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Squire's theorem for the magnetohydrodynamic sheet pinch

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Abstract

The stability of the quiescent ground state of an incompressible viscous fluid sheet bounded by two parallel planes, with an electrical conductivity varying across the sheet, and driven by an external electric field tangential to the boundaries is considered. It is demonstrated that irrespective of the conductivity profile, as magnetic and kinetic Reynolds numbers (based on the Alfvén velocity) are raised from small values, two-dimensional perturbations become unstable first.

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One of the basic configurations in magnetohydrodynamics (MHD) is the pinch, namely, an electrically conducting fluid confined by the action of an electric current passing through it, such that pressure gradients are balanced by the Lorentz force.

In the geometry of a plane sheet, the pinch with the fluid at rest is stable if the electrical conductivity is infinite, but may be destabilized by resistivity [1].

In this Brief Communication we demonstrate that for increasing Reynolds numbers the quiescent ground state of a voltage-driven resistive plane sheet pinch becomes first unstable to two-dimensional perturbations. This is a generalization of Squire's theorem in hydrodynamics [2,3]. For the special case of a spatially uniform resistivity and no dc magnetic field in the sheetwise or "toroidal" direction (the direction of the driving electric field) a proof was given in Dahlburg and Karpen [4].

If the resistivity is spatially uniform, the equilibrium current is also uniform and the equilibrium magnetic field must be a linear function of the cross-sheet coordinate (by equilibrium we mean a stationary state with the fluid at rest). To admit cross-sheet profiles of the equilibrium magnetic field deviating from the linear one, one has to allow for variation of the electrical conductivity across the sheet. This is important, since, as found recently [5], in a voltage-driven incompressible sheet pinch with spatially and temporally uniform kinematic viscosity and magnetic diffusivity and with impenetrable stress-free boundaries, the quiescent ground state with uniform current density and a linear profile of the magnetic field across the sheet remains stable, no matter how strong the driving electric field. This agrees with the observation made in studies of quasiequilibria, that is of states with a nonuniform current density in a fluid with uniform resistivity (these states thus decay resistively), that seemingly inflection points in the magnetic field (or current) profile are necessary for instabilities to appear [6].

We use the nonrelativistic, incompressible MHD equations,

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \rho\nu\nabla^2\mathbf{v} - \nabla p + \mathbf{J}\times\mathbf{B}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\eta \mu_0 \mathbf{J} - \mathbf{v} \times \mathbf{B}), \tag{2}$$

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot \mathbf{B} = 0, \tag{3}$$

where $\mathbf{v}$ is the fluid velocity, $\mathbf{B}$ the magnetic induction, $\mathbf{J}$ the electric current density ($= \nabla \times \mathbf{B}/\mu_0$, μ_0 denoting the magnetic permeability in a vacuum), ρ the mass density, p the mechanical pressure, ν the kinematic viscosity, and η the magnetic diffusivity [$(\mu_0 \eta)^{-1}$ is the electrical conductivity]. No externally applied force appears in Eq. (1). While ρ and ν are assumed constant, η is allowed to vary spatially:

$$\eta(\mathbf{x}) = \eta_0 \tilde{\eta}(\mathbf{x}), \tag{4}$$

where η_0 is a dimensional constant and $\tilde{\eta}(\mathbf{x})$ a nondimensional function of position.

Let L and B_0 denote arbitrary units of length and magnetic induction. Writing $v_A = B_0/\sqrt{\mu_0 \rho}$ for the Alfvén velocity corresponding to B_0, we transform to nondimensional quantities according to

$$\begin{gathered} \mathbf{x}/L \to \mathbf{x},\ \ \mathbf{B}/B_0 \to \mathbf{B},\ \ \mathbf{v}/v_A \to \mathbf{v},\ \ t/\frac{L}{v_A} \to t, \\ p/\rho v_A^2 \to p,\ \ \mathbf{J}/\frac{B_0}{\mu_0 L} \to \mathbf{J},\ \ \mathbf{E}/B_0 v_A \to \mathbf{E}. \end{gathered} \tag{5}$$

