Universität Potsdam

Michael Lukaschewitsch

Geoelectrical conductivity problems on unbounded domains

NLD Preprints ; 45

Geoelectrical conductivity Problems on unbounded Domains

Michael Lukaschewitsch
University of Potsdam

April 1998

Contents

Abstract

This paper deals with the electrical conductivity problem in geophysics. It is formulated as an elliptic boundary value problem of second order for a large class of bounded and unbounded domains. A special boundary condition, the so called "Complete Electrode Model", is used. Poincaré inequalities are formulated and proved in the context of weighted Sobolev spaces, leading to existence and uniqueness statements for the boundary value problem. In addition, a parameter-to-solution operator arising from the inverse conductivity problem in medicine (EIT) and geophysics is investigated mathematically and is shown to be smooth and analytic.

1 Introduction

1.1 Inverse problems in geophysics

A major application of geophysical methods is the exploration of the earth's interior by means of measurements on the boundary. There are many methods currently in use, based on the observation of a variety of physical effects: for example, geomagnetics, seismic approaches and geoelectric methods. Each such method is based on a special measurement process described by a specific physical model, and in each case the main task is a more or less sophisticated interpretation of the measured data. Mathematically, this interpretation is called the inversion of the operator representing the physical measurement process. In fact, the general theory of inverse problems has found applications in many fields, including geophysics, optics, acoustics, medical and industrial tomography. For a thorough introduction to the theory of inverse problems we recommend [Lou89] and [EHN96]. This paper deals with geoelectric methods, where the spatial distribution of electrical conductivity is the main subject of investigation. The most common geophysical application of geoelectric methods is in mineral exploration, but such methods also now play an ever greater role in geophysical engineering and environmental investigations.

The geoelectric measurement process consists of applying various surface current patterns and then measuring the corresponding surface potentials. The measured data depend on the (typically unknown) spatial distribution of electrical conductivity. In the context of inverse problems, the conductivity distribution is called the parameter distribution. The process which maps the parameter distribution to the measured data is called the foreward mapping (or the foreward operator) and is determined in geoelectrics by a special type of boundary value problem. It turns out that the foreward mapping depends nonlinearly on the conductivity distribution, and therefore we are faced with a nonlinear inverse problem. In order to solve this inverse problem, numerical and analytical investigations of the foreward mapping are necessary; the analytical investigations in a large class of bounded and unbounded domains are treated in Chapter 4. Chapters 1 - 4 introduce the concepts and technical tools needed for the analytical discussion. The numerical simulation of the foreward mapping by means of adaptive finite element methods is the subject of a forthcoming paper.

The well-known inversion of electrical measurements in medical applications is called "Electrical Impedance Tomography" (EIT) and aims to detect conductivity distributions inside the human body. The same partial differential equation governs the interior behaviour in both the geoelectric and medical applications of EIT. The difference arises when considering the spatial extension of the domains. The geophysical domain may be assumed to be unbounded, although the measurement on the boundary only has finite extension. In medical applications, the domain is bounded and the measurement covers all of the boundary. In both settings, experimental data result from measurements using a finite set of electrodes placed on the boundary. Various models describe the physical process of current flux beneath an electrode. In medical applications, it has turned out that the so-called "Complete Electrode Model" reproduces experimental data most accurately (see [CING89] and [SCI92]). In addition, this model leads to an exact description of the singularities arising at electrode edges and lends itself to the use of numerical methods. (see [PBP92], [CIP97] and [CIP96]). The present paper adapts this model to the geophysical setting.

2 Function spaces

2.1 Spaces on unbounded domains

2.1.1 Weight functions

A positive, locally Lipschitz continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_+$ is called a weight, used in subsequent sections to make functions square integrable by reducing their values at infinity. The Lipschitz continuity guarantees via the theorem of Rademacher (see for example [Zie89]) that ρ and its first partial derivatives are measurable and bounded on compact subsets. Additionally ρ has a total derivative almost everywhere. Concerning this paper, we are faced with the simple type of weight $\rho(x) = (1 + |x|)^{\alpha}$, where $\alpha > 1$. (The choice of α depends on the context.) It can easily be shown that this type of weight satisfies the following definition.

Definition 2.1 *A weight is called translation and dilation invariant if for all $K_0 \in \mathbb{R}_+$ and for all $x_0 \in \mathbb{R}^n$ there exists a real constant $\delta > 0$ and a constant $C \geq 1$ such that the inequality $C^{-1}\rho(x) \leq \rho(K(x-y)) \leq C\rho(x)$ for all $x \in \mathbb{R}^n$ holds uniformly for $|K - K_0| < \delta$ and $|y - y_0| < \delta$.*

This property is used in theorems which we cite from [Jan86], and encounters explicitly only in Appendix A.

2.1.2 Basic definitions

When dealing with the Neumann problem

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0 \quad \text{in} \quad \Omega \\ \sigma \frac{\partial u}{\partial n} &= f \quad \text{on} \quad \partial\Omega, \end{aligned}$$

which represents the physical process of injecting a current f into a domain Ω, where Ω is supposed to be unbounded, one cannot use the usual Sobolev space $W^1(\Omega)$, because in general the solution u for this problem is not square integrable in Ω.

Example 2.2 *Let Ω be the half-space R^n_- $(n \geq 3)$, set $\sigma \equiv 1$ and $f(s) = (1 + \|s\|^2)^{-\frac{3}{2}}$ for $s \in \partial\Omega = \mathbb{R}^{n-1} \times \{0\}$. A solution is then $u(\vec{x}) = \|(x_1, \cdots, x_{n-1}, x_n - 1)\|^{-1}$, which is not square integrable in $\mathbb{R}^n_-$.*

This problem can be circumvented by introducing weighted Sobolev spaces. We now introduce some basic concepts of such spaces.

Definition 2.3 *Let $\Omega \subseteq \mathbb{R}^n$ an open set. We use the following standard notation*

$$\begin{aligned} C_0^\infty(\Omega) &:= \{f:\Omega\to\mathbb{R} \mid \text{f infinitely often differentiable in } \Omega,\ \text{supp(f) is compact}\}, \\ C^\infty(\Omega) &:= \{f:\Omega\to\mathbb{R} \mid \text{f infinitely often differentiable in } \Omega\}, \\ \partial_i f &:= \frac{\partial f}{\partial x_i}, \quad i=1,\cdots,n, \\ \partial^\alpha f &:= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}, \quad \text{where} \quad \alpha=(\alpha_1,\cdots,\alpha_\mathrm{n})\in\mathbb{N}^n, \quad |\alpha| = \sum_{\mathrm{i}=1}^{\mathrm{n}} \alpha_\mathrm{i}. \end{aligned}$$

Definition 2.4 *Let $\rho:\mathbb{R}^n\to\mathbb{R}_+$ be a weight. On the sets C_0^∞ and C^∞ the weighted scalar product*

$$\langle f,g\rangle_{1,\rho} := \int_\Omega \rho(x)^{-2} f(x)g(x)dx + \sum_{i=1}^n \int_\Omega \partial_i f(x)\partial_i g(x)dx, \tag{1}$$

induces the norm

$$\|f\|_{1,\rho} := \Big(\langle f,f\rangle_{1,\rho}\Big)^{1/2} = \Big(\|\rho^{-1}f\|^2_{L^2(\Omega)} + \||\nabla f|\|^2_{L^2(\Omega)}\Big)^{1/2}. \tag{2}$$

Note that only the function f itself not its derivative is weighted.

Now we give two definitions of weighted Sobolev spaces on (possibly unbounded) domains, which correspond to the classical inhomogenious Sobolev spaces. It is well known that the two definitions are equivalent for bounded domains. In Appendix A we give a proof of this fact in the case of weighted spaces.

The first definition, which can be found in [Jan86], uses infinitely differentiable functions which have a finite $\|\cdot\|_{1,\rho}$-norm and builds the completion of these functions. The second one uses the concept of weak derivatives of integrable functions.

Definition 2.5 *The norm (2) is used to define the Sobolev spaces*

$$\begin{aligned} H^{1,\rho}(\Omega) &:= \overline{\Big\{f\in C^\infty(\Omega) \mid \|f\|_{1,\rho}<\infty\Big\}}^{\|\cdot\|_{1,\rho}}, &(3)\\ H_c^{1,\rho}(\Omega) &:= \overline{C_0^\infty(\mathbb{R}^n)|_\Omega}^{\|\cdot\|_{1,\rho}}, &(4)\end{aligned}$$

where $C_0^\infty(\mathbb{R}^n)|_\Omega$ denotes restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$.

Definition 2.6 *The set $W^{1,\rho}(\Omega)$ of all measurable functions u such that $\rho^{-1}u\in L^2(\Omega)$ and all distributional derivatives satisfy $\partial_i u\in L^2(\Omega)$ together with the scalar product (1) is called weighted Sobolev space of order 1.*

Definition 2.7 *i) The set $W^1(\Omega)$ of all measurable functions u such that $u\in L^2(\Omega)$ and all distributional derivatives satisfy $\partial_i u\in L^2(\Omega)$ together with the scalar product $\langle f,g\rangle_1 := \int_\Omega f(x)g(x)dx + \sum_{i=1}^n \int_\Omega \partial_i f(x)\partial_i g(x)dx$ is called (inhomogenious) Sobolev space of order 1.*

ii) The space $H^1(\Omega)$ is defined as closure of smooth functions with respect to the $\|\cdot\|_1$-norm.

It is well known that the last two definitions are equivalent, i.e., $W^1(\Omega) = H^1(\Omega)$.

Remark 2.8 *The relation $H_c^{1,\rho}(\Omega) \subset H^{1,\rho}(\Omega)$ holds, and the functions of both spaces need not vanish at the boundary $\partial\Omega$. They are even pointwise well defined there like in the case of the unweighted Sobolev spaces (see Theorem 4.1). For functions in $H_c^{1,\rho}(\Omega)$ there is a kind of growth condition at infinity.*
If the domain Ω is bounded and ρ is strictly positive on Ω, then the sets $H^{1,\rho}(\Omega)$ and $H^1(\Omega)$ are equal and the corresponding norms $\|\cdot\|_{1,\rho}$ and $\|\cdot\|_1$ are equivalent.

2.2 Lipschitz Boundaries

2.2.1 Definition

In the geophysical context one encounters domains similar to a half-space but with some topographical deformations. Later on, for the sake of simplicity we assume that such deformations have local support. In real life, deformations arising from rocks or electrodes stuck into the earth are not continuously differentiable and therefore we assume that our domain Ω has a Lipschitz-boundary and we take from [Jan86] and [Alt92] the following definition.

Definition 2.9 *Let Ω be a (possibly unbounded) domain. Its boundary $\partial\Omega$ is called Lipschitz if there exist numbers $\alpha, \beta > 0$ and a covering $\{U_i\}_{i\in\mathbb{N}}$ such that*

i) for each U_i, there exists an orthogonal transformation $A_i : U_i \to \mathbb{R}^n$, $(x_1, \cdots, x_n) \mapsto (y_1, \cdots, y_n)$, and a Lipschitz continuous map $a_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that for the set $U_\alpha := \left\{(y_1, \cdots, y_{n-1}) \in \mathbb{R}^{n-1} \mid |y_j| < \alpha,\ 1 \le j \le n-1\right\}$ holds:

a) $A_i(U_i \cap \partial\Omega) = \{\vec{y} \in \mathbb{R}^n \mid (y_1, \cdots, y_{n-1}) \in U_\alpha,\ y_n = a_i(y_1, \cdots, y_{n-1})\}$

b) $A_i(U_i \cap \Omega) = \{\vec{y} \in \mathbb{R}^n \mid (y_1, \cdots, y_{n-1}) \in U_\alpha,$
$\qquad a_i(y_1, \cdots, y_{n-1}) < y_n < a_i(y_1, \cdots, y_{n-1}) + \beta\}$

c) $A_i(U_i \cap (\mathbb{R}^n \setminus \Omega)) = \{\vec{y} \in \mathbb{R}^n \mid (y_1, \cdots, y_{n-1}) \in U_\alpha,$
$\qquad a_i(y_1, \cdots, y_{n-1}) - \beta < y_n < a_i(y_1, \cdots, y_{n-1})\}.$

The boundary $\partial\Omega$ is called locally finite Lipschitz if there exist numbers $L, \alpha, \beta > 0$, a positive integer N and a covering $\{U_i\}_{i\in\mathbb{N}}$ such that i) holds,

ii) at most N different U_i have nonempty intersection

and

iii) the maps a_i are all Lipschitz continuous with respect to the same constant L.

2.2.2 Flattening of the boundary

Remark 2.10 *When considering domains Ω with Lipschitz boundary $\partial\Omega$, the families of maps $\{A_i\}_{i\in\mathbb{N}}$ and $\{a_i\}_{i\in\mathbb{N}}$ may be used to define bijective in both directions Lipschitz continuous maps from the set of euclidean coordinates $Q_\alpha := U_\alpha \times (-\beta, \beta)$ to the sets U_i. The map $K_i : Q_\alpha \to \mathbb{R}^n$, defined by $(y_1, \cdots, y_n) \mapsto (y_1, \cdots, y_{n-1}, a_i(y_1, \cdots, y_{n-1}) + y_n)$, leads to the bijection $h_i := A_i^{-1} \circ K_i : Q_\alpha \to U_i$ such that the following properties hold :*

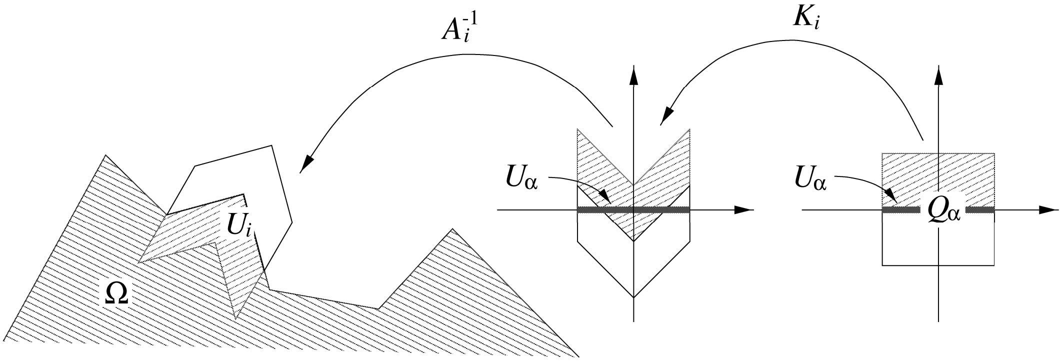


Figure 1: Flattening of the boundary

$$
\begin{array}{lrcl}
i) & h_i\left(Q_\alpha \cap \{\vec{y} \in \mathbb{R}^n \mid y_n > 0\}\right) & = & U_i \cap \Omega \\
ii) & h_i\left(Q_\alpha \cap \{\vec{y} \in \mathbb{R}^n \mid y_n = 0\}\right) & = & U_i \cap \partial\Omega \\
iii) & h_i\left(Q_\alpha \cap \{\vec{y} \in \mathbb{R}^n \mid y_n < 0\}\right) & = & U_i \cap \left(\mathbb{R}^n \setminus \Omega\right)
\end{array}
$$

iv) $h_i : Q_\alpha \to U_i$ *as well as* $h_i^{-1} : U_i \to Q_\alpha$ *is Lipschitz continuous.*

Proof: i) We have to show $A_i\left(U_i \cap \Omega\right) = K_i\left(Q_\alpha \cap \{\vec{y} \in \mathbb{R}^n \mid y_n > 0\}\right)$ which follows directly from the definitions. ii) and iii) are treated analoguously .
iv) Since A_i is an orthogonal transformation and therefore Lipschitz continuous, it suffices to prove that K_i and K_i^{-1} are Lipschitz continuous. We use the following estimates

$$
\begin{aligned}
\left\|K_i\left(\vec{y}\right) - K_i\left(\vec{z}\right)\right\|^2 &= \sum_{j=1}^{n-1}\left(y_j - z_j\right)^2 + \left(a_i\left(y_1, \cdots, y_{n-1}\right) - a_i\left(z_1, \cdots, z_{n-1}\right) + \left(y_n - z_n\right)\right)^2 \\
&\leq \sum_{j=1}^{n-1}\left(y_j - z_j\right)^2 + 2\left(a_i\left(y_1, \cdots, y_{n-1}\right) - a_i\left(z_1, \cdots, z_{n-1}\right)\right)^2 + \left(y_n - z_n\right)^2 \\
&\leq \sum_{j=1}^{n-1}\left(y_j - z_j\right)^2 + 2L^2\sum_{j=1}^{n-1}\left(y_j - z_j\right)^2 + \left(y_n - z_n\right)^2 \\
&\leq \left(1 + 2L^2\right)\left\|\vec{y} - \vec{z}\right\|^2
\end{aligned}
$$

to prove Lipschitz continuity for K_i.
The formula $K_i^{-1}\left(a\left(y_1, \cdots, y_n\right)\right) = \left(y_1, \cdots, y_{n-1}, y_n - a\left(y_1, \cdots, y_{n-1}\right)\right)$ leads to analoguous arguments for proving Lipschitz continuity of the mapping K_i^{-1}. ■

Remark 2.11 *If* Ω *has a locally finite Lipschitz boundary, the Lipschitz constant* $\sqrt{1 + 2L^2}$ *of the map* $h_i : Q_\alpha \to U_i$ *does not depend on i. Due to Lemma A.4, the maps* h_i *are in fact local isomorphisms, straightening out the boundary, which we use in subsequent sections. (See Figure 1.)*

3 Compact imbeddings

3.1 Compact imbeddings for general domains

Definition 3.1 *Let $\rho : \mathbb{R}^n \to \mathbb{R}_+$ be a weight. The mapping M_ρ is defined by the formula $M_\rho(f) := \rho^{-1} f$, for every measurable function $f : \Omega \to \mathbb{R}$.*

Theorem 3.2 *Let $\rho(x) = (1 + |x|)^\alpha$ be a weight and let $\alpha > \max\{1, n/2\}$. Then the mapping $M_\rho : H^{1,\rho}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ (resp. $M_\rho : H_c^{1,\rho}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$) is an injective, linear, continuous and compact operator with dense image, i.e a compact imbedding.*

Proof: See [Jan86]. ∎

Theorem 3.3 *Let $\rho(x) = (1 + |x|)^\alpha$ be a weight, let $\alpha > 1$ and the space dimension $n \geq 3$. Then the mapping $M_\rho : H_c^{1,\rho}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an injective, linear, continuous and compact operator with dense image.*

Proof: See [Jan86]. ∎

The transfer of these two theorems to general domains is achieved by using extension operators.

Definition 3.4 *Let $W(\Omega)$ any of the previously introduced function spaces (i.e $H^{1,\rho}(\Omega), W^{1,\rho}(\Omega), H^1(\Omega), \ldots$). A continuous operator $F : W(\Omega) \to W(\mathbb{R}^n)$ is called (continuous) extension operator if $F(u)|_\Omega = u$ for all $u \in W(\Omega)$.*

Lemma 3.5 *Let $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain such that there exists an extension operator $F : H^{1,\rho}(\Omega) \to H^{1,\rho}(\mathbb{R}^n)$ (resp. $F : H_c^{1,\rho}(\Omega) \to H_c^{1,\rho}(\mathbb{R}^n)$) in the sense of Definition 3.4.*
Furthermore, let $\rho(x) = (1 + |x|)^\alpha$ be a weight and let $R_\Omega : L^2(\mathbb{R}^n) \to L^2(\Omega)$ be the continuous and linear restriction map defined by $R_\Omega(f) := f|_\Omega$ for all $f \in L^2(\mathbb{R}^n)$.
If $\alpha > \max\{1, n/2\}$, then $M'_\rho := R_\Omega \circ M_\rho \circ F : H^{1,\rho}(\Omega) \to L^2(\Omega)$ (resp. $M'_\rho : H_c^{1,\rho}(\Omega) \to L^2(\Omega)$) is an injective, linear, continuous and compact operator with dense image. If $\alpha > 1$ and $n \geq 3$, then $M'_\rho : H_c^{1,\rho}(\Omega) \to L^2(\Omega)$ is an injective, linear, continuous and compact operator with dense image.

