On the Index of Differential Operators on Manifolds with Conical Singularities

B.-W. Schulze, B. Sternin, and V. Shatalov

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Bert-Wolfgang Schulze
Potsdam University
e-mail: schulze@mpg-ana.uni-potsdam.de
&
Boris Sternin* **and Victor Shatalov***
Moscow State University
e-mail: sternin@mpg-ana.uni-potsdam.de

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Abstract

The paper contains the proof of the index formula for manifolds with conical points. For operators subject to an additional condition of spectral symmetry, the index is expressed as the sum of multiplicities of spectral points of the conormal symbol (indicial family) and the integral from the Atiyah-Singer form over the smooth part of the manifold. The obtained formula is illustrated by the example of the Euler operator on a two-dimensional manifold with conical singular point.

Keywords: conical singularities, Mellin transform, pseudodifferential operators, ellipticity, Fredholm operators, regularizers, analytic index, trace class, Atiyah-Singer theorem, Euler characteristics

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Contents

Introduction

The index theory of elliptic operators on compact manifolds without boundary (see Atiyah-Singer [4]) have arized, in particular, as an answer to the question stated by Gel'fand [20] on homotopic classification of the set of elliptic opertors and on expressing the index in terms of topological invariants of this operator. While solving this problem, it was found out that the Atiyah-Singer index formula applied to the so-called group operators (that is, the operators associated with a G-structure of the manifold) establishes important differential-geometrical and topological relations. For example, it explains why the $\hat{A}$-genus as the index of (elliptic) Dirac operator is integer. So, there arised two directions in the index theory. The first is connected with the computation of the index of general (elliptic) Fredholm operators on different manifolds and, in particular, with the homotopy classification of the set of such operators. The second is connected with the investigation of classical differential-topological invariants: the signature, the $\hat{A}$-genus, and Euler characteristics with the help of representation of these invariants as indices of the group operators.

In the following years the index theory has been developed in both these directions. So, in the framework of the first direction there arised a natural problem of computation of the index for elliptic operators on manifolds with boundary, that is, of the index for boundary value problems. The important fact which simplified the solution of this problem was that up to this time the analytic apparatus for investigation of the Fredholm property of such problems was worked out. Namely, it was known that for a boundary value problem to determine a Fredholm operator, it is nesessary and (in the Sobolev space scale) sufficient that some special conditions (the so-called Shapiro-Lopatinskii conditions or coercitivity conditions) are fulfilled (see, e.g. [25]). For operators subject to these conditions the index formula have been also found and proved, and, in particular, the homotopy classification of some natural extension of the algebra of pseudodifferential operators, the Boutet be Monvel's algebra (see [15], [42]), have been given.

At the same time, it was found that not any elliptic operator admits boundary operators subject to the Shapiro-Lopatinskii conditions. Moreover, the topological obstruction on the symbol of the main operator necessary for the latter to admit a coercitive boundary value problem was given [3].

Unfortunately, this obstruction does not vanish for the important geometric operators: the signature operator and the Dirac operator. Hence, for these operators a boundary value problem subject to the Shapiro-Lopatinskii conditions does not exist and the general index formula for elliptic operators on manifolds with boundary is inapplicable.

In this connection, there arises the natural question, can one generalize the signature and the $\hat{A}$-genus (the index of the Dirac operator) formulas to the case of manifold with boundary. This problem was solved in the series of papers by Atiyah, Patodi, and Singer [2]. Namely, it was found that the formula for the signature

$$\operatorname{sign} X = \int_X L(p)$$

(here L is the Hirzebruch polynomial, and p are the Pontrjagin forms), being valid on a manifold without boundary, fails on a manifold with boundary. More exactly, the difference

$$\operatorname{sign} X - \int_X L(p)$$

is a nonvanishing function of the boundary $Y = \partial X$ of the manifold X. To compute this correction, the application of the index theory for a special operator on the manifold X turns out to be useful. Namely, this operator is the signature operator with special (nonlocal) homogeneous boundary value conditions which can also be interpreted as an operator in the space L_2 on the noncompact manifold $\hat{X}$ obtained by attaching an infinite cylinder to the boundary of the initial manifold M.

In such terms the mentioned difference is none more than the value at the origin of the so-called η-function of some operator associated with the signature operator.

So,

$$\operatorname{sign} X = \int\limits_X L(p) - \eta(0). \tag{1}$$

A similar formula for the Hirzebruch $\hat{A}$-genus

$$\operatorname{index} \hat{D} = \int\limits_X \hat{A}(p) - \frac{h + \eta(0)}{2} \tag{2}$$

relates the index of the Dirac operator on a manifold with boundary with the eta invariant (the reader will find the details in the above cited paper [2]). We emphasize that each summand involved into the right-hand side of (1) or (2) is not a topological invariant.

The paper by Atiyah, Patodi, and Singer, together with the evident interpretation of the manifold with infinite cylinder as a manifold with conical point, was a starting point of a wide range of investigations devoted to connections between different invariants of manifolds with singularities: topological (such as signature), differential geometric (such as the integral of the Hirzebruch polynomial), and spectral (eta invariant), as well as to the computation of the L_2-index of the Dirac operator, which is strongly connected with the above problems. Among the papers on this subject we mention first the papers [11], [12], [13] by J. Cheeger on spectral asymmetry, the books [33] by R. Melrose and [21], as well as the papers [5, 6, 7] by M. Bismuth and J. Cheeger, [8] by J. Brüning, [10] by J. Brüning and R. Seeley, [14] by A. W. Chou, [53] by X. Dai, [16] by H. Donnely, [17] by H. Donnely and C. Fefferman, [18] by R. D. Douglas and K. P. Wojciechowski, [38, 39] by W. Müller, [23] by G. Grubb, [24] by G. Grubb and R. Seeley, [26] by S. Klimek and K. Woiciechowski, [27] by E. Leichtnam and P. Piazza, [28] by M. Lesch, [30] by J. Lott, [31, 32] by R. Mazzeo and R. Melrose, [36] by H. Moscovici and R. Stanton, and others.

So far to the connections between differential geometric, topological and spectral invariants established by the group operators. At the same time, there is a natural problem of finding the conditions for the Fredholm property to be valid (the proof of the finiteness theorem), homotopy classification, and the computation of the index in topological terms in the *general* theory of elliptic equations on compact manifolds with singularities. This theory was intensively developed during last decades and essentially completed, at least in its analytical part. In fact, the finiteness theorem (Fredholm property) on manifolds with conical singularities in (weighted) Sobolev spaces was proved, the corresponding algebra of operators was constructed (see, e.g. [45]), and the structure of kernel and cokernel for corresponding operators was investigated. Also, the *homotopy classification* of the set of elliptic operators on manifolds with conical singularities was given (see [44]). In this paper, it is shown that the stable homotopy classes of elliptic principal symbols can be described as the direct sum

$$K\left(\tilde{T}^*\right) \oplus \mathbf{Z}^N, \tag{3}$$

where $\tilde{T}^*$ is the compressed cotangent bundle of the manifold M and N is the number of conical points of the underlying manifold.

Formula (3) determines the structure of the topological index for an elliptic operator on manifolds with conical singularities. Namely, the expression for the topological index has to satisfy the following requirements:

1) it has to be a sum of terms corresponding to the smooth part of the manifold and to its singular points;

2) each term on the right in this expression has to be an (integer) topological invariant;

3) the topological index has to be expressed via the principal symbol of the operator in the sense of elliptic theory on manifolds with conical singularities (see, e.g., [45]).

Many authors tried to find a solution of this problem. Let us mention, in particular, the papers by B. Plamenevskii and G. Rozenblyum [41], P. Piazza [40], as well as the recent paper [35] by R. Melrose and V. Nistor, where the generalization of the Atiyah-Patodi-Singer theory to operators of orders different from one is considered. Naturally, the generality of the situation (compared with that considered in [2]) leads to serious problems in obtaining explicit, final (and fine) index formulas. So, the index formula obtained by P. Piazza (as it was remarked by the author) cannot be viewed as a final one, since the contribution from the smooth part of the manifold is written down in the implicit form using zeta function of the operator in question. The formula by Plamenevskii-Rozenblyum has, in fact, the same disadvantage. In this paper, the index is expressed via traces of n-th component of the symbol of some pseudodifferential operator containing the composition of the operator and its regularizer. The index formula obtained in the paper [35] by R. Melrose and V. Nistor expresses the index of an elliptic operator via Wodzicki residue trace [52]. In the paper [35], the authors interpret functionals involved into the index formula as cycles of the Hochschild algebra of pseudodifferential operators on manifolds with cusp-type singularities of order one.

Let us mention here also the recent papers by B. Fedosov and B.-W. Schulze [19] and G. Rozenblyum [43] where the index of zero-order operators on the model cone is computed. In fact, these authors have computed the index for pseudodifferential operators of zero order with the operator-valued symbol on the (half-)line. Clearly, the answers in this case were obtained in terms of the index of a Fredholm family corresponding to such an operator.

Let us pass now to the description of the results of the present paper. First of all, we consider general elliptic (pseudo)differential operators on an arbitrary manifold with singularities of conical type subject to some additional condition of symmetry of the conormal symbol with respect to the point $p = i\gamma_0$ for some real value of γ_0. For operators from this class we obtain the index formula as the sum of multiplicities of some spectral points of the index family (conormal symbol) and the integral from Atiyah-Singer form over the smooth part of the manifold. In particular, if the weight line $\operatorname{Im} p = \gamma_0$ is free from the spectral points of the mentioned family, we obtain that the index of the operator in the weighted Sobolev space corresponding to this weight line is simply the integral from the corresponding Atiyah-Singer form.

