A Calculus of Boundary Value Problems in Domains with Non-Lipschitz Singular Points

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Abstract

The paper is devoted to pseudodifferential boundary value problems in domains with singular points on the boundary. The tangent cone at a singular point is allowed to degenerate. In particular, the boundary may rotate and oscillate in a neighbourhood of such a point. We show a criterion for the Fredholm property of a boundary value problem and derive estimates of solutions close to singular points.

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Contents

Introduction

In 1967, Kondrat'ev published his paper [Kon67] on elliptic boundary value problems in domains with conical points on the boundary. Although concrete examples had already been treated by other authors in the early '60s (see Gokhberg and Krupnik [GK70], Grisvard [Gri85] and references there), he was the first who studied these questions systematically. After the work of Kondrat'ev, the Fredholm property and asymptotic behaviour of solutions to boundary value problems in domains with conical points and edges were investigated by Eskin [Esk73], Leguillon and Sanches-Palencia [LSP75], Maz'ya (see [MKR97] for the complete bibliography), Melrose and Mendoza [MM83], Nazarov and Plamenevskii [NP91], Schulze [Sch91, Sch94, Sch97], Grisvard [Gri85], Schrohe and Schulze [SS94, SS95] and other authors.

In this paper we consider elliptic boundary value problems in domains with singular points on the boundary which are more intricate than the conical singularities. The tangent cone to the boundary is allowed to degenerate at singular points. In particular, the boundary may rotate and oscillate in a neighbourhood of a singular point. We derive a criterion for the Fredholm property of a boundary value problem and show estimates of the solutions near singular points.

In subsequent papers we will apply these results to boundary value problems of the elasticity theory in domains with such singular points.

Boundary value problems in domains with cusps on the boundary were earlier considered by Feigin [Fei71], Bagirov and Feigin [BF73], Maz'ya and Plamenevskii [MP77, MP78]. In these papers the neighbourhood of a singular point was mapped onto a cylinder with the help of a change of variables depending on the structure of the singularity. The papers [Fei71, BF73, MP78] contain elliptic estimates and Fredholm property for general boundary value problems in weighted Sobolev spaces. In [MP77], an asymptotic formula for solutions of the Dirichlet problem is shown. Feigin [Fei72] also investigated boundary value problems in domains with cusps concentrated along an edge on the boundary.

The study of elliptic boundary value problems in domains with logarithmic whirl points originates with the papers [Rab94, Rab95a, Rab97]. An algebra of pseudodifferential operators on a compact closed manifold with rather general singular points was constructed in [ST96]. Under an additional condition on the smoothness of symbols near singular points, the Fredholm property in Sobolev spaces with explicit asymptotics was proved for elliptic operators in the algebra and an index formula of Fedosov's type was shown. The assumption in question actually corresponds to the setting of the cone algebra as for the purposes of index theory this class is sufficient. In [ST97] it was shown how to dispense with this assumption in the case of power-like cusps. This latter case was studied by Schulze, Sternin and Shatalov in [SSS96], where functions of non-commuting operators are used to construct a pseudodifferential algebra. An operator algebra corresponding to power-like cusps of order 1 is treated in Melrose and Nistor [MN96] where the Hochschild homology of such an algebra are described. Note that the general results of Maz'ya and Plamenevskii [MP72, Pla73] still apply to derive asymptotic expansions of solutions of elliptic differential equations on a closed manifold with cusps.

The class of singularities we allow includes: 1) whirl points, in which case the tangent cone does not exist and the boundary oscillates and rotates close to the point; 2) cusps, in which case the tangent cone degenerates; and 3) cusp-like whirl points, in which case we have an intricate combination of the singularities of first two types.

The following example sheds some light on the behaviour of the domain close to a singular point in case $n = 2$, n being the dimension of the space. Consider a domain $\mathcal{D}$ in $\mathbb{R}^2$ given close to the singular point $0 \in \partial\mathcal{D}$ by

$$\mathcal{D} = \{(r,\omega) \in (0,\varepsilon) \times [0,2\pi) \colon r^{\nu} f_1(r) < \omega < r^{\nu} f_2(r) + r^{-\nu} f_3(r)\} \tag{0.0.1}$$

with some $\nu \geq 0$, where (r,ω) are polar coordinates in $\mathbb{R}^2$ with centre at the origin and f_i are real-valued functions on $(0,\varepsilon)$ satisfying

$$\begin{array}{rcl} |(r^{\nu+1}\partial/\partial r)^j f_i(r)| & \leq & c_{i,j} \ \text{ for all } j = 0,1,\ldots, \\ \lim\limits_{r\to 0}(r^{\nu+1}\partial/\partial r) f_i(r) & = & 0, \\ \inf\limits_{r\in(0,\varepsilon)} (f_2(r) - f_1(r)) & > & 0. \end{array} \tag{0.0.2}$$

As but one instance of functions $f_i(r)$ which fulfil the conditions (0.0.2) we show $f_1(r) = \sin r^{-\nu_1}$, $f_2(r) = 2 - \cos r^{-\nu_1}$ and $f_3(r) = \sin r^{-\nu_2}$, where $0 < \nu_1, \nu_2 < \nu$. If $\nu = 0$, we replace (0.0.1) with

$$\mathcal{D} = \{(r,\omega) \in (0,\varepsilon) \times [0,2\pi) \colon r^{\nu} f_1(r) < \omega < r^{\nu} f_2(r) + \log r f_3(r)\}. \tag{0.0.3}$$

The functions $f_1(r) = \sin(\log r)^{\nu_1}$, $f_2(r) = 2 - \cos(\log r)^{\nu_1}$, $f_3(r) = \sin(\log r)^{\nu_2}$ are easily verified to meet conditions (0.0.2) for $\nu = 0$, provided $0 < \nu_1, \nu_2 < 1$. If the domain $\mathcal{D}$ is given by (0.0.1) with all the $f_i(r)$, $i = 1,2,3$, constant, then the origin is a cusp on the boundary of $\mathcal{D}$. On the other hand, under conditions (0.0.2), the origin is a logarithmic whirl point for the domain $\mathcal{D}$ specified by

(0.0.3). In the general case, by (0.0.1) and (0.0.3) is defined a wide class of singular points at which the boundary is not Lipschitz.

Our approach to the study of boundary value problems in domains with singular points is based on a calculus of pseudodifferential operators with operator-valued symbols. It takes into account the behaviour of the boundary close to singular points. In the framework of this calculus we construct local inverse operators (*regularisers*) for a boundary value problem at singular points. The existence of local regularisers at singular points along with the ellipticity of the problem away from the singular points implies the Fredholm property of the problem in a familiar way. Moreover, the construction of local regularisers allows us to derive also estimates of solutions near singular points and asymptotic expansions of solutions provided that f_i behave "well" in a neighbourhood of the singularity.

Let us briefly describe the structure of the paper. In Chapter 1 we develop a calculus of pseudodifferential operators with operator-valued symbols to encompass the problems we study. In Chapter 2 we indicate how these techniques can be used to treat pseudodifferential operators on closed manifolds with singular points. In particular, we show a condition which is necessary and sufficient in order that an operator be Fredholm. In final Chapter 3 we study differential boundary value problems in domains with singular points on the boundary. We restrict our attention to domains with cusps; more complicated singularities will be considered in forthcoming papers. The coefficients of the differential operators under study are allowed to have oscillating discontinuities at the cusps, the degree of the oscillation depending on the structure of the cusps. In fact, we proceed in much the same way as in the case of manifolds without boundary, thus presenting a unified approach to treating these problems. As mentioned, our results apply to boundary value problems of the elasticity theory in domains with singular points.

The authors are greatly indebted to V. A. Kondrat'ev for many stimulating conversations during the preparation of the paper. He draws the author's attention to the fact that the techniques elaborated here may be used to study general boundary value problems for parabolic equations in a compact domain, such as the Dirichlet problem for the heat equation. The main difficulty in treating such problems is that the boundary contains points where the tangent hyperplane is orthogonal to the axis t. At these points the boundary surface is characteristic for the differential operator. In order to control the smoothness of solutions at these boundary points, one invokes weighted Sobolev spaces similar to those of Section 3.5 (cf. [Kon66]).

Chapter 1

Pseudodifferential Operators with Operator-Valued Symbols

1.1 Weight operator-valued functions

Let H, $\tilde{H}$ be complex Hilbert spaces. We always assume that they are separable. Denote by $L(H, \tilde{H})$ the space of all continuous linear operators from H to $\tilde{H}$.

Definition 1.1.1 *By $\Lambda(H, \tilde{H})$ we mean the space of all functions $\lambda(\xi)$ defined on the real axis $\mathbb{R}$ and with values in $L(H, \tilde{H})$ such that, for each $\xi \in \mathbb{R}$, there exists the inverse $\lambda^{-1}(\xi)$ and it satisfies*

$$\|\lambda(\xi + \eta)\lambda^{-1}(\xi)\|_{L(\tilde{H}, \tilde{H})} \leq c \langle \eta \rangle^{\epsilon} \quad \textit{for all} \quad \eta \in \mathbb{R}, \tag{1.1.1}$$

with some $c, \epsilon \in \mathbb{R}$ independent of ξ and η, where $\langle \eta \rangle = (1 + |\eta|^2)^{1/2}$.

The elements of $\Lambda(H, \tilde{H})$ will be referred to as the *weight operator-valued functions.* Let us show two important examples of such functions.

Example 1.1.2 Suppose M is a smooth compact closed manifold and V a smooth vector bundle over M. Given any $m \in \mathbb{R}$, set

$$\mathcal{R}_V^m(\xi) = (1 + \xi^2 + \Delta_V)^{m/2}, \quad \xi \in \mathbb{R}.$$

where $\Delta_V = \nabla_V^* \nabla_V$ is the *Laplace-Beltrami operator* on M associated with a connection ∇_V for V. It is well-known that $\mathcal{R}_V^m(\xi)$ extends to a topological isomorphism $H^s(M, V) \to H^{s-m}(M, V)$, for each $s \in \mathbb{R}$ and $\xi \in \mathbb{R}$. Pick $s \in \mathbb{R}$. We claim that

$$\lambda(\xi) = \mathcal{R}_V^s(\xi), \quad \xi \in \mathbb{R},$$

fulfills (1.1.1) with $H = H^s(M, V)$, $\tilde{H} = L^2(M, V)$ and $\epsilon = |s|$. Indeed, the operator Δ_V is selfadjoint and it has an orthonormal system of eigenfunctions

$(e_i)_{i=1,2,\dots}$ which is complete in $L^2(M,V)$. Let (μ_i) be the system of corresponding eigenvalues, each μ_i being non-negative. It is evident that

$$\lambda(\xi+\eta)\lambda^{-1}(\xi)u = \sum_{i=1}^{\infty} \lambda_i(\xi+\eta)\lambda_i^{-1}(\xi)\,(u,e_i)e_i, \tag{1.1.2}$$

where $\lambda_i(\xi) = (1+\xi^2+\mu_i)^{s/2}$ and $(\cdot,\cdot)$ is a scalar product in $L^2(M,V)$. From (1.1.2) it follows that

$$\begin{aligned} &\|\lambda(\xi+\eta)\lambda^{-1}(\xi)\|_{L(L^2(M,V),L^2(M,V))} \\ &\quad \le \sup_{i,\xi}(1+(\xi+\eta)^2+\mu_i)^{s/2}(1+\xi^2+\mu_i)^{-s/2} \\ &\quad \le 2^{|s|/2}\langle\eta\rangle^{|s|}, \end{aligned}$$

the last estimate being a consequence of a well-known elementary inequality. This is the required assertion.

□

Example 1.1.3 Let M be a compact smooth manifold with boundary ∂M. Denote by $2M$ the doubled manifold, i.e., the smooth compact closed manifold obtained by gluing two copies of M together along ∂M. Each smooth vector bundle V over M is the restriction of a smooth vector bundle $\tilde{V}$ over $2M$. If $s \in \mathbb{R}$, we write $H^s(M,V)$ for the restriction of the Sobolev space $H^s(2M,\tilde{V})$ to the interior of M. Given any $m \in \mathbb{Z}$, there is a parameter-dependent elliptic pseudodifferential operator $R^m(\xi)$ of order m and of type $V \to V$ on $2M$, such that

- $R^m(\xi)$ has a symbol in $\mathcal{S}^m_{1,0}(T^*2M;\mathbb{R})$ bearing the transmission property with respect to ∂M;
- the operator $\mathcal{R}^m_V(\xi) = r_+ R^m(\xi) e_+$ extends to a topological isomorphism $H^s(M,V) \to H^{s-m}(M,V)$ for all $s \in \mathbb{R}$ with $s > -\frac{1}{2}$, provided $|\xi|$ is large enough; its inverse is also pseudodifferential.

Here, e_+ denotes extension by zero to $2M$ and r_+ restriction to M. Such operators $\mathcal{R}^m_V(\xi)$ are known as *order-reducing* operators in the theory of boundary value problems. These types of operators have been used throughout: Boutet de Monvel [BdM71], Rempel and Schulze [RS84, 3.3], Grubb [Gru86, 2.5], and so on. For an explicit construction we refer the reader to Schrohe and Schulze [SS94, 2.3.10]. In [SS94, 5.3] it is even proven that the order-reducing operators occuring there are classical. Pick $s \in \mathbb{Z}_+$. By including an additional parameter we may actually assume that $\mathcal{R}^s_V(\xi) : H^s(M,V) \to L^2(M,V)$ is a topological isomorphism for all $\xi \in \mathbb{R}$ (cf. [SS94, 3.1.1]). Set

$$\lambda(\xi) = \mathcal{R}^s_V(\xi), \quad \xi \in \mathbb{R},$$

then $\lambda \in \Lambda(H^s(M,V), L^2(M,V))$ and

$$\|\lambda(\xi+\eta)\lambda^{-1}(\xi)\|_{L(L^2(M,V),L^2(M,V))} \le c\, \langle \eta \rangle^s$$

for all $\xi, \eta \in \mathbb{R}$, with c a constant independent of ξ and η.

□

1.2 Symbol classes

Fix

$$\begin{aligned} \lambda_1 &\in \Lambda(H_1, \tilde{H}_1), \\ \lambda_2 &\in \Lambda(H_2, \tilde{H}_2). \end{aligned}$$

Definition 1.2.1 *By $\mathcal{S}(\lambda_1,\lambda_2)$ we mean the class of C^∞ functions $a(x,\xi)$ on $\mathbb{R}\times\mathbb{R}$ with values in $L(H_1,H_2)$ such that, for each $\alpha,\beta \in \mathbb{Z}_+$, there is a constant $c_{\alpha,\beta}(a)$ with the property that*

$$\|\lambda_2(\xi)(D_x^\beta D_\xi^\alpha a(x,\xi))\lambda_1^{-1}(\xi)\|_{L(\tilde{H}_1,\tilde{H}_2)} \le c_{\alpha,\beta}(a) \quad \text{for all} \quad x,\xi \in \mathbb{R}. \qquad (1.2.1)$$

The space $\mathcal{S}(\lambda_1,\lambda_2)$ is given a Fréchet topology by the best constants $c_{\alpha,\beta}(a)$ in (1.2.1).

The elements of $\mathcal{S}(\lambda_1,\lambda_2)$ will be referred to as *operator-valued symbols.* To any symbol $a \in \mathcal{S}(\lambda_1,\lambda_2)$ we assign a pseudodifferential operator $A = \mathrm{op}(a)$ by

$$Au\,(x) = \frac{1}{2\pi}\int d\xi \int e^{i(x-x')\xi} a(x,\xi)u(x')dx', \quad x \in \mathbb{R},$$

where $u \in C^\infty_{comp}(\mathbb{R},H_1)$ and the integral is over all of $\mathbb{R}\times\mathbb{R}$.

Let us denote by $\mathcal{OP}\,\mathcal{S}(\lambda_1,\lambda_2)$ the space of all operators $A = \mathrm{op}(a)$ with symbols $a \in \mathcal{S}(\lambda_1,\lambda_2)$. For these operators we will occasionally write $a = \sigma_A$.

Definition 1.2.2 *$\mathcal{S}_d(\lambda_1,\lambda_2)$ stands for the class of C^∞ functions $a(x,x',\xi)$ on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ with values in $L(H_1,H_2)$ satisfying*

$$\|\lambda_2(\xi)(D_x^\beta D_{x'}^\gamma D_\xi^\alpha a(x,x',\xi))\lambda_1^{-1}(\xi)\|_{L(\tilde{H}_1,\tilde{H}_2)} \le c_{\alpha,\beta,\gamma}(a), \quad x,x',\xi \in \mathbb{R},$$

for each $\alpha,\beta,\gamma \in \mathbb{Z}_+$, with $c_{\alpha,\beta,\gamma}(a)$ a constant independent of x, x' and ξ.

The elements of $\mathcal{S}_d(\lambda_1,\lambda_2)$ are said to be *double* operator-valued symbols. To any double symbol $a(x,x',\xi)$ there corresponds a pseudodifferential operator $A = \mathrm{op}(a)$ by

$$Au\,(x) = \frac{1}{2\pi}\int d\xi \int e^{i(x-x')\xi} a(x,x',\xi)u(x')dx', \quad x \in \mathbb{R}, \qquad (1.2.2)$$

for $u \in C^\infty_{comp}(\mathbb{R},H_1)$. The class of such operators is denoted by $\mathcal{OP}\,\mathcal{S}_d(\lambda_1,\lambda_2)$.

Pseudodifferential operators with operator-valued symbols obeying estimates based on reductions of orders with parameters were first considered by Schulze [Sch89]. The idea to use weight operator-valued functions satisfying (1.1.1) is due to Rabinovich [Rab95a].

1.3 Oscillatory integrals

Suppose we are given a C^∞ function $a(y,\eta)$ on $\mathbb{R}\times\mathbb{R}$ with values in $L(H_1,H_2)$ satisfying, for some $\epsilon\in\mathbb{R}$, the estimates

$$\|D_y^\beta D_\eta^\alpha a(y,\eta)\|_{L(H_1,H_2)} \le c_{\alpha,\beta}(a)\ \langle\eta\rangle^\epsilon, \quad y,\eta\in\mathbb{R}, \tag{1.3.1}$$

for all $\alpha,\beta\in\mathbb{Z}_+$.

To $a(y,\eta)$ we assign the operator-valued integral

$$I(a)=\lim_{\varepsilon\to 0}\iint \chi(\varepsilon y,\varepsilon\eta)e^{iy\eta}a(y,\eta)dyd\eta, \tag{1.3.2}$$

where $\chi(y,\eta)$ is a cut-off function, i.e., $\chi\in C^\infty_{comp}(\mathbb{R}^2)$ and $\chi(y,\eta)=1$ in a neighbourhood of the origin.

Proposition 1.3.1 *Limit (1.3.2) exists in the norm of $L(H_1,H_2)$ and does not depend on the particular choice of the cut-off function χ. It fact, it is given by*

$$I(a)=\iint e^{iy\eta}\langle y\rangle^{-2N_1}\langle D_\eta\rangle^{2N_1}\left(\langle\eta\rangle^{-2N_2}\langle D_y\rangle^{2N_2}a(y,\eta)\right)dyd\eta,$$

where N_1, N_2 are non-negative integers satisfying $2N_1>1$, $2N_2>\epsilon+1$. Moreover, the value of $I(a)$ is independent of N_1 and N_2 in the above range and the norm of $I(a)$ is estimated by

$$\|I(a)\|_{L(H_1,H_2)}\le c\sum_{\substack{0\le\alpha\le 2N_1\\ 0\le\beta\le 2N_2}} c_{\alpha,\beta}(a). \tag{1.3.3}$$

Proof. In the scalar case this proposition is well-known and the proof thereof can be found in Kumano-go [Kg81] and elsewhere. The proof for operator-valued symbols is quite analogous.

□

1.4 Composition formulas for pseudodifferential operators

To be short we begin with the main result of this section.

