Synchronization in Active Networks

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Tiago Pereira
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Fainting with longing for my hearts own desire, my wife Daniele.

Contents

1 Introduction

The Newtonian determinism states that the present state of the universe determines its future precisely. The belief in the Newton determinism was well summarized by Laplace: *We ought to regard the present state of the universe as the effect of the past and the cause of the future* . This credo was based on Newton's equation of motion, which has the property that initial conditions determine the solutions forward and backward in time.

The works of Poincaré, Birkhoff, Smalle and others, and consequently their legacy on dynamics, have shown that many believes on the natural phenomena can be misguided. The deeper analysis of the equations underlying Newton laws shed a light in the prediction of long term behavior of dynamical systems. Many systems in nature present a sensitive dependence on the initial conditions; orbits of typical nearby points move away exponentially fast under the evolution of the dynamics. Therefore, the prediction of the future dynamics is impossible for large time intervals. Such a dynamical behavior characterizes deterministic chaos. Chaotic behavior has been extensively studied in several areas of physical sciences [1, 2, 3, 4], economy [5], ecology [6] and applied engineering [7].

In nature, though, one commonly finds interacting chaotic oscillators that through the coupling scheme form small and large networks, e.g., neural networks. Surprisingly, even though chaotic systems possess an exponential divergency of nearby trajectories, they can synchronize, still preserving the chaotic behavior [8, 9, 10]. The emergency of collective behavior among interacting systems is a rather common phenomenon being found in many branches of science [11, 12, 13, 14], as ecology [15], neuroscience [16, 17], and lasers [18, 19].

Synchronization ought to imply a collapse of the overall evolution onto a subspace of the system attractor, reducing the dimensionality of the system. That is, one is able to understand the dynamics of one oscillator by means of the other. Synchronization can be enhanced at different levels, that is, the constraints on which the synchronization appears. Those can be in the trajectory amplitude, requiring the amplitudes of both oscillators to be equal, giving place to complete synchronization. Conversely, the constraint could also be in a function of the trajectory, e.g. the phase, giving place to phase synchronization (PS). In this case, one requires the phase difference between both oscillators to be finite for all times, while the trajectory amplitude may be uncorrelated. Whereas the former case requires relatively strong coupling strengths, the latter can arise for very small coupling strengths. PS is relevant to important technological problems such as communication with chaos [20, 21], new insights into the collective behavior in networks of coupled chaotic oscillators [22, 17], pattern formation [23, 12], Parkinson disease [24], epilepsy [25], as well as behavioral activities [26].

In this work, we have analyzed the phenomenon of phase synchronization showing that: (i) for a broad class of attractors a phase based on the tangent field can be introduced having a physical meaning, i.e., it gives the correct average period; (ii) for a given phase definition, the upper bound for the absolute phase difference can be calculated, and is equal to the product of the average increasing of the phase in typical cycles with average period; (iii) for a class of oscillators endowed with proper rotations, we have proven that, in PS regimes, observations of the trajectory of one subsystem at specific times when the trajectory of the other subsystem crosses a Poincaré section gives places to localized sets in the state-space of the observed subsystem; (iv) the latter approach can be generalized to a much broader class of attractors, which has no coherent motion and no proper rotation. We achieve this results by demonstrating that the observations of trajectory of one subsystem can be done by means of any typical physical event occurrence. In PS regimes these observations give place to a localized set. (v) We demonstrate that PS is invariant under time-coordinate changes. (vi) Finally, we analyze a scenario of torus breakdown via a global bifurcation giving place to a new transition to chaos. We perform a detailed experiment and numerical investigations

