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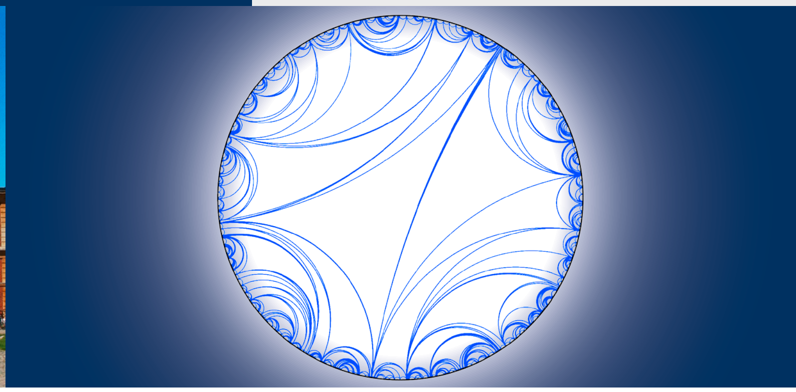

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A Radó theorem for the porous medium equation

Dmitry Fedchenko and Nikolai Tarkhanov

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This paper is dedicated to Lev Aizenberg on the occasion of his 80 *th birthday.*

Abstract We prove that if u is a locally Lipschitz continuous function on an open set $\mathcal{X} \subset \mathbb{R}^{n+1}$ satisfying the nonlinear heat equation $\partial_t u = \Delta(|u|^{p-1}u)$, $p > 1$, weakly away from the zero set $u^{-1}(0)$ in $\mathcal{X}$, then u is a weak solution to this equation in all of $\mathcal{X}$.

Keywords Quasilinear equations · removable sets · porous medium equation

Mathematics Subject Classification (2000) 35J60 · 35B60 · 31C45 · 30C62

Contents

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Introduction

The problem under study lies in the following. Suppose A is a nonlinear differential operator on an open set $\mathcal{X}$ in $\mathbb{R}^{n+1}$ and $\mathcal{S}$ is a closed subset of $\mathcal{X}$. For a given class $\mathcal{F}$ of functions on $\mathcal{X} \setminus \mathcal{S}$, the set $\mathcal{S}$ is said to be removable for $\mathcal{F}$ with respect to A if each function $u \in \mathcal{F}$ satisfying $A(u) = 0$ on $\mathcal{X} \setminus \mathcal{S}$ extends to a solution of this equation on the whole set $\mathcal{X}$. What balance between the growth of functions in $\mathcal{F}$ near $\mathcal{S}$ and the "smallness" of $\mathcal{S}$ is sufficient in order that $\mathcal{S}$ be removable for $\mathcal{F}$ relative to A?

The first result of this type is perhaps the Riemann theorem on the removability of one-point singularities for bounded holomorphic functions. For linear differential operators with C^∞ coefficients the problem was studied in [1], [6], etc. The paper [6] is of special importance for it singles out the crucial step in the study of removable singularities. To wit, on assuming $\mathcal{F}$ to be a class of functions on all of $\mathcal{X}$ one asks if any weak solution u to $A(u) = 0$ in $\mathcal{X} \setminus \mathcal{S}$ satisfies this equation weakly in all of $\mathcal{X}$. This paper facilitated considerable progress in the study of removable sets for solutions of linear equations, see [17, Ch. 1] and the references given there.

A starting point for nonlinear differential equations is the pioneering work on the local behaviour of solutions of quasilinear equations by Serrin [16]. The comparatively recent book [19] presents in a unified way the development of the theory of singularities for solutions of second order elliptic or parabolic quasilinear equations starting from the linear equations and the work [16]. As but one motivation of our previous paper [12] we mention that the book [19] does not contain any reference to [6] while the approach of the latter article may be undoubtedly of use for nonlinear equations, too. For general nonlinear equations there is no reasonable concept of a weak solution, however, one gets it immediately by turning to a variational setting and relaxing the initial equation into the Euler-Lagrange equation.