Proposition 1.4.1

1) *If $A\in\mathcal{OP}\,\mathcal{S}(\lambda_1,\lambda_2)$ and $B\in\mathcal{OP}\,\mathcal{S}(\lambda_2,\lambda_3)$, then $BA\in\mathcal{OP}\,\mathcal{S}(\lambda_1,\lambda_3)$, the symbol of BA is*

$$\sigma_{BA}(x,\xi)=\frac{1}{2\pi}\iint e^{-iy\eta}\sigma_B(x,\xi+\eta)\sigma_A(x+y,\xi)dyd\eta \tag{1.4.1}$$

and the corresponding mapping $\mathcal{S}(\lambda_1,\lambda_2)\times\mathcal{S}(\lambda_2,\lambda_3)\to\mathcal{S}(\lambda_1,\lambda_3)$ *is continuous.*

2) If $A\in\mathcal{OP}\,\mathcal{S}_d(\lambda_1,\lambda_2)$ *is an operator with double symbol* $a(x,x',\xi)$*, then* $A\in\mathcal{OP}\,\mathcal{S}(\lambda_1,\lambda_2)$*, the symbol of* A *is*

$$\sigma_A(x,\xi)=\frac{1}{2\pi}\iint e^{-iy\eta}a(x,x+y,\xi+\eta)dyd\eta \tag{1.4.2}$$

and the corresponding mapping $\mathcal{S}_d(\lambda_1,\lambda_2)\to\mathcal{S}(\lambda_1,\lambda_2)$ *is continuous.*

Note that the double integrals in (1.4.1) and (1.4.1) have to be regarded as oscillatory integrals.

Proof. Formulas (1.4.1) and (1.4.2) for the symbols are obtained in the same way as in the scalar case (cf. [Kg81]). We are thus left with the task to verify that the integrals (1.4.1) and (1.4.2) belong to the appropriate symbol classes.

Let us check it for the symbol $\sigma_{BA}(x,\xi)$. To this end, set $a(x,\xi)=\sigma_A(x,\xi)$ and $b(x,\xi)=\sigma_B(x,\xi)$.

From Definition 1.1.1 it follows that if $\lambda\in\Lambda(H,\tilde{H})$, then

$$\begin{aligned}\|\lambda(\eta)\|_{L(H,\tilde{H})} &\le \|\lambda(\eta)\lambda^{-1}(0)\|_{L(H,H)}\|\lambda(0)\|_{L(H,\tilde{H})}\\ &\le c\,\langle\eta\rangle^{\epsilon},\quad \eta\in\mathbb{R},\end{aligned} \tag{1.4.3}$$

with some $d\in\mathbb{R}$. Analogously,

$$\|\lambda^{-1}(\eta)\|_{L(\tilde{H},H)}\le c\,\langle\eta\rangle^{\epsilon},\quad \eta\in\mathbb{R}. \tag{1.4.4}$$

Set $c(x,y,\xi,\eta)=b(x,\xi+\eta)a(x+y,\xi)$. Given any $\alpha,\beta\in\mathbb{Z}_+$, we estimate the derivative $D_y^\beta D_\eta^\alpha c(x,y,\xi,\eta)$ for fixed x and ξ. For this purpose, we invoke Definition 1.2.1 and estimates (1.4.3) and (1.4.4) obtaining

$$\begin{aligned}\|D_y^\beta D_\eta^\alpha c(x,y,\xi,\eta)\|_{L(H_1,H_3)} &\le \|\lambda_3^{-1}(\xi+\eta)\|_{L(\tilde{H}_3,H_3)}\\ &\quad \|\lambda_3(\xi+\eta)(D_\eta^\alpha b(x,\xi+\eta))\lambda_2^{-1}(\xi+\eta)\|_{L(\tilde{H}_2,\tilde{H}_3)}\\ &\quad \|\lambda_2(\xi+\eta)\lambda_2^{-1}(\xi)\|_{L(\tilde{H}_2,\tilde{H}_2)}\\ &\quad \|\lambda_2(\xi)(D_y^\beta a(x+y,\xi))\lambda_1^{-1}(\xi)\|_{L(\tilde{H}_1,\tilde{H}_2)}\\ &\quad \|\lambda_1(\xi)\|_{L(H_1,\tilde{H}_1)}\\ &\le c\,\langle\xi+\eta\rangle^{\epsilon_3}\langle\eta\rangle^{\epsilon_2}\langle\xi\rangle^{\epsilon_1},\end{aligned} \tag{1.4.5}$$

for each $y,\eta\in\mathbb{R}$. Here, ϵ_i is the number of Definition 1.1.1 corresponding to λ_i, $i=1,2,3$. Proposition 1.3.1 now shows that the oscillatory integral in (1.4.1) is well-defined.

Let us prove that $\sigma_{BA} \in \mathcal{S}(\lambda_1, \lambda_3)$. To do this, we write $\sigma_{BA}(x,\xi)$ as an oscillatory integral

$$\sigma_{BA}(x,\xi) = \frac{1}{2\pi} \iint e^{-iy\eta} \langle y \rangle^{-2N_1} \langle D_\eta \rangle^{2N_1} \left(\langle \eta \rangle^{-2N_2} \langle D_y \rangle^{2N_2} c(x,y,\xi,\eta) \right) dy d\eta, \tag{1.4.6}$$

where $2N_1 > 1$, $2N_2 > \epsilon_2 + \epsilon_3 + 1$. From (1.4.6) we deduce that $\sigma_{BA}(x,\xi)$ is a finite sum of terms

$$s(x,\xi) = \frac{1}{2\pi} \iint e^{-iy\eta} \langle y \rangle^{-2N_1} \langle \eta \rangle^{-2N} D_\eta^\alpha b(x, \xi + \eta) D_y^\beta a(x+y, \xi) dy d\eta,$$

where $N_2 \leq N \leq N_1 + N_2$ and $\alpha \leq 2(N - N_2)$, $\beta \leq 2N_2$. Writing

$$\begin{aligned} &\lambda_3(\xi) s(x,\xi) \lambda_1^{-1}(\xi) \\ &\quad = \frac{1}{2\pi} \iint e^{-iy\eta} \langle y \rangle^{-2N_1} \langle \eta \rangle^{-2N} \lambda_3(\xi) \lambda_3^{-1}(\xi+\eta) \lambda_3(\xi+\eta) D_\eta^\alpha b(x, \xi+\eta) \\ &\qquad\qquad \lambda_2^{-1}(\xi+\eta) \lambda_2(\xi+\eta) \lambda_2^{-1}(\xi) \lambda_2(\xi) D_y^\beta a(x+y,\xi) \lambda_1^{-1}(\xi) dy d\eta \end{aligned}$$

and applying inequality (1.1.1) for the weight functions λ_2 and λ_3, we arrive at the estimate

$$\begin{aligned} \sup_{x,\xi} \|\lambda_3(\xi) s(x,\xi) \lambda_1^{-1}(\xi)\|_{L(\tilde{H}_1, \tilde{H}_3)} &\leq c \iint \langle y \rangle^{-2N_1} \langle \eta \rangle^{-2N + \epsilon_2 + \epsilon_3} dy d\eta \\ \times \sup_{x,\xi} \|\lambda_3(\xi) D_\xi^\alpha b(x,\xi) \lambda_2^{-1}(\xi)\|_{L(\tilde{H}_2, \tilde{H}_3)} &\sup_{x,\xi} \|\lambda_2(\xi) D_x^\beta a(x,\xi) \lambda_1^{-1}(\xi)\|_{L(\tilde{H}_1, \tilde{H}_2)}. \end{aligned} \tag{1.4.7}$$

As $-2N_1 < -1$ and $-2N + \epsilon_2 + \epsilon_3 < -1$, the integral on the right-hand side of (1.4.7) converges. Therefore,

$$\|\lambda_3(\xi) \sigma_{BA}(x,\xi) \lambda_1^{-1}(\xi)\|_{L(\tilde{H}_1, \tilde{H}_3)} \leq c \left(\sum_{\alpha=0}^{2N_1} c_{\alpha,0}(b) \right) \left(\sum_{\beta=0}^{2N_2} c_{0,\beta}(a) \right).$$

The estimates of the derivatives of $\sigma_{BA}(x,\xi)$ are proved in a similar way. Hence it follows that $\sigma_{BA} \in \mathcal{S}(\lambda_1, \lambda_3)$ and the mapping

$$\mathcal{S}(\lambda_1, \lambda_2) \times \mathcal{S}(\lambda_2, \lambda_3) \to \mathcal{S}(\lambda_1, \lambda_3)$$

given by $(\sigma_A, \sigma_B) \mapsto \sigma_{BA}$ is continuous.

This completes the proof of the first part of the proposition. The proof of the second part is analogous.

$\square$

1.5 Pseudodifferential operators with symbols slowly varying at infinity

Pseudodifferential operators with slowly varying scalar symbols were first introduced in the paper [Gru70] by Grushin.

Definition 1.5.1 *A symbol $a(x,\xi) \in \mathcal{S}(\lambda_1,\lambda_2)$ is said to vary slowly as $x \to +\infty$ if*

$$\lim_{x\to+\infty} \sup_{\xi\in\mathbb{R}} \|\lambda_2(\xi)(D_x^\beta D_\xi^\alpha a(x,\xi))\lambda_1^{-1}(\xi)\|_{L(\tilde{H}_1,\tilde{H}_2)} = 0, \tag{1.5.1}$$

for each $\alpha \in \mathbb{Z}_+$ and each $\beta \in \mathbb{Z}_+$ with $\beta \neq 0$.

We shall say that a double symbol $a(x,x',\xi) \in \mathcal{S}_d(\lambda_1,\lambda_2)$ *varies slowly* as $x \to +\infty$ if

$$\lim_{x\to+\infty} \sup_{\substack{y\in K\\ \xi\in\mathbb{R}}} \|\lambda_2(\xi)(D_x^\beta D_y^\gamma D_\xi^\alpha a(x,x+y,\xi))\lambda_1^{-1}(\xi)\|_{L(\tilde{H}_1,\tilde{H}_2)} = 0,$$

for each compact set $K \subset \mathbb{R}$, each $\alpha \in \mathbb{Z}_+$ and each $\beta,\gamma \in \mathbb{Z}_+$ with $\beta+\gamma \neq 0$.

Let $\mathcal{S}_{\mathrm{sv}}(\lambda_1,\lambda_2)$ and $\mathcal{S}_{d,\mathrm{sv}}(\lambda_1,\lambda_2)$ stand for the classes of symbols slowly varying as $x \to +\infty$.

We also distinguish the subclass $\mathcal{S}_0(\lambda_1,\lambda_2)$ of $\mathcal{S}_{\mathrm{sv}}(\lambda_1,\lambda_2)$ consisting of those symbols $a(x,\xi)$ which obey estimate (1.5.1) for all $\alpha,\beta \in \mathbb{Z}_+$ (thus including $\beta = 0$).

Proposition 1.5.2

1) *If $A \in \mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(\lambda_1,\lambda_2)$ and $B \in \mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(\lambda_2,\lambda_3)$, then $BA \in \mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(\lambda_1,\lambda_3)$ and the symbol of BA is of the form*

$$\sigma_{BA}(x,\xi) = \sigma_B(x,\xi)\sigma_A(x,\xi) + r(x,\xi), \tag{1.5.2}$$

where $r(x,\xi) \in \mathcal{S}_0(\lambda_1,\lambda_3)$.

2) *If $A \in \mathcal{OP}\,\mathcal{S}_{d,\mathrm{sv}}(\lambda_1,\lambda_2)$ is an operator with double symbol $a(x,x',\xi)$, then $A \in \mathcal{OP}\,\mathcal{S}_{\mathrm{sv}}(\lambda_1,\lambda_2)$ and the symbol of A is of the form*

$$\sigma_A(x,\xi) = a(x,x,\xi) + r(x,\xi), \tag{1.5.3}$$

where $r(x,\xi) \in \mathcal{S}_0(\lambda_1,\lambda_3)$.

Proof. To prove (1.5.2) we make use of formula (1.4.1). Set $a = \sigma_A$ and $b = \sigma_B$. Substituting the Lagrange expansion of $a(x+y,\xi)$ at $y=0$ to (1.4.1) gives

$$\begin{aligned}\sigma_{BA}(x,\xi) &= \left(\frac{1}{2\pi}\iint e^{-iy\eta}b(x,\xi+\eta)dyd\eta\right)a(x,\xi)\\ &\quad+\int_0^1 d\vartheta\frac{1}{2\pi}\iint ye^{-iy\eta}b(x,\xi+\eta)\frac{\partial a}{\partial x}(x+\vartheta y,\xi)dyd\eta\\ &= b(x,\xi)a(x,\xi)+r(x,\xi),\end{aligned}$$

where $r(x,\xi)=\int\limits_0^1 I_\vartheta(x,\xi)d\vartheta$ and

$$I_\vartheta(x,\xi)=\frac{1}{2\pi i}\iint e^{-iy\eta}\frac{\partial b}{\partial\xi}(x,\xi+\eta)\frac{\partial a}{\partial x}(x+\vartheta y,\xi)dyd\eta.$$

Our next task is to estimate the oscillatory integral $I_\vartheta(x,\xi)$. To this end, we write it in the form

$$\begin{aligned}I_\vartheta(x,\xi)=\frac{1}{2\pi i}\iint e^{-iy\eta}&\\ \langle y\rangle^{-2N_1}\langle D_\eta\rangle^{2N_1}&\left(\langle\eta\rangle^{-2N_2}\langle D_y\rangle^{2N_2}\frac{\partial b}{\partial\xi}(x,\xi+\eta)\frac{\partial a}{\partial x}(x+\vartheta y,\xi)\right)dyd\eta,\end{aligned}$$

where $2N_1-1>1$ and $2N_2>\epsilon_2+\epsilon_3+1$. Once again, $I_\vartheta(x,\xi)$ is a finite sum of terms

$$s(x,\xi)=\frac{1}{2\pi i}\iint e^{-iy\eta}\langle y\rangle^{-2N_1}\langle\eta\rangle^{-2N}D_\xi^\alpha b(x,\xi+\eta)D_y^\beta a(x+y,\xi)dyd\eta,$$

with $N_2\le N\le N_1+N_2$ and

$$\begin{aligned}1\le\alpha\le 2N_1+1,\\ 1\le\beta\le 2N_2+1.\end{aligned}$$

Analysis similar to that in the proof of Proposition 1.4.1 shows that there is a constant c independent of x, such that

$$\begin{aligned}&\sup_\xi\|\lambda_3(\xi)I_\vartheta(x,\xi)\lambda_1^{-1}(\xi)\|_{L(\tilde H_1,\tilde H_3)}\\ &\quad\le c\sum_{\alpha=1}^{2N_1+1}\sup_\xi\|\lambda_3(\xi)D_\xi^\alpha b(x,\xi)\lambda_2^{-1}(\xi)\|_{L(\tilde H_2,\tilde H_3)}\\ &\quad\quad\times\sum_{\beta=1}^{2N_2+1}\sup_{y,\xi}\|\lambda_2(\xi)\langle y\rangle^{-1}D_x^\beta a(x+\vartheta y,\xi)\lambda_1^{-1}(\xi)\|_{L(\tilde H_1,\tilde H_2)}\end{aligned}\tag{1.5.4}$$

for all $x\in\mathbb{R}$. Since $a\in\mathcal{S}_{\mathrm{sv}}(\lambda_1,\lambda_2)$, it is a simple matter to see that the second factor in the right-hand side of (1.5.4) tends to zero, as $x\to+\infty$. Moreover, the limit is achieved uniformly with respect to $\vartheta \in [0,1]$. The same argument applies to the derivatives of $I_\vartheta(x,\xi)$ whence it follows that $r \in \mathcal{S}_0(\lambda_1,\lambda_3)$, as required.

The second part of the proposition is proved in a similar way.

□

1.6 Sobolev spaces and boundedness of pseudodifferential operators

In what follows we assume that the weight operator-valued functions under consideration are of class C^∞ on $\mathbb{R}$ and satisfy

$$\sup_\xi \|(D^\alpha \lambda(\xi))\lambda^{-1}(\xi)\|_{L(\tilde{H},\tilde{H})} \le c_\alpha \quad \text{for all} \quad \alpha \in \mathbb{Z}_+, \tag{1.6.1}$$

with c_α a constant depending on λ.

If $\lambda \in \Lambda(H,\tilde{H})$ meets these conditions, then it is easily seen that $\lambda \in \mathcal{S}(\lambda, \mathrm{Id}_{\tilde{H}})$. Here we have an identification of the function $\lambda(\xi)$ on $\mathbb{R}$ and the function $a(x,\xi) := \lambda(\xi)$ on $\mathbb{R} \times \mathbb{R}$. The converse is also true, and so our assumption on λ just amounts to saying that λ is a "constant" symbol with respect to x.

Proposition 1.6.1 *Let $\lambda \in \Lambda(H,\tilde{H})$. Then, from $\lambda \in \mathcal{S}(\lambda, \mathrm{Id}_{\tilde{H}})$ it follows that $\lambda^{-1} \in \mathcal{S}(\mathrm{Id}_{\tilde{H}}, \lambda)$.*

Proof. Indeed, using the equality

$$D\lambda^{-1} = -\lambda^{-1}(D\lambda)\lambda^{-1},$$

we obtain

$$\begin{aligned} \|\lambda(\xi)D\lambda^{-1}(\xi)\|_{L(\tilde{H},\tilde{H})} &= \|-(D\lambda(\xi))\lambda^{-1}(\xi)\|_{L(\tilde{H},\tilde{H})} \\ &\le c_1 \end{aligned}$$

for all $\xi \in \mathbb{R}$. Further,

$$\lambda D^2\lambda^{-1} = 2((D\lambda)\lambda^{-1})^2 - (D^2\lambda)\lambda^{-1}$$

whence

$$\begin{aligned} &\|\lambda(\xi)D^2\lambda^{-1}(\xi)\|_{L(\tilde{H},\tilde{H})} \\ &\quad \le 2\,\|(D\lambda(\xi))\lambda^{-1}(\xi)\|^2_{L(\tilde{H},\tilde{H})} + \|(D^2\lambda(\xi))\lambda^{-1}(\xi)\|_{L(\tilde{H},\tilde{H})} \\ &\quad \le 2\,c_1^2 + c_2. \end{aligned}$$

In the general case, for $\alpha \neq 0$, we have

$$\lambda D^\alpha \lambda^{-1} = p((D^1\lambda)\lambda^{-1}, \ldots, (D^\alpha\lambda)\lambda^{-1}),$$

p being a polynomial with integer coefficients of non-commuting operators in the space $\tilde{H}$. In fact,

$$p((D^1\lambda)\lambda^{-1}, \ldots, (D^\alpha\lambda)\lambda^{-1}) = \alpha!((D^1\lambda)\lambda^{-1})^\alpha + \ldots - (D^\alpha\lambda)\lambda^{-1}.$$

Hence the desired statement follows.

□

For a function $u \in C^\infty_{comp}(\mathbb{R}, H)$, we denote by $\hat{u}(\xi) = \int e^{-i\xi x} u(x)dx$ the Fourier transform of u.

Definition 1.6.2 *Let $\lambda \in \Lambda(H, \tilde{H})$. By $H(\lambda)$ is meant the completion of the space $C^\infty_{comp}(\mathbb{R}, H)$ with respect to the norm*

$$\|u\|_{H(\lambda)} = \left(\int \|\lambda(\xi)\hat{u}(\xi)\|^2_{\tilde{H}} d\xi \right)^{1/2}. \tag{1.6.2}$$

From the Parseval identity it follows that norm (1.6.2) coincides with the norm

$$\|u\|_{H(\lambda)} = \left(\int \|\mathrm{op}(\lambda)u\|^2_{\tilde{H}} dx \right)^{1/2}.$$

Proposition 1.6.3 *Each operator $A \in \mathcal{OP}\,\mathcal{S}(\lambda_1, \lambda_2)$ extends to a continuous linear mapping $H(\lambda_1) \to H(\lambda_2)$. Moreover,*

$$\|Au\|_{H(\lambda_2)} \leq c\,(\sum_{\alpha+\beta\leq N} c_{\alpha,\beta}(\sigma_A))\, \|u\|_{H(\lambda_1)}, \quad u \in H(\lambda_1), \tag{1.6.3}$$

the constants $c > 0$ and $N \in \mathbb{Z}_+$ being independent of A.