2 Synchronization

Time plays a major role for biological and physical systems. Their dynamical behavior is governed by cycles of different periods which determines their intrinsic activity. There are a variety of physical and biological processes which require precise timing between oscillators for a proper functioning [15, 16]. A phenomenon able to provide such a timing is synchronization [8, 9, 10, 11, 12, 13, 14]. Several types of synchronization may arise depending on the nature of the oscillator and on the coupling properties. Given two identical oscillators $\mathbf{x}_1$ and $\mathbf{x}_2$ properly coupled, for strong enough coupling strengths complete synchronization can be achieved. This means that both trajectories present the same behavior:

$$\lim_{t\to\infty} |\mathbf{x}_1(t) - \mathbf{x}_2(t)| = 0. \tag{1}$$

Synchronization in this case is associated with a transition of the largest transverse Lyapunov exponent of the subspace $\mathbf{x}_1 = \mathbf{x}_2$, also known as synchronization manifold, from positive to negative values. In general, complete synchronization is only possible if the interacting oscillators are identical. If, however, they present a mismatch parameter the states can be close $\mathbf{x}_1 \approx \mathbf{x}_2$, but not equal.

For nonidentical oscillators other types of synchronization can appear, for example the lag synchronization, in which the trajectories of $\mathbf{x}_1$ and $\mathbf{x}_2$ present the same behavior unless a lag τ in time, which means:

$$\lim_{t\to\infty} |\mathbf{x}_1(t+\tau) - \mathbf{x}_2(t)| \approx 0. \tag{2}$$

A more complex type of synchronization is the generalized synchronization, where $\mathbf{x}_2$ presents the same behavior of $\mathbf{x}_1$ after being transformed by a function ψ:

$$\mathbf{x}_2 = \psi(\mathbf{x}_1), \tag{3}$$

Note that complete synchronization is a particular case of generalized synchronization where $\psi = \mathbf{1}$. One should carefully state what requirements ψ must fulfill. The main idea, however, is that the entrance in a small ϵ_1-ball in $\mathbf{x}_1$ implies the entrance of $\mathbf{x}_2$ in a small ϵ_2-ball. This means that neighborhoods in $\mathbf{x}_1$ are mapped into neighborhoods in $\mathbf{x}_2$, which implies the collapse of the solution to a subspace of the full attractor. Therefore, we are able to predict $\mathbf{x}_2$ by knowing $\mathbf{x}_1$ and ψ only.

While generalized synchronization, in general, requires relatively large coupling strength, since the synchronization manifest itself in the trajectories for a small coupling strength another kind of synchronization may arise, the phase synchronization. In such a case, the trajectories can be uncorrelated while the phase dynamics are synchronized. Denoting, $\vartheta_{1,2}(t)$ the phase of $\mathbf{x}_{1,2}$ the condition for phase synchronization is given by:

$$|\vartheta_1(t) - \vartheta_2(t)| \leq \varrho, \tag{4}$$

where ϱ is a finite real number. The next problem is how to define a phase. Despite of the large interest of the community driven by the large number of application with PS, there are still many open questions in the field, namely : (*i*) Which are the properties that a function must fulfill to be consider a phase? (*ii*) What is the boundedness condition for the phase difference, in other words, the upper bound for the phase difference? (*iii*) Is it always necessary to define a phase to measure PS? (*iv*) What is the relation between PS and communication in networks? (*v*) Can a time-coordinate change destroy PS? Some of these questions have been addressed, namely (*iii*) has been addressed with recurrence plots techniques [27], and (*iv*) by relating the mutual information between oscillators and their conditional Lyapunov exponent [28]. In this work, we address all these questions by exploring the natural link between synchronization and recurrence.

3 Phase of attractors

While in an autonomous nonlinear oscillator perturbations in the trajectory may grow or shrink, perturbations in the phase neither grow nor shrink, due to the fact that phase is associated with the zero Lyapunov exponent [12]. Therefore, one expect the phase to exist for a general oscillator. However, there is no general and unambiguous phase definition for a general attractor. Thus, we ought to study classes of compact attractors which have suitable properties, such that the notion of phase can be developed. We shall highlight two of them given by the two following definitions.

