Institut für Physik
Arbeitsgruppe “Statistische Physik/Chaostheorie”

Coherence and Synchronization of Noisy-Driven Oscillators

Dissertation
zur Erlangung des akademischen Grades
“doctor rerum naturalium”
(Dr. rer. nat.)
in der Wissenschaftsdisziplin “Theoretische Physik”

eingereicht an der
Mathematisch–Naturwissenschaftlichen Fakultät
der Universität Potsdam

von
Denis S. Goldobin
geboren am 30. September 1981 in Perm (Rußland)

Potsdam, den Mai 2007

Elektronisch veröffentlicht auf dem
Publikationsserver der Universität Potsdam:
http://opus.kobv.de/ubp/volltexte/2007/1504/
urn:nbn:de:kobv:517-opus-15047
[http://nbn-resolving.de/urn:nbn:de:kobv:517-opus-15047]

Abstract

In the present dissertation paper we study problems related to synchronization phenomena in the presence of noise which unavoidably appears in real systems. One part of the work is aimed at investigation of utilizing delayed feedback to control properties of diverse chaotic dynamic and stochastic systems, with emphasis on the ones determining predisposition to synchronization. Other part deals with a constructive role of noise, i.e. its ability to synchronize identical self-sustained oscillators.

First, we demonstrate that the coherence of a noisy or chaotic self-sustained oscillator can be efficiently controlled by the delayed feedback. We develop the analytical theory of this effect, considering noisy systems in the Gaussian approximation. Possible applications of the effect for the synchronization control are also discussed.

Second, we consider synchrony of limit cycle systems (in other words, self-sustained oscillators) driven by identical noise. For weak noise and smooth systems we proof the purely synchronizing effect of noise. For slightly different oscillators and/or slightly non-identical driving, synchrony becomes imperfect, and this subject is also studied. Then, with numerics we show moderate noise to be able to lead to desynchronization of some systems under certain circumstances. For neurons the last effect means "antireliability" (the "reliability" property of neurons is treated to be important from the viewpoint of information transmission functions), and we extend our investigation to neural oscillators which are not always limit cycle ones.

Third, we develop a weakly nonlinear theory of the Kuramoto transition (a transition to collective synchrony) in an ensemble of globally coupled oscillators in presence of additional time-delayed coupling terms. We show that a linear delayed feedback not only controls the transition point, but effectively changes the nonlinear terms near the transition. A purely nonlinear delayed coupling does not effect the transition point, but can reduce or enhance the amplitude of collective oscillations.

Zusammenfassung

In der vorliegenden Dissertation werden Synchronisationsphänomene im Vorhandensein von Rauschen studiert. Ein Ziel dieser Arbeit besteht in der Untersuchung der Anwendbarkeit verzögerter Rückkopplung zur Kontrolle von bestimmten Eigenschaften chaotischer oder stochastischer Systeme. Der andere Teil beschäftigt sich mit den konstruktiven Eigenschaften von Rauschen. Insbesondere wird die Möglichkeit, identische selbsterregte Oszillatoren zu synchronisieren untersucht.

Als erstes wird gezeigt, dass Kohärenz verrauschter oder chaotischer Oszillatoren durch verzögertes Rückkoppeln kontrolliert werden kann. Es wird eine analytische Beschreibung dieses Phänomens in verrauschten Systemen entwickelt. Außerdem werden mögliche Anwendungen im Zusammenhang mit Synchronisationskontrolle vorgestellt und diskutiert.

Als zweites werden Oszillatoren unter dem Einfluss von identischem Rauschen betrachtet. Für schwaches Rauschen und genügend glatte Systeme wird bewiesen, das Rauschen zu Synchronisation führt. Für leicht unterschiedliche Oszillatoren und leicht unterschiedliches Rauschen wird die Synchronisation unvollständig. Dieser Effekt wird auch untersucht. Dann wird mit Hilfe von Numerik gezeigt, dass moderates Rauschen zur Desynchronisierung von bestimmten Systemen führen kann. Dieser Effekt wird auch in neuronalen Oszillatoren untersucht, welche nicht unbedingt Grenzzyklen besitzen müssen.

Im dritten Teil wird eine schwache nichtlineare Theorie des Kuramoto-Übergangs, dem Übergang zur kollektiven Synchronisation, in einem Ensemble von global gekoppelten Oszillatoren mit zusätzlichen zeitverzögerten Kopplungstermen entwickelt. Es wird gezeigt, dass lineare Rückkopplung nicht nur den Übergangspunkt bestimmt, sondern auch die nichtlinearen Terme in der Nähe des Übergangs entscheidend verändert. Eine rein nichtlineare Rückkopplung verändert den Übergang nicht, kann aber die Amplitude der kollektiven Oszillationen vergrößern oder verringern.

Acknowledgements

First of all, I want to thank Prof. Dr. Arkady Pikovsky for introducing me into the theory of stochastic processes (and not only this one), his friendly guidance (not only in scientific matters), and also possibility to complete my thesis in his group. He not only provided excellent conditions for my research work, but always offered (and offers) the investigation subjects very attractive for me and with account for my interests and scientific biases and fancies.

I am also thankful to Dr. habil. Michael Rosenblum for interesting discussions and fruitful collaborations, and to Dr. habil. Michael Zaks, "who knows everything", for discussions and comments on my works. Here, I want to note also less intense, but not less interesting collaboration with Prof. Dr. Jürgen Kurths and Dr. Alexei Zaikin.

For nice atmosphere in the working group (and in the institute) I want to thank Markus Abel, Arthur Straube, Natalia Tukhlina, Dr. Rudi Hachenberger, Karsten Ahnert, Konstantin Mergenthaler, Andreas Pavlik and Malte Siefer.

The acknowledgement list would be not complete without Prof. Sergey Kuznetsov, Sergey Shklyaev, Prof. Alexander Neiman, Lev Tsimring, Alexander Balanov, Natalia Janson and Olexandr Popovych.

Contents

Introduction

Synchronization phenomena are extremely wide spread in environment (ranging from nature over engineering to social life) and are attracting great attention not only of scientists, but also engineers. The development of general theories of dynamical systems and stochastic processes and their mathematical tools has provided recently (in a historical perspective) opportunities for a systematic and quantitative study of these intriguing phenomena (see, for instance, [1]).

In spite of the attention to the phenomenon and intense investigations in the field, some problems still remain opened. Those related to the role of noise which is unavoidably present in real systems, and to the delayed feedback which provides possibilities of control of diverse dynamical systems, are subject of this dissertation paper. In details, the paper is organized as follows.

In chapter 2 we consider utilizing delayed feedback to control coherence (persistence of oscillation frequency) of stochastic limit cycle systems and deterministic chaotic ones [2, 3]. Coherence is quantified by virtue of the phase diffusion constant. The main point is that we do not intend to suppress chaos, but to control phase diffusion, and, therefore, use quite weak feedback.

First, the effect is demonstrated in numerical simulations, it appears to be especially pronounced for the Lorenz system. Also, the role of coherence for predisposition of systems to synchronization is illustrated with the Lorenz system entrained by an external periodic forcing. Then, using the Gaussian approximation, we develop an analytical description of the effect for stochastic limit cycle systems. The results of the analytic theory are compared to numerics.

In chapter 3 we turn our attention to the possible constructive role of noise: the phenomenon of synchronization of oscillators by common (white Gaussian) noise is investigated. For identical oscillators subject to identical noisy driving, the synchrony is measured by the Lyapunov exponent: oscillators are perfectly synchronous, when the Lyapunov exponent is negative, and nonsynchronous otherwise.

First, we derive the expression for the Lyapunov exponent for limit cycle oscillators within the framework of the phase approximation, and demonstrate that weak noise synchronizes systems with a smooth limit cycle [4, 5]. Then we study the role of small nonidentities either in noisy driving or in oscillators [6] which distort perfect synchrony observed for a negative Lyapunov exponent in the ideal situation. The results of analytical description are compared to numerics for a noisy Van der Pol–Duffing oscillator. Numerical simulation also reveals occurrence of a positive Lyapunov exponent for some systems under certain circumstances for moderate noise strength, what means desynchronization. Then, in order to check whether the effects observed are specific for withe Gaussian noise or general for different noises, we extend our investigation to the case of a telegraph noise [7] and find a good qualitative agreement between results for these two kinds of noise, providing these results to be general.

Further, we extend our investigation to the case of neuronal oscillators which are not always limit cycle ones. Here we show possibility of "antireliability" ("reliability" is the property of a single neuron to give identical responses under repeatedly applied weak input fluctuations of a prerecorded waveform [39]) and present an analytical model of the phenomenon [8].

In chapter 4 we consider phenomena related to synchronization in large populations of coupled noisy oscillators (independent noise for each oscillator). A transition to collective synchrony in an ensemble of globally coupled oscillators is known as the Kuramoto transition. An important application of the theory is collective dynamics of neuronal populations. Indeed, synchronization of individual neurons is believed to play the crucial role in the emergence of pathological rhythmic brain activity in Parkinsons disease, essential tremor, and epilepsies. One approach to suppress such an activity is to apply to the system a negative feedback loop [66, 67, 68]. Specifically, we develop a weakly nonlinear theory of the Kuramoto transition in the presence of linear and nonlinear time-delayed coupling terms [9]. This theory allows us to determine the Kuramoto transition point and the order parameter near the transition.

Chapter 5 ends the thesis with discussion of results and open questions.

Chapter 2

Coherence of Oscillators with Delayed Feedback

Coherence, or stability of oscillation frequency, is one of the main characteristics of self-sustained systems. This property determines the quality of clocks, electronic generators, lasers, etc. Quite often the improvement of the coherence is one of the major goals in the design of such oscillators. In terms of the phase dynamics, the coherence of a noisy limit cycle oscillator is quantified by the phase diffusion constant; it is proportional to the width of the spectral peak of oscillations. Many chaotic oscillators also admit phase dynamics description, and, hence, their coherence can be quantified by virtue of the phase diffusion constant as well [1].

In this chapter we demonstrate that the coherence of oscillations is essentially influenced by an external delayed feedback, thus offering a possibility for its effective control [2, 3]. Delayed feedback is widely used to achieve a qualitative change in the dynamics, e.g. to make chaotic oscillators to operate periodically (Pyragas' control method [10]) or to suppress space-time chaos [11, 12, 13]. Additionally to Pyragas' method utilizing a single delay feedback, there are methods employing multiple delay feedback control: they can be used either to stabilize fixed points [14, 15] or to make chaotic systems operating periodically [16, 17]. In our study we concentrate on the quantitative effect of a (single) delayed feedback on the phase diffusion properties of noisy periodic and chaotic oscillators.

Investigation of effects of irregularities and noise in systems with delay is a complicated problem, because one cannot apply here such well established tools as the Fokker-Planck equation, valid for Markov processes. In the case of delay the process is non-Markov and therefore the problems are treated by *ad hoc* statistical methods. This has been accomplished recently for bistable oscillators [18], see also [19, 20, 21]. Below we present a theory describing the effect of a delayed feedback on noisy self-sustained oscillations. It is based on the phase approximation of the dynamics, what means that the noise and delayed feedback are assumed to be weak. On the other hand, we consider a full nonlinear phase dynamics problem, and therefore our approach goes beyond the statistical analysis of linear stochastic delay-differential equations [22, 23].

2.1 Control of coherence: numerical results

In this section we present a numerical evidence for a possibility to control the diffusion constant by a delayed feedback. We begin by presenting the results of numerical simulation for noisy Van der Pol oscillator:

$$\ddot{x} - \mu(1-x^2)\dot{x} + \Omega_0^2 x = k(\dot{x}(t-\tau) - \dot{x}(t)) + \zeta(t) \ , \qquad \langle \zeta(t)\zeta(t')\rangle = 2d^2\delta(t-t') \ . \quad (2.1)$$

The l.h.s. represents the Van der Pol equation; in the absence of noise and delay ($k = d = 0$) and for small nonlinearity μ this model has a limit cycle solution $x_0 \approx 2\cos\phi$, $\dot{x}_0 \approx -2\Omega_0 \sin\phi$ with a uniformly growing phase $\phi(t) \approx \Omega_0 t + \phi_0$ [24]. Under the influence of noise and in the absence of feedback ($k = 0$, $d > 0$), $\phi(t)$ diffuses according to $\langle(\phi(t) - \langle\phi(t)\rangle)^2\rangle \propto D_0 t$; the diffusion constant D_0 is proportional to the intensity of noise d^2 [see Eq. (2.4) below for an exact relation].

We expect that in the presence of feedback the diffusion constant D generally differs from D_0; this is confirmed by the numerical results, shown in Fig. 2.1 for $\Omega_0 = 1$, $d = 0.1$, and $\mu = 0.7$. One can see that diffusion can be suppressed or enhanced, depending on the feedback strength k and the delay time τ. In this chapter the main our goal is to describe this picture theoretically.

Another numerical example demonstrates the effect of delayed feedback on phase diffusion in the chaotic Lorenz model

$$\begin{aligned} \dot{x} &= \sigma(y - x) \ , \\ \dot{y} &= rx - y - xz \ , \\ \dot{z} &= -bz + xy + k(z(t-\tau) - z(t)) \ , \end{aligned} \qquad (2.2)$$

where $\sigma = 10$, $r = 32$, and $b = 8/3$. The phase of the Lorenz system is well-defined if one uses a projection of the phase space on the plane ($u = \sqrt{x^2 + y^2}, z$) (see [1] and Fig. 2.3 below):

$$\phi(t) = \arctan\frac{z(t) - z_0}{u(t) - u_0} + \pi n, \qquad t \in [t_n, t_{n+1}),$$

where the point $\{u_0 = \sqrt{2b(r-1)},\ z_0 = r - 1\}$ corresponds to the nontrivial fixed points of the Lorenz system, t_k is the moment of the k-th trajectory passing through u_0 (for odd k leftwards and for even k rightwards, or conversely — depending on convention). Notice that there is no noise term in Eqs. (2.2): because of chaos the phase of the autonomous system grows non-uniformly, with a non-zero diffusion constant.

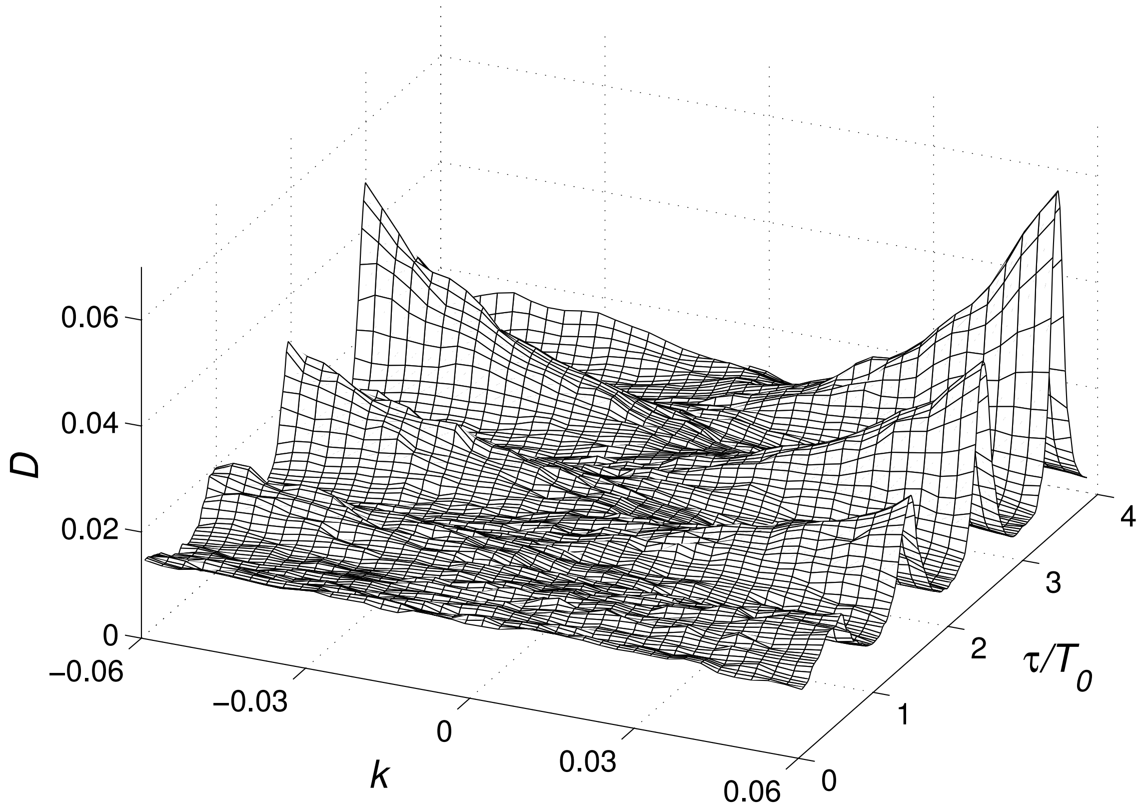

Figure 2.1: Diffusion constant D for the phase of the noise-driven Van der Pol oscillator with delayed feedback (2.1) as the function of τ/T_0 and k; $T_0 \approx 6.61$ is the oscillation period without delay.

The dependence of the diffusion constant D of the phase on the feedback parameters k and τ is shown in Fig. 2.2.

Qualitatively this dependence is similar to that for the Van der Pol model. However, there is an important distinction: the diffusion has a very deep minimum for positive feedback constant k and the delay time close to the mean oscillation period; here the rotation of the phase point along the trajectory of the Lorenz system becomes highly coherent.

Another representation of the effect of the delayed feedback on the coherence of the process is given by the power spectrum. Indeed, the power spectrum of an oscillatory observable has a peak at frequency Ω_0, and the width of the peak is proportional to the diffusion constant D. In Fig. 2.3 we show how the feedback changes the spectrum of the Lorenz system for the cases of maximal enhancement and maximal suppression of the diffusion constant. In this figure we also demonstrate that the effect is not related to suppression of chaos: large variations of the diffusion constant (more than 10 times) are not reflected in the topology of the strange attractor; also the calculated Lyapunov exponents are very close to those without feedback (Fig. 2.4). This suggests that the effect

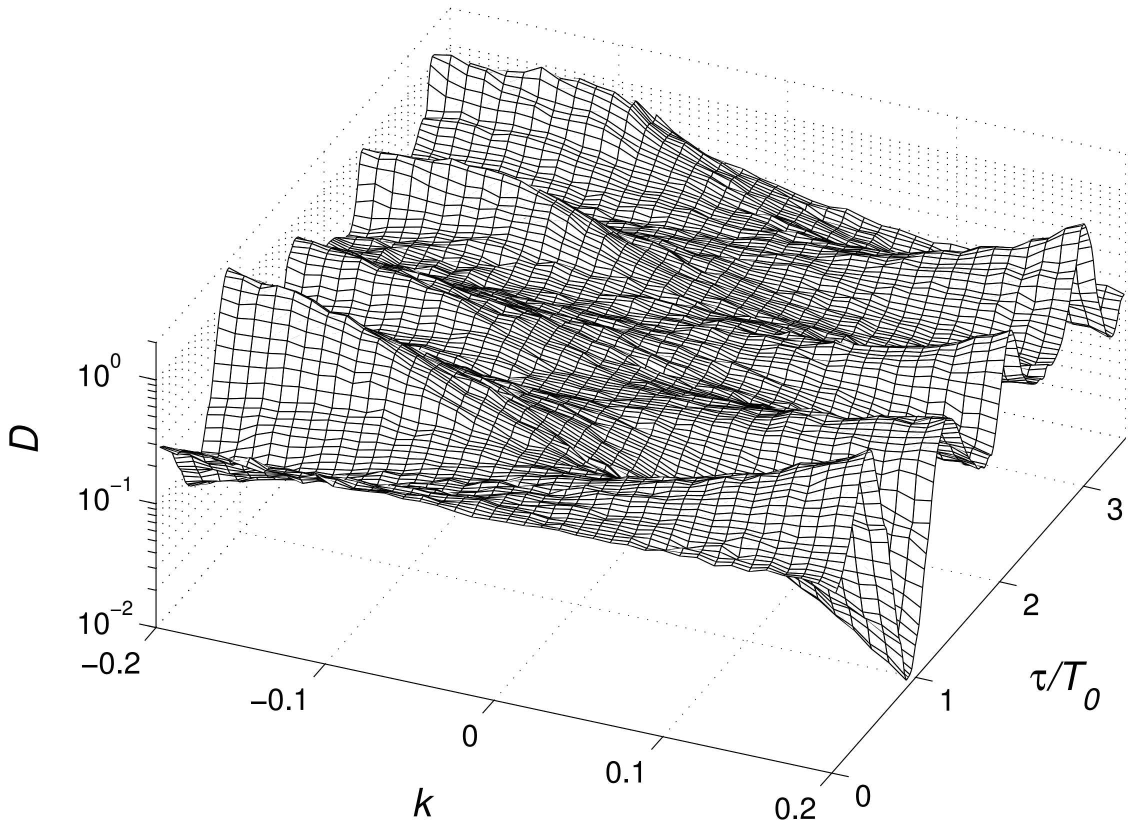

Figure 2.2: Diffusion constant D for the Lorenz system (2.2) as the function of τ/T_0 and k. $T_0 \approx 0.69$ is the average oscillation period without delay. Note the logarithmic scale of the D-axis.

of feedback on the coherence can be described in the framework of phase approximation to the dynamics (this approximation has been used in [25] to describe phase synchronization of chaotic oscillators).

One of the implications of the coherence control is a possibility to govern synchronization properties of an oscillator. Indeed, the ability of an oscillator to be entrained directly depends on the phase diffusion constant, thus improving coherence means improving of the synchronization ability [1]. We illustrate this by consideration of the phase synchronization of the Lorenz system by a periodic force $E \sin \nu t$ added to the equation for the variable z (Fig. 2.5). In the absence of the feedback the force is too weak to entrain the system, while the coherent oscillator demonstrates synchronization.

2.2 Basic phase model

According to a general theory (see, e.g., [28]), external force acting on a limit cycle oscillator in the first approximation affects the phase variable, but not the amplitudes, because the phase is free and can be adjusted by a very weak action, while the amplitude

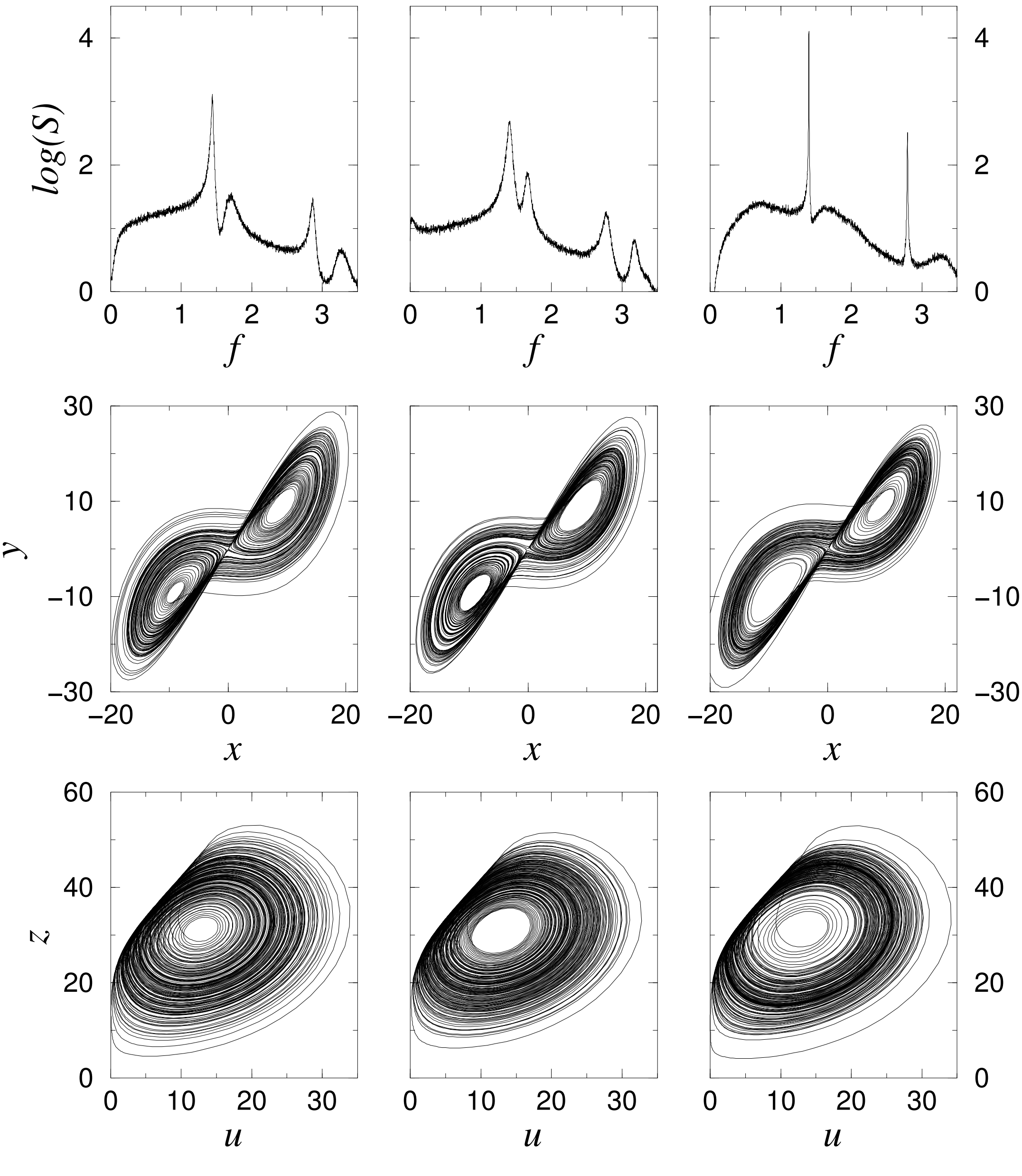

Figure 2.3: Spectra of the z component of the Lorenz system and projections of the phase portrait for the system in the absence of delayed feedback (left column) and in the presence of feedback with delay $\tau = 0.3$ (middle column) and $\tau = 0.65$ (right column); feedback strength $k = 0.2$. Note that feedback makes the spectral peak essentially more broad (enhanced diffusion, middle column) or more narrow (suppressed diffusion, right column), whereas practically no changes can be seen in the phase portraits.

variables are stable and thus change only slightly. We follow this idea to derive below our basic theoretical phase model starting from Van der Pol model (2.1) in the case of small nonlinearity $\mu \ll 1$. For small feedback and noise we can use the perturbation theory,

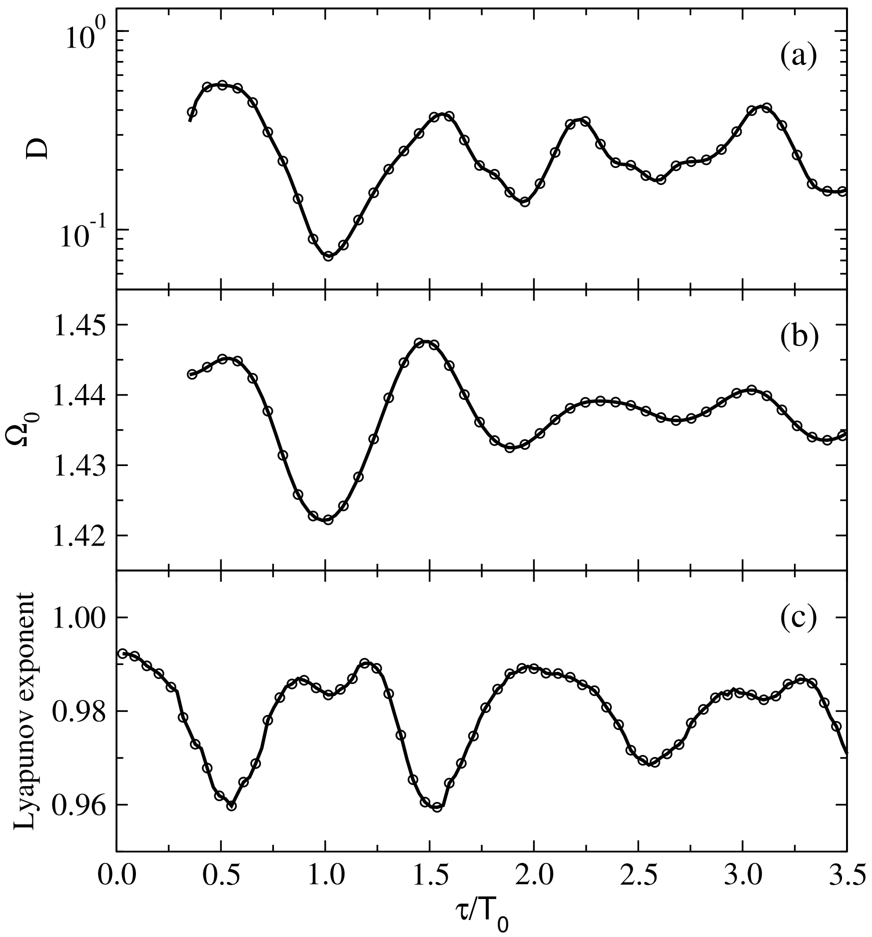

Figure 2.4: Effect of delayed feedback with strength $k = 0.1$ on the miscellaneous properties of the Lorenz system with $\sigma = 10$, $r = 32$, and $b = 8/3$ ($T_0 \approx 0.69$). Noteworthy, while the diffusion constant D varies by 10 times (a), the variation altitudes of the mean frequency (b) and the Lyapunov exponent (c) are 2% and 4%, respectively.

valid in the vicinity of the limit cycle (see, e.g., [28, 1]). We rewrite Eq. (2.1) as a system

$$\dot{x} = \Omega_0 y \; , \qquad \dot{y} = -\Omega_0 x + \mu(1-x^2)y + k(y(t-\tau) - y(t)) + \frac{1}{\Omega_0}\zeta(t)$$

and obtain according to [28, 1]

$$\dot{\phi} = \Omega_0 + \frac{\partial \phi}{\partial y_0}[k(y_0(t-\tau) - y_0(t)) + \frac{1}{\Omega_0}\zeta(t)] \; ,$$

where $x_0 = 2\cos\phi$, $y_0 = -2\sin\phi$ is the limit cycle solution related to the phase as $\phi = -\arctan(y_0/x_0)$; therefore $\dfrac{\partial \phi}{\partial y_0} = -\dfrac{x_0}{x_0^2 + y_0^2}$. Substituting the variables x_0, y_0 on the r.h.s. by ϕ we obtain

$$\dot{\phi} = \Omega_0 + k(\sin\phi(t-\tau) - \sin\phi(t))\cos(\phi(t)) + \frac{1}{2\Omega_0}\zeta(t)\cos(\phi) \; . \qquad (2.3)$$

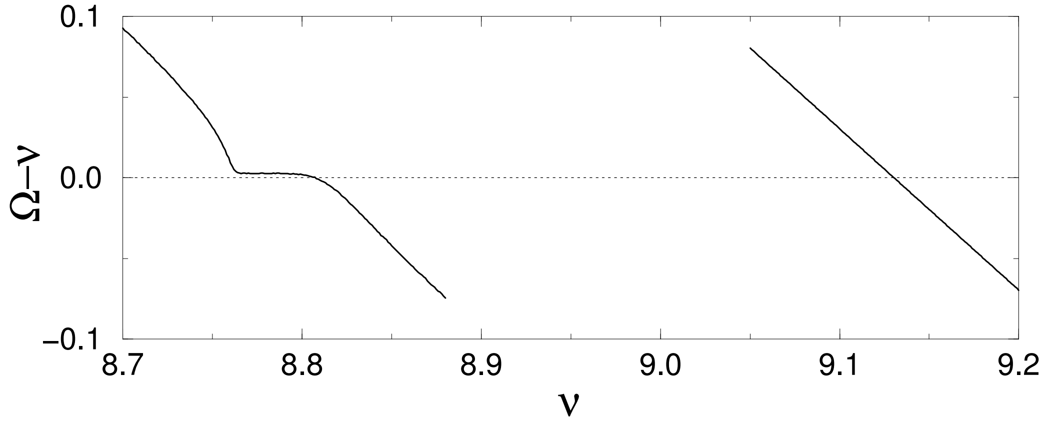

Figure 2.5: Entrainment of the Lorenz system by a harmonic force with $E = 2$. Right graph: without feedback the mean oscillator frequency Ω is not locked to the driving frequency ν. Left graph: the feedback with $k = 0.2$, $\tau = 0.65$ makes the oscillator coherent, what results in the appearance of the synchronization region $\Omega \approx \nu$ (cf. [26, 27]). Note also that the mean frequency is shifted by the feedback; this effect is theoretically explained below.