Specifically we discuss a Radó type theorem for solutions of the porous medium equation on an open set $\mathcal{X}$ in $\mathbb{R}^{n+1}$ which are defined to be weak solutions $u \in L^p_{\mathrm{loc}}(\mathcal{X})$ of the quasilinear equation $\partial_t u = \Delta(|u|^{p-1}u)$, where $p > 1$. The operator $A(u) = \partial_t u - \Delta(|u|^{p-1}u)$ is called the porous medium operator of index p. It is parabolic away from the zero points of u. The classical Radó theorem states that if u is a continuous function on an open set $\mathcal{X}$ in the complex plane which is holomorphic away from the set of zeroes then u is actually holomorphic in all of $\mathcal{X}$, see [14]. By the very nature, this is a result on removable sets for the class of continuous functions with respect to the Cauchy-Riemann operator in the plane. In 1983 Král extended the Radó theorem to harmonic functions showing that each C^1 function on an open set $\mathcal{X}$ in $\mathbb{R}^{n+1}$, which is harmonic away from the set of its zeroes, is actually harmonic on all of $\mathcal{X}$, see [9]. The paper [5] contains a Radó type theorem for the so-called generalised Cauchy-Riemann equations in $\mathbb{R}^{n+1}$. The notice [8] presents a Radó type theorem for p-harmonic functions on the plane. For a deeper discussion of Radó theorems for solutions of both linear and nonlinear differential equations we refer the reader to the monograph [17, 1.3.4] and the recent notice [12].

We now dwell on the contents of the paper. In Section 1 we remind of the porous medium equation, which is of great importance in applications. The particular structure of this nonlinear equation allows one to introduce a natural concept of a weak solution in the space of Lipschitz continuous functions on $\mathcal{X}$ to the equation. In Section 2 we specify the notion of a removable set for solutions of quasilinear partial differential equations. In Section 3 we adduce a fundamental lemma of [6] which is of key importance in the study of removable sets for solutions of linear equations. We show that the lemma is still useful in characterising removable singularities for solutions of quasilinear equations. Section 4 deals with removable sets for solutions of Sobolev classes while Section 5 does with removable sets for classes of C^s solutions to the porous medium equation. The results of Section 5 apply to study removability of the zero sets in Section 6. In Section 7 we discuss shortly a Radó type theorem for solutions of the porous medium equation.

The paper [7] rises immediately from [6] to introduce a notion of capacity which characterizes removable sets for solutions of linear equations. In [4], a concept of nonlinear capacity related to a nonlinear operator is applied to blow-up problems for diverse nonlinear partial differential equations including those with nonlocal nonlinearities.

1 The porous medium equation

The heat equation is one of the three classical linear second order equations which constitute the basis of any elementary introduction to partial differential equations. A number of related equations have been proposed both by applied and pure mathematicians as objects of study. The linear theory enjoyed much progress but it was soon observed that the most equations modelling physical phenomena without excessive simplification are nonlinear. The difficulty of building a theory for nonlinear equations had made it impossible to achieve any significant progress until the development of functional analysis in the first half of the 20 th century led to elaborating the nonlinear theory with mathematical rigor. This happened in particular in the area of quasilinear parabolic equations in divergence form where the theory reached maturity presented for instance in [10]. Still the nonlinear parabolic equations like the classical Navier-Stokes equations have remained a challenge for mathematicians, let alone the regularity of solutions.

The aim of the present paper is to study removable singularities of solutions to the nonlinear heat equation

$$\partial_t u = \Delta(|u|^{p-1} u), \tag{1}$$

where $p > 1$. We think tacitly of p as an integer while most of the theory extends with slight modifications to all real $p > 1$. Equation (1) is called the porous medium equation. This is an evolution equation in the n-dimensional

Euclidean space $\mathbb{R}^n$, by Δ being meant the Laplace operator acting in the space variables $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$. In the particular case $p = 2$ equation (1) is called Boussinesq's equation.