Definition 1 *Let $\mathcal{A}$ be a compact attractor. $\mathcal{A}$ is said to have a proper rotation if its trajectory has a defined direction of rotation (i.e. either clockwise or counterclockwise) and an unique center of rotation.*

Definition 2 *Assume that the compact attractor $\mathcal{A}$ admits a phase Φ. Let t^i be the ith return time of the trajectory to the section $\Phi = 0$. Then, $\mathcal{A}$ is said to be coherent if:*

$$|t^i - i\langle T\rangle| < \eta \ll \langle T\rangle, \tag{5}$$

where $\langle T\rangle$ is the average return time to the section $\Phi = 0$.

For a class of attractors endowed with coherent properties, it is possible to transform the original equation of motion to a new equation of motion that carries the information of the radius and the phase [29]. This result is given by:

Theorem 1 *Let the system $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$, where $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{G} : \mathbb{R}^m \to \mathbb{R}^m$, have a compact attractor $\mathcal{A}$ on which a T-periodic phase coordinate Φ is defined. Assume $\mathbf{G}$ to be differentiable and $\dot{\Phi} > 0$. Then, for any $\epsilon > 0$ there exist a coordinate change $\Phi \to \vartheta$ in a neighborhood N of $\mathcal{A}$ such that ϑ is T periodic and*

$$\begin{aligned} \dot{\mathbf{R}} &= \mathbf{G}(\mathbf{R}, \vartheta), && (6)\\ \dot{\vartheta} &= 1 + \delta(\mathbf{R}, \vartheta) && (7) \end{aligned}$$

where $\mathbf{R} : N \to \mathbb{R}^{m-1}$ and $\vartheta : N \to S^1$, such that $(\mathbf{R}, \vartheta)$ are the new coordinates, and $|\delta| < \eta + \epsilon$, except for ϑ in a set of measure less than η, where η is given by Eq. (5).

Unfortunately or fortunately, the phase and the radius coordinates are not unique, which means that one can consistently define more than one phase for $\mathcal{A}$. Indeed, if ϑ and $\tilde{\vartheta}$ are two phase definitions for the attractor $\mathcal{A}$, so that, $\dot{\vartheta} = 1 + \delta(\mathbf{R}, \vartheta)$, and $\dot{\tilde{\vartheta}} = 1 + \tilde{\delta}(\tilde{\mathbf{R}}, \tilde{\vartheta})$, for a sufficiently coherent attractor they give equivalent results. Indeed, note that:

$$\vartheta - \tilde{\vartheta} = \int_0^t [\delta(\mathbf{R}, \vartheta) - \tilde{\delta}(\tilde{\mathbf{R}}, \tilde{\vartheta})]dt. \tag{8}$$

Then, $|\vartheta - \tilde{\vartheta}| \leq |\int_0^t \delta(\mathbf{R}, \vartheta)dt| + |\int_0^t \tilde{\delta}(\tilde{\mathbf{R}}, \tilde{\vartheta})dt|$, therefore, we get: $|\vartheta - \tilde{\vartheta}| \leq 2Tmax(|\delta(\mathbf{R}, \vartheta)|, |\tilde{\delta}(\tilde{\mathbf{R}}, \tilde{\vartheta})|)$. The term $max(|\delta_1(\mathbf{R}, \vartheta_1)|, |\delta_2(\tilde{\mathbf{R}}, \vartheta_2)|)$ can be made small enough.

3.1 General phase definition

The former theorem guarantees the existence of a phase coordinate. However, as we have discussed this phase coordinate is not unique. Indeed, many phases have been introduced, for example: (*i*) the phase θ based on the angular displacement of the vector position; (*ii*) the phase ϕ based on the angular displacement of the vector velocity; (*iii*) the phase introduced by Hilbert transformer [12], (*iv*) the phase given by the interpolation between events where the phase is assumed to increase 2π.

As a consequence, a question is raised: *Which could be a general phase for a compact attractor?* Up to now, it seems to be hopeless the efforts to answer this questions. However, we have pursued a positive answer to some classes of attractors. Since we want to construct an approach exclusively dependent on the equations of motion we only consider the phases which are vector-field-dependent, e.g., ϕ and θ.