Proof. The boundedness of the operator $A \colon H(\lambda_1) \to H(\lambda_2)$ is equivalent to the boundedness of the operator $\tilde{A} \colon L^2(\mathbb{R}, \tilde{H}_1) \to L^2(\mathbb{R}, \tilde{H}_2)$, where $\tilde{A} = \mathrm{op}(\lambda_2)A\mathrm{op}(\lambda_1^{-1})$. By Proposition 1.6.1,

$$\begin{array}{rcl} \mathrm{op}(\lambda_1^{-1}) & \in & \mathcal{OP}\,\mathcal{S}(\mathrm{Id}_{\tilde{H}_1}, \lambda_1), \\ \mathrm{op}(\lambda_2) & \in & \mathcal{OP}\,\mathcal{S}(\lambda_2, \mathrm{Id}_{\tilde{H}_2}), \end{array}$$

therefore Proposition 1.4.1 yields $\tilde{A} \in \mathcal{OP}\,\mathcal{S}(\mathrm{Id}_{\tilde{H}_1}, \mathrm{Id}_{\tilde{H}_2})$. As the symbols in $\mathcal{S}(\mathrm{Id}_{\tilde{H}_1}, \mathrm{Id}_{\tilde{H}_2})$ obey the estimate

$$\sup_{x,\xi} \|D^\beta_x D^\alpha_\xi \tilde{a}(x,\xi)\|_{L(\tilde{H}_1, \tilde{H}_2)} \leq c_{\alpha,\beta}(\tilde{a}), \quad \text{for each} \quad \alpha, \beta \in \mathbb{Z}_+,$$

we may invoke the Calderon-Vaillancourt theorem (cf. Taylor [Tay83]) to conclude that $\tilde{A}$ extends to a bounded operator $L^2(\mathbb{R},\tilde{H}_1) \to L^2(\mathbb{R},\tilde{H}_2)$ and

$$\|\tilde{A}\|_{L(L^2(\mathbb{R},\tilde{H}_1),L^2(\mathbb{R},\tilde{H}_2))} \le \tilde{c} \sum_{\alpha+\beta\le\tilde{N}} c_{\alpha,\beta}(\sigma_{\tilde{A}}),$$

with $\tilde{c} > 0$ and $\tilde{N} \in \mathbb{Z}_+$ constants independent of $\tilde{A}$. Combining this with Proposition 1.4.1 (cf. 1), we arrive at estimate (1.6.3), as required.

□

1.7 Local invertibility of pseudodifferential operators at the point at infinity

We begin with two auxiliary propositions.

Proposition 1.7.1 *Let $\chi \in C^\infty_{comp}(\mathbb{R})$ and $\chi_R(x) = \chi(\frac{x}{R})$, $R > 0$. Then, for each operator $A \in \mathcal{OP}\,\mathcal{S}(\lambda_1,\lambda_2)$, we have*

$$\lim_{R\to\infty} \|[\chi_R, A]\|_{L(H(\lambda_1),H(\lambda_2))} = 0. \qquad (1.7.1)$$

where $[\chi_R, A] = \chi_R A - A\chi_R$.

Proof. Set $a = \sigma_A$. In just the same way as in the proof of Proposition 1.5.2 (cf. 1) we obtain that

$$\sigma_{A\chi_R}(x,\xi) = \chi_R(x)a(x,\xi) + r_R(x,\xi),$$

where

$$r_R(x,\xi) = \frac{1}{R}\int_0^1 d\vartheta \frac{1}{2\pi i}\iint e^{-iy\eta}\frac{\partial a}{\partial \xi}(x,\xi+\eta)\frac{\partial \chi}{\partial x}(\frac{x+\vartheta y}{R})dyd\eta. \qquad (1.7.2)$$

Interpreting the right-hand side of (1.7.2) as an oscillatory integral we deduce easily that

$$\lim_{R\to\infty}\sup_{x,\xi} \|\lambda_2(\xi)D_x^\beta D_\xi^\alpha r_R(x,\xi)\lambda_1^{-1}(\xi)\|_{L(\tilde{H}_1,\tilde{H}_2)} = 0.$$

As $[\chi_R, A] = -\mathrm{op}(r_R)$, equality (1.7.1) follows from Proposition 1.6.3, which completes the proof.

□

Proposition 1.7.2 *Suppose that $\chi \in C^\infty(\mathbb{R})$ satisfies $\chi(x) = 0$ for $x \le 1$ and $\chi(x) = 1$ for $x \ge 2$. Set $\chi_R(x) = \chi(\frac{x}{R})$, $R > 0$. Then, for each $A \in \mathcal{OP}\,\mathcal{S}_0(\lambda_1, \lambda_2)$, we have*

$$\begin{aligned} \lim_{R\to\infty} \|\chi_R A\|_{L(H(\lambda_1),H(\lambda_2))} &= 0, \\ \lim_{R\to\infty} \|A\chi_R\|_{L(H(\lambda_1),H(\lambda_2))} &= 0. \end{aligned} \tag{1.7.3}$$

Proof. Set $a = \sigma_A$. It is evident that

$$\sigma_{\chi_R A}(x,\xi) = \chi_R(x)a(x,\xi).$$

Since $a \in \mathcal{S}_0(\lambda_1, \lambda_2)$, we can assert that

$$\lim_{R\to\infty} \sup_{x,\xi} \|\lambda_2(\xi)(D_x^\beta D_\xi^\alpha(\chi_R(x)a(x,\xi)))\lambda_1^{-1}(\xi)\|_{L(\tilde{H}_1,\tilde{H}_2)} = 0$$

for all $\alpha, \beta \in \mathbb{Z}_+$. Combining this with Proposition 1.6.3 gives the first equality of (1.7.3).

On the other hand, since

$$\sigma_{A\chi_R}(x,\chi) = \chi_R(x)a(x,\xi) - \sigma_{[\chi_R,A]}(x,\xi),$$

the second equality in (1.7.3) follows from the first one with the help of Proposition 1.7.1.

□

The function χ of Proposition 1.7.2 gives rise to what we shall call the *cut-off function* at the point $+\infty$. By such a function we mean any $\chi \in C^\infty(\mathbb{R})$ equal to 1 in a neighbourhood of $+\infty$ and vanishing for $x < a$, where $a > 0$. Clearly, Proposition 1.7.2 still holds for arbitrary cut-off functions χ.

In the sequel χ_R stands for the function of Proposition 1.7.2. The following definition is basic in our theory.

Definition 1.7.3 *An operator $A \in L(H(\lambda_1), H(\lambda_2))$ is said to be locally invertible from the left (right) at the point $+\infty$ if there exist a number $R > 0$ and an operator $B \in L(H(\lambda_2), H(\lambda_1))$ such that $BA\chi_R = \chi_R$ ($\chi_R AB = \chi_R$), respectively.*

An operator A is called *locally invertible* at the point $+\infty$ if it is locally invertible both from the left and from the right at this point.

Theorem 1.7.4 *Suppose $A = \mathrm{op}(a)$, where $a \in \mathcal{S}_{\mathrm{sv}}(\lambda_1, \lambda_2)$. Then the operator $A : H(\lambda_1) \to H(\lambda_2)$ is locally invertible at the point $+\infty$ if and only if there exists a number $R > 0$ such that the operator-valued function $a(x,\xi)\colon H_1 \to H_2$ is invertible for all $(x,\xi) \in (R,+\infty) \times \mathbb{R}$ and*

$$\sup_{(R,+\infty)\times\mathbb{R}} \|\lambda_1(\xi)a^{-1}(x,\xi)\lambda_2^{-1}(\xi)\|_{L(\tilde{H}_2,\tilde{H}_1)} < \infty. \tag{1.7.4}$$

Proof. We first prove that an operator $A \in \mathcal{OP}\,\mathcal{S}(\lambda_1, \lambda_2)$ is locally invertible at the point $+\infty$ as an operator acting from $H(\lambda_1)$ to $H(\lambda_2)$ if and only if the operator $\tilde{A} = \mathrm{op}(\lambda_2) A\, \mathrm{op}(\lambda_1^{-1})$ is locally invertible at the point $+\infty$ as an operator acting from $L^2(\mathbb{R}, \tilde{H}_1)$ to $L^2(\mathbb{R}, \tilde{H}_2)$. We give the proof only for the local invertibility from the left at the point $+\infty$; similar arguments apply to the case of local invertibility from the right.

To this end, we assume that $A \colon H(\lambda_1) \to H(\lambda_2)$ is locally invertible from the left at the point $+\infty$. By definition, there are an $R > 0$ and an operator $B \in L(H(\lambda_2), H(\lambda_1))$ such that $BA\chi_R = \chi_R$. We can rewrite this equality as

$$BA\, \mathrm{op}(\lambda_1^{-1})\, \mathrm{op}(\lambda_1)\chi_R = \chi_R$$

or $T\, \mathrm{op}(\lambda_1)\chi_R = \chi_R$, where $T = BA\, \mathrm{op}(\lambda_1^{-1})$. Since

$$\begin{array}{rcl} T\, \mathrm{op}(\lambda_1)\chi_R\, \mathrm{op}(\lambda_1^{-1}) & = & T\chi_R - T[\chi_R, \mathrm{op}(\lambda_1)]\, \mathrm{op}(\lambda_1^{-1}), \\ \chi_R\, \mathrm{op}(\lambda_1^{-1}) & = & \mathrm{op}(\lambda_1^{-1})\chi_R - [\chi_R, \mathrm{op}(\lambda_1^{-1})], \end{array}$$

we deduce that

$$\mathrm{op}(\lambda_1)T\chi_R - \mathrm{op}(\lambda_1)T[\chi_R, \mathrm{op}(\lambda_1)]\, \mathrm{op}(\lambda_1^{-1}) = \chi_R - \mathrm{op}(\lambda_1)[\chi_R, \mathrm{op}(\lambda_1^{-1})].$$

Put

$$S_R = \mathrm{op}(\lambda_1)T[\chi_R, \mathrm{op}(\lambda_1)]\, \mathrm{op}(\lambda_1^{-1}) - \mathrm{op}(\lambda_1)[\chi_R, \mathrm{op}(\lambda_1^{-1})],$$

the operator acting in $L^2(\mathbb{R}, \tilde{H}_1)$. By Proposition 1.7.1, the norm of this operator tends to 0 when $R \to \infty$.

Let $\tilde{\chi}$ be another cut-off function at the point $+\infty$, such that $\tilde{\chi}\chi = \tilde{\chi}$. Then,

$$\mathrm{op}(\lambda_1)T\tilde{\chi}_R = (\mathrm{Id} + S_R)\tilde{\chi}_R,$$

where Id stands for the identity operator in $L^2(\mathbb{R}, \tilde{H}_1)$. Choose an $R > 0$ with the property that $\|S_R\| \le 1/2$. Then the inverse $(\mathrm{Id} + S_R)^{-1}$ exists and

$$(\mathrm{Id} + S_R)^{-1}\, \mathrm{op}(\lambda_1)T\tilde{\chi}_R = \tilde{\chi}_R.$$

Substituting $T = BA\, \mathrm{op}(\lambda_1^{-1})$ to this equality yields

$$(\mathrm{Id} + S_R)^{-1}\, \mathrm{op}(\lambda_1)B\, \mathrm{op}(\lambda_2^{-1})\tilde{A}\tilde{\chi}_R = \tilde{\chi}_R,$$

and consequently the operator $\tilde{A} = \mathrm{op}(\lambda_2)A\, \mathrm{op}(\lambda_1^{-1})$ is locally invertible from the left at the point $+\infty$.

In the same manner we can see that if the operator $\tilde{A}$ has a left locally inverse operator at the point $+\infty$, then A does so.

Let $A \in \mathcal{OP}\,\mathcal{S}_{\rm sv}(\lambda_1, \lambda_2)$. Then $\tilde{A} \in \mathcal{OP}\,\mathcal{S}_{\rm sv}(\mathrm{Id}_{\tilde{H}_1}, \mathrm{Id}_{\tilde{H}_2})$, as is easy to see. In particular, the symbol $\tilde{a}$ of $\tilde{A}$ obeys the estimates

$$\sup_{x,\xi} \|D_x^\beta D_\xi^\alpha \tilde{a}(x,\xi)\|_{L(\tilde{H}_1,\tilde{H}_2)} \le c_{\alpha,\beta}(\tilde{a}), \tag{1.7.5}$$

for $\alpha, \beta \in \mathbb{Z}_+$. Moreover, Proposition 1.5.2 shows that

$$\tilde{a}(x,\xi) = \lambda_2(\xi) a(x,\xi) \lambda_1^{-1}(x,\xi) + r(x,\xi),$$

where $r \in \mathcal{S}_0(\mathrm{Id}_{\tilde{H}_1}, \mathrm{Id}_{\tilde{H}_2})$. Hence it follows, by Proposition 1.7.2, that

$$\lim_{R\to+\infty} \|\chi_R \,\mathrm{op}(r)\|_{L(\tilde{H}_1,\tilde{H}_2)} = 0, \tag{1.7.6}$$

for each cut-off function χ at the point $+\infty$.

From (1.7.6) we conclude that the operator $\mathrm{op}(r)$ is not essential for the local invertibility of $\tilde{A}$ at the point $+\infty$. We are thus reduced to proving Theorem 1.7.4 for the operators with symbols $\lambda_2(\xi)a(x,\xi)\lambda_1^{-1}(x,\xi)$ satisfying estimates (1.7.5). These have been treated in the paper of Rabinovich [Rab95b] where a criterion of local invertibility at the point $+\infty$ is proved. The desired statement now follows immediately from Theorem 1.1 in [Rab95b].

□

The important point to note here is the nature of the local inverse operator under the condition of Theorem 1.7.4. Namely $B \in \mathcal{OP}\,\mathcal{S}(\lambda_2, \lambda_1)$, which is clear from the arguments in the proof of Theorem 1.1 in [Rab95b]

1.8 Pseudodifferential operators in classes with exponential weights

For a number $\gamma \in \mathbb{R}$, we denote by $H(\lambda;\gamma)$ the completion of $C^\infty_{comp}(\mathbb{R}, H)$ with respect to the norm

$$\|u\|_{H(\lambda;\gamma)} = \|e^{\gamma x} u\|_{H(\lambda)}.$$

An operator $A = \mathrm{op}(a)$ acting in these Sobolev spaces with exponential weights can be written in the form

$$Au\,(x) = \frac{1}{2\pi} \int_{\mathbb{R}+i\gamma} dz \int_{\mathbb{R}} e^{i(x-x')z} a(x,z) u(x') dx', \quad x \in \mathbb{R},$$

for $u \in C^\infty_{comp}(\mathbb{R}, H_1)$, where $z = \xi + i\gamma$, $\xi \in \mathbb{R}$.

We assume that $a(x, \xi + i\gamma)$ obeys estimates (1.2.1), the constants $c_{\alpha,\beta}(a)$ being allowed to depend on γ. Let us denote by $\mathcal{S}(\lambda_1, \lambda_2; \gamma)$ the class of all such symbols. As in Section 1.5, we distinguish the subclass $\mathcal{S}_{\rm sv}(\lambda_1, \lambda_2; \gamma)$ consisting of those symbols $a \in \mathcal{S}(\lambda_1, \lambda_2; \gamma)$ which vary slowly at the point $+\infty$.

The results of this chapter extend to the operators of class $\mathcal{OP}\,\mathcal{S}(\lambda_1, \lambda_2; \gamma)$ acting from $H(\lambda_1; \gamma)$ to $H(\lambda_2; \gamma)$. The proofs are actually the same.

Chapter 2

Weighted Pseudodifferential Operators

2.1 Weighted Fourier transform

Let $x = \delta(t)$ be a diffeomorphism of $\mathbb{R}_+$ onto $\mathbb{R}$, such that $\delta'(t) < 0$ for all $t \in \mathbb{R}_+$. Thus,

$$\begin{aligned} \lim_{t \to 0+} \delta(t) &= +\infty, \\ \lim_{t \to +\infty} \delta(t) &= -\infty. \end{aligned}$$

Set

$$\begin{aligned} \delta_* u\,(x) &= u(\delta^{-1}(x)), \qquad x \in \mathbb{R}; \\ \delta^* f\,(t) &= f(\delta(t)), \qquad t \in \mathbb{R}_+. \end{aligned}$$

Then,

$$\begin{aligned} \delta_* : \ C^\infty_{comp}(\mathbb{R}_+, H) &\to C^\infty_{comp}(\mathbb{R}, H), \\ \delta^* : \ C^\infty_{comp}(\mathbb{R}, H) &\to C^\infty_{comp}(\mathbb{R}_+, H) \end{aligned}$$

are inverse to each other and extend to isomorphisms of Hilbert spaces

$$\begin{aligned} \delta_* : \ L^2(\mathbb{R}_+, dm, H) &\to L^2(\mathbb{R}, H), \\ \delta^* : \ L^2(\mathbb{R}, H) &\to L^2(\mathbb{R}_+, dm, H), \end{aligned}$$

where $dm = |\delta'(t)|dt$.

For a function $u \in C^\infty_{comp}(\mathbb{R}_+, H)$, we define the weighted Fourier transform by

$$\begin{aligned} \mathbf{F}u\,(\tau) &= \int_{\mathbb{R}} e^{-i\tau x} \delta_* u(x) dx \\ &= \int_{\mathbb{R}_+} e^{-i\tau\delta(t)} u(t) dm(t), \qquad \tau \in \mathbb{R}. \end{aligned}$$

From the properties of the usual Fourier transform it follows that $\mathbf{F}$ extends to an isomorphism $L^2(\mathbb{R}_+, m, H) \to L^2(\mathbb{R}, H)$. Moreover, the inverse transform is given by the formula

$$\mathbf{F}^{-1} f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\delta(t)\tau} f(\tau) d\tau, \quad t \in \mathbb{R}_+,$$

for $f \in C^{\infty}_{comp}(\mathbb{R}, H)$.

The construction of the weighted Fourier transform goes back as far as Glushko and Savchenko [GS87] (cf. also Hirschmann [Hir90], Schulze and Tarkhanov [ST96]).

Example 2.1.1 Let $\delta(t) = -\log t$. In this case the weighted Fourier transform is actually the *Mellin transform*

$$\begin{aligned} \mathbf{F}u(\tau) &= \int_{\mathbb{R}_+} t^{-i\tau} u(t) \frac{dt}{t} \\ &= Mu(-i\tau), \quad \tau \in \mathbb{R}. \end{aligned}$$

Recall that the inversion formula for the Mellin transform reads

$$M^{-1} f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} t^{-\tau} f(\tau) d\tau, \quad t \in \mathbb{R}_+.$$

$\square$

Example 2.1.2 For a $p > 0$, consider the function

$$\delta(t) = \begin{cases} \frac{1}{pt^p}, & t \in (0,1]; \\ -t, & t \in [2, +\infty). \end{cases}$$

We extend δ to the interval $(1,2)$ so as to obtain a diffeomorphism of $\mathbb{R}_+$ onto $\mathbb{R}$ with negative derivative. Then, the corresponding weighted Fourier transform $\mathbf{F}$ can be regarded as a "correction" to the p-Borel transform.

$\square$

Example 2.1.3 Consider the function

$$\delta(t) = \begin{cases} e^{\frac{1}{t}}, & t \in (0,1]; \\ -t, & t \in [2, +\infty). \end{cases}$$

Once again, we extend δ to the interval $(1,2)$ so as to arrive at a diffeomorphism of $\mathbb{R}_+$ onto $\mathbb{R}$ with negative derivative. In this case, the weighted Fourier transform $\mathbf{F}$ is of transcendental nature.

$\square$

In what follows we are interested in the weighted Fourier transform for functions $x = \delta(t)$ defined in a small interval $(0, \varepsilon]$, $\varepsilon > 0$, and arbitrarily extended to the whole semiaxis so as to be diffeomorphisms of $\mathbb{R}_+$ onto $\mathbb{R}$ with negative derivative.