This simple variation of the heat equation is of great interest, for the theory of the porous medium equation contains deep and often sophisticated developments of nonlinear analysis. The book [18] settles the existence, uniqueness, stability and regularity theorems and asymptotic behaviour. There are a number of physical applications where a simple porous medium equation appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. Perhaps the best known of them is the description of the flow of an isentropic gas through a porous medium modelled independently by Leibenzon [11] and Muscat [13] around 1930. An earlier application is found in the studyÂt'of groundwater infiltration by Boussinesq in 1903, see [2]. Other applications occur in mathematical biology, boundary layer theory, and other theories.

The full form of the porous medium equation consists of adding a forcing term f in the right-hand side of (1). It makes the natural framework of the abstract functional theory for the porous medium equation, and has also received much attention when $f = f(u)$ represents the effects of reaction or absorption. The full form is also referred to as the porous medium equation with a source term.

Equation (1) is but one example of partial differential equations in the realm of what is called nonlinear diffusion. It is a nonlinear evolution equation of formally parabolic type. Its complete version in divergence form looks like

$$\partial_t u = \operatorname{div}\,(D(u)u') + f, \tag{2}$$

which is called the diffusion equation. Here, $u' = (u'_{x^1}, \ldots, u'_{x^n})$ stands for the complete derivitive of u in x, and usually $f \equiv 0$. The function $D(u)$ of $u \in \mathbb{R}$ is called the diffusion coefficient. For the porous medium equation it reduces to $D(u) = p|u|^{p-1}$, and the condition of nonnegativity of D is needed to make the equation formally parabolic. Whenever $D(u) = 0$ for some $u \in \mathbb{R}$, we say that equation (2) degenerates at that u-level, since it ceases to be strictly parabolic.

2 Removable sets for solutions of the diffusion equation

Assume that $\mathcal{X}$ is an open set in the space $\mathbb{R}^{n+1}$ of the coordinates (x,t), where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The diffusion equations give rise to a broad class of nonlinear operators of the form

$$A(u) := \partial_t u - \operatorname{div}\,(D(u)u') \tag{3}$$

acting on functions u of $(x,t) \in \mathcal{X}$. The following assumption on the diffusion coefficients will be needed throughout the paper. The Nemytskii operator $N_D(u) = D(u)$ is required to be a continuous selfmapping of $L^\infty_{\mathrm{loc}}(\mathcal{X})$. (We write $L^\infty_{\mathrm{loc}}(\mathcal{X})$ to emphasize an obvious generalisation for $D(x,u)$ depending on $x \in \mathcal{X}$.) As usual, the designations "loc" and "comp" specify the "local" and "with compact support" versions of the corresponding global Lebesgue or Sobolev spaces in $\mathcal{X}$.

In order to introduce weak solutions to the equation $A(u) = 0$ in $\mathcal{X}$ we give the operator A the domain $W^{1,\infty}_{\mathrm{loc}}(\mathcal{X})$. By the Rademacher theorem, this space coincides with the space of all locally Lipschitz continuous functions on $\mathcal{X}$, i.e., $C^{0,1}_{\mathrm{loc}}(\mathcal{X})$. If $u \in W^{1,\infty}_{\mathrm{loc}}(\mathcal{X})$, then $A(u)$ can be specified within the framework of distributions on $\mathcal{X}$ by setting

$$\langle A(u), g\rangle := -\langle u, \partial_t g\rangle + \langle D(u)u', g'\rangle \tag{4}$$

for all $g \in C^\infty_{\mathrm{comp}}(\mathcal{X})$. In fact, the right-hand side of (4) defines a continuous linear functional on the space $C^1_{\mathrm{comp}}(\mathcal{X})$, and so the operator A maps $W^{1,\infty}_{\mathrm{loc}}(\mathcal{X})$ continuously into the dual of $C^1_{\mathrm{comp}}(\mathcal{X})$.