In our work *Phase and average period of chaotic oscillators* [30] we analyze which phase could be regarded as the most general one. We have done some contributions towards this direction:

1. Using basic concepts of differential geometry, we have analyzed the geometrical meaning of the phase ϕ. We showed that ϕ is equal to the length of the Gauss map, the generator of the curvature in differential geometry. Such a phase definition can be interpreted as follows: the center of rotation is the trajectory itself. Thus, given the trajectory at a time $t+\Delta t$ the center of rotation is the trajectory at a time t. Therefore, one avoids the need of a proper rotation

2. We demonstrate, for attractors with proper rotations, that PS is invariant under the phase definition. Moreover, we discuss to which classes of oscillators the defined phases can be used to calculate quantities as the average frequency and the average period of oscillators.

3. Since the phase ϕ allows negative frequencies, it is not an one-to-one function with the trajectory. We overcome this problem by introducing a phase ψ, where $\dot{\psi} = |\dot{\phi}|$.

As a consequence of the positiveness of ψ its average increasing per cycle might be bigger than 2π. However, we have shown that, for oscillators with proper rotations, this deviation can be obtained from an equation, which allows the use of ψ as well.

4. *Homo(hetero)clinic Chaotic Attractor:* For such attractors the phases ϕ and ψ cannot be used to calculate the average period. That is so, due to the fact that the trajectory of these attractors gets arbitrarily close to the "rest" state, i.e. near the unstable homoclinic point. The phase depends on the derivatives of the trajectory which vanishes in the homoclinic points causing the phase to diverge, misleading the results. We have shown that this problem can be overcome by a translation of the attractor on the velocity space $(\dot{x}, \dot{y})$, such that after the translation the trajectory has a proper rotation in the velocity space.

4 Upper bounds for phase synchronization

PS implies that the phase difference is bounded, that is, there is a number $\varrho > 0$ such that the phase difference is always smaller than this number. In order to detect PS, one must analyze the boundedness condition in the phase difference. The main difficulties rely on the fact that the phase definition is not general and ϱ is arbitrary. This means that after introducing a phase and given threshold c, which bounds the phase difference, and a number $\alpha > 1$, $\varrho = \alpha \times c$ also bounds the phase difference. In computer and lab experiments one wishes to know the upper bound of the phase difference c, so, computation time in the PS detection can be saved.

Therefore, the natural questions is whether one could estimate the smaller value of the number c that bounds the phase difference given a phase definition. Since there are many phase definitions, one should estimate the minimum bound for a given phase definition.

For weak coherent attractors, that is, *coherent attractors disregarding its topology and the number of time-scales; given that at least one time-scale is coherent*, it is possible to define an event, such that the time t^i at which the ith event occurs is coherent. Note that t^i is the return times for a given time-scale. Therefore, given two weak coherent oscillators, namely $\mathbf{x}_1$ and $\mathbf{x}_2$, we have $|t_1^i - i\langle t_1\rangle| \ll \langle T_1\rangle$ and $|t_2^i - i\langle T_2\rangle| \ll \langle T_2\rangle$. If the two oscillators are in PS, then, $\langle T_1\rangle = \langle T_2\rangle$, and we have: $|t_1^i - t_2^i| \ll \langle T_{1,2}\rangle$. This condition guarantees that the number of events can differ at most by one, either both events occur at the same time, or one event occurs shortly after the other. Taking this into account we were able to estimate the bound constant c for the given phase definition

In our work *Upper Bounds for Phase Synchronization in Weak Coherent Attractors* [31] we have given an upper bound value for the absolute phase difference in Eq. (4), in terms of a defined phase. We have shown that this upper bound value $\langle r\rangle$ is the average growing of the phase per cycle in one of the subsystems. Particularly, $\langle r\rangle = \langle W\rangle \times \langle T\rangle$, where $\langle W\rangle$ is the average angular frequency associated to a subsystem $\mathbf{x}_1$, and $\langle T\rangle$ is the average returning time of trajectories in this same subsystem, calculated from the recurrence of events of the chaotic trajectory.