$\mathbf{E}$ is the electric field. Eqs. (1) and (2) then become

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} + R^{-1} \nabla^2 \mathbf{v} - \nabla p + \mathbf{J} \times \mathbf{B}, \tag{6}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (S^{-1} \tilde{\eta} \mathbf{J} - \mathbf{v} \times \mathbf{B}), \tag{7}$$

where

$$R = \frac{v_A L}{\nu} \text{ and } S = \frac{v_A L}{\eta_0} \tag{8}$$

are Reynolds numbers based on the Alfvén velocity, namely R the kinetic Reynolds number and S the Lundquist number. The nondimensional Ohm's law reads

$$S^{-1}\tilde{\eta}\mathbf{J} = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \tag{9}$$

We use Cartesian coordinates x_1, x_2, x_3 and consider our magnetofluid in the slab $0 < x_1 < 1$. In the x_2 and x_3 directions periodic boundary conditions are applied.

We assume that there is no mass flow and no magnetic flux through the boundary planes, i.e.,

$$v_1 = B_1 = 0 \quad \text{at } x_1 = 0, 1. \tag{10}$$

To allow for nontrivial time-asymptotic states, the system is driven by an electric field of strength E^* in the x_3 direction, which can be prescribed only on the boundary. Condition (10) implies that the tangential components of $\mathbf{v} \times \mathbf{B}$ on the boundary planes vanish, so that according to Eq. (9)

$$J_2 = 0, \quad J_3 = \frac{E^* S}{\tilde{\eta}_b} \quad \text{at } x_1 = 0, 1. \tag{11}$$

$\tilde{\eta}_b$ is the value of $\tilde{\eta}$ on the boundaries. The boundary conditions for the tangential components of $\mathbf{B}$ then become ($\mathbf{J} = \nabla \times \mathbf{B}$ in the nondimensional units)

$$\frac{\partial B_2}{\partial x_1} = \frac{E^* S}{\tilde{\eta}_b}, \quad \frac{\partial B_3}{\partial x_1} = 0 \quad \text{at } x_1 = 0, 1. \tag{12}$$

Stationary state with the fluid at rest are given as solutions of the equations

$$-\nabla p + \mathbf{J} \times \mathbf{B} = \mathbf{0}, \tag{13}$$

$$\nabla \times (\tilde{\eta}\mathbf{J}) = \mathbf{0}. \tag{14}$$

We assume $\tilde{\eta}$ to depend only on the coordinate x_1. Eq. (14) and the boundary conditions are then satisfied with

$$\mathbf{J} = \mathbf{J}^e = (0, 0, \tilde{\eta}^{-1} E^* S), \tag{15}$$

$$\mathbf{B} = \mathbf{B}^e = (0, E^* S I_0(x_1) + \overline{B_2^e}, \overline{B_3^e}), \tag{16}$$

where overbars denote spatial averages and $I_0(x_1) = \int \tilde{\eta}^{-1}\,\mathrm{d}x_1 - \overline{\int \tilde{\eta}^{-1}\,\mathrm{d}x_1}$.

E^* can be formally eliminated by the still free choice of the magnetic field unit, B_0. Let, for instance, $\tilde{\eta}$ be symmetric about the midplane $x_1 = 0.5$, so that in the case of $\overline{B_2^e} = 0$, B_2^e is correspondingly antisymmetric. We have the freedom to choose B_0 in such a way that then $|B_2^e| = 1$ on the boundary planes, and consequently

$$E^* = [SI_0(1)]^{-1}. \tag{17}$$

There is a Lorentz force in the x_1 direction,

$$\mathbf{J}^e \times \mathbf{B}^e = (-B_2^e J_3, 0, 0) = (-B_2^e \frac{\partial B_2^e}{\partial x_1}, 0, 0) \tag{18}$$

and Eq. (13) is satisfied with

$$p = p^e = -\frac{\mathbf{B}^{e2}}{2}. \tag{19}$$

So the quiescent ground state is exactly defined.

We use the decomposition

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{1}{2}\nabla \mathbf{B}^2 \tag{20}$$

and write

$$P = p + \frac{1}{2}\mathbf{B}^2, \quad \mathbf{b} = \mathbf{B} - \mathbf{B}^e, \quad \mathbf{j} = \mathbf{J} - \mathbf{J}^e. \tag{21}$$

$\mathbf{v}$ and $\mathbf{b}$ are our dynamical variables.