2.2 Weighted pseudodifferential operators

In this section we introduce classes of pseudodifferential operators which are based on the weighted Fourier transform $\mathbf{F}$ just in the same way as the classes $\mathcal{S}(\lambda_1, \lambda_2)$ and $\mathcal{S}_{\mathrm{sv}}(\lambda_1, \lambda_2)$ are based on the usual Fourier transform. The crucial fact is that all results concerning weighted pseudodifferential operators are obtained from the usual theory by means of the evident change of variables $x = \delta(t)$.

We begin with the observation that the weighted Fourier transform is related to the derivative

$$\mathbf{D} = \frac{1}{\delta'(t)} \frac{1}{i} \frac{\partial}{\partial t}$$

in the same manner as the usual Fourier transform to the derivative $D = \frac{1}{i} \frac{\partial}{\partial t}$ (cf. [ST96]).

Fix

$$\lambda_1 \in \Lambda(H_1, \tilde{H}_1),$$
$$\lambda_2 \in \Lambda(H_2, \tilde{H}_2).$$

Definition 2.2.1 *We denote by $\mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$ the class of C^∞ functions $a(t, \zeta)$ defined on $\mathbb{R}_+ \times (\mathbb{R} + i\gamma)$ and taking their values in $L(H_1, H_2)$ such that, for each $\alpha, \beta \in \mathbb{Z}_+$, there is a constant $c_{\alpha,\beta}(a)$ with the property that*

$$\|\lambda_2(\tau)(\mathbf{D}_t^\beta D_\tau^\alpha a(t, \tau + i\gamma))\lambda_1^{-1}(\tau)\|_{L(\tilde{H}_1, \tilde{H}_2)} \le c_{\alpha,\beta}(a) \ \ \text{for all} \ \ (t, \tau) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.2.1}$$

We emphasise that the constants $c_{\alpha,\beta}(a)$ in (2.2.1) are allowed to depend on γ.

To any symbol $a \in \mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$ we assign a weighted pseudodifferential operator $A = \mathrm{op}(a)$ by setting

$$Au\,(t) = \frac{1}{2\pi} \int_{\mathbb{R}+i\gamma} d\zeta \int_{\mathbb{R}_+} e^{i(\delta(t) - \delta(t'))\zeta} a(t, \zeta) u(t') dm(t'), \quad t \in \mathbb{R}_+, \tag{2.2.2}$$

for $u \in C^\infty_{comp}(\mathbb{R}_+, H_1)$, where $dm(t') = |\delta'(t')| dt'$.

Example 2.2.2 Consider an ordinary differential operator with operator-valued coefficients

$$Au(t) = \sum_{j=0}^m a_j(t) \mathbf{D}^j u(t), \quad t \in \mathbb{R}_+,$$

where $a_j \in C^\infty(\mathbb{R}_+, L(H_1, H_2))$ fulfill the estimates

$$\|\mathbf{D}^\beta a_j(t)\|_{L(H_1, H_2)} \le c_{j,\beta}, \quad \text{for each} \quad \beta \in \mathbb{Z}_+.$$

This operator can be written in the form (2.2.2) with

$$a(t,\zeta) = \sum_{j=0}^{m} a_j(t)\zeta^j, \quad (t,\zeta) \in \mathbb{R}_+ \times (\mathbb{R} + i\gamma),$$

as is easy to check. In order to get $a \in \mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$, it is necessary to put some further restrictions on λ_1 and λ_2.

□

The class of all pseudodifferential operators with symbols in $\mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$ is denoted by $\mathcal{OP}\,\mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$. In particular, if $\delta(t) = -\log t$, we arrive at the class of Mellin pseudodifferential operators (cf. for instance Schulze [Sch91], Rabinovich [Rab95a]).

In much the same way we can introduce the class $\mathcal{OP}\,\mathcal{S}_{w,d}(\lambda_1, \lambda_2; \gamma)$ of weighted pseudodifferential operators with double symbols.

We shall say that a symbol $a(t,\zeta) \in \mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$ *varies slowly* at the point $t = 0$ if

$$\lim_{t\to 0+} \sup_{\tau\in\mathbb{R}} \|\lambda_2(\tau)(\mathbf{D}_t^\beta D_\tau^\alpha a(t, \tau + i\gamma))\lambda_1^{-1}(\tau)\|_{L(\tilde{H}_1, \tilde{H}_2)} = 0, \tag{2.2.3}$$

for each $\alpha \in \mathbb{Z}_+$ and each $\beta \in \mathbb{Z}_+$ with $\beta \neq 0$.

The class of such symbols is denoted by $\mathcal{S}_{w,\mathrm{sv}}(\lambda_1, \lambda_2; \gamma)$. We distinguish the subclass $\mathcal{S}_{w,0}(\lambda_1, \lambda_2; \gamma)$ in $\mathcal{S}_{w,\mathrm{sv}}(\lambda_1, \lambda_2; \gamma)$ consisting of those symbols $a(t,\zeta)$ which obey estimates (2.2.3) for all $\alpha, \beta \in \mathbb{Z}_+$ (i.e., including $\beta = 0$).

In a similar way we define the class $\mathcal{S}_{w,d,\mathrm{sv}}(\lambda_1, \lambda_2; \gamma)$ of slowly varying double weighted symbols. The corresponding classes of pseudodifferential operators are denoted by

$$\begin{array}{l} \mathcal{OP}\,\mathcal{S}_{w,\mathrm{sv}}(\lambda_1, \lambda_2; \gamma), \\ \mathcal{OP}\,\mathcal{S}_{w,0}(\lambda_1, \lambda_2; \gamma), \\ \mathcal{OP}\,\mathcal{S}_{w,d,\mathrm{sv}}(\lambda_1, \lambda_2; \gamma). \end{array}$$

Example 2.2.3 Let M be a smooth compact closed manifold of dimension n. Consider a differential operator A of order m on the semicylinder $\mathbb{R}_+ \times M$ over M. In each local chart U on M, we have

$$A = \sum_{j+|\alpha|\le m} a_{j,\alpha}(t,x)\mathbf{D}^j D_x^\alpha,$$

where $a_{j,\alpha} \in C^\infty(\mathbb{R}_+ \times U)$. Suppose that the coefficients $a_{j,\alpha}$ satisfy the estimates

$$\sup_{t\in\mathbb{R}_+} |\mathbf{D}_t^\beta D_x^\gamma a_{j,\alpha}(t,x)| \le c_{\beta,\gamma}, \quad \beta \in \mathbb{Z}_+,\ \gamma \in \mathbb{Z}_+^n, \tag{2.2.4}$$

uniformly in x on compact subsets of U. Then, for any fixed $s, \gamma \in \mathbb{R}$, we have $A \in \mathcal{OP}\,\mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$, where

$$\begin{aligned} \lambda_1(\tau) &= (1 + \tau^2 + \Delta_M)^{\frac{s}{2}}, \\ \lambda_2(\tau) &= (1 + \tau^2 + \Delta_M)^{\frac{s-m}{2}}, \end{aligned}$$

Δ_M being a non-negative Laplace-Beltrami operator on M (cf. Example 1.1.2 in case $V = M \times \mathbb{C}$). If, in addition to (2.2.4), $a_{j,\alpha}$ bear

$$\lim_{t \to 0+} \mathbf{D}_t a_{j,\alpha}(t, x) = 0$$

uniformly in x on compact subsets of U, then $A \in \mathcal{OP}\,\mathcal{S}_{w,\mathrm{sv}}(\lambda_1, \lambda_2; \gamma)$.

□

The coefficients $(a_{j,\alpha})$ fulfilling (2.2.4) need not be smooth up to $t = 0$. In particular, they are allowed to behave like $e^{-p\delta(t)}(\delta(t))^{\mu} c(x)$, where $\Re p > 0$ and $\mu \in \mathbb{R}$.

By private communication we learned that T. Hirschmann (1994, *unpublished*) has independently studied a particular subclass for the case of closed manifolds M. In his setting the coefficients in the symbols are supposed to have asymptotics of conormal type, for $t \to 0$. The algebra thus obtained is an extension of the cone algebra of Schulze [Sch91] and it is closed under parametrix construction for elliptic elements.

Example 2.2.4 Let M be a compact smooth manifold with boundary ∂M and let V, $\tilde{V}$ (W, $\tilde{W}$) be smooth vector bundles over M (∂M), respectively. For an $m \in \mathbb{Z}$ and $d \in \mathbb{Z}_+$, we denote by $\mathrm{Alg}^{m,d}(V, \tilde{V}; W, \tilde{W})$ the algebra of Boutet de Monvel's operators of order m and type d between sections of the vector bundles in question (cf. Boutet de Monvel [BdM71], Rempel and Schulze [RS82], Grubb [Gru86], Schulze [Sch97]). An operator $A \in \mathrm{Alg}^{m,d}(V, \tilde{V}; W, \tilde{W})$ extends to a continuous mapping

$$A\colon \begin{array}{c} H^s(M, V) \\ \oplus \\ H^s(\partial M, W) \end{array} \to \begin{array}{c} H^{s-m}(M, \tilde{V}) \\ \oplus \\ H^{s-m}(\partial M, \tilde{W}), \end{array}$$

for every $s \in \mathbb{R}$ with $s > d - \frac{1}{2}$. For $m \in \mathbb{Z}$, pick a family of order-reducing isomorphisms

$$\mathcal{R}^m_{V,W}(\tau)\colon \begin{array}{c} H^s(M, V) \\ \oplus \\ H^s(\partial M, W) \end{array} \to \begin{array}{c} H^{s-m}(M, V) \\ \oplus \\ H^{s-m}(\partial M, W) \end{array}, \quad s > -\frac{1}{2},$$

within the algebra $\mathrm{Alg}^{m,0}(V, V; W, W)$, parametrised by $\tau \in \mathbb{R}$. In fact, we can always choose $\mathcal{R}^m_{V,W}$ of being without *potential* and *trace* conditions, i.e., of the form $\mathcal{R}^m_{V,W} = \mathcal{R}^m_V \oplus \mathcal{R}^m_W$ (cf. Examples 1.1.2 and 1.1.3). For an $s \in \mathbb{Z}_+$, set

$$\begin{array}{rcl} \lambda_1(\tau) & = & \mathcal{R}^s_{V,W}(\tau), \\ \lambda_2(\tau) & = & \mathcal{R}^{s-m}_{\tilde{V},\tilde{W}}(\tau), \end{array} \tag{2.2.5}$$

where $\tau \in \mathbb{R}$. Hence it follows that

$$\begin{array}{rcl} \lambda_1 & \in & \Lambda(H^s(M,V) \oplus H^s(\partial M, W), L^2(M,V) \oplus L^2(\partial M, W)), \\ \lambda_2 & \in & \Lambda(H^{s-m}(M,\tilde{V}) \oplus H^{s-m}(\partial M, \tilde{W}), L^2(M,\tilde{V}) \oplus L^2(\partial M, \tilde{W})). \end{array}$$

Now, given any $s \in \mathbb{Z}_+$ and $\gamma \in \mathbb{R}$, we can consider weighted pseudodifferential operators (2.2.2) with symbols $a \in \mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$ taking their values in the algebra $\mathrm{Alg}^{m,d}(V, \tilde{V}; W, \tilde{W})$, where λ_1, λ_2 are weight functions (2.2.5). In this way we obtain what is a significant ingredient of the algebra of boundary value problems on a manifold with singular points on the boundary (cf. Schrohe and Schulze [SS94, SS95]).

$\square$

2.3 Function spaces related to weighted pseudodifferential operators

In what follows we assume that $\lambda \in \Lambda(H, \tilde{H})$ is a symbol "with constant coefficients."

Definition 2.3.1 *By $H_w(\lambda)$ is meant the completion of $C^\infty_{comp}(\mathbb{R}_+, H)$ with respect to the norm*

$$\|u\|_{H_w(\lambda)} = \left(\int_{\mathbb{R}} \|\lambda(\tau) \mathbf{F} u(\tau)\|^2_{\tilde{H}} d\tau \right)^{1/2}. \tag{2.3.1}$$

From the Parseval identity it follows that norm (2.3.1) coincides with the norm

$$\|u\|_{H_w(\lambda)} = \left(\int_{\mathbb{R}_+} \|\mathrm{op}(\lambda) u\|^2_{\tilde{H}} dm(t) \right)^{1/2},$$

where $\mathrm{op}(\lambda) = \mathbf{F}^{-1} \lambda(\tau) \mathbf{F}$.

We now proceed similarly to Section 1.8. For $\gamma \in \mathbb{R}$, we denote by $H_w(\lambda; \gamma)$ the space of all distributions u on $\mathbb{R}_+$ with values in H, such that $e^{\gamma \delta(t)} u \in H_w(\lambda)$. This space is topologised under the norm

$$\|u\|_{H_w(\lambda;\gamma)} = \|e^{\gamma\delta(t)} u\|_{H_w(\lambda)}.$$

We check at once that

$$\|u\|_{H_w(\lambda;\gamma)} = \left(\int_{\mathbb{R}+i\gamma} \|\lambda(\Re \zeta) \mathbf{F} u(\zeta)\|^2_{\tilde{H}} d\zeta \right)^{1/2},$$

for each $u \in H_w(\lambda;\gamma)$.

In the sequel we use also two-parameter spaces $H_w(\lambda;\gamma,\mu)$, for $\gamma,\mu \in \mathbb{R}$. These consist of all distributions u on $\mathbb{R}_+$ with values in H, such that $e^{\gamma\delta(t)}(\delta'(t))^\mu u \in H_w(\lambda)$. We endow $H_w(\lambda;\gamma,\mu)$ with the norm

$$\begin{aligned} \|u\|_{H_w(\lambda;\gamma,\mu)} &= \|e^{\gamma\delta(t)}(\delta'(t))^\mu u\|_{H_w(\lambda)} \\ &= \|(\delta'(t))^\mu u\|_{H_w(\lambda;\gamma)}. \end{aligned}$$

In general, $H_w(\lambda;\gamma,\mu)$ do not behave properly under action of weighted pseudodifferential operators. In order to get asymptotic results it is necessary to put some further restrictions to δ (cf. Section 3.1).

2.4 Composition formulas

Let $a(s,\varsigma)$ be a C^∞ function on $\mathbb{R}_+ \times \mathbb{R}$ taking values in $L(H_1,H_2)$ and satisfying, for some $\epsilon \in \mathbb{R}$, the estimates

$$\|\mathbf{D}_s^\beta D_\varsigma^\alpha a(s,\varsigma)\|_{L(H_1,H_2)} \le c_{\alpha,\beta}(a)\ \langle\varsigma\rangle^\epsilon, \quad (s,\varsigma) \in \mathbb{R}_+ \times \mathbb{R}, \tag{2.4.1}$$

for all $\alpha,\beta \in \mathbb{Z}_+$.

To $a(s,\varsigma)$ we assign the operator-valued integral

$$I(a) = \lim_{\varepsilon\to 0} \iint_{\mathbb{R}_+\times\mathbb{R}} \chi(\varepsilon\delta(s),\varepsilon\varsigma)e^{i\delta(s)\varsigma}a(s,\varsigma)dm(s)d\varsigma,$$

where $\chi(y,\varsigma)$ is a cut-off function, i.e., $\chi \in C^\infty_{comp}(\mathbb{R}^2)$ and $\chi(y,\varsigma) = 1$ in a neighbourhood of the origin.

With the help of the change of variables $y = \delta(s)$ it is easy to see that $I(a)$ exists and is independent of the particular choice of χ. Moreover,

$$I(a) = \iint_{\mathbb{R}_+\times\mathbb{R}} e^{i\delta(s)\varsigma}\langle\delta(s)\rangle^{-2N_1}\langle D_\varsigma\rangle^{2N_1}\left(\langle\varsigma\rangle^{-2N_2}\langle\mathbf{D}_s\rangle^{2N_2}a(s,\varsigma)\right)dm(s)d\varsigma, \tag{2.4.2}$$

where N_1, N_2 are non-negative integers satisfying $2N_1 > 1$, $2N_2 > \epsilon + 1$.

To derive composition formulas for weighted pseudodifferential operators on the semiaxis, we pull back the group structure from $\mathbb{R}$ to $\mathbb{R}_+$ via the diffeomorphism δ. Namely, we set

$$t \circ s = \delta^{-1}(\delta(t)+\delta(s)), \quad t,s \in \mathbb{R}_+,$$

then $(\mathbb{R}_+,\circ)$ is a locally compact commutative group with invariant measure dm. In the case of Mellin pseudodifferential operators, we have $\delta(t) = -\log t$, $t \circ s = ts$ and $dm(t) = \frac{dt}{t}$.

Proposition 2.4.1
1) *Suppose that* $A \in \mathcal{OP}\,\mathcal{S}_w(\lambda_1, \lambda_2; 0)$ *and* $B \in \mathcal{OP}\,\mathcal{S}_w(\lambda_2, \lambda_3; 0)$. *Then,* $BA \in \mathcal{OP}\,\mathcal{S}_w(\lambda_1, \lambda_3; 0)$, *the symbol of* BA *is*

$$\sigma_{BA}(t, \tau) = \frac{1}{2\pi} \iint_{\mathbb{R}_+ \times \mathbb{R}} e^{-i\delta(s)\varsigma} \sigma_B(t, \tau + \varsigma) \sigma_A(t \circ s, \tau) dm(s) d\varsigma \tag{2.4.3}$$

and the corresponding mapping $\mathcal{S}_w(\lambda_1, \lambda_2; 0) \times \mathcal{S}_w(\lambda_2, \lambda_3; 0) \to \mathcal{S}_w(\lambda_1, \lambda_3; 0)$ *is continuous.*

2) *Suppose that* $A \in \mathcal{OP}\,\mathcal{S}_{w,d}(\lambda_1, \lambda_2; 0)$ *is an operator with double symbol* $a(t, t', \tau)$. *Then,* $A \in \mathcal{OP}\,\mathcal{S}_w(\lambda_1, \lambda_2; 0)$, *the symbol of* A *is*

$$\sigma_A(t, \tau) = \frac{1}{2\pi} \iint_{\mathbb{R}_+ \times \mathbb{R}} e^{-i\delta(s)\varsigma} a(t, t \circ s, \tau + \varsigma) dm(s) d\varsigma \tag{2.4.4}$$

and the corresponding mapping $\mathcal{S}_{w,d}(\lambda_1, \lambda_2; 0) \to \mathcal{S}_w(\lambda_1, \lambda_3; 0)$ *is continuous.*

We emphasise that the integrals in (2.4.3) and (2.4.4) are regarded as oscillatory integrals in the sense of formula (2.4.2).

Proposition 2.4.2
1) *Suppose that* $A \in \mathcal{OP}\,\mathcal{S}_{w,\mathrm{sv}}(\lambda_1, \lambda_2; 0)$ *and* $B \in \mathcal{OP}\,\mathcal{S}_{w,\mathrm{sv}}(\lambda_2, \lambda_3; 0)$. *Then,* $BA \in \mathcal{OP}\,\mathcal{S}_{w,\mathrm{sv}}(\lambda_1, \lambda_3; 0)$ *and the symbol of* BA *is of the form*

$$\sigma_{BA}(t, \tau) = \sigma_B(t, \tau) \sigma_A(t, \tau) + r(t, \tau),$$

where $r(t, \tau) \in \mathcal{S}_{w,0}(\lambda_1, \lambda_3; 0)$.

2) *Suppose that* $A \in \mathcal{OP}\,\mathcal{S}_{d,\mathrm{sv}}(\lambda_1, \lambda_2; 0)$ *is an operator with double symbol* $a(t, t', \tau)$. *Then,* $A \in \mathcal{OP}\,\mathcal{S}_{w,\mathrm{sv}}(\lambda_1, \lambda_2; 0)$ *and the symbol of* A *is of the form*

$$\sigma_A(t, \tau) = a(t, t, \tau) + r(t, \tau),$$

where $r(t, \tau) \in \mathcal{S}_{w,0}(\lambda_1, \lambda_3; 0)$.