5 Phase synchronization detection

In order to state the existence of PS, one has to introduce a phase $\phi(t)$ for the chaotic oscillator, what is not a straightforward task. Therefore, approaches to PS detection that overcome the need of a phase are required if one wants to detect PS to a general oscillator. First, we focus our attention on attractors whose phases are well defined, namely attractors with proper rotations and coherent motion, to develop a technique to PS detection that does not require explicitly a phase.

There is an interesting approach, for periodically driven oscillators, very useful and easy to implement that overcomes the need of a phase, the stroboscopic map technique. It consists in sampling the chaotic trajectory at times nT_0, where n is an integer and T_0 is the period of the driver. The stroboscopic map was used to detect PS [12, 32, 33]. The basic idea is that if the stroboscopic map is localized in the attractor, there is PS.

Inspired by the stroboscopic map technique we developed its generalization to coupled chaotic oscillators. The stroboscopic map is generalized to the **Conditional Poincaré Map**. Given two oscillators the conditional Poincaré map is constructed by collecting points in one oscillator at the moment at which the other completes one period. Note that for an attractor with proper rotation this is equivalent to observe one subsystem whenever the other crosses a Poincaré section. *If the set generated by this conditional Poincaré map does not visit any arbitrary region of the subsystem, then, there is PS.*

This generalization not only allows us to detect PS in coupled chaotic oscillator, without the introduction of a phase, but also allows us to explore the natural link between synchronization and the ergodic properties of one parameter family of transformations. In our work *Non-Transitive Maps in phase synchronization* [34] we have done the following contributions:

1. We formally introduce the conditional Poincaré map and the sets generated by it. We show that phase synchronization induces the lost of transitiveness of the conditional Poincaré map in the attractor of the subsystem.

2. We show that for coherent dynamics the localized sets exist if, and only if, there is PS. In other words, PS implies the existence of localized sets and vice-versa.

3. We illustrate how the conditional Poincaré map can be used to detect PS, without actually having to measure the phase, in the forced Chua's Circuit, experimentally and numerically, and in the coupled Rössler oscillator.

6 General framework for phase synchronization detection

The results concerning the conditional Poincaré Map, as introduced in the previous section, hold only for oscillators with proper rotations and coherent motion. That is so, because the conditional Poincaré map requires the definition of a particular Poincaré section, in which the trajectory crosses only once per cycle, such that the phase can be meant to increase 2π

between the crossings. The definition of such a section is always possible for attractors with proper rotation. However, there is a broad class of oscillators that does not have a proper rotation, and such sections cannot be defined. Therefore, a generalization of the conditional Poincaré Map is needed.

In our works *Detecting phase synchronization by localized maps: Application to neural networks* [35] and *General Framework for Phase Synchronization through localized sets* [36], we have extended our previous results by demonstrating that localized sets can be constructed while in PS by means of the observation of *any typical* event, which has a strong impact in the field of experimental physics, since in the laboratory measurements are restricted to the limitations of the experiment. Our results demonstrate that for two coupled oscillators $\mathbf{x}_1$ and $\mathbf{x}_2$, if one defines a typical event in an oscillator $\mathbf{x}_1$ and then observes the oscillator $\mathbf{x}_2$ whenever this event occurs, these observations give rise to a localized set in the accessible phase space of $\mathbf{x}_2$, if PS exists. We apply our theory to investigate the mechanism for losing PS in oscillators that possess multiple time-scales as well as the PS onset in networks of such oscillators. Finally, we relate the localized sets from our theory to the information exchange between coupled chaotic oscillators. These topics are further described in more details.