If $u \in W^{1,\infty}_{\mathrm{loc}}(\mathcal{X})$, then the image $A(u)$ is a distributions on $\mathcal{X}$. In this way, a function $u \in W^{1,\infty}_{\mathrm{loc}}(\mathcal{X})$ is said to satisfy the equation $A(u) = 0$ on an open set $U \subset \mathcal{X}$ if $A(u) = 0$ in the sense of distributions in U, i.e., $\langle A(u), g\rangle = 0$ for all $g \in C^\infty_{\mathrm{comp}}(U)$. Hence, by solutions of $A(u) = 0$ are meant weak solutions. This allows one to extend the definition of removable sets, introduced in [6] for linear differential operators A, to solutions of nonlinear diffusion equations (2).

Definition 1 Let S be a closed subset of $\mathcal{X}$ and $\mathcal{F}$ a class of functions in $W^{1,\infty}_{\mathrm{loc}}(\mathcal{X})$. The set S is called removable for $\mathcal{F}$ relative to the differential operator A if any function $u \in \mathcal{F}$ satisfying $A(u) = 0$ in $\mathcal{X} \setminus S$ actually satisfies $A(u) = 0$ in all of $\mathcal{X}$.

One may ask what conditions on the "size" of S are sufficient for S to be a removable set for $\mathcal{F}$ relative to A. For a survey of results on removable singularities we refer the reader to [17, Ch. 1] and [19]. For the most extensively studied classes $\mathcal{F}$ and differential operators A there have been known sharp sufficient conditions on removable sets in terms of the Hausdorff measure of $\mathcal{S}$. For both necessary and sufficient conditions on removable sets one appeals to the so-called capacity, see [7].

Example 1 Under a mild condition on the diffusion coefficient $D(u)$, any set $S \subset \mathcal{X}$ of measure zero is removable for $W^{2,\infty}_{\mathrm{loc}}(\mathcal{X})$ relative to A. Indeed, for any $u \in W^{2,\infty}_{\mathrm{loc}}(\mathcal{X})$, the distribution $A(u)$ is actually regular and coincides with the locally bounded function $A(u)$ evaluated almost everywhere in $\mathcal{X}$. Hence, if $A(u) = 0$ holds weakly in $\mathcal{X} \setminus S$, then $A(u)$ vanishes as distribution in all of $\mathcal{X}$.

3 A fundamental lemma

In order to characterize the removable sets in terms of the Hausdorff measure one uses a fundamental lemma of [6]. We first recall the definition of the Hausdorff measure.

For $0 \le d \le n+1$ we set

$$h_{d,\varepsilon}(S) := \inf \sum_{\nu} v_d \, r_{\nu}^d,$$

where the infimum is taken over all countable coverings $\{B_\nu\}$ of the set S by balls with radii $r_\nu \le \varepsilon$, and v_d is the volume of the unit ball in $\mathbb{R}^d$. Obviously, $h_{d,\varepsilon}(S)$ is a monotone decreasing function of $\varepsilon \to 0+$, and so it has a limit as $\varepsilon \to 0+$. The number

$$h_d(S) = \lim_{\varepsilon \to 0+} h_{d,\varepsilon}(S)$$

is called the d-dimensional Hausdorff measure of the set S.

Hausdorff measure is a regular metric outer measure on $\mathbb{R}^{n+1}$. Therefore, $h_d(S) = 0$ if and only if $h_d(K) = 0$ for each compact subset $K \subset S$. Note that h_{n+1} agrees with the standard Lebesgue measure in $\mathbb{R}^{n+1}$. In most cases one is interested only in whether the measure $h_d(S)$ is zero, finite, or infinite. From this point of view, instead of coverings by balls in the definition of h_d, we may use coverings by cubes or arbitrary (convex) sets of diameter $2r_v$, because all such coverings lead to equivalent measures.