The proofs of Propositions 2.4.1 and 2.4.2 are quite analogous to those of Propositions 1.4.1 and 1.5.2, respectively.

2.5 Boundedness

The continuity of weighted pseudodifferential operators is established by our next proposition.

Proposition 2.5.1 *Each operator* $A \in \mathcal{OP}\,\mathcal{S}_w(\lambda_1, \lambda_2; 0)$ *extends to a continuous linear mapping* $H_w(\lambda_1) \to H_w(\lambda_2)$. *Moreover,*

$$\|A\|_{L(H_w(\lambda_1), H_w(\lambda_2))} \le c \sum_{\alpha+\beta \le N} c_{\alpha,\beta}(\sigma_A),$$

where the constants $c > 0$ *and* $N \in \mathbb{Z}_+$ *are independent of* A.

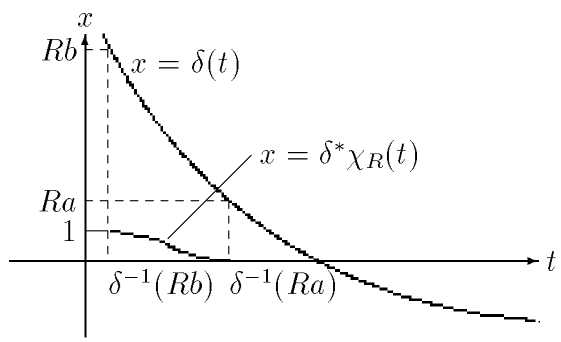

Fig. 2.1: A cut-off function at the point $t = 0$.

Proof. This follows by the same method as in the proof of Proposition 1.6.3.

□

Let χ be a cut-off function at the point $+\infty$, i.e., let $\chi \in C^\infty(\mathbb{R})$ vanish for $x < a$ and be equal to 1 for $x > b$, where $0 < a < b < \infty$. As in Section 1.7, set $\chi_R(x) = \chi(\frac{x}{R})$, for $R > 0$. This is again a cut-off function at $+\infty$. The pull-back $\delta^*\chi_R$ of χ_R under the mapping $x = \delta(t)$ is then what we shall call the *cut-off function* at the point $t = 0$ (cf. Fig. 2.1).

Proposition 2.5.2 *For each $A \in \mathcal{OP}\,\mathcal{S}_{w,0}(\lambda_1, \lambda_2; 0)$, it follows that*

$$\begin{aligned} \lim_{R\to\infty} \|\delta^*\chi_R\, A\|_{L(H_w(\lambda_1),H_w(\lambda_2))} &= 0, \\ \lim_{R\to\infty} \|A\, \delta^*\chi_R\|_{L(H_w(\lambda_1),H_w(\lambda_2))} &= 0. \end{aligned}$$

Proof. This is just a restatement of Proposition 1.7.2 in terms of weighted pseudodifferential operators.

□

Let us mention an important consequence of this proposition which states that the perturbations by operators in $\mathcal{OP}\,\mathcal{S}_{w,0}(\lambda_1, \lambda_2; 0)$ do not affect the local invertibility at the point $t = 0$, provided this is defined by invoking the cut-off functions $\delta^*\chi_R$.

Corollary 2.5.3 *Suppose $S \in \mathcal{OP}\,\mathcal{S}_{w,0}(\lambda_1, \lambda_2; 0)$. Then, an operator A in $L(H_w(\lambda_1), H_w(\lambda_2))$ is locally invertible at the point $t = 0$ if and only if so is $A + S$.*

Proof. The proof is immediate from Proposition 2.5.2.

□

Thus, the operators of $\mathcal{OP}\,\mathcal{S}_{w,0}(\lambda_1, \lambda_2; 0)$ are unessential in the problem of local invertibility at the point $t = 0$, for operators in $\mathcal{OP}\,\mathcal{S}_{w}(\lambda_1, \lambda_2; 0)$.

2.6 Local invertibility of weighted pseudodifferential operators at the singular point

An operator $A \in L(H_w(\lambda_1), H_w(\lambda_2))$ is said to be locally invertible from the left (right) at the point $t = 0$ if there are an $R > 0$ and an operator $B \in L(H_w(\lambda_2), H_w(\lambda_1))$ such that $BA\,\delta^*\chi_R = \delta^*\chi_R$ (*resp.* $\delta^*\chi_R\,AB = \delta^*\chi_R$).

An operator A is called *locally invertible* at the point $t = 0$ if it is locally invertible both from the left and from the right at this point.

We are now in a position to formulate the main result of this chapter which provides a criterion for the local invertibility of operators with symbols slowly varying at $t = 0$.

Theorem 2.6.1 *Let $A = \mathrm{op}(a)$, where $a \in \mathcal{S}_{w,\mathrm{sv}}(\lambda_1, \lambda_2; 0)$. Then the operator $A : H_w(\lambda_1) \to H_w(\lambda_2)$ is locally invertible at the point $t = 0$ if and only if there exists a number $\varepsilon > 0$ such that the operator-valued function $a(t, \tau)\colon H_1 \to H_2$ is invertible for all $(t, \tau) \in (0, \varepsilon) \times \mathbb{R}$ and*

$$\sup_{(0,\varepsilon)\times\mathbb{R}} \|\lambda_1(\tau) a^{-1}(t, \tau) \lambda_2^{-1}(\tau)\|_{L(\tilde{H}_2, \tilde{H}_1)} < \infty. \tag{2.6.1}$$

Proof. Cf. Theorem 1.7.4.

□

Under the condition of this theorem, the local inverse operator B is guaranteed to exist within the space $\mathcal{OP}\,\mathcal{S}_w(\lambda_2, \lambda_1)$ (cf. the remark after the proof of Theorem 1.7.4).

2.7 Weighted pseudodifferential operators in classes with exponential weights

If $a \in \mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$, $\gamma \in \mathbb{R}$, then the operator $A = \mathrm{op}(a)$ given by (2.2.2) extends to a continuous mapping $H_w(\lambda_1; \gamma) \to H_w(\lambda_2; \gamma)$. Theorem 2.6.1 is carried over to this setting in a slightly different form.

Theorem 2.7.1 *Let $A = \mathrm{op}(a)$, where $a \in \mathcal{S}_{w,\mathrm{sv}}(\lambda_1, \lambda_2; \gamma)$, $\gamma \in \mathbb{R}$. Then the operator $A\colon H_w(\lambda_1; \gamma) \to H_w(\lambda_2; \gamma)$ is locally invertible at the point $t = 0$ if and only if there exists an $\varepsilon > 0$ such that the operator-valued function $a(t, \tau + i\gamma)\colon H_1 \to H_2$ is invertible for all $(t, \tau) \in (0, \varepsilon) \times \mathbb{R}$ and*

$$\sup_{(0,\varepsilon)\times\mathbb{R}} \|\lambda_1(\tau) a^{-1}(t, \tau + i\gamma) \lambda_2^{-1}(\tau)\|_{L(\tilde{H}_2, \tilde{H}_1)} < \infty. \tag{2.7.1}$$

Denote by H_τ the space H equipped with the norm $u \mapsto \|\lambda(\tau)u\|_{\tilde{H}}$ depending on the parameter $\tau \in \mathbb{R}$. Then, another way of stating (2.7.1) is to say that

$$\sup_{(0,\varepsilon)\times\mathbb{R}} \|a^{-1}(t, \tau + i\gamma)\|_{L(H_{2,\tau}, H_{1,\tau})} < \infty.$$

The theorem gains in interest if we realise that, for the symbols $a(t,\zeta)$ sufficiently smooth up to $t = 0$, condition (2.7.1) can be replaced by that at $t = 0$. To do this, let us specify the meaning of being "sufficiently smooth."

A symbol $a \in \mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$ is said to be *regular* up to $t = 0$ if the limit

$$\lim_{t\to 0} a(t,\zeta) = a(0,\zeta) \tag{2.7.2}$$

exists in the sense that, for each $\alpha, \beta \in \mathbb{Z}_+$, we have

$$\lim_{t\to 0} \sup_{\tau\in\mathbb{R}} \|\lambda_2(\tau)\mathbf{D}_t^\beta D_\tau^\alpha (a(t, \tau + i\gamma) - a(0, \tau + i\gamma))\lambda_1^{-1}(\tau)\|_{L(\tilde{H}_1, \tilde{H}_2)} = 0.$$

Obviously, each symbol $a \in \mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$ regular up to $t = 0$ varies slowly at this point. On the other hand, for a symbol $a \in \mathcal{S}_{w,\mathrm{sv}}(\lambda_1, \lambda_2; \gamma)$, equality (2.7.2) reduces to

$$\lim_{t\to 0} \sup_{\tau\in\mathbb{R}} \|\lambda_2(\tau) D_\tau^\alpha (a(t, \tau + i\gamma) - a(0, \tau + i\gamma))\lambda_1^{-1}(\tau)\|_{L(\tilde{H}_1, \tilde{H}_2)} = 0$$

for all $\alpha \in \mathbb{Z}_+$.

Corollary 2.7.2 *Suppose $A \in \mathcal{S}_w(\lambda_1, \lambda_2; \gamma)$, $\gamma \in \mathbb{R}$, is an operator with a symbol a regular up to $t = 0$. Then the operator $A\colon H_w(\lambda_1; \gamma) \to H_w(\lambda_2; \gamma)$ is locally invertible at the point $t = 0$ if and only if the operator-valued function $a(0, \zeta)\colon H_1 \to H_2$ is invertible for all $\zeta \in \mathbb{R} + i\gamma$ and*

$$\sup_{\tau\in\mathbb{R}} \|\lambda_1(\tau) a^{-1}(0, \tau + i\gamma)\lambda_2^{-1}(\tau)\|_{L(\tilde{H}_2, \tilde{H}_1)} < \infty. \tag{2.7.3}$$

Proof. Indeed, the equality (2.7.2) just amounts to saying that the difference $a(t,\zeta) - a(0,\zeta)$ is of class $\mathcal{S}_{w,0}(\lambda_1, \lambda_2; \gamma)$. Applying Corollary 2.5.3 completes the proof.

$\square$

Chapter 3

Differential Operators in Domains with Cusps

3.1 Cusps

Let Ω be a domain in $\mathbb{R}^n$ star-shaped with respect to the origin and let $x = S(\omega)$ be a diffeomorphism of Ω onto an open subset of the unit sphere S^n in $\mathbb{R}^{n+1}$.

For a point $x^0 \in \mathbb{R}^{n+1}$, the set of all points $x = x^0 + rS(\omega)$ with $r \in \mathbb{R}_+$ and $\omega \in \Omega$ is an open cone with vertex at x^0. We call (r, ω) the *polar coordinates* of the point x.

As but one instance of this, we show

$$S(\omega) = \left(\omega_1, \ldots, \omega_n, \sqrt{1 - \omega_1^2 - \ldots - \omega_n^2}\right),$$

S being a diffeomorphism of the unit ball in $\mathbb{R}^n$ with center at the origin onto the upper half-sphere $\{x \in S^n : x_{n+1} > 0\}$ in $\mathbb{R}^{n+1}$.

In the sequel, $B(x^0, \varepsilon)$ stands for the ball with centre x^0 and radius $\varepsilon > 0$ in $\mathbb{R}^{n+1}$.

Consider a closed domain $\mathcal{D}$ in $\mathbb{R}^{n+1}$ given near a singular point $x^0 \in \partial\mathcal{D}$ by

$$\mathcal{D} \cap (B(x^0, \varepsilon) \setminus \{x^0\}) = \{x^0 + rS(f(r)\theta) : r \in (0, \varepsilon),\, \theta \in M\} \qquad (3.1.1)$$

with some $\varepsilon > 0$, where f is a positive function on the interval $(0, \varepsilon)$ and M a compact subdomain of Ω with smooth boundary.

To specify the function f in (3.1.1) we begin with a transparent geometric example.

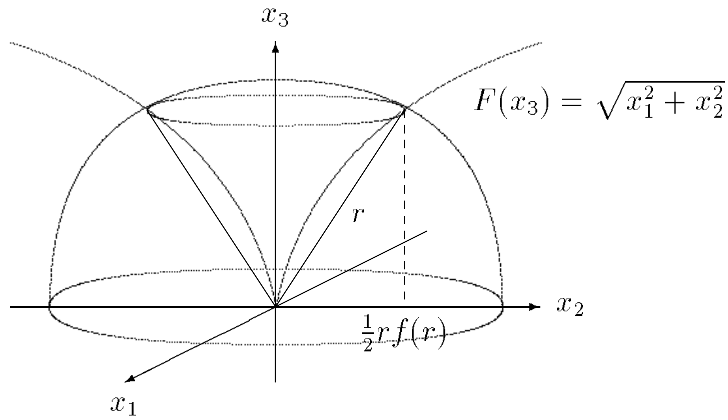

Fig. 3.1: A domain with a cusp at the origin.

Example 3.1.1 Suppose

$$\mathcal{D} = \{x \in \mathbb{R}^3 \colon x_3 \geq 0,\ F(x_3) \geq \sqrt{x_1^2 + x_2^2}\},$$

where F is a positive smooth function on the half-line $\mathbb{R}_+$ satisfying $F(0+) = 0$ (cf. Fig. 3.1). Let $f(r)$ be an implicit function of $r > 0$ defined by the equation

$$F\left(\sqrt{r^2 - \left(\tfrac{1}{2}rf\right)^2}\right) = \tfrac{1}{2}rf. \tag{3.1.2}$$

As is easy to see, $f(r)$ is uniquely defined unless F is rather intricate. Moreover, we have $|f(r)| < 2$ for all $r > 0$. Then,

$$\begin{aligned} &\mathcal{D} \cap (B(x^0, \varepsilon) \setminus \{x^0\}) \\ &\quad = \{r(f(r)\theta_1, f(r)\theta_2, \sqrt{1-(f(r)\theta_1)^2-(f(r)\theta_2)^2}) \colon r \in (0, \varepsilon), \theta_1^2 + \theta_2^2 \leq \left(\tfrac{1}{2}\right)^2\}. \end{aligned}$$

Since $F(0+) = 0$, it follows from (3.1.2) that

$$\lim_{r \to 0} \frac{f(r)}{\sqrt{4 - (f(r))^2}} = F'(0+)$$

or, equivalently,

$$\lim_{r \to 0} f(r) = \frac{2F'(0+)}{\sqrt{1 + (F'(0+))^2}} \tag{3.1.3}$$

provided the derivative $F'(0+)$ exists. If the boundary of $\mathcal{D}$ is smooth (i.e., possesses a tangential plane) at the origin, then $F'(0+) = +\infty$ and so (3.1.3) gives $f(0+) = 2$. If the origin is a conical point on the boundary (i.e., the tangential cone to $\partial\mathcal{D}$ at $x = 0$ is non-degenerate), then $F'(0+)$ is finite and different from zero. In this case $f(0+)$ is in the interval $(0, 2)$. Finally, if the boundary of $\mathcal{D}$ has a cusp at the origin (i.e., the tangential cone to the boundary at $x = 0$ degenerates), then $F'(0+) = 0$, whence $f(0+) = 0$. In particular, consider $F(t) = t^p$, where $p > 0$, then (3.1.2) becomes $(\frac{r}{2})^{2(p-1)}(4 - f^2)^p = f^2$. The boundary of $\mathcal{D}$ is smooth at the origin, if $0 < p < 1$, has a conical point at the origin, if $p = 1$, and has a power-like cusp at the origin, if $p > 1$.

□

Having disposed of this preliminary step, we can now return to the function f in (3.1.1). Unless otherwise stated, we assume that $f \in C^\infty(0, \varepsilon]$ is positive and bounded.

Set

$$\delta(t) = \int_t^\varepsilon \frac{dr}{rf(r)}, \quad \text{for} \quad t \in (0, \varepsilon], \tag{3.1.4}$$

then δ fulfills the requirements of Section 2.1.

Let us consider the 'totally characteristic' derivative

$$\begin{aligned} \mathbf{D} &= \frac{1}{\delta'(t)} D \\ &= (-tf(t))D, \end{aligned}$$

the last equality being a consequence of (3.1.4). By the above, the coefficient $-tf(t)$ is infinitesimal as $t \to 0$. The following technical result sheds light on the coefficients of $\mathbf{D}^j$, for $j \in \mathbb{Z}_+$.

Proposition 3.1.2 *For each $j = 0, 1, \ldots$, we have*

$$\mathbf{D}^j = (-tf)^j D^j + \sum_{\iota=1}^{j-1} p_{j,j-\iota}(f, tDf, \ldots, t^\iota D^\iota f)(-tf)^{j-\iota} D^{j-\iota}, \tag{3.1.5}$$

where $p_{j,j-\iota}$ is a polynomial of degree ι with (integer) complex coefficients.

Proof. Since

$$\begin{aligned} \mathbf{D}\left((-tf)^\nu u\right) &= (-tf)^\nu \mathbf{D}u - \nu(-tf)^\nu(\tfrac{1}{i}f + tDf)u, \\ \mathbf{D}\left(t^\nu D^\nu f\right) &= -\nu \tfrac{1}{i} f(t^\nu D^\nu f) - f(t^{\nu+1} D^{\nu+1} f), \end{aligned}$$

equality (3.1.5) follows by induction in j.

□

Equality (3.1.5) suggests a condition on the behaviour of f close to $t = 0$ which guarantees that the coefficients of the powers $\mathbf{D}^j$ are infinitesimal as $t \to 0$. Namely,

$$\sup_{t \in (0,\varepsilon]} |t^j D^j f(t)| \le c_j, \quad j = 0, 1, \ldots. \tag{3.1.6}$$

Let us show two typical choices of f satisfying (3.1.6).

Example 3.1.3 We set $f(r) = r^p$, where $p \geq 0$. This corresponds to a conical point, if $p = 0$, and a power-like cusp, if $p > 0$.

$\square$

Example 3.1.4 We set $f(r) = e^{-1/r}$. This corresponds to an exponential cusp.

$\square$

3.2 Differential operators

Let $A = \sum_{|\beta| \leq m} a_\beta(x) D^\beta$ be a differential operator with C^∞ coefficients in $\mathcal{D} \cap (B(x^0, \varepsilon) \setminus \{x^0\})$. We are looking for an expression for A in the "polar" coordinates $(r, \theta) \in (0, \varepsilon) \times M$, where $x = x^0 + rS(f(r)\theta)$. To this end, we make use of the following proposition.

Set

$$\left(\frac{\partial S}{\partial \omega} \right)^{-1} (\omega) = \left(\frac{\partial S^T}{\partial \omega} \frac{\partial S}{\partial \omega} \right)^{-1} \frac{\partial S^T}{\partial \omega}, \quad \omega \in \Omega,$$

where $\frac{\partial S}{\partial \omega} = \left(\frac{\partial S_\iota}{\partial \omega_j} \right)$ is the Jacobian matrix of S and the superscript 'T' means the transposed matrix. Since $\mathrm{rank}_{\mathbb{R}} \frac{\partial S}{\partial \omega} = n$ in the domain Ω, the inverse of $\frac{\partial S^T}{\partial \omega} \frac{\partial S}{\partial \omega}$ exists and is smooth in Ω. It follows that $\left(\frac{\partial S}{\partial \omega} \right)^{-1}$ is a left inverse for $\frac{\partial S}{\partial \omega}$.

Proposition 3.2.1 *For every $j = 1, \ldots, n+1$, we have*

$$\frac{\partial}{\partial x_j} = S_j(f\theta) \frac{\partial}{\partial r} + \frac{1}{rf} \sum_{\iota=1}^{n} \left(\left(\frac{\partial S}{\partial \omega} \right)^{-1}_{\iota j} (f\theta) - r \frac{\partial f}{\partial r} S_j(f\theta)\, \theta_\iota \right) \frac{\partial}{\partial \theta_\iota}, \qquad (3.2.1)$$

$\left(\frac{\partial S}{\partial \omega} \right)^{-1}_{\iota j}$ *being the (ι, j)-entry of* $\left(\frac{\partial S}{\partial \omega} \right)^{-1}$.