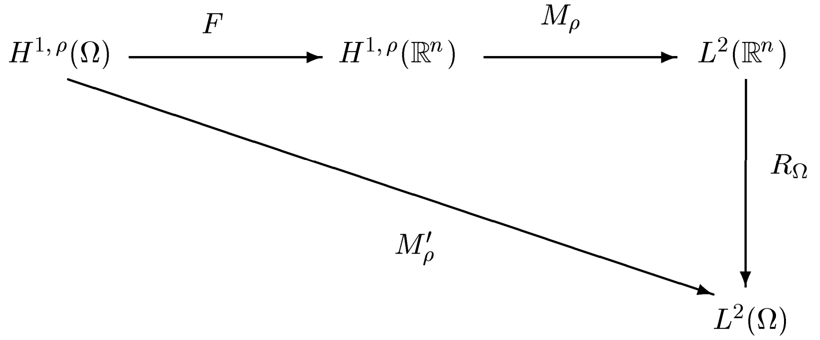
Proof: The mapping

$$\begin{aligned} M'_\rho := R_\Omega \circ M_\rho \circ F \quad &: \quad H^{1,\rho}(\Omega) \to L^2(\Omega) \\ (\text{resp. } M'_\rho := R_\Omega \circ M_\rho \circ F \quad &: \quad H_c^{1,\rho}(\Omega) \to L^2(\Omega)) \end{aligned} \tag{5}$$

is linear, continuous and compact by Theorem 3.2 (resp. by Theorem 3.3). Injectivity is immediately verified.
To prove the density of the image we have to show that every test function $\phi \in \mathcal{D}(\Omega)$ lies in the set $M_\rho(H_c^{1,\rho}(\Omega))$, because then it follows

$$L^2(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{L^2(\Omega)}} \subset \overline{M_\rho\left(H_c^{1,\rho}(\Omega)\right)}^{\|\cdot\|_{L^2(\Omega)}} \subset \overline{M_\rho\left(H^{1,\rho}(\Omega)\right)}^{\|\cdot\|_{L^2(\Omega)}}.$$

Therefore assume $\phi \in \mathcal{D}(\Omega)$ and define $u(x) := \rho(x)\phi(x)$ for all $x \in \Omega$. Then $M_\rho(u) = u\rho^{-1} = \phi$ and hence $\phi \in M_\rho(H_c^{1,\rho}(\Omega))$ if we can show $u \in H_c^{1,\rho}(\Omega)$.
Since $\partial_i\rho$, as well as ρ, is bounded on compact sets it follows that $\|u\|_{1,\rho} = \int_\Omega \rho^{-2}u^2dx + \sum_{i=1}^n \int_\Omega (\partial_i u)^2 dx < \infty$ by means of the formula $\partial_i u = \partial_i\rho\phi + \rho\partial_i\phi$. This formula is valid for every Distribution ρ and every infinitely differentiable function ϕ (see for example [Rud91] or [MG92]). Now corollary A.9 finishes the proof.



■

3.2 Application to geophysical domains

3.2.1 Extension operators for geophysical domains

For geophysical problems we consider domains which essentially look like the lower half-space $\mathbb{R}^n_-$ for $n \geq 2$. To be more precise we introduce the concept of a simplified geophysical domain.

Definition 3.6 *A domain $\Omega \subset \mathbb{R}^n$ for which there exists a compact subset $S \subset \mathbb{R}^n$ such that*

i) $\Omega \cap (\mathbb{R}^n \setminus S) = \mathbb{R}^n_- \setminus S$ and

ii) $\Omega \cap S$ has a Lipschitz boundary (see definition 2.9)

is called geophysical domain.

An illustration of such a domain is presented in figure 2.

Lemma 3.7 *Let Ω be a geophysical domain in the sense of Definition 3.6. Furthermore, let ρ be a translation and dilation invariant weight with respect to Definition 2.1. Then there exists an extension operator $F_\Omega : H^{1,\rho}(\Omega) \to H^{1,\rho}(\mathbb{R}^n)$ (resp. $F_\Omega : H_c^{1,\rho}(\Omega) \to H_c^{1,\rho}(\mathbb{R}^n)$) in the sense of Definition 3.4.*

Proof: From the assumptions, property ii) of definition 3.6 and definition 2.9 follows the existence of a finite covering $\{U_i\}_{i\in\{1,\cdots,n_0\}}$ such that $\partial\Omega \cap S \subset \bigcup_{i=1}^{n_0} U_i$. Then, by means of remark 2.10, there are bijective in both directions Lipschitz continuous maps $\{h_i\}_{i\in\{1,\cdots,n_0\}}$ supported in the compact set $V \subset \mathbb{R}^n$, where the set $V := \bigcup_{i=1}^{n_0} U_i$ is bounded and contains all of the disfigured boundary.
Let $U_0 := U_\varepsilon(\mathbb{R}^n \setminus V)$ an ε - environment of the set $\mathbb{R}^n \setminus V$, where $\varepsilon := \frac{1}{2}\mathrm{dist}\,((\partial\Omega \cap \mathrm{S}), \partial\mathrm{V})$. Then $\bigcup_{i=0}^{n_0} U_i = \mathbb{R}^n$ is an open covering of $\mathbb{R}^n$ and Theorem 6.20 of [Rud91] provides a partition of one $\{\Psi_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ which is subordinated to $\{U_i\}_{i\in\{0,\cdots,n_0\}}$. Additionally it satisfies the property that for every

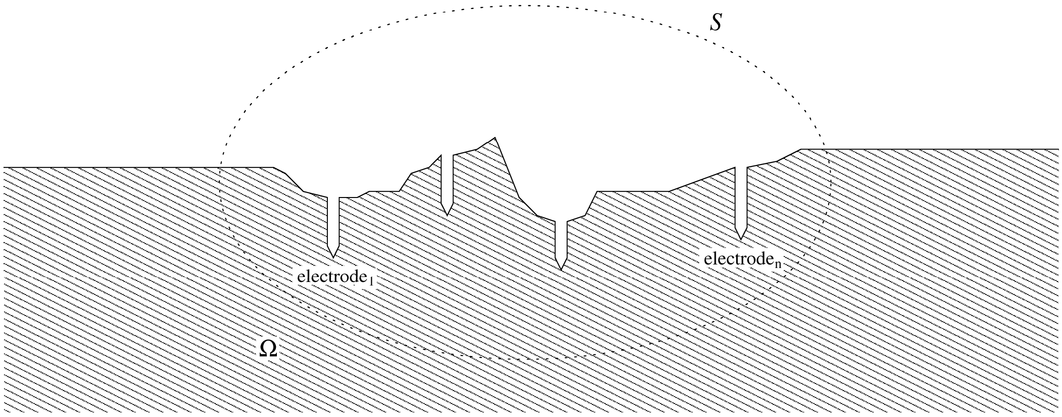


Figure 2: simplified geophysical domain

compact set $K \subset \mathbb{R}^n$ there exists an open set $W \supset K$ and a number $j_0 \in \mathbb{N}$ such that $\Psi_1(x) + \cdots + \Psi_{j_0}(x) = 1$ for all $x \in W$, which particularely implies locally finiteness of $\{\Psi_j\}_{j\in\mathbb{N}}$. Now let

$$\phi_0(x) := \sum_{\mathrm{supp}(\Psi_j)\subset U_0} \Psi_j(x) \qquad \forall x \in \mathbb{R}^n \tag{6}$$

and for each $i \in \{1, \cdots, n_0\}$ let

$$\phi_i(x) := \sum_{\mathrm{supp}(\Psi_j)\subset U_i \,\wedge\, \mathrm{supp}(\Psi_j)\not\subset\bigcup_{k=0}^{i-1} U_k} \Psi_j(x) \qquad \forall x \in \mathbb{R}^n. \tag{7}$$

Then it is clear that

i) the system $\{U_0, U_1, \cdots, U_{n_0}\}$ is an open covering of $\mathbb{R}^n$,

ii) every ϕ_i lies in $\mathcal{D}(U_i)$,

iii) $0 \leq \phi_i(x) \leq 1$ for all $x \in \mathbb{R}^n$,

iv) $\phi_0(x) = 1$ for all $x \in \mathbb{R}^n \setminus V$ and

v) $\sum_{i=0}^{n_0} \phi_i(x) = 1$ for all $x \in \mathbb{R}^n$.

Now we decompose every function $f \in \mathbb{C}^\infty(\Omega)$ into a finite sum : $f = \sum_{i=0}^{n_0} f\phi_i$. Let $S' \subset \mathbb{R}^n$ be a compact subset such that $V \cup S \subset S'$ holds. Since iv) above implies $\partial_j\phi_i(x) = 0$ for all $x \in \mathbb{R}^n \setminus S'$, we have the following estimates :

$$\begin{aligned}
\|f\phi_i\|^2_{H^{1,\rho}(U_i\cap\Omega)} &= \int_{U_i\cap\Omega} (f(x)\phi_i(x))^2 \rho(x)^{-2} dx + \sum_{j=1}^n \int_{U_i\cap\Omega} (\partial_j (f(x)\phi_i(x)))^2 dx \\
&\leq \|f\rho^{-1}\|^2_{L^2(\Omega)} + 2\sum_{j=1}^n \int_\Omega (\partial_j f(x)\phi_i(x))^2 dx + 2\sum_{j=1}^n \int_{S'\cap\Omega} (f(x)\partial_j\phi_i(x))^2 dx \\
&\leq \|f\rho^{-1}\|^2_{L^2(\Omega)} + 2\sum_{j=1}^n \int_\Omega (\partial_j f(x))^2 dx + 2nC \int_{S'\cap\Omega} f(x)^2 dx
\end{aligned}$$

$$\begin{aligned} &\leq \|f\rho^{-1}\|^2_{L^2(\Omega)} + \||\nabla f|\|^2 + 2nCR^2 \int_{S'\cap\Omega} \left(f\rho^{-1}\right)^2 dx \\ &\leq C'\|f\|_{1,\rho}, \end{aligned}$$

where $R := \sup\{\rho(x)\,|\,x \in S' \cap \Omega\}$ and C, C' do not depend on f. Therefore the mapping $M_i : H^{1,\rho}(\Omega) \to H^{1,\rho}(\Omega \cap U_i)$, defined by $M_i(f) := \phi_i f$, is continuous for $i \in \{0, \cdots, n_0\}$.

Lemma A.4 states that $*h_i$ and $*h_i^{-1}$ provide the isomorphisms $*h_i : H^{1,\rho}(\Omega \cap U_i) \cong H^{1,\rho}(Q_\alpha \cap \{\vec{y}\,|\,y_n > 0\})$ and $*h_i^{-1} : H^{1,\rho}(Q_\alpha) \cong H^{1,\rho}(U_i)$ for each $i \in \{1, \cdots, n_0\}$. Therefore we can apply the method of Hestenes (see for example [Fol95] Lemma 6.43) to extend the function $*h_i \circ M_i(f)$ from $Q_\alpha \cap \{\vec{y}\,|\,y_n > 0\}$ to Q_α. In the following we, denote this extension operator by F_{Q_α}. We have $\text{supp}\,(F_{Q_\alpha} \circ *h_i \circ M_i(f)) \subset Q_\alpha$ which implies $\text{supp}\left(*h_i^{-1} \circ F_{Q_\alpha} \circ *h_i \circ M_i(f)\right) \subset U_i$ and $*h_i^{-1} \circ F_{Q_\alpha} \circ *h_i \circ M_i(f) \in H^{1,\rho}(U_i)$. Therefore this mapping may be extended trivially to $\mathbb{R}^n$ by zero.

So far, we have constructed operators to extend the functions $\phi_i f$ for all $i \in \{1, \cdots, n_0\}$. To deal with $\phi_0 f$, which is supported on the unbounded set U_0, we note that $\partial\Omega \cap U_0 \subset \mathbb{R}^{n-1} \times \{0\} = \partial\mathbb{R}^n_-$. Therefore the method of Hestenes extends the function $\phi_0 f$ along the boundary $\partial\Omega \cap U_0$. $\phi_0 f$ vanishes on $\partial\Omega \cap (\mathbb{R}^n \setminus U_0)$ since $\text{supp}\,(\phi_0 f) \subset U_0 \cap \Omega$ and therefore we extend it along $\partial\Omega \cap (\mathbb{R}^n \setminus U_0)$ by zero. We denote the extension operator resulting from the last two steps by $F_{\mathbb{R}^n}$. We summarize these results by defining the extension of a function $f \in C^\infty(\Omega)$ by means of the formula

$$F_\Omega(f) := F_{\mathbb{R}^n} \circ M_0(f) + \sum_{i=1}^{n_0} *h_i^{-1} \circ F_{Q_\alpha} \circ *h_i \circ M_i(f). \tag{8}$$

Since each operator on the right hand side of Equation (8) is linear and continuous, $F_\Omega : H^{1,\rho}(\Omega) \cap C^\infty(\Omega) \to H^{1,\rho}(\mathbb{R}^n)$ is linear and continuous, too. It is immediately verified that for $x \in \Omega$ and $f \in H^{1,\rho}(\Omega) \cap C^\infty(\Omega)$ we have $(F_\Omega(f))\,(x) = x$. Therefore, the unique linear extension of F_Ω from $H^{1,\rho}(\Omega) \cap C^\infty(\Omega)$ to $H^{1,\rho}(\Omega)$ provides the desired continuous extension operator.
The extension $F_\Omega : H^{1,\rho}_c(\Omega) \to H^{1,\rho}_c(\mathbb{R}^n)$ is constructed analoguously. ■

3.2.2 Compact imbeddings for unbounded geophysical domains

Now we are able to formulate and prove an imbedding theorem for unbounded geophysical domains.

Theorem 3.8 *Let Ω be a geophysical domain in the sense of Definition 3.6. Furthermore, let $\rho(x) = (1 + |x|)^\alpha$ be a weight. If $\alpha > \max\{1, n/2\}$, then there exists a compact imbedding $M : H^{1,\rho}(\Omega) \to L^2(\Omega)$ (resp. $M : H^{1,\rho}_c(\Omega) \to L^2(\Omega)$). If $\alpha > 1$ and $n \geq 3$, then there exists a compact imbedding $M : H^{1,\rho}_c(\Omega) \to L^2(\Omega)$.*

Proof: Take the extension operator F_Ω of Lemma 3.7 and apply Lemma 3.5 to construct the extension operator $M := R_\Omega \circ M_\rho \circ F_\Omega$ having the desired properties.
■

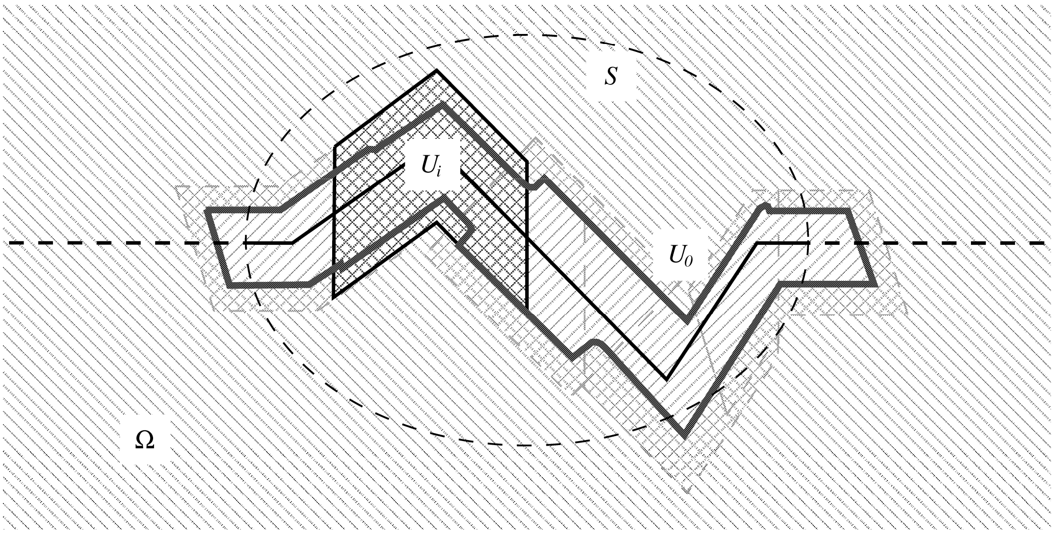


Figure 3: Covering of $\mathbb{R}^n$

4 Trace theorems

Theorem 4.1 *Let Ω be a (possibly unbounded) domain with a locally finite Lipschitz boundary $\partial\Omega$. Further, let ρ be a translation and dilation invariant weight in the sense of Definition 2.1 such that $\rho(x) \geq 1$ for all $x \in \mathbb{R}^n$.*

Then there exists a linear and continuous map $\gamma : H^{1,\rho}(\Omega) \to L^{2,\rho^{-1}}(\partial\Omega)$ (resp. $\gamma : H_c^{1,\rho}(\Omega) \to L^{2,\rho^{-1}}(\partial\Omega)$) such that $\gamma(u) = u|_{\partial\Omega}$ for all $u \in C^1(\overline{\Omega}) \cap H^{1,\rho}(\Omega)$, where $L^{2,\rho^{-1}}(\Omega)$ is the space of measurable functions u, such that $u\rho^{-1} \in L^2(\Omega)$.

Additionally, if there exists an extension operator (see Definition 3.4), then the trace operator $\gamma : H^{1,\rho}(\Omega) \to L^{2,\rho^{-1}}(\partial\Omega)$ (resp. $\gamma : H_c^{1,\rho}(\Omega) \to L^{2,\rho^{-1}}(\partial\Omega)$) is compact.

Proof: The theorem, except the compactness assertion, can be found in [Jan86] Th.4.6. For proving compactness, we use the inequality

$$\beta \int_{U_\alpha} \frac{\left(u \circ A_i^{-1}\left(y', a_i(y')\right)\right)^2}{\left(\rho \circ A_i^{-1}\left(y', a_i(y')\right)\right)^2} dy' \leq c_1\beta^2 \int_{U_\alpha} \int_{a_i(y')}^{a_i(y')+\beta} \left(\partial_n \left(u \circ A_i^{-1}\right)(y', y_n)\right)^2 dy_n dy' + c_2 \int_{U_\alpha} \int_{a_i(y')}^{a_i(y')+\beta} \frac{\left(u \circ A_i^{-1}(y', y_n)\right)^2}{\left(\rho \circ A_i^{-1}(y', y_n)\right)^2} dy_n dy', \tag{9}$$

which holds for functions $u \in C^1(\overline{\Omega}) \cap H^{1,\rho}(\Omega)$ and can be found in the proof of [Jan86] Th.4.6.

Here, y' denotes the vector of the $n-1$ euclidean coordinates $y' \in U_\alpha$ arising from the orthogonal coordinate transform $A_i : \vec{x} \mapsto \vec{y}$ corresponding to Definition 2.9. Integration is done with respect to the new coordinates y', y_n. Let $\{\phi_i\}_{i\in I}$ be a partition of unity subordinated to the covering $\{U_i\}_{i\in I}$. Due to Definition B.1 we have

$$\int_{\partial\Omega} u^2\rho^{-2} d\mathcal{S} = \sum_{i\in I} \int_{U_\alpha} \left(\phi_i u^2 \rho^{-2}\right) \circ A_i^{-1}(y', a_i(y')) \sqrt{1 + |\nabla a_i(y')|^2} dy'. \tag{10}$$

Since there exists a fixed real and positive constant L such that $\partial_i a_i(y') \leq L$ for almost every $y' \in U_\alpha$, it follows from Equation (9) and (10) :

$$\begin{aligned}
\int_{\partial\Omega} u^2\rho^{-2} d\mathcal{S} &\leq C\sum_{i\in I}\int_{U_\alpha}\left(u^2\rho^{-2}\right)\circ A_i^{-1}(y', a_i(y'))dy' \\
&\leq C'\beta\sum_{i\in I}\int_{U_\alpha}\int_{a_i(y')}^{a_i(y')+\beta}\left(\partial_n\left(u\circ A_i^{-1}\right)(y', y_n)\right)^2 dy_n dy' + \\
&\quad \frac{C'}{\beta}\sum_{i\in I}\int_{U_\alpha}\int_{a_i(y')}^{a_i(y')+\beta}\frac{\left(u\circ A_i^{-1}(y', y_n)\right)^2}{\left(\rho\circ A_i^{-1}(y', y_n)\right)^2} dy_n dy' \\
&\leq C'\beta\sum_{i\in I}\int_{U_i\cap\Omega}|\nabla u(x)|^2 dx + \frac{C'}{\beta}\sum_{i\in I}\int_{U_i\cap\Omega} u(x)^2\rho(x)^{-2}dx \\
&\leq C''\beta\int_\Omega |\nabla u(x)|^2 dx + \frac{C''}{\beta}\int_\Omega u(x)^2\rho(x)^{-2}dx, \qquad (11)
\end{aligned}$$

where the last inequality results from the locally finiteness of the Lipschitz boundary (see Definition 2.9). Further, we have used the fact that the Jacobian of the orthogonal transformation A_i equals 1, i.e $|J_{A_i}| = |J_{A_i^{-1}}| = 1$. Now let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $H^{1,\rho}(\Omega)$ (resp. in $H_c^{1,\rho}(\Omega)$) such that $u_j \rightharpoonup g$ for $n \to \infty$, where $g \in H^{1,\rho}(\Omega)$ (resp. $g \in H_c^{1,\rho}(\Omega)$) without loss of generality may assumed to be the zero function. Since $H^{1,\rho}(\Omega)$ is a Hilbert-space and therefore reflexive, we have to show $\gamma(u_j) \to \gamma(g) = 0$ for $n \to \infty$.