Let us summarize briefly the method used in this paper to obtain the index formula. The main idea (though quite natural) is to construct the regularizer for the considered operator with the help of the so-called Green operator on the singular part of the manifold and standard regularizer on the smooth part of the underlying manifold. Note that similar technique (for different purposes) was used in the well-known paper by M. Gromov and H. B. Lawson [22] (see also N. Anghel [1].)

In conclusion, let us mention some consequences and generalizations of our main theorem. First, as a trivial consequence we obtain the relative formula connecting indices of one and the same operator acting in different weighted Sobolev spaces.

Second, the index formula obtained in the paper for conical singularities, is also valid for manifolds with *cusp-type* singularities. The corresponding elliptic theory was developed by the authors in the series of recent papers [46, 47, 49]. In particular, in the paper [49] it is shown that any elliptic operator on a manifold with singularities of cuspidal type is homotopic to some elliptic operators on a (topologically equivalent) manifold with conical singularities. Hence, the computation of the index on manifolds with cusp-type singularities is reduced to that on manifolds with singularities of conical type. Let us mention in this connection we mention the papers [37], [29], [9], where the index for geometrical operators on manifolds with cuspidal singularities is computed, and, clearly, the above cited paper [35].

Finally, note that the proof of the index theorem presented in this paper is quite elementary and does not use any complicated algebraic-topological constructions.

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1 Statement of the problem

The aim of this paper is to establish an index formula for elliptic differential operators on a manifold M with conical-like singularities. Recall that the definition of a manifold with conical singularities is as follows:

Let M be a Hausdorff topological space with a fixed finite set of points $\{\alpha_1, \ldots, \alpha_N\}$ such that:

i) the space $M \setminus \{\alpha_1, \ldots \alpha_N\}$ possesses a structure of a smooth manifold of dimension n;

ii) for each α_j, $j = 1, \ldots, N$ there exists a neighborhood U_j, a smooth compact manifold Ω_j, and a homeomorphism

$$\varphi_j \; : \; U_j \; \to \; K_j = ([0,1) \times \Omega_j) \, / \, (\{0\} \times \Omega_j)$$

of U_j on the open cone K_j with base Ω_j such that the image of the vertex of the cone is α_j and the restriction

$$\varphi_j|_{((0,1)\times\Omega_j)} \; : \; ((0,1) \times \Omega_j) \; \to \; U_j \setminus \{\alpha_j\} \tag{4}$$

is a diffeomorphism (we consider $((0,1) \times \Omega_j)$ as a smooth manifold with standard smooth structure of the direct product);

iii) the structure ring $\mathrm{Diff}(M)$ of differential operators (see [48]) near each point α_j on M consists of operators representable in the form

$$\hat{D} = r^{-m} \sum_{l=0}^{m} \hat{a}_l\,(r) \left(ir \frac{d}{dr} \right)^l \tag{5}$$

with $\hat{a}_l\,(r) \in \mathrm{Diff}^{\,m-l}\,(\Omega_j)$. So, $\hat{a}_l\,(r)$ are differential operators on Ω_j with smooth coefficients locally expressed in the form

$$\hat{a}_l\,(r) = \sum_{|\alpha| \le m-l} a_{l\alpha}\,(r, \omega) \left(\frac{\partial}{\partial \omega} \right)^\alpha ,$$

where $\omega = (\omega^1, \ldots, \omega^{n-1})$ are local coordinates on Ω_j with (r, ω) being used as local coordinates in U_j due to diffeomorphism (4).

To be short, we shall consider manifolds with only one singular point α, so that $N = 1$ (and we omit the corresponding index j). Clearly, it leads to no loss of generality.

Consider an elliptic differential operator $\hat{D}$ on M acting in sections of vector bundles E_1 and E_2 on M:

$$\hat{D} \; : \; C^{\infty}(M, E_1) \; \to \; C^{\infty}(M, E_2) \,. \tag{6}$$

Recall (see, e. g. [45]) that the definition of the ellipticity of operator (6) is as follows:

Definition 1 Operator (6) is called *elliptic*, if:

i) the operator is *elliptic* in the usual sense on $M \setminus \{\alpha\}$*;*

ii) the operator family

$$\hat{D}(p) = \sum_{l=0}^{m} \hat{a}_l(0)\, p^l \tag{7}$$

is an *elliptic family* of differential operators on Ω with parameter p.

It is known that under this requirement family (7) is meromorphically invertible in the whole plane $\mathbf{C}$ with coordinate p. Besides, each strip $a \le \operatorname{Im} p \le b$ contains at most a finite number of poles of the family $\hat{D}^{-1}(p)$, and the estimate

$$\left\| \left(1 + |p|^2 + \Delta_\Omega\right)^{-m/2} \hat{D}^{-1}(p) \right\|_{H^s(\Omega) \to H^s(\Omega)} \le C \tag{8}$$

takes place on each horizontal line containing no poles of $\hat{D}^{-1}(p)$. Here Δ_Ω is a positive Beltrami-Laplace operator on Ω.

The operator (6) can be realized as a bounded operator

$$\hat{D}_\gamma \; : \; H^{s,\gamma}(M, E_1) \; \to \; H^{s-m,\gamma-m}(M, E_2) \,, \tag{9}$$

where $H^{s,\gamma}(M, E_j)$, $j = 1, 2$ are weighted Sobolev spaces with weight γ of sections of bundles E_j defined (locally) by the norm

$$||f||^2_{s,\gamma} = \int_0^\infty r^{-2\gamma} \left\| \left(1 + \left(ir\frac{d}{dr}\right)^2 + \Delta_\Omega\right)^{s/2} f \right\|^2_{L_2(\Omega)} \frac{dr}{r}.$$

The finiteness theorem claims that (9) is a Fredholm operator for all $s \in \mathbf{R}$ provided the line $\operatorname{Im} p = \gamma$ is free from spectral points of the family $\hat{D}(p)$. The spaces

Ker D_γ and Coker D_γ do not depend on s (but depend on γ!)[1]. So the analytic index of this operator

$$\operatorname{index}_a \hat{D} = \dim \operatorname{Ker} \hat{D} - \dim \operatorname{Coker} \hat{D}$$

is well-defined. In this paper, we present an index formula for operator (9) in terms of a symbol of an operator on manifolds with singularities. Recall that the symbol of such an operator is a pair

$$(\operatorname{smbl}_\psi(\hat{D}), \operatorname{smbl}_c(\hat{D})),$$

where $\operatorname{smbl}_\psi(\hat{D})$ is the usual symbol of the pseudodifferential operator viewed as a function on the nonsingular part of the manifold, and $\operatorname{smbl}_c(\hat{D})$ is its conormal symbol (or indicial family in the notation of [33]) defined by the behavior of the operator at the singular points of the manifold (see [45]).

Remark 1 As it was shown in the paper [49] the computation of the index of an operators on a manifold with cusp-type singularities can be reduced to that on a manifold with conical points. So, the results obtained in this paper are also applicable to operators on manifolds with cusp-type singularities of arbitrary (positive) order.

2 Construction of a regularizer

In the sequel, we make use of the well-known formula

$$\operatorname{index}_a \hat{D} = \operatorname{trace}\left(1 - \hat{R}\hat{D}\right) - \operatorname{trace}\left(1 - \hat{D}\hat{R}\right) \tag{10}$$

for the analytic index, where $\hat{R}$ is a regularizer for the operator $\hat{D}$, that is, an operator

$$\hat{R} \;:\; H^{s-m,\gamma}(M, E_2) \;\to\; H^{s,\gamma}(M, E_1)$$

such that the operators

$$\hat{Q}' = 1 - \hat{R}\hat{D} \;\text{ and }\; \hat{Q} = 1 - \hat{D}\hat{R}$$

[1]Below, we shall omit the superscript γ of the operator $\hat{D}$ having in mind that the number γ is fixed.

are both of trace class[2]; from now on we omit the inessential factor r^{-m} in local expression (5) of the operator $\hat{D}$. In this section, we shall derive an explicit expression for the operator $\hat{R}$ constructed in a special way by means of a local regularizer near the singular point and of a pseudodifferential one on the remaining part of the manifold M. To this end we introduce the following additional requirement:

Condition 1 The symbol of the operator $\hat{D}$ does not depend on the variable r in a neighborhood of the singularity point α, say, for $r \in [0, \varepsilon]$.

Remark 2 The latter requirement leads to no loss of generality. Actually, one can deform the initial operator in the class of elliptic operators in such a way that this requirement will be valid. The deformation can be constructed as follows:

Let

$$\hat{D} = \hat{D}\left(r, ir\frac{d}{dr}\right)$$

be local representation (5) of the operator $\hat{D}$. Consider the function $\nu(r)$ defined on the segment $[0, 3\varepsilon]$ (we suppose that $\varepsilon < 1/3$, so that this interval is in the domain of action of coordinates (r, ω)) such that (see Figure 1):

- this function is infinitely smooth in $[0, 3\varepsilon]$;
- it vanishes identically in the segment $[0, \varepsilon]$ and coincides with r on $[2\varepsilon, 3\varepsilon]$;
- it increases monotonously on $[\varepsilon, 2\varepsilon]$.

Then the operator $\hat{D}_1$ equal to $\hat{D}(\nu(r), ird/dr)$ in $[0, 3\varepsilon]$ and to $\hat{D}$ outside, is a smooth differential operator on M which possesses all needed properties. Besides, this operator is homotopic to $\hat{D}$ in the class of elliptic operators, the homotopy being given by

$$\hat{D}_t = \hat{D}\left(t\nu(r) + (1-t)r, ir\frac{d}{dr}\right), \ t \in [0,1],$$

so that the index of the operator $\hat{D}$ coincides with that of $\hat{D}_1$.