6.1 Lost of localization

We have shown that the lost of PS is caused by the existence of non-locked *unstable periodic orbits* (UPOs). This result can be proven by means of the localized sets. First, one defines an event, e.g. the entrance in the ϵ-ball, close enough to an UPO in $\mathbf{x}_1$, and in $\mathbf{x}_2$ one chooses an initial condition close enough to another UPO. The dynamics of both chaotic flows are governed by the UPOs for a time inversely proportional to the largest eigenvalue of the UPOs. By observing the oscillator $\mathbf{x}_2$ whenever the event in $\mathbf{x}_1$ happens (we have the same dynamics as observing $\mathbf{x}_2$ every period of the UPO of $\mathbf{x}_1$), our results show that if the UPOs are unlocked the observations spread over the UPO of $\mathbf{x}_2$, implying that the sets are nonlocalized. This result corroborates the numerical analysis of Ref. [12].

6.2 PS in bursting neurons connected via chemical synapses

Bursting neurons present naturally a non-coherent dynamics due to the existence of two time-scales provided by the bursting and spiking dynamics. As we coupled them via a chemical inhibitory synapses, they undergo to an even more non-coherent state, due to the competition between the multi-scale dynamics. Therefore, it is unclear how to introduce a phase.

The majority of approaches to detect PS is based on the increasing in the phase of 2π between two bursts or between two spikes. Moreover, when chemical synapses are used the bursts and spikes amplitudes are modulate by the coupling. Thus, it is rather difficult to define a threshold to detect the burst or spike occurrence. In general, bursts are missed, which can mislead the detection of PS by using approaches that are threshold dependent.

Since our approach is not threshold dependent we can analyze the onset of PS in both scales, the bursting and the spiking scales. This is done by defining a threshold in the membrane potential in one neuron, and then, observing the other whenever the membrane potential reaches the threshold. In our work *Onset of Phase Synchronization in Neurons Connected via Chemical Synapses* [37] we have shown that PS is a common behavior in Hindmarsh-Rose neurons with inhibitory chemical synapses.

6.3 Clusters of phase synchronization in networks

The ideas introduced herein are also useful to analyze the onset of synchronization in neural networks. We consider networks up to 100 non-identical HR neurons coupled via excitatory synapses. Our numerical analysis has shown that it is possible to achieve clusters of PS for parameters far smaller than the ones needed to achieve PS in the whole network. We have found that clusters of PS appear for 10% of the coupling strength necessary to have PS in the whole net.

Such clusters may offer a suitable environment for information exchanging mainly for two reasons: (*i*) each cluster may be used to transmit information in a particular bandwidth, which may provide a multiplexing processing of information. (*ii*) They provide a multichannel communication, that is, one can integrate a large number of neurons (chaotic oscillators) into a single communication system, and information can arrive simultaneously at different places of the network. This scenario may have technological applications, e.g. in digital communication [20, 21], and it may also guide us towards a better understanding of information processing in real neural networks [24, 25, 26].

6.4 Communication and localized sets

We have also analyzed the relationship between the localized sets and the capacity of information transmission between chaotic oscillators. The amount of information that two oscillators $\mathbf{x}_1$ and $\mathbf{x}_2$ can exchange is given by the mutual information $I(\mathbf{x}_1, \mathbf{x}_2)$ [38]:

$$I(\mathbf{x}_1, \mathbf{x}_2) = H(\mathbf{x}_1) - H(\mathbf{x}_1|\mathbf{x}_2), \tag{9}$$

where $H(\mathbf{x}_1)$ is the entropy of the oscillator $\mathbf{x}_1$ and $H(\mathbf{x}_1|\mathbf{x}_2)$ is the conditional entropy between $\mathbf{x}_1$ and $\mathbf{x}_2$, which measures the ambiguity of the received signal, roughly speaking the errors in the transmission.