Equation (11) extends by continuity to all of $H_c^{1,\rho}(\Omega)$ and due to Lemma A.11 also to all of $H^{1,\rho}(\Omega)$. Therefore we have

$$\begin{aligned}
\|\gamma(u_j)\|_{L^{2,\rho^{-1}}(\partial\Omega)} &= \int_{\partial\Omega} u_j^2\rho^{-2}d\mathcal{S} \leq \underbrace{C''\beta\int_\Omega |\nabla u_j(x)|^2 dx}_{I} \\
&\quad + \underbrace{\frac{C''}{\beta}\int_\Omega u_j(x)^2\rho(x)^{-2}dx}_{II}. \qquad (12)
\end{aligned}$$

Since the mapping $M_\rho : H^{1,\rho}(\Omega) \to L^2(\Omega)$ (resp. $M_\rho : H_c^{1,\rho}(\Omega) \to L^2(\Omega)$) is compact, (see Theorem 3.2 and Lemma 3.5) we get $\|M_\rho(u_j)\|_{L^2(\Omega)} = \int_\Omega (u_j(x))^2 \rho(x)^{-2}dx \to 0$, i.e. term II of Equation (12) converges to 0 as $j \to \infty$ for every $\beta > 0$.

Since weakly convergent sequences are bounded, we have $\int_\Omega |\nabla u_j(x)|^2 dx \leq \|u_j\|_{1,\rho} \leq K$ for all $j \in \mathbb{N}$ and for a fixed positive constant K. This implies convergence of Term I to zero as β approaches zero. Therefore, the left hand side of Equation (12) must converge to zero as $j \to \infty$. ∎

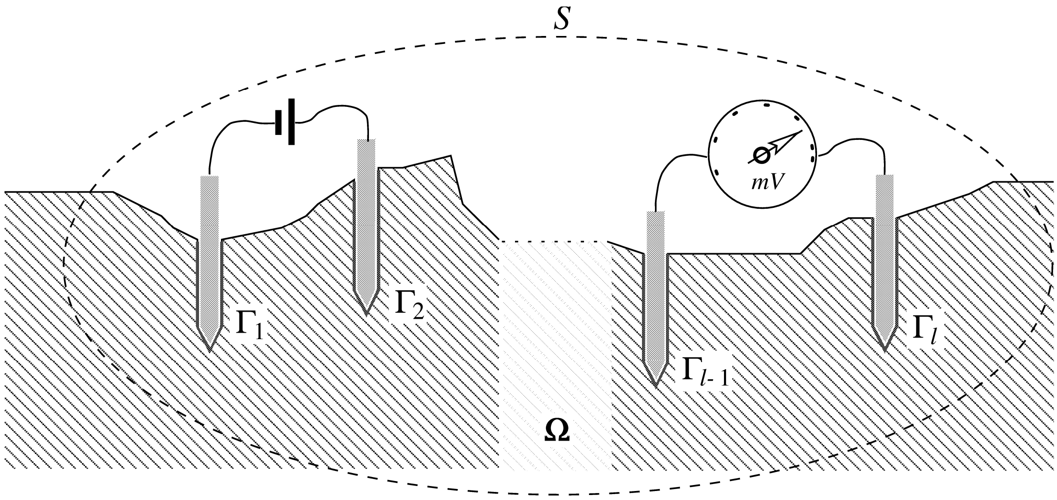


Figure 4: Complete Electrode Model

5 The Complete Electrode Model in geophysics

In medical impedance tomography different physical models were investigated and it turned out that the so-called "Complede Electrode Model" reproduces experimental data most accurate (see [SCI92] or [JS97]).
We want to generalize this model in some directions :

- switching from bounded domains to unbounded geophysical domains,
- using arbitrary contact impedance layers instead of constant layers,
- providing more general normalizations for solutions .

5.1 Definition

In this chapter Ω denotes an unbounded geophysical domain (see Definition 3.6), i.e, one which has a Lipschitz boundary (see Definition 2.9) and l electrodes stucked into it. In the following, the part of the surface which is beneath the i^{th}-electrode is referred to as Γ_i .

First of all, we give a description of the boundary value problem within the classical formulation. Therefore let $\sigma : \Omega \to \mathbb{R}_+$ be a conductivity distribution , $u : \Omega \to \mathbb{R}$ the potential function , $(I_1, \cdots, I_l)$ the discrete vector of applied currents and $(u_1, \cdots, u_l)$ the discrete vector of measured voltages .
i) The interior differential equation is the usual one :

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega, \tag{13}$$

where $\nabla\cdot$ denotes the well known "div" operator from vector analysis. Physically, Equation (13) states that inside the domain there are no current sources.
ii) The total current applied to the i^{th}-electrode is

$$\int_{\Gamma_i} \sigma \partial_\nu u d\mathcal{S} = I_i, \tag{14}$$

where $\partial\nu$ denotes the derivative in outer normal direction.
iii) No current crosses the surface area which hasn't any contact with electrodes :

$$\sigma\partial_\nu u = 0 \quad \text{on } \partial\Omega \setminus \bigcup_{i=1}^{l} \Gamma_i. \tag{15}$$

iv) There exists a thin resistance layer $z_i : \Gamma_i \to \mathbb{R}_+$ between the i$^{\text{th}}$-electrode and the surface $\partial\Omega$, coupling the potential distribution of the interior function u to the measured electrode voltage U_i :

$$u + z_i\sigma\partial_\nu u = u_i \quad \text{on } \Gamma_i. \tag{16}$$

In the following, $z_i : \Gamma_i \to \mathbb{R}_+$ is called contact impedance .

5.2 Weak formulation of the boundary-value problem

In [SCI92] a weak formulation for the Complete Electrode Model was derived, assuming bounded domains Ω and constant contact impedances, i.e., $z_i(s) = const$ for all $s \in \Gamma_i$.

A solution to the (weak) boundary value problem with respect to the Complete Electrode Model consists of a differentiable potential function and a discrete vector of measured electrode voltages. Therefore we introduce the following spaces of functions (similar to those defined in [SCI92]) :

Definition 5.1 *The space of solutions to the boundary value problem with respect to the Complete Electrode Model is the direct sum of a space of differentiable functions and the vector space of discrete electrode voltages :*

$$H_l^{\rho}(\Omega) := H^{1,\rho}(\Omega) \oplus \mathbb{R}^l \tag{17}$$

$$H_l^{\rho,c}(\Omega) := H_c^{1,\rho}(\Omega) \oplus \mathbb{R}^l, \tag{18}$$

where the order of smoothness subscript 1 *is omitted, because in the following, we do not encounter spaces of higher order differentiable functions. The scalar product on these spaces which induces the natural product topology is the following :*

$$\begin{aligned} \langle (u,\vec{u}),(v,\vec{v})\rangle_l &:= \langle u,v\rangle_{1,\rho} + \sum_{i=1}^{l} u_i v_i \\ &= \int_\Omega \rho^{-2} uv dx + \int_\Omega \nabla u \nabla v dx + \sum_{i=1}^{l} u_i v_i. \end{aligned} \tag{19}$$

The space $H_l^{\rho}(\Omega)$ (resp. $H_l^{\rho,c}(\Omega)$) as a product of Hilbert spaces is a Hilbert space, too. The induced norm reads as follows :

$$\|(u,\vec{u})\|_l := \left(\|u\|_{1,\rho}^2 + \sum_{i=1}^{l} u_i^2 \right)^{\frac{1}{2}} = \left(\|u\|_{1,\rho}^2 + |\vec{u}|_2^2 \right)^{\frac{1}{2}}. \tag{20}$$

Before presenting the Hilbert space theory of boundary value problems, we introduce the following frequently used linear and bilinear forms.

Definition 5.2 *For $(u,\vec{u}),(v,\vec{v}) \in H_l^\rho(\Omega)$ (resp. $(u,\vec{u}),(v,\vec{v}) \in H_l^{\rho,c}(\Omega)$) the bilinear form $b : H_l^\rho(\Omega) \times H_l^\rho(\Omega) \to \mathbb{R}$ (resp. $b : H_l^{\rho,c}(\Omega) \times H_l^{\rho,c}(\Omega) \to \mathbb{R}$) is defined by*

$$\begin{aligned} b\left((u,\vec{u}),(v,\vec{v})\right) \ &:= \int_\Omega \sigma(x)\nabla u(x)\nabla v(x)dx \ + \\ &\sum_{i=1}^{l}\int_{\Gamma_i} z_i^{-1}(s)(u(s)-u_i)(v(s)-v_i)d\mathcal{S}. \end{aligned} \tag{21}$$

For each vector of applied currents $\vec{I}=(I_1,\cdots,I_l)$ the linear form $g_{\vec{I}}$ is defined by :

$$g_{\vec{I}}\left((u,\vec{u})\right) := \sum_{i=1}^{l} I_i u_i \tag{22}$$

.

Now we are sufficiently prepared to generalize the concept in [SCI92] of weak solutions to boundary value problems with respect to the Complete Electrode Model.

Definition 5.3 *An element $(u,\vec{u}) \in H_l^\rho(\Omega)$ (resp. $(u,\vec{u}) \in H_l^{\rho,c}(\Omega)$) is called weak solution to the boundary value problem* (13), (14), (15) *and* (16) *if the equation*

$$b\left((u,\vec{u}),(v,\vec{v})\right) = g_{\vec{I}}\left((v,\vec{v})\right) \tag{23}$$

holds for all $(v,\vec{v}) \in H_l^\rho(\Omega)$ (resp. $(v,\vec{v}) \in H_l^{\rho,c}(\Omega)$).

Lemma 5.4 *Let Ω be a (possibly unbounded) domain, $\sigma \in C^1(\overline{\Omega})$ and $z_i \in C(\Gamma_i)$ for each $i \in \{1,\cdots,l\}$. Further, let $(u,\vec{u}) \in \left(C^2(\Omega)\cap C^1(\overline{\Omega})\right) \oplus \mathbb{R}^l$ be a classical solution to the boundary value problem* (13), (14), (15) *and* (16). *Then it is also a weak solution with respect to Definition 5.3, i.e., it satisfies the weak formulation*

$$b\left((u,\vec{u}),(v,\vec{v})\right) = g_{\vec{I}}\left((v,\vec{v})\right) \ \forall (v,\vec{v}) \in H_l^{\rho,c}(\Omega). \tag{24}$$

Furthermore, if Ω is a bounded domain the sum of currents necessarily vanishes, i.e.,

$$\sum_{j=1}^{l} I_j = 0. \tag{25}$$

Proof: The starting point for such proofs is usually the Gauss-theorem (see Theorem B.3) :

$$\int_{\partial\Omega'} \sigma(s)v(s)\partial_\nu u(s)d\mathcal{S} - \int_{\Omega'} \sigma(x)\nabla u(x)\nabla v(x)dx = \int_{\Omega'} v(x)\underbrace{\nabla\cdot(\sigma(x)\nabla u(x))dx}_{=0} = 0,$$

where $v \in C_0^\infty(\mathbb{R}^n)|_\Omega$ and $\Omega' \subset \Omega$ is any bounded open subset such that $\mathrm{supp}(v)\setminus\partial\Omega \subset \Omega'$. Since v vanishes on that part of $\partial\Omega'$ lying inside Ω, the last equation can be written as

$$\int_{\partial\Omega} \sigma(s)v(s)\partial_\nu u(s)d\mathcal{S} - \int_{\Omega} \sigma(x)\nabla u(x)\nabla v(x)dx = 0. \tag{26}$$

Boundary condition (15) and (16) together with equation (26) provides

$$\sum_{i=1}^{l} \int_{\Gamma_i} z_i^{-1}(s)(u(s) - u_i)v(s)d\mathcal{S} + \int_{\Omega} \sigma(x)\nabla u(x)\nabla v(x)dx = 0. \tag{27}$$

Boundary condition (16) together with condition (14) yields

$$\int_{\Gamma_i} z_i^{-1}(s)(u_i - u(s))d\mathcal{S} = \int_{\Gamma_i} \sigma(s)\partial_\nu u(s)d\mathcal{S} = I_i,$$

which leads directly to

$$\sum_{i=1}^{l} \int_{\Gamma_i} v_i z_i^{-1}(s)(u_i - u(s))d\mathcal{S} = \sum_{i=1}^{l} I_i v_i,$$

and together with equation (27) we conclude

$$b\left((u, \vec{u}), (v, \vec{v})\right) = g_{\vec{I}}\left((v, \vec{v})\right), \tag{28}$$

which extends by continuity of the biliner form (see Theorem 5.7) to all $(v, \vec{v}) \in H_l^{\rho,c}(\Omega)$. To prove $\sum_{j=1}^{l} I_j = 0$ for bounded Ω let $v \equiv 1$. Equation (26), (14) and (15) immediately provides $\sum_{j=1}^{l} I_j = \int_{\partial\Omega} \sigma(s)\partial_\nu u(s)d\mathcal{S} = 0$. ■

5.3 Poincaré inequalities

Ellipticity is a key criterion in proving the existence and uniquenes of solutions to boundary value problems raised by operators of second order. Inequalities of Poincaré-type play an essential role in proving the ellipticity of such operators. In the paper [SCI92] quotient spaces were used to obtain a normalization which provides unique solutions. We prefer to show appropriate Poincaré-type inequalities, valid even if the domain Ω is unbounded, involving terms which allow us later to introduce a suitable normalization.

Theorem 5.5 *Let Ω be a (possibly unbounded) domain with a locally finite Lipschitz boundary (see Definition 2.9) disfigured by l electrodes stucked into Ω such that there exists an extension operator $F : H^{1,\rho}(\Omega) \to H^{1,\rho}(\mathbb{R}^n)$ (resp. $F : H_c^{1,\rho}(\Omega) \to H_c^{1,\rho}(\mathbb{R}^n)$) (see Definition 3.4). Furthermore, let $\rho(x) := (1 + |x|)^\alpha$ be a weight and denote the measure of the surface area $\int_{\Gamma_i} 1 d\mathcal{S}$ beneath the i^{th}-electrode by $|\Gamma_i|$. Then the following conditions lead to Poincaré inequalities :*

i) If $\alpha > n/2$ and $\{i_1, \cdots, i_{l_0}\} \subset \{1, \cdots, l\}$, then there exists a constant $C > 0$ such that

$$\|(u, \vec{u})\|_l^2 \leq C \left(\||\nabla u|\|_{L^2(\Omega)}^2 + \sum_{i=1}^{l} \int_{\Gamma_i} (u(s) - u_i)^2 d\mathcal{S} + \left| \sum_{j=1}^{l_0} |\Gamma_{i_j}| u_{i_j} \right|^2 \right) \tag{29}$$

is valid for all $(u, \vec{u}) \in H_l^\rho(\Omega)$ (resp. $(u, \vec{u}) \in H_l^{\rho,c}(\Omega)$).

ii) If $n \geq 3$, $1 < \alpha \leq n/2$ and Ω is really an unbounded domain such that $\rho^{-1} \notin L^2(\Omega)$, then there exists a constant $C > 0$ such that

$$\|(u, \vec{u})\|_l^2 \leq C \left(\| |\nabla u| \|_{L^2(\Omega)}^2 + \sum_{i=1}^{l} \int_{\Gamma_i} (u(s) - u_i)^2 d\mathcal{S} \right) \tag{30}$$

holds for all $(u, \vec{u}) \in H_l^{\rho, c}(\Omega)$.

Proof: i) If we assume the assertion is false, then we can find a sequence $(u_n, \vec{u}^n) \in H_l^\rho(\Omega)$, such that for each $n \in \mathbb{N}$ the inequality $\|(u_n, \vec{u}^n)\|_l^2 > n \left(\| |\nabla u_n| \|_{L^2(\Omega)}^2 + \sum_{i=1}^{l_0} \int_{\Gamma_i} (u_n(s) - u_i^n)^2 d\mathcal{S} + \left| \sum_{j=1}^{l_0} |\Gamma_{i_j}| u_{i_j}^n \right|^2 \right)$ holds. Dividing both sides by $\|(u_n, \vec{u}^n)\|_l^2$ we can summarize the assumption to

$$1 = \|(u_n, \vec{u}^n)\|_l^2 = \|\rho^{-1} u_n\|_{L^2(\Omega)}^2 + \| |\nabla u_n| \|_{L^2(\Omega)}^2 + \sum_{i=1}^{l} (u_i^n)^2 \tag{31}$$

and

$$\frac{1}{n} > \| |\nabla u_n| \|_{L^2(\Omega)}^2 + \sum_{i=1}^{l} \int_{\Gamma_i} (u_n(s) - u_i^n)^2 d\mathcal{S} + \left| \sum_{j=1}^{l_0} |\Gamma_{i_j}| u_{i_j}^n \right|^2 \tag{32}$$

for each $n \in \mathbb{N}$.

Equation (31) leads to $\sum_{i=1}^{l} (u_i^n)^2 \leq 1$ for all $n \in \mathbb{N}$ and therefore exists a subsequence (denoted again by $\{(u_n, \vec{u}^n)\}_{n \in \mathbb{N}}$) and an element $\vec{g} \in \mathbb{R}^l$ such that $\lim_{n \to \infty} \sum_{i=1}^{l} (u_i^n - g_i)^2 = 0$.

The next step is reached using the fact that $\lim_{n \to \infty} \| |\nabla u_n| \|_{L^2(\Omega)} = 0$ and $\|u_n\|_{1,\rho} \leq 1$ for all $n \in \mathbb{N}$. Theorem 3.2 and Lemma 3.5 can be applied to provide a subsequence (denoted again by $\{(u_n, \vec{u}^n)\}_{n \in \mathbb{N}}$) such that $M_\rho(u_n) = \rho^{-1} u_n$ is L^2-convergent to some element in $L^2(\Omega)$. Therefore, the sequence $\{u_n\}_{n \in \mathbb{N}}$ is a Cauchy-sequence in $H^{1,\rho}(\Omega)$, and there exists an element $g \in H^{1,\rho}(\Omega)$ such that $\lim_{n \to \infty} \|u_n - g\|_{1,\rho} = 0$. Since $\lim_{n \to \infty} \| |\nabla u_n| \|_{L^2(\Omega)} = 0$ we have $\partial_i g = 0$ almost everywhere in Ω for each $i \in \{1, \cdots, n\}$. Corollary A.10 applied to g states that there exists a constant $c_0 \in \mathbb{R}$ such that $g(x) = c_0$ for almost every $x \in \Omega$ and $(\gamma(g))(s) = c_0$ for almost every $s \in \partial\Omega$.