As it was mentioned above, we construct the regularizer for the operator $\hat{D}$ as a combination of a local regularizer near the singular point α and the pseudodifferential regularizer on the rest part of the manifold.

[2]By operators of the trace class we mean the operators whose Schwartz kernel is integrable on the diagonal with respect to the measure $dv_\Omega\, dr/r$ (dv_Ω being a smooth volume element on the base Ω of the cone). In fact, we use below only operators which are projectors on finite-dimensional space, and operators with Schwartz kernels vanishing near the singular points.

Figure 1. The function used in the deformation process.

2.1 Construction of the local regularizer

According to the above modification, the operator $\hat{D}$ can be represented in the form

$$\hat{D} = \hat{D}\left(ir\frac{d}{dr}\right)$$

in some neighborhood of the singular point. For the technical reasons which will be clarified below, we shall realize this operator as an operator $\hat{D}_0$ on the (infinite) model cone

$$K = ([0,\infty) \times \Omega) \,/\, (\{0\} \times \Omega)$$

acting as a bounded operator in the function spaces

$$\hat{D}_0 = \hat{D}\left(ir\frac{d}{dr}\right) \;:\; H^{s,\gamma,0}(K) \to H^{s-m,\gamma,0}(K), \tag{11}$$

where the spaces $H^{s,\gamma_-,\gamma_+}(K)$ are defined as a completion of the space $C_0^\infty(\mathbf{R}_+ \times \Omega)$ with respect to the norm (in the situation of the infinite cylinder $(-\infty,+\infty) \times \Omega$ such spaces were introduced and used by B. Sternin [50], [51]; our presentation of the local regularizer is a slight modification of results obtained in this book)

$$\|u\|^2_{s,\gamma_-,\gamma_+} = \int\limits_0^\infty \psi_{\gamma_-,\gamma_+}(r)\left\|\left(1+\left(ir\frac{d}{dr}\right)^2+\Delta_\Omega\right)^{s/2}u\right\|^2_{L_2(\Omega)}\frac{dr}{r},$$

where the weight function $\psi_{\gamma_-,\gamma_+}(r)$ is a strictly positive smooth function on $\mathbf{R}_+$ such that

$$\begin{aligned}\psi_{\gamma_-,\gamma_+}(r) &= r^{-2\gamma_-} \text{ near } r=0,\\ \psi_{\gamma_-,\gamma_+}(r) &= r^{-2\gamma_+} \text{ near } r=\infty.\end{aligned}$$

The construction of the local regularizer is different in the cases $\gamma<0$, $\gamma=0$, and $\gamma>0$ (clearly, we suppose that the lines $\operatorname{Im} p=\gamma$ and $\operatorname{Im} p=0$ do not contain poles of the family $\hat{D}^{-1}(p)$; if this is not so, we can use a suitable change of the unknown).

Case 1: $\gamma<0$. Let $\varphi_1(r)$, $\varphi_2(r)$ be a partition of unity such that $\varphi_1(r)\equiv 1$ near $r=0$, and $\varphi_2(r)\equiv 1$ near $r=\infty$. Consider the operator

$$\begin{aligned}\hat{R}_1[f] &= \mathcal{M}_\gamma^{-1}\left(\hat{D}^{-1}(p)\left[\mathcal{M}_\gamma(\varphi_1 f)\right](p)\right)\\ &+\mathcal{M}_0^{-1}\left(\hat{D}^{-1}(p)\left[\mathcal{M}_0(\varphi_2 f)\right](p)\right),\end{aligned}\tag{12}$$

where the γ-Mellin transform $\mathcal{M}_\gamma(f)$ of a function $f(r)$ is defined as

$$\tilde{f}(p)=\mathcal{M}_\gamma(f)=\int_0^\infty r^{ip}f(r)\frac{dr}{r}$$

considered as a function on the line $\mathcal{L}_\gamma=\{\operatorname{Im} p=\gamma\}$, and the inverse $\mathcal{M}_\gamma^{-1}$ is given by

$$f(r)=\mathcal{M}_\gamma^{-1}\left(\tilde{f}\right)=\frac{1}{2\pi}\int_{\mathcal{L}_\gamma} r^{-ip}\tilde{f}(p)\,dp.$$

Let us evaluate the $H^{s,\gamma,0}$-norm of (12) provided the function f belongs to the space $H^{s-m,\gamma,0}(K)$. First observe that the multiplication by φ_1 is a bounded operator from the space $H^{s-m,\gamma,0}(K)$ to $H^{s-m,\gamma}(K)=H^{s-m,\gamma,\gamma}(K)$. Second, the operator

$$\mathcal{M}_\gamma^{-1}\circ\hat{D}^{-1}(p)\circ\mathcal{M}_\gamma\;:\;H^{s-m,\gamma}(K)\to H^{s,\gamma}(K)$$

is bounded due to estimates (8). Third, the space $H^{s,\gamma}(K)$ for $\gamma<0$ is continuously embedded into the space $H^{s,\gamma,0}(K)$. Hence, the first summand

$$\mathcal{M}_\gamma^{-1}\circ\hat{D}^{-1}(p)\circ\mathcal{M}_\gamma\circ\varphi_1\;:\;H^{s-m,\gamma,0}(K)\to H^{s,\gamma,0}(K)$$

is continuous. Similar considerations show that the second summand

$$\mathcal{M}_0^{-1}\circ\hat{D}^{-1}(p)\circ\mathcal{M}_0\circ\varphi_2\;:\;H^{s-m,\gamma,0}(K)\to H^{s,\gamma,0}(K)$$

is also continuous and, hence, the operator $\hat{R}_1$ is a continuous operator in the space scale $H^{s,\gamma,0}(K)$ of order $-m$. Therefore, the estimate

$$\left\|\hat{R}_1 f\right\|_{s,\gamma,0} \leq C \left\|f\right\|_{s-m,\gamma,0}$$

is valid.

Now, let us compute the compositions of the constructed operator $\hat{R}_1$ with the operator $\hat{D}(ird/dr)$. First, we have

$$\begin{aligned}
\hat{D}_0 \circ \hat{R}_1[f] &= \hat{D}_0 \mathcal{M}_\gamma^{-1}\left(\hat{D}^{-1}(p)\left[\mathcal{M}_\gamma(\varphi_1 f)\right](p)\right) \\
&\quad + \hat{D}_0 \mathcal{M}_0^{-1}\left(\hat{D}^{-1}(p)\left[\mathcal{M}_0(\varphi_2 f)\right](p)\right) \\
&= \mathcal{M}_\gamma^{-1}\left(\mathcal{M}_\gamma(\varphi_1 f)(p)\right) + \mathcal{M}_0^{-1}\left(\mathcal{M}_0(\varphi_2 f)(p)\right) \\
&= f
\end{aligned}$$

due to the well-known properties of the Mellin transform. So,

$$\hat{D}_0 \circ \hat{R}_1 = \mathbf{1},$$

where $\mathbf{1}$ stands for the identity operator in $H^{s-m,\gamma,0}(K)$. Second, we have

$$\begin{aligned}
&\mathcal{M}_\gamma^{-1} \circ \hat{D}^{-1}(p) \circ \mathcal{M}_\gamma \circ \varphi_1 \circ \hat{D}_0 = \varphi_1 + \\
&\mathcal{M}_\gamma^{-1} \circ \hat{D}^{-1}(p) \circ \mathcal{M}_\gamma \circ \left[\varphi_1, \hat{D}_0\right],
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{M}_0^{-1} \circ \hat{D}^{-1}(p) \circ \mathcal{M}_0 \circ \varphi_2 \circ \hat{D}_0 = \varphi_2 + \\
&\mathcal{M}_0^{-1} \circ \hat{D}^{-1}(p) \circ \mathcal{M}_0 \circ \left[\varphi_2, \hat{D}_0\right].
\end{aligned}$$

Taking into account the relation $\varphi_1 + \varphi_2 = 1$, we arrive at the relation

$$\hat{R}_1 \circ \hat{D}_0 = \mathbf{1} - \hat{Q}_1',$$

with

$$\hat{Q}_1' = \left(\mathcal{M}_\gamma^{-1} \circ \hat{D}^{-1}(p) \circ \mathcal{M}_\gamma - \mathcal{M}_0^{-1} \circ \hat{D}^{-1}(p) \circ \mathcal{M}_0\right) \circ \left[\varphi_2, \hat{D}_0\right].$$

The operator $\left[\varphi_2, \hat{D}_0\right]$ is a differential operator in ird/dr of order $m-1$

$$\left[\varphi_2, \hat{D}_0\right] = \sum_{j=0}^{m-1} b_j(r)\left(ir\frac{d}{dr}\right)^j$$

with $b_j(r) \in C_0^\infty(\mathbf{R}_+)$. Hence,

$$\hat{Q}_1' u = \frac{1}{2\pi} \int\limits_{\mathcal{L}_\gamma} r^{-ip} \hat{D}^{-1}(p) F(p)\, dp - \frac{1}{2\pi} \int\limits_{\mathcal{L}_0} r^{-ip} \hat{D}^{-1}(p) F(p)\, dp, \tag{13}$$

where

$$F(p) = \mathcal{M}_\gamma \left(\left[\varphi_2, \hat{D}_0 \right] u \right) = \mathcal{M}_0 \left(\left[\varphi_2, \hat{D}_0 \right] u \right)$$

(these two functions are given by one and the same analytic expression and are considered as the restriction of one and the same analytic function on lines $\mathcal{L}_\gamma$ and $\mathcal{L}_0$, respectively.) From the above considerations, it follows that $F(p)$ is an entire function in p and that $\hat{D}^{-1}(p) F(p)$ decreases as $\operatorname{Im} p \to \infty$ uniformly in $\operatorname{Re} p$ in compact sets. So we can apply the residue formula to the right-hand part of (13) and obtain

$$\hat{Q}_1' u(r) = i \sum \operatorname{Res}_{p_j} \left[r^{-ip} \hat{D}^{-1}(p) \int\limits_0^\infty r_1^{ip} \sum_{j=0}^{m-1} b_j(r_1) \left(i r_1 \frac{d}{dr_1} \right)^j u(r_1)\, dr_1 \right], \tag{14}$$

where the sum is taken over all poles of the operator function $\hat{D}^{-1}(p)$ lying in the strip $\gamma < \operatorname{Im} p < 0$ (recall that there exists at most a finite number of such points). Since all the residues of this operator function are finite-dimensional operators, so is the operator $\hat{Q}_1'$.

