-\delta)} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(k+1)}$$
$$\times \exp(\varepsilon\langle\xi\rangle^{1/\theta}) g_\varepsilon(t,\xi) \left\{ \sum_{|\gamma|=k+1} \sum_{\alpha_1\leq\alpha} \sum_{\beta_1} \left(\frac{C}{C_1}\right)^{|\alpha_1+\beta_1|} \left(\frac{CC_1}{C_2}\right)^{|\gamma|} \frac{1}{\gamma!} \frac{(k+1)!}{l!} \right.$$
$$\times \frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!} \frac{(\gamma+\alpha_1)!(|\alpha-\alpha_1|)!}{(|\alpha|+k+1)!} \frac{\beta!}{\beta_1!(\beta-\beta_1)!} \left(\frac{\beta_1!(|\beta-\beta_1+\gamma|)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)} \sum_{m=1}^{\infty} \frac{(Ig)^m}{m!}$$
$$+ \sum_{l=1}^{k} \sum_{|\gamma|=k+1-l} \sum_{\alpha_1\leq\alpha} \sum_{\beta_1} \left(\frac{C}{C_1}\right)^{|\alpha_1+\beta_1|} \left(\frac{CC_1}{C_2}\right)^{|\gamma|} \frac{1}{\gamma!} \frac{(k+1)!}{l!}$$
$$\times \frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!} \frac{(\gamma+\alpha_1)!(|\alpha-\alpha_1|+l)!}{(|\alpha|+k+1)!}$$

$$\times \frac{\beta!}{\beta_1!(\beta-\beta_1)!}\left(\frac{\beta_1!(|\beta-\beta_1+\gamma|+l)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)} Ig(1+Ig)^{l-1}\sum_{m=1}^{\infty}\frac{(Ig)^m}{m!}\Bigg\}$$

Due to choice C_1, C_2 we get for all $0 \geq l \leq k$

$$\sum_{|\gamma|=k+1-l}\sum_{\alpha_1\leq\alpha}\sum_{\beta_1\leq\beta}\left(\frac{C}{C_1}\right)^{|\alpha_1+\beta_1|}\left(\frac{CC_1}{C_2}\right)^{|\gamma|}\frac{1}{\gamma!}\frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!}\frac{(k+1)!}{l!}$$
$$\times\frac{(\gamma+\alpha_1)!(|\alpha-\alpha_1|+l)!}{(|\alpha|+k+1)!}\frac{\beta!}{\beta_1!(\beta-\beta_1)!}\left(\frac{\beta_1!(|\beta-\beta_1+\gamma|+l)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)}\leq 1\,. \quad (3.9)$$

Indeed, it is easily seen that

$$\begin{aligned}
&\frac{\beta!}{\beta_1!(\beta-\beta_1)!}\left(\frac{\beta_1!(|\beta-\beta_1+\gamma|+l)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)}\\
\leq\ &\frac{|\beta|!}{|\beta_1|!|\beta-\beta_1|!}\left(\frac{|\beta_1|!(|\beta-\beta_1|+k+1)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)}\\
\leq\ &\frac{|\beta|!(|\beta-\beta_1|+k+1)!}{(|\beta|+k+1)!|\beta-\beta_1|!}\left(\frac{|\beta_1|!(|\beta-\beta_1|+k+1)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)-1}\\
\leq\ &\frac{(|\beta-\beta_1|+1)\cdots(|\beta-\beta_1|+k+1)}{(|\beta|+1)\cdots(|\beta|+k+1)}\left(\frac{|\beta_1|!(|\beta-\beta_1|+k+1)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)-1}\leq 1\,.
\end{aligned}$$

Further we have

$$\begin{aligned}
&\frac{1}{\gamma!}\frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!}\frac{(\gamma+\alpha_1)!(|\alpha-\alpha_1|+l)!}{(|\alpha|+k+1)!}\frac{(k+1)!}{l!}\\
=\ &\frac{(\alpha_1+\gamma)!}{\alpha_1!\gamma!}\frac{\alpha!}{(\alpha-\alpha_1)!}\frac{(|\alpha-\alpha_1|+l)!}{(|\alpha|+k+1)!}\frac{(k+1)!}{l!}\\
\leq\ & n^{|\alpha_1+\gamma|}\frac{\alpha!}{\alpha_1!(\alpha-\alpha_1)!}\frac{\alpha_1!}{|\alpha_1|!}\frac{|\alpha_1|!(|\alpha-\alpha_1|+l)!}{(|\alpha|+k+1)!}\frac{(k+1)!}{l!}\\
\leq\ & n^{|\alpha_1+\gamma|}\frac{|\alpha|!}{|\alpha_1|!|\alpha-\alpha_1|!}\frac{|\alpha_1|!(|\alpha-\alpha_1|+l)!}{(|\alpha|+k+1)!}\frac{(k+1)!}{l!}\\
\leq\ & n^{|\alpha_1+\gamma|}\frac{|\alpha|!(|\alpha-\alpha_1|+l)!}{|\alpha-\alpha_1|!(|\alpha|+k+1)!}\frac{(k+1)!}{l!}\\
\leq\ & n^{|\alpha_1+\gamma|}\frac{(|\alpha-\alpha_1|+|\alpha_1|)\cdots(|\alpha-\alpha_1|+1)(|\alpha-\alpha_1|+l)\cdots(l+1)}{(|\alpha_1|+|\alpha-\alpha_1|+k+1)\cdots(k+1)}\\
\leq\ & n^{|\alpha_1+\gamma|}\,.
\end{aligned}$$