Hölder's inequality leads to

$$\left| \sum_{j=1}^{l_0} |\Gamma_{i_j}| u_{i_j}^n - \sum_{j=1}^{l_0} |\Gamma_{i_j}| g_{i_j} \right| \leq \left(\sum_{i=1}^{l} |\Gamma_i|^2 \right)^{\frac{1}{2}} \underbrace{\left(\sum_{i=1}^{l} (u_i^n - g_i)^2 \right)^{\frac{1}{2}}}_{\to 0 \text{ as } n \to \infty} \tag{33}$$

and together with inequality (32) and the triangle inequality we conclude

$$\left| \sum_{j=1}^{l_0} |\Gamma_{i_j}| g_{i_j} \right| = 0. \tag{34}$$

In order to estimate $|\sum_{i=1}^{l}\int_{\Gamma_i}(g(s)-g_i)d\mathcal{S}|$ we execute the following steps. Use Theorem 4.1 to get

$$\begin{aligned}\int_{\bigcup_{i=1}^{l}\Gamma_i}(g(s)-u_n(s))^2 d\mathcal{S} &\leq \max_{s\in\bigcup_{i=1}^{l}\Gamma_i}\rho(s)^2\int_{\bigcup_{i=1}^{l}\Gamma_i}\rho(s)^{-2}(g(s)-u_n(s))^2 d\mathcal{S}\\ &\leq C'\|g-u_n\|^2_{L^{2,\rho^{-1}}(\partial\Omega)} \leq C''\underbrace{\|g-u_n\|^2_{1,\rho}}_{\to 0 \text{ as } n\to\infty}. \end{aligned} \tag{35}$$

Furthermore, we get the decomposition

$$\begin{aligned}\sum_{i=1}^{l}\int_{\Gamma_i}(g(s)-g_i)^2 d\mathcal{S} \leq\ & \underbrace{3\sum_{i=1}^{l}\int_{\Gamma_i}(g(s)-u_n(s))^2 d\mathcal{S}}_{I}+ \\ & \underbrace{3\sum_{i=1}^{l}\int_{\Gamma_i}(u_n(s)-u_i^n)^2 d\mathcal{S}}_{II}+\underbrace{3\sum_{i=1}^{l}|\Gamma_i|(u_i^n-g_i)^2}_{III}\end{aligned} \tag{36}$$

and the estimate (due to Hölder's inequality)

$$\begin{aligned}\left|\sum_{j=1}^{l_0}\int_{\Gamma_{i_j}}(g(s)-g_{i_j})d\mathcal{S}\right| &\leq \sum_{i=1}^{l}\int_{\Gamma_i}|(g(s)-g_i)|d\mathcal{S} \\ &\leq \sum_{i=1}^{l}|\Gamma_i|^{\frac{1}{2}}\left(\int_{\Gamma_i}(g(s)-g_i)^2 d\mathcal{S}\right)^{\frac{1}{2}}.\end{aligned} \tag{37}$$

Inequality (35) leads to the conclusion that term I vanishes as $n\to\infty$. For the second term II this holds, too, due to inequality (32). The last term III approaches zero as $n\to\infty$ because $\lim_{n\to\infty}\sum_{i=1}^{l}(u_i^n-g_i)^2=0$. Therefore inequality (36) and (37) lead to $\left|\sum_{j=1}^{l_0}\int_{\Gamma_{i_j}}(g(s)-g_{i_j})d\mathcal{S}\right|=0$ and we conclude $\sum_{j=1}^{l_0}\int_{\Gamma_{i_j}}g(s)d\mathcal{S}=\sum_{j=1}^{l_0}|\Gamma_{i_j}|g_{i_j}$. Further, we have $\sum_{j=1}^{l_0}\int_{\Gamma_{i_j}}g(s)d\mathcal{S}=\sum_{j=1}^{l_0}\int_{\Gamma_{i_j}}c_0 d\mathcal{S}=c_0\sum_{j=1}^{l_0}|\Gamma_{i_j}|$. From inequality (34) we get $\left|\sum_{j=1}^{l_0}|\Gamma_{i_j}|g_{i_j}\right|=0$, and therefore $c_0=0$. Finally, inequality (36) directly implies $\int_{\Gamma_i}(g(s)-g_i)^2 d\mathcal{S}=0$ for each $i\in\{1,\cdots,l\}$. Hence $g_i=g(s)=c_0=0$ for almost all $s\in\Gamma_i$, so that $g_i=0$ for each $i\in\{1,\cdots,l\}$. In summary, the previous calculations lead to the following contradiction :

$$\|(g,\vec{g})\|_l^2=\|g\|_{1,\rho}^2+\sum_{i=1}^{l}g_i^2=0$$

$$1=\|(u_n,\vec{u}^n)\|_l^2 \quad\text{and}\quad \lim_{n\to\infty}\|(u_n,\vec{u}^n)-(g,\vec{g})\|_l^2=0.$$

ii) Like in the proof of i) we start out from the negation of assertion ii). Then we get a sequence $(u_n,\vec{u}^n)\in H_l^{\rho,c}(\Omega)$ such that for each $n\in\mathbb{N}$

$$1=\|(u_n,\vec{u}^n)\|_l^2=\|\rho^{-1}u_n\|^2_{L^2(\Omega)}+\||\nabla u_n|\|^2_{L^2(\Omega)}+\sum_{i=1}^{l}(u_i^n)^2 \tag{38}$$

and

$$\frac{1}{n} > \||\nabla u_n|\|^2_{L^2(\Omega)} + \sum_{i=1}^{l} \int_{\Gamma_i} (u_n(s) - u_i^n)^2 d\mathcal{S} \tag{39}$$

holds. The same conclusions like in i) lead to the fact that there exists a $(g, \vec{g}) \in H_l^{\rho,c}(\Omega)$ such that $\lim_{n\to\infty} \|(u_n, \vec{u}^n) - (g, \vec{g})\|_l = 0$ and $\nabla g = 0$ almost everywhere wich implies $g(x) = c_0$ for almost every $x \in \Omega$. Since $\rho^{-1} \notin L^2(\Omega)$, the constant c_0 must vanish and therefore g vanishes almost everywhere in Ω, too. Since the derivation of equation (36) remains true in this case and the terms I, II and III reach zero, too, as $n \to \infty$, we also have $g_i(s) = c_0 = 0$ for almost every $s \in \Gamma_i$ and all $i \in \{1, \cdots, l\}$ and get the same contradiction like in case i). ∎

Corollary 5.6 *Let Ω be a geophysical domain (see Definition 3.6). Under the assumptions i) of Theorem 5.5 there exists for each $i_0 \in \{1, \cdots, l\}$ a constant $C_{i_0} > 0$ such that*

$$\|(u, \vec{u})\|_l^2 \leq C_{i_0} \left(\||\nabla u|\|^2_{L^2(\Omega)} + \sum_{i=1}^{l} \int_{\Gamma_i} (u(s) - u_i)^2 d\mathcal{S} + |\Gamma_{i_0}|^2 u_{i_0}^2 \right) \tag{40}$$

holds for all $(u, \vec{u}) \in H_l^\rho(\Omega)$.

Proof: Theorem 5.5 i). ∎

5.4 Existence and uniqueness theorems

Lemma 5.7 *Let Ω be a (possibly unbounded) domain with a Lipschitz boundary disfigured by l electrodes stucked into Ω. Then the bilinear form $b : H_l^\rho(\Omega) \times H_l^\rho(\Omega) \to \mathbb{R}$ (resp. $b : H_l^{\rho,c}(\Omega) \times H_l^{\rho,c}(\Omega) \to \mathbb{R}$) from Definition 5.2 is continuous .*

Proof: Let $(u, \vec{u}), (v, \vec{v}) \in H_l^\rho(\Omega)$ (resp. $(u, \vec{u}), (v, \vec{v}) \in H_l^{\rho,c}(\Omega)$). Then

$$\begin{aligned} |b((u, \vec{u}), (v, \vec{v}))| &\leq \|\sigma\|_\infty \||\nabla u|\|_{L^2} \||\nabla v|\|_{L^2} + \\ &\max_{i\in\{1,\cdots,l\}} \left\{ \sup_{s\in\Gamma_i} \{z_i^{-1}(s)\} \right\} \underbrace{\left(\sum_{i=1}^{l} \int_{\Gamma_i} |u(s) - u_i||v(s) - v_i| d\mathcal{S} \right)}_{I}. \end{aligned} \tag{41}$$

We decompose term I of equation (41) into the sum

$$\begin{aligned} I \leq &\underbrace{\sum_{i=1}^{l} \int_{\Gamma_i} |u(s)||v(s)| d\mathcal{S}}_{II} + \\ &\underbrace{\sum_{i=1}^{l} \int_{\Gamma_i} |u(s)||v_i| d\mathcal{S}}_{III} + \underbrace{\sum_{i=1}^{l} \int_{\Gamma_i} |u_i||v(s)| d\mathcal{S}}_{IV} + \underbrace{\sum_{i=1}^{l} \int_{\Gamma_i} |u_i||v_i| d\mathcal{S}}_{V}, \end{aligned} \tag{42}$$

and estimate each term separately.

$$\begin{aligned} II &\leq \max_{s\in\bigcup_{i=1}^{l}\Gamma_i}\{\rho(s)^2\}\int_{\bigcup_{i=1}^{l}\Gamma_i}\frac{|u(s)|}{\rho(s)}\frac{|v(s)|}{\rho(s)}d\mathcal{S} \\ &\leq C\|u\|_{L^{2,\rho^{-1}}(\partial\Omega)}\|v\|_{L^{2,\rho^{-1}}(\partial\Omega)} \\ &\leq C'\|u\|_{1,\rho}\|v\|_{1,\rho}, \end{aligned} \tag{43}$$

where the last equation results from the trace theorem 4.1.

$$\begin{aligned} III &= \int_{\bigcup_{i=1}^{l}\Gamma_i}|u(s)|\left(\sum_{i=1}^{l}\chi_{\Gamma_i}(s)|v_i|\right)d\mathcal{S} \\ &\leq \max_{s\in\bigcup_{i=1}^{l}\Gamma_i}\{\rho(s)\}\left(\int_{\bigcup_{i=1}^{l}\Gamma_i}\sum_{i=1}^{l}\chi_{\Gamma_i}(s)|v_i|^2 d\mathcal{S}\right)^{\frac{1}{2}}\left(\int_{\bigcup_{i=1}^{l}\Gamma_i}\frac{|u(s)^2|}{\rho(s)^2}d\mathcal{S}\right)^{\frac{1}{2}} \\ &\leq C\left(\sum_{i=1}^{l}|\Gamma_i|v_i^2\right)^{\frac{1}{2}}\|u\|_{L^{2,\rho^{-1}}(\partial\Omega)} \\ &\leq C'\left(\sum_{i=1}^{l}v_i^2\right)^{\frac{1}{2}}\|u\|_{1,\rho}. \end{aligned} \tag{44}$$

Term IV can be estimated analoguous to III.

$$V = \sum_{i=1}^{l}|\Gamma_i||u_i||v_i| \leq \max_{i\in\{1,\cdots,l\}}|\Gamma_i|\left(\sum_{i=1}^{l}|u_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{l}|v_i|^2\right)^{\frac{1}{2}}. \tag{45}$$

It follows immediately from Definition 5.1 and Equations (41) - (45) that each term of the decomposition (41) and (42) can be estimated by $C\|(u,\vec{u})\|_l\|(v,\vec{v})\|_l$. for a fixed positive constant C. To be more precise, the estimate

$$|b((u,\vec{u}),(v,\vec{v}))| \leq C'\max\{\|\sigma\|_\infty, \max_{i\in\{1,\cdots,l\}}\|z_i^{-1}\|_{L^\infty(\Gamma_i)}\}\|(u,\vec{u})\|_l\|(v,\vec{v})\|_l$$

is valid for a fixed positive constant C'. ■

If constant functions are allowed to belong to the Hilbert space of solutions, then the boundary value problem is solved by the sum of any solution and any constant function. In order to get existence and uniquenes results for solutions to the boundary value problem, we have to establish appropriate normalization . Therefore we introduce the following subspaces.

Definition 5.8 *Let* $\Lambda := \{i_1,\cdots,i_{l_0}\} \subset \{1,\cdots,l\}$. *Then define the spaces of solutions*

$$H^\rho_{l,\Lambda}(\Omega) := \left\{(u,\vec{u}) \in H^\rho_l(\Omega) \;\middle|\; \sum_{j=1}^{l_0}|\Gamma_{i_j}|u_{i_j} = 0\right\} \tag{46}$$

and

$$H_{l,\Lambda}^{\rho,c}(\Omega) := \left\{ (u,\vec{u}) \in H_l^{\rho,c}(\Omega) \;\middle|\; \sum_{j=1}^{l_0} |\Gamma_{i_j}| u_{i_j} = 0 \right\}, \tag{47}$$

where the Hilbert spaces $H_l^\rho(\Omega)$ and $H_l^{\rho,c}(\Omega)$ are defined in 5.1.

In the following, we verify the assumptions of Lax-Milgram's lemma . Bilinearforms satisfying ii) of Lemma C.1 are called coercive or elliptic .

Theorem 5.9 *Let Ω be a (possibly unbounded) domain with a locally finite Lipschitz boundary (see Definition 2.9) disfigured by l electrodes stucked into Ω such that there exists an extension operator $F : H^{1,\rho}(\Omega) \to H^{1,\rho}(\mathbb{R}^n)$ (resp. $F : H_c^{1,\rho}(\Omega) \to H_c^{1,\rho}(\mathbb{R}^n)$) (see Definition 3.4). Let $\rho(x) := (1+|x|)^\alpha$ be a weight and $\Lambda := \{i_1, \cdots, i_{l_0}\} \subset \{1, \cdots, l\}$. Furthermore, assume a conductivity distribution $\sigma : \Omega \to \mathbb{R}_+$ and contact impedances $z_i : \Gamma_i \to \mathbb{R}_+$ such that there exist constants $0 < c \leq C < \infty$, where*

$$\begin{aligned} c \leq \sigma(x) \leq C \quad & \forall x \in \Omega \quad \text{and} \\ c \leq z_i(s) \leq C \quad & \forall s \in \Gamma_i \quad \text{for each} \quad i \in \{1, \cdots, l\} \end{aligned} \tag{48}$$

holds. Then, if $\alpha > n/2$ the bilinear form $b : H_{l,\Lambda}^\rho(\Omega) \times H_{l,\Lambda}^\rho(\Omega) \to \mathbb{R}$ (resp. $b : H_{l,\Lambda}^{\rho,c}(\Omega) \times H_{l,\Lambda}^{\rho,c}(\Omega) \to \mathbb{R}$) from Definition 5.2 satisfies the assumptions of Lax-Milgram's lemma (Lemma C.1). If $n \geq 3$, $n/2 \geq \alpha > 1$ and Ω is such that $\rho^{-1} \notin L^2(\Omega)$, then $b : H_l^{\rho,c}(\Omega) \times H_l^{\rho,c}(\Omega) \to \mathbb{R}$ satisfies the assumptions of Lax-Milgram's lemma, too.

Proof: Theorem 5.5 leads in all cases to the inequality

$$\|(u,\vec{u})\|_l^2 \leq C' \left(\||\nabla u|\|_{L^2(\Omega)}^2 + \sum_{i=1}^{l} \int_{\Gamma_i} (u(s) - u_i)^2 d\mathcal{S} \right) \tag{49}$$

for all $(u,\vec{u}) \in H_{l,\Lambda}^\rho(\Omega)$ (resp. $(u,\vec{u}) \in H_{l,\Lambda}^{\rho,c}(\Omega)$) and a fixed constant $C' > 0$. Equation (48) and (49) lead directly to the coercivity esimate

$$\begin{aligned} \|(u,\vec{u})\|_l^2 &\leq C' \max\{\frac{1}{c}, C\} \left(\int_\Omega \sigma(x) |\nabla u(x)|^2 dx + \sum_{i=1}^{l} \int_{\Gamma_i} z_i^{-1}(s)(u(s) - u_i)^2 d\mathcal{S} \right) \\ &= C' \max\{\frac{1}{c}, C\} b((u,\vec{u}),(u,\vec{u})) \end{aligned} \tag{50}$$

and therefore condition ii) of Theorem C.1 is verified. Condition i) (i.e. continuity of the bilinearform b) is satisfied due to lemma 5.7. It remains to prove that $g_{\vec{I}}$ is a continuous functional on $H_{l,\Lambda}^\rho(\Omega)$ (resp. $H_{l,\Lambda}^{\rho,c}(\Omega)$) which follows easily from the Cauchy-Schwarz inequality. ■

Applying Lax-Milgram's lemma, now we are able to prove existence and uniqueness for solutions to boundary value problems with respect to the Complete Electrode Model for a large class of bounded and unbounded domains. Nevertheless in the following, we would like to restrict the theory to (unbounded) geophysical domains in order to prevent the formulations from unnecessarily blowing up. It should be stressed that the following results remain true in the case of bounded domains.

Theorem 5.10 *Let Ω be a (possibly unbounded) geophysical domain and $\Lambda := \{i_1, \cdots, i_{l_0}\} \subset \{1, \cdots, l\}$. Further, let $\vec{I} \in \mathbb{R}^l$ satisfy Equation* (25)*. Then there exists a unique weak $H^\rho_{l,\Lambda}(\Omega)$-solution to the boundary value problem* (13), (14), (15) *and* (16).

In addition, there exists a mapping $L : \mathbb{R}^l \to H^\rho_{l,\Lambda}(\Omega)$ defined by $L(\vec{I}) := (u, \vec{u})$, for all $\vec{I} \in \mathbb{R}^l$ satisfying Equation (25)*, where $(u, \vec{u})$ is the unique weak solution to the boundary value problem. The mapping L is continuous and bounded by $C' \max\{\frac{1}{c}, C\}$, where the constants are the same as in Equation* (50).

Remark 5.11 *By choosing a set $\Lambda := \{i_1, \cdots, i_{l_0}\} \subset \{1, \cdots, l\}$, we set up a normalization for solutions to the boundary value problem. Equation* (46) *and* (47) *state that the mean value of certain electrode voltages must vanish. The special case $\Lambda := \{i_0\}$ for a number $1 \leq i_0 \leq l$ leads to the normalization, where the voltage of one fixed electrode is set to zero. Such an electrode is called ground or reference electrode .*

If we consider the case $n \geq 3$, $n/2 \geq \alpha > 1$ where Ω is such that $\rho^{-1} \notin L^2(\Omega)$, then we get unique solutions in $H^{\rho,c}_l(\Omega)$ due to Theorem 5.9 and Lemma C.1, where every solution is normalized by setting its values to zero at infinity.

5.5 The parameter-to-solution map

In this subsection the parameter-to-solution map is introduced. This map is of special interest for solution techniques to the inverse problem and it maps conductivity distributions and contact impedances to weak solutions to the boundary value problem.

For the remainder of this section, we consider (unbounded) geophysical domains Ω with l electrodes stucked into and weights ρ such that the assumptions of Theorem 5.5 are satisfied.

First of all, we have to choose appropriate (space of function) domains for the parameter-to-solution map.

Definition 5.12 *Denote that part of the boundary, having contact with the* i$^{\text{th}}$ *electrode by Γ_i. Then define the parameter space*

$$\begin{aligned} P_\Lambda := \{&(\sigma, z_1, \dots, z_l) \in L^\infty(\Omega) \times L^\infty(\Gamma_1) \times \cdots \times L^\infty(\Gamma_l) \,|\, \exists\, s > 0 : \\ &s < \sigma \text{ a. e. in } \Omega \text{ and } s < z_i \text{ a. e. in } \Gamma_i \text{ for each } i \in \{1, \dots, l\}\} \end{aligned} \tag{51}$$

and the norm

$$\|(\sigma, z_1, \dots, z_l)\|_\Lambda := \max\left\{\|\sigma\|_{L^\infty(\Omega)}, \|z_1\|_{L^\infty(\Gamma_1)}, \cdots, \|z_l\|_{L^\infty(\Gamma_l)}\right\}. \tag{52}$$

Furthermore, we use the abbreviation

$$(\sigma, Z) := (\sigma, z_1, \dots, z_l).$$

Now we are ready to define the parameter-to-solution mapping. Therefore, we fix some $\vec{I} \in \mathbb{R}^l$ such that $\sum_{j=1}^l I_j = 0$ and we select an arbitrary set of parameters $(\sigma, Z) \in P_\Lambda$. This data constitutes a boundary value problem and the weak solution to the problem is chosen to be the image of (σ, Z) under the parameter-to-solution mapping.