The ranges of the operators involved into the sum on the right of (14) are clearly independent (they consist of functions with conormal asymptotics at $r = 0$ with power p_j); the dimensions of these ranks will be called *multiplicities* of the points p_j. These multiplicities can be computed in terms of the main part of the Laurent expansion of the operator $\hat{D}^{-1}(p)$ at the corresponding point p_j.

Let us show how it can be done on the example when $\hat{D}^{-1}(p)$ has only simple poles. In this case we have

$$\hat{D}^{-1}(p) = \frac{A}{p - p_j} + B(p),$$

where A is a finite-dimensional operator, and $B(p)$ is holomorphic near p_j. We have

$$\begin{aligned} & \operatorname{Res}_{p_j} \left[r^{-ip} \hat{D}^{-1}(p) \int\limits_0^\infty r_1^{ip} \sum_{j=0}^{m-1} b_j(r_1) \left(i r_1 \frac{d}{dr_1} \right)^j u(r_1)\, dr_1 \right] \\ = \; & r^{-ip_j} A \left[\int\limits_0^\infty r_1^{ip_j} \sum_{j=0}^{m-1} b_j(r_1) \left(i r_1 \frac{d}{dr_1} \right)^j u(r_1)\, dr_1 \right], \end{aligned}$$

and the multiplicity of the considered summand equals the dimension of the rank of A.

We arrive at the following result:

Proposition 1 *If $\gamma < 0$, then the operator* (11) *is an epimorphism with finite-dimensional kernel. There exists a continuous operator*

$$\hat{R}_1 : H^{s-m,\gamma,0}(K) \to H^{s,\gamma,0}(K)$$

such that

$$\begin{aligned} \hat{D}_0 \circ \hat{R}_1 &= \mathbf{1}, \\ \hat{R}_1 \circ \hat{D}_0 &= \mathbf{1} - \hat{Q}_1', \end{aligned}$$

where $\hat{Q}_1'$ is the projector on the kernel of operator (11)*. The operators $\hat{R}_1$ and $\hat{Q}_1'$ are given by relations* (12) *and* (14)*, respectively.*

Proof. The only fact left to be proved is that the operator $\hat{Q}_1'$ is a projector on the kernel of the operator (11). First, due to the relation

$$\hat{R}_1 \circ \hat{D}_0 \circ \hat{R}_1 \circ \hat{D}_0 = \hat{R}_1 \circ \hat{D}_0,$$

the operator $\hat{R}_1 \circ \hat{D}_0$ is a projector, and, hence, so is $\hat{Q}_1' = \mathbf{1} - \hat{R}_1 \circ \hat{D}_0$. Second, we have

$$\hat{D}_0 \circ \hat{Q}_1' = \hat{D}_0 \circ \left[\mathbf{1} - \hat{R}_1 \circ \hat{D}_0\right] = 0,$$

so that $\operatorname{Im} \hat{Q}_1' \subset \operatorname{Ker} \hat{D}_0$. Finally, if $u \in \operatorname{Ker} \hat{D}_0$, then

$$0 = \hat{R}_1 \circ \hat{D}_0 u = u - \hat{Q}_1' u,$$

and $u \in \operatorname{Im} \hat{Q}_1'$. This completes the proof.

Case 2: $\gamma > 0$. Let $\varphi_1(r)$, $\varphi_2(r)$ be a partition of unity introduced above. Denote

$$\begin{aligned} \hat{R}_1[f] &= \varphi_1 \mathcal{M}_\gamma^{-1}\left(\hat{D}^{-1}(p)\left[\mathcal{M}_\gamma(f)\right](p)\right) \\ &\quad + \varphi_2 \mathcal{M}_0^{-1}\left(\hat{D}^{-1}(p)\left[\mathcal{M}_0(f)\right](p)\right), \end{aligned} \tag{15}$$

and

$$\hat{Q}_1 u(r) = -i \sum \left[\sum_{j=0}^{m-1} b_j(r)\left(ir\frac{d}{dr}\right)^j\right] \operatorname{Res}_{p_j}\left[r^{-ip}\hat{D}^{-1}(p)\,\mathcal{M}(u)\right], \tag{16}$$

where the outer sum is taken over all poles of $\hat{D}^{-1}(p)$ inside the strip $0 < \operatorname{Im} p < \gamma$. The proof of the following result is quite similar to that in the first case (in fact, the proof can also be obtained from Proposition 1 from the duality reason):

Proposition 2 *If $\gamma > 0$, then the operator* (11) *is a monomorphism with finite-dimensional cokernel. There exists a continuous operator*

$$\hat{R}_1 \,:\, H^{s-m,\gamma,0}(K) \to H^{s,\gamma,0}(K)$$

such that

$$\begin{aligned} \hat{D}_0 \circ \hat{R}_1 &= \mathbf{1} - \hat{Q}_1, \\ \hat{R}_1 \circ \hat{D}_0 &= \mathbf{1}, \end{aligned}$$

where $\hat{Q}_1$ is the projector on the cokernel of operator (11). *The operators $\hat{R}_1$ and $\hat{Q}_1$ are given by relations* (15) *and* (16), *respectively.*

Case 3: $\gamma = 0$. In this case operator (11) is clearly an isomorphism, and the inverse operator is given by

$$\hat{R}_1 f = \mathcal{M}_0^{-1}\left(\hat{D}^{-1}(p)\left[\mathcal{M}_0(f)\right](p)\right).$$

2.2 Construction of the pseudodifferential regularizer

This section is devoted to the construction of the regularizer apart from the singular point of the manifold M. Since in this region the operator $\hat{D}$ is an elliptic differential operator with smooth coefficients, it is natural to construct its regularizer as a classical pseudodifferential operator. The construction of such a regularizer is well-known, and we shall not present all the calculations here, the more that in the sequel we shall not need the explicit form of this regularizer. We restrict ourselves to the formulation of the corresponding result and some comments on the coincidence of the constructed regularizer and the local one constructed in the previous subsection.

Proposition 3 *There exists a classical pseudodifferential operator*

$$\hat{R}_2 \,:\, H^{s-m}_{\mathrm{comp}}(M \setminus \{\alpha\}) \to H^{s}_{\mathrm{loc}}(M \setminus \{\alpha\})$$

such that[3]

$$\begin{aligned} \hat{R}_2 \circ \hat{D} u &= \left(\mathbf{1} - \hat{Q}_2'\right) u, \\ \hat{D} \circ \hat{R}_2 u &= \left(\mathbf{1} - \hat{Q}_2\right) u \end{aligned}$$

for $u \in C_0^\infty(M \setminus \{\alpha\})$ with infinitely smoothing operators $\hat{Q}_2$ and $\hat{Q}_2'$.

[3] In the relations below we omit the standard cut-off functions.

Figure 2. Cut-off functions.

We remark that the two regularizers $\hat{R}_1$ and $\hat{R}_2$ coincide modulo infinitely smoothing operators on functions whose support do not intersect the support of the function φ_1 above. Actually, on such functions

$$\hat{R}_1 f = \mathcal{M}_0^{-1}\left(\hat{D}^{-1}(p)\left[\mathcal{M}_0(f)\right](p)\right),$$

and the simple change of variables $t = \ln r^{-1}$ reduces this operator to a pseudodifferential operator (since $\hat{D}^{-1}(p)$ is a pseudodifferential operator on Ω which is a full regularizer for $\hat{D}(p)$) being a full regularizer for $\hat{D}$. In this consideration the fact that in the last formula the integration is taken over the real axis is essential. *This is exactly the reason why we have to use the spaces $H^{s,\gamma_-,\gamma_+}(K)$ with $\gamma_+ = 0$ in the construction of a local regularizer.*

2.3 Construction of a global regularizer

Let us define functions f, ψ_1, $\psi_2 \in C^\infty\left(\overline{\mathbf{R}}_+\right)$ in such a way that (see Figure 2)

- the function $f(r)$ is a positive smooth function equal 1 in a neighborhood of the origin and 0 for $r \geq \varepsilon - \delta$ (here δ is some positive number, $\delta < \varepsilon$);
- $\psi_1|_{\operatorname{supp} f} = 1$ or $\psi_1 f = f$, and $\psi_1 \equiv 0$ for $r \geq \varepsilon$;

- $\psi_2|_{\operatorname{supp}(1-f)} = 1$ or $\psi_1 (1-f) = 1-f$, $\psi_1 \equiv 0$ for $r \le \delta$.

First, applying the weight shift $\hat{D} \mapsto \hat{D}^{(\gamma)} = r^{-\gamma} \hat{D} r^{\gamma}$ we reduce our investigation to the case $\gamma = 0$. The symbol of the operator $\hat{D}^{(\gamma)}$ equals

$$[\mathrm{smbl} D^{(\gamma)}](p) = [\mathrm{smbl} \hat{D}](p + i\gamma).$$

To be short, we temporarily omit the superscript (γ).