The spatial means of v_2, v_3, B_2, and B_3 are independent of time. For the case of spatially uniform η this is shown in Ref. [5], and the arguments given there are easily generalized to the case of nonuniform η. Without loss of generality we restrict ourselves to the case of $\overline{v_2} = \overline{v_3} = 0$, since any mean flow can be removed by a Galilean transformation. The mean values $\overline{B_2}$ and $\overline{B_3}$ are parameters of the equilibrium field.

We now Fourier expand in the x_2 and x_3 directions. Let $v_{i\mathbf{k}}$ and $b_{i\mathbf{k}}$ denote the Fourier coefficients of v_i and b_i for wavenumber $\mathbf{k} = (k_2, k_3)$. Linearizing about the static equilibrium, Eqs. (6) and (7) become

$$
\begin{aligned}
\dot{v}_{1\mathbf{k}} &= -P'_{\mathbf{k}} - R^{-1}(\mathbf{k}^2 - D^2)v_{1\mathbf{k}} + ik_2 B_2^e b_{1\mathbf{k}} + ik_3 \overline{B_3^e} b_{1\mathbf{k}}, \\
\dot{v}_{2\mathbf{k}} &= -ik_2 P_{\mathbf{k}} - R^{-1}(\mathbf{k}^2 - D^2)v_{2\mathbf{k}} + ik_2 B_2^e b_{2\mathbf{k}} \\
&\quad + ik_3 \overline{B_3^e} b_{2\mathbf{k}} + (B_2^e)' b_{1\mathbf{k}}, \\
\dot{v}_{3\mathbf{k}} &= -ik_3 P_{\mathbf{k}} - R^{-1}(\mathbf{k}^2 - D^2)v_{3\mathbf{k}} + ik_2 B_2^e b_{3\mathbf{k}} \\
&\quad + ik_3 \overline{B_3^e} b_{3\mathbf{k}}, \\
\dot{b}_{1\mathbf{k}} &= ik_2 B_2^e v_{1\mathbf{k}} + ik_3 \overline{B_3^e} v_{1\mathbf{k}} - S^{-1}[ik_2 \tilde{\eta} j_{3\mathbf{k}} - ik_3 \tilde{\eta} j_{2\mathbf{k}}], \\
\dot{b}_{2\mathbf{k}} &= ik_2 B_2^e v_{2\mathbf{k}} + ik_3 \overline{B_3^e} v_{2\mathbf{k}} - (B_2^e)' v_{1\mathbf{k}} \\
&\quad - S^{-1}[ik_3 \tilde{\eta} j_{1\mathbf{k}} - (\tilde{\eta} j_{3\mathbf{k}})'], \\
\dot{b}_{3\mathbf{k}} &= ik_2 B_2^e v_{3\mathbf{k}} + ik_3 \overline{B_3^e} v_{3\mathbf{k}} \\
&\quad - S^{-1}[(\tilde{\eta} j_{2\mathbf{k}})' - ik_2 \tilde{\eta} j_{1\mathbf{k}}],
\end{aligned} \tag{22}
$$

where a prime denotes differentiation with respect to x_1 and $D = \partial/\partial x_1$.

Generalizing Squire's transformation of ordinary hydrodynamics [2,3] to the magnetohydrodynamic case, we define

$$
\begin{aligned}
\tilde{k} = (k_2^2 + k_3^2)^{1/2}, \quad \tilde{P}_{\tilde{k}}/\tilde{k} = P_{\mathbf{k}}/k_2 \\
\tilde{k}\tilde{v}_{2\tilde{k}} = k_2 v_{2\mathbf{k}} + k_3 v_{3\mathbf{k}}, \quad \tilde{v}_{1\tilde{k}} = v_{1\mathbf{k}}, \quad \tilde{k}\tilde{R} = k_2 R, \\
\tilde{k}\tilde{b}_{2\tilde{k}} = k_2 b_{2\mathbf{k}} + k_3 b_{3\mathbf{k}}, \quad \tilde{b}_{1\tilde{k}} = b_{1\mathbf{k}}, \quad \tilde{k}\tilde{S} = k_2 S.
\end{aligned} \tag{23}
$$

We can assume $k_2 \neq 0$ here, since modes with $k_2 = 0$ cannot grow and are bound to decay if not in addition D and k_3 vanish. This can be seen from the system (22), where the only driving terms are those with B_2^e (as demonstrated below, a nonvanishing $\overline{B_3^e}$ does not influence growth rates). In the case of $D = k_2 = k_3 = 0$ also dissipation is absent, so the corresponding perturbations are neutrally stable.