Definition 5.13 *Fix $\vec{I} \in \mathbb{R}^l$ such that $\sum_{j=1}^{l} I_j = 0$ and let $(\sigma, Z) = (\sigma, z_1, \ldots, z_l) \in P_\Lambda$. Then the boundary value problem*

$$g_{\vec{I}}((v,\vec{v})) = b((u,\vec{u}),(v,\vec{v})) = \int_\Omega \sigma(x)\nabla u(x)\nabla v(x)dx + \sum_{i=1}^{l}\int_{\Gamma_i} z_i^{-1}(s)(u(s)-u_i)(v(s)-v_i)dS \quad (53)$$

for all $(v,\vec{v}) \in H^\rho_{l,\Lambda}(\Omega)$ has a unique solution $(u,\vec{u}) \in H^\rho_{l,\Lambda}(\Omega)$ (due to Theorem 5.10). The map

$$A_{\vec{I}}((\sigma,Z)) := (u,\vec{u}), \qquad A_{\vec{I}} : P_\Lambda \to H^\rho_{l,\Lambda}(\Omega) \quad (54)$$

is called parameter-to-solution map . In the following, we need a slightly modified version of the map $A_{\vec{I}}$. If we replace z_i^{-1} by z_i in equation (53), the unique solvability of the boundary value problem is preserved and we can define another parameter-to-solution map where the contact impedances appear in terms of conductivities and not in terms of resistances.

$$B_{\vec{I}}((\sigma,Z)) := (u,\vec{u}), \qquad B_{\vec{I}} : P_\Lambda \to H^\rho_{l,\Lambda}(\Omega), \quad (55)$$

where $(u,\vec{u}) \in H^\rho_{l,\Lambda}(\Omega)$ is the unique solution of

$$g_{\vec{I}}((v,\vec{v})) = \int_\Omega \sigma(x)\nabla u(x)\nabla v(x)dx + \sum_{i=1}^{l}\int_{\Gamma_i} z_i(s)(u(s)-u_i)(v(s)-v_i)dS, \quad (56)$$

for all $(v,\vec{v}) \in H^\rho_{l,\Lambda}(\Omega)$.

5.5.1 Fréchet-differentiability of the parameter-to-solution map

To calculate the Fréchet-derivative of $A_{\vec{I}}$ we have to decompose the map $A_{\vec{I}}$ into $B_{\vec{I}}$ and a simple map which provides the transition from z_i to z_i^{-1}. Then we have to calculate the derivatives for each map separately. The resulting formula for the Fréchet-derivative, where Ω is a bounded domain with smooth boundary, can be found in [JS97] without proof. For calculating the Fréchet-derivative of the parameter-to-solution map $B_{\vec{I}} : P_\Lambda \to H^\rho_{l,\Lambda}(\Omega)$, we use a technique introduced in [Cal80]. First of all, we fix some notation for the isomorphism from $H^\rho_{l,\Lambda}(\Omega)$ to its dual $H^\rho_{l,\Lambda}(\Omega)^\star$, induced by the bilinear form, determined by the right hand side of Equation (56). To simplify notation, when using operator norms, we ommit subscripts indicating the type of an operator norm.

Definition 5.14 *Let the general assumptions on the domain and the weight be satisfied and let $(\sigma, Z) \in P_\Lambda$. For every $(u,\vec{u}) \in H^\rho_{l,\Lambda}(\Omega)$ define the linear functional $L_{(\sigma,Z)}(u,\vec{u})$ by the formula*

$$\left(L_{(\sigma,Z)}(u,\vec{u})\right)(v,\vec{v}) := \int_\Omega \sigma(x)\nabla u(x)\nabla v(x)dx + \sum_{i=1}^{l}\int_{\Gamma_i} z_i(s)(u(s)-u_i)(v(s)-v_i)dS \quad (57)$$

for all $(v,\vec{v}) \in H^{\rho}_{l,\Lambda}(\Omega)$*. Note that the terms* z_i *(not* z_i^{-1}*) occur in Equation* (57)*.*

Furthermore, define the linear operator $L_{(\sigma,Z)} : H^{\rho}_{l,\Lambda}(\Omega) \to H^{\rho}_{l,\Lambda}(\Omega)^{\star}$ *by* $(u,\vec{u}) \mapsto L_{(\sigma,Z)}(u,\vec{u})$*. The inspection of the proof of Lemma 5.7 provides the estimate*

$$\left|\left(L_{(\sigma,Z)}(u,\vec{u})\right)(v,\vec{v})\right| \le C\|(\sigma,Z)\|_{\Lambda}\|(u,\vec{u})\|_{l}\|(v,\vec{v})\|_{l}. \tag{58}$$

Therefore, $L_{(\sigma,Z)}(u,\vec{u}) \in H^{\rho}_{l,\Lambda}(\Omega)^{\star}$ *and its operator norm can be estimated from above :*

$$\left(\|L_{(\sigma,Z)}(u,\vec{u})\|_{H^{\rho}_{l,\Lambda}(\Omega)^{\star}} =\right) \|L_{(\sigma,Z)}(u,\vec{u})\| \le C\|(\sigma,Z)\|_{\Lambda}\|(u,\vec{u})\|_{l}. \tag{59}$$

Remark 5.15 *The operator* $L_{(\sigma,Z)} : H^{\rho}_{l,\Lambda}(\Omega) \to H^{\rho}_{l,\Lambda}(\Omega)^{\star}$ *is continuously invertible on* $H^{\rho}_{l,\Lambda}(\Omega)^{\star}$*. Its inverse is denoted in the following by* $L^{-1}_{(\sigma,Z)}$*. Furthermore, the estimate*

$$\left\|L^{-1}_{(\sigma,Z)}\right\| \le C'\left\|\left(\frac{1}{\sigma},\frac{1}{z_1},\dots,\frac{1}{z_l}\right)\right\|_{\Lambda} \tag{60}$$

is satisfied.

Proof: Theorem 5.5 provides the estimate

$$\|(u,\vec{u})\|_l^2 \le C'\left(\||\nabla u|\|^2_{L^2(\Omega)} + \sum_{i=1}^{l}\int_{\Gamma_i}(u(s)-u_i)^2 d\mathcal{S}\right), \tag{61}$$

which leads to

$$\begin{aligned} \|(u,\vec{u})\|_l^2 &\le C'\left(\left\|\frac{1}{\sigma}\right\|_{\infty}\|\sigma|\nabla u|\|^2_{L^2(\Omega)} + \sum_{i=1}^{l}\left\|\frac{1}{z_i}\right\|_{\infty}\int_{\Gamma_i} z_i(s)(u(s)-u_i)^2 d\mathcal{S}\right) \\ &\le C'\left\|\left(\frac{1}{\sigma},\frac{1}{z_1},\dots,\frac{1}{z_l}\right)\right\|_{\Lambda}\left(L_{(\sigma,Z)}(u,\vec{u})\right)(u,\vec{u}). \end{aligned} \tag{62}$$

Now an application of Lemma C.1 finishes the proof. ■

Lemma 5.16 *The linear operator* $L_{(\cdot,\cdot)} : P_{\Lambda} \to L(H^{\rho}_{l,\Lambda}(\Omega), H^{\rho}_{l,\Lambda}(\Omega)^{\star})$ *is continuous as well as the nonlinear operator* $L^{-1}_{(\cdot,\cdot)} : P_{\Lambda} \to L(H^{\rho}_{l,\Lambda}(\Omega)^{\star}, H^{\rho}_{l,\Lambda}(\Omega))$ *.*

Proof: Equation (59) provides $\|L_{(\sigma,Z)}\| \le C\|(\sigma,Z)\|_{\Lambda}$. Let $(\sigma,Z),(\sigma',Z') \in P_{\Lambda}$. Then we have the following estimate :

$$\begin{aligned} \|L^{-1}_{(\sigma,Z)} - L^{-1}_{(\sigma',Z')}\| &= \|L^{-1}_{(\sigma',Z')}\left(L_{(\sigma',Z')} - L_{(\sigma,Z)}\right)L^{-1}_{(\sigma,Z)}\| \\ &\le \|L^{-1}_{(\sigma',Z')}\|\|L_{(\sigma',Z')} - L_{(\sigma,Z)}\|\|L^{-1}_{(\sigma,Z)}\| \\ &\le C'^2\left\|\left(\frac{1}{\sigma},\frac{1}{z_1},\dots,\frac{1}{z_l}\right)\right\|_{\Lambda}\left\|\left(\frac{1}{\sigma'},\frac{1}{z'_1},\dots,\frac{1}{z'_l}\right)\right\|_{\Lambda} C\|(\sigma',Z')-(\sigma,Z)\|_{\Lambda}, \end{aligned} \tag{63}$$

where we have used Equation (60). If (σ',Z') and (σ,Z) are uniformly strictly positive, i.e., if there exists a real constant $0 < c_0$ such that $c_0 \le \sigma, z_1, \dots, z_l$ and $c_0 \le \sigma', z'_1, \dots, z'_l$, then the last equation is sufficent for proving continuity of $L^{-1}_{(\cdot,\cdot)}$.
■

Lemma 5.17 *Let $(\sigma, Z) \in P_\Lambda$ be a point and let $(u, \vec{u}) = B_{\vec{I}}(\sigma, Z)$. Then the mapping*

$$\left(\mathcal{D}B_{\vec{I}}(\sigma, Z)\right)(\cdot,\cdot) := -L^{-1}_{(\sigma,Z)} L_{(\cdot,\cdot)}(u, \vec{u}) \tag{64}$$

is the Fréchet-derivative of $B_{\vec{I}}$ at (σ, Z) .

Proof: Continuity of the mapping $L_{(\cdot,\cdot)}(u, \vec{u}) : P_\Lambda \to H^\rho_{l,\Lambda}(\Omega)^\star$ is due to Equation (59) and therefore the estimate (60) now provides continuity of the composition $-L^{-1}_{(\sigma,Z)} L_{(\cdot,\cdot)}(u, \vec{u})$. Proving linearity of $\left(\mathcal{D}B_{\vec{I}}(\sigma, Z)\right)(\cdot,\cdot)$ is trivial and it remains to show that the asymptotic estimate (123) of Definition C.2 holds.

Let $\varepsilon > 0$ be such that the open Ball $B_\varepsilon(\sigma, Z) = \{(\tau, Y) \in P_\Lambda \mid \|(\tau, Y) - (\sigma, Z)\|_\Lambda < \varepsilon\}$ is contained in P_Λ. Let $(\tau, Y) \in B_\varepsilon(\sigma, Z)$ and $(\delta, \Delta) := (\tau, Y) - (\sigma, Z)$. Note that $(u, \vec{u}) = L^{-1}_{(\sigma,Z)}(g_{\vec{I}})$, set $(y, \vec{y}) := L^{-1}_{(\tau,Y)}(g_{\vec{I}})$ and let $(w, \vec{w}) := (y, \vec{y}) - (u, \vec{u})$. Then it follows

$$\begin{aligned} L_{(\sigma,Z)}(u, \vec{u}) &= L_{(\tau,Y)}(u + w, \vec{u} + \vec{w}) = \\ &\quad L_{(\sigma+\delta,Z+\Delta)}(u, \vec{u}) + L_{(\sigma+\delta,Z+\Delta)}(w, \vec{w}) = \\ &\quad L_{(\sigma,Z)}(u, \vec{u}) + L_{(\sigma,Z)}(w, \vec{w}) + L_{(\delta,\Delta)}(u, \vec{u}) + L_{(\delta,\Delta)}(w, \vec{w}), \end{aligned} \tag{65}$$

where we have used the linearity of the operator $L_{(\cdot,\cdot)}$. Therefore we get

$$0 = L_{(\delta,\Delta)}(u, \vec{u}) + L_{(\sigma,Z)}(w, \vec{w}) + L_{(\delta,\Delta)}(w, \vec{w}), \tag{66}$$

which actually is an operator equation in $H^\rho_{l,\Lambda}(\Omega)^\star$. Since $L_{(\sigma,Z)} : H^\rho_{l,\Lambda}(\Omega) \to H^\rho_{l,\Lambda}(\Omega)^\star$ is continuously invertible, (see Remark 5.15), we are allowed to apply $L^{-1}_{(\sigma,Z)}$ on both sides of Equation (66). Therefore the identity

$$\left(Id_{W^\rho_{l,\Lambda}(\Omega)} + L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)}\right)(w, \vec{w}) = -L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)}(u, \vec{u}) \tag{67}$$

holds. If we assume $\|L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)}\| < 1$, where $\|\cdot\|$ denotes the operator norm in $L(H^\rho_{l,\Lambda}(\Omega))$, then the left hand side of Equation (67) is invertible and we get a Neumann series representation :

$$\begin{aligned} (w, \vec{w}) &= \left(\sum_{j=0}^{\infty} \left(-L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)}\right)^j\right) \left(-L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)}\right)(u, \vec{u}) \\ &= \left(\sum_{j=1}^{\infty} \left(-L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)}\right)^j\right)(u, \vec{u}). \end{aligned} \tag{68}$$

Since $\|L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)}\| \le \|L^{-1}_{(\sigma,Z)}\| C \|(\delta, \Delta)\|_\Lambda$, due to equation (59), we see that for sufficiently small ε the Neumann series converges and we are able to verify the estimate :

$$\begin{aligned} &\left\|B_{\vec{I}}(\sigma + \delta, Z + \Delta) - B_{\vec{I}}(\sigma, Z) - \left(\mathcal{D}B_{\vec{I}}(\sigma, Z)\right)(\delta, \Delta)\right\|_l \\ = &\left\|(u + w, \vec{u} + \vec{w}) - (u, \vec{u}) + L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)}(u, \vec{u})\right\|_l \end{aligned}$$

$$
\begin{aligned}
&= \left\| \left(\sum_{j=2}^{\infty} \left(-L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)} \right)^j \right) (u,\vec{u}) \right\|_l \\
&\leq \left\| L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)} \right\|^2 \left(\sum_{j=0}^{\infty} \left\| L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)} \right\|^j \right) \|(u,\vec{u})\|_l \\
&\leq C^2 \|L^{-1}_{(\sigma,Z)}\|^2 \|(\delta,\Delta)\|^2_\Lambda \underbrace{\frac{\|(u,\vec{u})\|_l}{1 - C\|L^{-1}_{(\sigma,Z)}\| \|(\delta,\Delta)\|_\Lambda}}_{I} . \qquad (69)
\end{aligned}
$$

Since fraction I remains bounded as $\|(\delta,\Delta)\|_\Lambda \to 0$, we conclude that the left hand side of the estimate (69) is of type $o((\delta,\Delta))$, thereby establishing Fréchet-differentiability of the operator $B_{\vec{I}}$. ■

Now we consider the mapping μ which provides the transition from z_i to z_i^{-1} :

$$
\begin{aligned}
\mu : P_\Lambda &\to P_\Lambda \\
\mu(\sigma, Z) &:= (\sigma, z_1^{-1}, \cdots, z_l^{-1}). \qquad (70)
\end{aligned}
$$

Proposition 5.18 *The mapping $\mu : P_\Lambda \to P_\Lambda$ (defined by Equation (70)) is differentiable if all z_i are strictly positive. .*

Proof: It suffices to prove that each component μ_i of the mapping μ is differentiable (see Lemma C.5). Furthermore, each component is differentiable if and only if all partial derivatives $\mathcal{D}_j\mu_i$ exist and are continuous (see Lemma C.6). Therefore we calculate the partial derivatives.

i) $i \neq j$: $\Rightarrow \mathcal{D}_i\mu_j = 0$;

ii) $i = 1$, $j = 1$: $\mu_1(\sigma, z_1, \ldots, z_l) = \sigma \Rightarrow \mathcal{D}_1\mu_1 = Id$;

iii) $i > 1$, $i = j$: $\Rightarrow \mathcal{D}_i\mu_i = -z_i^{-2}$, where $-z_i^{-2}$ has to be understand as a linear mapping which takes a function $h \in L^\infty(\Gamma_i)$ to $-h/z_i^2 \in L^\infty(\Gamma_i)$. It is an easy exercise to verify that the mapping $-h/z_i^2$ is the Fréchet-derivative of $1/z_i$ with respect to the $\|\cdot\|_\infty$-norm (alternatively one may use Lemma C.8) and that $z_i \mapsto -h/z_i^2 : L^\infty(\Omega) \to L(L^\infty(\Omega))$ is a continuous mapping if in both cases z_i is strictly positive.

■

To summarize the previous calculations, we write down the formula for the Fréchet-derivative $\mathcal{D}\mu$:

$$
(\mathcal{D}\mu(\sigma, Z))(\delta, \Delta) = \left(\delta, -\delta_1 z_1^{-2}, \ldots, -\delta_l z_l^{-2}\right). \qquad (71)
$$

We have $A_{\vec{I}} = B_{\vec{I}} \circ \mu$ and the Fréchet-derivative of $A_{\vec{I}}$ is calculated by applying the chain rule (Theorem C.4) .

$$
\begin{aligned}
\left(\mathcal{D}A_{\vec{I}}(\sigma, Z)\right)(\delta, \Delta) &= \left(\mathcal{D}B_{\vec{I}}(\mu(\sigma, Z))\right) \circ (\mathcal{D}\mu(\sigma, Z))(\delta, \Delta) \\
&= \left(\mathcal{D}B_{\vec{I}}(\sigma, z_1^{-1}, \ldots, z_l^{-1})\right)\left(\delta, -\delta_1 z_1^{-2}, \ldots, -\delta_l z_l^{-2}\right) \\
&= -L^{-1}_{(\sigma, z_1^{-1}, \ldots, z_l^{-1})} L_{\left(\delta, -\delta_1 z_1^{-2}, \ldots, -\delta_l z_l^{-2}\right)}(u, \vec{u}). \qquad (72)
\end{aligned}
$$

To calculate a perturbation of first order $(u^\delta, \vec{u}^\delta) = \left(\mathcal{D}A_{\vec{I}}(\sigma, Z)\right)(\delta, \Delta)$, we therefore have to solve the problem

$$L_{(\sigma, z_1^{-1}, \dots, z_l^{-1})}(u^\delta, \vec{u}^\delta) = -L_{(\delta, -\delta_1 z_1^{-2}, \dots, -\delta_l z_l^{-2})}(u, \vec{u}), \quad (73)$$

which is equivalent to

$$b\left((u^\delta, \vec{u}^\delta), (v, \vec{v})\right) = \quad (74)$$
$$-\int_\Omega \delta \nabla u \nabla v dx + \sum_{i=1}^{l} \int_{\Gamma_i} \frac{\delta_i(s)}{z_i(s)^2}(u(s) - u_i)(v(s) - v_i) d\mathcal{S},$$

for all $(v, \vec{v}) \in H^\rho_{l,\Lambda}(\Omega)$ and agrees with the Fréchet-derivative presented in [JS97] for bounded domains and constant contact impedances. Note that $(u, \vec{u}) \in H^\rho_{l,\Lambda}(\Omega)$ is the solution of the unpurturbed boundary value problem, i.e. it satisfies the equation

$$b\left((u, \vec{u}), (v, \vec{v})\right) = g_{\vec{I}}\left((v, \vec{v})\right),$$

for all $(v, \vec{v}) \in H^\rho_{l,\Lambda}(\Omega)$.

5.5.2 Analyticity of the parameter-to-solution map

In fact, we are able to show that the maps $\mu, B_{\vec{I}}$ and $A_{\vec{I}}$ are infinitely often differentiable (see Definition C.3). For $B_{\vec{I}}$ we show analyticity (see Definition C.10).

Proposition 5.19 *The mapping $\mu : P_\Lambda \to P_\Lambda$ (defined by Equation (70)) is of class C^∞ in P_Λ (see Definition C.3). Let $\delta^1, \dots, \delta^n \in P_\Lambda$ and $n \geq 2$. Then the n^{th} derivative of μ at $(z_1, \dots, z_l)$ is the n-linear mapping*

$$D^n\mu \; : \; P_\Lambda \to L^n_s(P_\Lambda, P_\Lambda) \quad (75)$$
$$\left(D^n\mu(\sigma, z_1, \dots, z_l)\right)(\delta^1, \dots, \delta^n) = \left(0, \frac{n!(-1)^n \prod_{i=1}^n \delta^i_1}{z_1^{n+1}}, \dots, \frac{n!(-1)^n \prod_{i=1}^n \delta^i_l}{z_l^{n+1}}\right),$$

where $L^n_s(P_\Lambda, P_\Lambda)$ denotes the Banach space of all n-linear, symmetric and continuous functions from $\prod_{i=1}^n P_\Lambda$ to P_Λ. Note that the first derivative is represented by Equation (71).