Define the operator

$$\hat{R} \ : \ H^{s-m,0}(M) \ \to \ H^{s,0}(M) \tag{17}$$

by the formula

$$\hat{R} = \psi_1 \hat{R}_1 f + \psi_2 \hat{R}_2 (1-f). \tag{18}$$

Then we have

$$\begin{aligned} \hat{D}\hat{R} \ &= \ \hat{D}\psi_1 \hat{R}_1 f + \hat{D}\psi_2 \hat{R}_2 (1-f) = \psi_1 \hat{D} \hat{R}_1 f + \left[\hat{D}, \psi_1\right] \hat{R}_1 f \\ &\quad + \psi_2 \hat{D} \hat{R}_2 (1-f) + \left[\hat{D}, \psi_2\right] \hat{R}_2 (1-f), \end{aligned}$$

where $[\cdot,\cdot]$ stands for a commutator of operators. Taking into account that the operator $\hat{R}_1$ is an isomorphism for $\gamma = 0$, we arrive at the relations

$$\begin{aligned} &\hat{D}\hat{R} = \psi_1 f + \psi_2 \left(\mathbf{1} - \hat{Q}_2\right)(1-f) + \left[\hat{D}, \psi_1\right] \hat{R}_1 f \\ &+ \left[\hat{D}, \psi_2\right] \hat{R}_2 (1-f) = \mathbf{1} - \Big\{ \psi_2 \hat{Q}_2 (1-f) \\ &- \left[\hat{D}, \psi_1\right] \hat{R}_1 f - \left[\hat{D}, \psi_2\right] \hat{R}_2 (1-f) \Big\} = \mathbf{1} - \hat{Q} \end{aligned}$$

with

$$\hat{Q} = \psi_2 \hat{Q}_2 (1-f) - \left[\hat{D}, \psi_1\right] \hat{R}_1 f - \left[\hat{D}, \psi_2\right] \hat{R}_2 (1-f). \tag{19}$$

Similar, with the help of Propositions 1 and 3

$$\begin{aligned} &\hat{R}\hat{D} = \psi_1 \hat{R}_1 f \hat{D} + \psi_2 \hat{R}_2 (1-f) \hat{D} = \psi_1 \hat{R}_1 \hat{D} f + \psi_1 \hat{R}_1 \left[f, \hat{D}\right] \\ &+ \psi_2 \hat{R}_2 \hat{D} (1-f) - \psi_2 \hat{R}_2 \left[f, \hat{D}\right] = \mathbf{1} - \Big\{ \psi_2 \hat{Q}_2' (1-f) \\ &+ \left(\psi_2 \hat{R}_2 - \psi_1 \hat{R}_1\right) \left[f, \hat{D}\right] \Big\} = \mathbf{1} - \hat{Q}' \end{aligned}$$

with

$$\hat{Q}' = \psi_2 \hat{Q}_2' (1-f) + \left(\psi_2 \hat{R}_2 - \psi_1 \hat{R}_1\right) \left[f, \hat{D}\right]. \tag{20}$$

We remark that one of the operators $\hat{Q}_1$, $\hat{Q}'_1$ vanishes depending on the sign of the number γ.

In view of the pseudolocality principle for pseudodifferential operators and of the last remark of the preceding subsection the operators $\hat{Q}$ and $\hat{Q}'$ are infinitely smoothing ones, so that operator (18) is a global regularizer for the operator

$$\hat{D} \,:\, H^{s,\gamma}(M, E_1) \to H^{s-m,\gamma}(M, E_2). \tag{21}$$

So, we have arrived at the following statement:

Theorem 1 *The operator $\hat{R}$ given by* (18) *is a regularizer for operator* (21)*, that is, the relations*

$$\hat{D}\hat{R} = \mathbf{1} - \hat{Q}$$

and

$$\hat{R}\hat{D} = \mathbf{1} - \hat{Q}'$$

take place with infinitely smoothing operators $\hat{Q}$, $\hat{Q}'$ given by (25), (20)*, respectively.*

3 Analytic index (preliminary computation)

In this section, we present the explicit computation of the traces in formula (10). In view of the results obtained in the previous section, this formula can be rewritten as

$$\operatorname{index}_a \hat{D} = \operatorname{trace} \hat{Q}' - \operatorname{trace} \hat{Q},$$

where the operators $\hat{Q}'$ and $\hat{Q}$ are given by formulas (25), (20), respectively.

We have

$$\operatorname{trace} \hat{Q}' = \operatorname{trace} \left\{ \psi_2 \hat{Q}'_2 (1-f) + \left(\psi_2 \hat{R}_2 - \psi_1 \hat{R}_1 \right) \left[f, \hat{D} \right] \right\}.$$

Consider the first summand on the right in the latter relation. This summand can be rewritten in the form

$$\psi_2 \hat{Q}'_2 (1-f) = Q'_2(1-f) + (\psi_2 - 1)\, Q'_2(1-f).$$

Let $K(r,\omega; r', \omega')$ be the Schwartz kernel of the operator Q'_2 so that

$$\hat{Q}'_2 u = \int\limits_K K\left(r,\omega; r',\omega'\right) u\left(r',\omega'\right)\, d\omega' \frac{dr'}{r'}.$$

Hence,

$$\operatorname{trace}\left\{(\psi_2-1)\,\hat{Q}_1'(1-f)\right\}=\int\limits_K(\psi_2(r)-1)\,K(r,\omega;r,\omega)\,(1-f(r))\,d\omega\frac{dr}{r}=0$$

(recall that the definition of ψ_2 and $(1-f)$ implies $(\psi_2(r)-1)(1-f(r))=0$), and we obtain

$$\operatorname{trace}\left\{\psi_2\hat{Q}_2'(1-f)\right\}=\operatorname{trace}\left\{\hat{Q}_2'(1-f)\right\}.$$

Similarly, we have

$$\begin{aligned}&\operatorname{trace}\left\{\left(\psi_2\hat{R}_2-\psi_1\hat{R}_1\right)\left[f,\hat{D}\right]\right\}\\ =\;&\operatorname{trace}\left\{\left(\hat{R}_2-\hat{R}_1\right)\left[f,\hat{D}\right]\right\}+\operatorname{trace}\left\{(\psi_2-1)\,\hat{R}_2\left[f,\hat{D}\right]\right\}\\ &-\operatorname{trace}\left\{(\psi_1-1)\,\hat{R}_1\left[f,\hat{D}\right]\right\}\\ =\;&\operatorname{trace}\left\{\left(\hat{R}_2-\hat{R}_1\right)\left[f,\hat{D}\right]\right\}.\end{aligned}$$

So, we have arrived at the relation

$$\operatorname{trace}\hat{Q}'=\operatorname{trace}\left\{\hat{Q}_2'(1-f)\right\}+\operatorname{trace}\left\{\left(\hat{R}_2-\hat{R}_1\right)\left[f,\hat{D}\right]\right\}.$$

Let us compute the last summand on the right in the obtained formula, taking into account the fact that the operators $\hat{R}_1$, $\hat{R}_2$, and $\hat{D}$ do not depend on r on the supports of functions from the range of the operator $\left[f,\hat{D}\right]$. We have

$$\hat{D}_0=\sum_{j=0}^{m}a_j\left(ir\frac{d}{dr}\right)^j,\ a_j\in\operatorname{Diff}^{m-j}(\Omega),$$

and, hence,

$$\begin{aligned}&\hat{D}_0(fu)\\ =\;&\sum_{j=0}^{m}a_j\left(ir\frac{d}{dr}\right)^j(fu)\\ =\;&\sum_{j=0}^{m}a_j\sum_{l=0}^{j}C_j^l\left[\left(ir\frac{d}{dr}\right)^{j-l}f(r)\right]\left(ir\frac{d}{dr}\right)^l u\\ =\;&f(r)\sum_{j=0}^{m}a_j\left(ir\frac{d}{dr}\right)^j u+\sum_{j=1}^{m}a_j\sum_{l=0}^{j-1}C_j^l\left[\left(ir\frac{d}{dr}\right)^{j-l}f(r)\right]\left(ir\frac{d}{dr}\right)^l u.\end{aligned}$$

So, for the commutator $\left[f,\hat{D}\right]$ we obtain

$$\left[f,\hat{D}\right]=-\sum_{l=0}^{m-1}\left(\sum_{j=l+1}^{m}C_j^l a_j\left(ir\frac{d}{dr}\right)^{j-l}f\left(r\right)\right)\left(ir\frac{d}{dr}\right)^{l}.$$

Moreover, setting $\hat{R}=\hat{R}_2-\hat{R}_1$, we have

$$\begin{aligned}\hat{R}\left[f,\hat{D}\right]u &= -\int_0^\infty K\left(r,r_1\right)\sum_{l=0}^{m-1}\left(\sum_{j=l+1}^{m}C_j^l a_j\left(ir_1\frac{d}{dr_1}\right)^{j-l}f\left(r_1\right)\right)\left(ir_1\frac{d}{dr_1}\right)^{l}\\ &\quad\times u\left(r_1\right)\frac{dr_1}{r_1}=-\sum_{l=0}^{m-1}\sum_{j=l+1}^{m}\left(-1\right)^l C_j^l\int_0^\infty\left\{\left(ir_1\frac{d}{dr_1}\right)^{l}\right.\\ &\quad\times\left.\left[K\left(\frac{r}{r_1}\right)a_j\left(ir_1\frac{d}{dr_1}\right)^{j-l}f\left(r_1\right)\right]\right\}u\left(r_1\right)\frac{dr_1}{r_1},\end{aligned}$$

where $K\left(r,r_1\right)=K\left(r/r_1\right)$ is the (operator-valued) Schwartz kernel of the operator $\hat{R}$. Hence,

$$\begin{aligned}\operatorname{trace}\hat{R}\left[f,\hat{D}\right] &= -\sum_{l=0}^{m-1}\sum_{j=l+1}^{m}\left(-1\right)^l C_j^l\int_0^\infty\operatorname{trace}\left\{\left(ir_1\frac{d}{dr_1}\right)^{l}\left[K\left(\frac{r}{r_1}\right)\right.\right.\\ &\quad\times\left.\left.\left. a_j\left(ir_1\frac{d}{dr_1}\right)^{j-l}f\left(r_1\right)\right]\right\}\right|_{r_1=r}\frac{dr}{r}.\end{aligned}$$