Proof. We have $x = x^0 + rS(f(r)\theta)$, whence

$$\frac{\partial x}{\partial (r, \theta)} = \left(\begin{array}{cc} S(f(r)\theta) + \frac{\partial f}{\partial r} \sum\limits_{\iota=1}^{n} \frac{\partial S}{\partial \omega_\iota}(f\theta)\theta_\iota & rf\, \frac{\partial S}{\partial \omega}(f\theta) \end{array} \right),$$

S being regarded as a column vector.

Moreover, since $S_1^2(\omega) + \ldots + S_{n+1}^2(\omega) = 1$ for all $\omega \in \Omega$, it follows that

$$S^T \tfrac{\partial S}{\partial \omega} \equiv 0 \quad \text{in} \quad \Omega,$$

whence

$$\det \frac{\partial S[\iota]}{\partial \omega} = (-1)^{\iota - 1} S_\iota \det \left(S \ \ \frac{\partial S}{\partial \omega} \right)$$

for each $\iota = 1, \ldots, n+1$. Here $S[\iota]$ denotes the mapping S with the ι th component omitted.

By Cramer's rule, the inverse matrix for $\frac{\partial x}{\partial(r,\theta)}$ is of the form

$$\left.\frac{\partial(r,\theta)}{\partial x}\right|_{x=x^0+rS(f\theta)} = \begin{pmatrix} S^T(f\theta) \\ \frac{1}{rf(r)}\left(\frac{\partial S}{\partial \omega}\right)^{-1}(f\theta) \end{pmatrix} - \frac{r\frac{\partial f}{\partial r}}{rf}\begin{pmatrix} 0 \\ S^T(f\theta)\theta_1 \\ \ldots \\ S^T(f\theta)\theta_n \end{pmatrix},$$

where $\left(\frac{\partial S}{\partial \omega}\right)^{-1}$ is the left inverse for $\frac{\partial S}{\partial \omega}$.

Thus, the chain rule yields

$$\begin{aligned} \frac{\partial}{\partial x_j} &= \frac{\partial r}{\partial x_j}\frac{\partial}{\partial r} + \sum_{\iota=1}^{n}\frac{\partial \theta_\iota}{\partial x_j}\frac{\partial}{\partial \theta_\iota} \\ &= S_j(f\theta)\frac{\partial}{\partial r} + \frac{1}{rf}\sum_{\iota=1}^{n}\left(\left(\frac{\partial S}{\partial \omega}\right)^{-1}_{\iota j}(f\theta) - r\frac{\partial f}{\partial r}S_j(f\theta)\,\theta_\iota\right)\frac{\partial}{\partial \theta_\iota}, \end{aligned}$$

for $j = 1, \ldots, n+1$, as required.

$\square$

Using matrix conventions, we can rewrite (3.2.1) as

$$D_x = \frac{1}{-rf}\left(S(f\theta)\mathbf{D}_r - \left(\left(\frac{\partial S}{\partial \omega}\right)^{-1}(f\theta)\right)^T D_\theta + r\frac{\partial f}{\partial r}\,S(f\theta)\,\theta D_\theta\right),$$

where $\theta D_\theta = \sum_{\iota=1}^{n}\theta_\iota \frac{1}{i}\frac{\partial}{\partial \theta_\iota}$. Since

$$\begin{aligned} \mathbf{D}\frac{1}{-rf} &= \frac{1}{-rf}\left(\frac{1}{i}f + rDf\right), \\ \mathbf{D}\,S(f\theta) &= -\left(\sum_{\iota=1}^{n}\frac{\partial S}{\partial \omega_\iota}(f\theta)\,f\theta_i\right)rDf, \\ D_\theta\,S(f\theta) &= fD_\omega S\,(f\theta), \end{aligned}$$

we deduce that, under the change of variables $\pi(r,\theta) = x^0 + rS(f(r)\theta)$, the differential operator A transforms into an operator

$$\pi^\sharp A = (\delta'(r))^m \sum_{j+|\alpha|\le m}\left(\sum_{j+|\alpha|\le|\beta|\le m}(\delta'(r))^{|\beta|-m}\; p^{(\beta)}_{j,\alpha}\;\pi^* a_\beta(r,\theta)\right)\mathbf{D}^j D^\alpha_\theta \tag{3.2.2}$$

on the cylinder $(0,\varepsilon)\times M$ over M, where $p^{(\beta)}_{j,\alpha}$ are polynomials with integer coefficients of $r^\iota f^{(\iota)}$ $(\iota = 0, 1, \ldots, |\beta| - j)$, θ and elements of the matrices $D^I_\omega S$ and $D^I_\omega\left(\frac{\partial S}{\partial \omega}\right)^{-1}$, $|I| \le |\beta| - j - |\alpha|$, with $\omega = f\theta$. (The operator $\pi^\sharp A$ is called the *pull-back* of A under π.) The polynomials $p^{(\beta)}_{j,\alpha}$ can be computed from (3.2.1).

Note that the pull-backs $\pi^* a_\beta(r,\theta)$ behave "well" on the cylinder $[0,\varepsilon) \times \Omega$ if so do the coefficients of A near the point x^0.

Set

$$a_{j,\alpha}(r,\theta) = \sum_{j+|\alpha| \le |\beta| \le m} (\delta'(r))^{|\beta|-m} \, p^{(\beta)}_{j,\alpha} \, \pi^* a_\beta(r,\theta),$$

for $(r,\theta) \in (0,\varepsilon) \times M$. We require $a_{j,\alpha}$ to fulfill the estimates

$$\sup_{r \in (0,\varepsilon)} |\mathbf{D}_r^\beta D_\theta^\gamma a_{j,\alpha}(r,\theta)| \le c_{\beta,\gamma}, \quad \beta \in \mathbb{Z}_+, \ \gamma \in \mathbb{Z}_+^n, \tag{3.2.3}$$

uniformly in $\theta \in M$ (cf. (2.2.4)). Proposition 3.1.2 shows that, under condition (3.1.6), $a_{j,\alpha}$ satisfy (3.2.3) if so do $\pi^* a_\beta(r,\theta)$, i.e.,

$$|D^B a_\beta(x)| \le c_B\,(a_\beta)\left(-\delta'(|x-x^0|)\right)^{|B|} \tag{3.2.4}$$

for all multi-indices $B \in \mathbb{Z}_+^{n+1}$.

Estimates (3.2.3) just amount to saying that $(\delta')^{-m}\pi^\sharp A$ is a weighted differential operator in the sense of Section 2.2, with δ given by (3.1.4). Moreover, $(\delta')^{-m}\pi^\sharp A$ is a differential operators with a symbol slowly varying at the point $r=0$ if, in addition to (3.2.3), the coefficients $a_{j,\alpha}$ bear

$$\lim_{r\to 0} \mathbf{D}_r a_{j,\alpha}(r,\theta) = 0 \tag{3.2.5}$$

uniformly in $\theta \in M$.

As but one instance of a function $a(r,\theta)$ satisfying (3.2.3) and (3.2.5) we show $e^{i(\delta(r))^\mu} c(\theta)$, where $c \in C^\infty(M)$ and $\mu \in (0,1)$.

Proposition 3.2.2 *Suppose that*

$$\lim_{r\to 0+} r^j D^j f(r) = 0, \quad \textit{for each} \quad j = 1,2,\ldots. \tag{3.2.6}$$

Let (3.2.4) hold. Then, $a_{j,\alpha}$ satisfy (3.2.5) if so do $\pi^* a_\beta(r,\theta)$, i.e.,

$$\lim_{x\to x^0} D_{x_j} a_\beta(x)/\delta'(|x-x^0|) = 0 \tag{3.2.7}$$

for every $j = 1,\ldots,n+1$.

Proof. Indeed, condition (3.2.7) implies

$$\lim_{r\to 0} \mathbf{D}_r\, \pi^* a_\beta(r,\theta) = 0$$

uniformly in $\theta \in M$. Moreover,

$$\mathbf{D}_r\ (\delta'(r))^\mu = -i\,\mu\ (\delta'(r))^\mu\,(f + rf')$$

vanishes as $r \to 0$, for each $\mu \le 0$. It remains to evaluate the derivative of $p^{(\beta)}_{j,\alpha}$ when $r \to 0$. To this end, set $\xi_\iota = r^\iota f^{(\iota)}$, $\iota = 0, 1, \ldots, |\beta| - j$, and let η_ν, $\nu = 1, \ldots, N$, be an indexing of the elements of both matrices $D^I_\omega S$ and $D^I_\omega \left(\frac{\partial S}{\partial \omega}\right)^{-1}$, $|I| \le |\beta| - j - |\alpha|$, where $\omega = f\theta$. By the chain rule, we get

$$\mathbf{D}_r\, p^{(\beta)}_{j,\alpha} = \sum_{\iota=0}^{|\beta|-j} \frac{\partial p^{(\beta)}_{j,\alpha}}{\partial \xi_\iota}\, \mathbf{D}\left(r^\iota f^{(\iota)}\right) + \sum_{\nu=1}^{N} \frac{\partial p^{(\beta)}_{j,\alpha}}{\partial \eta_\nu}\, \mathbf{D}_r\, \eta_\nu(f\theta)$$

and

$$\begin{aligned} \mathbf{D}\left(r^\iota f^{(\iota)}\right) &= if\left(\iota r^\iota f^{(\iota)} + r^{\iota+1} f^{(\iota+1)}\right), \\ \mathbf{D}_r\, \eta_\nu(f\theta) &= -\left(\sum_{\iota=1}^{n} \omega_\iota \frac{\partial \eta_\nu}{\partial \omega_\iota}\right)|_{\omega=f\theta}\, rDf, \end{aligned}$$

whence

$$\lim_{r\to 0} \mathbf{D}_r\, p^{(\beta)}_{j,\alpha} = 0$$

uniformly in $\theta \in M$. This completes the proof.

□

The choice of f meeting (3.2.6) seems to be the best adapted to our theory. Note that this condition is stronger than (3.1.6).

Let us mention yet another advantage of using functions f satisfying (3.2.6). To this end, we denote by $q^{(\beta)}_{j,\alpha}$ the polynomial obtained from $p^{(\beta)}_{j,\alpha}$ via replacing $rf', \ldots, r^{|\beta|-j} f^{(|\beta|-j)}$ by zeroes. It is easy to see that $q^{(\beta)}_{j,\alpha}$ depends on r and $\omega = f\theta$ rather than on θ directly. Set

$$a^{(0)}_{j,\alpha}(r,\theta) = \sum_{|\beta|=m} q^{(\beta)}_{j,\alpha}\, \pi^* a_\beta(r,\theta),$$

and write

$$\pi^\sharp A = (\delta')^m \sum_{j+|\alpha|\le m} a^{(0)}_{j,\alpha}(r,\theta)\mathbf{D}^j D^\alpha_\theta + (\delta')^m S. \tag{3.2.8}$$

Proposition 3.2.3 *Under condition (3.2.6), if moreover a_β fulfill (3.2.4), then the coefficients of the differential operator S in (3.2.8) are infinitesimal as $r \to 0$.*

Proof. Indeed, we have

$$S = \sum_{j+|\alpha|\le m} \delta_{j,\alpha} \mathbf{D}^j D^\alpha_\theta,$$

with

$$\begin{aligned} \delta_{j,\alpha} &= a_{j,\alpha}(r,\theta) - a^{(0)}_{j,\alpha}(r,\theta) \\ &= \sum_{|\beta|=m} \left(p^{(\beta)}_{j,\alpha} - q^{(\beta)}_{j,\alpha}\right) \pi^* a_\beta(r,\theta) + \sum_{j+|\alpha|\le|\beta|<m} (\delta'(r))^{|\beta|-m}\, p^{(\beta)}_{j,\alpha}\, \pi^* a_\beta(r,\theta). \end{aligned}$$

If $j = |\beta|$, then $p^{(\beta)}_{j,\alpha} - q^{(\beta)}_{j,\alpha} = 0$ by the very definition. For fixed j, α and β with $j < |\beta| = m$, set $J = |\beta| - j$. Using Taylor's expansion for the polynomial $p^{(\beta)}_{j,\alpha}$ gives

$$p^{(\beta)}_{j,\alpha} - q^{(\beta)}_{j,\alpha} = \sum_{\substack{\gamma \in \mathbb{Z}_+^{J+2+N} \setminus \{0\} \\ \gamma_0 = \gamma_{J+1} = \ldots = \gamma_{J+1+N} = 0}} \frac{1}{\gamma!} \partial^{\gamma}_{\xi,\theta,\eta} p^{(\beta)}_{j,\alpha} |_{\xi_1 = \ldots = \xi_J = 0} (rf')^{\gamma_1} \ldots (r^J f^{(J)})^{\gamma_J}.$$

Combining this with (3.2.6) and (3.2.4), we deduce that the first sum in the expression for $\delta_{j,\alpha}$ vanishes when $r \to 0$.

On the other hand, if $|\beta| < m$, then $(\delta'(r))^{|\beta| - m} \to 0$ as $r \to 0$. This shows that the second term of $\delta_{j,\alpha}$ also vanishes when $r \to 0$. Hence the desired conclusion follows.

□

We show below that the operator $(\delta')^m S$ has a small local norm in suitable function spaces and is thus unessential in the problem of local invertibility at the point $r = 0$.

The class of coefficients satisfying (3.2.4) and (3.2.7) contains some functions rapidly oscillating near the cusp (i.e., close to $r = 0$).

Example 3.2.4 For each $0 < p < 1$ and $c \in C^\infty(\Omega)$, the function

$$a(r, \omega) = e^{i(\delta(r))^p} c(\omega)$$

fulfills both (3.2.4) and (3.2.7).

□

3.3 Differential operators on manifolds with cusps

Let $\mathcal{M}$ be a (topological) submanifold of $\mathbb{R}^{n+1}$ of dimension $d + 1$, where $0 \le d < n$.

A point $x^0 \in \mathcal{M}$ is said to be a *singular point* of this manifold if $\mathcal{M}$ is given close to x^0 in the form

$$\mathcal{M} \cap (B(x^0, \varepsilon) \setminus \{x^0\}) = \{x^0 + rS(f(r)\vartheta) \colon r \in (0, \varepsilon), \vartheta \in M\}, \qquad (3.3.1)$$

with some $\varepsilon > 0$, where f is a positive function on the interval $(0, \varepsilon)$ and M is a smooth compact closed submanifold of Ω of dimension d.

We may specify the function f in (3.3.1) in the same way as in Section 3.1, thus specifying various kinds of singular points.

We will restrict our attention to local coordinate charts on M with local coordinates

$$\begin{cases} \vartheta_1 &= \vartheta_1(\theta), \\ & \dots \\ \vartheta_n &= \vartheta_n(\theta), \end{cases}$$

where rank $\left(\frac{\partial \vartheta_i}{\partial \theta_j}\right) = d$ for all $\theta = (\theta_1, \dots, \theta_d)$ varying over a domain $\Theta \subset \mathbb{R}^d$.

Under the change of variables $\pi(r, \vartheta) = x^0 + rS(f(r)\vartheta)$, a differential operator A on the "smooth part" $\mathcal{M} \cap (B(x^0, \varepsilon) \setminus \{x^0\})$ of $\mathcal{M}$ close to x^0 transforms into an operator $\pi^\sharp A$ on the cylinder $(0, \varepsilon) \times M$ over M. Moreover, analysis similar to that in Section 3.2 actually shows that $\pi^\sharp A$ is of the form (3.2.2) in local coordinates $\vartheta = \vartheta(\theta)$ on M.

We are thus led to typical differential operators on the manifold $\mathcal{M}$ close to the singular point x^0. When written in the polar coordinates with center x^0, these are differential operators on the cylinder $(0, \varepsilon) \times M$ over M of the form

$$A = \left(\frac{1}{-rf}\right)^m \sum_{j=0}^{m} a_j(r) \mathbf{D}^j,$$

where $a_j \in C^\infty((0, \varepsilon), \mathrm{Diff}^{m-j}(M))$ and $\mathbf{D} = (-rf)D_r$. Thus, in every local chart $U \overset{\cong}{\to} \Theta \subset \mathbb{R}^d$ on M, we have

$$a_j(r) = \sum_{|\alpha| \le m-j} a_{j,\alpha}(r, \theta) D_\theta^\alpha,$$

$a_{j,\alpha}$ being C^∞ functions in $(0, \varepsilon) \times \Theta$.

We shall make two standing assumptions on the coefficients $a_{j,\alpha}$ under consideration (cf. Example 2.2.3). Namely, they are required to fulfill both (3.2.3) and (3.2.5) uniformly on compact subsets of Θ. It is immaterial which covering of M by local charts we choose to define our class of operators as long as M is compact.

3.4 Canonical domains and surfaces with cusps

Fix a bounded positive function $f \in C^\infty(0, \varepsilon]$. From now on we tacitly assume that f bears estimates (3.1.6).

We extend f to a positive C^∞ function on the entire semiaxis $\mathbb{R}_+$ which is constant for $r \ge R > \varepsilon$. It will cause no confusion if we use the same letter to designate f and its extension.

If $f(r) < c_0$ for $r \in (0, \varepsilon]$, which we may assume, then there is a desired extension of f with values in $(0, c_0]$.

Suppose M be a subset of Ω, such that $c_0 M \subset \Omega$. Let

$$C_{x^0} = \{x^0 + rS(f(r)\theta) \colon r \in \mathbb{R}_+, \theta \in M\}, \tag{3.4.1}$$

the right-hand side being well-defined due to the fact that Ω is star-shaped with respect to the origin.

Note that the part of C_{x^0} lying outside the ball $B(x^0, R)$ is a conical set, i.e., if $x \in C_{x^0}$ and $|x - x^0| \geq R$, then $x^0 + \lambda(x - x^0) \in C_{x^0}$ for all $\lambda \geq 1$.

If M is a compact domain with smooth boundary in Ω, then we call C_{x^0} the *canonical domain* with a singular point on the boundary.

If M is a smooth compact closed submanifold of Ω of dimension d, with $0 \leq d < n$, then C_{x^0} is said to be a *canonical surface* with a singular point.

Remark 3.4.1 *Obviously, if C_{x^0} is a canonical domain with a singular point on the boundary, then the boundary of C_{x^0} is a canonical surface with a singular point.*

3.5 Function spaces in a canonical domain

Consider a canonical domain with a singular point on the boundary given by (3.4.1), M being a compact domain with smooth boundary in Ω.

Let

$$\delta(t) = \int_t^\varepsilon \frac{dr}{rf(r)}, \quad \text{for} \quad t \in \mathbb{R}_+$$

(cf. (3.1.4)). From the properties of the function f it follows that δ is a diffeomorphism of $\mathbb{R}_+$ onto the entire real axis.

For $s \in \mathbb{Z}_+$ and $\gamma, \mu \in \mathbb{R}$, the space $H^{s,\gamma,\mu}(C_{x^0})$ is defined to be the completion of $C^\infty_{comp}(C_{x^0})$ with respect to the norm

$$\|u\|_{H^{s,\gamma,\mu}(C_{x^0})} = \left(\int_{\mathbb{R}_+} e^{2\gamma\delta(r)} (\delta'(r))^{2\mu} \Big(\sum_{j=0}^{s} \|\mathbf{D}^j \pi^* u\|^2_{H^{s-j}(M)}\Big)\, dm(r) \right)^{\frac{1}{2}}, \tag{3.5.1}$$

where $\pi^* u$ stands for the pull-back of u under the mapping $\pi \colon \mathbb{R}_+ \times M \to C_{x^0}$ given by $\pi(r,\theta) = x^0 + rS(f(r)\theta)$. (Note that $\pi^* u \in C^\infty_{comp}(\mathbb{R}_+, C^\infty(M))$ provided $u \in C^\infty_{comp}(C_{x^0})$.)