Proof: For $n = 1$ consider Equation (71) and assume the assertion, i.e., Equation (75) is true for $n > 1$. We consider each component $(D^n\mu)_i$ of the mapping $D^n\mu$ separately. It is sufficient, due to Lemma C.6, to calculate the derivatives of the partial mappings. We have to distinguish the following three different cases :

i) $i \neq j$: $\Rightarrow \mathcal{D}_j(\mathcal{D}^n\mu)_i = 0$ for all $n \in \mathbb{N}$.

ii) $i = 1, j = 1$: Equation (71) shows that the first derivative $\mathcal{D}^1\mu$ does not depend on σ and therefore all partial derivatives of higher order must vanish, i.e., particularly $\mathcal{D}_1(\mathcal{D}^n\mu) = 0$ for $n \geq 1$.

iii) $i > 1$, $i = j$: We abbreviate the notation for the partial mappings

$$\begin{aligned} [\mathcal{D}^n \mu]_i &: P_\Lambda \to L_s^n(\Gamma_i, \Gamma_i) \\ [\mathcal{D}^n \mu]_i(t) &:= (\mathcal{D}^n \mu)_i(\sigma, z_1, \ldots, z_{i-1}, t, z_{i+1}, \ldots, z_l). \end{aligned}$$

From Equation (75) and Equation (71) it follows that

$$[\mathcal{D}^n \mu]_i(t) = \frac{n!(-1)^n \prod_{k=1}^n \delta_i^k}{t^{n+1}} \tag{76}$$

and since the i$^{\text{th}}$ partial derivative of the i$^{\text{th}}$ component is given by

$$\mathcal{D}_i \left((\mathcal{D}^n \mu)_i \right) (\sigma, z_1, \ldots, z_l) = \mathcal{D}[\mathcal{D}^n \mu]_i(z_i), \tag{77}$$

we have to calculate the derivative of the right hand side of Equation (76). The expression there is a product of the following two mappings : $t \mapsto \prod_{k=1}^n \delta_i^k \in L_s^n(\Gamma_i, \Gamma_i)$ and $t \mapsto (n!(-1)^n)/t^{n+1} \in L^\infty(\Gamma_i)$, where the first is a constant mapping and therefore has a vanishing derivative. The derivative of the product is calculated using the product rule of differentiation, Lemma C.7. Since the derivative of the second mapping is $(n+1)!(-1)^{n+1}\delta/t^{n+2}$ (which follows by induction from Lemma C.7 and Lemma C.8), the derivative of the product is

$$\frac{(n+1)!(-1)^{n+1}\delta \prod_{k=1}^n \delta_i^k}{t^{n+2}}. \tag{78}$$

Setting $\delta_i^{n+1} := \delta$, the formula for the total (n + 1)$^{\text{th}}$ derivative drops out immediately.

■

Definition 5.20 *Let E, F be Banach spaces, $L^k(E,F)$ the space of k-linear continuous maps of $\prod_{i=1}^k E$ to F. A k-linear map $M \in L^k(E,F)$ is called symmetric if for all permutations $\sigma \in S_k$*

$$\sigma M = M \tag{79}$$

holds, where σM is defined by $(\sigma M)(x_1, \cdots, x_k) := M(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$. $L_s^k(E,F) \subset L^k(E,F)$ denotes the subspace of all symmetric k-linear maps. The symmetrization operator $S^k : L^k(E,F) \to L_s^k(E,F)$ is defined by

$$S^k M = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma M \tag{80}$$

and has operator norm 1.

Lemma 5.21 *The mapping*

$$L_{(\cdot,\cdot)}^{-1} L_{(\cdot,\cdot)} : P_\Lambda \times P_\Lambda \to L(H_{l,\Lambda}^\rho(\Omega)) \tag{81}$$

is continuous as well as

$$L_{(\cdot,\cdot)}^{-1} L_{(\cdot,\cdot)}(u, \vec{u}) : P_\Lambda \times P_\Lambda \to H_{l,\Lambda}^\rho(\Omega). \tag{82}$$

Proof: Let $(\sigma, Z), (\sigma', Z'), (\delta, \Delta), (\delta', \Delta') \in P_\Lambda$ and assume $(\sigma, Z), (\sigma', Z')$ to be uniformly strictly positive. Then the estimates used in the proof of Lemma 5.16 provide

$$\begin{aligned}
\|L^{-1}_{(\sigma',Z')}L_{(\delta',\Delta')} - L^{-1}_{(\sigma,Z)}L_{(\delta,\Delta)}\| &\leq \|L^{-1}_{(\sigma',Z')}\|\|L_{(\delta',\Delta')} - L_{(\delta,\Delta)}\| + \\
&\quad \|L_{(\delta,\Delta)}\|\|L^{-1}_{(\sigma,Z)} - L^{-1}_{(\sigma',Z')}\| \\
&\leq C'\|(\delta', \Delta') - (\delta, \Delta)\|_\Lambda + \\
&\quad \|(\delta, \Delta)\|_\Lambda C'' \big\|(\sigma', Z') - (\sigma, Z)\big\|_\Lambda, \qquad (83)
\end{aligned}$$

which is sufficient for proving continuity of the operator $L^{-1}_{(\cdot,\cdot)}L_{(\cdot,\cdot)}$. The second assertion follows immediately from the first since we have $\|M(u, \vec{u})\|_l \leq \|M\|\|(u, \vec{u})\|_l$ for every continuous linear mapping $M : H^\rho_{l,\Lambda}(\Omega) \to H^\rho_{l,\Lambda}(\Omega)$. ■

Lemma 5.22 *Let E be a Banach space and F be a Banach algebra. Furthermore, let $f_1, \dots, f_n : E \to F$ be continuous mappings. Then the mapping*

$$\begin{aligned}
f_1 \cdot \ldots \cdot f_n &: E \to F \\
x &\mapsto f_1(x) \cdot \ldots \cdot f_n(x)
\end{aligned}$$

is continuous, too.

Proof: Since the mapping $x \mapsto (f_1(x), \dots, f_n(x))$ is continuous, the assertion follows from the continuity of the product. ■

Corollary 5.23 *Let j be a natural number. Then the mapping*

$$\left(L^{-1}_{(\cdot,\cdot)}L_{(\cdot,\cdot)}\right)^j : P_\Lambda \times P_\Lambda \to L(H^\rho_{l,\Lambda}(\Omega)) \qquad (84)$$

as well as

$$\left(L^{-1}_{(\cdot,\cdot)}L_{(\cdot,\cdot)}\right)^j (u, \vec{u}) : P_\Lambda \times P_\Lambda \to H^\rho_{l,\Lambda}(\Omega) \qquad (85)$$

is continuous.

Proof: Lemma 5.21 together with Lemma 5.22. The second assertion follows from the first and the fact that $\|M(u, \vec{u})\|_l \leq \|M\|\|(u, \vec{u})\|_l$ holds for every continuous linear mapping $M : H^\rho_{l,\Lambda}(\Omega) \to H^\rho_{l,\Lambda}(\Omega)$. ■

Lemma 5.24 *Let E, F be Banach spaces and n be a positive natural number. Furthermore, let $f, f' : E \to F$ and $g_1, \dots, g_n : F \to E$ be continuous linear mappings. Then the estimate*

$$\|fg_1 \cdots fg_n - f'g_1 \cdots f'g_n\| \leq \left(\sum_{i=0}^{n-1} \|f\|^i \|f'\|^{n-1-i}\right) \|f - f'\|\|g_1\| \cdots \|g_n\| \qquad (86)$$

holds, where $\|\cdot\|$ denotes the appropriate operator norm.

Proof: We use induction. The case n=1 follows immediately :

$$\|fg_1 - f'g_1\| \leq \|f - f'\|\|g_1\|.$$

Assume the assertion is true for $n-1$. We have the estimate

$$\begin{aligned} \|fg_1 \cdots fg_n - f'g_1 \cdots f'g_n\| &\leq \\ & \underbrace{\|fg_1 \cdots fg_{n-1}f - f'g_1 \cdots f'g_{n-1}f'\|}_{I} \|g_n\|. \end{aligned} \tag{87}$$

Term I can be estimated separately by

$$\begin{aligned} I &= \|(f-f')g_1fg_2 \cdots fg_{n-1}f + f'g_1fg_2 \cdots fg_{n-1}f \\ & \quad -f'g_1 \cdots f'g_{n-1}(f'-f) - f'g_1f'g_2 \cdots f'g_{n-1}f\| \\ &\leq \|f-f'\|\|g_1\| \cdots \|g_{n-1}\|(\|f\|^{n-1} + \|f'\|^{n-1}) + \\ & \quad \|f\|\|f'\|\|g_1\| \underbrace{\|fg_2 \cdots fg_{n-1} - f'g_2 \cdots f'g_{n-1}\|}_{II}. \end{aligned} \tag{88}$$

Using the assertion for $n-2$ we get

$$II \leq \left(\sum_{i=0}^{n-3} \|f\|^i \|f'\|^{n-3-i} \right) \|f - f'\|\|g_2\| \cdots \|g_{n-1}\|. \tag{89}$$

Since

$$\|f\|^{n-1} + \|f'\|^{n-1} + \|f\|\|f'\| \sum_{i=0}^{n-3} \|f\|^i \|f'\|^{n-3-i} = \sum_{i=0}^{n-1} \|f\|^i \|f'\|^{n-1-i}, \tag{90}$$

we conclude from Equation (87), (88) and (89) that the assertion holds even for n.
■

Definition 5.25 *Let j be a positive integer. Then define the mappings*

$$\begin{aligned} P_\Lambda \times \prod_{k=1}^{j} P_\Lambda &\rightarrow L(H^\rho_{l,\Lambda}(\Omega)) \\ \Phi_j\left((\sigma,Z),(\delta_1,\Delta_1),\ldots,(\delta_j,\Delta_j)\right) &:= \prod_{k=1}^{j} \left(-L^{-1}_{(\sigma,Z)} L_{(\delta_k,\Delta_k)}\right) \qquad \text{and} \\ P_\Lambda \times \prod_{k=1}^{j} P_\Lambda &\rightarrow H^\rho_{l,\Lambda}(\Omega) \\ \Phi_j^{(u,\vec{u})}\left((\sigma,Z),(\delta_1,\Delta_1),\ldots,(\delta_j,\Delta_j)\right) &:= \left(\prod_{k=1}^{j} \left(-L^{-1}_{(\sigma,Z)} L_{(\delta_k,\Delta_k)}\right)\right)(u,\vec{u}) \end{aligned} \tag{91}$$

Lemma 5.26 *Let j be a positive integer. Then the mappings $\Phi_j(\sigma, Z) : \prod_{k=1}^{j} P_\Lambda \to L(H^\rho_{l,\Lambda}(\Omega))$ and $\Phi_j^{(u,\vec{u})}(\sigma, Z) : \prod_{k=1}^{j} P_\Lambda \to H^\rho_{l,\Lambda}(\Omega)$ (see Definition (5.25)) are continuous. Furthermore, if $(\sigma, Z), (\sigma', Z') \in P_\Lambda$ are uniformly strictly positive, then the estimate*

$$\begin{aligned} &\|\Phi_j\left((\sigma, Z), (\delta_1, \Delta_1), \ldots, (\delta_j, \Delta_j)\right) - \Phi_j\left((\sigma', Z'), (\delta_1, \Delta_1), \ldots, (\delta_j, \Delta_j)\right)\| \\ &\quad \leq C\|(\sigma, Z) - (\sigma', Z')\|_\Lambda \|(\delta_1, \Delta_1)\|_\Lambda \cdots \|(\delta_j, \Delta_j)\| \end{aligned} \tag{92}$$

is valid for a fixed positive constant C. Therefore the mappings

$$\begin{aligned} \Phi_j &: P_\Lambda \to L^j(P_\Lambda, L(H^\rho_{l,\Lambda}(\Omega))) \\ S^j\Phi_j &: P_\Lambda \to L^j_s(P_\Lambda, L(H^\rho_{l,\Lambda}(\Omega))) \\ \Phi_j^{(u,\vec{u})} &: P_\Lambda \to L^j(P_\Lambda, H^\rho_{l,\Lambda}(\Omega)) \\ S^j\Phi_j^{(u,\vec{u})} &: P_\Lambda \to L^j_s(P_\Lambda, H^\rho_{l,\Lambda}(\Omega)) \end{aligned} \tag{93}$$

are continuous.

Proof: Continuity of the mapping $\Phi_j(\sigma, Z)$ follows from Equation (59) and Equation (60). Due to Lemma 5.24 we have

$$\|L^{-1}_{(\sigma,Z)} L_{(\delta_1,\Delta_1)} \cdots L^{-1}_{(\sigma,Z)} L_{(\delta_j,\Delta_j)} - L^{-1}_{(\sigma',Z')} L_{(\delta_1,\Delta_1)} \cdots L^{-1}_{(\sigma',Z')} L_{(\delta_j,\Delta_j)}\| \tag{94}$$
$$\leq \left(\sum_{i=0}^{j-1} \|L^{-1}_{(\sigma,Z)}\|^i \|L^{-1}_{(\sigma',Z')}\|^{n-1-i}\right) \|L^{-1}_{(\sigma,Z)} - L^{-1}_{(\sigma',Z')}\| \|L_{(\delta_1,\Delta_1)}\| \cdots \|L_{(\delta_j,\Delta_j)}\|.$$

Since $(\sigma, Z), (\sigma', Z') \in P_\Lambda$ are supposed to be uniformly strictly positive, it follows that $\left\|\left(\frac{1}{\sigma}, \frac{1}{z_1}, \ldots, \frac{1}{z_l}\right)\right\|_\Lambda, \left\|\left(\frac{1}{\sigma'}, \frac{1}{z'_1}, \ldots, \frac{1}{z'_l}\right)\right\|_\Lambda$ are uniformly bounded by a positive constant C. Then Equation (63), (59) and (60) lead to the assertion. If $(\sigma', Z') \in P_\Lambda$ is sufficiently close to (σ, Z), then $(\sigma, Z), (\sigma', Z')$ are uniformly strictly positive and therefore the mappings defined by Equation (93) are continuous, too. To provide the previous results for the mappings $\Phi_j^{(u,\vec{u})}(\sigma, Z)$ and $\Phi_j^{(u,\vec{u})}$, additionally we have to apply the estimate $\|M(u, \vec{u})\|_l \leq \|M\| \|(u, \vec{u})\|_l$, which holds for every continuous linear mapping $M : H^\rho_{l,\Lambda}(\Omega) \to H^\rho_{l,\Lambda}(\Omega)$. It is immediately verified, that the estimate (92) holds for the mapping $S^j\Phi_j$, too. ■

Lemma 5.27 *The mapping*

$$\mathcal{S} : P_\Lambda \times P_\Lambda \to H^\rho_{l,\Lambda}(\Omega) \tag{95}$$

$$\mathcal{S}\left((\sigma, Z), (\delta, \Delta)\right) := \sum_{j=0}^{\infty} \left(-L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)}\right)^j (u, \vec{u}) \tag{96}$$

is continuous.

Proof: Let $(\sigma_0, Z_0) \in P_\Lambda$, $R > 0$ sufficiently small, i.e., such that for all $(\sigma, Z) \in \mathcal{B}_R\left((\sigma_0, Z_0)\right)$ and for all $(\delta, \Delta) \in \mathcal{B}_R\left((0,0)\right)$ $\|L^{-1}_{(\sigma,Z)}\| \|L_{(\delta,\Delta)}\| < c_0 < 1$ holds for a real constant c_0. Then we get the estimate

$$\left\|\sum_{j=0}^{\infty} \left(-L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)}\right)^j (u, \vec{u})\right\|_l \leq \sum_{j=0}^{\infty} \left\|L^{-1}_{(\sigma,Z)} L_{(\delta,\Delta)}\right\|_l^j \|(u, \vec{u})\|_l \leq \|(u, \vec{u})\|_l \sum_{j=0}^{\infty} c_0^j. \tag{97}$$

Therefore the series converges uniformly in $\mathcal{B}_R((\sigma_0, Z_0)) \times \mathcal{B}_R((0,0))$. Since each term is continuous (due to Corollary 5.23), it follows from standard arguments that the limit of the series is continuous, too. ■

In order to apply the converse Taylor theorem (Lemma C.9), we choose the symmetric maps $S^j\Phi_j$ for assembling the appropriate series. Since $S^jM(x,\ldots,x) = M(x,\ldots,x)$ for any j-linear map M, it follows immediately from Equation (68) that

$$\begin{aligned}
(y,\vec{y}) &= (u,\vec{u}) + \sum_{j=1}^{n} S^j\Phi_j^{(u,\vec{u})}((\sigma,Z),\underbrace{(\delta,\Delta),\ldots,(\delta,\Delta)}_{\text{j times}}) + \\
&\qquad \left(-L^{-1}_{(\sigma,Z)}L_{(\delta,\Delta)}\right)^{n+1}\left(\sum_{j=0}^{\infty}\left(-L^{-1}_{(\sigma,Z)}L_{(\delta,\Delta)}\right)^j\right)(u,\vec{u}) \\
&= (u,\vec{u}) + \sum_{j=1}^{n} S^j\Phi_j^{(u,\vec{u})}((\sigma,Z),\underbrace{(\delta,\Delta),\ldots,(\delta,\Delta)}_{\text{j times}}) + \\
&\qquad \underbrace{S^n\Phi_n((\sigma,Z),\underbrace{(\delta,\Delta),\ldots,(\delta,\Delta)}_{\text{n times}})\left(-L^{-1}_{(\sigma,Z)}L_{(\delta,\Delta)}\right)\mathcal{S}((\sigma,Z),(\delta,\Delta))}_{=R((\sigma,Z),(\delta,\Delta))((\delta,\Delta),\ldots,(\delta,\Delta))},
\end{aligned} \tag{98}$$

where the mapping

$$\begin{aligned}
R &: P_\Lambda \times P_\Lambda \to L_s^n(P_\Lambda, H^\rho_{l,\Lambda}(\Omega)) \qquad \text{is defined by} \\
R &\;((\sigma,Z),(\delta,\Delta))((\delta_1,\Delta_1),\ldots,(\delta_n,\Delta_n)) := \\
&S^n\Phi^n_{(\sigma,Z)}((\delta_1,\Delta_1),\ldots,(\delta_n,\Delta_n))\underbrace{\left(-L^{-1}_{(\sigma,Z)}L_{(\delta,\Delta)}\right)\mathcal{S}((\sigma,Z),(\delta,\Delta))}_{=:\Psi_{(\sigma,Z),(\delta,\Delta)}}.
\end{aligned} \tag{99}$$

Using the abbreviation $\Psi_{(\cdot,\cdot),(\cdot,\cdot)}$, we get the estimate

$$\begin{aligned}
&\left\|\left(R((\sigma,Z),(\delta,\Delta)) - R((\sigma',Z'),(\delta',\Delta'))\right)((\delta_1,\Delta_1),\ldots,(\delta_n,\Delta_n))\right\|_l \\
&\leq \Big\|S^n\Phi^n_{(\sigma,Z)}((\delta_1,\Delta_1),\ldots,(\delta_n,\Delta_n))\,\Psi_{(\sigma,Z),(\delta,\Delta)} - \\
&\qquad S^n\Phi^n_{(\sigma',Z')}((\delta_1,\Delta_1),\ldots,(\delta_n,\Delta_n))\,\Psi_{(\sigma',Z'),(\delta',\Delta')}\Big\|_l \\
&\leq \left\|S^n\Phi^n_{(\sigma,Z)}((\delta_1,\Delta_1),\ldots,(\delta_n,\Delta_n))\right\|\left\|\Psi_{(\sigma,Z),(\delta,\Delta)} - \Psi_{(\sigma',Z'),(\delta',\Delta')}\right\|_l + \\
&\qquad \left\|\Psi_{(\sigma',Z'),(\delta',\Delta')}\right\|_l\left\|\left(S^n\Phi^n_{(\sigma,Z)} - S^n\Phi^n_{(\sigma',Z')}\right)((\delta_1,\Delta_1),\ldots,(\delta_n,\Delta_n))\right\| \\
&\leq C\left\|\left(\frac{1}{\sigma},\frac{1}{z_1},\ldots,\frac{1}{z_l}\right)\right\|^n_\Lambda \|(\delta_1,\Delta_1)\|_\Lambda\cdots\|(\delta_n,\Delta_n)\|_\Lambda \underbrace{\left\|\Psi_{(\sigma,Z),(\delta,\Delta)} - \Psi_{(\sigma',Z'),(\delta',\Delta')}\right\|_l}_{I} + \\
&\qquad \left\|\Psi_{(\sigma',Z'),(\delta',\Delta')}\right\|_l C' \underbrace{\left\|(\sigma,Z)-(\sigma',Z')\right\|_\Lambda}_{II}\|(\delta_1,\Delta_1)\|_\Lambda\cdots\|(\delta_n,\Delta_n)\|_\Lambda.
\end{aligned} \tag{100}$$

Note that the map $\Psi_{(\cdot,\cdot),(\cdot,\cdot)}$ as composition of two continuous maps is continuous, too. Since Term I and II converge to zero as $(\sigma',Z') \to (\sigma,Z)$ and

$(\delta', \Delta') \to (\delta, \Delta)$, we conclude, after dividing both sides of the estimate (100) by $\|(\delta_1, \Delta_1)\|_\Lambda \cdots \|(\delta_n, \Delta_n)\|_\Lambda$ and taking the supremum for $(\delta_i, \Delta_i) \neq 0$, that the mapping $R\,((\sigma', Z'), (\delta', \Delta'))$ converges to $R\,((\sigma, Z), (\delta, \Delta))$ in $L^n_s(P_\Lambda, H^\rho_{l,\Lambda}(\Omega))$ as $(\sigma', Z') \to (\sigma, Z)$ and $(\delta', \Delta') \to (\delta, \Delta)$. Note that $\|\Psi_{(\sigma',Z'),(\delta',\Delta')}\|$ is uniformly bounded for sufficiently small (δ', Δ') and for (σ', Z') sufficiently close to (σ, Z). Therefore the mapping (99) is continuous. Moreover, the estimate

$$\begin{aligned} &\|R\,((\sigma, Z), (\delta, \Delta))\,((\delta_1, \Delta_1), \ldots, (\delta_n, \Delta_n))\| \\ &\qquad \le C\|(\delta_1, \Delta_1)\|_\Lambda \cdots \|(\delta_n, \Delta_n)\|_\Lambda C'\|(\delta, \Delta)\|_\Lambda \frac{\|(u, \vec{u})\|_l}{1 - \|L^{-1}_{(\sigma,Z)}\|\|L_{(\delta,\Delta)}\|} \end{aligned} \tag{101}$$

together with the uniform boundednes of the fraction in Equation (101) for sufficiently small (δ, Δ) provides $\|R\,((\sigma, Z), (\delta, \Delta))\| \to 0$ as (δ, Δ) approaches 0. Therefore, letting $d_j := j!S^j\Phi^{(u,\vec{u})}_j$ for $j \in \{1, \ldots, n\}$, we have verified all assumptions of the converse Taylor theorem (see Lemma C.9), which leads to the following results.