Since the inequalities

$$\frac{Cn}{C_1} \le \frac{1}{3}, \qquad \frac{CC_1 n}{C_2} \le \frac{1}{3}$$

imply

$$\sum_\alpha \left(\frac{Cn}{C_2}\right)^{|\alpha|} \le 2^n \, .$$

Consequently, we obtain

$$\begin{aligned}
& \sup_{x\in K} \|D_\xi^\alpha D_x^\beta r_{k+1}\| \\
\le \;& c_\varepsilon C_1^{|\alpha+\beta|} C_2^{k+1} \frac{(|\alpha|+k+1)!}{(k+1)!} \left((|\beta|+k+1)!\right)^{\theta(\rho-\delta)} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(k+1)} \\
& \times \exp(\varepsilon\langle\xi\rangle^{1/\theta}) g_\varepsilon(t,\xi) \left\{1 + \sum_{l=1}^{k} Ig(1+Ig)^{l-1}\right\} \sum_{m=1}^{\infty} \frac{(Ig)^m}{m!}
\end{aligned}$$

We consider now

$$\begin{aligned}
1 + \sum_{l=1}^{k} Ig(1+Ig)^{l-1} &= 1 + Ig + Ig(1+Ig) + \cdots + Ig(1+Ig)^{k-1} \\
&= (1+Ig)\left(1 + Ig + Ig(1+Ig) + \cdots + Ig(1+Ig)^{k-2}\right) \\
&= (1+Ig)^2\left(1 + Ig + Ig(1+Ig) + \cdots + Ig(1+Ig)^{k-3}\right) \\
&= (1+Ig)^3\left(1 + Ig + Ig(1+Ig) + \cdots + Ig(1+Ig)^{k-4}\right) \\
&= (1+Ig)^k
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\sup_{x\in K} \|D_\xi^\alpha D_x^\beta r_{k+1}\| \;\le\;& C_\varepsilon C_1^{|\alpha+\beta|} C_2^{k+1} \frac{(|\alpha|+k+1)!}{(k+1)!} (|\beta|+k+1)!^{\theta(\rho-\delta)} \\
& \times \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(k+1)} \exp(\varepsilon\langle\xi\rangle^{1/\theta}) g_\varepsilon(t,\xi)(1+Ig)^k \sum_{m=1}^{\infty} \frac{(Ig)^m}{m!} \, .
\end{aligned}$$

Thus, (3.6) is proved. We consider now q_{k+1}:

$$q_{k+1} = -i\int_s^t r_{k+1}(s_1)ds_1 + \sum_{l=2}^{\infty}(-i)^l \int_s^t ds_1 \ldots \int_s^{s_{l-1}} ds_l r(s_1)\ldots r(s_{l-1}) r_{k+1}(s_l). \quad (3.10)$$