We are mostly interested in the long-time behavior of the phase, therefore we average the r.h.s. over the period of oscillations. As a result, the r.h.s. contains only the terms depending on the phase differences. Next, we use that ζ is delta-correlated and independent of ϕ, so that

$$\langle \zeta(t)\zeta(t') \cos\phi(t) \cos\phi(t') \rangle \approx \langle \zeta(t)\zeta(t') \rangle \langle \cos\phi(t) \cos\phi(t') \rangle = d^2 \delta(t-t') \ .$$

Finally we obtain our basic phase equation

$$\dot{\phi} = \Omega_0 + a(\sin(\phi(t-\tau) - \phi(t)) + \xi(t) \ , \tag{2.4}$$

where $a = k/2$ is the renormalized strength of the feedback and $\xi(t)$ is the effective noise satisfying $\langle \xi(t)\xi(t') \rangle = \frac{d^2}{4\Omega_0^2}\delta(t-t')$.

We emphasize that, although we derived Eq. (2.4) for the Van der Pol equation, a similar equation can be obtained for any limit cycle oscillator (if the assumption of weak perturbations is valid) – the only difference may be in a more complex dependence on the phase difference, containing not only one sine function, but its harmonics as well. Moreover, as the phase dynamics of chaotic oscillators is qualitatively similar to the dynamics of noisy periodic oscillators (see [1]), Eq. (2.4) can serve as a model for chaotic oscillators in the presence of the feedback loop. In the latter case the term $\xi(t)$ reflects the irregularity of chaotic amplitudes. Note that Eq. (2.4) has been used in [21] to describe the evolution of the phase of an optical field in a laser with a weak optical feedback.

2.3 Statistical analysis of the phase model

As the first step in the theoretical analysis of the model equation (2.4) we separate the phase growth into the average growth and the fluctuations, according to $\phi = \Omega t + \psi$, where Ω is the unknown mean frequency and ψ is the slow phase. For the fluctuating instantaneous frequency $v(t) = \dot{\psi}$ we obtain from Eq. (2.4)

$$v(t) = \Omega_0 - \Omega + \xi(t) - a \sin \Omega\tau \cos\left[\psi(t-\tau) - \psi(t)\right] + a \cos \Omega\tau \sin\left[\psi(t-\tau) - \psi(t)\right] \ . \quad (2.5)$$

In the following we analyze this equation using different approximations.

2.3.1 Noise-free case: multistability in oscillation frequency

We begin our consideration with the noise-free case, $\xi = \psi = v = 0$, when Eq. (2.5) reduces to

$$\Omega + a \sin \Omega\tau = \Omega_0 \ . \quad (2.6)$$

Thus, the delayed feedback changes the frequency of the oscillator. The transcendent Eq. (2.6) has a unique solution for any Ω_0, if $|a\tau| < 1$, and multiple solutions otherwise. The latter case is especially difficult and will be considered elsewhere. (Numerical simulation of the effect of the noise on the multistable states in Eq. (2.4) was performed in [21].) Below we will consider a situation with weak delayed feedback only, when no multistability occurs. We will also show that noise can destroy multistability, so that in its presence the condition $|a\tau| < 1$ can be weakened [see Eq. (2.11) below].

2.3.2 Linear approximation

Here we assume that the fluctuations of the phase are small, i.e. $\psi(t) - \psi(t-\tau) \ll 2\pi$. In this first order in ψ approximation we obtain from (2.5) with account of Eq. (2.6)

$$v(t) = \dot{\psi} = \xi(t) + a \cos \Omega\tau (\psi(t-\tau) - \psi(t)) \ , \quad (2.7)$$

where Ω is a solution of (2.6). This linear equation can be easily solved in the Fourier domain:

$$f_v(\omega) = \frac{f_\xi(\omega)}{1 - a \cos \Omega\tau \dfrac{e^{-i\omega\tau} - 1}{i\omega}} \ ,$$

where $v(t) = \int_{-\infty}^{+\infty} f_v(\omega) e^{i\omega t} d\omega$, $\xi(t) = \int_{-\infty}^{+\infty} f_\xi(\omega) e^{i\omega t} d\omega$. As a result the power spectrum of frequency fluctuations $S_v(\omega)$ can be related to the power spectrum of noise $S_\xi(\omega)$ (note that no further assumption on the noise statistics is needed):

$$S_v(\omega) = \frac{\omega^2 S_\xi(\omega)}{\omega^2 + 2\omega a \sin \omega\tau \cos \Omega\tau + 2(1 - \cos \omega\tau) a^2 \cos^2 \Omega\tau} \ .$$

The diffusion constant can be obtained by considering the limit $\omega \to 0$:

$$S_v(0) = \frac{S_\xi(0)}{(1 + a\tau \cos \Omega\tau)^2} .$$

Thus, the diffusion constant $D = 2\pi S_v(0)$ is obtained in the linear approximation as

$$D = \frac{D_0}{(1 + a\tau \cos \Omega\tau)^2} , \tag{2.8}$$

where $D_0 = 2\pi S_\xi(0)$ is the diffusion of the "no control" oscillator.

Below we will obtain a more precise expression for the diffusion constant, however the simple formula (2.8) allows us to give a qualitative explanation of the numerical results presented in Figs. 2.1,2.2. As it follows from (2.8), the feedback term can compensate or amplify the fluctuations in the phase growth, in dependence on the sign of the product $a \cos \Omega\tau$ (for small feedback this term can be estimated as $a \cos \Omega_0\tau$), because this product appears in Eq. (2.7) as the effective strength of the feedback regulating the fluctuations of the phase. This explains the oscillatory dependence of the diffusion constant on the delay time τ.

2.3.3 Gaussian approximation

Our main statistical approach in the treatment of full nonlinear Eq. (2.4) is based on the Gaussian approximation for $\psi(t)$. We also assume the noisy term $\xi(t)$ to be Gaussian. However, contrary to the numerical simulation, where the noise is white, we consider a general spectrum of the noise. Averaging Eq. (2.5) for the fluctuations of the instantaneous frequency $v(t) = \dot{\psi}$ (which is also Gaussian), we come to the equation for the mean frequency Ω:

$$0 = \Omega_0 - \Omega - a \sin \Omega\tau \left\langle \cos[\psi(t-\tau) - \psi(t)] \right\rangle . \tag{2.9}$$

The phase difference $\eta(t) = \psi(t-\tau) - \psi(t)$ is Gaussian with zero average, hence $\langle \cos \eta \rangle = \exp[- \langle \eta^2 \rangle /2]$. The phase difference η can be represented as an integral of the instantaneous frequency

$$\eta(t) = - \int_{t-\tau}^{t} v(s) ds ,$$

what gives for the variance of η

$$\left\langle \eta^2 \right\rangle = 2 \int_0^\tau (\tau - s) V(s) ds \equiv 2R . \tag{2.10}$$

Here we have introduced the autocorrelation function of the instantaneous frequency

$$V(u) = \langle v(t) v(t+u) \rangle .$$

Using the notation introduced in Eq. (2.10) we rewrite Eq. (2.9) for the average frequency as

$$\Omega = \Omega_0 - ae^{-R} \sin \Omega\tau \ . \tag{2.11}$$

We note that it is similar to Eq. (2.6), but contains an additional factor e^{-R} which describes the mentioned above partial suppression of the effect of the delayed feedback due to phase diffusion.

To obtain equations for the autocorrelation function $V(u)$ we introduce also the autocorrelation function of the noise $C(u)$ and the cross-correlation function $S(u)$, defined according to

$$C(u) = \langle \xi(t)\xi(t+u) \rangle \ , \qquad S(u) = \langle \xi(t)v(t+u) \rangle \ .$$

Let us multiply Eq. (2.5) with $v(t+u)$ and $\xi(t+u)$ and average:

$$V(u) = \langle v(t)v(t+u) \rangle = \langle \xi(t)v(t+u) \rangle - a \sin \Omega\tau \left\langle v(t+u) \cos \left(\int_{t-\tau}^{t} v(s)ds \right) \right\rangle$$
$$- a \cos \Omega\tau \left\langle v(t+u) \sin \left(\int_{t-\tau}^{t} v(s)ds \right) \right\rangle \ ,$$

$$S(u) = \langle v(t)\xi(t+u) \rangle = \langle \xi(t)\xi(t+u) \rangle - a \sin \Omega\tau \left\langle \xi(t+u) \cos \left(\int_{t-\tau}^{t} v(s)ds \right) \right\rangle$$
$$- a \cos \Omega\tau \left\langle \xi(t+u) \sin \left(\int_{t-\tau}^{t} v(s)ds \right) \right\rangle \ .$$

To accomplish the averaging we use the Furutsu-Novikov formula [29, 30] for zero-mean Gaussian variables x, y:

$$\langle xF(y) \rangle = \langle F'(y) \rangle \langle xy \rangle \ .$$

For the case under consideration this means that averages of all terms having the form $\langle x \cos y \rangle$ vanish, while other terms of type $\langle x \sin y \rangle$ yield

$$\left\langle v(t+u) \sin \left(\int_{t-\tau}^{t} v(s)ds \right) \right\rangle = \left\langle \cos \left(\int_{t-\tau}^{t} v(s)ds \right) \right\rangle \left\langle v(t+u) \int_{t-\tau}^{t} v(s)ds \right\rangle =$$
$$= e^{-R} \int_{-\tau}^{0} V(s-u)ds \ ,$$

$$\left\langle \xi(t+u) \sin \left(\int_{t-\tau}^{t} v(s)ds \right) \right\rangle = \left\langle \cos \left(\int_{t-\tau}^{t} v(s)ds \right) \right\rangle \left\langle \xi(t+u) \int_{t-\tau}^{t} v(s)ds \right\rangle =$$
$$= e^{-R} \int_{-\tau}^{0} S(s-u)ds \ .$$

This leads to the equations:

$$V(u) = S(u) - ae^{-R}\cos\Omega\tau \int_0^\tau V(s+u)ds \ , \qquad (2.12)$$

$$S(u) = C(u) - ae^{-R}\cos\Omega\tau \int_0^\tau S(u-s)ds \ . \qquad (2.13)$$

Together with Eq. (2.11) and the definition of quantity R given by Eq. (2.10), they constitute a closed system.

To proceed it is convenient to consider the spectra according to

$$\mathcal{V}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du V(u) e^{-i\omega u} \ ,$$

and similarly for $\mathcal{S}$ and $\mathcal{C}$. Then Eqs. (2.12), (2.13) yield

$$\mathcal{V}(\omega) = \mathcal{S}(\omega) - ae^{-R}\cos\Omega\tau \frac{e^{i\omega\tau}-1}{i\omega} \ , \qquad (2.14)$$

$$\mathcal{S}(\omega) = \mathcal{C}(\omega) - ae^{-R}\cos\Omega\tau\mathcal{S}(\omega)\frac{1-e^{-i\omega\tau}}{i\omega} \ , \qquad (2.15)$$

what allows us to exclude $\mathcal{S}(\omega)$ and obtain

$$\mathcal{V}(\omega) = \frac{\mathcal{C}(\omega)}{1 + 2a\tau e^{-R}\cos\Omega\tau\frac{\sin\omega\tau}{\omega\tau} + a^2\tau^2 e^{-2R}\cos^2\Omega\tau\frac{2-2\cos\omega\tau}{\omega^2}} \ . \qquad (2.16)$$

Eq. (2.10) in the spectral form reads

$$R = \int_{-\infty}^{\infty} \frac{1-\cos\omega\tau}{\omega^2}\mathcal{V}(\omega)d\omega \ ; \qquad (2.17)$$

here we have used that $\mathcal{V}(\omega)$ is an even function. The system (2.16), (2.17) is still hard to solve in the general form, due to integration in (2.17).

The quantity of our main interest is the diffusion constant D of the phase ψ. D is related to the spectral density of the frequency fluctuations at zero frequency: $D = 2\pi\mathcal{V}(0)$. Using Eq. (2.16) we obtain for this quantity:

$$D = \frac{D_0}{(1 + a\tau e^{-R}\cos\Omega\tau)^2} \ , \qquad (2.18)$$

where $D_0 = 2\pi\mathcal{C}(0)$ is the "no control" diffusion constant in the absence of the feedback. To obtain a closed system for the determination of D we further assume that the spectrum of the frequency fluctuations $\mathcal{V}(\omega)$ is very broad. One can expect this if the spectrum of noise $\mathcal{C}(\omega)$ is broad, i.e. if the noise is nearly δ-correlated. More precisely, we assume that the correlation time of frequency fluctuation is much smaller than the delay time τ, so that the integral (2.17) can be approximated as

$$R \approx \int_{-\infty}^{\infty} \frac{1-\cos\omega\tau}{\omega^2}\mathcal{V}(0)d\omega = \frac{\tau D}{2} \ . \qquad (2.19)$$

As a result we obtain a closed system of equations – the main result of our analysis –

$$D = \frac{D_0}{(1 + a\tau e^{-\frac{\tau D}{2}} \cos\Omega\tau)^2} \,, \tag{2.20}$$

$$\Omega = \Omega_0 - a e^{-\frac{\tau D}{2}} \sin\Omega\tau \,, \tag{2.21}$$

relating the diffusion constant D in the presence of the feedback to the "no control" diffusion constant D_0 and to the parameters of the feedback τ and a, as well as to the "no control" frequency Ω_0. This is a nonlinear system of two equations for two variables D and Ω, which can be solved numerically for a given set of parameters. In the case of small noise, $D_0\tau \ll 1$, we can set $e^{-\frac{\tau D}{2}} \approx 1$ and end with Eqs. (2.6), (2.8), obtained above in the linear approximation.

Another useful approximation is that of small feedback, then we can approximate the diffusion constant in (2.19) by its "no control" value, this gives

$$D = \frac{D_0}{(1 + a\tau e^{-\frac{\tau D_0}{2}} \cos\Omega\tau)^2} \,, \qquad \Omega = \Omega_0 - a e^{-\frac{\tau D_0}{2}} \sin\Omega\tau \,. \tag{2.22}$$

Now only the equation for Ω is implicit, while the diffusion constant depends on the parameters in an explicit way.

We compare the theoretical results given by Eqs. (2.20), (2.21) with the direct numerical simulations in Figs. 2.6, 2.7. In Fig. 2.6 we present numerical results for the phase model (2.4). The presented case of relatively strong noise demonstrates a good correspondence with theory. Furthermore, one can see that the effect of delayed feedback decreases with τ, because of the diffusion. Physically, it can be explained as follows. The feedback either compensates or amplifies the deviations from the uniform phase growth. If the diffusion constant is large, then during a large delay time the phases $\phi(t)$ and $\phi(t-\tau)$ are practically uncorrelated, thus the feedback reduces to a random term which neither compensates nor amplifies the fluctuations.

Figure 2.7 demonstrates the results for the Van der Pol model (2.1). The only parameter we have fitted here is the "no control" frequency $\Omega_0 \approx 0.95$. Here the correspondence with theory is good for small τ, but fails for large τ. The reason is that in this case the effective noise is small and therefore the feedback control is effective even for large delays. However, for large $a\tau$ Eq. (2.21) exhibits multistability, what results in an enhancement of the diffusion; here neither the linear approximation for small noise [Eqs. (2.6), (2.8)] nor the Gaussian approximation used in derivation of (2.20), (2.21) is valid.

2.4 Summary and discussion

In summary, we have presented the effect of the coherence control by means of the delayed feedback. The control is possible for noisy limit cycles oscillators as well as for chaotic

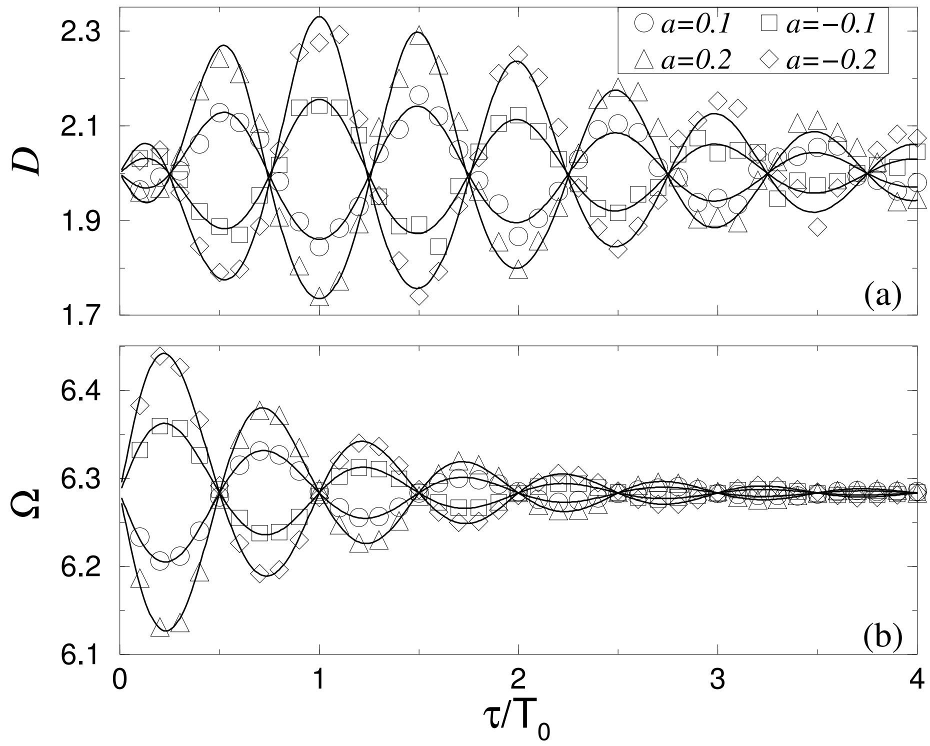

Figure 2.6: Diffusion constant D (a) and mean frequency Ω (b) as functions of delay τ for the model (2.4) with $\langle \xi(t)\xi(t+t')\rangle = 2\delta(t')$ and $\Omega_0 = 2\pi$, and different values of feedback strength. Symbols present the results of the direct numerical simulation of the model (2.4); solid lines shows theoretical results according to Eqs. (2.20), (2.21).

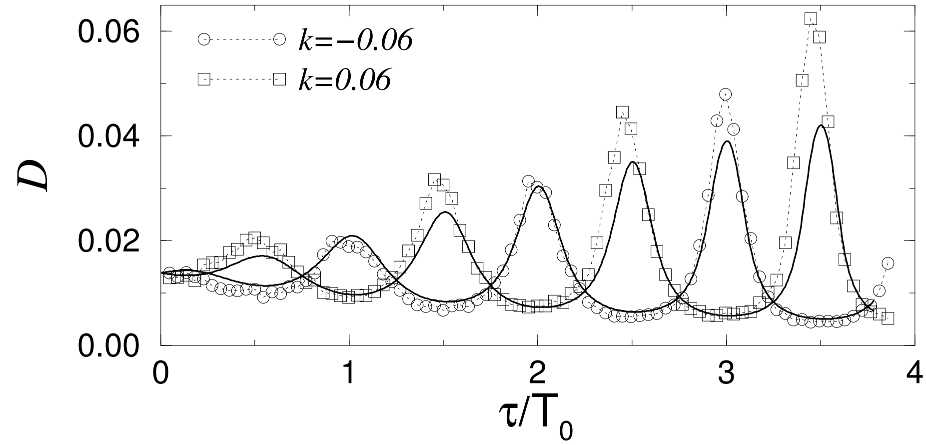

Figure 2.7: Diffusion constant D of the Van der Pol model with delayed feedback [parameters are the same as in Fig. (2.1)]. Symbols present the results of the direct numerical simulation; solid lines show the corresponding theoretical results according to Eqs. (2.20), (2.21). The delay time is normalized by the average period $T_0 = 2\pi/0.95$.

systems, admitting computation of the phase. Next, we have developed a statistical theory of phase diffusion under the influence of a delayed feedback. Using the Gaussian approximation, we have derived a closed system of equations for the diffusion constant and the mean frequency for the case of short-time correlations of the instantaneous frequency. The theory works if the feedback is not very strong, or if the noise is strong enough to suppress multistability in mean frequency. An opposite situation, where effects of multistability are dominant, will be considered elsewhere.

Noteworthy, formally the equations describing the control are the same as in the Pyragas method of chaos control. However, in our case the delay time τ is not necessarily equal to the period of some unstable cycle, embedded in chaos. Moreover, we consider the situation when the feedback is so small that no stabilization of periodic orbits occur. For the Lorenz system, e.g., such a stabilization by the simplest Pyragas method is not possible for some cycles due to a special symmetry of the system. The main difference to the Pyragas approach is that we do not intend to suppress chaos, but to control uniformity – coherence – of phase growth in a chaotic system.

Note also that our method differs from other possibilities to control the diffusion properties of the phase. For example, synchronization of oscillations by a periodic external force reduces or even completely suppresses the diffusion (the relevant model is the noisy Adler equation [1], or, equivalently, an equation of motion of an overdamped noise-driven particle in a periodic potential, see [31, 32] for calculation of the diffusion for the latter problem). In our method no periodic force is needed and the system remains autonomous, preserving full symmetry with respect to time shifts. In other words, the power spectrum of the delay-controlled oscillations does not contain delta-peaks but is continuous.

A direction of the future development of this work is aimed at detailed understanding of the particular features of the control of chaotic systems. Indeed, in this case our theory provide only qualitative explanation of the effect. This limitation of the theory is related to the statistical properties of the effective noise in a chaotic system that definitely cannot be considered as weak or Gaussian. (We remind that effective noise here describes the effect of irregular, although deterministic, amplitudes, on the phase dynamics.) Particularly, it is known that for the Lorenz system this noise is not symmetric and possesses nontrivial correlation properties [26, 27]. Our preliminary numerical investigations show that the feedback significantly affects these correlations. We illustrate this in the Fig. 2.8, where we present the autocorrelation function of the Poincaré return times in the Lorenz system. It is seen that for the case of feedback with $\tau = 0.65 \approx T_0$, the successive return times become essentially anticorrelated, what apparently accounts for unusually high (by factor ≈ 30) suppression of the phase diffusion. We have demonstrated that this effect is of particular importance for the control of synchronization. In fact, the delayed feedback has a twofold effect on synchronization properties. On one hand, the feedback shifts the

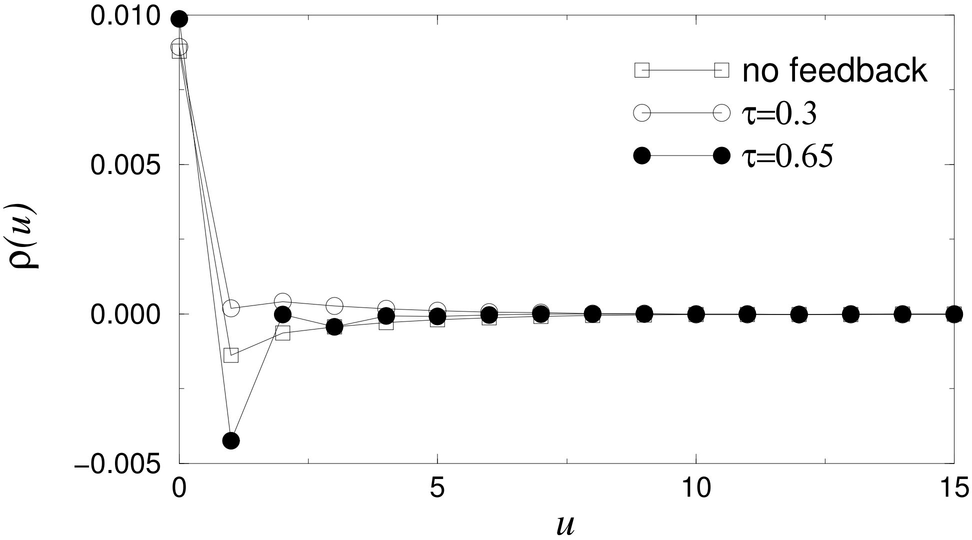

Figure 2.8: Correlation functions $\rho(u)$ for the sequences of the Poincaré return times in the Lorenz system, in the absence and in the presence of the delayed feedback with $k = 0.2$. Note that variances of the return times, given by $\rho(0)$, are practically unchanged, whereas the anticorrelation between two successive intervals is either decreased (for $\tau = 0.3$) or increased (for $\tau = 0.65$).

oscillation frequency, thus giving a possibility to facilitate or impede the entrainment (this effect is important for periodic oscillators as well). On the other hand, synchronization can be suppressed or enhanced by the regulation of the coherence.

Chapter 3

Synchronization of Oscillators by Common Noise

The main effect of noise on periodic self-sustained oscillations is phase diffusion: the oscillations are no more periodic but possess finite correlations [33, 34]. However, noise can play also an ordering role, e.g., it can lead to a synchronization. If two identical (or slightly different) systems are driven by the same noise, then their states can be synchronized by this action. This effect depends on the sign of the largest LE which measures the exponential growth rate of trajectory perturbations by the given realization of noisy signal. For autonomous deterministic periodic self-sustained oscillations the largest LE is zero, it corresponds to a perturbation along the trajectory in the phase space, and there are no synchronization. In driven systems the largest LE may become negative, what would lead to a synchronization: both systems driven by the same noise forget their initial conditions and eventually evolve to a same state.

There are several fields where this effect has been observed, although under different names. In neurophysiology one describes identical responses of a neuron to a repeated noisy driving of a prerecorded waveform as "reliability" [37, 38, 39, 40]. In recent experiments with noise-driven neodymium-doped yttrium aluminum garnet (Nd:YAG) lasers [41, 42] this synchronization was called "consistency". When the driving is not noisy but chaotic, one speaks on generalized synchronization [43, 42].

This problem was first formulated in [44, 45], where the LE has been calculated for a self-sustained quasiharmonic oscillator driven by a random sequence of pulses (recently, such kind of noise again has attracted attention, and the problem of phase synchronization between uncoupled limit-cycle oscillators induced by common random impulsive forcing has been anew addressed in [46]). In this chapter, we consider general dynamical systems driven by Gaussian white noise (some results are reproduced for telegraph noise). We note that the effect considered is a particular case of synchronization in noisy systems (more general aspects of this phenomenon are presented in [47, 48]).

We start with developing an analytical description of noisy self-sustained oscillators (limit cycle systems) [5, 6]. The approach used is based on the reduction of the dynamics to a phase equation. This is valid if the action of noise on the oscillation amplitude is small. We derive the Langevin equation for the phase and find a stationary distribution of it. The LE is represented via an integral of this distribution. In particular, we explicitly demonstrate that for small noise the exponent is negative, i.e., weak noise always leads to synchrony.

While in the perfect case of identical oscillators and identical driving, a negative LE leads to perfect synchrony of systems, in real situations this perfect synchrony is distorted by nonidentities. We pay a particular attention to non-perfect situations [6, 8], when there is small nonidentity either (i) in systems subject to identical noisy driving or (ii) in noise driving identical systems (intrinsic noise). The analytical results obtained within the framework of the phase approximation (for weak noise) are underpinned by numerical simulation.

For a finite noise strength, an analytical investigation of the problem is generally not possible and numerical simulation is needed. This simulation discovers a positive LE to appear for some systems under certain circumstances [6, 7, 8], what means desynchronization of oscillators.

In the main part of investigation we assume noise to be white Gaussian [5, 6, 8]. Another standard kind of noise remarkably different to white Gaussian one in its properties is a telegraph one. The diversity between these kinds of noise gives opportunity to check wether the effects observed are robust and general or specific for white Gaussian noise. Moreover, telegraph noise has an evident periodic analogue: a piece-constant 2τ-periodic signal, where τ is the mean switching time for corresponding telegraph noise. Comparison of effects of these two piece-constant signals on self-sustained oscillators could give an additional insight to the mechanism of synchronization by common noise, and has been performed [7]. An additional investigation of a similar problem can be also found in [49].