Theorem 5.28 *The mapping $B_{\vec{I}} : P_\Lambda \to H^\rho_{l,\Lambda}(\Omega)$ is of class C^n for every natural number n, i.e., it is of class C^∞. The n^{th} derivative of $B_{\vec{I}}$ at $(\sigma, Z) \in P_\Lambda$ is the mapping $j!S^j\Phi^{(u,\vec{u})}_j : P_\Lambda \to L^j_s(P_\Lambda, H^\rho_{l,\Lambda}(\Omega))$. Furthermore, $B_{\vec{I}}$ is analytic.*

Proof: It remains to prove analyticity, which follows from the decomposition

$$(y, \vec{y}) \;=\; (u, \vec{u}) + \sum_{j=1}^{\infty} S^j\Phi^{(u,\vec{u})}_j((\sigma, Z)\,\underbrace{(\delta, \Delta), \ldots, (\delta, \Delta)}_{\text{j times}}) \tag{102}$$

and the fact that

$$\left\| S^j\Phi^{(u,\vec{u})}_j((\sigma, Z)\,\underbrace{(\delta, \Delta), \ldots, (\delta, \Delta)}_{\text{j times}}) \right\| \le \|(u, \vec{u})\| \left(\left\| L^{-1}_{(\sigma,Z)} \right\| \left\| L_{(\delta,\Delta)} \right\| \right)^j. \tag{103}$$

■

Corollary 5.29 *The mapping $A_{\vec{I}} : P_\Lambda \to H^\rho_{l,\Lambda}(\Omega)$ is of class C^n for every natural number n, i.e., it is of class C^∞.*

Proof: Theorem 5.28, Proposition 5.19 together with Theorem C.4. ■

A Properties of the function spaces

Lemma A.1 *For any Lipschitz continuous transformation $T : \Omega \to \mathbb{R}^n$, whose inverse $T^{-1} : T(\Omega) \to \Omega$ exists, and for all measurable functions $f : \mathbb{R}^n \to \mathbb{R}_+$ the change of variables formula*

$$\int_\Omega f \circ T\, |J_T| dx \quad = \quad \int_{T(\Omega)} f dx \tag{104}$$

is valid, where $\Omega \subset \mathbb{R}^n$ denotes an open subset and J_T denotes the jacobian of T, i.e., $J_T = \det T'(x)$.

Proof: [Rud87] Th. 7.26, Lemma 7.25 together with Rademacher's theorem (see [Zie89] Th. 2.2.1). ■

Lemma A.2 *Let $\Omega, \Omega' \subset \mathbb{R}^n$ be domains and $T : \Omega \to \Omega'$ a bijective map where T and T^{-1} are Lipschitz continuous. Then the pullback operators*

$$\begin{aligned} *T: &\quad H^{1,\rho}(\Omega') \to H^{1,\rho\circ T}(\Omega), \quad u \mapsto u \circ T \ \text{ and} \\ *T^{-1}: &\quad H^{1,\rho\circ T}(\Omega) \to H^{1,\rho}(\Omega'), \quad u \mapsto u \circ T^{-1} \end{aligned}$$

are continuous and therefore $H^{1,\rho}(\Omega') \cong H^{1,\rho\circ T}(\Omega)$. An analoguous statement for the spaces $H_c^{1,\rho}(\Omega)$ holds, too.

Proof: Let $\phi \in C^\infty(\Omega)$ and $\|\phi\|_{1,\rho} \leq \infty$. We estimate each component of the norm

$$\|{*T\phi}\|^2_{1,\rho\circ T} = \underbrace{\|\phi \circ T\, (\rho \circ T)^{-1}\|^2_{L^2(\Omega)}}_{I} + \underbrace{\||\nabla\, (\phi \circ T)|\|^2_{L^2(\Omega)}}_{II} \tag{105}$$

separately. For the first term Lemma A.1 is applied :

$$\begin{aligned} I = \int_\Omega \left(\phi^2\rho^{-2}\right) \circ T(x) dx &= \int_{T(\Omega)} \left(\phi^2\rho^{-2}\right)(x) |J_{T^{-1}}(x)| dx \\ &\leq C_n \mathrm{Lip}\{T^{-1}\}^n \left\|\phi\rho^{-1}\right\|^2_{L^2(\Omega')}. \end{aligned}$$

As an easy consequence of Rademacher's theorem (see for example [Zie89] Th. 2.2.1) we conclude that T_i, $\partial_j T_i$ are measurable, $\partial_j T_i$ exists and $|\partial_j T_i| \leq \mathrm{Lip}\{T\}$ almost everywhere. It follows also from Rademacher's theorem that $\partial_i\left(\phi \circ T(x)\right) = \sum_{j=1}^n \partial_j\phi(T(x))\partial_i T_j$ holds pointwise almost everywhere in Ω. Therefore we get the estimate

$$\begin{aligned} \int_\Omega \partial_i\left(\phi \circ T(x)\right)^2 dx &\leq n \sum_{j=1}^n \int_\Omega \partial_j\phi\left(T(x)\right)^2 \partial_i T_j(x)^2 dx \\ &\leq n \mathrm{Lip}\{T\}^2 \sum_{j=1}^n \int_{T(\Omega)} \partial_j\phi(x)^2 |J_{T^{-1}}(x)| dx \\ &\leq n C_n \mathrm{Lip}\{T\}^2 \mathrm{Lip}\{T^{-1}\}^n \||\nabla\phi|\|^2_{L^2(\Omega')} \end{aligned}$$

which leads to

$$I + II \leq C(n,T)\|\phi\|_{1,\rho}\,, \tag{106}$$

where the constant $C(n,T)$ depends only on the space dimension n and the Lipschitz constant of T. Now the inequality

$$\|{*T}\phi\|_{1,\rho\circ T} \leq C(n,T)\|\phi\|_{1,\rho} \tag{107}$$

extends by continuity to all functions $\phi \in H^{1,\rho}(\Omega)$ (resp. $\phi \in H_c^{1,\rho}(\Omega)$). ■

The following two Lemmas can be found in [Jan86]. For convenience we present entire proofs for the assertions made there.

Lemma A.3 *Let $\Omega' \subset \mathbb{R}^n$ be a domain, ρ a translation and dilation invariant weight (see Definition 2.1) and $T(x) := K_0(x - x_0)$ for a constant $K_0 > 0$ and some $x_0 \in \mathbb{R}^n$. Then the pullback operator $*T$ provides the isomorphism*

$$*T : H^{1,\rho}(\Omega) \cong H^{1,\rho}(T(\Omega)),$$
$$(resp.\ *T : H_c^{1,\rho}(\Omega) \cong H_c^{1,\rho}(T(\Omega))).$$

Proof: Lemma A.2 yields the isomorphism $H^{1,\rho}(T(\Omega)) \cong H^{1,\rho\circ T}(\Omega)$ (resp. $H_c^{1,\rho}(T(\Omega)) \cong H_c^{1,\rho\circ T}(\Omega)$) and it remains to prove $H^{1,\rho\circ T}(\Omega) \cong H^{1,\rho}(\Omega)$ (resp. $H_c^{1,\rho\circ T}(\Omega) \cong H_c^{1,\rho}(\Omega)$) which follows directly from the translation and dilation invariance of the weight ρ (see definition 2.1). ■

Lemma A.4 *Let $\Omega, \Omega' \subset \mathbb{R}^n$ be domains, $\rho(x) = (1 + |x|)^\alpha$ a weight ($\alpha > 1$) and $T : \Omega \to \Omega'$ a bijective map such that T and T^{-1} are Lipschitz continuous.*
Then the pullback operators

$$\begin{aligned} *T: &\quad H^{1,\rho}(\Omega') \to H^{1,\rho}(\Omega), \quad u \mapsto u \circ T \ \text{ and} \\ *T^{-1}: &\quad H^{1,\rho}(\Omega) \to H^{1,\rho}(\Omega'), \quad u \mapsto u \circ T^{-1} \end{aligned}$$

are continuous and therefore the relation $H^{1,\rho}(\Omega') \cong H^{1,\rho}(\Omega)$ holds. The statement remains true if the spaces $H^{1,\rho}(\Omega)$ are replaced by $H_c^{1,\rho}(\Omega)$.

Proof: Lemma A.3 allows us to assume without loss of generality $0 \in \Omega \cap \Omega'$. Because T and T^{-1} are Lipschitz continuous we can find a constant $M > 0$ such that $M^{-1}\|x\| \leq \|T(x)\| \leq M\|x\|$ holds for all $x \in \Omega$. Since ρ is increasing in $\|x\|$ (which means $\rho(x) < \rho(y)$ if $\|x\| < \|y\|$), we have

$$\rho\left(M^{-1}x\right) \leq \rho\left(T(x)\right) \leq \rho\left(Mx\right) \text{ for all } x \in \Omega.$$

The translation and dilation invariance of ρ provides constants K_1, K_2 such that

$$\rho(x) \leq K_1\rho\left(M^{-1}x\right) \quad \text{and} \quad K_2^{-1}\rho(Mx) \leq \rho\left(x\right) \text{ for all } x \in \Omega.$$

For $K := \max\{K_1, K_2\}$ it follows

$$K^{-1}\rho(x) \leq \rho\left(T(x)\right) \leq K\rho(x) \text{ for all } x \in \Omega.$$

Due to the last inequality we conclude $H^{1,\rho\circ T}(\Omega) \cong H^{1,\rho}(\Omega)$ (resp. $H_c^{1,\rho\circ T}(\Omega) \cong H_c^{1,\rho}(\Omega)$). The case T^{-1} is treated analoguously and Lemma A.2 finishes the proof.
■

Lemma A.5 *Let Ω be a domain. For $u \in H^{1,\rho}(\Omega)$ (resp. $u \in H_c^{1,\rho}(\Omega)$) (see Definition 2.5) the following statements hold :*

i) u is a measurable function and $u\rho^{-1} \in L^2(\Omega)$,

ii) there exist functions $\partial_i u \in L^2(\Omega)$ such that

$$\int_\Omega (\partial_i u)\phi dx = -\int_\Omega (\partial_i \phi) u dx \quad \textit{for all } \phi \in \mathcal{D}(\Omega) \tag{108}$$

for each $i \in \{1, \cdots, n\}$ (i.e. the $\partial_i u$ are the distributional derivatives of u).

Proof: Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy-sequence (with respect to the norm $\|\cdot\|_{1,\rho}$) of infinitely differentiable functions. Since $\|(u_n - u_m)\rho^{-1}\|_{L^2(\Omega)} \to 0$ and $\||\nabla(u_n - u_m)|\|_{L^2(\Omega)} \to 0$ as $m, n \to \infty$, we know that there exist functions $g, g_i \in L^2(\Omega)$ such that $\|u_n\rho^{-1} - g\|_{L^2(\Omega)} \to 0$ and for each $i \in \{1, \cdots, n\}$ $\|\partial_i u_n - g_i\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. The function $u := g\rho \in L^2_{loc}$ is a regular Distribution (since ρ is measurable and bounded on compact subsets). For any compact subset $K \subset \Omega$ holds

$$\left(\max_{x\in K}\{\rho(x)\}\right)^{-2} \int_K (u_n - u)^2 \le \int_K (u_n - u)^2 \rho^{-2} dx \le \|u_n\rho^{-1} - g\|_{L^2(\Omega)}$$

and therefore we have

$$\|u_n - u\|_{L^2(K)} \to 0 \quad \text{as } n \to \infty$$

for every compact subset $K \subset \Omega$. Letting $\phi \in \mathcal{D}(\Omega)$ and $K := \mathrm{supp}(\phi)$ we finally conclude

$$\int_\Omega u(\partial_i\phi)dx = \lim_{n\to\infty}\int_\Omega u_n(\partial_i\phi)dx = -\lim_{n\to\infty}\int_\Omega (\partial_i u_n)\phi dx = -\int_\Omega g_i\phi dx.$$

Setting $\partial_i u := g_i$ therefore finishes the proof. ■

Lemma A.6 *Let Ω be a domain, $\{\phi_\epsilon\}_{\epsilon>0}$ a Dirac-sequence (i.e. $\phi_\epsilon \in C_0^\infty(B_\epsilon(0))$, $\phi_\epsilon \ge 0$ and $\int_{\mathbb{R}^n}\phi_\epsilon(x)dx = 1$ for all $\epsilon > 0$), $K \subset \Omega$ a compact subset, $0 < \tau < 1/2\min\{1, \mathrm{dist}(K, \partial\Omega)\}$, $K_\tau := \overline{\mathcal{U}_\tau(K)}$ the closed τ-environment of K and*

$$u^\epsilon_{K_\tau}(x) := \left(\chi_{K_\tau}u\right) * \phi_\epsilon(x) = \int_{\mathbb{R}^n} \phi_\epsilon(x-y)\left(\chi_{K_\tau}(y)u(y)\right)dy. \tag{109}$$

Then for $u \in W^{1,\rho}(\Omega)$ (see Definition 2.6) we have $u^\epsilon_{K_\tau} \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $\partial_i u^\epsilon_{K_\tau} \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $\lim_{\epsilon\to 0}\|u^\epsilon_{K_\tau} - u\|_{W^{1,\rho}(K)} \to 0$, $\lim_{\epsilon\to 0}\|u^\epsilon_{K_\tau} - u\|_{W^1(K)} \to 0$ and

$$\partial_i u^\epsilon_{K_\tau}(x) = \left(\chi_{K_\tau}\partial_i u\right) * \phi_\epsilon(x) \quad \text{for all } x \in K. \tag{110}$$

Proof: The assumption $u\rho^{-1} \in L^2(\Omega)$ and the inequality

$$\int_{K_\tau} u^2(x)dx \le \left(\max_{x\in K_\tau}\{\rho(x)\}\right)^2 \|\rho^{-1}u\|_{L^2(\Omega)}.$$

provides $\chi_{K_\tau} u \in L^2(\mathbb{R}^n)$. The well known convolution theorem therefore states $u^\epsilon_{K_\tau} \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\partial_i u^\epsilon_{K_\tau}(x) = \left(\chi_{K_\tau} u\right) * (\partial_i \phi_\epsilon)(x)$.
Now let $\epsilon < \tau$ and $x \in K$. Then the function $\phi_\epsilon(x - \cdot) \in \mathcal{D}(\Omega)$, $\mathrm{supp}(\phi_\epsilon(x - \cdot)) \subset K_\tau$ and because $\partial_i u$ is the distributional derivative of u we are justified to apply integration by parts :

$$\begin{aligned} \partial_i u^\epsilon_{K_\tau}(x) &= \int_{\mathbb{R}^n} (\partial_i \phi_\epsilon(x-y)) \left(\chi_{K_\tau}(y) u(y)\right) dy \\ &= -\int_\Omega \partial_i \left(\phi_\epsilon(x - \cdot)\right)(y) u(y) dy \\ &= \int_\Omega \phi_\epsilon(x-y) \partial_i u(y) dy \\ &= \int_{\mathbb{R}^n} \phi_\epsilon(x-y) \left(\chi_{K_\tau}(y) \partial_i u(y)\right) dy \\ &= \left(\chi_{K_\tau} \partial_i u\right) * (\phi_\epsilon)(x). \end{aligned}$$

Since $\partial_i u \in L^2(\Omega)$, it follows $\partial_i u^\epsilon_{K_\tau} \in L^2(K) \cap C^\infty(\mathbb{R}^n)$. Furthermore, we have

$$\begin{aligned} \|u^\epsilon_{K_\tau} - u\|_{L^2(K)} &= \|\left(\chi_{K_\tau} u\right) * \phi_\epsilon - \chi_{K_\tau} u\|_{L^2(K)} \\ &\leq \|\left(\chi_{K_\tau} u\right) * \phi_\epsilon - \chi_{K_\tau} u\|_{L^2(\mathbb{R}^n)} \to 0 \end{aligned}$$

and similarely

$$\|\partial_i u^\epsilon_{K_\tau} - \partial_i u\|_{L^2(K)} \leq \|\left(\chi_{K_\tau} \partial_i u\right) * \phi_\epsilon - \chi_{K_\tau} \partial_i u\|_{L^2(\mathbb{R}^n)} \to 0$$

as $\epsilon \to 0$ which leads to both

$$\lim_{\epsilon \to 0} \left\| u^\epsilon_{K_\tau} - u \right\|_{W^1(K)} = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} \left\| u^\epsilon_{K_\tau} - u \right\|_{W^{1,\rho}(K)} = 0,$$

because we have $\|\rho^{-1}(u^\epsilon_{K_\tau} - u)\|_{L^2(K)} \leq C_K^{-2} \|(u^\epsilon_{K_\tau} - u)\|_{L^2(K)}$ for $C_K := \min_{x \in K} \{\rho(x)\}$. ■

Theorem A.7 *For an arbitrary domain Ω the relation $H^{1,\rho}(\Omega) = W^{1,\rho}(\Omega)$ (see Definition 2.5 and Definition 2.6) holds.*

Proof: i) $H^{1,\rho}(\Omega) \subset W^{1,\rho}(\Omega)$: This is essentially the statement of Lemma A.5. ii) $W^{1,\rho}(\Omega) \subset H^{1,\rho}(\Omega)$: Using lemma A.6 the proof works analoguous to a standard proof for unweighted Sobolev spaces, therefore we present only a brief sketch :

Cover Ω by at most countably many bounded sets U_i such that $\overline{U_i} \subset \Omega$. Then there exists a locally finite partition of unity $\{\eta_i\}_{i \in \mathbb{N}}$ subordinated to the covering $\{U_i\}_{i \in \mathbb{N}}$. Lemma A.6 provides functions $u_{i,\epsilon} \in C^\infty(\mathbb{R}^n)$ such that

$$\|u_{i,\epsilon} - u\|_{W^{1,\rho}(\overline{U_i})}, \ \|u_{i,\epsilon} - u\|_{W^1(\overline{U_i})} \leq \epsilon \cdot 2^{-i} \cdot \max\{1, \|\eta_i\|^{-1}_{C^1(\overline{\Omega})}\}$$

holds. For $u_\epsilon := \sum_{i \in \mathbb{N}} \eta_i u_{i,\epsilon} \in C^\infty(\mathbb{R}^n)$ we get

$$\|\rho^{-1}(u_\epsilon - u)\|_{L^2(\Omega)} \leq \sum_{i \in \mathbb{N}} \|\rho^{-1} \eta_i (u_{i,\epsilon} - u)\|_{L^2(\overline{U_i})} \leq \epsilon$$

and

$$\begin{aligned}\|\partial_i (u_\epsilon - u)\|_{L^2(\Omega)} &\leq \sum_{i\in\mathbb{N}} \|\partial_i (\eta_i (u_{i,\epsilon} - u))\|_{L^2(\overline{U_i})} \\ &\leq \sum_{i\in\mathbb{N}} \|\partial_i\eta_i (u_{i,\epsilon} - u)\|_{L^2(\overline{U_i})} + \|\eta_i\partial_i (u_{i,\epsilon} - u)\|_{L^2(\overline{U_i})} \leq 2\epsilon,\end{aligned}$$

which leads to $\|u_\epsilon - u\|_{W^{1,\rho}(\Omega)} \leq (\sqrt{4n+1})\epsilon$. ■

Corollary A.8 *Let $u \in H_c^{1,\rho}(\Omega)$. Then the claims of lemma A.6 concerning the regularization of u remain true.*

Proof: Theorem A.7 provides $H^{1,\rho}(\Omega) = W^{1,\rho}(\Omega)$. Since $H_c^{1,\rho}(\Omega) \subset H^{1,\rho}(\Omega)$, Lemma A.6 can be applied to $u \in H_c^{1,\rho}(\Omega)$, too. ■

Corollary A.9 *Let $u \in W^{1,\rho}(\Omega)$ (see Definition 2.6) and* $\operatorname{supp}(u)$ *a compact subset of Ω. Then $u \in H_c^{1,\rho}(\Omega)$.*

Proof: Let $\tau := 1/2 \min\{1, \operatorname{dist}(\operatorname{supp}(u), \partial\Omega)\}$, $K := U_\tau(\operatorname{supp}(u))$ and apply lemma A.6 to generate a sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \in C_0^\infty(\Omega)$, $\operatorname{supp}(u_n) \subset K \subset \Omega$ and $\lim_{n\to\infty} \|u - u_n\|_{W_c^{1,\rho}(\Omega)} = 0$. ■

Corollary A.10 *Let Ω be an arbitrary domain, $u \in H^{1,\rho}(\Omega)$ or $u \in H_c^{1,\rho}(\Omega)$ and $\partial_i u(x) = 0$ for almost every $x \in \Omega$ and for all $i \in \{1, \cdots, n\}$.*

Then there exists a constant $c_0 \in \mathbb{R}$ such that $u(x) = c_0$ for almost every $x \in \Omega$. Additionally, if Ω has a locally finite Lipschitz boundary $\partial\Omega$, then the well defined boundary values are also constant : $(\gamma(u))(s) = c_0$ for almost every $s \in \partial\Omega$. (For properties of the trace operator γ see Theorem 4.1).