As pointed out in Ref. [28] the mutual information can also be estimated through the conditional exponents associated to the synchronization manifold. Thus, the mutual information is given by:

$$I(\mathbf{x}_1, \mathbf{x}_2) = \sum \lambda_{\parallel}^{+} - \sum \lambda_{\perp}^{+} \tag{10}$$

where $\lambda_{\parallel}^{+}$ are the positive conditional Lyapunov exponents associated to the synchronization manifold, the information produced by the synchronous trajectories, and $\lambda_{\perp}^{+}$ are the positive conditional Lyapunov exponents transversal to the synchronization manifold, related with the errors in the information transmission. In general, we expect $\sum \lambda_{\parallel}^{+} \leq \sum \lambda^{+}$, where λ^{+} are the positive Lyapunov exponents. Thus $I(S, R) \leq \sum \lambda^{+} - \sum \lambda_{\perp}^{+}$. In order to estimate an upper bound for $I(S, R)$, we need to estimate $\lambda_{\perp}^{+}$, what can be done directly from the localized sets.

The conditional transversal exponent can be estimated from the localized sets by a simple geometric analysis. Let t_1^i be the time at which the oscillator $\mathbf{x}_1$ reaches the Poincaré plane at x_1^* while the oscillator $\mathbf{x}_2$ is at $\mathbf{x}_2^i = \mathbf{x}_2(t_1^i)$. Conversely t_2^i is the time at which the oscillator $\mathbf{x}_2$ reaches the Poincaré plane at x_2^* while the oscillator $\mathbf{x}_1$ is at $\mathbf{x}_1^i = \mathbf{x}_1(t_2^i)$. Our analysis has shown that the local transversal exponent is given by:

$$\lambda_{\perp} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{|t_1^i - t_2^i|} \ell n \left| \frac{\mathbf{x}_1^* - \mathbf{x}_2^i}{\mathbf{x}_2^* - \mathbf{x}_1^i} \right| , \tag{11}$$

where we use the convention $0 \times log 0 = 0$. Of course, we only estimate a local conditional exponent, close to the defined event. Using this approach, we have shown in our paper [39] that if two neurons are synchronous in the bursting scale, information about one neuron can be retrieved using both the bursting and spiking scale with minimal amounts of errors

7 Phase synchronization is time coordinate invariant

Properties of synchronization between dynamical systems with a reparametrizable time-coordinate are important not only in physical theories without an absolute time, but also in biological oscillators which have their own intrinsic time, and technological applications that require their own time coordinate. After a time coordinate change a fundamental question arises: *Can synchronization be destroyed or can one suitablely define a new time where the timing properties are even more precise?* Time reparametrizations cause no change in the topological dynamics, the direction of the flow does not change and the paths of the orbits remain unaltered, but the duration of the cycles can be drastically modified.

In our paper *Phase Synchronization is time coordinate invariant* [40] we have investigated the effect of general time reparametrizations on the phase synchronization phenomenon. We have shown that for general dynamical oscillators it is not possible either to introduce or to lose PS through such a transformation. Furthermore, we have discussed possible application of these ideas to important technological problems such as nonlinear digital communication [20, 21]. Moreover, we have illustrated these results namely in unsynchronized oscillators, showing that the enhancement of zero phase diffusion does not enhance PS. In a second example we showed that breaking the boundedness condition imposed on λ PS can be enhanced. However such a transformation has no physical meaning. Finally, in our last example we have shown that the time reparametrization can introduce the presence of distinct time scales, which can feign PS.

8 Torus destruction via global bifurcations

Nonlinear phenomena are abundant in nature. The behavior of a nonlinear oscillator may change as we vary a control parameter; the oscillator undergoes a bifurcation, causing a change in the topological picture of the solution in the phase space, thus altering the dynamics. Bifurcations can be local (changing only locally the behavior of the oscillator in the phase space) or global (changing the behavior of phase space as a whole). The understanding of the bifurcation scenario for an oscillator is particularly important for the characterization of its dynamical behavior. Surprisingly, there are just a few bifurcation scenarios in which a nonlinear oscillator initially presenting a periodic behavior can undergo transition to chaos. Among the routes to chaos are the bifurcations of a quasi-periodic attractor with 2 incommensurate frequencies, also known as torus T^2.