As mentioned above, one of important applications of the theory is "reliability" of neurons [39]. In our terms, a "reliable" neuron is a neuron with a negative LE. As neuronal oscillators possess some specific properties, we reproduce for them our investigation performed for limit cycle oscillators [8]. The effects are characterized not only via calculations of the LE, but also the event synchronization correlations [61]. We construct a theory that explains the observed in numerics "antireliability" as a combined effect of a high sensitivity to noise of some stages of the dynamics and nonisochronicity of oscillations. Geometrically, the antireliability is described by a random noninvertible one-dimensional map.

3.1 Limit cycle systems: White Gaussian noise

3.1.1 Phase approximation

We start with general stochastic equations for the dynamics of an N-dimensional oscillatory system x_j, $j = 1, \ldots, N$, in presence of uncorrelated forces $\xi_k(t)$, $k = 1, \ldots, M \le N$:

$$\frac{dx_j}{dt} = f_j(\mathbf{x}) + \sum_{k=1}^{M} Q_{jk}(\mathbf{x})\xi_k(t) \ . \tag{3.1}$$

If in the noiseless system there exists a limit cycle $\mathbf{x}^0 = \mathbf{x}^0(t + 2\pi/\omega_0)$, it can be parameterized by the phase variable $\varphi(\mathbf{x}^0)$ [28] which grows linearly in time: $\dot{\varphi} = \omega_0$. For a stable limit cycle the phase, satisfying the same equation, can be introduced also in its vicinity. In presence of noise the evolution of the phase in a small vicinity of the cycle is governed by equations

$$\frac{d\varphi}{dt} = \omega_0 + \sum_{j=1}^{N}\sum_{k=1}^{M} \frac{\partial\varphi(\mathbf{x})}{\partial x_j} Q_{jk}(\mathbf{x})\bigg|_{\mathbf{x}=\mathbf{x}^0(\varphi)} \xi_k(t). \tag{3.2}$$

The deviations from the cycle are small in two cases: (i) if the noise intensity is small, or (ii) if the leading negative Lyapunov exponent (LE) is large whereas the noise is moderate. Below we normalize time in such a way that the frequency of the limit cycle is one. A particular form of the stochastic equation for the phase depends on how the noise enters the original system (3.1). If there is a single noise source, i.e. only for one k $\xi_k \neq 0$, then

$$\frac{d\varphi}{dt} = 1 + \varepsilon f(\varphi)\xi(t), \tag{3.3}$$

where $\xi(t)$ is a δ-correlated Gaussian noise with $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t'+t)\rangle = 2\delta(t')$, parameter ε describes the noise intensity (as a result of time normalization $\varepsilon \sim \omega_0^{-1/2}$). $f(\varphi)$ is a normalized periodic function of the phase: $f(\varphi) = f(\varphi + 2\pi)$, $\int_0^{2\pi} f^2(\varphi)d\varphi = 2\pi$. A more complex equation appears if there are several noise sources in the original system (we call this multi-component case):

$$\frac{d\varphi}{dt} = 1 + \sum_{k=1}^{M} \varepsilon_k\, f_k(\varphi)\,\xi_k(t), \qquad \langle\xi_i(t)\,\xi_k(t'+t)\rangle = 2\,\delta(t')\,\delta_{ik}. \tag{3.4}$$

Our goal in this section is the analytical analysis of stability of solutions of stochastic equations (3.3), (3.4) (cf. [50]). For this we consider the linearized Eq. (3.3) for a small deviation α:

$$\frac{d\alpha}{dt} = \varepsilon\alpha f'(\varphi)\xi(t). \tag{3.5}$$

The LE measuring the average exponential growth rate of α can be obtained by averaging the corresponding velocity

$$\lambda = \langle \frac{d}{dt} \ln \alpha \rangle = \langle \varepsilon f'(\varphi) \xi(t) \rangle. \tag{3.6}$$

For the multi-component noise the corresponding expression reads

$$\lambda = \sum_{k=1}^{M} \langle \varepsilon_k f_k'(\varphi) \xi_k(t) \rangle. \tag{3.7}$$

Note that the LE determines the asymptotic behavior of small perturbations, and in our case describes whether close initial points diverge or converge in course of the evolution. This process must not be monotonous, i.e. close trajectories can diverge at some time intervals while demonstrating asymptotic convergence, and vice versa.

3.1.2 Fokker-Planck equation and its stationary solution

The Fokker-Planck equation for the stochastic Eq. (3.3), interpreted in Stratonovich sense, reads [51, 52]

$$\frac{\partial W(\varphi, t)}{\partial t} + \frac{\partial}{\partial \varphi} \left(W(\varphi, t) - \varepsilon^2 f(\varphi) \frac{\partial}{\partial \varphi} \left(f(\varphi) \, W(\varphi, t) \right) \right) = 0. \tag{3.8}$$

In a stationary state the probability flux S is constant:

$$W(\varphi) - \varepsilon^2 f(\varphi) \frac{d}{d\varphi} \left(f(\varphi) \, W(\varphi) \right) = S. \tag{3.9}$$

This allows us to express the solution for periodic boundary conditions as

$$W(\varphi) = C \int_{\varphi}^{\varphi + 2\pi} \frac{d\psi}{f(\varphi) \, f(\psi)} \exp \left(- \frac{1}{\varepsilon^2} \int_{\varphi}^{\psi} \frac{d\theta}{f^2(\theta)} \right), \tag{3.10}$$

where C is determined by the normalization condition:

$$C^{-1} = \int_{0}^{2\pi} d\varphi \int_{\varphi}^{\varphi + 2\pi} d\psi \, \frac{\exp \left(- \frac{1}{\varepsilon^2} \int_{\varphi}^{\psi} \frac{d\theta}{f^2(\theta)} \right)}{f(\varphi) \, f(\psi)} . \tag{3.11}$$

The probability flux reads

$$S = \left(1 - \exp \left(- \frac{1}{\varepsilon^2} \int_{0}^{2\pi} \frac{d\theta}{f^2(\theta)} \right) \right) C. \tag{3.12}$$

The analogous expression for the multi-component noise is

$$\frac{\partial W(\varphi,\, t)}{\partial t} + \frac{\partial}{\partial \varphi}\left(W(\varphi,\, t) - \sum_{k=1}^{M} \varepsilon_k^2\, f_k(\varphi) \frac{\partial}{\partial \varphi}\left(f_k(\varphi)\, W(\varphi,\, t) \right) \right) = 0. \tag{3.13}$$

Remarkably, this formula is equivalent to the single-component one (3.8), if one sets

$$f^2(\varphi) = \frac{\sum\limits_{k=1}^{M} \varepsilon_k^2\, f_k^2(\varphi)}{\sum\limits_{k=1}^{M} \varepsilon_k^2} \qquad\qquad \varepsilon^2 = \sum_{k=1}^{M} \varepsilon_k^2. \tag{3.14}$$

Thus, the stationary solution presented above is valid in this case as well.

3.1.3 Lyapunov exponent

For the calculation of the LE (3.6),(3.7) we have to find averages of the type $\langle F(\varphi)\xi(t)\rangle$. Such expressions for stochastic equations (3.3), (3.4) with delta-correlated noise can be calculated using the Novikov-Furutsu formula:

$$\langle F(\varphi)\xi(t)\rangle = \varepsilon \langle F'(\varphi) f(\varphi)\rangle. \tag{3.15}$$

Writing the average as the integral over the stationary phase distribution we obtain for the single-component case

$$\lambda = \varepsilon^2 \langle f''(\varphi) f(\varphi)\rangle = \varepsilon^2\, C \int_0^{2\pi} d\varphi \int_{\varphi}^{\varphi+2\pi} d\psi\, \frac{\frac{\partial^2 f(\varphi)}{\partial \varphi^2}}{f(\psi)}\, \exp\left(-\frac{1}{\varepsilon^2} \int_{\varphi}^{\psi} \frac{d\theta}{f^2(\theta)} \right). \tag{3.16}$$

The corresponding result for the multi-component noise reads

$$\lambda = \sum_{k=1}^{M} \varepsilon_k^2 \int_0^{2\pi} \frac{d^2 f_k(\varphi)}{d\varphi^2} f_k(\varphi)\, W(\varphi) d\varphi. \tag{3.17}$$

Prior to the analysis of the obtained expressions we mention that in the limit of small noise the LE is always negative: in the leading order in ε

$$\lambda \approx -\sum_{k=1}^{M} \frac{\varepsilon_k^2}{2\pi} \int_0^{2\pi} \left(\frac{d f_k(\varphi)}{d\varphi} \right)^2 d\varphi < 0\,. \tag{3.18}$$

(M can be equal to 1, what corresponds to a single-component noise.)

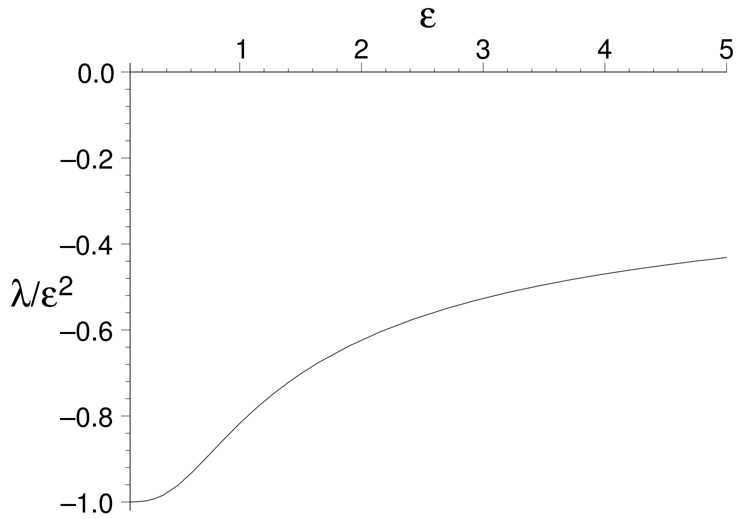

Figure 3.1: Linearly polarized noise. Dependence of the Lyapunov exponent (normalized by ε^2) on the noise amplitude ε. For small and large noises λ_0 quadratically depends on ε, with different coefficients.

Example: Linearly polarized homogeneous noise

If in the original system the noise is additive and forces only one variable of the system, and the limit cycle is nearly a circle with nearly constant phase velocity on it, then one obtains a single-component stochastic phase equation with $f(\varphi) = \sqrt{2}\sin\varphi$. In this case

$$\int_{\varphi}^{\varphi+2\pi} \frac{d\psi}{f(\psi)} \exp\left(-\frac{1}{\varepsilon^2}\int_{\varphi}^{\psi}\frac{d\theta}{f^2(\theta)}\right) = \int_{\varphi}^{\pi+\pi\left[\frac{\varphi}{\pi}\right]} d\psi\, \frac{\exp\left(\dfrac{\cot\psi - \cot\varphi}{2\varepsilon^2}\right)}{\sqrt{2}\sin\psi},$$

where [...] denotes the integer part. For the chosen function $f(\varphi)$ the distribution has period π and for $\varphi \in [0,\, \pi)$

$$W(\varphi) = \frac{C}{2}\int_{\varphi}^{\pi} d\psi\, \frac{\exp\left(\dfrac{\cot\psi - \cot\varphi}{2\varepsilon^2}\right)}{\sin\psi\,\sin\varphi};$$

$$S = C = \left(\int_{-\infty}^{\infty} dy \int_{-\infty}^{y} dx \frac{\exp\left(\dfrac{x-y}{2\varepsilon^2}\right)}{\sqrt{1+x^2}\sqrt{1+y^2}}\right)^{-1},$$

$$\lambda = -\frac{\varepsilon^2\, C}{2}\int_{-\infty}^{\infty} dy \int_{-\infty}^{y} dx \frac{\exp\left(\dfrac{x-y}{2\varepsilon^2}\right)}{(1+x^2)^{\frac{1}{2}}\,(1+y^2)^{\frac{3}{2}}}. \tag{3.19}$$

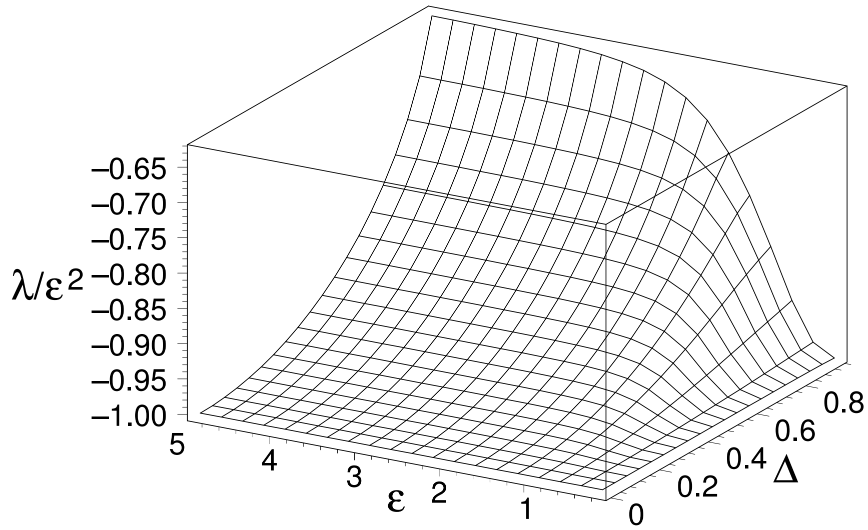

Figure 3.2: Superposition of two independent linearly polarized noise terms. Dependence of the Lyapunov exponent (normalized by ε^2) on the noise amplitude ε and on the normalized ratio of noise intensities Δ.

In the transformation to last expression, which makes the convergence of the integrals clear, the ansatz $(x,\, y) = (\cot\psi,\, \cot\varphi)$ has been used. A further simplification appears to be not possible, and formula (3.19) has been used for numerical calculation. The obtained dependence of the LE on the noise intensity is presented in Fig. 3.1.

Example: superposition of two independent linearly polarized noise terms

If we use the same conditions as above (nearly circular limit cycle with uniform rotation on it) but consider the effect of two independent noisy forces acting on two variables shifted in phase by $\pi/2$, then naturally we get an equation with a multi-component noise with $f_1(\varphi) = \sqrt{2}\sin\varphi$ and $f_2(\varphi) = \sqrt{2}\cos\varphi$. The effective coupling function $f(\varphi) = \sqrt{1+\Delta\cos 2\varphi}$ and the noise intensity $\varepsilon^2 = \varepsilon_1^2 + \varepsilon_2^2$, where $\Delta \equiv \dfrac{\varepsilon_2^2 - \varepsilon_1^2}{\varepsilon_2^2 + \varepsilon_1^2}$ (evidently $\Delta \in [-1,\, 1]$) should be inserted in the stationary distribution (3.10)–(3.12). From the expression for $f(\varphi)$ follows the symmetry $(\Delta,\, \varphi) \leftrightarrow (-\Delta,\, \varphi + \pi/2)$. In this case

$$\int \frac{d\theta}{f^2(\theta)} = \frac{1}{\sqrt{1-\Delta^2}} \left(\pi \left[\frac{\theta}{\pi} \right] - \arctan\left(\cot\theta \sqrt{\frac{1+\Delta}{1-\Delta}} \right) \right),$$

what gives the following expressions for the probability density, flux, and the normalization constant

$$W(\varphi)=C\int\limits_{\varphi}^{\varphi+\pi}d\psi\frac{\exp\left(-\frac{1}{\varepsilon^2\sqrt{1-\Delta^2}}\left(\pi\left[\frac{\theta}{\pi}\right]-\arctan\left(\cot\theta\sqrt{\frac{1+\Delta}{1-\Delta}}\right)\right)\Big|_{\varphi}^{\psi}\right)}{\sqrt{1+\Delta\cos2\varphi}\sqrt{1+\Delta\cos2\psi}},$$

$$C=\left(2\int\limits_{0}^{\pi}d\varphi\int\limits_{\varphi}^{\varphi+\pi}d\psi\frac{\exp\left(-\frac{1}{\varepsilon^2\sqrt{1-\Delta^2}}\left(\pi\left[\frac{\theta}{\pi}\right]-\arctan\left(\cot\theta\sqrt{\frac{1+\Delta}{1-\Delta}}\right)\right)\Big|_{\varphi}^{\psi}\right)}{\sqrt{1+\Delta\cos2\varphi}\sqrt{1+\Delta\cos2\psi}}\right)^{-1},$$

$$S=\left(1-\exp\left(-\frac{\pi}{\varepsilon^2\sqrt{1-\Delta^2}}\right)\right)C.$$

The final expression for the LE reads

$$\begin{aligned}\lambda &= -2\varepsilon^2C\int\limits_{0}^{\pi}d\varphi\int\limits_{\varphi}^{\varphi+\pi}d\psi\,\frac{\sqrt{1+\Delta\cos2\varphi}}{\sqrt{1+\Delta\cos2\psi}}\\ &\times\exp\left(-\frac{1}{\varepsilon^2\sqrt{1-\Delta^2}}\left(\pi\left[\frac{\theta}{\pi}\right]-\arctan\left(\cot\theta\sqrt{\frac{1+\Delta}{1-\Delta}}\right)\right)\Big|_{\varphi}^{\psi}\right).\end{aligned}\tag{3.20}$$

The dependence on the noise intensity and the essential parameter Δ is shown in Fig. 3.2.

3.1.5 Non-perfect cases: Different oscillators

Above we have evaluated the Lyapunov exponent (LE) which completely describes synchrony of oscillators in the perfect case (identical oscillators under identical driving): oscillators are perfectly synchronous for a negative LE. But in real situations this perfect synchrony is distorted by nonidentities in oscillators and noisy forces. Here we consider the role of small nonidentities.

Deriving LE, we do not assume noise to be necessarily small, and note that the phase approximation can be still valid for limit cycle systems with large leading negative LE even for moderate noise. Nevertheless, henceforth, using the phase approximation, we restrict ourselves to the case of weak noise because in this case the approximation used is most reliable.

Within the framework of this approximation the evolution of N slightly different limit cycle oscillators can be described by the following generalization of (3.3)

$$\dot\varphi_j=\omega+\sigma_j+\varepsilon f(\varphi_j)\xi(t),\qquad j=1,2,...,N,\tag{3.21}$$

where σ_j are deviations of frequencies from the mean frequency, $\sum_{j=1}^N\sigma_j=0$. Note that the differences in functions f can be neglected due to smallness of ε. We expect the states of the oscillators to be close if the mismatch is small compared to the LE $|\sigma_j| \ll |\lambda| \ll 1$, then it is appropriate to introduce new variables $\varphi = N^{-1}\sum_{j=1}^{N}\varphi_j$ and $\theta_j = \varphi_j - \varphi$, $j = 1, 2, ..., N-1$. Then system (3.21) for small θ_j can be written as

$$\dot{\varphi} = \omega + \varepsilon f(\varphi)\xi(t) \,, \tag{3.22}$$

$$\dot{\theta}_j = \sigma_j + \varepsilon f'(\varphi)\theta_j\xi(t). \tag{3.23}$$

Noting that the deviations θ_j with different j are independent, we can study the evolution of each deviation θ_j separately and drop index j. Thus the evolution of φ and θ is the same as for two slightly different oscillators.

The following from (3.22), (3.23) Fokker–Plank equation for the probability density distribution $W(\varphi,\, \theta,\, t)$ reads

$$\frac{\partial W}{\partial t} + \omega\frac{\partial W}{\partial \varphi} + \sigma\frac{\partial W}{\partial \theta} - \varepsilon^2\hat{Q}^2W = 0, \tag{3.24}$$

where $\hat{Q}g \equiv \dfrac{\partial}{\partial\varphi}\left(f(\varphi)g\right) + \dfrac{\partial}{\partial\theta}\left(f'(\varphi)\theta\, g\right)$. Performing expansion of the stationary solution in powers of ε^2 [we can here consider σ as of the same order as ε^2, due to a possibility to renormalize θ in (3.24)] we obtain in the zeroth order $W_0 = w(\theta)$ and in the first order

$$\omega\frac{\partial W_1}{\partial\varphi} + \sigma\frac{\partial w}{\partial\theta} - \varepsilon^2\hat{Q}^2w = 0. \tag{3.25}$$

Substituting for $\hat{Q}^2w(\theta)$ and integrating Eq. (3.25) over $\varphi \in [0,\, 2\pi)$, we obtain (due to 2π-periodicity of W_1 in φ)

$$\sigma\frac{dw}{d\theta} = \varepsilon^2\,\overline{f'^2}\left(\theta^2\frac{d^2w}{d\theta^2} + 4\theta\frac{dw}{d\theta} + 2w\right). \tag{3.26}$$

For $\sigma = 0$ the solution of (3.26) is a δ-function. When $\sigma \neq 0$, this equation can be rewritten as

$$x^2\frac{d^2w}{dx^2} + (4x-1)\frac{dw}{dx} + 2w = 0, \tag{3.27}$$

where $x \equiv \varepsilon^2\,\overline{f'^2}\sigma^{-1}\theta = |\lambda|\sigma^{-1}\theta$. Solving this differential equation by virtue of the substitution $w(x) = h(x)/x^2$ and accounting for the normalization condition $\int_{-\infty}^{+\infty}w(\theta)d\theta = (2\pi)^{-1}$, we find

$$w(\theta) = \begin{cases} \dfrac{|\sigma|}{2\pi|\lambda|\theta^2}\exp\left(-\dfrac{\sigma}{|\lambda|\theta}\right), & \sigma\theta > 0; \\ 0, & \sigma\theta \leq 0. \end{cases} \tag{3.28}$$

This function is infinitely smooth at $\theta = 0$. Noteworthy, for any pair of oscillators driven by the same noise, the phase of the faster oscillator never lags behind that of the slower one.

One can also evaluate the moments

$$\langle|\theta|^k\rangle = \left(\frac{|\sigma|}{|\lambda|}\right)^k \Gamma(1-k), \tag{3.29}$$

and the most probable value $\theta_{\rm mp} = \sigma/(2|\lambda|)$. From this formula we see again that the phase difference θ is small provided $|\sigma| \ll |\lambda|$. Formula (3.29) gives finite moments for $k < 1$ only. Higher moments diverge due to the power-law distribution of θ; to obtain finite moments one has to go beyond the linear in θ approximation even for small mismatches σ.

In the thermodynamical limit $N \to \infty$ one can also evaluate the ensemble averages for moments of $|\theta|$

$$\langle|\theta|^k\rangle_{\rm ens} = \frac{\Gamma(1-k)}{|\lambda|^k} \int\limits_{-\infty}^{+\infty} |\sigma|^k F(\sigma)\, d\sigma.$$

Here $F(\sigma)$ is the distribution of σ_j.

3.1.6 Non-perfect cases: Different noises

Quite often N identical systems that are driven by a common external noise $\xi(t)$ experience also influences of different independent (e.g., thermal) noises $\eta_j(t)$. The phase dynamics in this case is given by

$$\dot\varphi_j = \omega + \varepsilon f(\varphi_j)\xi(t) + \gamma_j g_j(\varphi_j)\eta_j(t), \tag{3.30}$$

where $j = 1, 2, ..., N$, the functions f and g_j are normalized $\overline{f^2} = \overline{g_j^2} = 1$, ε and γ_j are the noise amplitudes, and $\langle\xi(t)\,\xi(t+t')\rangle = 2\delta(t')$, $\langle\eta_j(t)\,\eta_k(t+t')\rangle = 2\delta_{jk}\,\delta(t')$, and $\langle\xi(t)\,\eta_j(t+t')\rangle = 0$. Similar to the case of mismatch, we can introduce a phase φ satisfying (3.22) and obtain for small deviations θ_j

$$\dot\theta = \varepsilon f'(\varphi)\theta\xi(t) + \gamma g(\varphi)\eta(t), \tag{3.31}$$

where we omitted index j. In this case the relevant Fokker–Plank equation takes the form

$$\frac{\partial W}{\partial t} + \omega\frac{\partial W}{\partial\varphi} - \gamma^2 g^2(\varphi)\frac{\partial^2 W}{\partial\theta^2} - \varepsilon^2\hat Q^2 W = 0. \tag{3.32}$$

The stationary distribution can be found with the same approximative method as that of Eq. (3.24). Instead of (3.26) we now obtain

$$\gamma^2\frac{d^2w}{d\theta^2} + \varepsilon^2\,\overline{f'^2}\left(\theta^2\frac{d^2w}{d\theta^2} + 4\theta\frac{dw}{d\theta} + 2w\right) = 0, \tag{3.33}$$

where due to the condition $\overline{g^2} = 1$ the dependence on the function g disappears. With rescaling $x \equiv \varepsilon\gamma^{-1}\sqrt{\overline{f'^2}}\,\theta = \sqrt{|\lambda|}\,\gamma^{-1}\theta$ the last equation can be rewritten as

$$(x^2+1)\frac{d^2w}{dx^2} + 4x\frac{dw}{dx} + 2w = 0, \tag{3.34}$$

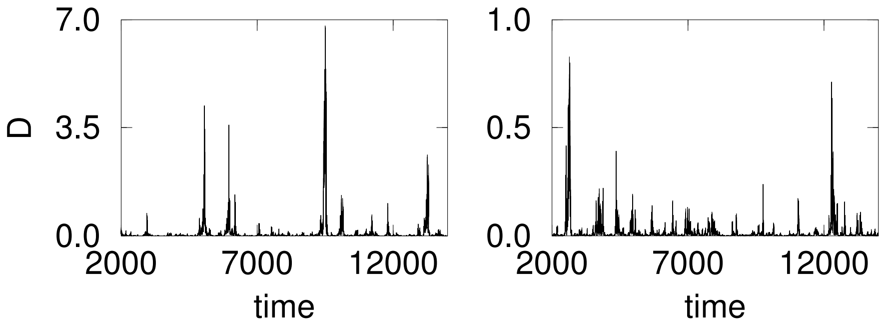

Figure 3.3: The time dependencies of the difference $D \equiv \sqrt{(x_1 - x_2)^2 + (\dot{x}_1 - \dot{x}_2)^2}$ between two Van der Pol–Duffing oscillators (3.37). Left panel: common white noise acts on oscillators with small mismatch [Eq. (3.38) with $\sigma = 10^{-4}$]. Right panel: two identical oscillators are driven by slightly different noises [Eq. (3.39) with $\gamma/\varepsilon = 2{\cdot}10^{-4}$]. Parameters: $\mu = 0.2$, $b = 1$, $\varepsilon = 0.2$.

and solved by virtue of the same substitution $w(x) = h(x)/x^2$. Accounting for the normalization condition, we find the solution

$$w(\theta) = \frac{\gamma}{2\pi^2\sqrt{|\lambda|}}\left[1 + \frac{|\lambda|}{\gamma^2}\theta^2\right]^{-1} \tag{3.35}$$

in the form of the Cauchy distribution. Similar to (3.28) it has a power-law tail what indicates on large fluctuations even for small values of γ. In both cases of small mismatch and small nonidentity of noise these fluctuations have a form of intermittent bursts (see Fig. 3.3, cf. [4]), similar to other cases of imperfect synchronization [1].

One can evaluate the moments

$$\langle|\theta|^k\rangle_{\mathrm{ens}} = \frac{1}{|\lambda|^{\frac{k}{2}}\cos(\pi k/2)}\int_0^{+\infty}\gamma^k G(\gamma)\,d\gamma\,, \tag{3.36}$$

and in the thermodynamical limit $N \to \infty$ the ensemble averages for moments of $|\theta|$ are

$$\langle|\theta|^k\rangle_{\mathrm{ens}} = \frac{1}{|\lambda|^{\frac{k}{2}}\cos(\pi k/2)}\int_0^{+\infty}\gamma^k G(\gamma)\,d\gamma\,.$$

Here $G(\gamma)$ is the distribution of γ_j.

3.1.7 Desynchronization by strong noise

Although a small noise in all considered cases synchronizes the self-sustained oscillators, a desynchronization is possible for large noise intensities. This has been demonstrated in [44, 45] for a noise in the form of a sequence of random pulses. In [4] a positive Lyapunov exponent (LE) has been reported for a discontinuous integrate-and-fire neural model. Here we demonstrate that a desynchronization by noise is possible for white Gaussian noise source and a smooth oscillator, provided the latter has a sufficient degree of non-isochronicity.

As a model we use a standard Van der Pol–Duffing oscillator

$$\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + x + bx^3 = \varepsilon\xi(t), \tag{3.37}$$

where $\xi(t)$ is normalized white Gaussian noise. Here μ describes closeness to the Hopf bifurcation point and the "Duffing parameter" b describes nonisochronicity of oscillations. In Fig. 3.4 we show the dependencies of the LE on the noise amplitude ε for $\mu = 0.2$ and different values of b. One can see that at $b \gtrsim 0.5$ (of course, this critical value depends on μ) positive LEs appear in a certain range of ε, while the asymptotic law $\lim_{\varepsilon\to 0} \lambda/\varepsilon^2 = const < 0$ is valid for all b. The region of positive LEs extends for large b.