Proof: Let $\{K_i\}_{i\in\mathbb{N}}$ be a countable compact exhaustion of Ω, i.e., $K_i \subset\subset \Omega$, $K_i \subset K_{i+1}$, K_i compact, and $\bigcup_{i\in\mathbb{N}} K_i = \Omega$. Then lemma A.6, corollary A.8 and theorem A.7 provide for each K_i constant regularizers :

$$u_i^\epsilon \in C^\infty(\mathbb{R}^n),\ u_i^\epsilon(x) = \mathit{const}\ \forall x \in K_i,\ \lim_{\epsilon\to 0} \|u_i^\epsilon - u\|_{L^2(K_i)} = 0, \tag{111}$$

since Equation (110) implies $\partial_j u_i^\epsilon = 0$ for all $j \in \{1, \cdots, n\}$. Therefore, we conclude that there exists a constant c_0 such that $u(x) = c_0$ for almost all $x \in K_i$. Further, we conclude by induction that c_0 is the same constant for all K_i, $i \in \mathbb{N}$ and therefore $u(x) = c_0$ for almost all $x \in \Omega$.

To deal with the trace operator, we consider the sequence $u_i(x) := c_0$ for all $x \in \Omega$ and verify immediately $\|u_i - u\|_{1,\rho} = 0$. Hence, Theorem 4.1 implies $\|\gamma(u_i) - \gamma(u)\|_{L^{2,\rho^{-1}}(\Omega)} = \|c_0 - \gamma(u)\|_{L^{2,\rho^{-1}}(\Omega)} \leq \|u_i - u\|_{1,\rho} = 0$. Particularly, this leads to $\|\rho^{-1}(c_0 - \gamma(u))\|_{L^2(\partial\Omega)} = 0$ and therefore $(\gamma(u))(s) = c_0$ for almost every $s \in \partial\Omega$. ■

Lemma A.11 *Let Ω be an arbitrary domain such that there exists an extension operator $F : H^{1,\rho}(\Omega) \to H^{1,\rho}(\mathbb{R}^n)$ (see Definition* (3.4)*). Then*

$$H^{1,\rho}(\Omega) = \overline{\left\{f \in C^\infty(\overline{\Omega}) \mid \|f\|_{1,\rho} < \infty\right\}}^{\|\cdot\|_{1,\rho}}. \tag{112}$$

Proof: It is clear that the inclusion $\overline{\left\{f \in C^\infty(\overline{\Omega}) \mid \|f\|_{1,\rho} < \infty\right\}}^{\|\cdot\|_{1,\rho}} \subset H^{1,\rho}(\Omega)$ is valid. Therefore let $u \in H^{1,\rho}(\Omega)$ and apply the extension operator to provide $F(u) \in H^{1,\rho}(\mathbb{R}^n)$. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \in C^\infty(\mathbb{R}^n)$ and $\lim_{n\to\infty} \|F(u) - u_n\|_{W^{1,\rho}(\mathbb{R}^n)} = 0$. It follows immediately that $\lim_{n\to\infty} \|u - u_n|_\Omega\|_{W^{1,\rho}(\Omega)} = 0$. ∎

B Boundary-Integrals, Gauss' theorem

Corresponding to [Alt92] we use the following definition of integrals along Lipschitz boundaries.

Definition B.1 *Let Ω be a domain with a Lipschitz boundary $\partial\Omega$ (see Definition 2.9), $\{U_i\}_{i\in\mathbb{N}}$ the corresponding covering of $\partial\Omega$ and $\{\phi_i\}_{i\in\mathbb{N}}$ a partition of unity subordinated to the boundary covering, i.e., such that* $\operatorname{supp}\{\phi_i\} \subset U_i$ *for each $i \in \mathbb{N}$. A function $f : \partial\Omega \to \mathbb{R}$ is called measurable resp. locally integrable if the functions*

$$y' \mapsto f \circ A_i^{-1}(y', a_i(y')) \text{ for } y' \in U_\alpha \tag{113}$$

are measurable resp. integrable, where we have used the notation $y' := (y_1, \cdots, y_{n-1})$ and A_i from Definition 2.9. The boundary integral is defined by

$$\int_{\partial\Omega} f d\mathcal{S} := \sum_{i\in I} \int_{\partial\Omega} \phi_i f d\mathcal{S}. \tag{114}$$

If $\operatorname{supp}(f) \subset U_i$ *the boundary integral is defined by*

$$\int_{\partial\Omega} f d\mathcal{S} := \int_{\mathbb{R}^{n-1}} f \circ A_i^{-1}(y', a_i(y'))\sqrt{1 + |\nabla a_i(y')|^2} dy'. \tag{115}$$

Spaces of integrable functions on the boundary $\partial\Omega$ are defined by

$$L^2(\partial\Omega) := \left\{f : \partial\Omega \to \mathbb{R} \,|\, f \text{ measurable and } \|f\|_{L^2(\partial\Omega)} < \infty\right\} \tag{116}$$

where

$$\|f\|_{L^2(\partial\Omega)} := \left(\int_{\partial\Omega} |f|^2 d\mathcal{S}\right)^{\frac{1}{2}}. \tag{117}$$

Remark B.2 *i) Since the map $|\nabla a_i(\cdot)| : U_\alpha \to \mathbb{R}_+$ is bounded and measurable, the same holds for the term $\sqrt{1 + |\nabla a_i(y')|^2}$ in Equation* (115)*. Therefore the integral in Equation* (115) *is well defined.*
ii) The definition of the boundary integral by means of Formula (115) *is independent of the boundaries' representation by the Lipschitz function a_i. If there is another function b_i and another coordinate transformation B_i such that*

$$\begin{aligned}(U_i \cap \partial\Omega) &= A_i^{-1}\left(\{\vec{y} \in \mathbb{R}^n \mid y' \in U_\alpha,\ y_n = a_i(y')\}\right)\\ &= B_i^{-1}\left(\{\vec{y} \in \mathbb{R}^n \mid y' \in U_{\alpha'},\ y_n = b_i(y')\}\right),\end{aligned}$$

(see Definition 2.9) then for each f for which holds $\operatorname{supp}(f) \subset U_i$*, the equation*

$$\int_{U_\alpha} f \circ A_i^{-1}(y', a_i(y'))\sqrt{1 + |\nabla a_i(y')|^2}dy' = \int_{U_{\alpha'}} f \circ B_i^{-1}(y', b_i(y'))\sqrt{1 + |\nabla b_i(y')|^2}dy' \tag{118}$$

is valid. A proof can be found for example in [Alt92].

Theorem B.3 *Let* Ω *be a bounded domain with a Lipschitz boundary* $\partial\Omega$ *and* $u, v \in H^{1,\rho}(\Omega)$*. If* ρ *is strictly positive, then for each* $i \in \{1, \cdots, n\}$ *holds :*

$$\int_\Omega (u\partial_i v + v\partial_i u)dx = \int_{\partial\Omega} uv\nu_i d\mathcal{S}. \tag{119}$$

Proof: Theorem A 5.9 in [Alt92] combined with Remark (2.8). ∎

C Functional analytic tools

The following lemma of Lax-Milgram can be found in various books. For example in [Alt92], [MR93].

Lemma C.1 *Let* X *be a Hilbert space and let*

$$b : X \times X \to \mathbb{R} \tag{120}$$

be a bilinear form such that there exist positive constants $c_1, c_2 \in \mathbb{R}$ *for which the following is satisfied :*

i) $|b(x, y)| \leq c_1\|x\|_X\|y\|_X$ *for all* $x \in X$ *and* $y \in X$*,*

ii) $c_2\|x\|_X^2 \leq b(x, x)$ *for all* $x \in X$*.*

Then for every $g \in X^\star$ *(i.e., for every functional* g *on* X*) there exists a unique* $x \in X$ *such that*

$$b(x, y) = g(y) \quad \text{for all } y \in X. \tag{121}$$

The mapping $L : X^\star \to X$*, defined by* $L(g) := x$*, where* x *is the unique element in* X *satisfying (121), is linear, continuous and continuously invertible. Additionally, the following estimates for the operator norms are valid :*

$$\|L\| \leq \frac{1}{c_2} \quad \text{and} \quad \|L^{-1}\| \leq c_1. \tag{122}$$

Definition C.2 *Let* X, Y *be two Banach spaces,* $U \subset X$ *an open subset and* $f : U \to Y$ *a mapping. Then* f *is called (Fréchet-)differentiable at* $x \in U$ *if there esists a continuous linear map* $\mathcal{D}f(x) \in L(X, Y)$ *with the property that*

$$\lim_{h \to 0} \frac{\|f(x + h) - f(x) - (\mathcal{D}f(x))(h)\|_Y}{\|h\|_X} = 0. \tag{123}$$

If f *is differentiable at each* $x \in U$ *and the mapping* $x \mapsto \mathcal{D}f(x) : U \to L(X, Y)$ *is continuous,* f *is called continuously differentiable or of class* C^1*.*

Definition C.3 *Let X, Y be two Banach spaces, $U \subset X$ an open subset, $f : U \to Y$ a mapping and $n \geq 1$ a natural number. Then f is called of class C^n if*

$$D^n := D(D^{n-1}f) : U \to L^n(E, F) \tag{124}$$

exists and is continuous. Note that the identification $L^n(E, F) \cong L(E, L^{n-1}(E, F))$ holds. The space of n-linear maps (see Definition 5.20) is normed by

$$\|A\| := \sup \left\{ \frac{\|A(x_1, \ldots, x_n)\|}{\|x_1\| \cdots \|x_n\|} \middle| x_1, \ldots, x_n \neq 0 \right\}. \tag{125}$$

If f is of class C^n for each $n \in \mathbb{N}$, then f is called of class C^∞. Further details can be found in various textbooks on functional analysis, for example in [RA83] and [Lan93].

Theorem C.4 *Let X, Y, Z be Banach spaces, $U \subset X$ open, $V \subset Y$ open and $x \in U$. Assume that $f : U \to V$ is (Fréchet-)differentiable at x and $g : V \to Z$ is (Fréchet-)differentiable at $f(x)$. Then $g \circ f$ is (Fréchet-)differentiable at x and*

$$\mathcal{D}(g \circ f)(x) = \mathcal{D}g\,(f(x)) \circ \mathcal{D}f(x). \tag{126}$$

Note that $\circ$ in this formula denotes the composition of linear maps.
Moreover, if f, g are of class C^n for a natural number n, then $g \circ f$ is of class C^n, too.

Proof: See [Lan93] Chain Rule in chapt. XIII, §3 and Theorem 6.5 in §6. ■

Lemma C.5 *Let $X, Y_1, \ldots, Y_m$ be Banach spaces, $U \subset X$ an open subset, $x \in U$, $f_i : U \to Y_i$ maps and $f := (f_1, \ldots, f_m)$. Then $f : U \to Y_1 \times \cdots \times Y_m$ is (Fréchet-)differentiable at x if and only if each f_i is (Fréchet-)differentiable at x. In this case the formula*

$$\mathcal{D}f(x) = (\mathcal{D}f_1(x), \ldots, \mathcal{D}f_m(x)) \tag{127}$$

is valid.

Proof: See [Lan93] Chain Rule in chapt. XIII, §3. ■

Lemma C.6 *Let $X_1, \ldots, X_m, Y$ be Banach spaces, $U_i \subset X_i$ open subsets and $f : U_1 \times \cdots \times U_m \to Y$ a mapping. Then f is of class C^1, i.e., particularly (Fréchet-)differentiable, if and only if all partial derivatives $\mathcal{D}_i f : U_1 \times \cdots \times U_m \to L(X_i, Y)$ (i.e. the derivatives of the partial maps $x_i \mapsto f(x_1, \ldots, x_i, \ldots, x_m)$) exist and are continuous.*

Proof: See [Lan93] theorem 7.1. ■

Lemma C.7 *Let X, Y_1, Y_2, Y be Banach spaces and let $\cdot : Y_1 \times Y_2 \to Y$ be a continuous bilinear map. Furthermore, let $U \subset X$ an open subset, $f : U \to Y_1$ and $g : U \to Y_2$ be (Fréchet-)differentiable at $x \in U$. Then the product map $f \cdot g$ is (Fréchet-)differentiable at x and*

$$\mathcal{D}(f \cdot g)(x) = \mathcal{D}f(x) \cdot g(x) + f(x) \cdot \mathcal{D}g(x) \tag{128}$$

Proof: See [Lan93] Product in chapt. XIII, §3. ■

Lemma C.8 *Let A be a Banach algebra with unit e, and let U be the open set of invertible elements. Then the map $x \mapsto x^{-1}$ is (Fréchet-)differentiable on U. Its derivative at $x_0 \in U$ is the linear map*

$$h \mapsto -x_0^{-1} h {x_0}^{-1}. \tag{129}$$

Proof: See [Lan93] Quotient in chapt. XIII, §3. ■

Now we present the very important Taylor theorem in functional analytic context together with its converse.

Lemma C.9 *Let E, F be Banach spaces and $U \subset E$ an open subset. A map $f : U \to F$ is of class C^n if and only if there are continuous mappings*

$$d_j : U \to L_s^j(E, F), \qquad \text{for } j = 1, \ldots, n \tag{130}$$

and

$$R : U \times \mathcal{B}_\varepsilon(0) \to L_s^n(E, F), \tag{131}$$

for a sufficiently small $\varepsilon > 0$, such that

$$\begin{aligned} f(x+h) = f(x) + \frac{d_1(x)(h)}{1!} + \frac{d_2(x)(h,h)}{2!} + \cdots \\ + \frac{d_n(x)(h,\ldots,h)}{n!} + R(u,h)(h,\ldots,h), \end{aligned} \tag{132}$$

where $R(u, 0) = 0$. Moreover, if f is C^n, then necessarily $d_j = \mathcal{D}^j f$ for $j \in 1, \ldots, n$.

Proof: See [RA83] Taylor's Theorem and the Converse Taylor's Theorem. ■

Definition C.10 *Let X, Y be two Banach spaces, $U \subset X$ an open subset, $f : U \to Y$ a mapping which is of class C^∞. Then f is called analytic if the Taylor series*

$$f(x) + \frac{d_1(x)(h)}{1!} + \frac{d_2(x)(h,h)}{2!} + \cdots \frac{d_n(x)(h,\ldots,h)}{n!} + \cdots$$

converges for all $h \in \mathcal{B}_\varepsilon(0)$ for some real $\varepsilon > 0$.

Index

References

[Alt92] Hans Wilhelm Alt. *Lineare Funktionalanalysis.* Springer-Verlag, Berlin Heidelberg New York, 1992.

[Cal80] A.P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and Its Applications to Continuum Physics*, pages 67–73, Rio de Janeiro, 1980. Sociedade Brasileira de Matematica.

[CING89] K.-S. Cheng, D. Isaacson, J.C. Newell, and D.G. Gisser. Electrode models for electric current computed tomography. *IEEE Trans. Biomed. Engrg. 3*, (36):918–924, 1989.

[CIP96] Sorin Ciulli, Simona Ispas, and Michael Pidcock. Anomalous thresholds and edge singularities in electrical impedance tomography. *J. Math Phys.*, Vol. 37 No. 6:4388–4417, 1996.

[CIP97] Sorin Ciulli, Simona Ispas, and Michael Pidcock. Boundary modelling in electrical impedance tomography. In *Proceedings of the Conference Inverse Problems in Medical Imaging and Nondestructive Testing in Oberwolfach, FRA, February 4-10, 1996*, pages 188–195, Wien New York, 1997. Springer-Verlag.

[EHN96] H. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems.* Kluwer Acad. Publ., Dordrecht, 1996.

[Fol95] Gerald B. Folland. *Introduction to partial differential equations.* Princeton University Press, Princeton, New Jersey, 1995.

[Jan86] R. Janssen. Elliptic problems on unbounded domains. *SIAM J. MATH. ANAL. Vol. 17 No. 6.*, 1986.

[JS97] S. Järvenpää and E. Somersalo. Impedance imaging and electrode models. In *Proceedings of the Conference Inverse Problems in Medical Imaging and Nondestructive Testing in Oberwolfach, FRA, February 4-10, 1996*, pages 65–74, Wien New York, 1997. Springer-Verlag.

[Lan93] Serge Lang. *Real and Functional Analysis.* Springer-Verlag, New York, Berlin, Heidelberg, 1993.

[Lou89] Alfred K. Louis. *Inverse und schlecht gestellte Probleme.* Teubner, Stuttgart, 1989.

[MG92] M.A.Al-Gwaiz. *Theory of distributions.* Marcel Dekker Inc., New York, Basel, Hong Kong, 1992.

[MR93] R. C. Rogers M. Renardy. *An Introduction to Partial Differential Equations.* Springer Verlag, New York, Berlin, Heidelberg, 1993.

[PBP92] Kevin Paulson, William Breckon, and Michael Pidcock. Electrode modelling in electrical impedance tomography. *SIAM J. Appl. Math.*, Vol. 52, No. 4:1012–1022, 1992.

[RA83] T. Ratiu R. Abraham, J.E. Marsden. *Manifolds, Tensor Analysis, and Applications*. Addison-Wesley Publishing Company, Inc., London, Amsterdam, Sydney, Tokyo, 1983.

[Rud87] Walter Rudin. *Real and complex analysis*. Mc Graw Hill, 1987.

[Rud91] Walter Rudin. *Functional Analysis*. Mc Graw Hill, 1991.

[SCI92] E. Somersalo, M. Cheney, and D. Isaacson. Existence and uniqueness for electrode models for electric current computed tomography. *SIAM J. Appl. Math.*, 52, No. 4:1023–1040, 1992.

[Zie89] William P. Ziemer. *Weakly Differentiable Functions*. Springer, New York, 1989.