Then, we obtain

$$
\begin{aligned}
\sup_{x\in K}\|D_\xi^\alpha D_x^\beta q_{k+1}\| &\le \sum_{l=1}^{\infty}\sum_{\substack{\alpha_1,\dots,\alpha_{l-1}\\ \alpha_1\le\alpha,\dots,\alpha_{l-1}\le\alpha}}\sum_{\substack{\beta_1,\dots,\beta_{l-1}\\ \beta_1\le\beta,\dots,\beta_{l-1}\le\beta}}\frac{\alpha!}{\alpha_1!\cdots\alpha_{l-1}!(\alpha-\alpha_1-\cdots-\alpha_{l-1})!}\\
&\quad\times\frac{\beta!}{\beta_1!\cdots\beta_{l-1}!(\beta-\beta_1-\cdots-\beta_{l-1})!}\\
&\quad\times\|D_\xi^{\alpha_1}D_x^{\beta_1}Ir\|\cdots\|D_\xi^{\alpha_{l-1}}D_x^{\beta_{l-1}}Ir\|\|D_\xi^{\alpha-\sum_{j=1}^{l-1}\alpha_j}D_x^{\beta-\sum_{j=1}^{l-1}\beta_j}Ir_{k+1}\|\\
&\le c_\varepsilon\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(k+1)}\exp(\varepsilon\langle\xi\rangle^{1/\theta})\\
&\quad\times\sum_{l=1}^{\infty}\sum_{\substack{\alpha_1,\dots,\alpha_{l-1}\\ \alpha_1\le\alpha,\dots,\alpha_{l-1}\le\alpha}}\sum_{\substack{\beta_1,\dots,\beta_{l-1}\\ \beta_1\le\beta,\dots,\beta_{l-1}\le\beta}}\frac{\alpha!}{\alpha_1!\cdots\alpha_{l-1}!(\alpha-\alpha_1-\cdots-\alpha_{l-1})!}\\
&\quad\times\frac{\beta!}{\beta_1!\cdots\beta_{l-1}!(\beta-\beta_1-\cdots-\beta_{l-1})!}\\
&\quad\times C^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}C_1^{|\alpha-\alpha_1-\cdots-\alpha_{l-1}+\beta-\beta_1-\cdots-\beta_{l-1}|}C_2^{k+1}\\
&\quad\times\alpha_1!\cdots\alpha_{l-1}!\frac{(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+k+1)!}{(k+1)!}\\
&\quad\times(\beta_1!\cdots\beta_{l-1}!(|\beta-\beta_1-\cdots-\beta_{l-1}|+k+1)!)^{\theta(\rho-\delta)}\\
&\quad\times(1+Ig)^k\underbrace{Ig\cdots Ig}_{l}\sum_{m=1}^{\infty}\frac{(Ig)^m}{m!}\\
&\le c_\varepsilon C_1^{|\alpha+\beta|}C_2^{k+1}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(k+1)}\exp(\varepsilon\langle\xi\rangle^{1/\theta})\\
&\quad\times\frac{(|\alpha|+k+1)!}{(k+1)!}((|\beta|+k+1)!)^{\theta(\rho-\delta)}(1+Ig)^k\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{(Ig)^{m+l}}{(m+l)!}\\
&\quad\times\sum_{\substack{\alpha_1,\dots,\alpha_{l-1}\\ \alpha_1\le\alpha,\dots,\alpha_{l-1}\le\alpha}}\sum_{\substack{\beta_1,\dots,\beta_{l-1}\\ \beta_1\le\beta,\dots,\beta_{l-1}\le\beta}}\left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}\\
&\quad\times\frac{\alpha!}{\alpha_{l-1}!(\alpha-\alpha_1-\cdots-\alpha_{l-1})!}\frac{(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+k+1)!}{(|\alpha|+k+1)!}\\
&\quad\times\frac{\beta!}{\beta_1!\cdots\beta_{l-1}!(\beta-\beta_1-\cdots-\beta_{l-1})!}\\
&\quad\times\left(\frac{\beta_1!\cdots\beta_{l-1}!(|\beta-\beta_1-\cdots-\beta_{l-1}|+k+1)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)}.
\end{aligned}
$$

Let us prove that by means of the appropriate choice of the constants C_1 and C_2 we can get an inequality

$$\sum_{\substack{\alpha_1,\cdots,\alpha_{l-1}\\ \alpha_1\le\alpha,..,\alpha_{l-1}\le\alpha}}\ \sum_{\substack{\beta_1,\cdots,\beta_{l-1}\\ \beta_1\le\beta,..,\beta_{l-1}\le\beta}}\left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}$$
$$\times\frac{\alpha!}{\alpha_{l-1}!(\alpha-\alpha_1-\cdots-\alpha_{l-1})!}\frac{(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+k+1)!}{(|\alpha|+k+1)!}$$
$$\times\frac{\beta!}{\beta_1!\cdots\beta_{l-1}!(\beta-\beta_1-\cdots-\beta_{l-1})!}$$
$$\times\left(\frac{\beta_1!\cdots\beta_{l-1}!(|\beta-\beta_1-\cdots-\beta_{l-1}|+k+1)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)}\le 1\,.$$

Indeed

$$\sum_{\substack{\alpha_1,\cdots,\alpha_{l-1}\\ \alpha_1\le\alpha,..,\alpha_{l-1}\le\alpha}}\ \sum_{\substack{\beta_1,\cdots,\beta_{l-1}\\ \beta_1\le\beta,..,\beta_{l-1}\le\beta}}\left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}$$
$$\times\frac{\alpha!}{\alpha_{l-1}!(\alpha-\alpha_1-\cdots-\alpha_{l-1})!}\frac{(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+k+1)!}{(|\alpha|+k+1)!}$$
$$\times\frac{\beta!}{\beta_1!\cdots\beta_{l-1}!(\beta-\beta_1-\cdots-\beta_{l-1})!}$$
$$\times\left(\frac{\beta_1!\cdots\beta_{l-1}!(|\beta-\beta_1-\cdots-\beta_{l-1}|+k+1)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)}$$
$$\le\sum_{\substack{\alpha_1,\cdots,\alpha_{l-1}\\ \alpha_1\le\alpha,..,\alpha_{l-1}\le\alpha}}\ \sum_{\substack{\beta_1,\cdots,\beta_{l-1}\\ \beta_1\le\beta,..,\beta_{l-1}\le\beta}}\left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}$$
$$\times\frac{\alpha!}{\alpha_1!\cdots\alpha_{l-1}!(\alpha-\alpha_1-\cdots-\alpha_{l-1})!}\frac{\alpha_1}{|\alpha_1|!}\cdots\frac{\alpha_{l-1}}{|\alpha_{l-1}|!}$$
$$\times\frac{|\alpha_1|!\cdots|\alpha_{l-1|!}(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+k+1)!}{(|\alpha|+k+1)!}$$
$$\times\frac{|\beta|!}{|\beta_1|!\cdots|\beta-\beta_1-\cdots-\beta_{l-1}|!}$$
$$\times\left(\frac{|\beta_1|!\cdots|\beta_{l-1}|!(|\beta-\beta_1-\cdots-\beta_{l-1}|+k+1)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)}$$
$$\le\sum_{\substack{\alpha_1,\cdots,\alpha_{l-1}\\ \alpha_1\le\alpha,..,\alpha_{l-1}\le\alpha}}\ \sum_{\substack{\beta_1,\cdots,\beta_{l-1}\\ \beta_1\le\beta,..,\beta_{l-1}\le\beta}}\left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}$$