To characterize the synchronization-desynchronization transition in system (3.37) quantitatively, we have performed numerical simulation of two weakly nonidentical Van der Pol–Duffing oscillators under common white Gaussian noise

$$\ddot{x}_{1,2} - \mu\left(1 - x_{1,2}^2\right)\dot{x}_{1,2} + (1 \pm \sigma)\, x_{1,2} + bx_{1,2}^3 = \varepsilon\xi(t)\ , \tag{3.38}$$

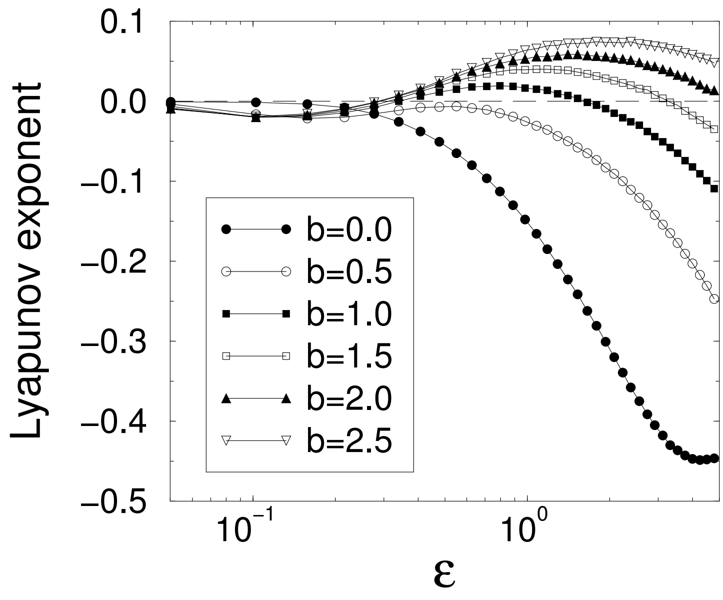

Figure 3.4: For the Van der Pol–Duffing oscillator (3.37) driven by white Gaussian noise, the dependencies of the Lyapunov exponent on the noise amplitude ε are plotted for $\mu = 0.2$ and different values of b.

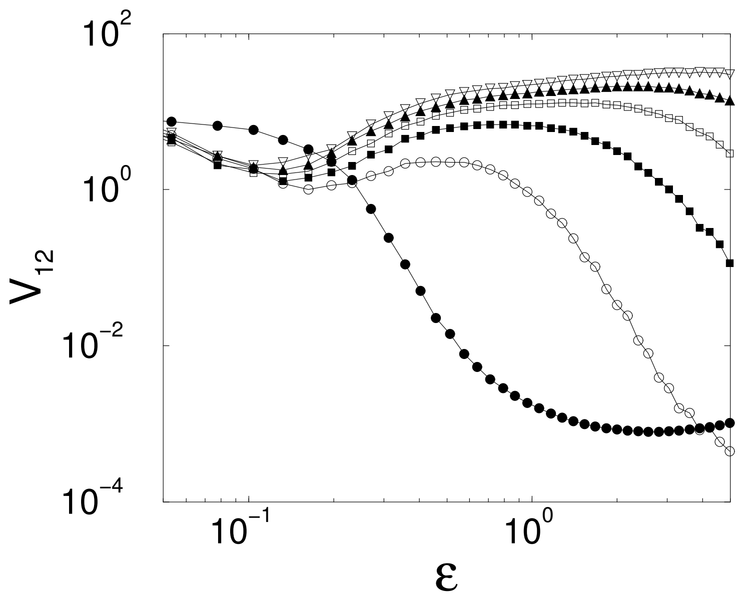

Figure 3.5: The dependencies $V_{12}(\varepsilon)$ are plotted for $\mu = 0.2$ and $\sigma = 0.002$ for the pair of nonidentical Van der Pol–Duffing oscillators under common white Gaussian noise (3.38). The values of b are marked as in the Fig. 3.4.

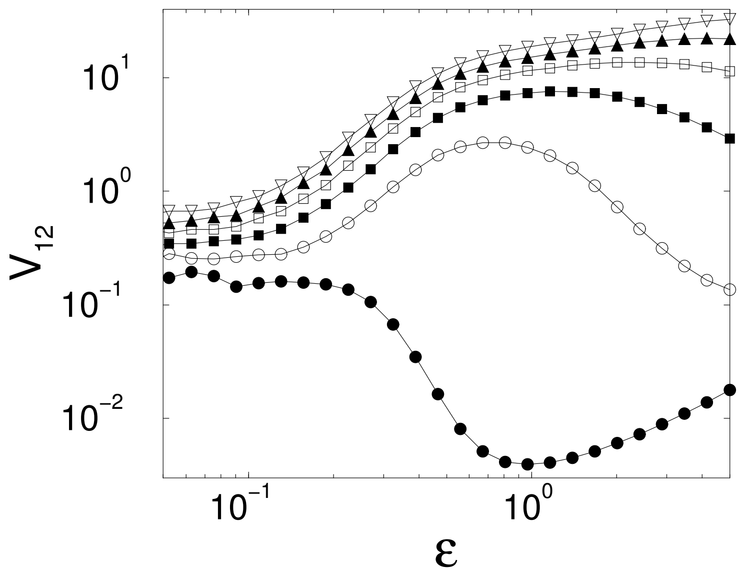

Figure 3.6: The dependencies $V_{12}(\varepsilon)$ are plotted for $\mu = 0.2$ and $\gamma/\varepsilon = 0.01$ for the pair of identical Van der Pol–Duffing oscillators driven by different white Gaussian noises (3.39). The values of b are marked as in the Fig. 3.4.

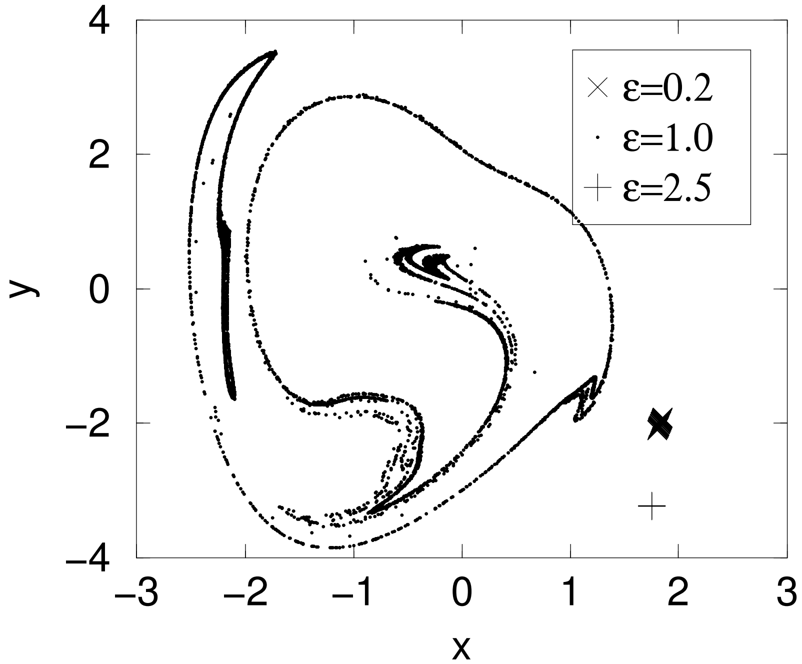

Figure 3.7: The snapshots of the ensemble of 10000 Van der Pol–Duffing oscillators whit homogeneous distribution of σ_j within $[-0.01; 0.01]$ under common white Gaussian noise are presented at $\mu = 0.2$ and $b = 1$. The three chosen values of noise amplitude ε correspond to negative [$\varepsilon = 0.2$, the states in the vicinity of the point $(1.82; -2.07)$; and $\varepsilon = 2.5$, the states in the vicinity of the point $(1.76; -3.29)$] and positive ($\varepsilon = 1$) LEs.

and of two identical Van der Pol–Duffing oscillators driven with slightly different noisy forces

$$\ddot{x}_{1,2} - \mu\left(1 - x_{1,2}^2\right)\dot{x}_{1,2} + x_{1,2} + bx_{1,2}^3 = \varepsilon\xi(t) \pm \gamma\eta(t)\ . \tag{3.39}$$

The quality of synchronization have been measured by the average difference $V_{12} = \langle (x_1 - x_2)^2 + (\dot{x}_1 - \dot{x}_2)^2 \rangle$. In dependence on the noise amplitude ε this quantity has a maximum in the region of positive values of the LE, see Figs. 3.5, 3.6.

We have also performed simulations with a large ensemble of slightly different oscillators driven by the same noise. Here the distribution of the systems states on the plane $(x,\, \dot{x})$ at a certain moment of time is concentrated for a negative LE and is extended for a positive LE, see Fig. 3.7. These distributions correspond to different types of snapshot attractors in system (3.37), see [54].

3.2 Limit cycle systems: Telegraph noise

While in [44, 45] the Lyapunov exponent (LE) was calculated for oscillators driven by a random sequence of pulses, and in the first section of this chapter (see also [1, 4, 5, 6]) the white Gaussian noise was considered, here a noise of other nature is considered, the telegraph one. By a normalized telegraph noise we mean the signal having values ± 1 and switching instantaneously between these values time to time. The distribution of time intervals between consequent switchings is exponential with the average value τ. The case of telegraph noise may be interesting not only because it completely differs from the previous two, but also because it allows to "touch" the question of relations between periodic and stochastic driving, e.g. to compare results for telegraph noise with the average switching time τ and the piece-constant periodic signal of the same amplitude and the period 2τ. This is why we consider synchronization by telegraph noise here.

A limit cycle oscillator subject to a small external force is again described by the phase approximation [28], where only dynamics of the system on the limit cycle of the noiseless system is considered:

$$\dot{\varphi} = \omega + \varepsilon f(\varphi)\xi(t)\,. \tag{3.40}$$

Here $2\pi/\omega$ is the period of the limit cycle in the noiseless system, ε is the amplitude of noise, $f(\varphi)$ is the normalized sensitivity of the system to noise [$(2\pi)^{-1}\int_0^{2\pi} f^2(\varphi)d\varphi = 1$], and ξ is a normalized telegraph noise.

3.2.1 Master equation and its stationary solution

Studying statistical properties of the system under consideration, one can introduce two probability density functions $W_\pm(\varphi,\,t)$ defining the probability to locate the system in vicinity of φ with $\xi = \pm 1$, correspondingly, at the moment t. Then the Master equations of the system reads

$$\frac{\partial W_+(\varphi,\,t)}{\partial t} + \frac{\partial}{\partial\varphi}\left[(\omega + \varepsilon f(\varphi))W_+(\varphi,\,t)\right] = \frac{1}{\tau}W_-(\varphi,\,t) - \frac{1}{\tau}W_+(\varphi,\,t), \tag{3.41}$$

$$\frac{\partial W_-(\varphi,\,t)}{\partial t} + \frac{\partial}{\partial\varphi}\left[(\omega - \varepsilon f(\varphi))W_-(\varphi,\,t)\right] = \frac{1}{\tau}W_+(\varphi,\,t) - \frac{1}{\tau}W_-(\varphi,\,t). \tag{3.42}$$

In the terms of $W \equiv W_+ + W_-$ and $V \equiv W_+ - W_-$ the last system takes the form of

$$\dot{W} = -\omega W_\varphi - \varepsilon\left(f\,V\right)_\varphi, \qquad \dot{V} = -\omega V_\varphi - \varepsilon\left(f\,W\right)_\varphi - \frac{2}{\tau}V. \tag{3.43}$$

For steady distributions the probability flux S is constant:

$$S = \omega W(\varphi) + \varepsilon f(\varphi)\,V(\varphi);$$

and system (3.43) with periodic boundary conditions has the solution

$$V(\varphi) = -\frac{\varepsilon\omega\,C}{\omega^2 - \varepsilon^2 f^2(\varphi)} \int\limits_{\varphi}^{\varphi+2\pi} d\psi\, f'(\psi)\exp\left(\frac{2}{\tau}\int\limits_{\varphi}^{\psi}\frac{d\theta}{\omega^2 - \varepsilon^2 f^2(\theta)}\right), \tag{3.44}$$

where C is defined by the normalization condition:

$$\begin{aligned} C^{-1} &= 2\pi\left(\exp\left(\frac{2}{\tau}\int\limits_0^{2\pi}\frac{d\theta}{\omega^2-\varepsilon^2 f^2(\theta)}\right)-1\right) \\ &+\varepsilon^2\int\limits_0^{2\pi}d\varphi\int\limits_{\varphi}^{\varphi+2\pi}d\psi\frac{f(\varphi)\,f'(\psi)}{\omega^2-\varepsilon^2 f^2(\varphi)}\exp\left(\frac{2}{\tau}\int\limits_{\varphi}^{\psi}\frac{d\theta}{\omega^2-\varepsilon^2 f^2(\theta)}\right). \end{aligned} \tag{3.45}$$

The probability flux reads

$$S=\omega\left(\exp\left(\frac{2}{\tau}\int\limits_0^{2\pi}\frac{d\theta}{\omega^2-\varepsilon^2 f^2(\theta)}\right)-1\right)C.$$

3.2.2 Lyapunov exponent

Studying stability of solutions of the stochastic equation (3.40), one has to consider behavior of a small perturbation α:

$$\dot{\alpha}=\varepsilon f'(\varphi)\alpha\,\xi(t).$$

The LE measuring the average exponential growth rate of α can be obtained by averaging the corresponding velocity

$$\begin{aligned} \lambda &= \langle\frac{d}{dt}\ln\alpha\rangle=\langle\varepsilon f'(\varphi)\xi(t)\rangle=\varepsilon\int\limits_0^{2\pi}f'(\varphi)V(\varphi)d\varphi \\ &= -\varepsilon^2\omega\,C\int\limits_0^{2\pi}d\varphi\int\limits_{\varphi}^{\varphi+2\pi}d\psi\,\frac{f'(\varphi)\,f'(\psi)}{\omega^2-\varepsilon^2 f^2(\varphi)}\exp\left(\frac{2}{\tau}\int\limits_{\varphi}^{\psi}\frac{d\theta}{\omega^2-\varepsilon^2 f^2(\theta)}\right). \end{aligned} \tag{3.46}$$

When $\tau\ll 1$ or $\varepsilon\ll\omega$, the eq. (3.46) can be simplified:

$$\lambda_{\text{app}}=-\frac{\varepsilon^2}{2\pi\omega}\left(\exp\left(\frac{4\pi}{\tau\omega^2}\right)-1\right)^{-1}\int\limits_0^{2\pi}d\varphi\int\limits_0^{2\pi}d\psi\,f'(\varphi)f'(\psi+\varphi)\,\exp\frac{2\psi}{\tau\omega^2}. \tag{3.47}$$

The last expression is strictly negative. Indeed, in the Fourier space it reads

$$\lambda_{\text{app}}=-\omega\tau\varepsilon^2\sum_{k=1}^{\infty}|C_k|^2\frac{k^2}{1+(k\tau\omega^2/2)^2},$$

where $C_k=(2\pi)^{-1}\int\limits_0^{2\pi}f(\varphi)\,e^{-ik\varphi}d\varphi$.

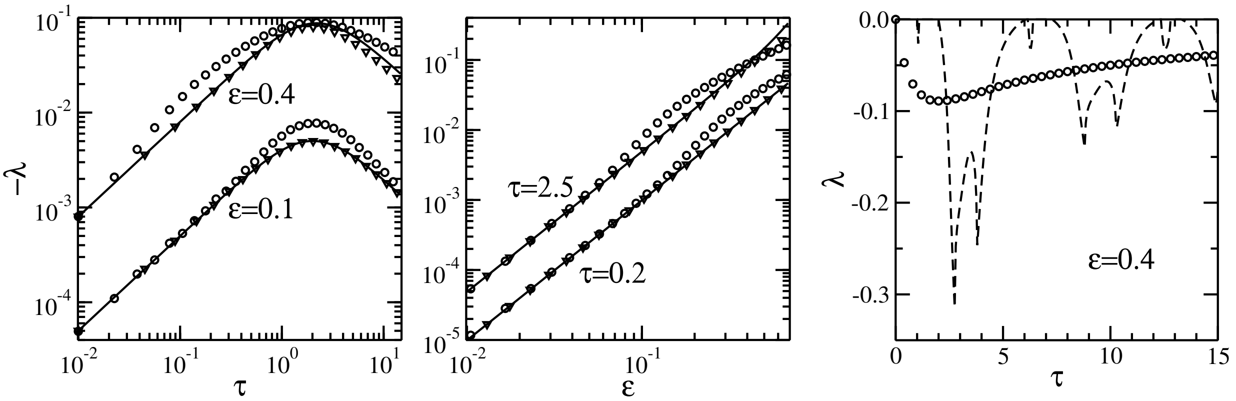

Figure 3.8: Samples of dependencies $\lambda(\varepsilon,\, \tau)$ for the modified Van der Pol oscillator (3.48) at $\mu = 0.1$. The solid lines present the analytical results of phase description, the triangles plot results of the approximation (3.47), the circles correspond to numerical simulation of the noisy modified Van der Pol oscillator, and the dashed line corresponds to numerical simulation of the periodically driven one.

3.2.3 Comparison to numerical simulation

We found that both for weak noise and frequent switching the LE is negative regardless to the properties of the smooth function $f(\varphi)$ (as for weak white Gaussian noise in similar systems considered in the previous section, cf. [4, 5]). Also in the previous section (cf. [6]), moderate white Gaussian noise is shown to be able to lead to instability even in smooth systems. The following issues appear to be interesting (i) *what is the region of validity of the analytical theory developed here*, (ii) *whether there are some footprints of the synchronization by periodic forcing in the stochastic synchronization*, and (iii) *whether telegraph noise can desynchronize oscillators.*

To address the first two questions we performed simulation of a modified Van der Pol

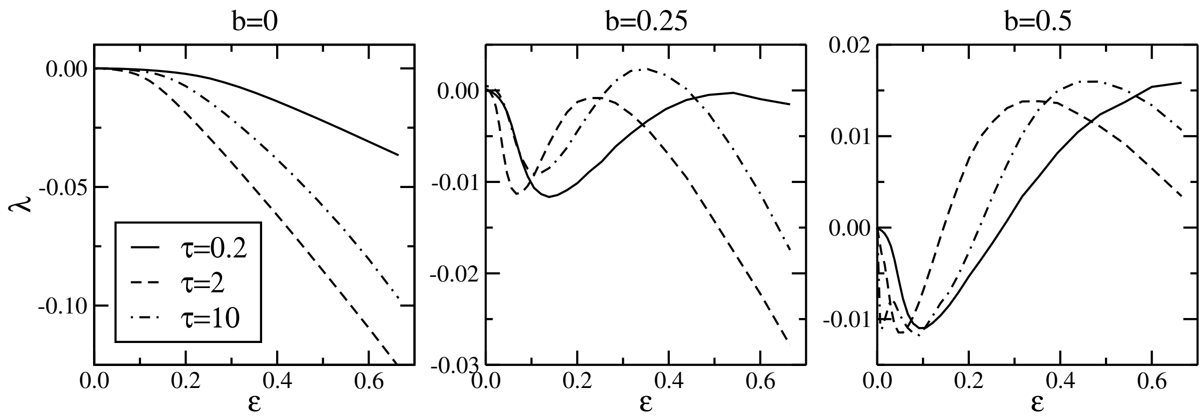

Figure 3.9: Samples of dependencies $\lambda(\varepsilon,\, \tau)$ for the Van der Pol–Duffing oscillator (3.49) at $\mu = 0.1$. The values of the parameters b and τ are indicated above the plots.

oscillator:

$$\ddot{x} - \mu(1 - x^2 - \dot{x}^2)\dot{x} + x = \varepsilon\sqrt{2}\xi(t), \tag{3.48}$$

where $\xi(t)$ is either a telegraph noise with the average switching time τ or a periodic stepwise signal with the period 2τ, i.e. the constant switching time τ. The forcing-free modified Van der Pol oscillator has the round stable limit cycle of the unit radius for all $\mu > 0$. Nevertheless, the phase equation (3.40) with $\omega = 1$ and the simple function $f(\varphi) = \sqrt{2}\cos\varphi$ may be correctly adopted only if the phase speed is near-constant all over the limit cycle, which is valid at small μ only.

In Fig. 3.8 one can see that our analytical theory is fortunately in good agreement with the results of analytical simulation not only for weak noise; and the dependence $\lambda(\varepsilon, \tau)$ for the stochastic driving has no footprints of the one for the periodic driving.

While the dynamical system (3.48) does not exhibit positive LEs at any noise intensity and any μ, they can be observed for a Van der Pol–Duffing model (a similar situation occurs for white Gaussian noise):

$$\ddot{x} - \mu(1 - 2x^2)\dot{x} + x + 2bx^3 = \varepsilon\sqrt{2}\xi(t), \tag{3.49}$$

where the “Duffing parameter” b describes nonisochronicity of oscillations. In Fig. 3.9 one can see that at large enough b positive LE appears in a certain range of parameters.

Having considered the phenomenon of synchronization of limit cycle oscillators by common telegraph noise, we may summarize:

— Both for weak noise and frequent switching the Lyapunov exponent is negative;

— For some systems, the phase model gives quite adequate results even for moderate noise levels and values of the average switching time;

— The dependence $\lambda(\varepsilon, \tau)$ for stochastic driving appears to have no footprints of the one for periodic driving;

— In some systems, moderate telegraph noise can desynchronize oscillations.

Here we do not present results for the non-perfect situations (like in the previous section, cf. [6]): slightly nonidentical oscillators driven by an identical noise signal, and identical oscillators driven by slightly nonidentical noise signals. The reason is that for weak noise these results appear to be the same as in the previous section but with λ_{app} given by Eq. (3.47) instead of λ.

3.3 Antireliability of neural oscillators

Recently, the reliability property of spiking neurons has attracted large attention [39]. The effect appears as a coincidence of responses of a single neuron subject to repeatedly applied weak input fluctuations. From the theoretical viewpoint, reliability is a manifestation of

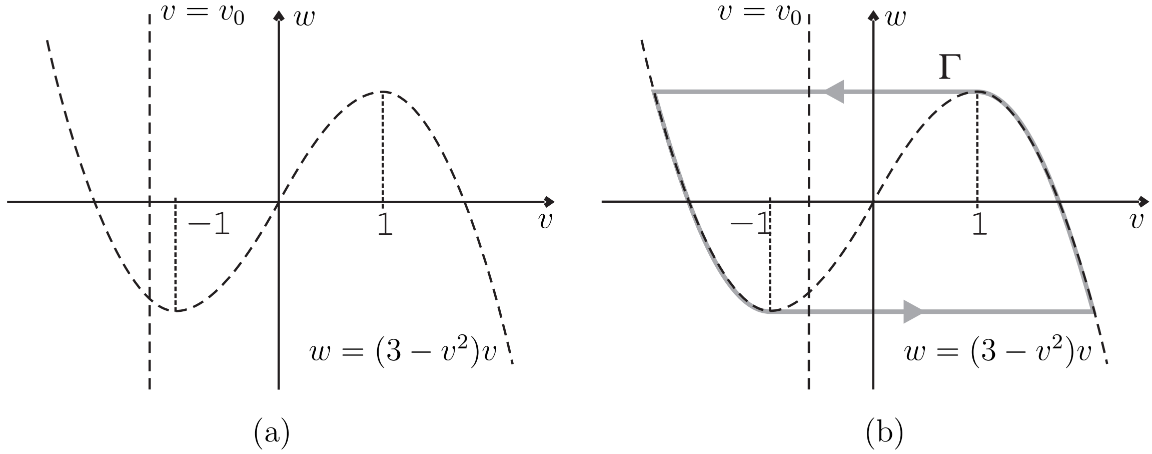

Figure 3.10: The phase space of the FitzHugh–Nagumo system (3.50) for (a) $v_0 < -1$, and (b) $v_0 > -1$.

the synchronization of nonlinear systems by common noisy driving [44, 45, 53, 55, 56, 57, 4, 5, 6, 7]. Indeed, a usual protocol in the reliability experiment, when a particular fluctuating waveform is repeatedly used to force a neuron, is equivalent to the driving of an ensemble of identical neurons by the common fluctuating force. The intrinsic noise is a source of non-identity, and may lead to a non-perfect reliability.

Reliability means that the response of a nonlinear system on the fluctuating forcing is stable. As mentioned above, quantitatively, this stability is measured by the largest Lyapunov exponent (LE) in the presence of noise. For a limit cycle in a smooth dynamical system we have shown that a small noise results in a negative LE, thus leading to synchronization and reliability (cf. [44, 45, 4, 5]), while a larger noise can result in a positive LE (cf. [44, 45, 58, 6, 7]; a positive exponent was also reported for a non-smooth system in [4]). Noteworthy, neurons are limit cycle oscillators only in the regime of periodic spiking, thus requiring a particular investigation close to the transition from excitable behavior to periodic spiking. (Additionally, the dynamics of a spiking system is crucially characterized by the time series of firing events, what makes peculiarity of neural systems more pronounced.)

Here we perform such an investigation and demonstrate an *antireliability*, i.e. a transition to a positive LE, for a model of a neuron in an excitable state. We show that a FitzHugh–Nagumo (FHN) neuron responds to a fluctuating forcing of a certain intensity in a non-reliable manner, while reliability is observed for very small and very large intensities of driving noise. Furthermore, we develop an analytical approach, allowing us to calculate the LE for moderate noise amplitudes. We explain the transition to antireliability geometrically as a chaotic transition due to random stretchings and foldings of the phase mapping.

3.3.1 Simulation for FHN

The basic model we use to describe a spiking neuron is the FHN system [59, 60]:

$$\dot{v} = \alpha^{-1}[(3 - v^2)v - w] + \xi(t), \qquad \dot{w} = v - v_0, \tag{3.50}$$

where α is a small parameter (below in numerical simulations we fix $\alpha = 0.05$), and $\xi(t)$ is a white Gaussian noise: $\langle \xi(t)\xi(t + t')\rangle = 2\varepsilon^2\delta(t')$. For $v_0 < -1$ the only attractor in the noiseless system is a stable fixed point (see Fig. 3.10a), i.e. the system is in the excitable regime, and the LE is negative: $\lim_{\varepsilon\to 0} \lambda = const < 0$. For $v_0 > -1$ this fixed point becomes unstable, and the stable limit cycle appears (see Fig. 3.10b), i.e. periodic spiking takes place, and $\lim_{\varepsilon\to 0} \lambda = 0$. Due to smallness of parameter α the oscillation transition is very sharp, and already for $v_0 > -1 + O(\alpha^{1/2})$ the cycle takes a characteristic for relaxation oscillations form, not depending on v_0.

Lyapunov exponent

In a vicinity of the transition value $v_0 \lesssim -1$ the system is mostly sensitive to external noise, which evokes a spike train. The latter can be more or less regular, but here we focus on the stability properties of the dynamics, and characterize them in Fig. 3.11 with the largest LE. One can see that the region of moderate noise intensities, where LE is positive, exists both when the dynamics is excitable ($v_0 < -1$ and LE is negative for vanishing noise) and when the system is oscillating ($v_0 \gtrsim -1$ and LE vanishes for vanishing noise). Only outside the vicinity of the transition (for $|v_0 + 1| \gtrsim 0.005$) the LE remains negative for all ε.

We denote the regime with positive LE as an antireliable one, and illustrate it in Fig. 3.12. Here the same realization of noise drives 10 uncoupled identical neurons. While for $\lambda < 0$ the perfect synchrony of spikes is observed, for $\lambda > 0$ one can see an alternation between the epochs of asynchronous and relatively synchronous behavior. The latter epochs, which look in middle row of Fig. 3.12 as vertical stripes, are in fact not perfectly synchronous, but slightly different in the spike timings ($\Delta t_i \approx 0.01 - 0.1$). We will give an explanation for this intermittency below.

In real situations, the perfect synchrony (for $\lambda < 0$) is distorted by small nonidentities in the oscillators or in the noisy driving (e.g. by an additional noise specific for each oscillator, in the context of neuron reliability one speaks on intrinsic noise). We illustrate this imperfect synchrony in Fig. 3.12, right column.

Event synchronization approach

Fig. 3.12 provides a qualitative frame for observations of antireliability in experiments, because there typically the same protocol as above and the same representation of data is

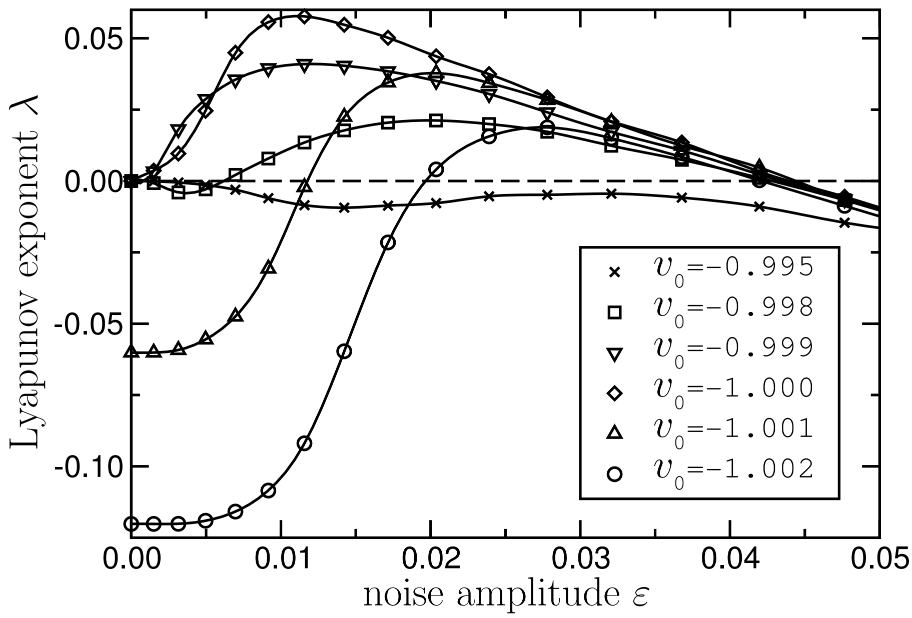

Figure 3.11: The Lyapunov exponent vs the noise amplitude for the FHN system (3.50) for different values of v_0.

used. To characterize the reliability and antireliability quantitatively, we adopt the event synchronization approach [61]. For the observed sequence of spikes, one can introduce the "reduction" function $x^\tau(t) = c\sum_{j=1}^{M}(\Theta(t-t_j) - \Theta(t-t_j-\tau))$, where Θ is the Heaviside function, t_j is the time of j-th firing event, and c is a normalization constant defined by $\langle x^\tau \rangle = 1$. The synchrony of firing events for two systems with the reduction functions $x^\tau(t)$ and $y^\tau(t)$ can be quantified by the event synchronization correlation function $C^\tau_{xy}(t') = \langle x^\tau(t) y^\tau(t+t')\rangle$. In the case of a perfect event synchrony,

$$C^\tau_{xy}(t) = \begin{cases} 1 - |t/\tau|, & |t| < \tau \ ; \\ 0, & |t| \geq \tau \ . \end{cases} \tag{3.51}$$

We present the calculations of the event synchronization in Fig. 3.13. In panel (a) identical neurons are considered. One can see that the regions of perfect event synchrony ($C^\tau(0) = 1$) coincide with the ones of negative LE in Fig. 3.11. For regions of antireliability ($\lambda > 0$), the event correlation function $C^\tau(0)$ is small, but does not vanish: this is due to intermittent synchronous epochs seen in Fig. 3.12. The persistence of synchrony against the intrinsic noise nonidentity may be estimated from Fig. 3.13b.