If $f(r) \equiv c_0$, i.e., x^0 is a conical point, then $\delta(r) = \frac{1}{c_0} \log \frac{\varepsilon}{r}$, and hence $e^{\delta(r)} = (\frac{\varepsilon}{r})^{\frac{1}{c_0}}$ is a power of $\delta'(r) = -\frac{1}{c_0}\frac{1}{r}$ up to an unessential constant factor. In this case we can restrict the discussion to two-parameter spaces $H^{s,\gamma}(C_{x^0})$, as is customary in the cone theory (cf. Schulze [Sch91, Sch94, Sch97]). In the general case e^δ is by no means a multiple of a power of δ'.

Example 3.5.1 Let $f(r) = r^p$ for $r \in (0, \varepsilon]$, where $p > 0$. Then we have

$$\delta(r) = \frac{1}{p} \left(\frac{1}{r^p} - \frac{1}{\varepsilon^p} \right)$$

on $(0, \varepsilon]$. It follows that $e^{\delta(r)}$ grows exponentially as $r \to 0$, while $\delta'(r) = -r^{-p-1}$ is of power order of growth.

$\square$

For the general case, we note that only the spaces $H^{s,0,0}(C_{x^0})$ are of independent interest while $H^{s,\gamma,\mu}(C_{x^0})$ can be derived from these as spaces with a weight factor.

Proposition 3.5.2 *Suppose f fulfills estimates (3.1.6). Then,*

$$\|u\|_{H^{s,\gamma,\mu}(C_{x^0})} \sim \|\pi_*(e^{\gamma\delta}(\delta')^{\mu})\, u\|_{H^{s,0,0}(C_{x^0})} \quad \textit{for} \quad u \in C^{\infty}_{comp}(C_{x^0}),$$

where the equivalence of two norms means that their ratio is bounded both above and below by positive constants independent of u.

The estimate we obtain on the course of proof seems to be of independent interest.

Proof. Pick $u \in C^{\infty}(\mathbb{R}_+, H)$, H being a Hilbert space. An easy computation shows that

$$\mathbf{D} \left(e^{\gamma\delta}(\delta')^{\mu}\, u \right) = e^{\gamma\delta}(\delta')^{\mu}\, \mathbf{D}u + \left(\tfrac{1}{i}\gamma + \mu(\tfrac{1}{i}f + rDf) \right) e^{\gamma\delta}(\delta')^{\mu}\, u.$$

We now proceed by induction in $j = 1, 2, \ldots$, thus obtaining

$$\mathbf{D}^j(e^{\gamma\delta}(\delta')^{\mu}\, u) = e^{\gamma\delta}(\delta')^{\mu}\, \mathbf{D}^j u + \sum_{\iota=1}^{j} p_{j,j-\iota}(f, rDf, \ldots, r^{\iota} D^{\iota} f)\, e^{\gamma\delta}(\delta')^{\mu}\, \mathbf{D}^{j-\iota} u, \tag{3.5.2}$$

where $p_{j,j-\iota}$ is a polynomial of degree ι with coefficients depending on γ and μ (cf. the proof of Proposition 3.1.2). From (3.5.2) we deduce in turn that

$$e^{\gamma\delta}(\delta')^{\mu}\, \mathbf{D}^j u = \mathbf{D}^j(e^{\gamma\delta}(\delta')^{\mu}\, u) + \sum_{\iota=1}^{j} \tilde{p}_{j,j-\iota}(f, rDf, \ldots, r^{\iota} D^{\iota} f)\, \mathbf{D}^{j-\iota}(e^{\gamma\delta}(\delta')^{\mu}\, u), \tag{3.5.3}$$

$\tilde{p}_{j,j-\iota}$ being a polynomial of degree ι with coefficients depending on γ and μ.

Combining (3.5.2) with (3.5.3) and invoking estimates ((3.1.6)), we conclude that

$$c \sum_{\iota=0}^{j} \|\mathbf{D}^{\iota}(e^{\gamma\delta}(\delta')^{\mu}\, u)\|_H \le e^{\gamma\delta}(-\delta')^{\mu} \sum_{\iota=0}^{j} \|\mathbf{D}^{\iota} u\|_H \le C \sum_{\iota=0}^{j} \|\mathbf{D}^{\iota}(e^{\gamma\delta}(\delta')^{\mu}\, u)\|_H$$

for all $t \in \mathbb{R}_+$, where c and C are positive constants depending only on f but not on u. This gives the desired conclusion when substituted to (3.5.1).

$\square$

Under the pull-back mapping $u \mapsto \pi^* u$ the space $H^{s,\gamma,0}(C_{x^0})$ is topologically isomorphic to the Hilbert space $H_w(\lambda;\gamma)$ (cf. Section 2.3), where $\lambda(\tau) = \mathcal{R}^s_{M\times\mathbb{C}}(\tau)$ is the order-reducing family of Example 1.1.3. For a proof of this fact we refer the reader to Schrohe and Schulze [SS94, 3.1.9].

Combining this with Proposition 3.5.2 and with what has been said in Section 2.3, we see that

$$\begin{aligned} \|u\|_{H^{s,\gamma,\mu}(C_{x^0})} &\sim \|\pi_*(\delta')^\mu\, u\|_{H^{s,\gamma,0}(C_{x^0})} \\ &\sim \|(\delta')^\mu\, \pi^* u\|_{H_w(\mathcal{R}^s_{M\times\mathbb{C}};\gamma)} \\ &= \left(\int_{\mathbb{R}+i\gamma} \|\mathcal{R}^s_{M\times\mathbb{C}}(\Re\zeta)\mathbf{F}((\delta')^\mu\, \pi^* u)(\zeta)\|^2_{L^2(M)} d\zeta\right)^{1/2}. \end{aligned} \tag{3.5.4}$$

For an integer $s > 0$, we denote by $H^{s-\frac{1}{2},\gamma,\mu}(\partial C_{x^0})$ the function space on the boundary of C_{x^0} consisting of the restrictions to ∂C_{x^0} of elements in $H^{s,\gamma,\mu}(C_{x^0})$. (Note that the boundary of C_{x^0} does not contain x^0 and hence is smooth.) The space $H^{s-\frac{1}{2},\gamma,\mu}(\partial C_{x^0})$ is topologised under the quotient norm.

3.6 Function spaces on a canonical surface

Function spaces on a canonical surface with a singular point are introduced similarly to those on a canonical domain.

Namely, let C_{x^0} is a canonical surface with a singular point. For $s \in \mathbb{Z}_+$ and $\gamma, \mu \in \mathbb{R}$, the space $H^{s,\gamma,\mu}(C_{x^0})$ is defined to be the completion of $C^\infty_{comp}(C_{x^0})$ under the norm (3.5.1), where M is now a smooth compact closed manifold.

Proposition 3.5.2 still holds in this setting. Moreover, under the pull-back mapping $u \mapsto \pi^* u$ the space $H^{s,\gamma,0}(C_{x^0})$ is topologically isomorphic to the Hilbert space $H_w(\lambda;\gamma)$ where $\lambda(\tau) = \mathcal{R}^s_{M\times\mathbb{C}}(\tau)$ is the order-reducing family of Example 1.1.2 (cf. Schulze and Tarkhanov [ST96, 1.5]).

Since this latter space $H_w(\lambda;\gamma)$ is in fact defined for all real s, we can extend the definition of $H^{s,\gamma,0}(C_{x^0})$ to $s \in \mathbb{R}$. Then, applying Proposition 3.5.2 we arrive at the scale of function spaces $H^{s,\gamma,\mu}(C_{x^0})$ over all $s \in \mathbb{R}$. Namely, $H^{s,\gamma,\mu}(C_{x^0})$ is defined to be the completion of $C^\infty_{comp}(C_{x^0})$ with respect to the norm

$$\|u\|_{H^{s,\gamma,\mu}(C_{x^0})} = \|(\delta')^\mu\, \pi^* u\|_{H_w(\mathcal{R}^s_{M\times\mathbb{C}};\gamma)}. \tag{3.6.1}$$

For $s \in \mathbb{Z}_+$ this gives what we have already defined above, up to an equivalent norm.

Formulas (3.5.4) remain valid for all $s \in \mathbb{R}$ in case C_{x^0} is a canonical surface with a singular point.

Moreover, if C_{x^0} is a canonical domain with a singular point on the boundary, then the definitions of $H^{s-\frac{1}{2},\gamma,\mu}(\partial C_{x^0})$, for $s \in \mathbb{Z}_+$, at the end of Section 3.5 and by (3.6.1) result in the same space up to an equivalent norm.

3.7 Local invertibility of a differential operator at a cusp

We are now prepared to apply Theorem 2.6.1 to the problem of local invertibility of differential operators on a closed manifold $\mathcal{M}$ with singular points.

Suppose $\mathcal{M}$ is written close to a singular point $x^0 \in \mathcal{M}$ in the form (3.3.1), with M a smooth compact closed manifold. We assume that the function f fulfills estimates (3.1.6).

Let A be a differential operator of order m on $\mathcal{M} \cap (B(x^0,\varepsilon) \setminus \{x^0\})$, where $\varepsilon > 0$. As described in Section 3.3, after the change of variables

$$\pi(x,\vartheta) = x^0 + rS(f(r)\vartheta)$$

the operator A takes the form

$$\pi^\sharp A = (\delta')^m \sum_{j=0}^{m} a_j(r)\mathbf{D}^j, \tag{3.7.1}$$

where $a_j \in C^\infty((0,\varepsilon), \mathrm{Diff}^{m-j}(M))$. The "coefficients" $a_{j,\alpha}(r,\theta)$, $|\alpha| \le m-j$, of these differential operators are required to satisfy estimates (3.2.3) and (3.2.5) uniformly on small balls in M.

We have $\pi^*(Au) = (\pi^\sharp A)\,\pi^* u$ or, equivalently,

$$Au = \pi_*(\pi^\sharp A)\,\pi^* u \tag{3.7.2}$$

on functions u defined in a punctured neighbourhood of x^0 on $\mathcal{M}$.

Proposition 3.7.1 *For each $s, \gamma, \mu \in \mathbb{R}$ and each $\varphi, \psi \in C^\infty_{comp}[0,\varepsilon)$, there is a continuous extension*

$$(\pi_*\varphi)\,A\,(\pi_*\psi)\colon H^{s,\gamma,\mu}(C_{x^0}) \to H^{s-m,\gamma,\mu-m}(C_{x^0}).$$

Proof. We first observe that

$$(\pi_*\varphi)\,A\,(\pi_*\psi) = \pi_*\,\varphi(\pi^\sharp A)\psi\,\pi^*,$$

which is due to (3.7.2).

Pick $u \in H^{s,\gamma,\nu}(C_{x^0})$. Proposition 3.5.2 yields

$$\begin{aligned}&\|(\pi_*\varphi)\, A\, (\pi_*\psi)u\|_{H^{s-m,\gamma,\mu-m}(C_{x^0})}\\ &\qquad \leq c\, \|\pi_* \left(e^{\gamma\delta}(\delta')^{\mu-m}\varphi(\pi^\sharp A)\psi\, \pi^* u\right)\|_{H^{s-m,0,0}(C_{x^0})},\end{aligned}$$

with c a constant independent of u. On the other hand, combining (3.7.1) and (3.5.3) we conclude that

$$e^{\gamma\delta}(\delta')^{\mu-m}\varphi(\pi^\sharp A)\psi\, \pi^* u = \sum_{j=0}^{m} \tilde{a}_j \mathbf{D}^j \left(e^{\gamma\delta}(\delta')^{\mu}\pi^* u\right),$$

where $\tilde{a}_j \in C^\infty(\mathbb{R}_+, \mathrm{Diff}^{m-j}(M))$ vanish away from the interval $(0,\varepsilon)$. Moreover, the "coefficients" of $\tilde{a}_j$ still fulfill estimates (3.2.3) (and (3.2.5)) uniformly on small balls in M.

Set

$$\tilde{a}(r,\tau) = \sum_{j=0}^{m} \tilde{a}_j(r)\tau^j,$$

then $\tilde{a} \in \mathcal{S}_w(\mathcal{R}^s_{M\times\mathbb{C}}, \mathcal{R}^{s-m}_{M\times\mathbb{C}}; 0)$ (cf. Example 2.2.3). We now invoke equality (3.6.1) to obtain

$$\begin{aligned}\|\pi_* \left(e^{\gamma\delta}(\delta')^{\mu-m}\varphi(\pi^\sharp A)\psi\, \pi^* u\right)\|_{H^{s-m,0,0}(C_{x^0})} &= \|\mathrm{op}(\tilde{a})\left(e^{\gamma\delta}(\delta')^{\mu}\pi^* u\right)\|_{H_w(\mathcal{R}^{s-m}_{M\times\mathbb{C}})}\\ &\leq c\, \|e^{\gamma\delta}(\delta')^{\mu}\pi^* u\|_{H_w(\mathcal{R}^{s}_{M\times\mathbb{C}})}\end{aligned}$$

with some new constant c independent of u, the last estimate being a consequence of Proposition 2.5.1. Repeated application of (3.6.1) completes the proof.

$\square$

Since A is a differential operator, the local invertibility of A at the point x^0 is completely determined by the restriction of A to a punctured neighbourhood of x^0 on $\mathcal{M}$. Thus, we can assume that the coefficients of A vanish away from a compact subset of $\mathcal{M} \cap B(x^0,\varepsilon)$, for if not, we replace A by $(\pi_*\omega)A$ with any cut-off function $\omega \in C^\infty_{comp}[0,\varepsilon)$. Proposition 3.7.1 makes it obvious that the operator $A\colon H^{s,\gamma,\mu}(C_{x^0}) \to H^{s-m,\gamma,\mu-m}(C_{x^0})$ is locally invertible at the point x^0 if and only if $(\delta')^{-m}\pi^\sharp A : H_w(\mathcal{R}^s_{M\times\mathbb{C}}; \gamma,\mu) \to H_w(\mathcal{R}^{s-m}_{M\times\mathbb{C}}; \gamma,\mu)$ is locally invertible at the point $r = 0$. This latter operator has symbol

$$a(r,\zeta) = \sum_{j=0}^{m} a_j(r)\zeta^j$$

which is of class $\mathcal{S}_{w,\mathrm{sv}}(\mathcal{R}^s_{M\times\mathbb{C}}, \mathcal{R}^{s-m}_{M\times\mathbb{C}}; \gamma)$. Hence Theorem 2.7.1 is applicable and we arrive at the following result.

Theorem 3.7.2 *The operator* $A: H^{s,\gamma,0}(C_{x^0}) \to H^{s-m,\gamma,-m}(C_{x^0})$ *is locally invertible at the point* x^0 *if and only if there exists a number* $\epsilon > 0$ *such that the operator-valued function* $a(r, \tau + i\gamma): H^s(M) \to H^{s-m}(M)$ *is invertible for all* $(r, \tau) \in (0, \epsilon) \times \mathbb{R}$ *and*

$$\sup_{(0,\epsilon)\times\mathbb{R}} \|\mathcal{R}^s_{M\times\mathbb{C}}(\tau) a^{-1}(r, \tau + i\gamma) \mathcal{R}^{m-s}_{M\times\mathbb{C}}(\tau)\|_{L(L^2(M), L^2(M))} < \infty. \tag{3.7.3}$$

If $a(r, \tau + i\gamma)$ is an elliptic operator on M with parameter $\tau \in \mathbb{R}$, uniformly in $r \in (0, \varepsilon)$ (see for instance Agranovich and Vishik [AV64]), then Theorem 3.7.2 reads as follows.

Corollary 3.7.3 *Let* $\tau_i(r, \gamma)$, $i = 1, 2, \ldots,$ *be the eigenvalues of the operator pencil* $a(r, \tau + i\gamma)$, *and let* $s, \gamma \in \mathbb{R}$. *Then, in order that the operator* $A: H^{s,\gamma,0}(C_{x^0}) \to H^{s-m,\gamma,-m}(C_{x^0})$ *be locally invertible at the point* x^0, *it is necessary and sufficient that*

$$\lim_{\epsilon\to 0} \inf_{r\in(0,\epsilon)} |\Im \tau_i(r, \gamma)| > 0, \tag{3.7.4}$$

for each $i = 1, 2, \ldots$.

Condition (3.7.4) means that there exists an open strip in the complex plane, which contains the real axis and which is free of the eigenvalues $\tau_i(r, \gamma)$ for $r \in (0, \epsilon_0)$, where ϵ_0 is small enough.

Proof. Consider the composition

$$\psi(r, \tau) = \mathcal{R}^{s-m}_{M\times\mathbb{C}}(\tau) a(r, \tau + i\gamma) \mathcal{R}^{-s}_{M\times\mathbb{C}}(\tau).$$

By assumption, $\psi(r, \tau)$ is an elliptic pseudodifferential operator of order zero on M with parameter $\tau \in \mathbb{R}$, uniformly in $r \in (0, \epsilon)$. Hence it follows, by a theorem of Agranovich and Vishik [AV64], that there exists an $R > 0$ such that $\psi(r, \tau)$ is invertible for all $r \in (0, \epsilon)$ and $\tau \in \mathbb{R}$ with $|\tau| > R$. On the other hand, since both $\mathcal{R}^{-s}_{M\times\mathbb{C}}(\tau)$ and $\mathcal{R}^{s-m}_{M\times\mathbb{C}}(\tau)$ are families of isomorphisms, condition (3.7.4) is equivalent to the fact that to every $R > 0$ there corresponds an $\epsilon_0 \in (0, \epsilon]$ such that $\psi(r, \tau)$ is invertible whenever $r \in (0, \epsilon_0)$ and $|\tau| \le R$. We thus conclude that $a(r, \tau + i\gamma)$ is invertible for all $r \in (0, \epsilon_0)$ and $\tau \in \mathbb{R}$. Moreover, the inverse $\psi^{-1}(r, \tau) = \mathcal{R}^s_{M\times\mathbb{C}}(\tau) a^{-1}(r, \tau + i\gamma) \mathcal{R}^{m-s}_{M\times\mathbb{C}}(\tau)$ meets the estimate (3.7.3). Indeed, if $|\tau| \le R$, then (3.7.3) follows from the continuity of $\psi^{-1}(r, \tau)$ in τ. If $|\tau| > R$, then (3.7.3) is a consequence of norm estimates for pseudodifferential operators with parameter in Sobolev spaces, cf. *ibid.* This completes the proof, when combined with Theorem 3.7.2.

□

To extend Theorem 3.7.2 and Corollary 3.7.3 to arbitrary weight exponents μ (not merely $\mu = 0$) we impose an additional condition on f, namely

$$\lim_{r\to 0+} r^j D^j f(r) = 0, \quad \text{for each} \quad j = 0, 1, \ldots. \tag{3.7.5}$$

(cf. (3.2.6)). As described in Section 3.1, this corresponds to the case where x^0 is a cusp.

Corollary 3.7.4 *Suppose that f fulfills (3.7.5). Let $s, \gamma, \mu \in \mathbb{R}$. Then, the operator $A\colon H^{s,\gamma,\mu}(C_{x^0}) \to H^{s-m,\gamma,\mu-m}(C_{x^0})$ is locally invertible at the point x^0 if and only if there exists an $\epsilon > 0$ such that the operator-valued function $a(r, \tau + i\gamma)\colon H^s(M) \to H^{s-m}(M)$ is invertible for all $(r,\tau) \in (0,\epsilon) \times \mathbb{R}$, and its inverse meets (3.7.3).*

Proof. Indeed, by (3.5.2) (or (3.5.2)),

$$\mathbf{D}^j \left((\delta')^\mu u \right) = (\delta')^\mu \left(\mathbf{D}^j u + \textstyle\sum_{\iota=1}^j p_{j,j-\iota}(f, rDf, \ldots, r^\iota D^\iota f)\, \mathbf{D}^{j-\iota} u \right),$$

$p_{j,j-\iota}$ being a polynomial of degree ι with coefficients depending on μ, such that $p_{j,j-\iota}(0) = 0$. Hence it follows that, under assumption (3.7.5), the operator $\mathbf{D}$ commutes with the weight factor $(\delta')^\mu$ up to an operator which is unessential for the local invertibility. Thus, applying Theorem 3.7.2 yields the desired conclusion.