$$\times \frac{|\alpha|!}{|\alpha_1|!\cdots|\alpha-\alpha_1-\cdots-\alpha_{l-1}|!} \frac{|\alpha_1|!\cdots|\alpha_{l-1}|!(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+k+1)!}{(|\alpha|+k+1)!}$$
$$\times \frac{|\beta|!(|\beta-\beta_1-\cdots-\beta_{l-1}|+k+1)!}{(|\beta|+k+1)!|\beta-\beta_1-\cdots-\beta_{l-1}|!}$$
$$\times \left(\frac{|\beta_1|!\cdots|\beta_{l-1}|!(|\beta-\beta_1-\cdots-\beta_{l-1}|+k+1)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)-1}$$
$$\leq \sum_{\substack{\alpha_1,\dots,\alpha_{l-1} \\ \alpha_1\leq\alpha,\dots,\alpha_{l-1}\leq\alpha}} \sum_{\substack{\beta_1,\dots,\beta_{l-1} \\ \beta_1\leq\beta,\dots,\beta_{l-1}\leq\beta}} \left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}$$
$$\times \frac{|\alpha|!(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+k+1)!}{(|\alpha|+k+1)!|\alpha-\alpha_1-\cdots-\alpha_{l-1}|!}$$
$$\times \frac{|\beta|!(|\beta-\beta_1-\cdots-\beta_{l-1}|+k+1)!}{(|\beta|+k+1)!|\beta-\beta_1-\cdots-\beta_{l-1}|!}$$
$$\times \left(\frac{|\beta_1|!\cdots|\beta_{l-1}|!(|\beta-\beta_1-\cdots-\beta_{l-1}|+k+1)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)-1}$$
$$\leq \sum_{\substack{\alpha_1,\dots,\alpha_{l-1} \\ \alpha_1\leq\alpha,\dots,\alpha_{l-1}\leq\alpha}} \sum_{\substack{\beta_1,\dots,\beta_{l-1} \\ \beta_1\leq\beta,\dots,\beta_{l-1}\leq\beta}} \left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|}$$
$$\times \frac{(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+1)\cdots(|\alpha-\alpha_1-\cdots-\alpha_{l-1}|+k+1)}{(|\alpha|+1)\cdots(|\alpha|+k+1)}$$
$$\times \frac{(|\beta-\beta_1-\cdots-\beta_{l-1}|+1)\cdots(|\beta-\beta_1-\cdots-\beta_{l-1}|+k+1)}{(|\beta|+1)\cdots(|\beta|+k+1)}$$
$$\times \left(\frac{|\beta_1|!\cdots|\beta_{l-1}|!(|\beta-\beta_1-\cdots-\beta_{l-1}|+k+1)!}{(|\beta|+k+1)!}\right)^{\theta(\rho-\delta)-1}$$
$$\leq \sum_{\substack{\alpha_1,\dots,\alpha_{l-1} \\ \alpha_1\leq\alpha,\dots,\alpha_{l-1}\leq\alpha}} \sum_{\substack{\beta_1,\dots,\beta_{l-1} \\ \beta_1\leq\beta,\dots,\beta_{l-1}\leq\beta}} \left(\frac{C}{C_1}\right)^{|\alpha_1+\cdots\alpha_{l-1}+\beta_1+\cdots+\beta_{l-1}|} \leq 1$$

if $|\alpha+\beta|\neq 0$. (For $\alpha=\beta=0$, the estimate is easily seen from (2.2), (2.3)). Thus, we obtain

$$\sup_{x\in K}\|D_\xi^\alpha D_x^\beta q_{k+1}\| \leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^{k+1} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(k+1)} \frac{(|\alpha|+k+1)!}{(k+1)!}$$
$$\times((|\beta|+k+1)!)^{\theta(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta})(1+Ig)^k \sum_{l=1}^\infty \sum_{m=1}^\infty \frac{(Ig)^{m+l}}{(m+l)!}$$