3.3.2 Mechanism of antireliability and analytical model

We now turn to an analytical description of the effect and to revealing its mechanism. First, we give a general argument that a positive LE cannot be explained within the one-dimensional phase approximation [28] to the oscillation dynamics (a statistical evidence

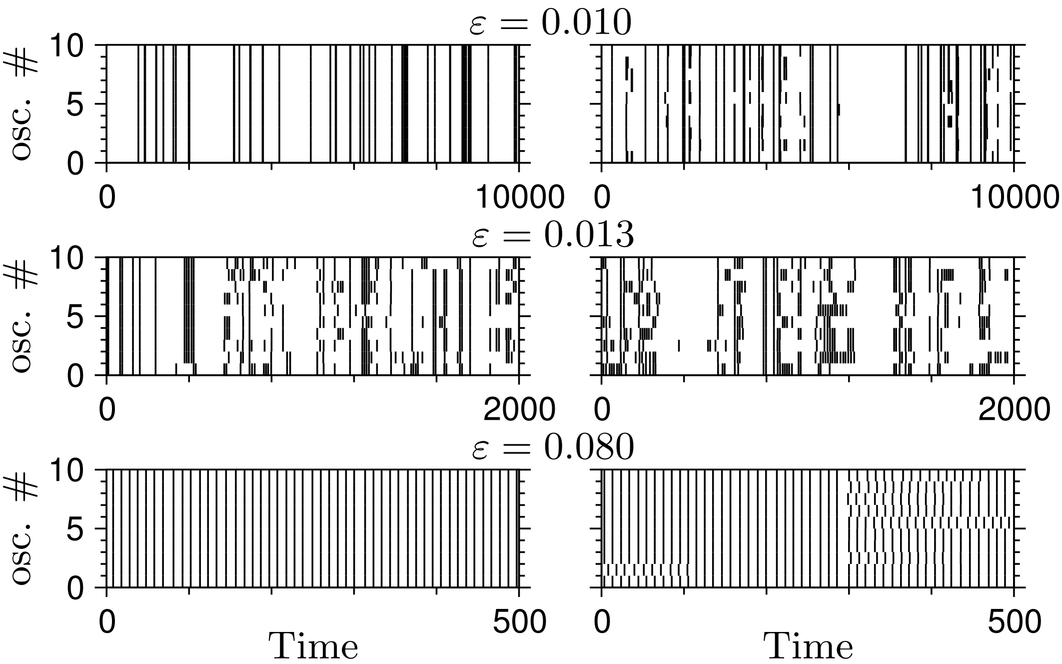

Figure 3.12: Samples of firing patterns for the ensemble of 10 neurons (3.50) driven by common noise. Each spike is depicted as a short vertical stripe, so a long vertical stripe corresponds to a synchronous, reliable firing. Left column: the neurons are perfectly identical, right column: there is small intrinsic noise with amplitude $\varepsilon_{\text{int}} = 5 \cdot 10^{-5}$. For $\varepsilon = 0.01$ and $\varepsilon = 0.08$, the LE is negative (reliability), for $\varepsilon = 0.013$ the LE is positive (antireliability).

of this fact has been presented in the first section of this chapter; see also [4, 5]). Indeed, a time-continuous evolution of the phase (and of any one-dimensional variable) under arbitrary forcing, on a finite time interval can be reduced to a monotonous transformation of the phase. Because an attracting set of a monotonous transformation has a negative LE, a positive LE is excluded. Therefore, we have to go beyond the usual one-dimensional phase approximation for the dynamics of perturbed oscillatory systems. This makes the problem nontrivial, because in higher dimensions one cannot obtain the LE by a plain averaging. For simplicity, we assume that the noise-free system is periodic, and model the two-dimensional perturbed dynamics with the system

$$\dot{\varphi} = \omega + a(\varphi)r, \quad \dot{r} = -\gamma(\varphi)r + f(\varphi)\xi(t). \tag{3.52}$$

Here φ is the oscillation phase and r is a transversal deviation from the limit cycle (hereafter referred as an amplitude), $\omega = 2\pi/T$ is the oscillation frequency. We have introduced three functions: $a(\varphi)$ describes the nonisochronicity of the system, $\gamma(\varphi)$ is the relaxation rate of the amplitude perturbations, $f(\varphi)$ is the sensitivity to noise; all these functions of φ are 2π-periodic. We have omitted noise in the equation for the phase because of its,

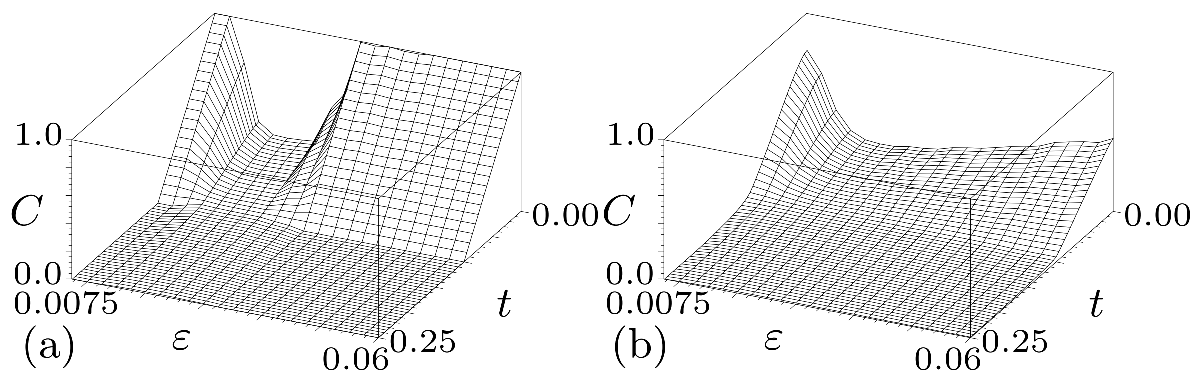

Figure 3.13: The correlation function $C^{\tau}_{xy}(t)$ for two FHN systems driven by (a) identical noise signals, and (b) by nonidentical ones, with the intrinsic noise of amplitude $\varepsilon_{\text{int}} = 5 \cdot 10^{-5}$. Parameters $v_0 = -1.001$, $\tau = 0.1$.

already mentioned, purely stabilizing effect.

For a relatively small noise the dynamics is close to the limit cycle, on which, at the noiseless limit, $\varphi(t) = \varphi_0(t) = \varphi(0) + \omega t$ and $r(t) = 0$. The infinitesimally small perturbations obey the linearized equations which, to the main order with respect to noise, take the form

$$\delta\dot{\varphi} = a(\varphi_0)\delta r, \quad \delta\dot{r} = -\gamma(\varphi_0)\delta r + f'(\varphi_0)\xi(t)\delta\varphi. \tag{3.53}$$

We now make two assumptions that are typically valid for spiky systems under consideration. First, let us suppose that the system is especially sensitive to noise on some short part of the limit circle near $\varphi = \Phi$, and neglect the effect of noise for the rest of the cycle. This means that $f'(\varphi_0)$ is non-zero only in some interval $[\varphi^-, \varphi^+] \ni \Phi$. For the FHN system with $v_0 \approx -1$ this is exactly the region near the tip of the slow branch $v = -1$, $w = -2$. Here the trajectory slowly passes close to the unstable steady state and is highly sensitive to perturbations. The next assumption is that the relaxation rate γ is large at least on some pieces of the limit cycle. For the FHN system this is ensured due to the separation of slow and fast motions.

As a result of these two assumptions, we can separate the dynamics of perturbations in two stages: (i) a noise-induced evocation in a vicinity of Φ, and (ii) a relaxation. Prior to stage (i) we take a phase perturbation, i.e. $\delta r = 0$, $\delta\varphi = \delta\varphi_0$. During the evocation we can neglect all terms in (3.53) except the noisy one, which yields a perturbation in the amplitude

$$\delta r_0 = S\delta\varphi_0, \qquad S = \int_{\varphi^-/\omega}^{\varphi^+/\omega} f'(\omega t)\xi(t)\mathrm{d}t.$$

As $\xi(t)$ is a Gaussian white noise, S is a Gaussian random variable with zero average and

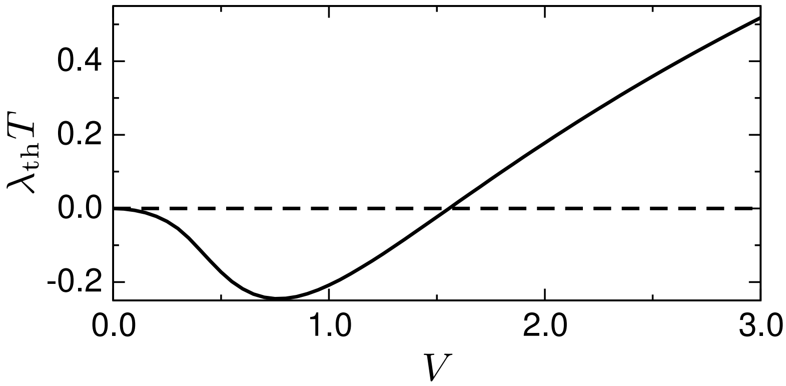

Figure 3.14: The theoretical dependence $\lambda_{\rm th}(V)$.

the variance

$$\langle S^2 \rangle = \frac{2\varepsilon^2}{\omega} \int_{\varphi^-}^{\varphi^+} [f'(\varphi)]^2 \mathrm{d}\varphi \; .$$

The next, relaxation stage, where the effect of noise can be neglected, starts with the perturbation δr_0, $\delta\varphi_0$ at the time t_0. According to Eqs. (3.53)

$$\begin{aligned} \delta r(t) &= \delta r_0 \mathrm{e}^{-\int_{t_0}^{t} \gamma(\omega t')\mathrm{d}t'} \approx \delta r_0 e^{-\gamma(\Phi)(t-t_0)}, \\ \delta\varphi(t) &\approx \delta\varphi_0 + \delta r_0 \int_{t_0}^{t} a(\omega t') e^{-\gamma(\Phi)(t'-t_0)} \mathrm{d}t' \\ &\approx \delta\varphi_0 + \delta r_0 \frac{a(\Phi)}{\gamma(\Phi)} \left(1 - e^{-\gamma(\Phi)(t-t_0)}\right) . \end{aligned}$$

Thus, the amplitude relaxes to zero, and for the phase perturbation we obtain the mapping:

$$\delta\varphi_{n+1} = \delta\varphi_n \, (1 + R) \, , \tag{3.54}$$

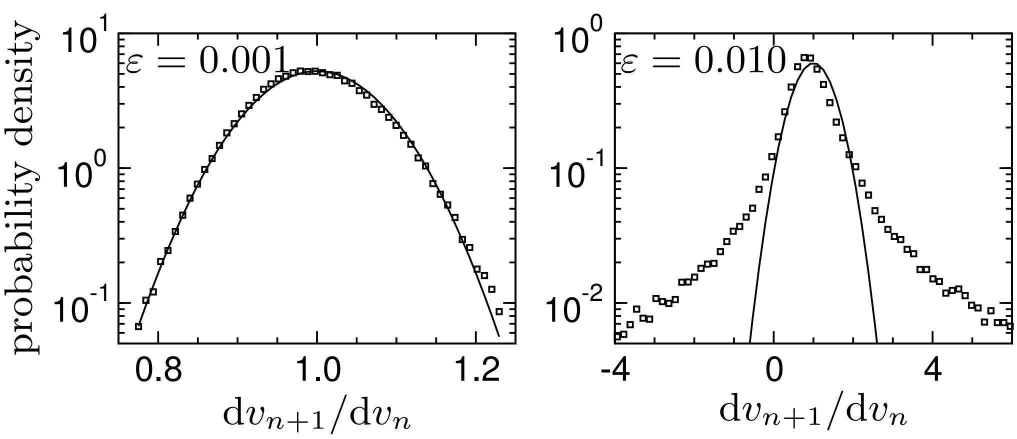

Figure 3.15: Samples of the distribution of the derivatives $\mathrm{d}v_{n+1}/\mathrm{d}v_n$ for the FHN system at $v_0 = -0.998$. The squares present results of numerical simulation, the solid line fits them with a Gaussian distribution.

where we have introduced index n indicating repetitive passages through the noise-sensitive region. The quantity $R = Sa(\Phi)/\gamma(\Phi)$ is a zero-mean Gaussian variable with the variance

$$V^2 = \frac{2\varepsilon^2 a^2(\Phi)}{\omega\gamma^2(\Phi)} \int_{\varphi^-}^{\varphi^+} [f'(\varphi)]^2 \mathrm{d}\varphi. \tag{3.55}$$

The LE for the random mapping (3.54) is

$$\begin{aligned} \lambda_{\mathrm{th}} &= T^{-1}\langle \ln|1+R| \rangle \\ &= \frac{1}{\sqrt{2\pi}TV} \int_{-\infty}^{+\infty} \ln|1+R| e^{-\frac{R^2}{2V^2}} \mathrm{d}R. \end{aligned} \tag{3.56}$$

In Fig. 3.14 we depict the dependence $\lambda_{\mathrm{th}}(V)$. The LE changes sign at $V \approx 1.5560$, what corresponds to the onset of desynchronization and antireliability. According to formula (3.55), V is proportional to the amplitude of noise ε, to the sensitivity of the dynamics to the noise $\propto [f'(\varphi)]^2$, and to nonisochronicity of the oscillations a, and inverse proportional to the relaxation rate γ.

Above we have assumed that the neuron is in the oscillating regime, and that there is no noise acting directly on the phase in (3.53). A violation of both these conditions leads to an additional contraction of the phase (which is, of course, much stronger for a neuron in excitable state with small noise, because there the trajectory spends a lot of time in a vicinity of the stable fixed point). Thus, the resulting curve Fig. 3.14 should be shifted down. It becomes then similar to numerically observed dependencies of Fig. 3.11. A negative LE for very large noise intensities, observed in Fig. 3.11, cannot be explained by the theory above, as the underlying assumptions are not valid for strong noise.

We now compare the theoretical predictions with the numerics. To check the mapping for the phase perturbations (3.54), we fix a region on the branch of slow motions of system near $v_* \approx -\sqrt{3}$, $w_* \approx 0$. Here, due to the strong contraction of the fast variable v, only perturbations along the slow branch are present. We characterize these perturbations with their projection on coordinate v. In Fig. 3.15 we present the histograms of the derivatives $\mathrm{d}v_{n+1}/\mathrm{d}v_n$ for pieces of trajectories starting at (v_*, w_*) and returning to its vicinity. These quantities, which are the multipliers for infinitesimal perturbations, according to the theory above correspond to the quantities $1+R$ in (3.54). One can see that the distribution of these multipliers is nearly symmetric around $\mathrm{d}v_{n+1}/\mathrm{d}v_n \approx 1$. For small noise the Gaussian distribution fits very well, while for larger noise one observes "heavy tails" $[\propto (\mathrm{d}v_{n+1}/\mathrm{d}v_n - 1)^{-2}]$, which are presumably due to violations of the assumptions used in the derivation of (3.54).

In order to clarify the geometric nature of the transition to positive LE, we have followed the evolution of finite but small segments of the slow branch, starting in a vicinity of (v_*, w_*). All the points evolve under the same realization of noise for a fixed time

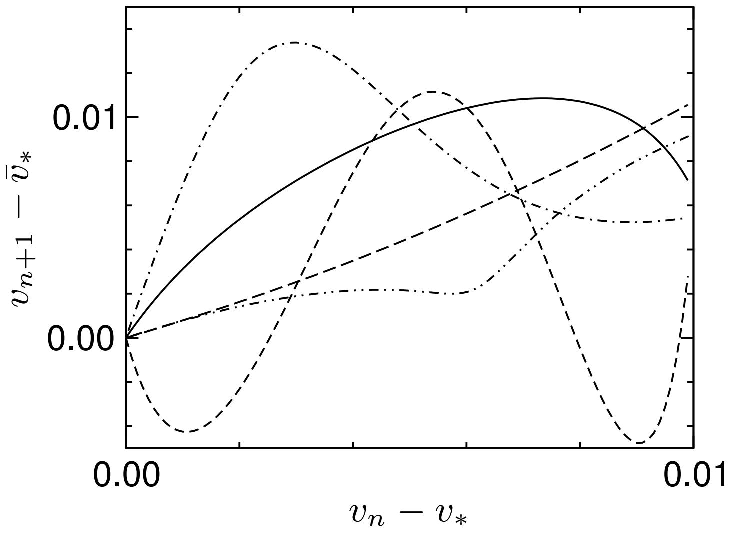

Figure 3.16: Sample mappings $v_n \to v_{n+1}$ for finite segments of the slow branch for $v_0 = -0.998$, $\varepsilon = 0.01$ (positive LE). An offset on the vertical axis is arbitrary.

interval $\approx T$. The resulting mappings of the segment are shown in Fig. 3.16. Of course, the mapping is random, therefore we have different realizations that depend on the noise waveform. The crucial point is that many of these mappings are not one-to-one. This reveals the geometrical mechanism of chaotization: during the evolution, the segments of the cycle are stretched and folded, so that the resulting mapping is non-monotonous.

The distribution of multipliers Fig. 3.15 allows us also to explain the intermittent character of the anti-reliability. According to eq. (3.54) and Fig. 3.15, there is a finite probability to observe a vanishing multiplier $\mathrm{d}\varphi_{n+1}/\mathrm{d}\varphi_n$ (geometrically, these events correspond to extrema of the random mapping Fig. 3.16). If such an event happens, the states of different identical neurons become very close to each other, thus they fire nearly simultaneously. Only after a certain number of cycles with large multipliers $|\mathrm{d}\varphi_{n+1}/\mathrm{d}\varphi_n| > 1$, the close states diverge and the difference in the responses of identical neurons to the common noise becomes visible.

3.4 Summary and discussion

In this chapter we have considered synchronization of oscillators by common noise (white Gaussian or telegraph). The effect of synchronization is quantified by the Lyapunov exponent (LE). Identical (or slightly different) oscillators are synchronized by common noisy driving when the LE is negative, and desynchronized otherwise. For limit cycle oscillators the LE vanishes in the noiseless limit and is proven to be negative for weak noise: $\lim_{\varepsilon\to 0} \lambda/\varepsilon^2 = const < 0$ (ε is the noise amplitude). For this purpose the approach of the phase approximation has been used.

While for a negative LE, identical oscillators are perfectly synchronous under identical driving, nonidentities distort perfect synchrony. For limit cycle oscillators driven by weak noise the role of nonidentities either (i) in oscillators or (ii) in noise is studied analytically. This study has revealed a crucial role of finite deviations between the states of subsystems even for arbitrary small nonidentities, and a strongly intermittent character of synchrony.

The analytical results obtained within the framework of the phase approximation are underpinned by the results of numerical simulation for the Van der Pol–Duffing oscillator. This simulation also allows us to consider the effect of finite amplitude noise, where no analytical investigation is possible. In some systems under certain circumstances, moderate noise appears to be able to lead to desynchronization: a positive LE is observed.

The results obtained for limit cycle oscillators appear to be general for different kinds of noise, because they do not diverse much even for such different noises as white Gaussian and telegraph ones.

Neural oscillators being interesting from the practical viewpoint (for them the reliability property [39], directly related to the phenomena we consider, appears to be important) can not be always considered as limit cycle ones. This fact makes a particular consideration of such systems necessary.

We have performed such an investigation and have shown, within the framework of the noise-driven FitzHugh–Nagumo model, that identical neurons can respond antireliably to the noisy evocation. Antireliability, which manifests itself as a non-correlation of spikes, is observed when the neurons are close to the transition excitability–oscillations, where the dynamics is mostly sensitive to perturbations. Quantitatively, the antireliability is characterized as a state with a positive largest LE. The latter is purely noise-induced, as the noiseless FitzHugh–Nagumo system does not posses even a transient chaos. We explain the transition to the antireliability within the approximate analytical theory for small noise-induced deviations from the deterministic trajectory, which goes beyond the one-dimensional phase approximation. This is crucial, because only due to the evocation of transversal to trajectory perturbation, the stretchings and foldings that lead to chaos, can occur. The final formulae (3.55), (3.56) give an explicit dependence of the LE on physical properties of the neuron, such as nonisochronicity and sensitivity, thus guiding an experimental search for the effect. The theoretical expression for the random multiplier of phase deviations explains also the intermittent character of the antireliable state: during the epochs where the multiplier is close to zero, a temporarily synchronous firing of neurons is observed.

Chapter 4

Effects of Delayed Feedback on Kuramoto Transition

A transition to collective synchrony in an ensemble of globally coupled oscillators is known as the Kuramoto transition [63]. An important application of the theory of this transition is collective dynamics of neuronal populations. Indeed, synchronization of individual neurons is believed to play the crucial role in the emergence of pathological rhythmic brain activity in Parkinson's disease, essential tremor, and epilepsies (for instance, see [64, 65, 71]). One approach to suppress such an activity is to apply to the system a negative feedback loop [66, 67, 68, 69, 70].

In [66, 67] a linear delayed feedback has been employed in order to stabilize the absolutely nonsynchronous state. Another approach has been proposed in [68], where the authors have suggested a nonlinear suppression of collective oscillations. There giant values of the nonlinear feedback coefficient are used resulting in a crucial diminishing of the amplitude of collective oscillations. In such circumstances the stability of the absolutely nonsynchronous state becomes out of the author interest.

In this chapter we develop a weakly nonlinear theory of the Kuramoto transition in the presence of linear and nonlinear time-delayed coupling terms [9]. First, we establish the relationships between phase models and original limit cycle systems, of particular interest are physically relevant feedback terms within the framework of phase dynamics. Second, we treat a linear delayed feedback, and show that such a feedback not only controls the transition point, but effectively changes the nonlinear terms near the transition. Third, we study a purely nonlinear delayed coupling, which does not effect the transition point,[1] but can reduce or enhance the amplitude of collective oscillations.

We heavily rely in our analysis on the corresponding treatment of the system without

[1]We do not use large values of the nonlinear feedback coefficient like in [68]; therefore have to take care of linear stability properties.

delay by Crawford [62].

4.1 From limit cycle systems to phase models

Here we introduce our basic model — an ensemble of autonomous oscillators subject to different types of global coupling. We take individual oscillators as Van der Pol ones and write the model as

$$\ddot{x}_i - \mu(1 - x_i^2)\dot{x}_i + \omega_i^2 x_i = 2\sqrt{2}\omega_i \xi_i(t) + \varepsilon' F(\overline{x}, \overline{y}) \tag{4.1}$$

where $\xi_i(t)$ is a δ-correlated Gaussian noise: $\langle \xi_i(t)\xi_j(t-t')\rangle = 2D\,\delta_{ij}\,\delta(t')$. The ensemble averages are defined as

$$\overline{x} = \frac{1}{N}\sum_{j=1}^{N} x_j \ , \qquad \overline{y} = \frac{1}{N}\sum_{j=1}^{N} \frac{\dot{x}_j}{\omega_j} \ .$$

In the reduction to phase equations we use the smallness of parameters μ and ε', and suppose the natural frequencies ω_i to be distributed in a relatively close vicinity of the mean frequency $\omega_0 \equiv N^{-1}\sum_{j=1}^{N}\omega_j$. Because $\mu \ll \omega_i$, the solution of the autonomous Van der Pol oscillator can be written as $x_i(t) \approx A_i(t)\cos(\varphi_i(t))$ where on the limit cycle $A_i \approx 2$ and $\dot{\varphi}_i = \omega_i$. Because $\varepsilon' \ll \mu$, coupling does not affect the amplitude (which remains ≈ 2), but only the phase. It is convenient to introduce the complex order parameter

$$R(t) = |R|e^{i\theta(t)} = \frac{1}{2}(\overline{x} + i\overline{y}) = \frac{1}{N}\sum_j e^{i\varphi_j(t)} \tag{4.2}$$

and to represent the global coupling in terms of R. The absolute value of the order parameter is close to zero for nearly uniform, nonsynchronized distributions, and reaches 1 for strongly synchronized states.

Below we will be interested in linear coupling with and without time delay [66, 67], and in a nonlinear coupling:

$$\varepsilon' F(\overline{x}, \overline{y}) = 2\omega_0 \varepsilon \overline{y}(t) + 2\omega_0 \varepsilon_f \overline{y}(t-T) + \frac{d}{dt}(\overline{x}^2(t-T))(K_x \overline{x}(t) + K_y \overline{y}(t)) \ .$$

Here we use the simplest form of nonlinear coupling which does not effect linear stability of the absolutely nonsynchronous state. As a result, the phase equations for the oscillators read

$$\begin{aligned} \dot{\varphi}_i \ &= \ \omega_i + \frac{\varepsilon}{N}\sum_{j=1}^{N}\sin(\varphi_j(t) - \varphi_i(t)) + \frac{\varepsilon_f}{N}\sum_{j=1}^{N}\sin(\varphi_j(t-T) - \varphi_i(t)) \\ &\quad + \varepsilon_{of}|R(t-T)|^2|R(t)|\sin[\,2\theta(t-T) - \theta(t) - \varphi_i(t) + \nu] + \xi_i(t), \end{aligned} \tag{4.3}$$

where $\varepsilon_{of} e^{i\nu} = 2(K_x + iK_y)$. Here three coupling parameters describe different types of coupling: ε describes collective linear coupling without delay, as in the original Kuramoto model; ε_f describes linear coupling with delay, as has been proposed in [66, 67]; ε_{of} describes nonlinear coupling with delay.

4.1.1 Thermodynamic limit

In the thermodynamic limit $N \to \infty$ we can introduce a distribution of natural frequencies $g(\omega)$ and rewrite system (4.3) as

$$\begin{aligned}\dot{\varphi}(\omega) = \omega + \varepsilon \int_{-\infty}^{+\infty} g(\omega') \sin\left(\varphi(\omega',\, t) - \varphi(\omega,\, t)\right)\, d\omega' \\ +\varepsilon_f \int_{-\infty}^{+\infty} g(\omega') \sin\left(\varphi(\omega',\, t-T) - \varphi(\omega,\, t)\right)\, d\omega' \\ +\varepsilon_{of} |R(t-T)|^2 |R(t)| \sin[\, 2\theta(t-T) - \theta(t) - \varphi(\omega, t) + \nu] + \xi(\omega,\, t)\ .\end{aligned} \tag{4.4}$$

For a statistical description we introduce a probability density $\rho(\omega,\, \varphi,\, t)$ (normalized as $\int_0^{2\pi} \rho(\omega,\, \varphi,\, t)\, d\varphi = 1$) that is governed by the Fokker-Planck equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \varphi} (\rho\, v) - D \frac{\partial^2 \rho}{\partial \varphi^2} = 0, \tag{4.5}$$

where

$$\begin{aligned}v(\omega) = \omega + \varepsilon \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \sin(\theta - \varphi) \rho(\omega',\, \theta,\, t) \\ +\varepsilon_f \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \sin(\theta - \varphi) \rho(\omega',\, \theta,\, t-T) \\ +\varepsilon_{of} |R(t-T)|^2 |R(t)| \sin\left(2\theta(t-T) - \theta(t) - \varphi + \nu\right)\ .\end{aligned} \tag{4.6}$$

The order parameter introduced in (4.2) now takes the form

$$R(t) = \frac{1}{N} \sum_j e^{i\varphi_j(t)} = \int_{-\infty}^{+\infty} d\omega\, g(\omega) \int_0^{2\pi} d\varphi\, \rho(\omega,\, \varphi,\, t)\, e^{i\varphi}. \tag{4.7}$$

Mathematically, our problem is the problem of behavior of small perturbations ρ_1 of the absolutely nonsynchronous state

$$\rho_0 = \frac{1}{2\pi} \tag{4.8}$$

in the vicinity of the threshold where this state becomes linearly unstable and collective oscillations onset. Below we work with Eq.(4.5).

4.2 Linear delayed feedback

The case of linear delayed feedback is described by Eqs.(4.5),(4.6) with $\varepsilon_{of} = 0$.