Lemma 1 *Let K be a compact subset of $\mathbb{R}^{n+1}$. Then, for each $d = n+1-m$ and $\varepsilon > 0$, there is a C^∞ function χ_ε with compact support in $\mathbb{R}^{n+1}$ such that the support of χ_ε belongs to the ε-neighbourhood of K, $\chi_\varepsilon \equiv 1$ in a smaller neighbourhood of K, and*

$$\|\partial^\alpha \chi_\varepsilon\|_{L^1(\mathbb{R}^{n+1})} \le C_\alpha \, \varepsilon^{m-|\alpha|} \, (h_d(K) + \varepsilon)$$

for all α with $|\alpha| \le m$, where the constant C_α is independent of ε.

Proof This is a refinement of a well-known lemma of [1]. For a proof, see [6] or [17, 1.2.1].

It is worth pointing out that d is assumed to be nonnegative. Hence it follows that $n+1-m \ge 0$, i.e., $m \le n+1$.

4 Removable sets for Sobolev functions

In this section we characterize removable sets for the class $\mathcal{F} := W^{1,\infty}_{\mathrm{loc}}(\mathcal{X})$. As mentioned, this space is a localisation of Lipschitz continuous functions on $\mathcal{X}$.

Theorem 1 *If $h_n(S) = 0$, then the set S is removable for $W^{1,\infty}_{\mathrm{loc}}(\mathcal{X})$ relative to A.*

For general linear partial differential operators A, Theorem 1 is contained in [6].

Proof Let $u \in W^{1,\infty}_{\mathrm{loc}}(\mathcal{X})$ satisfy $A(u) = 0$ in $\mathcal{X} \setminus \mathcal{S}$. Pick any $g \in C^\infty_{\mathrm{comp}}(\mathcal{X})$ and set $K = S \cap \operatorname{supp} g$. Then

$$\langle A(u), g\rangle = \langle A(u), \chi_\varepsilon g\rangle + \langle A(u), (1-\chi_\varepsilon)g\rangle$$

for all $\varepsilon > 0$, where χ_ε is the function of Lemma 1 with $m = 1$.

Since $A(u) = 0$ in $\mathcal{X} \setminus \mathcal{S}$ and the support of $(1-\chi_\varepsilon)g$ is a compact subset of $\mathcal{X} \setminus \mathcal{S}$, it follows that

$$\begin{aligned}\langle A(u), g\rangle &= \langle A(u), \chi_\varepsilon g\rangle \\ &= -\langle u, \partial_t(\chi_\varepsilon g)\rangle + \langle D(u)u', (\chi_\varepsilon g)'\rangle \\ &= -\langle u, \partial_t \chi_\varepsilon g\rangle - \langle u, \chi_\varepsilon \partial_t g\rangle + \langle D(u)u', \chi'_\varepsilon g\rangle + \langle D(u)u', \chi_\varepsilon g'\rangle\end{aligned}$$

for all $\varepsilon > 0$. Consequently, by the Hölder inequality and Lemma 1, we readily obtain

$$\begin{aligned}|\langle A(u), g\rangle| \le\ & C_1\ (h_n(K)+\varepsilon)\, \|u\|_{L^\infty(K_\varepsilon)}\, \|g\|_{L^\infty(\mathcal{X})} \\ &+ C_0\, \varepsilon\, (h_n(K)+\varepsilon)\, \|u\|_{L^\infty(K_\varepsilon)}\, \|\partial_t g\|_{L^\infty(\mathcal{X})} \\ &+ C_1\ (h_n(K)+\varepsilon)\, \|D(u)u'\|_{L^\infty(K_\varepsilon,\mathbb{R}^n)}\, \|g\|_{L^\infty(\mathcal{X})} \\ &+ C_0\, \varepsilon\, (h_n(K)+\varepsilon)\, \|D(u)u'\|_{L^\infty(K_\varepsilon,\mathbb{R}^n)}\, \|g'\|_{L^\infty(\mathcal{X},\mathbb{R}^n)}\end{aligned} \tag{5}$$

for all sufficiently small $\varepsilon > 0$, where K_ε stands for the ε-neighbourhood of K and C_0, C_1 are constants independent of ε.