$$
\begin{aligned}
&\leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^{k+1} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(k+1)} \frac{(|\alpha|+k+1)!}{(k+1)!} \\
&\quad \times ((|\beta|+k+1)!)^{\theta(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta})(1+Ig)^k \sum_{l=1}^{\infty}\sum_{k=l}^{\infty} \frac{(Ig)^{k+1}}{(k+1)!} \\
&\leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^{k+1} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(k+1)} \frac{(|\alpha|+k+1)!}{(k+1)!} \\
&\quad \times ((|\beta|+k+1)!)^{\theta(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta})(1+Ig)^k \sum_{k=1}^{\infty}\sum_{l=1}^{k} \frac{(Ig)^{k+1}}{(k+1)!} \\
&\leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^{k+1} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(k+1)} \frac{(|\alpha|+k+1)!}{(k+1)!} \\
&\quad \times ((|\beta|+k+1)!)^{\theta(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta})(1+Ig)^k \sum_{k=1}^{\infty} \frac{k(Ig)^{k+1}}{(k+1)!} \\
&\leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^{k+1} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)(k+1)} \frac{(|\alpha|+k+1)!}{(k+1)!} \\
&\quad \times ((|\beta|+k+1)!)^{\theta(\rho-\delta)} \exp(\varepsilon\langle\xi\rangle^{1/\theta}) Ig(1+Ig)^k \sum_{k=1}^{\infty} \frac{(Ig)^k}{k!}.
\end{aligned}
$$

The lemma is proved. □

The conclusion of the proof of existence. From (3.7) there follows that

$$
\begin{aligned}
\sup_{x\in K} \|D_\xi^\alpha D_x^\beta q_k(t,s,x,\xi)\| &\leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^{k} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)k} \frac{(|\alpha|+k)!}{k!} (\beta! k!)^{\theta(\rho-\delta)} \\
&\quad \times \exp(\varepsilon\langle\xi\rangle^{1/\theta})(\varepsilon\langle\xi\rangle^{1/\theta} + C\ln\langle\xi\rangle) \\
&\quad \times (1 + C\langle\xi\rangle^{1/\theta})^k \exp(\varepsilon\langle\xi\rangle^{1/\theta} + C\ln\langle\xi\rangle) \\
&\leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^{k} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)k+C} \frac{(|\alpha|+k)!}{k!} (\beta! k!)^{\theta(\rho-\delta)} \\
&\quad \times \exp(2\varepsilon\langle\xi\rangle^{1/\theta})(\varepsilon\langle\xi\rangle^{1/\theta} + C\ln\langle\xi\rangle)(1 + C\langle\xi\rangle^{1/\theta})^k \\
&\leq c_\varepsilon C_1^{|\alpha+\beta|} C_2^{k} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)k+C} \frac{(|\alpha|+k)!}{k!} (\beta! k!)^{\theta(\rho-\delta)} \\
&\quad \times \exp(2\varepsilon\langle\xi\rangle^{1/\theta})(\varepsilon\langle\xi\rangle^{1/\theta} + C\ln\langle\xi\rangle) 2^k C^k ((\langle\xi\rangle^{1/\theta})^k \\
&\leq c_\varepsilon C_1^{|\alpha+\beta|} \tilde{C}_2{}^{k} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|+(1/\theta-(\rho-\delta))k} \frac{(|\alpha|+k)!}{k!} \\
&\quad \times (\beta! k!)^{\theta(\rho-\delta)} \exp(3\varepsilon\langle\xi\rangle^{1/\theta})(\varepsilon\langle\xi\rangle^{1/\theta} + C\ln\langle\xi\rangle) \qquad (3.11)
\end{aligned}
$$

for all $k = 0, 1, \ldots,$ uniformly with respect to $(t,s) \in [0,T_1] \times [0,T_1]$. Thus, $q_k \in C([0,T_1] \times [0,T_1]; FS^{\infty,\theta}_{\rho_1,\delta_1}(\mathbb{R}^n))$, where $1/\theta - (\rho-\delta) < \rho_1 - \delta_1$. Last inequality is equivalent to (1.4). Therefore, from the Theorem 1.4, we get

$$q(t,s,x,\xi) \in C([0,T_1]\times[0,T_1]; FS^{\infty;\theta}_{\rho_1,\delta_1}(\Omega))$$

which equivalent to $\sum_{j\ge 0} p_j|_\Omega$ in the sense of Definition 1.4, where $\Omega\subset\mathbb{R}^n$. Thus existence of the solution $Q(t,s)$ is proved.

Uniqueness. We note that according to above consideration the Cauchy problem

$$\begin{cases} D_t\, Q(t,t') + Q(t,t')R(t,s) + R_0(t,s) \in C([0,T_1]\,;\, V_R^\theta(\mathbb{R}^n))\,,\\ Q(t',t') = 0, \qquad 0\le t'\le t\le T_1\,,\end{cases}$$

has a solution for every $t'\in[s,T_1]$. Further, for every $t'\in[s,T_1]$ a problem

$$\begin{cases} D_t\, V(t,t') + V(t,t')R^{\#}(t,s) \in C([0,T_1]\,;\, V_R^\theta(\mathbb{R}^n))\,,\\ V(t',t') = I, \qquad 0\le t\le t'\le T_1\,,\end{cases} \tag{3.12}$$

has a solution as well. Applying conjugation to (3.12) we obtain

$$\begin{cases} D_t\, V^{\#}(t,t') - R(t,s)V^{\#}(t,t') \in C([0,T_1]\,;\, V_R^\theta(\mathbb{R}^n)),\\ V^{\#}(t',t') = I, \qquad 0\le t\le t'\le T_1\,.\end{cases}$$