Multiplying the first of the equations (22) by $\tilde{k}/k_2$ we have

$$
\begin{aligned}
\frac{\tilde{k}}{k_2}\dot{\tilde{v}}_{1\tilde{k}} &= -\tilde{P}'_{\tilde{k}} - \tilde{R}^{-1}(\tilde{k}^2 - D^2)\tilde{v}_{1\tilde{k}} + i\tilde{k}B_2^e \tilde{b}_{1\tilde{k}} \\
&\quad + \frac{\tilde{k}}{k_2} ik_3 \overline{B_3^e} \tilde{b}_{1\tilde{k}},
\end{aligned} \tag{24}
$$

while the second and the third of these equations can be combined to give

$$\begin{aligned}\frac{\tilde{k}}{k_2}\dot{\tilde{v}}_{2\tilde{k}} = -i\tilde{k}\tilde{P}_{\tilde{k}} - \tilde{R}^{-1}(\tilde{k}^2 - D^2)\tilde{v}_{2\tilde{k}} + i\tilde{k}B_2^e\tilde{b}_{2\tilde{k}} \\ +\frac{\tilde{k}}{k_2}ik_3\overline{B_3^e}\tilde{b}_{2\tilde{k}} + (B_2^e)'\tilde{b}_{1\tilde{k}}.\end{aligned} \tag{25}$$

Similarly, on observing that

$$j_{3\mathbf{k}} = b'_{2\mathbf{k}} - ik_2b_{1\mathbf{k}}, \;\; j_{2\mathbf{k}} = ik_3b_{1\mathbf{k}} - b'_{3\mathbf{k}} \tag{26}$$

and consequently

$$\begin{aligned}k_2j_{3\mathbf{k}} - k_3j_{2\mathbf{k}} &= k_2b'_{2\mathbf{k}} - ik_2^2b_{1\mathbf{k}} - ik_3^2b_{1\mathbf{k}} + k_3b'_{3\mathbf{k}} \\ &= \tilde{k}\tilde{b}'_{2\tilde{k}} - i\tilde{k}^2\tilde{b}_{1\tilde{k}} \\ &= \tilde{k}\tilde{j}_{3\tilde{k}},\end{aligned} \tag{27}$$

one finds from the last three of the equations (22)

$$\frac{\tilde{k}}{k_2}\dot{\tilde{b}}_{1\tilde{k}} = i\tilde{k}B_2^e\tilde{v}_{1\tilde{k}} + \frac{\tilde{k}}{k_2}ik_3\overline{B_3^e}\tilde{v}_{1\tilde{k}} - \tilde{S}^{-1}i\tilde{k}\tilde{\eta}\tilde{j}_{3\tilde{k}} \tag{28}$$

and

$$\begin{aligned}\frac{\tilde{k}}{k_2}\dot{\tilde{b}}_{2\tilde{k}} = i\tilde{k}B_2^e\tilde{v}_{2\tilde{k}} + \frac{\tilde{k}}{k_2}ik_3\overline{B_3^e}\tilde{v}_{2\tilde{k}} - (B_2^e)'\tilde{v}_{1\tilde{k}} \\ +\tilde{S}^{-1}(\tilde{\eta}\tilde{j}_{3\tilde{k}})'.\end{aligned} \tag{29}$$

Now let first $\overline{B_3^e} = 0$. Then up to the factor $\tilde{k}/k_2$ on the left-hand sides, which can be removed by the additional transformation $\tilde{t} = (k_2/\tilde{k})t$, the equations (24), (25), (28) and (29) have the same mathematical form as the system (22) with the x_3 dependence dropped, $v_3 = b_3 = 0$ and $k_2 = \tilde{k}$. And if λ is an eigenvalue of the linear operator on the right-hand side of the original system (22), then $(\tilde{k}/k_2)\lambda$ is an eigenvalue of the linear operator on the right-hand side of the system (24), (25), (28), (29) (to see this replace on the left-hand side of the system (22) $\dot{v}_{i\mathbf{k}}$ and $\dot{b}_{i\mathbf{k}}$ by $\lambda v_{i\mathbf{k}}$ and $\lambda b_{i\mathbf{k}}$ and apply the manipulations described). Thus, if there is an unstable eigenmode of the original system, i.e. an eigenvalue λ with positive real part (giving the growth rate of the mode), then, since $\tilde{k}/k_2 \geq 1$, there is an unstable eigenmode of the derived two-dimensinal system with at least the same growth rate; if the eigenmode of the original system is really three-dimensional, that is $k_3 \neq 0$, then the eigenmode of the two-dimensional system grows indeed faster. Furthermore, and most important, $\tilde{R} \leq R$ and $\tilde{S} \leq S$. That is, if the Reynolds numbers are raised from small values, two-dimensional perturbations become unstable first.