4.2.1 Linear stability of the absolutely nonsynchronous state

Infinitesimal perturbations ρ_1 of the state (4.8) are governed by the linearization of Eq.(4.5):

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial v_1}{\partial \varphi} + v_0 \frac{\partial \rho_1}{\partial \varphi} - D \frac{\partial^2 \rho_1}{\partial \varphi^2} = 0, \tag{4.9}$$

with

$$\begin{aligned} \frac{\partial v_1}{\partial \varphi} = & -\varepsilon \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \cos(\theta - \varphi) \rho(\omega', \theta, t) \\ & -\varepsilon_f \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \cos(\theta - \varphi) \rho(\omega', \theta, t - T). \end{aligned} \tag{4.10}$$

Substituting $\rho_1 = \sum_k c_k(\omega) e^{ik\varphi + \lambda t}$ $(k \neq 0)$, one can find independent equations for different c_k:

$$(\lambda + ik\omega + Dk^2) c_k(\omega) = \frac{\varepsilon + \varepsilon_f e^{-\lambda T}}{2} (\delta_{k,1} + \delta_{k,-1}) C_k, \tag{4.11}$$

where $C_k = \int_{-\infty}^{+\infty} g(\omega) c_k(\omega)\, d\omega$. Modes with $|k| \neq 1$ always decay ($\lambda_k = -Dk^2 - ik\omega$). The case of $k = \pm 1$ (the case of $k = -1$ can be obtained via complex conjugation from the case $k = 1$) requires an analysis. We can find

$$c_1(\omega) = \frac{\varepsilon + \varepsilon_f e^{-\lambda T}}{2(\lambda + D + i\omega)} C_1. \tag{4.12}$$

Multiplying this equation by $g(\omega)$ and integrating over ω, we obtain

$$\left(1 - \frac{\varepsilon + \varepsilon_f e^{-\lambda T}}{2} \int_{-\infty}^{+\infty} \frac{g(\omega)\, d\omega}{D + \lambda + i\omega} \right) C_1 = 0.$$

So, the spectrum of the increments λ for the modes with $k = 1$ is formed by the roots of the "spectral function" $\Lambda(\lambda)$

$$\Lambda(\lambda) \equiv 1 - \frac{\varepsilon + \varepsilon_f e^{-\lambda T}}{2} \int_{-\infty}^{+\infty} \frac{g(\omega)\, d\omega}{D + \lambda + i\omega}. \tag{4.13}$$

Here two situations are possible: (i) a nonsymmetric distribution of natural frequencies $g(\omega) \neq g(-\omega)$, and (ii) a symmetric distribution of natural frequencies $g(\omega) = g(-\omega)$. In the first case $\Im\left(\int\limits_{-\infty}^{+\infty} \frac{g(\omega)\,d\omega}{D+i\omega}\right) = \int\limits_{-\infty}^{+\infty} \frac{\omega g(\omega)\,d\omega}{D^2+\omega^2} \neq 0$ (we do not consider any degenerated situations but the physically motivated ones, like $g(\omega) = g(-\omega)$); therefore real roots of $\Lambda(\lambda)$ (including $\lambda = 0$) are not admitted. In the second case $\Lambda^*(\lambda) = \Lambda(\lambda^*)$; therefore real roots are admitted, and complex roots appear in pairs $(\lambda,\ \lambda^*)$. So, the *critical perturbations* (henceforth, the term "critical perturbations" means the infinitesimal perturbations of the absolutely nonsynchronous state, growing when the state is unstable) have the following form:

(i) $g(\omega) \neq g(-\omega)$

for $\lambda = -i\Omega$, $\quad \rho_1 = \alpha(\omega)e^{i(\varphi-\Omega t)} + c.c.$ — Hopf bifurcation;

(ii) $g(\omega) = g(-\omega)$

for $\lambda = 0$, $\quad \rho_1 = \alpha(\omega)e^{i\varphi} + c.c.$ — steady-state bifurcation,

for $\lambda = \pm i\Omega$, $\quad \rho_1 = \alpha(\omega)e^{i(\varphi-\Omega t)} + \beta(\omega)e^{i(\varphi+\Omega t)} + c.c.$ — Hopf bifurcation (after Crawford [62]).

According to Crawford, the only symmetric distributions admitting the Hopf bifurcation in the absence of a delayed feedback are multimodal ones (for further details on behavior of ensembles with multimodal distributions consult [72, 73, 74]). The role of time-delayed feedback in populations with bimodal $g(\omega)$ has been studied in [75].

4.2.2 Weakly nonlinear analysis: Nonsymmetric distribution $g(\omega)$

In this section we use conventional multiple scale analysis to develop a weakly nonlinear theory of the synchronization transition, considering ε as a bifurcation parameter. We write $\varepsilon = \varepsilon_0 + \kappa^2\varepsilon_2$ where ε_0 is the critical value of ε, and κ is a formal small parameter, represent the probability density as

$$\rho(x,t) = \rho_0 + \kappa\rho_1 + \kappa^2\rho_2 + \kappa^3\rho_3 + \dots\,,$$

and introduce "slow times" t_k:

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t_0} + \kappa^2\frac{\partial}{\partial t_2} + \kappa^4\frac{\partial}{\partial t_4} + \dots\,.$$

Here only even powers of κ are used for ε and time scales due to the system symmetry.

Assuming

$$\rho_1 = \alpha_1(\omega,\, t_2, t_4, ...)e^{i(\varphi-\Omega t_0)} + c.c.$$

and substituting this in Eq.(4.5) we obtain in the order κ^2 (there is no secular terms in this order):

$$\frac{\partial \rho_2}{\partial t_0} + \rho_0 \frac{\partial v_2}{\partial \varphi} + \frac{\partial}{\partial \varphi}(\rho_1 v_1) + v_0 \frac{\partial \rho_2}{\partial \varphi} - D \frac{\partial^2 \rho_2}{\partial \varphi^2} = 0, \tag{4.14}$$

where

$$\begin{aligned} v_1 &= \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \sin(\theta - \varphi) \alpha_1(\omega') \left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right) e^{i(\theta - \Omega t_0)} + c.c. \\ &= i\pi \int_{-\infty}^{+\infty} \alpha_1(\omega') \left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right) e^{i(\varphi - \Omega t_0)} g(\omega')\, d\omega' + c.c. \\ &= i\pi \left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right) A_1 e^{i(\varphi - \Omega t_0)} + c.c.\,, \end{aligned} \tag{4.15}$$

$$A_j \equiv \int_{-\infty}^{+\infty} \alpha_j(\omega) g(\omega)\, d\omega\,. \tag{4.16}$$

Noteworthy, from (4.12) it follows that

$$\alpha_1(\omega) = \frac{\varepsilon_0 + \varepsilon_f e^{i\Omega T}}{2(D + i(\omega - \Omega))} A_1\,. \tag{4.17}$$

The "driving term" in (4.14) has the form:

$$\begin{aligned} \frac{\partial}{\partial \varphi}(\rho_1 v_1) &= \frac{\partial}{\partial \varphi}\left(i\pi \left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right) \alpha_1(\omega) A_1 e^{i2(\varphi - \Omega t_0)} + c.c. + \dots\right) \\ &= -2\pi \left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right) \alpha_1(\omega) A_1 e^{i2(\varphi - \Omega t_0)} + c.c.\,. \end{aligned}$$

Searching for solution of Eq.(4.14) in the form

$$\rho_2 = \alpha_2(\omega, t_2, t_4, ...) e^{2(\varphi - \Omega t_0)} + c.c.\,,$$

we obtain, making use of (4.17) and $v_2 = 0$ (due to $\int_0^{2\pi} e^{i\varphi} \rho_2(\omega, \varphi, t)\, d\varphi = 0$),

$$(-i2\Omega + i2\omega + 4D)\alpha_2(\omega) = 2\pi \left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right) \alpha_1(\omega) A_1 = \pi \frac{\left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right)^2}{D + i(\omega - \Omega)} A_1^2,$$

i.e.

$$\alpha_2(\omega) = \frac{\pi \left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right)^2 A_1^2}{2(D + i(\omega - \Omega))(2D + i(\omega - \Omega))}. \tag{4.18}$$

In the order κ^3 of Eq.(4.5), secular terms appear:

$$\frac{\partial \rho_3}{\partial t} + \frac{\partial \rho_1}{\partial t_2} + \rho_0 \frac{\partial v_3}{\partial \varphi} + \frac{\partial}{\partial \varphi}(\rho_1 v_2 + \rho_2 v_1) + v_0 \frac{\partial \rho_3}{\partial \varphi} - D \frac{\partial^2 \rho_3}{\partial \varphi^2} = 0. \tag{4.19}$$

As mentioned above $v_2 = 0$. For v_3 we obtain

$$v_3 = v_{\rho_3} + \varepsilon_2 \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \sin(\theta - \varphi) \rho_1(\omega', \theta, t)$$
$$+\varepsilon_f \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \sin(\theta - \varphi) \left((-T) \frac{\partial \alpha_1(\omega', t_2, ...)}{\partial t_2} e^{i(\theta - \Omega(t-T))} + c.c. \right)$$
$$= v_{\rho_3} + \left(i\pi \left(\varepsilon_2 A_1 - \varepsilon_f e^{i\Omega T} T \frac{\partial A_1}{\partial t_2} \right) e^{i(\varphi - \Omega t_0)} + c.c. \right),$$

where

$$v_{\rho_3} = \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \sin(\theta - \varphi) \left(\varepsilon_0 \rho_3(\omega', \theta, t) + \varepsilon_f \rho_3(\omega', \theta, t - T) \right),$$

and

$$\frac{\partial}{\partial \varphi} (\rho_2 v_1) = \frac{\partial}{\partial \varphi} \left(-i\pi \left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T} \right) A_1^* \alpha_2(\omega) e^{i(\varphi - \Omega t_0)} + c.c. + \dots \right)$$
$$= \pi \left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T} \right) A_1^* \alpha_2(\omega) e^{i(\varphi - \Omega t_0)} + c.c. .$$

Therefore Eq.(4.20) can be rewritten as

$$\left[\frac{\partial \alpha_1(\omega)}{\partial t_2} - \frac{\varepsilon_2}{2} A_1 + \frac{\varepsilon_f e^{i\Omega T}}{2} T \frac{\partial A_1}{\partial t_2} + \pi \left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T} \right) A_1^* \alpha_2(\omega) \right] e^{i(\varphi - \Omega t_0)} + c.c. + \dots = 0, \quad (4.20)$$

where "…" denotes terms a fortiori orthogonal to the solutions of the problem conjugated to (4.9), i.e. non-secular terms.

Defining scalar product of τ-time-periodic fields $f(\omega, \varphi, t_0)$ and $h(\omega, \varphi, t_0)$ as

$$\langle f, h \rangle \equiv \int_{-\infty}^{+\infty} d\omega\, g(\omega) \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_0^{\tau} \frac{dt_0}{\tau} f^*(\omega, \varphi, t_0)\, h(\omega, \varphi, t_0), \quad (4.21)$$

we find the problem conjugated to (4.9) in the Fourier space:

$$\left(-\lambda - ik\omega + Dk^2 \right) c_k^+(\omega) = \frac{\varepsilon_0 + \varepsilon_f e^{\lambda T}}{2} (\delta_{k,1} + \delta_{k,-1}) \int_{-\infty}^{+\infty} g(\omega') c_k^+(\omega')\, d\omega' \quad (4.22)$$

(c_k^+ are defined so that the solutions of the conjugated problem are $\rho^+ = \sum_k c_k^+(\omega) e^{ik\varphi + \lambda t}$, $k \neq 0$). Similarly to (4.12), the required eigensolution of the conjugated problem is

$$\rho^+(\omega, \varphi, t) = \frac{e^{i(\varphi - \Omega t)}}{D - i(\omega - \Omega)} \quad (4.23)$$

(normalization is not important).

Evaluating the scalar product of ρ^+ and Eq.(4.20)

$$\int_{-\infty}^{+\infty} \frac{g(\omega)\,d\omega}{D+i(\omega-\Omega)} \left[\frac{\partial \alpha_1(\omega)}{\partial t_2} - \frac{\varepsilon_2}{2} A_1 \right.$$
$$\left. + \frac{\varepsilon_f e^{i\Omega T}}{2} T \frac{\partial A_1}{\partial t_2} + \pi \left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T} \right) A_1^* \alpha_2(\omega) \right] = 0$$

and substituting for α_j, we obtain:

$$\left[\frac{(\varepsilon_0+\varepsilon_f e^{i\Omega T})^2}{2} \int_{-\infty}^{+\infty} \frac{g(\omega)\,d\omega}{(D+i(\omega-\Omega))^2} + \varepsilon_f e^{i\Omega T} T \right] \frac{\partial A_1}{\partial t_2}$$
$$-\varepsilon_2 A_1 + \frac{\pi^2}{2} \left| \varepsilon_0 + \varepsilon_f e^{i\Omega T} \right|^2 \left(\varepsilon_0 + \varepsilon_f e^{i\Omega T} \right)^2 \tag{4.24}$$
$$\times \int_{-\infty}^{+\infty} \frac{g(\omega)\,d\omega}{(D+i(\omega-\Omega))^2(2D+i(\omega-\Omega))} A_1 \left| A_1 \right|^2 = 0\,.$$

For the further analysis, it is convenient to introduce the function

$$G(z) \equiv \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{g(\omega)\,d\omega}{\omega - z}\,.$$

Now Eq.(4.24) can be rewritten in the final form:

$$\dot{A}_1 - \lambda_2(\varepsilon_0,\,\Omega) A_1 + P(\varepsilon_0,\,\Omega) A_1 \left| A_1 \right|^2 = 0\,, \tag{4.25}$$

where λ_2 is the linear growth rate

$$\lambda_2(\varepsilon,\,\Omega) = \frac{\varepsilon_2}{i\pi \left(\varepsilon + \varepsilon_f e^{i\Omega T} \right)^2 G'(\Omega + iD) + \varepsilon_f e^{i\Omega T} T}\,, \tag{4.26}$$

and

$$P(\varepsilon,\,\Omega) = \frac{\pi^2 \left| \varepsilon + \varepsilon_f e^{i\Omega T} \right|^2 \left(iDG'(\Omega+iD) - G(\Omega+i2D) + G(\Omega+iD) \right)}{D \left(iDG'(\Omega+iD) + \pi^{-1} D \varepsilon_f e^{i\Omega T} T \left(\varepsilon + \varepsilon_f e^{i\Omega T} \right)^{-2} \right)}\,. \tag{4.27}$$

Equation (4.25) and the expressions (4.26), (4.27) are the main result of our analysis for nonsymmetric frequency distributions. They give a full description of the effect of the delayed global feedback on the synchronization transition in the ensemble of oscillators. The linear part (4.26) has already been discussed in [66], and the expression (4.27) completes the description of the synchronization transition. Having determined the amplitude

A_1 from (4.25), one can find the establishing probability density

$$\rho(\omega,\,\varphi,\,t)=\frac{1}{2\pi}\left[1+\frac{\pi\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)}{D+i(\omega-\Omega)}A_1(t)e^{i(\varphi-\Omega t)}+c.c.\right.$$
$$\left.+\frac{\pi^2\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)^2}{(D+i(\omega-\Omega))(2D+i(\omega-\Omega))}A_1^2(t)e^{i2(\varphi-\Omega t)}+c.c.+O(A_1^3)\right], \qquad (4.28)$$

and the order parameter

$$R(t)=2\pi A_1^{*}e^{i\Omega t}(1+O(A_1^2))\,.$$

4.2.3 Weakly nonlinear analysis: Symmetric distribution $g(\omega)$ — steady-state bifurcation

In the case of $\lambda=0$, the critical value of coupling coefficient can be found in an explicit form:

$$\varepsilon_0=\frac{2}{D}\left[\int_{-\infty}^{+\infty}\frac{g(\omega)\,d\omega}{D^2+\omega^2}\right]^{-1}-\varepsilon_f. \qquad (4.29)$$

Actually, the case of symmetric distribution $g(\omega)$ providing $\Omega=0$ is a special case of the situation studied in the previous subsection. So, the previous analysis turns into the analysis for this case as soon as we set $\Omega=0$. Therefore

$$\dot{R}-\lambda_2(\varepsilon_0,\,0)R+(2\pi)^{-2}P(\varepsilon_0,\,0)R\,|R|^2+O(R^5)=0, \qquad (4.30)$$

$[\,\lambda_2(\varepsilon_0,\,0),\ P(\varepsilon_0,\,0)\in\mathbf{R}$, since for symmetric $g(\omega)$, $G^*(z)=G(-z^*)\,]$ and the probability density is

$$\rho(\omega,\,\varphi,\,t)=\frac{1}{2\pi}\left[1+\frac{\varepsilon_0+\varepsilon_f}{2(D+i\omega)}R^*(t)e^{i\varphi}+cc\right.$$
$$\left.+\frac{(\varepsilon_0+\varepsilon_f)^2}{4(D+i\omega)(2D+i\omega)}R^{*2}(t)e^{i2\varphi}+c.c.+O(R^3)\right]. \qquad (4.31)$$

Actually, in this case $\mathrm{Arg}(R)$ remains constant on the central manifold, nevertheless we keep it in a complex form for the reason of generality.

4.2.4 Weakly nonlinear analysis: Symmetric distribution $g(\omega)$ — Hopf bifurcation

Considering the perturbation

$$\rho_1=\alpha_1(\omega,\,t_2,t_4,...)e^{i(\varphi-\Omega t_0)}+\beta_1(\omega,\,t_2,t_4,...)e^{i(\varphi+\Omega t_0)}+c.c.$$

near the threshold, from Eq.(4.5) we obtain in the order κ^2 (there are no secular terms in this order) Eq.(4.14) with

$$v_1 = i\pi\left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right) A_1 e^{i(\varphi-\Omega t_0)} + i\pi\left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T}\right) B_1 e^{i(\varphi+\Omega t_0)} + c.c.\ . \tag{4.32}$$

From Eq.(4.12), $\alpha_1(\omega) = \dfrac{\varepsilon_0 + \varepsilon_f e^{i\Omega T}}{2(D + i(\omega-\Omega))} A_1$, and $\beta_1(\omega) = \dfrac{\varepsilon_0 + \varepsilon_f e^{-i\Omega T}}{2(D + i(\omega+\Omega))} B_1$. Therefore

$$\begin{aligned}
\frac{\partial}{\partial\varphi}(\rho_1 v_1) &= \frac{\partial}{\partial\varphi}\Big[i\pi\left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right) A_1\alpha_1(\omega) e^{i2(\varphi-\Omega t_0)} \\
&\quad + i\pi\left\{\left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T}\right)_1 B\alpha_1(\omega) + \left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right) A_1\beta_1(\omega)\right\} e^{i2\varphi} \\
&\quad + i\pi\left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T}\right) B_1\beta_1(\omega) e^{i2(\varphi+\Omega t_0)} + c.c.\Big] \\
&= -\pi\left[\frac{\left(\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right)^2}{D + i(\omega-\Omega)} A_1^2 e^{i2(\varphi-\Omega t_0)}\right. \\
&\quad + \left|\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right|^2 \left(\frac{1}{D + i(\omega-\Omega)} + \frac{1}{D + i(\omega+\Omega)}\right) A_1 B_1 e^{i2\varphi} \\
&\quad \left. + \frac{\left(\varepsilon_0 + \varepsilon_f e^{-i\Omega T}\right)^2}{D + i(\omega+\Omega)} B_1^2 e^{i2(\varphi+\Omega t_0)} + c.c.\right].
\end{aligned}$$

Searching for ρ_2 in the form

$$\rho_2 = \eta^{(-1)}(\omega,\, t_2,\, t_4, ...) e^{i2(\varphi-\Omega t_0)} + \eta^{(0)}(\omega,\, t_2,\, t_4, ...) e^{i2\varphi} + \eta^{(1)}(\omega,\, t_2,\, t_4, ...) e^{i2(\varphi+\Omega t_0)} + c.c.\ ,$$

we find

$$(i2l\Omega + i2\omega + 4D)\eta^{(l)}(\omega) = -\frac{\Omega}{\pi}\int_0^{\Omega^{-1}\pi} e^{-i2l\Omega t_0} \frac{\partial}{\partial\varphi}(\rho_1 v_1)\, dt_0\,.$$

So,

$$\eta^{(\pm 1)}(\omega) = \frac{\pi\left(\varepsilon_0 + \varepsilon_f e^{\mp i\Omega T}\right)^2}{2(D + i(\omega\pm\Omega))(2D + i(\omega\pm\Omega))} \begin{cases} B_1^2, & l = +1, \\ A_1^2, & l = -1; \end{cases} \tag{4.33}$$

$$\eta^{(0)}(\omega) = \frac{\pi}{2}\frac{\left|\varepsilon_0 + \varepsilon_f e^{i\Omega T}\right|^2}{2D + i\omega}\left(\frac{1}{D + i(\omega-\Omega)} + \frac{1}{D + i(\omega+\Omega)}\right) A_1 B_1\,. \tag{4.34}$$

In the order κ^3 Eq.(4.19) with $v_2 = 0$ and

$$v_3 = v_{\rho_3} + \left(i\pi\left(\varepsilon_2 A_1 - \varepsilon_f e^{i\Omega T} T\frac{\partial A_1}{\partial t_2}\right) e^{i(\varphi-\Omega t_0)} + i\pi\left(\varepsilon_2 B_1 - \varepsilon_f e^{-i\Omega T} T\frac{\partial B_1}{\partial t_2}\right) e^{i(\varphi+\Omega t_0)} + c.c.\right)$$

is valid. Here $v_{\rho_3} = \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega' g(\omega') \sin(\theta-\varphi) \left(\varepsilon_0 \rho_3(\omega',\theta,t) + \varepsilon_f \rho_3(\omega',\theta,t-T)\right)$; and

$$\begin{aligned}
\frac{\partial}{\partial\varphi}(\rho_2 v_1) &= \frac{\partial}{\partial\varphi}\Big(-i\pi\left[\left(\varepsilon_0+\varepsilon_f e^{-i\Omega T}\right)A_1^*\eta^{(-1)} + \left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)B_1^*\eta^{(0)}\right]e^{i(\varphi-\Omega t_0)} \\
&\quad -i\pi\left[\left(\varepsilon_0+\varepsilon_f e^{-i\Omega T}\right)A_1^*\eta^{(0)} + \left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)B_1^*\eta^{(1)}\right]e^{i(\varphi+\Omega t_0)} + c.c. + \dots\Big) \\
&= \pi\left(\left(\varepsilon_0+\varepsilon_f e^{-i\Omega T}\right)A_1^*\eta^{(-1)} + \left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)B_1^*\eta^{(0)}\right)e^{i(\varphi-\Omega t_0)} \\
&\quad +\pi\left(\left(\varepsilon_0+\varepsilon_f e^{-i\Omega T}\right)A_1^*\eta^{(0)} + \left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)B_1^*\eta^{(1)}\right)e^{i(\varphi+\Omega t_0)} + c.c. + \dots\,,
\end{aligned}$$

where "…" again denotes certainly non-secular terms [a fortiori orthogonal to the solutions of the problem conjugated to (4.9)].

So, Eq.(4.19) can be rewritten as

$$\begin{aligned}
&\left[\frac{\partial\alpha_1(\omega)}{\partial t_2} - \frac{\varepsilon_2}{2}A_1 + \frac{\varepsilon_f e^{i\Omega T}}{2}T\frac{\partial A_1}{\partial t_2}\right. \\
&\qquad \left. +\pi\left(\varepsilon_0+\varepsilon_f e^{-i\Omega T}\right)A_1^*\eta^{(-1)} + \pi\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)B_1^*\eta^{(0)}\right]e^{i(\varphi-\Omega t_0)} \\
&+\left[\frac{\partial\beta_1(\omega)}{\partial t_2} - \frac{\varepsilon_2}{2}B_1 + \frac{\varepsilon_f e^{-i\Omega T}}{2}T\frac{\partial B_1}{\partial t_2}\right. \\
&\qquad \left. +\pi\left(\varepsilon_0+\varepsilon_f e^{-i\Omega T}\right)A_1^*\eta^{(0)} + \pi\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)B_1^*\eta^{(1)}\right]e^{i(\varphi+\Omega t_0)} + c.c. + \dots = 0
\end{aligned} \tag{4.35}$$

(again "…" denotes certainly non-secular terms).

Evaluating the scalar products of Eq.(4.35) and the solutions of the conjugated problem $\rho_\alpha^+ = \dfrac{e^{i(\varphi-\Omega t)}}{D-i(\omega-\Omega)}$ and $\rho_\beta^+ = \dfrac{e^{i(\varphi+\Omega t)}}{D-i(\omega+\Omega)}$, successively, we obtain the governing equation for A_1:

$$\begin{aligned}
&\left[\frac{(\varepsilon_0+\varepsilon_f e^{i\Omega T})^2}{2}\int_{-\infty}^{+\infty}\frac{g(\omega)\,d\omega}{(D+i(\omega-\Omega))^2} + \varepsilon_f e^{i\Omega T}T\right]\frac{\partial A_1}{\partial t_2} \\
&\qquad -\varepsilon_2 A_1 + \frac{\pi^2}{2}\left|\varepsilon_0+\varepsilon_f e^{i\Omega T}\right|^2\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)^2 \\
&\quad \times\left[A_1|A_1|^2\int_{-\infty}^{+\infty}\frac{g(\omega)\,d\omega}{(D+i(\omega-\Omega))^2(2D+i(\omega-\Omega))}\right. \\
&\left. +A_1|B_1|^2\int_{-\infty}^{+\infty}\frac{g(\omega)\,d\omega}{(D+i(\omega-\Omega))(2D+i\omega)}\left(\frac{1}{D+i(\omega-\Omega)}+\frac{1}{D+i(\omega+\Omega)}\right)\right] = 0\,;
\end{aligned} \tag{4.36}$$

and the governing equation for B_1 into which Eq.(4.36) turns via the substitution $(A_1,\,\Omega)\leftrightarrow(B_1,\,-\Omega)$. Finally,

$$\dot{A}_1+[-\lambda_2(\varepsilon_0,\,\Omega)+\,P\,(\varepsilon_0,\,\Omega)\,|A_1|^2+\,Q\,(\varepsilon_0,\,\Omega)\,|B_1|^2]A_1=0\,, \tag{4.37}$$

$$\dot{B}_1+[-\lambda_2^*(\varepsilon_0,\,\Omega)+P^*(\varepsilon_0,\,\Omega)\,|B_1|^2+Q^*(\varepsilon_0,\,\Omega)\,|A_1|^2]B_1=0\,, \tag{4.38}$$

where we make use of the fact that for symmetric $g(\omega)$, $G^*(z)=G(-z^*)$, and, consequently, $\lambda_2(\varepsilon,\,-\Omega)=\lambda_2^*(\varepsilon,\,\Omega)$, $P(\varepsilon,\,-\Omega)=P^*(\varepsilon,\,\Omega)$, $Q(\varepsilon,\,-\Omega)=Q^*(\varepsilon,\,\Omega)$;

$$\begin{aligned}Q(\varepsilon,\,\Omega)\;&=\;\frac{\pi^2\left|\varepsilon+\varepsilon_f e^{i\Omega T}\right|^2}{iG'(\Omega+iD)+\pi^{-1}\varepsilon_f e^{i\Omega T}T\left(\varepsilon+\varepsilon_f e^{i\Omega T}\right)^{-2}}\\&\quad\times\left(\frac{G'(\Omega+iD)}{\Omega-iD}-\frac{(\Omega+iD)G(\Omega+iD)}{2\Omega(\Omega-iD)^2}\right.\\&\quad\left.+\frac{G(-\Omega+iD)}{2\Omega(\Omega+iD)}+\frac{i2DG(i2D)}{(\Omega^2+D^2)(\Omega-iD)}\right).\end{aligned} \tag{4.39}$$

The probability distribution is

$$\begin{aligned}\rho(\omega,\,\varphi,\,t)\;=\;&\frac{1}{2\pi}\left[1+\frac{\pi\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)}{D+i(\omega-\Omega)}A_1(t)e^{i(\varphi-\Omega t)}+c.c.\right.\\&+\frac{\pi\left(\varepsilon_0+\varepsilon_f e^{-i\Omega T}\right)}{D+i(\omega+\Omega)}B_1(t)e^{i(\varphi+\Omega t)}+c.c.\\&+\frac{\pi^2\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)^2}{(D+i(\omega-\Omega))(2D+i(\omega-\Omega))}A_1^2(t)e^{i2(\varphi-\Omega t)}+c.c.\\&+\frac{2\pi^2\left|\varepsilon_0+\varepsilon_f e^{i\Omega T}\right|^2(D+i\omega)}{(2D+i\omega)(D+i(\omega-\Omega))(D+i(\omega+\Omega))}A_1(t)B_1(t)e^{i2\varphi}+c.c.\\&+\frac{\pi^2\left(\varepsilon_0+\varepsilon_f e^{-i\Omega T}\right)^2}{(D+i(\omega+\Omega))(2D+i(\omega+\Omega))}B_1^2(t)e^{i2(\varphi+\Omega t)}+c.c.\\&\left.+O(A_1^3+B_1^3)\right],\end{aligned} \tag{4.40}$$

and the order parameter

$$R(t)=2\pi\left(A_1^*e^{i\Omega t}+B_1^*e^{-i\Omega t}\right)+O\big(A_1^3+B_1^3\big)\,.$$

Analysis of the amplitude equations (4.37), (4.38)

The amplitude equations (4.37),(4.38) coincide with a special case of the amplitude equations for the co-dimension 2 generalized Hopf–Hopf bifurcation. The peculiarity is that here the coefficients in Eq.(4.37) are not independent of the corresponding ones in Eq.(4.38) but conjugated to them. The generalized Hopf–Hopf bifurcation is well treated (see, for instance, [76]), but we represent here this equations analysis for our special case.

Evolution of the magnitudes $|A_1|$ and $|B_1|$ does not depend on the phases $\arg A_1$ and $\arg B_1$. Introducing $x = \Re(P)|A_1|^2$, $y = \Re(P)|B_1|^2$, $q = \Re Q/\Re P$, $r = \Re\lambda_2$, and rescaling time $2t \to t$, we can write

$$\begin{cases} \dot{x} + x^2 + qxy = rx, \\ \dot{y} + y^2 + qxy = ry. \end{cases} \tag{4.41}$$

For $\Re P > 0$ only the first quadrant of the (x, y)-plane makes sense, while for $\Re P < 0$ only the third one does. An absence of any limit cycles is guaranteed since the divergence

$$\nabla \cdot \left((xy)^{-\frac{2+q}{1+q}} \vec{F} \right) = -2(1+q)^{-1}(xy)^{-\frac{2+q}{1+q}} r$$

is of fixed sign, where $\vec{F}$ is the phase flow.

The investigation for fixed points yields:

Fixed point coordinates	Eigenvalues	Eigenvectors
$(0,\ 0)$	$\{r,\ r\}$	$\{(1,\ 0),\ (0,\ 1)\}$
$(0,\ r)$	$\{r(1-q),\ -r\}$	$\{(1,\ 0),\ (0,\ 1)\}$
$(r,\ 0)$	$\{-r,\ r(1-q)\}$	$\{(1,\ 0),\ (0,\ 1)\}$
$\left(\frac{r}{1+q}, \frac{r}{1+q}\right)$	$\left\{-r,\ -r\frac{1-q}{1+q}\right\}$	$\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}$

In order to investigate the asymptotical behavior of the system for large x and y, we perform the substitution $x = R^{-1}\cos\phi$, $y = R^{-1}\sin\phi$, $[\,t\,] = R$, and consider system behavior close to $R = 0$ (see [77]):

$$\dot{R} = R(\cos^3\phi + \sin^3\phi + q\sin\phi\cos\phi(\cos\phi + \sin\phi)) ,$$

$$\dot{\phi} = \frac{1}{2}\sin 2\phi(\cos\phi + \sin\phi)(1-q) .$$

The last system has the following fixed points: $\phi_* = 0,\ \pi/4,\ \pi/2,\ \pi,\ 5\pi/4,\ 3\pi/2$. The problem of their stability is quite trivial.