Define an operator G as follows

$$Gf(t') := \int_s^{t'} f(t)V^{\#}(t,t')\,dt\,.$$

If we set

$$f(t) := D_t u(t) + u(t)R(t,s)$$

then we obtain

$$Gf(t') = -iu(t') + iu(s)V^{\#}(s,t') - \int_s^{t'} u(t)\left(D_tV^{\#}(t,t') - R(t,s)V^{\#}(t,t')\right)\,dt\,.$$

Hence,

$$\begin{aligned} u(t') &= V^{\#}(s,t')u(s) + iG\left(D_tu(t) + R(t,s)u(t)\right)\\ &\quad + i\int_s^{t'} u(t)\left(D_tV^{\#}(t,t') - R(t,s)V^{\#}(t,t')\right)\,dt\,. \end{aligned} \tag{3.13}$$

Therefore, if $Q_1(t,s)$ and $Q_2(t,s)$ are two solutions to the problem (2.1), then putting in (3.13) a distribution-valued function

$$u(t) = \left(Q_1(t,s) - Q_2(t,s)\right)u_0\,, \qquad u_0\in G^{(\theta)'}(\mathbb{R}^n)\,,$$

we arrive at

$$Q_1(t,s) - Q_2(t,s) \in C_t^1([0,T_0]; V_R^\theta(\mathbb{R}^n))\,.$$

The theorem is proved. □

4 Applications

In this section we consider the Cauchy problem for the hyperbolic equation

$$D_t^2 u - \lambda^2(t)\sum_{i=1}^n D_{x_j}^2 u + a(t)\lambda^2(t)\Lambda(t)^{-\dfrac{\theta}{\theta-1}}\sum_{i=1}^n b_j(t)D_{x_j}u = 0, \tag{4.1}$$

$$u(s,x) = u_0(x),\ \ D_t u(s,x) = u_1(x),\ \ s,t\in[0,T],\ \ u_0(x),u_1(x)\in G_0^{(\theta)}(\mathbb{R})\,. \tag{4.2}$$

We describe the operator of (4.1) by means of the real-valued function $\lambda\in C^\infty([0,T])$ such that $\lambda(0)=\lambda'(0)=0$, $\lambda'(t)>0$ when $t\neq 0$. In the following λ' means the same as $d\lambda/dt$. For $\lambda(t)$ we define $\Lambda(t)=\int_0^t\lambda(r)\,dr$ and assume that the function $\lambda^2\Lambda^{\theta/(1-\theta)}$ belongs to $C^\infty([0,T])$ and that the following estimates

$$\begin{aligned} c\lambda(t)/\Lambda(t) \ &\le\ \lambda'(t)/\lambda(t)\ \le\ c_0\left(m(t)\lambda(t)\Lambda^{\theta/(1-\theta)}(t)\right) \quad \text{for all} \quad t\in(0,T]\,,\\ |\lambda^{(k)}(t)| \ &\le\ c_k\left(m(t)\lambda(t)\Lambda^{\theta/(1-\theta)}(t)\right)^{k-1}|\lambda'(t)| \quad \text{for all} \quad k=2,3,\ldots,\ \ t\in(0,T], \end{aligned}$$

are satisfied with positive constants c, c_0, c_k, where $c>\theta/(2(\theta-1))$. Here $m(t)\Lambda^{\theta/(1-\theta)}(t)$ is a monotone function and $m(t)$ tends to 0 as $t\to 0$.
Further we assume that

$$a(t)\lambda^2(t)\Lambda(t)^{-\dfrac{\theta}{\theta-1}}\sum_{i=1}^n b_j(t)\in C^\infty([0,t])\,, \tag{4.3}$$

$$|D_t^k a(t)| + |D_t^k b_j(t)| \le C_k\left(m(t)\lambda(t)\Lambda^{\theta/(1-\theta)}(t)\right)^k \quad \text{for all} \quad t\in(0,T]\,, \tag{4.4}$$

$$\tag{4.5}$$

$$a_1(t) := \max_{\tau\le t}|a(t)|\to 0 \quad \text{as} \quad t\to 0. \tag{4.6}$$

Our aim is the construction of real-valued phase functions Φ_k, $k,=1,2$, and of amplitude functions A_{jk} such that for a given $s\in J$ solution $u(t,x)$ of (4.1) can be represented as

$$u(t,x) = \frac{1}{(2\pi)^n}\sum_{j=0}^1\sum_{k=1}^2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - y\cdot\xi + \Phi_k(t,s,\xi))}A_{jk}(t,s,\xi)u_j(y)\,d\xi\,dy, \tag{4.7}$$

where the phase functions $\Phi_{kl}(t,s,\xi)$ are defined by

$$\Phi_k(t,s,\xi) = (-1)^k|\xi|\int_s^t\lambda(\tau)\,d\tau\,, \quad k=1,2. \tag{4.8}$$

The amplitude functions $A_{jkl}(t,s,\xi)$ are such that for every given positive number ε there exist positive constant K, such that, for every $j,\ k,\ l,\ p,\ r,\ \alpha,\ p+r\le 2$, the inequality

$$|D_t^p D_s^r D_\xi^\alpha A_{jk}(t,s,\xi)| \le C_{p,r,\alpha}\langle\xi\rangle^{K+(p+r)/m-|\alpha|(\theta-2)/\theta}\exp\left((p+r)\varepsilon\langle\xi\rangle^{1/\theta}\right) \qquad (4.9)$$

holds for all $s,t\in[0,T]$ $\xi\in\mathbb{R}^n$.