Curry and Yorke [41] showed that chaos, with broadband spectra, could emerge directly from the destabilization of a torus T^2. This scenario is characterized by a T^2 born from the destabilization of a stationary solution, followed by a successive series of phase-lockings and, eventually, the onset of chaos. The emergence of chaotic behavior is generally associated to the appearance of a localized folded and wrinkled structure, as reported in laser systems [42] and in line-coupled diodes [43]. A frequency characterization of this type of chaos shows that the frequencies present before the torus breaks are still dominant in the chaotic trajectory. A metric characterization shows the appearance of an extremely small and positive Lyapunov exponent.

Recently, it was shown the appearance of chaotic behavior by a transition directly from the torus T^2[44, 45], in which there is a collision of the torus with saddle points originating a type-II intermittency [46]. In this scenario, there is a mechanism of reinjection associated to a heteroclinic connection between the central unstable focus and the saddles, which takes the trajectory back to the vicinity of the focus. This trajectory evolves in a laminar fashion around the focus towards the saddles. It behaves chaotically in the saddles vicinity, and far away from the region delimited by these saddles, returning to the focus along the heteroclinic orbit. This alternance between this type of laminar and chaotic behavior characterizes the type-II intermittency.

The breakdown of the torus T^2 by a torus-saddle collision differs from the Curry-Yorke in the sense that it is associated to the appearance of many frequencies, where the old frequencies no longer play an important role. Additionally, a global bifurcation creates a robust heteroclinic cycle with the appearance of a very large positive Lyapunov exponent. As reported by Lethelier et al.[47], this phenomenon is not only restricted to continuous-time systems but it happens also in spatio-temporal systems.

In our works *Global Bifurcation Destroying The Experimental Torus* T^2 [48] and *A Scenario for Torus* T^2 *Destruction via a Global Bifurcation* [49], we have presented an experimental and numerical analysis to pursue a detailed picture of this scenario. We have shown mainly the following points: (*i*) The existence of the saddle points. They are experimentally observed by introducing a perturbation in the circuit, at a moment that the saddle points are close to the T^2 torus. They are numerically detected by the method introduced in [48]; (*ii*) The existence of the focus points. (*iii*) The manifolds of the saddle points. We also show evidences of the heteroclinic orbit between the saddle points and the focus, by iterating points at the unstable manifold of the saddle and verifying that they eventually are mapped in around the focus; (*iv*) The Fourier and the Lyapunov spectra suffer a sudden transition at the moment of the torus collision. (*v*) Verification of a power scaling law for the average laminar length, $\langle T \rangle$ with respect to the distance between the parameter, f_p (perturbing frequency), and the critical parameter, f_c, where the bifurcation takes place. We find that $\langle T \rangle \propto |f - f_c|^{-\mu}$, with $\mu = 0.96 \pm 0.05$ (experimentally) and $\mu = 0.98 \pm 0.08$ (numerically).

9 Conclusions

In this work we have carried out a dynamical study of synchronization. Through theoretical and numerical analysis we have pursued a general framework for phase synchronization analysis. Our main achievements are the following:

1. The analysis of a phase based on the tangent field. We have shown that such a phase has a physical meaning, i.e., it gives the correct average period of an oscillator.

2. We have shown that, for a given phase definition, the upper bound for the absolute phase difference can be calculated, and it is equal to the product of the average increasing of the phase in typical cycles with the average period.

3. We have shown that in PS regimes the observations of the trajectory of one subsystem, done by means of any typical physical event occurrence, generate a localized set. This set can be used to state the present of synchronization and to estimate the synchronization level. Moreover, we have analyzed the relationship between such sets and the amount of information transmission that can be exchange between coupled oscillators.

4. We have demonstrated that PS is invariant under time coordinate changes. Such a result may shed light into the synchronization analysis of physical systems, in which a real time is not an accessible information.

Finally, we have analyzed a scenario of torus breakdown giving place to a new transition to chaos. We have performed a detailed experiment and numerical investigations in order to characterize this scenario.

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