The results of the bifurcation analysis of the dynamical system (4.41) are summarized on Fig. 4.1 [note the system symmetry $(\Re\lambda_2,\ x,\ y,\ t) \leftrightarrow (-\Re\lambda_2,\ -x,\ -y,\ -t)$]. Non-trivial stable solutions are possible only for $\Re\lambda_2 > 0$, $\Re P > 0$, $\Re Q > -\Re P$. Specifically, there is one non-trivial stable solution $|A_1| = |B_1| = \sqrt{\dfrac{\Re\lambda_2}{\Re(P+Q)}}$, at $-\Re P < \Re Q < \Re P$, and

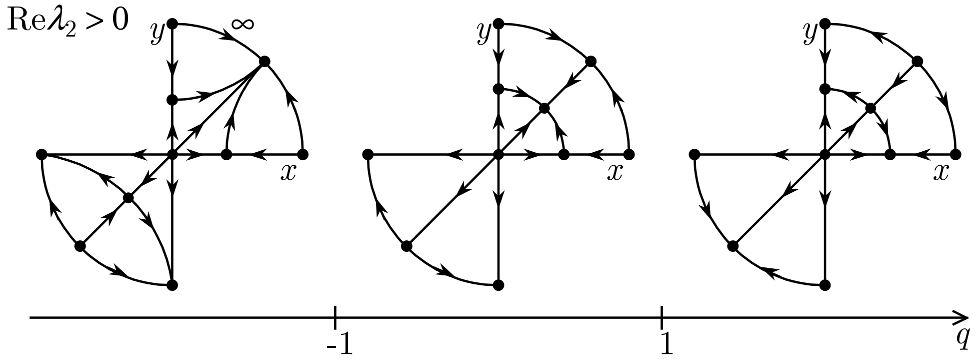

Figure 4.1: *Symmetric distribution* $g(\omega)$*.* The regime diagram for the Hopf bifurcation is plotted. Remember the symmetry $(\Re\lambda_2,\, x,\, y,\, t) \leftrightarrow (-\Re\lambda_2,\, -x,\, -y,\, -t)$. External arcs correspond to infinity.

there is bistability between $|A_1| = 0$, $|B_1| = \sqrt{\dfrac{\Re\lambda_2}{\Re P}}$, and $|A_1| = \sqrt{\dfrac{\Re\lambda_2}{\Re P}}$, $|B_1| = 0$, at $\Re P < \Re Q$.

4.2.5 Example: Nonsymmetric Lorentz distribution $g(\omega)$

Let us consider a non-symmetric Lorentz distribution as an example:

$$g(\omega) = \frac{\gamma}{\pi\left((\omega-\omega_0)^2+\gamma^2\right)}\,, \tag{4.42}$$

where γ is the characteristic width of the distribution, ω_0 is the average frequency (ω can be assumed positive, otherwise one should replace φ with $-\varphi$). Therefore

$$\begin{aligned} G(z) &= \frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{g(\omega)\,d\omega}{\omega-z} = \frac{i\gamma}{2\pi^2}\int_{-\infty}^{+\infty}\frac{d\omega}{(\omega-z)(\omega-\omega_0+i\gamma)(\omega-\omega_0-i\gamma)} \\ &= \frac{\gamma}{\pi}\frac{1}{(\omega-z)(\omega-\omega_0-i\gamma)}\bigg|_{\omega=\omega_0-i\gamma} = \frac{i}{2\pi}\frac{1}{\omega_0-i\gamma-z}\,, \end{aligned}$$

where $\Im z$ is assumed to be positive [this is fulfilled for (4.26), (4.27), (4.39) as $D > 0$].

Spectrum. The eq. $\Lambda(\mu - i\Omega) = 0$ takes the form:

$$1 + \frac{i\left(\varepsilon+\varepsilon_f e^{-\mu T+i\Omega T}\right)}{2(\omega_0-\Omega-i(\gamma+D+\mu))} = 0\,,$$

or

$$\omega_0 - \Omega - i(\gamma+D+\mu) + \frac{i}{2}\left(\varepsilon+\varepsilon_f e^{-\mu T+i\Omega T}\right) = 0\,.$$

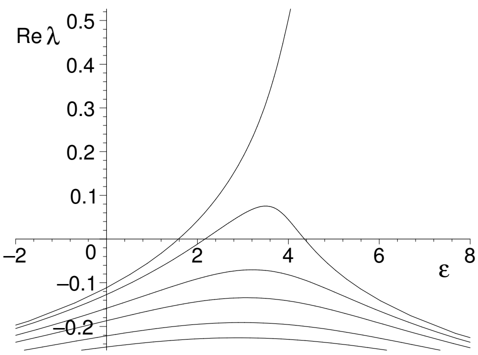

Figure 4.2: *Nonsymmetric Lorentz distribution* $g(\omega)$. The spectrum sample at $\omega_0 = 3$, $\varepsilon_f = 1.5$, $T = 5.7$, $\gamma + D = 1.5$ is plotted.

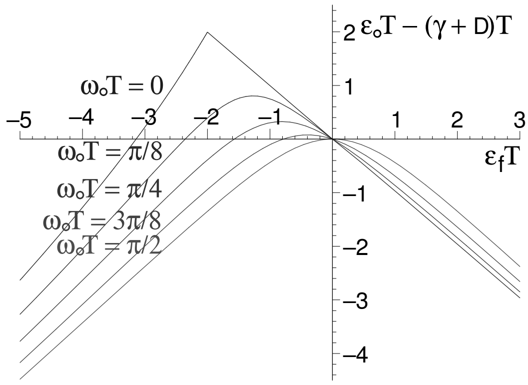

Figure 4.3: *Nonsymmetric Lorentz distribution* $g(\omega)$. The stability boundary is plotted for different values of ω_0.

Extracting the real and imaginary parts, we find

$$\Omega = \omega_0 - \frac{\varepsilon_f}{2} e^{-\mu T} \sin \Omega T \,, \tag{4.43}$$

$$\varepsilon = 2(\gamma + D + \mu) - \varepsilon_f e^{-\mu T} \cos \Omega T \,. \tag{4.44}$$

So, the increments can be parameterized by Ω: $\mu = \dfrac{1}{T} \ln \dfrac{\varepsilon_f \sin \Omega T}{2(\omega_0 - \Omega)}$, $\varepsilon = 2(\mu + \gamma + D) - \varepsilon_f e^{-\mu T} \cos \Omega T$. In Fig. 4.2 a sample of spectrum is presented.

$\gamma = 0.1$, $\omega_0 = 3$, $D = 0.1$ $\qquad$ $\gamma = 0.5$, $\omega_0 = 3$, $D = 1$

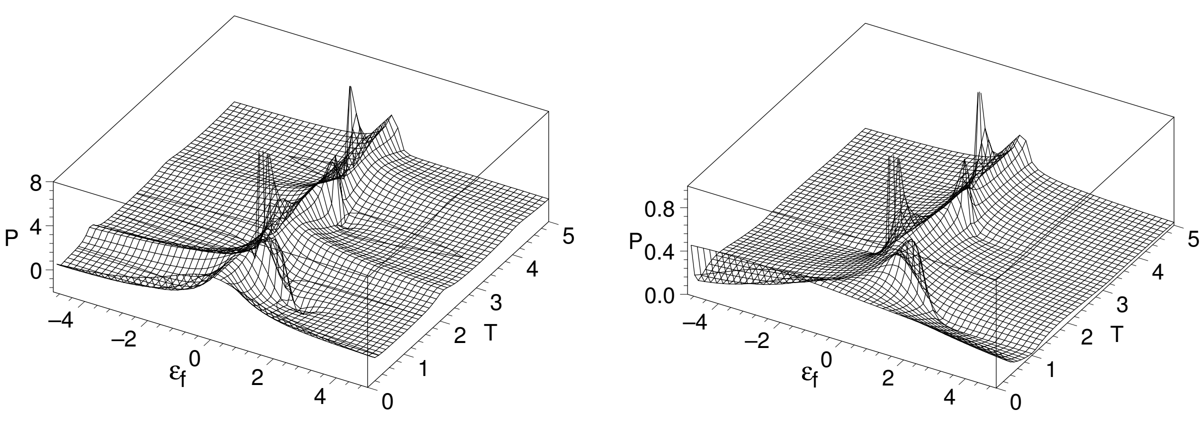

Figure 4.4: *Nonsymmetric Lorentz distribution* $g(\omega)$. Samples of the dependence of P on ε_f and T. In the left plot additional lines depict zeros of P.

Hopf bifurcation. The value of Ω providing the minimal critical value ε_0 belongs to the interval $\left(\frac{\pi}{T}\mathrm{Eint}\left(\frac{\omega_0 T}{\pi}\right), \frac{\pi}{T}\left(\mathrm{Eint}\left(\frac{\omega_0 T}{\pi}\right)+1\right)\right)$, where $\mathrm{Eint}(x)$ is the integer part of x, and satisfies to the equation system (4.43),(4.44) at $\mu=0$. Due to the symmetry of this system at $\mu=0$: $(\omega_0 T,\, \varepsilon_f) \to (\omega_0 T+\pi/2,\, -\varepsilon_f)$, it is enough to consider $\omega_0 \in (0,\, \pi/2]$. Stability boundaries for different values of ω_0 are plotted in Fig. 4.3.

According to the consideration above,

$$\begin{aligned}
\lambda_2(\varepsilon_0,\,\Omega) &= \frac{\varepsilon_2}{i\pi\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)^2 G'(\Omega+iD)+\varepsilon_f e^{i\Omega T}T} \\
&= \varepsilon_2\left[-\frac{1}{2}\left(\frac{\varepsilon_0+\varepsilon_f e^{i\Omega T}}{\omega_0-\Omega-i(\gamma+D)}\right)^2+\varepsilon_f e^{i\Omega T}T\right]^{-1} = \frac{\varepsilon_2}{2+\varepsilon_f T e^{i\Omega T}}, \quad (4.45)
\end{aligned}$$

$$\begin{aligned}
P(\varepsilon_0,\,\Omega) &= \frac{\pi^2\left|\varepsilon_0+\varepsilon_f e^{i\Omega T}\right|^2\left(iDG'(\Omega+iD)-G(\Omega+i2D)+G(\Omega+iD)\right)}{D\left(iDG'(\Omega+iD)+\pi^{-1}D\varepsilon_f e^{i\Omega T}T\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)^{-2}\right)} \\
&= \frac{\pi^2\left|\varepsilon_0+\varepsilon_f e^{i\Omega T}\right|^2\left(-\frac{D}{(\omega_0-\Omega-i(\gamma+D))^2}\right.}{D^2\left(-\frac{1}{(\omega_0-\Omega-i(\gamma+D))^2}\right.} \\
&\qquad \frac{\left.-\frac{i}{\omega_0-\Omega-i(\gamma+2D)}+\frac{i}{\omega_0-\Omega-i(\gamma+D)}\right)}{\left.+\frac{2\varepsilon_f e^{i\Omega T}T}{\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)^2}\right)} \\
&= -i\left[(\omega_0-\Omega-i(\gamma+2D))\right. \\
&\qquad \left.\times\left(1-2\varepsilon_f e^{i\Omega T}T\left(\frac{\omega_0-\Omega-i(\gamma+D)}{\varepsilon_0+\varepsilon_f e^{i\Omega T}}\right)^2\right)\right]^{-1} \\
&= 4\left[(\varepsilon_0+\varepsilon_f e^{i\Omega T}+2D)(2+\varepsilon_f T e^{i\Omega T})\right]^{-1}. \qquad (4.46)
\end{aligned}$$

Some examples of $P(\varepsilon_0,\Omega)$ are shown in Fig. 4.4. The non-smoothnesses of surfaces correspond to intersections of different bifurcation surfaces (co-dim. 2 bifurcation), and occur at $\omega_0 T=\pi\left(n-\frac{1-\mathrm{Sign}(\varepsilon_f)}{4}\right)$, $n=1,2,3\ldots$. In a vicinity of these intersections, the weakly nonlinear analysis performed is not sufficient. Indeed, the results of the analysis performed describe correctly the system's behavior on the two subspaces (the first one is the central manifold of the perturbations critical on one side of the intersection, the second one — on the other side), theis direct product forms the central manifold of the bifurcation related to this intersection, but does not describe an interplay between excited modes. At a relatively small noise intensity D for "narrow" distributions $g(\omega)$ (i.e. when γ/Ω_0 is small), parameter regions with negative P appear in close vicinities of some of above-mentioned intersections (note, negative P corresponds to subcritical bifurcation). These regions shrink as D and γ/ω_0 grow, and collapse when either D or γ/ω_0 is of the order of unity.

Summarizing, Fig. 4.3 shows that a delayed feedback is able to linearly stabilize the absolutely nonsynchronous state of the system under consideration, and to effectively change P either enhancing or diminishing the amplitude of collective oscillations close to the threshold. A strong enough delayed feedback can even turn a supercritical bifurcation into a subcritical one (simultaneously shifting the linear stability threshold to higher ε).

4.2.6 Example: Symmetric Lorentz distribution $g(\omega)$

One of the simplest examples of symmetric distributions is the Lorentz one:

$$g(\omega) = \frac{\gamma}{\pi\left(\omega^2 + \gamma^2\right)}. \tag{4.47}$$

For this distribution, $G(\lambda) = \dfrac{i}{2\pi}\displaystyle\int_{-\infty}^{+\infty} \frac{g(\omega)\, d\omega}{\omega - \lambda} = -\frac{i}{2\pi(\lambda + i\gamma)}$.

Spectrum. The equation $\Lambda(\lambda) = 0$ can be rewritten as

$$\varepsilon - 2(\gamma + D) + \varepsilon_f e^{-\lambda T} = 2\lambda. \tag{4.48}$$

The last equation evidently has only one real root for $\varepsilon_f \geq 0$, and zero or two real roots for $\varepsilon_f < 0$. Non-real increments can again be parameterized by Ω: $\mu = \dfrac{1}{T}\ln\left(-\dfrac{\varepsilon_f \sin\Omega T}{2\Omega}\right)$, $\varepsilon = 2(\mu + \gamma + D) - \varepsilon_f e^{-\mu T}\cos\Omega T$. In Fig. 4.5 samples of spectrum are presented. The real increment branch is depicted by a bold curve.

For positive ε_f the critical perturbation corresponds to $\lambda = 0$, whereas for negative ε_f the critical perturbations correspond to $\lambda = 0$ if $\varepsilon_f > \varepsilon_{f*}$, and to $\lambda = \pm i\Omega$ otherwise. Here ε_{f*} is ε_f at which $\dfrac{\partial\lambda}{\partial\varepsilon} = \infty$ for the real increment branch at $\lambda = 0$, in other words, $\lambda_2/\varepsilon_2 = \infty$ for the steady-state bifurcation.

Stability boundary is presented in Fig. 4.3 (the curve corresponding to $\omega_0 = 0$). The point of non-smoothness is the point of the intersection of the steady-state bifurcation curve (the left-hand branch) and the Hopf one (the right-hand branch).

$\varepsilon_f = -2,\, T = 3,\, \gamma + D = 1.5$ $\varepsilon_f = -0.3,\, T = 3,\, \gamma + D = 1.5$ $\varepsilon_f = 2,\, T = 3,\, \gamma + D = 1.5$

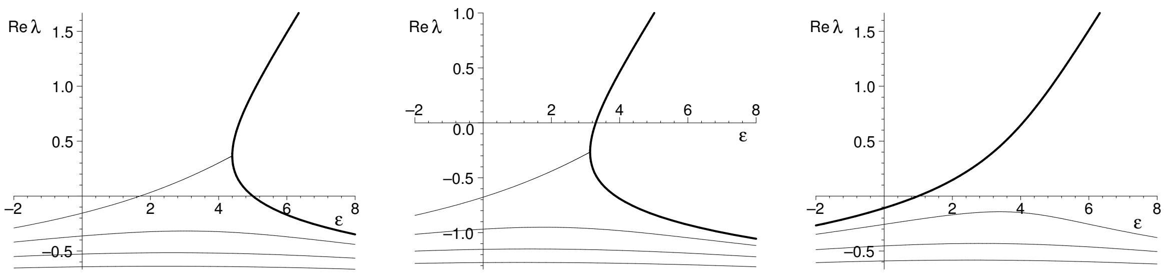

Figure 4.5: *Symmetric Lorentz distribution* $g(\omega)$*.* Samples of spectrum. The bold curve depicts the real increment branch.

Steady-state bifurcation. From Eq.(4.48) we have

$$\varepsilon_0 = 2(D + \gamma) - \varepsilon_f\,. \tag{4.49}$$

Substituting in (4.45),(4.46) for $\omega_0 = 0$, $\Omega = 0$, we obtain

$$\lambda_2(\varepsilon_0,\, 0) = \frac{\varepsilon_2}{2 + \varepsilon_f T}\,, \tag{4.50}$$

$$P(\varepsilon_0,\, 0) = \left[(2D + \gamma)\left(1 + \frac{\varepsilon_f T}{2}\right)\right]^{-1}. \tag{4.51}$$

From Eq.(4.50), $\varepsilon_{f*} = -2/T$; monotonous perturbations are critical for $\varepsilon_f > \varepsilon_{f*}$, and a small-amplitude unstable monotonous solution exists (within the "supercritical" range of parameters) for $\varepsilon_f < \varepsilon_{f*}$, $\varepsilon > \varepsilon_0$ defined by Eq.(4.49). From Eq.(4.51) it follows that, for steady-state bifurcation, P is positive (supercritical bifurcation) so far as monotonous perturbations are critical.

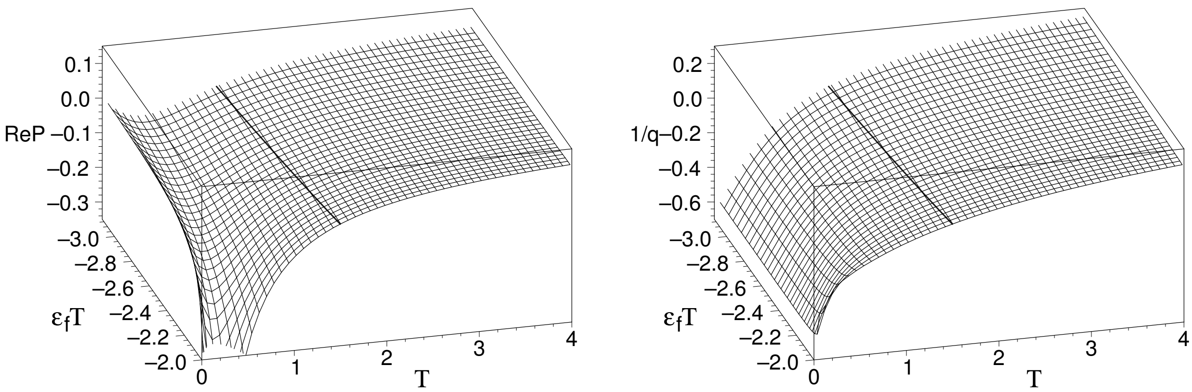

Figure 4.6: *Symmetric Lorentz distribution* $g(\omega)$*.* The Hopf bifurcation. The dependencies of $\Re P$ and q^{-1} on ε_f and T at $\gamma = 1.0$, $D = 0.5$. The additional bold lines depict zeros of the coefficients plotted.

Hopf bifurcation. For the Hopf bifurcation the eq. $\Lambda(\Omega)=0$ take the form

$$\Omega = -\frac{\varepsilon_f}{2}\sin\Omega T, \tag{4.52}$$

$$\varepsilon_0 = 2(\gamma+D) - \varepsilon_f \cos\Omega T. \tag{4.53}$$

The value of Ω providing the minimal critical value ε_0 belongs to the interval $(0,\, 2\pi/T)$ [more precisely, $(0,\, \pi/T)$ if $\varepsilon_f < 0$, $(\pi/T,\, 2\pi/T)$ otherwise]. As oscillatory perturbations are critical one for $\varepsilon_f < \varepsilon_{f*} = -2/T$, only this parameter range is of our interest.

The expression for $P(\varepsilon,\, \Omega)$ coincides with the one given by the final form of Eq.4.46, and

$$\begin{aligned} Q(\varepsilon_0,\, \Omega) &= \frac{\pi\left(\varepsilon_0+\varepsilon_f e^{i\Omega T}\right)^2}{2+\varepsilon_f e^{i\Omega T}T}\left[\frac{i}{2\pi(\Omega+i(\gamma+D))^2(\Omega-iD)}\right. \\ &\quad +\frac{i(\Omega+iD)}{4\pi\Omega(\Omega-iD)^2(\Omega+i(\gamma+D))}+\frac{i}{4\pi\Omega(\Omega+iD)(\Omega-i(\gamma+D))} \\ &\quad \left.+\frac{-i2D}{2\pi(\Omega^2+D^2)(\Omega-iD)(2D+\gamma)}\right] \\ &= -\frac{i}{\Omega(\Omega-iD)^2(2+\varepsilon_f e^{i\Omega T}T)}\left[3\Omega^2+i\Omega\gamma-D(\gamma+D)\right. \\ &\quad \left.+\frac{(\Omega+i(\gamma+D))^2}{\Omega-i(\gamma+D)}\left(iD+\frac{\gamma-2D}{\gamma+2D}\Omega\right)\right] \end{aligned} \tag{4.54}$$

Sample dependencies are shown in Fig. 4.6 [for $\varepsilon_f \in (-\pi/T, -2/T)$ where delayed feedback stabilizes the absolutely nonsynchronous state]. The surfaces do not undergo considerable deformation under changing γ and T. One can see, that the solution with $|A_1| = |B_2|$ is not stable — when the bifurcation is supercritical, the bistability between two oscillating regimes appears (the same is for all parameter values we considered).

Summarizing, the delayed feedback linearly stabilizes the absolutely nonsynchronous state of the system for $\varepsilon_f \in (-\pi/T, 0)$. Collective mode appears via a subcritical steady-state bifurcation for $\varepsilon_f < -2/T$, and via a Hopf one otherwise. For $\varepsilon_f \in (-2/T, 0)$, the ratio $|R|^2/(\varepsilon-\varepsilon_0)$ decreases as $|\varepsilon_f T|$ increases. At small T, the Hopf bifurcation is subcritical; at large enough T, it is supercritical, and there is bistability between the two "pure" oscillatory solutions.

4.3 Purely nonlinear delayed feedback

4.3.1 Fokker-Planck equation and linear stability of the absolutely nonsynchronous state

In this section we consider a purely delayed feedback in the ensemble of oscillators. We set $\varepsilon_f = 0$ in Eqs.(4.5),(4.6) and write the Fokker-Plank equation of the basic model in the thermodynamical limit $T \to \infty$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \varphi}(\rho\, v) - D\frac{\partial^2 \rho}{\partial \varphi^2} = 0\,, \tag{4.55}$$

where

$$\begin{gathered} v(\omega) = \omega + \varepsilon \int_0^{2\pi} d\varphi' \int_{-\infty}^{+\infty} d\omega' g(\omega') \sin(\varphi' - \varphi)\rho(\omega',\, \varphi',\, t) \\ +\varepsilon_{of}|R|^2(t-T)|R|(t) \sin\left(2\theta(t-T) - \theta(t) - \varphi + \nu\right)\,. \end{gathered} \tag{4.56}$$

Now the problem of the linear stability of the absolutely nonsynchronous state has the same form as for the previous case at $\varepsilon_f = 0$. Here oscillatory critical perturbations are degenerate as soon as $\exists\, \omega_0$: $g(\omega - \omega_0) = g(\omega_0 - \omega)$ (cf. [62]). And it is convenient to perform a substitution

$$(\varphi_i,\, \nu) \to (\varphi_i + \omega_0 t,\, \nu + 2\omega_0 T)\,, \tag{4.57}$$

where generally $\omega_0 = \int_{-\infty}^{+\infty} \omega\, g(\omega)\, d\omega$. Similarly to Sec. 4.2, the critical perturbations [after the substitution (4.57) has been performed] have the following form:

(i) $g(\omega) \neq g(-\omega)$

for $\lambda = -i\Omega$ $\rho_1 = \alpha(\omega) e^{i(\varphi - \Omega t)} + c.c.$ — Hopf bifurcation;

(ii) $g(\omega) = g(-\omega)$

for $\lambda = 0$ $\rho_1 = \alpha(\omega) e^{i\varphi} + c.c.$ — steady-state bifurcation,

for $\lambda = \pm i\Omega$ $\rho_1 = \alpha(\omega) e^{i(\varphi - \Omega t)} + \beta(\omega) e^{i(\varphi + \Omega t)} + c.c.$ — Hopf bifurcation (after Crawford).

4.3.2 Weakly nonlinear analysis

Let us consider now the Hopf bifurcation for $g(\omega) = g(-\omega)$. To obtain the case $g(\omega) \neq g(-\omega)$ from the one considered, it is enough to set the amplitude of the second perturbation $\beta(\omega) = 0$. The steady-state bifurcation for $g(\omega) = g(-\omega)$ can be obtained from the last case by setting $\Omega = 0$.