It is easy to see that the integrand on the right in the latter relation is a linear combination with constant coefficients of the derivatives

$$\left(ir\frac{d}{dr}\right)^{j}f\left(r\right),\ j\geq 1.$$

All terms containing these derivatives of orders larger than one will vanish after integration because of the particular form of the measure dr/r. The remaining terms are

$$\begin{aligned}\operatorname{trace}\hat{R}\left[f,\hat{D}\right] &= -\sum_{l=0}^{m-1}\left(-1\right)^l\left(l+1\right)\int_0^\infty\operatorname{trace}\left\{\left[\left(ir_1\frac{d}{dr_1}\right)^{l}\right.\right.\\ &\quad\times\left.\left.\left. K\left(\frac{r}{r_1}\right)a_j\right]\right|_{r/r_1=1}\right\}ir\frac{df\left(r\right)}{dr}\frac{dr}{r}=-iK_1\left(1\right),\end{aligned}$$

where $K_1(r/r_1)$ is the kernel of the operator

$$\hat{R}\frac{d\hat{D}}{dp}\left(ir\frac{d}{dr}\right).$$

Taking into account the relation between the Schwartz kernel of the Mellin operator and its symbol, we obtain

$$\operatorname{trace}\hat{R}\left[f,\hat{D}\right]=-\frac{i}{2\pi}\int_{-\infty}^{\infty}\operatorname{trace}\operatorname{smbl}\left[\hat{R}\frac{d\hat{D}}{dp}\right](p)\,dp,$$

and, finally,

$$\begin{aligned}\operatorname{trace}\hat{Q}' &= \operatorname{trace}\left\{\hat{Q}_2'(1-f)\right\}\\ &-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\operatorname{trace}\operatorname{smbl}\left[\left(\hat{R}_1-\hat{R}_2\right)\frac{d\hat{D}}{dp}\right](p)\,dp.\end{aligned}\tag{22}$$

Let us interpret the last term on the right in the latter formula. In essence, it is the regularized integral from the expression

$$\operatorname{trace}\left[\hat{R}_1(p)\frac{d\hat{D}(p)}{dp}\right]=\operatorname{trace}\left[\hat{D}^{-1}(p)\frac{d\hat{D}(p)}{dp}\right]\tag{23}$$

since, as it follows from the considerations below, the last term on the right in (22) does not depend on its pseudodifferential part $\hat{R}_2(p)$. Unfortunately, the integral from function (23) over the axis p diverges, and the pseudodifferential part regularizes this integral. We shall denote the regularized integral by

$$\text{reg-trace}\left[\hat{D}^{-1}(p)\right]=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\operatorname{trace}\operatorname{smbl}\left[\left(\hat{R}_1-\hat{R}_2\right)\frac{d\hat{D}}{dp}\right](p)\,dp$$

and call it the regularized trace of the family $\hat{D}^{-1}(p)$. To show the meaning of this regularized trace, we employ here the remarkable formula:

$$\operatorname{Res}_{p_j}\left(\operatorname{trace}\left[\hat{D}^{-1}(p)\frac{d\hat{D}(p)}{dp}\right]\right)=\dim\operatorname{Ker}\hat{D}(p_j)=\dim\operatorname{Coker}\hat{D}(p_j),$$

so that this term gives the correction to the first term of the index formula when the operator is modified by the change of the unknown $u=r^{\gamma}u_1$.

Similar, but more simple computations show that

$$\operatorname{trace} \hat{Q} = \operatorname{trace}\left\{\hat{Q}_2 (1-f)\right\}.$$

The final trace formula for the index of the operator $\hat{D}$ is

$$\begin{aligned} \operatorname{index}_a \hat{D} &= -\text{reg-trace}\left[\hat{D}^{-1}(p)\right] \\ &\quad + \operatorname{trace}\left\{\hat{Q}_2' (1-f)\right\} - \operatorname{trace}\left\{\hat{Q}_2 (1-f)\right\}. \end{aligned} \tag{24}$$

4 Proof of the main theorem

The aim of this section is to derive a formula expressing the index of an operator $\hat{D}$ on a manifold M with conical points via the Atiyah-Singer form. To this end, we introduce the following requirement on the operator in question.

Condition 2 Let $\hat{D}(p)$ be, as above, the operator-valued symbol of the operator $\hat{D}$ near the singular point α (we recall that, due to Condition 1 above, the operator $\hat{D}$ has constant in r coefficients in a neighborhood of the singular point, and that we suppose that the manifold M has only one singular point α; in particular, the bundles E_1 and E_2 do not depend on r.) Then there exists a number γ_0 and a pair (σ_1, σ_2) of bundle isomorphisms

$$\begin{aligned} \sigma_1 &: E_1 \to E_1, \\ \sigma_2 &: E_2 \to E_2 \end{aligned}$$

such that

$$\sigma_2 \hat{D}\left(\frac{d}{dt} + \gamma_0\right) = \hat{D}\left(-\frac{d}{dt} + \gamma_0\right)\sigma_1,$$

where $t = \ln r^{-1}$. So, the pair (σ_1, σ_2) covers the diffeomorphism $t \mapsto 2t_0 - t$ for any value of t_0 for the shifted operator $\hat{D}(d/dt + \gamma_0)$.

The introduced condition is fulfilled with $\gamma_0 = 0$ for all natural operators such as the exterior differentiation, the Beltrami-Laplace operator associated with a Riemannian metric independent of t, the Euler operator $d + \delta$, and so on. It is also fulfilled if the operator $\hat{D}$ contains derivatives only of even orders. In this case one can choose $\sigma_j = 1$, $j = 1, 2$. Let us show this on the example of the operator $d + \delta$, acting from forms of the even degree on the manifold M to that of odd degree.

Similarly to the above considerations, we suppose that the Riemannian metric determining the operator δ has the form

$$\frac{dr^2}{r^2} + \tilde{g} = dt^2 + \tilde{g}$$

near the singular point of the manifold M. Here $\tilde{g}$ is some metric on the manifold Ω independent of t (we use coordinate $t = \ln r^{-1}$ instead of r here.) Each form β on the cylinder $C = \mathbf{R} \times \Omega$ (which is a local model of the manifold M in a neighborhood of the singular point) can be represented in the form

$$\beta = \beta_0 (t) + dt \wedge \beta_1 (t) ,$$

where $\beta_0 (t)$ and $\beta_1 (t)$ are smooth forms on Ω of corresponding degree depending smoothly on t. The latter relation determines the bundle isomorphism

$$\Lambda^k (C) \approx \Lambda^k (\Omega) \oplus \Lambda^{k-1} (\Omega) .$$

It is easy to see that the matrix of the operator d in this direct decomposition is

$$\begin{pmatrix} \tilde{d} & 0 \\ \partial/\partial t & -\tilde{d} \end{pmatrix},$$

where tilde denotes the corresponding operations on Ω. Let us compute the matrix of the operator δ. We have

$$\begin{aligned} \delta\beta &= (-1)^n * d * [\beta_0 + dt \wedge \beta_1] = (-1)^n * d \left[dt \wedge \tilde{*}\beta_0 + \tilde{*}\beta_1\right] \\ &= (-1)^n * \left[\tilde{d}\tilde{*}\beta_1 + dt \wedge \left(\frac{\partial}{\partial t}\tilde{*}\beta_1 - \tilde{d}\tilde{*}\beta_0\right)\right] \\ &= (-1)^n \left[-\tilde{*}\tilde{d}\tilde{*}\beta_0 + (-1)^{n-1} \frac{\partial}{\partial t}\beta_1 + dt \wedge \tilde{*}\tilde{d}\tilde{*}\beta_1\right] \\ &= \tilde{\delta}\beta_0 - \frac{\partial}{\partial t}\beta_1 - dt \wedge \tilde{\delta}\beta_1. \end{aligned}$$

Hence, the matrix of the operator δ in the above decomposition is

$$\begin{pmatrix} \tilde{\delta} & -\partial/\partial t \\ 0 & -\tilde{\delta} \end{pmatrix},$$

and

$$d + \delta \mapsto \begin{pmatrix} \tilde{d} + \tilde{\delta} & -\partial/\partial t \\ \partial/\partial t & -\left(\tilde{d} + \tilde{\delta}\right) \end{pmatrix}.$$

Now, the diffeomorphism $t \mapsto -t$ induces the bundle isomorphism σ on differential forms which has the matrix

$$\sigma \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to the considered decomposition. Clearly, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \tilde{d} + \tilde{\delta} & -\partial/\partial t \\ \partial/\partial t & -\left(\tilde{d} + \tilde{\delta}\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \tilde{d} + \tilde{\delta} & \partial/\partial t \\ -\partial/\partial t & -\left(\tilde{d} + \tilde{\delta}\right) \end{pmatrix}.$$

The latter relation proves that Condition 2 is fulfilled for the considered operator with $\sigma_1 = \sigma_2 = \sigma$.

Let us consider first the case $\gamma_0 = 0$.