Below we give only few steps of the construction because one can find the omitted details in [21]. For a positive constant N, $N\ge 1$, we define $t_{\xi,\theta}$ as a root of the equation

$$\Lambda(t)^\theta\langle\xi\rangle^{\theta-1} = N^{\theta-1} \qquad (4.10)$$

with respect to t. Here $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. For a positive M we denote by $\mathbb{R}^n_M$ the set $\{\xi\in\mathbb{R}^n|\langle\xi\rangle\ge M\}$. Then, for every given positive numbers M and N we define

$$\begin{aligned} Z_{pd}(M,N,\theta) &= \{(t,\xi)\in J\times\mathbb{R}^n \ \ | \ \ \Lambda(t)^\theta\langle\xi\rangle^{\theta-1}\le N^{\theta-1}\ ,\langle\xi\rangle\ge M\}\,,\\ Z_{hyp}(M,N,\theta) &= \{(t,\xi)\in J\times\mathbb{R}^n \ \ | \ \ \Lambda(t)^\theta\langle\xi\rangle^{\theta-1}\ge N^{\theta-1}\ ,\langle\xi\rangle\ge M\}\,. \end{aligned}$$

Furthermore, the continuous roots of the equation

$$\tau^2-\lambda^2(t)\sum_{j=1}^n\xi_j^2+a(t)\lambda^2(t)\Lambda(t)^{-\dfrac{\theta}{\theta-1}}\sum_{j=1}^n b_j(t)\xi_j=0, \qquad (4.11)$$

(these are the zeros of the complete symbol) are denoted by $\tau_l(t,x)$, $l=1,2$.

Proposition 4.1 *There exist positive constants M, N, and δ_1 such that the zeros $\tau_1(t,\xi)$, $\tau_2(t,\xi)$, are smooth functions on the set $Z_h(M,N,\theta)$, $\tau_1,\tau_2\in C^\infty(Z_h(M,N,\theta))$, and for every k,α the inequalities*

$$\begin{aligned} |D_t^k D_\xi^\alpha\tau_l(t,\xi)| &\le C_{k,\alpha}\langle\xi\rangle^{1-|\alpha|}\lambda(t)\left(m(t)\lambda(t)\Lambda^{\theta/(1-\theta)}(t)\right)^k\,, && (4.12)\\ |\tau_1(t,\xi)-\tau_2(t,\xi)| &\ge \delta_1\lambda(t)\langle\xi\rangle\,, && (4.13)\\ |D_t^k D_\xi^\alpha \operatorname{Im}\tau_l(t,\xi)| &\le o(1)C_{k,\alpha}\langle\xi\rangle^{-|\alpha|}\lambda(t)\left(m(t)\lambda(t)\Lambda^{\theta/(1-\theta)}(t)\right)^k\Lambda(t)^{\theta/(1-\theta)} && (4.14) \end{aligned}$$

hold for all $(t,\xi)\in Z_h(M,N,\gamma)$ and all $l=1,2$. Here $o(1)\to 0$ as $t\to 0$

To construct fundamental solution to this problem one faces the above investigated problem (2.1) with the function

$$\begin{aligned} g_\varepsilon(t,\xi) \; &= \; \Big\{\chi\left(\Lambda(t)^\theta\langle\xi\rangle^{\theta-1}\varepsilon^{-\frac{2\theta(\theta-1)}{\theta-2}}/2\right)\left(\rho(t,\xi)+\frac{\rho_t(t,\xi)}{\rho(t,\xi)}\right) \\ &+ \left(1-\chi\left(2\Lambda(t)^\theta\langle\xi\rangle^{\theta-1}\varepsilon^{-\frac{2\theta(\theta-1)}{\theta-2}}\right)\right)\widetilde{a}(t)\lambda(t)\Lambda(t)^{\frac{\theta}{1-\theta}}\Big\} \\ &\times\chi\left(\widetilde{a}(t)\varepsilon^{\frac{\theta}{2-\theta}}\right) + C\left\{1-\chi\left(\widetilde{a}(t)\varepsilon^{\frac{\theta}{2-\theta}}\right)\right\}, \\ &\text{where } \widetilde{a}(t) := \max\{a_1(t), m(t)\}\,, \end{aligned} \tag{4.15}$$

where function $\rho(t,\xi)$ is defined as a positive root of the equation

$$\rho^2 - 1 - \langle\xi\rangle\lambda^2(t)\Lambda(t)^{-\frac{\theta}{\theta-1}} = 0\,,$$

while $\chi \in C_0^\infty(\mathbb{R})$ is a cutoff function such that $\chi(z) = 0$ if $|z| \geq 1$, and $\chi(z) = 1$ if $|z| \leq 1/2$.