Considering the critical perturbation

$$\rho_1 = \alpha_1(\omega,\, t_2, t_4, ...) e^{i(\varphi - \Omega t_0)} + \beta_1(\omega,\, t_2, t_4, ...) e^{i(\varphi + \Omega t_0)} + c.c.\,,$$

we may write down from Eq.(4.55) in the order κ^2 (there are no secular terms in this order):

$$\frac{\partial \rho_2}{\partial t_0} + \rho_0 \frac{\partial v_2}{\partial \varphi} + \frac{\partial}{\partial \varphi}(\rho_1 v_1) + v_0 \frac{\partial \rho_2}{\partial \varphi} - D\frac{\partial^2 \rho_2}{\partial \varphi^2} = 0\,, \tag{4.58}$$

where

$$v_1 = i\pi\varepsilon_0 A_1 e^{i(\varphi-\Omega t_0)} + i\pi\varepsilon_0 B_1 e^{i(\varphi+\Omega t_0)} + c.c.\,. \tag{4.59}$$

Now from Eq.(4.12), $\alpha_1(\omega) = \dfrac{\varepsilon_0\, A_1}{2(D+i(\omega-\Omega))}$, and $\beta_1(\omega) = \dfrac{\varepsilon_0\, B_1}{2(D+i(\omega+\Omega))}$. Therefore,

$$\begin{aligned}\frac{\partial}{\partial \varphi}(\rho_1 v_1) &= -\pi\varepsilon_0^2 \left[\frac{A_1^2 e^{i2(\varphi-\Omega t_0)}}{D+i(\omega-\Omega)}\right.\\ &\quad + \left(\frac{1}{D+i(\omega-\Omega)} + \frac{1}{D+i(\omega+\Omega)}\right) A_1 B_1 e^{i2\varphi}\\ &\quad \left. + \frac{B_1^2 e^{i2(\varphi+\Omega t_0)}}{D+i(\omega+\Omega)} + c.c.\right].\end{aligned}$$

Searching for ρ_2 in the form

$$\begin{aligned}\rho_2 = \eta^{(-1)}(\omega,\, t_2,\, t_4, ...)e^{i2(\varphi-\Omega t_0)} + \eta^{(0)}(\omega,\, t_2,\, t_4, ...)e^{i2\varphi}\\ + \eta^{(1)}(\omega,\, t_2,\, t_4, ...)e^{i2(\varphi+\Omega t_0)} + c.c.\,,\end{aligned}$$

we obtain

$$(i2l\Omega + i2\omega + 4D)\eta^{(l)}(\omega) = -\frac{\Omega}{\pi}\int\limits_0^{\Omega^{-1}\pi} e^{-i2l\Omega t_0} \frac{\partial}{\partial \varphi}(\rho_1 v_1)\; dt_0\,.$$

So,

$$\eta^{(\pm 1)}(\omega) = \frac{\pi\varepsilon_0^2}{2(D+i(\omega\pm\Omega))(2D+i(\omega\pm\Omega))}\begin{cases} B_1^2, & l=+1,\\ A_1^2, & l=-1;\end{cases} \tag{4.60}$$

$$\eta^{(0)}(\omega) = \frac{\pi\varepsilon_0^2}{2(2D+i\omega)}\left(\frac{1}{D+i(\omega-\Omega)} + \frac{1}{D+i(\omega+\Omega)}\right) A_1 B_1. \tag{4.61}$$

In the order κ^3 of Eq.(4.55), secular terms appear:

$$\frac{\partial \rho_3}{\partial t} + \frac{\partial \rho_1}{\partial t_2} + \rho_0 \frac{\partial v_3}{\partial \varphi} + \frac{\partial}{\partial \varphi}(\rho_1 v_2 + \rho_2 v_1) + v_0 \frac{\partial \rho_3}{\partial \varphi} - D\frac{\partial^2 \rho_3}{\partial \varphi^2} = 0\,, \tag{4.62}$$

where $v_2 = 0$ and

$$\begin{aligned} v_3 &= v_1(A_1,\, B_1 \to A_3,\, B_3) + v_1(\varepsilon_0 \to \varepsilon_2) \\ &\quad + \varepsilon_{of}|R|^2(t-T)|R|(t)\sin\left(2\theta(t-T) - \theta(t) - \varphi + \nu\right) \\ &= v_1(A_1,\, B_1 \to A_3,\, B_3) + v_1(\varepsilon_0 \to \varepsilon_2) \\ &\quad + \varepsilon_{of}\Im\left(R^2(t-T)R^*(t)e^{\nu-\varphi}\right) . \end{aligned}$$

As $R = 2\pi A_1^* e^{i\Omega t} + 2\pi B_1^* e^{-i\Omega t}$,

$$\begin{aligned} R^2(t-T)R^*(t) &= 8\pi^3(A_1^{*2}e^{i2\Omega(t-T)} + 2A_1^*B_1^* + B_1^{*2}e^{-i2\Omega(t-T)})(A_1e^{-i\Omega t} + B_1e^{i\Omega t}) \\ &= 8\pi^3\left(e^{-i2\Omega T}|A_1|^2A_1^*e^{i\Omega t} + 2|A_1|^2B_1^*e^{-i\Omega t}\right. \\ &\quad \left. + 2|B_1|^2A_1^*e^{i\Omega t} + |B_1|^2B_1^*e^{-i\Omega t} + \dots\right) , \end{aligned}$$

where "…" again denotes certainly non-secular terms. Therefore

$$\begin{aligned} v_3 &= \dots - 8\pi^3\varepsilon_{of}\Im\left[(2|B_1|^2 + |A_1|^2e^{i2\Omega T})A_1e^{i(\varphi-\Omega t-\nu)}\right. \\ &\quad \left. + (2|A_1|^2 + |B_1|^2e^{-i2\Omega T})B_1e^{i(\varphi+\Omega t-\nu)}\right] , \end{aligned}$$

and

$$\begin{aligned} \rho_0\frac{\partial v_3}{\partial\varphi} &= \dots - 4\pi^2\varepsilon_{of}\Re\left[(2|B_1|^2 + |A_1|^2e^{i2\Omega T})A_1e^{i(\varphi-\Omega t-\nu)}\right. \\ &\quad \left. + (2|A_1|^2 + |B_1|^2e^{-i2\Omega T})B_1e^{i(\varphi+\Omega t-\nu)}\right] . \end{aligned}$$

The term $\frac{\partial}{\partial\varphi}(\rho_2 v_1)$ can be taken from Sec. 4.2.4: its contribution to $P(\varepsilon,\, \Omega)$ is given by the formula (4.27) with $\varepsilon_f = 0$, and its contribution to $Q(\varepsilon,\, \Omega)$ is given by the formula (4.39) with $\varepsilon_f = 0$. Summing up these results, we find that Eqs.(4.37),(4.38) are relevant here, and

$$\lambda_2(\varepsilon,\, \Omega) = \frac{\varepsilon_2}{i\pi\varepsilon^2\, G'(\Omega + iD)} , \tag{4.63}$$

$$P(\varepsilon,\, \Omega) = \frac{\pi^2\varepsilon^2}{D}\left[1 + \frac{G(\Omega+iD) - G(\Omega+2iD)}{iDG'(\Omega+iD)}\right] + \frac{i4\pi\varepsilon_{of}e^{i(2\Omega T-\nu)}}{\varepsilon G'(\Omega+iD)} , \tag{4.64}$$

$$\begin{aligned} Q(\varepsilon,\, \Omega) = \frac{\pi^2\varepsilon^2}{iG'(\Omega+iD)}&\left[\frac{G'(\Omega+iD)}{\Omega - iD} - \frac{(\Omega+iD)\,G(\Omega+iD)}{2\Omega(\Omega-iD)^2}\right. \\ &\left. + \frac{G(-\Omega+iD)}{2\Omega(\Omega+iD)} + \frac{i2D\,G(i2D)}{(\Omega^2+D^2)(\Omega-iD)}\right] + \frac{i8\pi\varepsilon_{of}e^{-i\nu}}{\varepsilon G'(\Omega+iD)} . \end{aligned} \tag{4.65}$$

The probability density is

$$\begin{aligned}\rho(\omega,\,\varphi,\,t) &= \frac{1}{2\pi}\Bigg[1+\frac{\pi\varepsilon_0\,A_1(t)}{D+i(\omega-\Omega)}e^{i(\varphi-\Omega t)}+c.c.\\&+\frac{\pi\varepsilon_0\,B_1(t)}{D+i(\omega+\Omega)}e^{i(\varphi+\Omega t)}+c.c.\\&+\frac{\pi^2\varepsilon_0^2\,A_1^2(t)}{(D+i(\omega-\Omega))(2D+i(\omega-\Omega))}e^{i2(\varphi-\Omega t)}+c.c.\\&+\frac{2\pi^2\varepsilon_0^2(D+i\omega)\,A_1(t)B_1(t)}{(2D+i\omega)(D+i(\omega-\Omega))(D+i(\omega+\Omega))}e^{i2\varphi}+c.c.\\&+\frac{\pi^2\varepsilon_0^2\,B_1^2(t)}{(D+i(\omega+\Omega))(2D+i(\omega+\Omega))}e^{i2(\varphi+\Omega t)}+c.c.+O(A_1^3+B_1^3)\Bigg]\end{aligned} \tag{4.66}$$

and the order parameter

$$R(t)=2\pi\left(A_1^*e^{i\Omega t}+B_1^*e^{-i\Omega t}\right)+O(A_1^3+B_1^3)\,.$$

4.3.3 Example: Lorentz distribution $g(\omega)$

For the symmetric Lorentz distribution (4.47), $G(\lambda)=-\dfrac{i}{2\pi(\lambda+i\gamma)}$ (cf. Sec. 4.2.6).

Spectrum. The characteristic equation $\Lambda(\lambda)=0$ takes the form [Eq.(4.48) with $\varepsilon_f=0$]

$$\varepsilon-2(\gamma+D)=2\lambda, \tag{4.67}$$

and has only one real root. So, the only possible bifurcation of the absolutely nonsynchronous state is a steady-state one at $\varepsilon_0=2(\gamma+D)$ (cf. [62]).

Steady-state bifurcation. Omitting B_1 and setting $\Omega=0$ successively, we find

$$\lambda_2(\varepsilon_0,\,0)=\frac{\varepsilon_2}{2}\,, \tag{4.68}$$

$$P(\varepsilon_0,\,0)=\frac{1}{2D+\gamma}-4\pi^2\varepsilon_{of}e^{-i\nu}(\gamma+D)\,. \tag{4.69}$$

Summarizing: In the case of Lorentz distribution $g(\omega)$ the delayed feedback considered diminish or enhance the amplitude of the establishing collective mode depending on the value of ν:

$$|A_1|^2=\frac{\lambda_2}{\Re P}=\frac{\varepsilon_2}{2(2D+\gamma)^{-1}-8\pi^2\varepsilon_{of}(\gamma+D)\cos\nu}\,. \tag{4.70}$$

Moreover, for strong enough feedback with $\cos\nu > 0$, $\Re P$ can become negative, what means a subcritical Kuramoto transition. Also, a nonlinear shift of the rotation frequency of R in the counterclockwise direction appears

$$\omega_2 = \Im(P)\,|A_1|^2 = \frac{\varepsilon_2 \Im(P)}{2\,\Re(P)} = \frac{\varepsilon_2}{2}\frac{\tan\nu}{\left[4\pi^2\varepsilon_{of}(2D-\gamma)(D+\gamma)\cos\nu\right]^{-1} - 1}\,. \tag{4.71}$$

4.4 Multimodal distributions $g(\omega)$

Considering general cases in the two last sections, we paid considerable attention to the Hopf bifurcations with a 4-d central manifold. But considering examples of symmetric distribution $g(\omega)$, we did not faced such a bifurcation of the absolutely nonsynchronous state with the example used in Sec. 4.3.3, and found periodic oscillatory collective modes to be the only non-stationary collective modes, which can be excited, with the example used in Sec. 4.2.6. Stable regimes with modulated oscillations of the order parameter appear (for instance) for bimodal distributions (see [62]). But such kind of distributions is out of scope of this work.

4.5 Summary

We have developed a weakly nonlinear analysis of the effect of delayed feedback on the Kuramoto transition. The amplitude equations derived and their coefficients found are the main result of the current chapter as they give a full description of the effect of the delayed global feedback on the synchronization transition in the ensemble of oscillators:

Delayed feedback	Distribution of natural frequencies	Bifurcation	Amplitude equations	Coefficients of the amplitude equations
Linear feedback	$g(\omega) \neq g(-\omega)$	Hopf	(4.25)	(4.26), (4.27)
	$g(\omega) = g(-\omega)$	Steady-state	(4.30)	(4.26), (4.27) with $\Omega = 0$
	$g(\omega) = g(-\omega)$	Hopf	(4.37), (4.38)	(4.26), (4.27), (4.39)
Nonlinear feedback	$g(\langle\omega\rangle+\Delta\omega) \neq g(\langle\omega\rangle-\Delta\omega)$	Hopf	(4.25)	(4.63), (4.64)
	$g(\langle\omega\rangle+\Delta\omega) = g(\langle\omega\rangle-\Delta\omega)$	Steady-state	(4.25)	(4.63), (4.64) with $\Omega = 0$
	$g(\langle\omega\rangle+\Delta\omega) = g(\langle\omega\rangle-\Delta\omega)$	Hopf	(4.37), (4.38)	(4.63), (4.64), (4.65)

Conclusion

In this thesis we have addressed different aspects of synchronization of noisy oscillators. We have started with Chapter 2 where we consider effects of delayed feedback on the coherence of oscillations of noisy oscillators and deterministic chaotic systems (to some extend, the former one may be considered as the simplest approximation to the latter). Here coherence is quantified by virtue of the phase diffusion constant and qualitatively defines predisposition of oscillators to synchronization. In Chapter 3 the possible constructive role of noise in synchronization has been considered. Specifically, the phenomenon of synchronization of identical (or slightly different) oscillators by common noise ha been addressed. Additionally, in order to consider the "reliability" property of neurons [39], the theory developed has been extended to the case of neuron-like oscillators. And finally, in Chapter 4, we have turned our attention to the Kuramoto transition in an ensemble of globally coupled oscillators in presence of additional time-delayed coupling terms. We have developed a weakly nonlinear analysis of this transition.

In the following, we discuss the main results of the work and open questions.

Coherence of oscillators with delayed feedback

We have demonstrated the effect of the coherence control by means of the delayed feedback [2, 3]. The control is possible for noisy limit cycles oscillators as well as for chaotic systems, admitting introducing the phase-like variable [1]. Noteworthy, formally the equations describing the control are the same as in the Pyragas method of chaos control. However, in our case the delay time τ is not necessarily equal to the period of some unstable cycle, embedded in chaos. Moreover, we consider the situation when the feedback is so small that no stabilization of periodic orbits occurs. The main difference to the Pyragas approach is that we do not intend to suppress chaos, but to control uniformity – coherence – of phase growth in a chaotic system.

As a quantifier of coherence we have used the phase diffusion constant. (The relation of this constant to the predisposition of oscillators to synchronization has been also illustrated with the Lorenz system entrained by periodical driving: entrainment becomes more effective as the diffusion constant decreases.) The effect of the delayed feedback on phase diffusion appears to be especially pronounced for deterministic chaotic systems, e.g., the diffusion constant of the Lorenz system may be suppressed by factor 30 without any visible change in the topology of the strange attractor and with quite small variations of the Lyapunov exponent.

Next, we have developed a statistical theory of phase diffusion under the influence of a delayed feedback. Using the Gaussian approximation, we have derived a closed system of equations for the diffusion constant and the mean frequency for the case of short-time correlations of the instantaneous frequency. The theory works if the feedback is not very strong, or if the noise is strong enough to suppress multistability in mean frequency.

In principle, an opposite situation, where effects of multistability are dominant, gives some opportunities for an analytical investigation. Indeed, there quite rare events of spontaneous switchings between different phase growth rates take place and determine the phase diffusion process. But this question will be considered elsewhere.

Another possible direction of the future development of this work is aimed at detailed understanding of the particular features of the control of chaotic systems. Indeed, in this case our theory provide only qualitative explanation of the effect. This limitation of the theory is related to the statistical properties of the effective noise in a chaotic system that definitely cannot be considered as weak or Gaussian. (We remind that effective noise here describes the effect of irregular, although deterministic, amplitudes, on the phase dynamics.) Particularly, it is known that for the Lorenz system this noise is not symmetric and possesses nontrivial correlation properties [26, 27]. Our preliminary numerical investigations show that the feedback significantly affects these correlations.

The last (but not least) open question we would like to note here is the one of a multiple delay feedback control of coherence (afore the chaos suppression [14, 15, 16, 17]). Here especially the regions of superharmonic resonances: $n\tau + m_1\tau_1 + m_2\tau_2 = 0$, where n, $m_{1,2}$ are integer (the effect is expected to be most pronounced for $|n| = |m_{1,2}| = 1$), τ is the average return time of chaotic oscillations, and $\tau_{1,2}$ are the delay times, are of interest.

Synchronization of oscillators by common noise

We have considered synchronization of oscillators (uncoupled) by common noise (white Gaussian or telegraph). The effect of synchronization is quantified by virtue of the Lyapunov exponent (LE): identical (or slightly different) oscillators are synchronized by common noisy driving when the LE is negative, and desynchronized otherwise. For limit cycle oscillators the LE has been proven to be negative for weak noise: $\lim_{\varepsilon\to 0} \lambda/\varepsilon^2 = const < 0$

(ε is the noise amplitude) [4, 5]. For this purpose the approach of the phase approximation has been used.

While for a negative LE, identical oscillators are perfectly synchronous under identical driving, nonidentities distort perfect synchrony. For limit cycle oscillators driven by weak noise the role of nonidentities either (i) in oscillators or (ii) in noise has been studied analytically [6]. This study has reviled a crucial role of finite deviations between states of subsystems even for arbitrary small nonidentities and strongly intermittent character of synchrony.

The analytical results obtained within the framework of the phase approximation has been underpinned by the results of numerical simulation for the Van der Pol–Duffing oscillator. This simulation also allows to consider the effect of finite amplitude noise when no analytical investigation is possible. In some systems under certain circumstances, moderate noise appears to be able to lead to desynchronization: a positive LE is observed [6].

The results found for limit cycle oscillators appear to be general for different kinds of noise, because they have been demonstrated to be similar even for such different noises as white Gaussian and telegraph ones (compare [6] and [7]).

In order to study the "reliability" property of neurons [39] which, on the one hand, is directly related to the phenomena we have considered, and, on the other hand, appear to be important in neurosciences, we have extended our investigation to the neuron-like systems which are not always limit cycle ones. Doing so, we have shown, within the framework of the noise-driven FitzHugh–Nagumo model, that identical neurons can respond antireliably to the noisy evocation. Antireliability, which manifests itself as a non-correlation of spikes, is observed when the neurons are close to the transition excitability–oscillations, where the dynamics is mostly sensitive to perturbations. Quantitatively, the antireliability is characterized as a state with a positive largest LE. The latter is purely noise-induced, as the noiseless FitzHugh–Nagumo system does not posses even a transient chaos. We have explained the transition to the antireliability within the approximate analytical theory for small noise-induced deviations from the deterministic trajectory, which goes beyond the one-dimensional phase approximation. This is crucial, because only due to the evocation of transversal to trajectory perturbation, the stretchings and foldings that lead to chaos, can occur. The final formulae obtained give an explicit dependence of the LE on physical properties of the neuron, such as nonisochronicity and sensitivity, thus guiding an experimental search for the effect. The theoretical expression for the random multiplier of phase deviations explains also the intermittent character of the antireliable state: during the epochs where the multiplier is close to zero, a temporarily synchronous firing of neurons is observed.

In the course of this study the role of a non-localized distribution of the local Lyapunov exponents (LLE, "local" means calculated over finite time intervals) has been repeatedly exhibiting. Indeed, even when the (global) LE is positive, during the epoches of a negative LLE one observes relatively synchronous behavior of subsystems, and, on the other hand, even when the (global) LE is negative, during the epoches of a positive LLE one observers, in real situations, nonsynchronous behavior of subsystems due to extreme sensitivity (during these epoches) of subsystems to intrinsic noise. Remarkably, in systems with noise-induced bursting the LLE may be quite adequately treated as a nonsymmetric telegraph noise. This gives opportunities for a systematic investigation of the effects mentioned in this paragraph, underpinned by an analytical description. But this subject goes beyond the framework of the present thesis.

Effects of delayed feedback on Kuramoto transition

We have developed a weakly nonlinear theory of the Kuramoto transition in an ensemble of globally coupled oscillators in presence of additional time-delayed coupling terms. This theory allows us to determine the order parameter and the system's behavior near the transition point.

We have considered a linear delayed feedback and a purely nonlinear one. For both of them we treat the general case of a nonsymmetric distribution of natural frequencies when generally only a Hopf bifurcation is possible, and the case of a symmetric distribution of natural frequencies when generally a steady-state and a Hopf (according to the Crawford's terminology [62]) bifurcations are possible. For all these cases the transition point has been determined as well as the behavior of the order parameter in vicinity of this point.

Noteworthy, a linear feedback not only controls the transition point, but effectively changes the nonlinear terms near the transition. A purely nonlinear delayed coupling does not effect the transition point, but can reduce or enhance the amplitude of collective oscillations.

Bibliography

[1] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2003).

[2] D. Goldobin, M. Rosenblum, and A. Pikovsky *Controlling oscillator coherence by delayed feedback*, Phys. Rev. E **67**(6), 061119 (2003).

[3] D. Goldobin, M. Rosenblum, and A. Pikovsky *Coherence of noisy oscillators with delayed feedback*, Physica A **327**(1–2), 124–128 (2003).

[4] J. Teramae and D. Tanaka, *Robustness of the Noise-Induced Phase Synchronization in a General Class of Limit Cycle Oscillators*, Phys. Rev. Lett. **93**, 204103 (2004).

[5] D. S. Goldobin and A. Pikovsky, *Synchronization of self-sustained oscillators by common white noise*, Physica A **351**, 126 (2005).

[6] D. S. Goldobin and A. Pikovsky, *Synchronization and desynchronization of self-sustained oscillators by common noise*, Phys. Rev. E **71**(4), 045201(R) (2005).

[7] D. S. Goldobin, *Synchronization of Limit Cyrcle Oscillators by Telegraph Noise*, in *Unresolved Problems and Fluctuations: UPoN 2005*, edited by L. Reggiani *et al.*, AIP Conf. Proc. **800**(1), 394 (2005).

[8] D. S. Goldobin and A. Pikovsky, *Antireliability of noise-driven neurons*, Phys. Rev. E **73**(6), 061906 (2006).

[9] D. S. Goldobin and A. Pikovsky, *Effects of Delaed Feedback on Kuramoto Transition*, Prog. Theor. Phys. Suppl. **161**, 43–52 (2006).

[10] K. Pyragas, *Continuous control of chaos by self-controlling feedback*, Phys. Lett. A **170**, 421–428 (1992).

[11] G. Franceschini, S. Bose, and E. Schöll, *Control of chaotic spatiotemporal spiking by time-delay autosynchronization*, Phys. Rev. E **60**(5), 5426–5434 (1999).

[12] M. Bertram and A. S. Mikhailov, *Pattern formation in a surface chemical reaction with global delayed feedback*, Phys. Rev. E **63**(6), 066102 (2001).

[13] P. Parmananda and J. L. Hudson, *Controlling spatiotemporal chemical chaos using delayed feedback*, Phys. Rev. E **64**(3), 037201 (2001).

[14] A. Ahlborn, U. Parlitz, *Stabilizing Unstable Steady States Using Multiple Delay Feedback Control*, Phys. Rev. Lett. **93**(26), 264101 (2004).

[15] A. Ahlborn, U. Parlitz, *Controlling dynamical systems using multiple delay feedback control*, Phys. Rev. E **72**(1), 016206 (2005).

[16] A. Ahlborn, U. Parlitz, *Chaos Control using Notch Filter Feedback*, Phys. Rev. Lett. **96**(3), 034102 (2006).

[17] A. Ahlborn, U. Parlitz, *Laser stabilization with multiple-delay feedback control*, Opt. Lett. **31**(4), 465–467 (2006).

[18] L. S. Tsimring and A. Pikovsky, *Noise-Induced Dynamics in Bistable Systems with Delay*, Phys. Rev. Lett. **87**(25), 250602 (2001).

[19] S. Guillouzic, I. L'Heureux, and A. Longtin, *Small delay approximation of stochastic delay differential equations*, Phys. Rev. E **59**(4), 3970–3982 (1999).

[20] T. Ohira and T. Yamane, *Delayed stochastic systems*, Phys. Rev. E **61**(2), 1247–1257 (2000).

[21] C. Masoller, *Noise-Induced Resonance in Delayed Feedback Systems*, Phys. Rev. Lett. **88**(3), 034102 (2002).

[22] U. Küchler and B. Mensch, Stoch. Stoch. Rep. **40**, 23 (1992).

[23] T. D. Frank and P. J. Beek, *Stationary solutions of linear stochastic delay differential equations: Applications to biological systems*, Phys. Rev. E **64**(2), 021917 (2001).

[24] N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Gordon and Breach, New York, 1961).

[25] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, *From Phase to Lag Synchronization in Coupled Chaotic Oscillators*, Phys. Rev. Lett. **78**(22), 4193 (1997).

[26] M. Zaks, E.-H. Park, M. Rosenblum, and J. Kurths, *Alternating Locking Ratios in Imperfect Phase Synchronization*, Phys. Rev. Lett. **82**(21), 4228 (1999).

[27] E.-H. Park, M. A. Zaks, and J. Kurths, *Phase synchronization in the forced Lorenz system*, Phys. Rev. E **60**(6), 6627–6638 (1999).

[28] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer-Verlag, Berlin, 1984) [republisheb by Dover, New York, 2003].

[29] K. Furutsu, J. Res. Natl. Bur. Stand., Sect. D**667**, 303 (1963).

[30] Y. Novikov, Zh. Eksp. Teor. Fiz. **47**, 1919 (1964) [Sov. Phys. JETP **20**, 1290 (1965)].

[31] P. Reimann *et al.*, *Giant Acceleration of Free Diffusion by Use of Tilted Periodic Potentials*, Phys. Rev. Lett. **87**(1), 010602 (2001).

[32] B. Lindner, M. Kostur, and L. Schimansky-Geier, *Optimal diffusive transport in a tilted periodic potential*, Fluct. Noise Lett. **1**(1), R25–R39 (2001).

[33] R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).

[34] A. N. Malakhov, *Fluctuations in Self–Oscillatory Systems* (Nauka, Moscow, 1968, in Russian).

[35] L. Baroni, R. Livi, and A. Torcini, *Transition to stochastic synchronization in spatially extended systems*, Phys. Rev. E **63**, 036226 (2001).

[36] P. Khoury, M. A. Lieberman, and A. J. Lichtenberg, *Degree of synchronization of noisy maps on the circle*, Phys. Rev. E **54**, 3377 (1996).

[37] T. H. Bullock, *The Reliability of Neurons*, J. Gen. Physiol. **55**(5), 565-584 (1970).

[38] C. Allen and C. F. Stevens, *An Evaluation of Causes for Unreliability of Synaptic Transmission*, Proc. Natl Acad. Sci. USA **91**(22), 10380-10383 (1994).

[39] Z. F. Mainen and T. J. Sejnowski, *Reliability of Spike Timing in Neocortical Neurosns*, Science **268**, 1503 (1995).

[40] C. V. Rao, D. M. Wolf, and A. P. Arkin, *Control, exploitation and tolerance of intracellular noise*, Nature **420**(6912), 231-237 (2002).

[41] A. Uchida, R. McAllister, and R. Roy, *Consistency of nonlinear system response to complex drive signals*, Phys. Rev. Lett. **93**, 244102 (2004).

[42] R. McAllister, A. Uchida, R. Meucci, and R. Roy, *Generalized synchronization of chaos: experiments on a two-mode microchip laser with optoelectronic feedback*, Physica D **195**(3-4), 244-262 (2004).

[43] H. D. I. Abarbanel, N. F. Rulkov, and M. M. Suschik, *Generalized synchronization of chaos: The auxillary system approach*, Phys. Rev. E **53**, 4528 (1996).

[44] A. S. Pikovsky, *Synchronization and stochastization of the ensemble of autogenerators by external noise*, Radiophys. Quantum Electron. **27**(5), 576–581 (1984).

[45] A. S. Pikovsky, *Synchronization and stochastization of nonlinear oscillations by external noise*, in *Nonlinear and Turbulent Processes in Physics*, edited by R. Z. Sagdeev (Harwood Academic, Singapore, 1984), pp. 1601–1604.

[46] H. Nakao, K. Arai, K. Nagai, Y. Tsubo, Y. Kuramoto, *Synchrony of limit-cycle oscillators induced by random external impulses*, Phys. Rev. E **72**(2), 026220 (2005).

[47] J. Freund, L. Schimansky-Geier, P. Hänggi, *Frequency and phase synchronization in stochastic systems*, CHAOS **13**(1), 225–238 (2003).

[48] V. Anishchenko, A. Neiman, F. Moss, L. Schimansky-Geier, *Entrainment between heart rate and weak noninvasive forcing*, Physics Uspekhi **42**, 7–50 (1999).

[49] H. Nakao, K. Nagai, K. Arai, *Reproducibility of a Noisy Limit-Cycle Oscillator Induced by a Fluctuating Input*, Prog. Theor. Phys. Suppl. **161**, 294–297 (2006).

[50] L. Arnold, *Random Dynamical Systems* (Springer, Berlin, 1998).

[51] P. Hänggi, H. Thomas, *Stochastic processes: Time evolution, symmetries and linear response*, Physics Reports **88**(4), 207–319 (1982).

[52] H. Z. Risken, *The Fokker–Planck Equation* (Springer, Berlin, 1989).

[53] L. Yu, E. Ott, and Q. Chen, *Transition to Chaos for Random Dynamical Systems*, Phys. Rev. Lett. **65**, 2935 (1990).

[54] L. Yu, Q. Chen, and E. Ott, *Fractal distribution of floaters on a fluid surface and the transition to chaos for random maps*, Physica D **53**, 102 (1991).

[55] A. S. Pikovsky, *Statistics of trajectory separation in noisy dynamical systems*, Phys. Lett. A **165**, 33–36 (1992).

[56] P. Khoury, M. A. Lieberman, and A. J. Lichtenberg, *Degree of synchronization of noisy maps on the circle*, Phys. Rev. E **54**(4), 3377–3388 (1996).

[57] J. Ritt, *Evaluation of entrainment of a nonlinear neural oscillator to white noise*, Phys. Rev. E **68**(4), 041915 (2003).

[58] L. Schimansky-Geier and H. Herzel, *Positive Lyapunov exponents in the Kramers oscillator*, J. Stat. Phys. **70**(1–2), 141–147 (1993).

[59] R. A. FitzHugh, *Impulses and Physiological States in Theoretical Models of Nerve Membrane*, Biophys. J. **1**(6), 445 (1961).

[60] J. Nagumo, S. Arimoto, and S. Yoshizawa, *Active Pulse Transmission Line Simulating Nerve Axon*, Proc. IRE **50**(10), 2061 (1962).

[61] R. Quian Quiroga, T. Kreuz, P. Grassberger, *Event synchronization: A simple and fast method to measure synchronicity and time delay patterns*, Phys. Rev. E **66**(4), 041904 (2002).

[62] J. D. Crawford, *Amplitude Expansions for Instabilities in Populations of Globally-Coupled Oscillators*, J. Stat. Phys. **74**(5–6), 1047–1084 (1994).

[63] Y. Kuramoto, *Lecture Notes in Phys.* **39**, ed. H. Araki (Springer, New York, 1975), p. 420.

[64] D. Golomb, D. Hansel, and G. Mato, *Neuro-Informatics and Neural Modeling, Handbook of Biological Physics*, Vol. 4, ed. F. Moss and S. Gielen (Elsevier, Amsterdam, 2001), p. 887.

[65] *Epilepsy as a Dynamic Disease*, ed. J. Milton and P. Jung (Springer, Berlin, 2003).

[66] M. G. Rosenblum and A. S. Pikovsky, *Controlling Synchronization in an Ensemble of Globally Coupled Oscillators*, Phys. Rev. Lett. **92**(11), 114102 (2004).

[67] M. Rosenblum and A. Pikovsky, *Delayed feedback control of collective synchrony: An approach to suppression of pathological brain rhythms*, Phys. Rev. E **70**(4), 041904 (2004).

[68] O. V. Popovych, C. Hauptmann, and P. A. Tass, *Effective Desynchronization by Nonlinear Delayed Feedback*, Phys. Rev. Lett. **94**(16), 164102 (2005).

[69] M. Rosenblum, N. Tukhlina, A. Pikovsky, and L. Cimponeriu, *Delayed feedback suppression of collective rhythmic activity in a neuronal ensemble*, Int. J. Bifurcation Chaos Appl. Sco. Eng. **16**(7), 1989–1999 (2006).

[70] N. Tukhlina, M. Rosenblum, A. Pikovsky, and J. Kurths, *Feedback suppression of neural synchrony by vanishing stimulation*, Phys. Rev. E **75**(1), 011918 (2007).

[71] P. A. Tass, *Phase Resetting in Medicine and Biology, Stochastic Modelling and Data Analysis* (Springer, Berlin, 1999).

[72] K. Okuda and Y. Kuramoto, *Mutual Entrainment between Populations of Coupled Oscillators*, Progr. Theor. Phys. **86**(6), 1159–1176 (1991).

[73] L. L. Bonilla, J. C. Neu, and R. Spigler, *Nonlinear stability of incoherence and collective synchronization in a population of coupled oscillators*, J. Stat. Phys. **67**(1–2), 313–330 (1992).

[74] J. A. Acebrón, L. L. Bonilla, S. DeLeo, and R. Spigler, *Breaking the symmetry in bimodal frequency distributions of globally coupled oscillators*, Phys. Rev. E **57**(5), 5287–5290 (1998).

[75] E. Montbrió, D. Pazó, and J. Schmidt, *Time delay in the Kuramoto model with bimodal frequency distribution*, Phys. Rev. E **74**(5), 056201 (2006).

[76] Yu. A. Kuznetsov, *Elements of Applied Bifurcation Theory. Series: Applied Mathematical Sciences* Vol. 112, 3rd ed. (Springer, New York, 2004).

[77] A. M. Fridman, *Physics of Gravitating Systems: Equilibrium and Stability of Gravitating Systems* (Springer, New York, 1984).