Using formula (24) for the operator $\hat{D}_\gamma = r^{-\gamma}\hat{D}r^\gamma$, we obtain

$$\begin{aligned}\operatorname{index}_a \hat{D} &= -\frac{1}{2\pi i}\int\limits_{\operatorname{Im} p=\gamma} \operatorname{trace}\operatorname{smbl}\left[(\hat{R}_1-\hat{R}_2)\frac{d\hat{D}}{dp}\right](p)\,dp + \operatorname{trace}\{\hat{Q}_2'(1-f)\} \\ &\quad -\operatorname{trace}\{\hat{Q}_2(1-f)\}.\end{aligned}$$

Let us deform the integration contour $\operatorname{Im} p = \gamma$ to the contour $\operatorname{Im} p = 0$. During this process the contour will intersect all poles of the operator $\hat{R}_1(p) = \hat{D}^{-1}(p)$ lying in the strip between $\operatorname{Im} p = \gamma$ and $\operatorname{Im} p = 0$. The residue theorem gives us

$$\begin{aligned}\operatorname{index}_a D &= -\operatorname{sgn}\gamma\left[\sum_j m_j\right] - \text{reg-trace}[\hat{D}^{-1}(p)] + \operatorname{trace}\{\hat{Q}_2'(1-f)\} \\ &\quad -\operatorname{trace}\{\hat{Q}_2(1-f)\},\end{aligned}$$

where m_j are multiplicities of poles of $\hat{D}^{-1}(p)$, and the sum is taken over all poles between $\operatorname{Im} p = \gamma$ and $\operatorname{Im} p = 0$. Due to Propositions 2 and 3, we have

$$\operatorname{index}_a D == \operatorname{index}_a \hat{D}_0 - \text{reg-trace}[\hat{D}^{-1}(p)] + \operatorname{trace}\{\hat{Q}_2'(1-f)\} - \operatorname{trace}\{\hat{Q}_2(1-f)\}. \tag{25}$$

Let now t_0 be any point in the domain where the function f varies, and let U be a sufficiently small collar neighborhood of this point:

$$U = \{t_0 - \varepsilon' < t < t_0 + \varepsilon'\} \times \Omega$$

for sufficiently small ε'. Consider the manifold $M_{t_0,\varepsilon'}$ obtained by deleting the neighborhood $\{t \le t_0 - \varepsilon'\}$ from the manifold M. By $2M$ (the double of the manifold M), we denote the manifold obtained by gluing together two copies of the manifold $M_{t_0,\varepsilon'}$ with different orientations along the neighborhood U with the help of the diffeomorphism $t \mapsto 2t_0 - t$ (see Figure 3). Later on, identifying bundles E_1 and E_2 over U with the help of isomorphisms σ_1 and σ_2, respectively, we obtain bundles on the double $2M$ which we again denote by E_1 and E_2. Due to Condition 2, the operator $\hat{D}$ correctly defines a continuous operator

$$\hat{D}_2 \,:\, C^\infty(2M, E_1) \to C^\infty(2M, E_2).$$

The constructions similar to that above lead us to the following formula for the index of the latter operator (we remark that $\hat{D}_2$ is an elliptic operator on the smooth manifold M without boundary):

$$\operatorname{index}_a \hat{D}_2 = \operatorname{trace}\hat{Q}_2' - \operatorname{trace}\hat{Q}_2, \tag{26}$$

Figure 3. The double of the manifold.

where $\hat{Q}_2'$ and $\hat{Q}_2$ are obtained with the help of the regularizer $\hat{R}_2$ constructed above on the "smooth part" of the manifold M but considered now on the double $2M$. Clearly, one have to use the regularizer $\hat{R}_2$ satisfying the condition

$$\sigma_1 \hat{R}_2 (p) = \hat{R}_2 (-p) \, \sigma_2.$$

Let now dv be a (positive) volume element corresponding to some Riemannian metric g on the manifold M which is, as above, representable in the form

$$g = \frac{dr^2}{r^2} + \tilde{g} = dt^2 + \tilde{g}$$

in the neighborhood of the singular point, where $\tilde{g}$ is a metric on Ω independent of r (the concrete choice of such a metric is not essential for us in future.) Then the traces on the right in (26) can be written down as integrals over the double $2M$ of the corresponding local expressions $T\left(\hat{Q}_2'\right)$ and $T\left(\hat{Q}_2\right)$:

$$\begin{aligned} \operatorname{trace} \hat{Q}_2' &= \int\limits_{2M} T\left(\hat{Q}_2'\right) dv, \\ \operatorname{trace} \hat{Q}_2 &= \int\limits_{2M} T\left(\hat{Q}_2\right) dv. \end{aligned}$$

Due to the symmetry reasons, the expressions on the right in the latter formulas can be represented as integrals over manifold $M_{t_0} = M \setminus \{t \le t_0\}$:

$$\begin{aligned}
\operatorname{trace} \hat{Q}_2' &= 2 \int\limits_{M_{t_0}} T\left(\hat{Q}_2'\right)\, dv, \\
\operatorname{trace} \hat{Q}_2 &= 2 \int\limits_{M_{t_0}} T\left(\hat{Q}_2\right)\, dv,
\end{aligned}$$

so that index formula (26) becomes

$$\operatorname{index}_a \hat{D}_2 = 2 \left(\int\limits_{M_{t_0}} T\left(\hat{Q}_2'\right)\, dv - \int\limits_{M_{t_0}} T\left(\hat{Q}_2\right)\, dv \right). \tag{27}$$

On the other hand, we can use the Atiyah-Singer index theorem [4], thus obtaining

$$\operatorname{index}_a \hat{D}_2 = \int\limits_{2M} \omega_{as}\left(\hat{D}\right),$$

where $\omega_{as}\left(\hat{D}\right)$ is the Atiyah-Singer form defined by the (elliptic) operator $\hat{D}$. Again, for symmetry reasons, we get

$$\operatorname{index}_a \hat{D}_2 = 2 \int\limits_{M_{t_0}} \omega_{as}\left(\hat{D}\right) \tag{28}$$

Comparing expressions (27) and (28), we arrive at the relation

$$\int\limits_{M_{t_0}} T\left(\hat{Q}_2'\right)\, dv - \int\limits_{M_{t_0}} T\left(\hat{Q}_2\right)\, dv = \int\limits_{M_{t_0}} \omega_{as}\left(\hat{D}\right) \tag{29}$$

for any value of t_0. Later on, differentiation of formula (27) with respect to t_0 shows that the expressions

$$\int\limits_{t=t_0} T\left(\hat{Q}_2'\right)\, d\tilde{v} \text{ and } \int\limits_{t=t_0} T\left(\hat{Q}_2\right)\, d\tilde{v}$$

($\tilde{v}$ is a volume element on Ω defined by the metric $\tilde{g}$) are equal in the domain of variation of the function f, so that the expression on the right in (24) does not depend on the choice of this function. So, formula (19) can be rewritten in the form[4]

$$\operatorname{index}_a \hat{D} = \operatorname{index}_a \hat{D}_0 - \text{reg-trace}\left[\hat{D}^{-1}(p)\right] + \int\limits_M \omega_{as}\left(\hat{D}\right) \tag{30}$$

Now let us focus our attention on the case $\gamma_0 \neq 0$. In this case we can perform all the above considerations using the operator

$$\hat{D}_{\gamma_0} = \psi_{\gamma_0}^{-1}\hat{D}\psi_{\gamma_0} \tag{31}$$

with some function $\psi_{\gamma_0} \in C^\infty(M\backslash\alpha)$ equal to r^{γ_0} near the singular point. For such an operator we obtain formula (29) with regularizer $\hat{R}_2$ replaced by $\psi_{\gamma_0}^{-1}\hat{R}_2\psi_{\gamma_0}$. However, the conjugation by the function ψ_{γ_0} does not affect neither traces of operators on the left in (29) nor the main symbol of the operator $\hat{D}$ involved in the right-hand part of this formula. So, in this case formula (29) remains valid also for initial operators $\hat{Q}_2$, $\hat{Q}'_2$, and $\hat{D}$. This proves (30) in the general case.

Now let us compute the regularized trace on the right in (30) provided that Condition 2 is valid. First of all, without loss of generality we can assume that the number γ_0 in Condition 2 equals zero (in the opposite case we pass to the operator $\hat{D}_{\gamma_0}$ given by (31). We have to consider the following two cases.

Case 1. No pole of the family $\hat{D}^{-1}(p)$ lies on the real axis.
Case 2. There exist real poles of the family $\hat{D}^{-1}(p)$.

In Case 1 formula (30) is valid (recall that this formula is obtained under the requirement that the family $\hat{D}^{-1}(p)$ is regular on the real axis). First, in this case we have, due to Propositions 1 and 2

$$\operatorname{index}_a \hat{D}_0 = -\operatorname{sgn}\gamma \sum_j m_j,$$

where the sum is taken over all poles of the family $\hat{D}^{-1}(p)$ lying in the strip $0 < \operatorname{Im} p < 0$ for $\gamma > 0$ and $\gamma < \operatorname{Im} p < 0$ for $\gamma < 0$ (for $\gamma = 0$ this expression vanishes). Second, we have

$$\text{reg-trace}[\hat{D}^{-1}(p)] = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \operatorname{trace}\operatorname{smbl}\left[(\hat{R}_1 - \hat{R}_2)\frac{d\hat{D}}{dp}\right](p)\,dp.$$

[4] The last integral on the right in the relation below is understood as a limit

$$\int_M \omega_{as}\left(\hat{D}\right) = \lim_{t_0\to 0}\int_{M_{t_0}} \omega_{as}\left(\hat{D}\right);$$

this limit exists due to the formula (29).