By assumption, $h_n(K) = 0$ for each compact set $K \subset S$. Therefore, the right-hand side of (5) tends to zero as $\varepsilon \to 0+$. Thus, $\langle A(u), g\rangle = 0$ weakly in $\mathcal{X}$, as desired.

The arguments of [17, 1.2.2] show that the assumptions on S in Theorem 1 cannot be improved in terms of the Hausdorff measure.

Example 2 Assume that S is a closed subset of $\mathcal{X}$ satisfying $h_n(S) = 0$. Then the set S is removable for $W^{1,\infty}_{\mathrm{loc}}(\mathcal{X})$ relative to the porous medium operator with $p \ge 1$.

5 Removable sets for smooth functions

In this section we will be concerned with removable sets for $C^1_{\mathrm{loc}}(\mathcal{X})$ relative to A. As usual, for any integer $s \ge 0$, we denote by $C^s_{\mathrm{loc}}(\mathcal{X})$ the space of s times continuously differentiable functions on $\mathcal{X}$. If $s = 0$, it is customary to omit the index. In order to get substantial results, it is necessary to put some restrictions on the diffusion coefficient $D(u)$ of A. Since this work is intended as an attempt at motivating the Radó type theorem for solutions of the porous medium equation, we choose the abstract setting of the diffusion equation. To wit, assume that the Nemytskii operator $N_D(u) = D(u)$ is a continuous selfmapping of $C_{\mathrm{loc}}(\mathcal{X})$.

Theorem 2 *Suppose $h_n(K) < \infty$ for each compact set $K \subset S$. Then S is removable for $C^1_{\mathrm{loc}}(\mathcal{X})$ relative to A.*

Under a mild condition on the diffusion coefficient $D(u)$, the set S is removable for $C^2_{\mathrm{loc}}(\mathcal{X})$ relative to A provided that the interior of S is empty.

Proof Assume that $u \in C^1_{\mathrm{loc}}(\mathcal{X})$ satisfies $A(u) = 0$ weakly on $\mathcal{X} \setminus \mathcal{S}$. Let $g \in C^\infty_{\mathrm{comp}}(\mathcal{X})$, and let $K = S \cap \operatorname{supp} g$.

Since the support of $A(u)$ belongs to S, we obtain for the function χ_ε from Lemma 1 with $m = 1$ that

$$\begin{aligned}\langle A(u), g\rangle &= \langle A(u), \chi_\varepsilon g\rangle \\ &= -\langle u, \partial_t(\chi_\varepsilon g)\rangle + \langle D(u)u', (\chi_\varepsilon g)'\rangle \\ &= \langle -\partial_t \chi_\varepsilon u + \sum_{j=1}^n \partial_j \chi_\varepsilon D(u) \partial_j u, g\rangle - \langle u, \chi_\varepsilon \partial_t g\rangle + \langle D(u)u', \chi_\varepsilon g'\rangle\end{aligned}$$

for all $\varepsilon > 0$. On applying the Hölder inequality and Lemma 1 we obtain

$$\begin{aligned}&|-\langle u, \chi_\varepsilon \partial_t g\rangle + \langle D(u)u', \chi_\varepsilon g'\rangle| \\ &\le C_0\, \varepsilon\, (h_n(K) + \varepsilon)\, \|u\|_{L^\infty(K_\varepsilon)} \|\partial_t g\|_{L^\infty(\mathcal{X})} \\ &+ C_0\, \varepsilon\, (h_n(K) + \varepsilon)\, \|D(u)u'\|_{L^\infty(K_\varepsilon, \mathbb{R}^n)} \|g'\|_{L^\infty(\mathcal{X}, \mathbb{R}^n)}\end{aligned}$$

which is dominated by $C\, \varepsilon\, (h_n(K) + \varepsilon)$, where the constant C is independent of ε. Consequently, $A(u)$ is the limit of the net of continuous functions

$$-\partial_t \chi_\varepsilon u + \sum_{j=1}^n \partial_j \chi_\varepsilon D(u) \partial_j u \tag{6}$$

in the space of distributions on $\mathcal{X}$.