□

3.8 Local invertibility of a boundary value problem at a cusp

In this section we indicate how the above techniques may be used to treat a boundary value problem in a closed domain $\mathcal{D} \subset \mathbb{R}^{n+1}$ with a singular point x^0 on the boundary. Namely,

$$\left\{ \begin{array}{rcll} Au & = & f & \text{in} \quad \mathcal{D} \setminus \{x^0\}, \\ B_i u & = & u_i & \text{on} \quad \partial\mathcal{D} \setminus \{x^0\}, \end{array} \right. \tag{3.8.1}$$

where A is a differential operator in $\mathcal{D} \setminus \{x^0\}$ and (B_i) a system of differential operators defined in a neighbourhood of $\partial\mathcal{D} \setminus \{x^0\}$. We write m for the order of A and m_i for the order of B_i.

Suppose $\mathcal{D}$ is written close to x^0 in the form (3.1.1), where M is a compact domain with smooth boundary in Ω. We assume that f fulfills (3.1.6).

We are concerned with the problem of local invertibility of (3.8.1) at the singular point x^0. For this reason we restrict our attention to the punctured neighbourhood of x^0 given by (3.1.1). This is nothing but $C_{x^0} \cap B(x^0, \varepsilon)$ where C_{x^0} is a canonical domain with a singular point on the boundary (cf. (3.4.1)). There is no loss of generality in assuming that both A and (B_i) are defined on the entire domain C_{x^0} and vanish away from a ball with center x^0. We are thus led to a boundary value problem in the canonical domain C_{x^0}. Proposition 3.7.1 suggests us suitable function spaces to study the problem, namely

$$\begin{pmatrix} A \\ \oplus r_{\partial C_{x^0}} B_i \end{pmatrix} : \; H^{s,\gamma,\mu}(C_{x^0}) \to \begin{array}{c} H^{s-m,\gamma,\mu-m}(C_{x^0}) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2},\gamma,\mu-m_i}(\partial C_{x^0}) \end{array}, \tag{3.8.2}$$

where $r_{\partial C_{x^0}}$ means restriction to the boundary of C_{x^0} and s is any integer with $s > \max m_i$.

Under the change of variables $\pi(r,\theta) = x^0 + rS(f(r)\theta)$, the operators A and (B_i) transform into operators

$$\begin{array}{rcl} \pi^\sharp A &=& (\delta')^m \sum\limits_{j+|\alpha|\le m} a_{j,\alpha}(r,\theta)\mathbf{D}_r^j D_\theta^\alpha, \\ \pi^\sharp B_i &=& (\delta')^{m_i} \sum\limits_{j+|\alpha|\le m_i} b_{i,j,\alpha}(r,\theta)\mathbf{D}_r^j D_\theta^\alpha \end{array}$$

over the semicylinder $\mathbb{R}_+ \times M$ The coefficients $a_{j,\alpha}$ and $b_{i,j,\alpha}$ are required to satisfy (3.2.3) and (3.2.5) uniformly in $\theta \in M$.

Since the factor $\delta'(r)$ is different from zero for $r > 0$, operator (3.8.2) is locally invertible at the point x^0 if and only if the operator

$$\begin{pmatrix} (\delta')^{-m}\pi^\sharp A \\ \oplus r_{\partial M}(\delta')^{-m_i}\pi^\sharp B_i \end{pmatrix} : \; H_w(\mathcal{R}^s_{M\times\mathbb{C}};\gamma,\mu) \to \begin{array}{c} H_w(\mathcal{R}^{s-m}_{M\times\mathbb{C}};\gamma,\mu) \\ \oplus \\ \oplus H_w(\mathcal{R}^{s-m_i-\frac{1}{2}}_{\partial M\times\mathbb{C}};\gamma,\mu) \end{array} \tag{3.8.3}$$

is locally invertible at the point $r = 0$. Here $\mathcal{R}^s_{M\times\mathbb{C}}(\tau)$ stands for the order-reducing family of Example 1.1.3 and $\mathcal{R}^{s-m_i-\frac{1}{2}}_{\partial M\times\mathbb{C}}(\tau)$ for that of Example 1.1.2.

The advantage of using this reformulation of problem (3.8.2) lies in the fact that operator (3.8.3) fits in the theory of Part 2. It has symbol

$$a(r,\zeta) = \begin{pmatrix} \sum\limits_{j+|\alpha|\le m} a_{j,\alpha}(r,\theta)D_\theta^\alpha\zeta^j, \\ \oplus \sum\limits_{j+|\alpha|\le m_i} r_{\partial M} b_{i,j,\alpha}(r,\theta)D_\theta^\alpha\zeta^j \end{pmatrix} : \; H^s(M) \to \begin{array}{c} H^{s-m}(M) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2}}(\partial M) \end{array}$$

which is of class $\mathcal{S}_{w,\mathrm{sv}}(\lambda_1,\lambda_2;\gamma)$, with

$$\begin{array}{rcl} \lambda_1(\tau) &=& \mathcal{R}^s_{M\times\mathbb{C}}(\tau), \\ \lambda_2(\tau) &=& \mathcal{R}^{s-m}_{M\times\mathbb{C}}(\tau) \oplus \left(\oplus\mathcal{R}^{s-m_i-\frac{1}{2}}_{\partial M\times\mathbb{C}}(\tau)\right). \end{array}$$

It is worth emphasising that the symbol $a(r,\zeta)$ takes its values in the "algebra" of boundary value problems on the domain M.

We thus conclude that Theorem 2.7.1 is applicable which results in the following criterion of local solvability of problem (3.8.1) at the point x^0.

Theorem 3.8.1 *Let $s \in \mathbb{Z}_+$ satisfy $s > \max m_i$ and let $\gamma \in \mathbb{R}$, $\mu = 0$. The operator (3.8.2) is locally invertible at x^0 if and only if there exists an $\epsilon > 0$ such that the symbol $a(r, \tau + i\gamma)\colon H^s(M) \to H^{s-m}(M) \oplus (\oplus H^{s-m_i-\frac{1}{2}}(\partial M))$ is invertible for all $(r, \tau) \in (0, \epsilon) \times \mathbb{R}$ and*

$$\sup_{(0,\epsilon)\times\mathbb{R}} \|\lambda_1(\tau) a^{-1}(r, \tau + i\gamma) \lambda_2^{-1}(\tau)\|_{L(L^2(M)\oplus(\oplus L^2(\partial M)), L^2(M))} < \infty. \tag{3.8.4}$$

If $a(r, \tau + i\gamma)$ is an elliptic boundary value problem on M with parameter $\tau \in \mathbb{R}$, uniformly in $r \in (0, \varepsilon)$, then Theorem 3.8.1 reads just as Corollary 3.7.3.

Corollary 3.8.2 *Let $\tau_i(r, \gamma)$, $i = 1, 2, \ldots$, be the eigenvalues of the operator pencil $a(r, \tau + i\gamma)$. Then, in order that operator (3.8.2) (for $\mu = 0$) be locally invertible at the point x^0, it is necessary and sufficient that*

$$\lim_{\epsilon \to 0} \inf_{r \in (0,\epsilon)} |\Im \tau_i(r, \gamma)| > 0,$$

for each $i = 1, 2, \ldots$.

We now wish to arrange that these results hold for arbitrary $\mu \in \mathbb{R}$, not merely $\mu = 0$. To this end, we proceed as in Corollary 3.7.4.

Corollary 3.8.3 *Suppose f fulfills (3.7.5). Let $s \in \mathbb{Z}_+$ satisfy $s > \max m_i$ and let $\gamma, \mu \in \mathbb{R}$. Then, operator (3.8.2) is locally invertible at the point x^0 if and only if there exists a number $\epsilon > 0$ such that the operator-valued function $a(r, \tau + i\gamma) : H^s(M) \to H^{s-m}(M) \oplus (\oplus H^{s-m_i-\frac{1}{2}}(\partial M))$ is invertible for all $(r, \tau) \in (0, \epsilon) \times \mathbb{R}$, and its inverse meets (3.8.4).*

The symbol $a(r, \zeta)$ which controls the local solvability of problem (3.8.1) at the singular point x^0 is similar to that in case x^0 is a conical point (cf. Kondrat'ev [Kon67]). However, the case of pure cusps differs from the case of conical points to some extent. Namely, Corollary 3.8.3 shows that the weight factors $(\delta')^\mu$ do not influence the local solvability of problem (3.8.1); only the weight exponent γ enters condition (3.8.4). On the other hand, in the case of conical points δ' is a multiple of $1/r$ and it is well-known that condition (3.8.4) depends on the weight factors r^γ (*prohibited* weight exponents, cf. *ibid*). The reason of this is that the function $f \equiv const$ does not meet condition (3.7.5).

3.9 The Dirichlet and Neumann problems in a domain with cusps

In this section we consider the Dirichlet and Neumann problems for the Laplace equation in a closed domain $\mathcal{D} \subset \mathbb{R}^2$ with a singular point $x^0 \in \partial\mathcal{D}$. We assume that the portion of $\mathcal{D}$ in a small ball with centre x^0 is given by

$$\mathcal{D} \cap (B(x^0, \varepsilon) \setminus \{x^0\}) = \{x^0 + (r\cos(f(r)\theta), r\sin(f(r)\theta)) \colon r \in (0, \varepsilon), \theta \in M\},$$

where f is a positive function on the interval $(0, \varepsilon)$ and $M = [\theta_1, \theta_2]$ is a segment in $[0, 2\pi)$.

The function f is required to fulfill the following assumptions:

- $f \in C^\infty(0, \varepsilon]$;
- $\lim_{r \to 0+} r^j D^j f = 0$, for each $j = 0, 1, \ldots$

(thus, x^0 is a cusp).

We extend f to a smooth positive function on the whole semiaxis $\mathbb{R}_+$, such that $f(r) = const$ for r large enough. Then, we introduce the diffeomorphism $\delta \colon \mathbb{R}_+ \to \mathbb{R}$ by (3.1.4), for each $t \in \mathbb{R}_+$.

A trivial verification shows that

$$\begin{aligned} \frac{\partial}{\partial x_1} &= \cos(f\theta)\frac{\partial}{\partial r} + \frac{1}{rf}\left(-\sin(f\theta) - r\frac{\partial f}{\partial r}\theta\cos(f\theta)\right)\frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial x_2} &= \sin(f\theta)\frac{\partial}{\partial r} + \frac{1}{rf}\left(\cos(f\theta) - r\frac{\partial f}{\partial r}\theta\sin(f\theta)\right)\frac{\partial}{\partial \theta} \end{aligned}$$

(cf. (3.2.1)). It follows that the pull-back of the Laplace operator Δ under the change of variables $\pi(r, \theta) = x^0 + (r\cos(f(r)\theta), r\sin(f(r)\theta))$ is

$$\pi^\sharp \Delta = (\delta')^2 \left(-\mathbf{D}_r^2 - D_\theta^2 + S\right),$$

where the operator S has a small local norm at the singular point and is thus unessential in the problem of local invertibility. Moreover, since the unit outward vector to $\partial\mathcal{D}$ at the point (r, θ) is

$$\nu = \left(\cos\left(\frac{\pi}{2} + f\theta\right), \sin\left(\frac{\pi}{2} + f\theta\right)\right),$$

we have

$$\pi^\sharp \frac{\partial}{\partial \nu} = \delta'\left(-\frac{\partial}{\partial \theta}\right)$$

up to unessential operator.

From what has been proved in Section 3.8 we deduce that the local solvability at x^0, for the Dirichlet problem

$$\begin{pmatrix} \Delta \\ r_{\partial C_{x^0}} \end{pmatrix} \colon H^{s,\gamma,\mu}(C_{x^0}) \to \begin{array}{c} H^{s-2,\gamma,\mu-2}(C_{x^0}) \\ \oplus \\ H^{s-\frac{1}{2},\gamma,\mu}(\partial C_{x^0}) \end{array} \qquad (3.9.1)$$

with $s \in \mathbb{Z}_+$ and $s > \frac{1}{2}$, is controlled by the operator-valued symbol

$$\begin{pmatrix} -D_\theta^2 - \zeta^2 \\ r_{\partial M} \end{pmatrix} \colon H^s(M) \to \begin{array}{c} H^{s-2}(M) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial M) \end{array},$$

where $\zeta \in \mathbb{R} + i\gamma$. This symbol is easily verified to be an isomorphism unless $\Re\zeta = 0$ and $\gamma = \frac{\theta_2 - \theta_1}{\pi} j$, $j \in \mathbb{Z} \setminus \{0\}$. Corollary 3.8.3 now leads to the following result.

Theorem 3.9.1 *Let $\gamma \neq \frac{\theta_2 - \theta_1}{\pi} j$ for all $j \in \mathbb{Z} \setminus \{0\}$. Then, as defined by (3.9.1), the Dirichlet problem is locally invertible at the cusp.*

For an explicit algebra of pseudodifferential operators containing local parametrices of the Dirichlet problem in a plane domain with a conical point on the boundary, we refer the reader to a recent paper of Ueda [Ued96]. The same parametrix construction still goes for the Dirichlet problem for a general strongly elliptic operator, not merely for the Laplacian.

We now turn to the Neumann problem. As described in Section 3.8, the local solvability at x^0, for the Neumann problem

$$\begin{pmatrix} \Delta \\ r_{\partial C_{x^0}} \frac{\partial}{\partial \nu} \end{pmatrix} : \ H^{s,\gamma,\mu}(C_{x^0}) \to \begin{array}{c} H^{s-2,\gamma,\mu-2}(C_{x^0}) \\ \oplus \\ H^{s-\frac{3}{2},\gamma,\mu}(\partial C_{x^0}) \end{array} \tag{3.9.2}$$

with $s \in \mathbb{Z}_+$ and $s > \frac{3}{2}$, is controlled by the operator-valued symbol

$$\begin{pmatrix} -D_\theta^2 - \zeta^2 \\ r_{\partial M} \frac{\partial}{\partial \theta} \end{pmatrix} : \ H^s(M) \to \begin{array}{c} H^{s-2}(M) \\ \oplus \\ H^{s-\frac{3}{2}}(\partial M) \end{array},$$

where $\zeta \in \mathbb{R} + i\gamma$. This symbol is easily proved to be an isomorphism unless $\Re\zeta = 0$ and $\gamma = \frac{\theta_2 - \theta_1}{\pi} j$, $j \in \mathbb{Z}$. Corollary 3.8.3 now gives the following result.

Theorem 3.9.2 *Let $\gamma \neq \frac{\theta_2 - \theta_1}{\pi} j$ for all $j \in \mathbb{Z}$. Then, as defined by (3.9.2), the Neumann problem is locally invertible at the cusp.*

3.10 Fredholm property

Let $\mathcal{D}$ be a compact domain in $\mathbb{R}^{n+1}$ with a finite number of singular points on the boundary, $\operatorname{sing} \partial\mathcal{D} = \{x^1, \ldots, x^N\}$. Suppose that $\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}$ is a C^∞ manifold with boundary. Moreover, we require $x^1, \ldots, x^N$ to be cusps, as defined in Section 3.1.

Consider a boundary value problem in $\mathcal{D}$,

$$\left\{ \begin{array}{rcll} Au & = & f & \text{in} \quad \mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}, \\ B_i u & = & u_i & \text{on} \quad \partial\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}, \end{array} \right. \tag{3.10.1}$$

where A is a differential operator in $\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}$ and (B_i) a system of differential operators defined in a neighbourhood of $\partial\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}$. In the sequel, we denote by m the order of A and by m_i the order of B_i.

We first define appropriate function spaces to study problem (3.10.1). To this end, for each point x^ν, choose a ball $B(x^\nu, \varepsilon_\nu)$ such that the part of $\mathcal{D}$ lying in $B(x^\nu, \varepsilon_\nu)$ is given by (3.1.1), now all the objects S, f and M depending on ν. We can assume, by decreasing ε_ν if necessary, that the balls $B(x^\nu, \varepsilon_\nu)$ are pairwise non-overlapping. For each $\nu = 1, \ldots, N$, we fix a C^∞ function χ_ν with a compact support in $B(x^\nu, \varepsilon_\nu)$, such that $0 \le \chi_\nu \le 1$ and $\chi_\nu \equiv 1$ in a neigbourhood of the point x^ν. Moreover, we set $\chi_0 = 1 - \sum_{\nu=1}^N \chi_\nu$. Let $s \in \mathbb{Z}_+$ and let $\gamma = (\gamma_1, \ldots, \gamma_N)$, $\mu = (\mu_1, \ldots, \mu_N)$ be tuples of $\mathbb{R}^N$. Denote by $H^{s,\gamma,\mu}(\mathcal{D})$ the completion of the space $C^\infty_{comp}(\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D})$ with respect to the norm

$$\|u\|_{H^{s,\gamma,\mu}(\mathcal{D})} = \left(\|\chi_0 u\|^2_{H^s(\mathcal{D})} + \sum_{\nu=1}^N \|\chi_\nu u\|^2_{H^{s,\gamma_\nu,\mu_\nu}(C_{x^\nu})} \right)^{\frac{1}{2}}$$

(cf. (3.5.1)). It is a simple matter to see that $H^{s,\gamma,\mu}(\mathcal{D})$ is a Hilbert space. We write $H^{s-\frac{1}{2},\gamma,\mu}(\partial\mathcal{D})$ for the function space on the boundary of $\mathcal{D}$ which consists of the traces on the smooth part of $\partial\mathcal{D}$ of elements in $H^{s,\gamma,\mu}(\mathcal{D})$. This space is topologised under the quotient norm.

We require the coefficients of the operators A and (B_i) to satisfy conditions (3.2.3) and (3.2.5) close to each point x^ν, $\nu = 1, \ldots, N$. Then, for each integer $s > \max m_i$ and each $\gamma, \mu \in \mathbb{R}^N$, problem (3.10.1) gives rise to a continuous linear operator

$$\begin{pmatrix} A \\ \oplus r_{\partial\mathcal{D}} B_i \end{pmatrix} : \; H^{s,\gamma,\mu}(\mathcal{D}) \to \begin{array}{c} H^{s-m,\gamma,\mu-m}(\mathcal{D}) \\ \oplus \\ \oplus H^{s-m_i-\frac{1}{2},\gamma,\mu-m_i}(\partial\mathcal{D}) \end{array} \tag{3.10.2}$$

(cf. (3.8.2)).

To every singular point x^ν we assign an operator pencil $a_{x^\nu}(r,\zeta)$, as described in Section 3.8. Suppose, for each $\nu = 1, \ldots, N$, that $a_{x^\nu}(r, \tau + i\gamma_\nu)$ is an elliptic boundary value problem on M_ν with parameter $\tau \in \mathbb{R}$, uniformly in $r \in (0, \epsilon_\nu)$.

Theorem 3.10.1 *Let $s \in \mathbb{Z}_+$ satisfy $s > \max m_i$, and let $\gamma, \mu \in \mathbb{R}^N$. Then, the operator (3.10.2) is Fredholm if and only if:*

1) *the boundary value problem $(A, \oplus B_i)$ is elliptic at each point of $\mathcal{D} \setminus \operatorname{sing} \partial\mathcal{D}$; and*

2) *at each point $x^\nu \in \operatorname{sing} \partial\mathcal{D}$, we have*

$$\lim_{\epsilon \to 0} \inf_{r \in (0,\epsilon)} |\Im \tau_{x^\nu,i}(r, \gamma_\nu)| > 0,$$

for each $i = 1, 2, \ldots$, where $\tau_{x^\nu,i}(r,\gamma_\nu)$ are the eigenvalues of the operator pencil $a_{x^\nu}(r, \tau + i\gamma_\nu)$.

Proof. The proof is straightforward from Corollary 3.8.2 by invoking the standard machinery connected with the "pasting together" of a global regulariser from local regularisers by means of a special partition of unity on $\mathcal{D}$. $\square$

Theorem 3.10.1 applies, in particular, to the Dirichlet and Neumann problems treated in Section 3.9 (cf. also Maz'ya and Plamenevskii [MP78]).

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