Lemma 4.1 *Function $g_\varepsilon(t,\xi)$ of* (4.15) *satisfies conditions* (2.4) , (2.5) .

Proof. Let be given a positive number ε. Consider

$$\int_0^T g_\varepsilon(t,\xi)\,dt.$$

Firstly we evaluate an integral

$$\int_0^T \chi\left(\Lambda(t)^\theta\langle\xi\rangle^{\theta-1}\varepsilon^{-\frac{2\theta(\theta-1)}{\theta-2}}/2\right)\left(\rho(t,\xi)+\frac{\rho_t(t,\xi)}{\rho(t,\xi)}\right)dt \leq \int_0^{t_{\xi,\theta}}\left(\rho(t,\xi)+\frac{\rho_t(t,\xi)}{\rho(t,\xi)}\right)dt$$

where point $t_{\xi,\theta}$ is determined by

$$\Lambda(t_{\xi,\theta})^\theta\langle\xi\rangle^{\theta-1} = 2\varepsilon^{\frac{2\theta(\theta-1)}{\theta-2}} \tag{4.16}$$

that is by (4.10) with $2\varepsilon^{\frac{2\theta(\theta-1)}{\theta-2}} = N^{\theta-1}$. We have

$$\begin{aligned}\int_0^{t_{\xi,\theta}}\left(\rho(t,\xi)+\frac{\rho_t(t,\xi)}{\rho(t,\xi)}\right)dt &\le \int_0^{t_{\xi,\theta}}dt+\int_0^{t_{\xi,\theta}}\lambda(t)\langle\xi\rangle^{\frac{1}{2}}\Lambda(t)^{-\frac{\theta}{2(\theta-1)}}\,dt+\ln\rho(t_{\xi,\theta},\xi)\\ &\le 1+\ln\rho(t_{\xi,\theta},\xi)+\frac{2(\theta-1)}{\theta-2}\Lambda(t_{\xi,\theta})^{\frac{\theta-2}{2(\theta-1)}}\langle\xi\rangle^{\frac{1}{2}}\\ &\le C+C\ln\langle\xi\rangle+\frac{2^{\frac{2\theta^2-\theta-2}{2\theta(\theta-1)}}(\theta-1)}{\theta-2}\varepsilon\langle\xi\rangle^{\frac{1}{\theta}}\,. \end{aligned}\tag{4.17}$$

Further,

$$\begin{aligned}&\int_0^T\left(1-\chi\left(2\Lambda(t)^\theta\langle\xi\rangle^{\theta-1}\varepsilon^{-\frac{2\theta(\theta-1)}{\theta-2}}\right)\right)\widetilde{a}(t)\lambda(t)\Lambda(t)^{\frac{\theta}{1-\theta}}\chi\left(\widetilde{a}(t)\varepsilon^{\frac{\theta}{2-\theta}}\right)\,dt\\ &\le \int_{t'_{\xi,\theta}}^T\widetilde{a}(t)\lambda(t)\Lambda(t)^{\frac{\theta}{1-\theta}}\chi\left(\widetilde{a}(t)\varepsilon^{\frac{\theta}{2-\theta}}\right)\,dt\\ &\le \varepsilon^{\frac{\theta}{\theta-2}}\int_{t'_{\xi,\theta}}^T\lambda(t)\Lambda(t)^{\frac{\theta}{1-\theta}}\,dt\,,\end{aligned}\tag{4.18}$$

where point $t'_{\xi,\theta}$ is determined by

$$4\Lambda(t'_{\xi,\theta})^\theta\langle\xi\rangle^{\theta-1}=\varepsilon^{\frac{2\theta(\theta-1)}{\theta-2}}\,.\tag{4.19}$$

Hence

$$\begin{aligned}\varepsilon^{\frac{\theta}{\theta-2}}\int_{t'_{\xi,\theta}}^T\lambda(t)\Lambda(t)^{\frac{\theta}{1-\theta}}\,dt &= \varepsilon^{\frac{\theta}{\theta-2}}\frac{1}{1-\theta}\left(\Lambda(T)^{\frac{1}{1-\theta}}-\Lambda(t'_{\xi,\theta})^{\frac{1}{1-\theta}}\right)\\ &\le \varepsilon^{\frac{\theta}{\theta-2}}\frac{1}{\theta-1}\Lambda(t'_{\xi,\theta})^{\frac{1}{1-\theta}}\\ &\le \frac{1}{\theta-1}2^{\frac{2}{\theta(\theta-1)}}\varepsilon\langle\xi\rangle^{\frac{1}{\theta}}\end{aligned}\tag{4.20}$$

The lemma is proved. □

Acknowledgement: We wrote this paper during our visits to the Max-Planck-Arbeitsgruppe "Partielle Differentialgleichungen und Komplexe Analysis" at the University of Potsdam. The authors are very grateful to all members of the Max-Planck-Arbeitsgruppe Analysis for their warm and helpful hospitality. We owe special thanks to Professor B-.W.Schulze, the leader of the Max-Planck-Arbeitsgruppe who invited us to Potsdam and encouraged to write this paper.

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