To complete the proof, we note that also the conditions (3), which take the form

$$v'_{1\mathbf{k}} + ik_2 v_{2\mathbf{k}} + ik_3 v_{3\mathbf{k}} = 0, \tag{30}$$

$$b'_{1\mathbf{k}} + ik_2 b_{2\mathbf{k}} + ik_3 b_{3\mathbf{k}} = 0 \tag{31}$$

in Fourier space, are satisfied for the derived two-dimensional system: One finds

$$\tilde{v}'_{1\tilde{k}} + i\tilde{k}\tilde{v}_{2\tilde{k}} = 0, \;\; \tilde{b}'_{1\tilde{k}} + i\tilde{k}\tilde{b}_{2\tilde{k}} = 0. \tag{32}$$

A nonvanishing $\overline{B^e_3}$, finally, leads to Alfvénic oscillations, but does not influence the growth rates of unstable modes. This is most easily seen if Elsässer variables $\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{b}$ are used. If the system (22) is tranformed to these variables, on the right-hand sides of the equations for the $\dot{z}^{+}_{i\mathbf{k}}$ the term $+ik_3\overline{B^e_3}z^{+}_{i\mathbf{k}}$ appears, while there is a term $-ik_3\overline{B^e_3}z^{-}_{i\mathbf{k}}$ on the right-hand sides of the equations for the $\dot{z}^{-}_{i\mathbf{k}}$. Now let, for prescribed initial conditions at $t = 0$, $[\mathbf{z}^{+}_{\mathbf{k}}, \mathbf{z}^{-}_{\mathbf{k}}]$ be the solution for the case with $\overline{B^e_3}$ set equal to zero. Then the solution for nonvanishing $\overline{B^e_3}$ is given by $[\mathbf{z}^{+}_{\mathbf{k}} \exp\{ik_3\overline{B^e_3}t\}, \mathbf{z}^{-}_{\mathbf{k}} \exp\{-ik_3\overline{B^e_3}t\}]$, that is, the solution is merely modulated by an oscillation with frequency $k_3\overline{B^e_3}$. This applies equally to the solutions of the system (24), (25), (28), (29).

In conclusion we note that, though for increasing Reynolds numbers the equilibrium becomes first unstable to two-dimensional perturbations, this does not yet imply that also the bifurcating new time-asymptotic states are two-dimensional. Numerical simulations show that the initial growth of a two-dimensional perturbation can be followed by an evolution towards a three-dimensional final state [7]. It is not clear yet, however, whether the quiescent ground state loses its stability directly to three-dimensional (final) states or whether two-dimensional states with flow are stable in certain Reynolds number intervals close to the primary bifurcation point.

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REFERENCES

[1] H. P. Furth, J. Killeen, and M. N. Rosenbluth, Phys. Fluids **6**, 459 (1963).

[2] H. B. Squire, Proc. R. Soc. London Ser. A **142**, 621 (1933).

[3] T. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge University Press, Cambridge, England, 1981).

[4] R. B. Dahlburg and J. T. Karpen, J. Geophys. Res. **100**, 23489 (1995).

[5] N. Seehafer, E. Zienicke, and F. Feudel, Phys. Rev. E **54**, 2863 (1996).

[6] R. B. Dahlburg, T. A. Zang, D. Montgomery, and M. Y. Hussaini, Proc. Natl. Acad. Sci. USA **80**, 5798 (1983).

[7] R. B. Dahlburg, S. K. Antiochos, and T. A. Zang, Phys. Fluids B **4**, 3902 (1992).