Applying the substitution $p \mapsto -p$, we obtain

$$\text{reg-trace}[\hat{D}^{-1}(p)] = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \operatorname{trace} \operatorname{smbl} \left[(\hat{R}_1 - \hat{R}_2) \frac{d\hat{D}}{dp} \right] (-p)\, dp.$$

On the other hand, due to Condition 2 the following relations hold:

$$\hat{R}_j \left(-\frac{d}{dt} \right) = \sigma_1^{-1} \hat{R}_j \left(\frac{d}{dt} \right) \sigma_2, \quad j = 1, 2,$$

and

$$\frac{d\hat{D}}{dp} \left(-\frac{d}{dt} \right) = \sigma_2^{-1} \frac{d\hat{D}}{dp} \left(\frac{d}{dt} \right) \sigma_1.$$

Hence, we have

$$\begin{aligned} \text{reg-trace} \hat{D}^{-1}(p) &= -\frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \operatorname{trace} \operatorname{smbl} \left[\sigma_1^{-1} (\hat{R}_1 - \hat{R}_2) \sigma_2 \sigma_2^{-1} \frac{d\hat{D}}{dp} \sigma_1 \right] (p)\, dp \\ &= -\frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \operatorname{trace} \sigma_1^{-1} \operatorname{smbl} \left[(\hat{R}_1 - \hat{R}_2) \frac{d\hat{D}}{dp} \right] (p) \sigma_1\, dp \\ &= -\frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \operatorname{trace} \operatorname{smbl} \left[(\hat{R}_1 - \hat{R}_2) \frac{d\hat{D}}{dp} \right] (p)\, dp \\ &= -\text{reg-trace}\, \hat{D}^{-1}(p), \end{aligned}$$

and we have proved that the regularized trace vanishes. so, in Case 1 the index formula becomes

$$\operatorname{index}_a \hat{D} = -\operatorname{sgn}\gamma \sum_j m_j + \int\limits_M \omega_{as} \left(\hat{D} \right). \tag{32}$$

Let us pass to the consideration of Case 2. In this case we cannot use formula (30) directly but we can write down this formula for the operator $\hat{D}_\varepsilon$ given by (31) with $\gamma_0 = \varepsilon$. Suppose, for definiteness, that $\gamma > 0$. Then we use $\varepsilon > 0$ and arrive at the formula

$$\operatorname{index}_a \hat{D} = -\sum_j m_j + \frac{1}{2\pi i} \int\limits_{\operatorname{Im} p = \varepsilon} \operatorname{trace} \operatorname{smbl} \left[(\hat{R}_1 - \hat{R}_2) \frac{d\hat{D}}{dp} \right] (p)\, dp + \int\limits_M \omega_{as} \left(\hat{D} \right),$$

where, as above, the sum is taken over all poles of $D^{-1}(p)$ lying in the strip $0 <$ Imp $< \gamma$. If we pass to the limit as $\varepsilon \to +0$ in the first integral on the left, we obtain

$$\begin{aligned}\lim_{\varepsilon\to+0} \int\limits_{\operatorname{Im} p=\varepsilon} \operatorname{reg-trace}[\hat{D}^{-1}(p)] &= \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \operatorname{trace\,smbl}\left[(\hat{R}_1-\hat{R}_2)\frac{d\hat{D}}{dp}\right](p)\,dp \\ &= \frac{1}{2}\sum_j \operatorname{Res}_{p_j}\operatorname{trace}\left(\hat{D}^{-1}\frac{d\hat{D}}{dp}\right)(p) \\ &\quad+\frac{1}{2\pi i}\text{v.p.}\int\limits_{-\infty}^{\infty} \operatorname{trace\,smbl}\left[(\hat{R}_1-\hat{R}_2)\frac{d\hat{D}}{dp}\right](p)\,dp.\end{aligned}$$

The last integral on the right vanishes (the proof is quite the same as in Case 1). Taking into account that the residues at p_j of the operator family $\hat{D}^{-1}\frac{d\hat{D}}{dp}(p)$ are projectors on the kernel of the operator $\hat{D}(p_j)$, we arrive at the relation

$$\operatorname{index}_a\hat{D} = -\frac{1}{2}\sum_j{}' m_j - \sum_j{}'' m_j + \int\limits_M \omega_{as}\left(\hat{D}\right),$$

where Σ' is taken over all real poles of $D^{-1}(p)$, and Σ'' over all poles of $\hat{D}^{-1}(p)$ lying in the strip $0 < \operatorname{Im} p < \gamma$. The considerations in the case $\gamma < 0$ are quite similar.

The resulting index formula is

$$\operatorname{index}_a\hat{D} = -\operatorname{sgn}\gamma\left[\frac{1}{2}\sum_j{}' m_j + \sum_j{}'' m_j\right] + \int\limits_M \omega_{as}\left(\hat{D}\right), \tag{33}$$

where the first sum is taken over all real poles $\hat{D}^{-1}(p)$, and the second one over all poles of this family lying in the strip between the lines

$$\operatorname{Im} p = 0 \text{ and } \operatorname{Im} p = \gamma$$

(in the case $\gamma = 0$ this sum vanishes).

So, we have proved the following theorem:

Theorem 2 *Let M be a compact smooth manifold with the only conical point α. Let $\hat{D}$ be an elliptic differential operator on M (in the sense of Definition* 1 *above) acting in sections of vector bundles E_1, E_2:*

$$\hat{D}_\gamma : H^{s,\gamma}(E_1) \to H^{s-m,\gamma-m}(E_1) \tag{34}$$

satisfying Condition 2 above. Suppose that the line $\operatorname{Im} p = \gamma$ *is free from singular points of the meromorphic family* $\hat{D}^{-1}(p)$*. Then operator* (34) *is a Fredholm one and its index can be computed by formula* (33)*.*

Remark 3 The obtained result evidently coincides with the known results for the operators of first order having the form

$$\sigma\left(\frac{d}{dt} + \hat{A}\right)$$

with a self-adjoint operator $\hat{A}$ (see, e.g., [33]) in the case when the spectrum of the operator $\hat{A}$ is symmetric. Actually, in this case the η-invariant [2] vanishes and we arrive at formula (33).

Remark 4 As it follows from the above considerations, the last summand in the index formula

$$\int\limits_{M} \omega_{as}(\hat{D})$$

has left-integer values provided that the operator $\hat{D}$ satisfies Condition 2 above.

Corollary 1 The following formula compares indices of operators D_γ with different values of γ (see also [33], [40], [45], [34]):

$$\operatorname{index}_a \hat{D}_{\gamma_1} - \operatorname{index}_a \hat{D}_{\gamma_2} = \operatorname{index}_a \hat{D}_{0,\gamma_1,\gamma_2},$$

where $m(p_i)$ are multiplicities of poles to the family $\hat{D}^{-1}(p)$, and the sum is taken over all poles in the strip $\gamma_1 < \operatorname{Im} p < \gamma_2$. Here

$$\hat{D}_{0,\gamma_1,\gamma_2} \;:\; H^{s,\gamma_1,\gamma_2} \;\rightarrow\; H^{s-m,\gamma_1,\gamma_2}$$

is the operator on the model cone determined by the conormal symbol $\hat{D}_0$ of the operator $\hat{D}$ (for investigation of such operators see [50], [51].)

5 Example

As an example we now calculate the index of the Euler operator on a two-dimensional manifold with the only conical point which is topologically equivalent to a smooth two-dimensional manifold of the genus g (see Figure 3). In this case the matrix of the operator $i\,(d+\delta)$ corresponding to the decompositions

$$\begin{aligned} \beta &= f_0\,(t,\varphi) + f_2\,(t,\varphi)\,dt \wedge d\varphi, \\ \beta &= f_1^1\,(t,\varphi)\,dt + f_1^2\,(t,\varphi)\,d\varphi \end{aligned}$$

for even and odd forms, respectively, is

$$\begin{pmatrix} i\partial/\partial t & -i\partial/\partial\varphi \\ i\partial/\partial\varphi & i\partial/\partial t \end{pmatrix},$$

where φ is the angular coordinate on the circle.

The corresponding operator family on the circle is

$$\hat{D}\,(p) = \begin{pmatrix} -p & -i\partial/\partial\varphi \\ i\partial/\partial\varphi & -p \end{pmatrix},$$

and its inverse is given by

$$\hat{D}^{-1}\,(p) = \left(-\frac{d^2}{d\varphi^2} + p^2\right)^{-1} \begin{pmatrix} -p & i\partial/\partial\varphi \\ -i\partial/\partial\varphi & -p \end{pmatrix}.$$

From the latter formula, it is clear that the poles of the family $\hat{D}^{-1}\,(p)$ are the points $p = ik$, $k \in \mathbf{Z}$ of the complex p-plane. It is easy to see that the multiplicities of all poles except for $k = 0$ equal 1, and the multiplicity of the pole $k = 0$ equals 2. Applying Theorem 2 to the obtained operator, we get

$$\operatorname{index}_a^\gamma\,(d+\delta) = -\operatorname{sgn}\gamma(1+k_\gamma) + \int\limits_M \omega_{as}\left(\hat{D}\right),$$

where $\operatorname{index}_a^\gamma\,(d+\delta)$ is the index of the operator $d+\delta$ in the space $H^{s,\gamma}$, k_γ is the number of poles of the family $\hat{D}^{-1}\,(p)$ with imaginary parts between γ and 0. The last term on the right in the latter formula can easily be calculated by using the double $2M$ of the manifold M. Actually, we have

$$\int\limits_M \omega_{as}\left(\hat{D}\right) = \frac{1}{2}\int\limits_{2M} \omega_{as}\left(\hat{D}\right).$$

Using the well-known results for the operator $d+\delta$, we see that

$$\int\limits_{2M} \omega_{as}\left(\hat{D}\right) = 2 - 4g,$$

where g is the (topological) genus of the manifold M. Finally, we have

$$\operatorname{index}_a^\gamma\,(d+\delta) = -\operatorname{sgn}\gamma(1+k_\gamma) + (1-2g).$$

This completes the consideration of our example.

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