By Lemma 1, we have

$$\begin{aligned}\|\partial_t \chi_\varepsilon\|_{L^1(\mathcal{X})} &\le C_1\, (h_n(K) + \varepsilon)\,, \\ \|\partial_j \chi_\varepsilon\|_{L^1(\mathcal{X})} &\le C_1\, (h_n(K) + \varepsilon)\end{aligned}$$

for all positive $\varepsilon \le 1$ and $j = 1, \ldots, n$. Since $h_n(K) < \infty$, we can assert that the nets $\partial_t \chi_\varepsilon$ and $\partial_1 \chi_\varepsilon, \ldots, \partial_n \chi_\varepsilon$ are bounded in $L^1(\mathcal{X})$. Hence it follows that every of the nets has a subsequence which converges in the weak* topology of $C_{\mathrm{loc}}(\mathcal{X})'$. The limit of this subsequence is necessarily zero, for the net χ_ε, and so also the nets $\partial_t \chi_\varepsilon$ and $\partial_1 \chi_\varepsilon, \ldots, \partial_n \chi_\varepsilon$, converges to zero in the sense of distributions on $\mathcal{X}$. Multiplication by u or $D(u)\partial_1 u, \ldots, D(u)\partial_n u$, respectively, defines a continuous operator in $C_{\mathrm{loc}}(\mathcal{X})'$. Therefore, some subsequence of (6) converges to zero in the weak* topology of $C_{\mathrm{loc}}(\mathcal{X})'$. Since, however, the net itself converges to $A(u)$ in the space of distributions on $\mathcal{X}$, it follows that $A(u) = 0$ on $\mathcal{X}$, as desired.

Example 3 Suppose $h_n(K) < \infty$ for each compact set $K \subset S$. Then S is removable for continuously differentiable solutions of the porous medium equation in $\mathcal{X}$, with any $p > 1$.

6 A Radó theorem

A Radó type theorem for solutions of linear differential equations was first formulated in the monograph [17, 1.3.4] whose original Russian edition was published in 1991.

In order to formulate a Radó theorem in the context of nonlinear diffusion equations, we still keep the assumption that $N_D(u) = D(u)$ maps $C_{\mathrm{loc}}(\mathcal{X})$ continuously into itself. By Theorem 2, if S is a closed subset of $\mathcal{X}$ such that $h_n(K) < \infty$ for each compact set $K \subset S$, then S is removable for $C^1_{\mathrm{loc}}(\mathcal{X})$ relative to A.

Lemma 2 *Assume that S is a smooth hypersurface in $\mathcal{X}$. Then S is removable for $C^1_{\mathrm{loc}}(\mathcal{X})$ relative to A.*

Proof Indeed, when restricted to subsets of a smooth submanifold S of $\mathcal{X}$ of dimension d, the Hausdorff measure h_d is commensurable with the corresponding surface measure on S induced by the Lebesgue measure in $\mathbb{R}^{n+1}$. Hence it follows immediately that $h_n(K) < \infty$ for each compact set $K \subset S$, showing the desired assertion.

While Theorem 2 characterises those $S \subset \mathcal{X}$ which are removable for all solutions $u \in C^1_{\mathrm{loc}}(\mathcal{X})$ to $A(u) = 0$ in $\mathcal{X} \setminus S$, the Radó type theorems deal with individual solutions $u \in C^1_{\mathrm{loc}}(\mathcal{X})$ of this equation. As S one takes the preimage of a point by u, e.g., $S = u^{-1}(0)$ which is the set of all $x \in \mathcal{X}$ satisfying $u(x) = 0$. Then, a Radó theorem for solutions of the nonlinear equation $A(u) = 0$ states that if $u \in C^1_{\mathrm{loc}}(\mathcal{X})$ satisfies $A(u) = 0$ weakly in $\mathcal{X} \setminus u^{-1}(0)$ then $A(u) = 0$ is actually fulfilled in the sense of distributions in all of $\mathcal{X}$.

Theorem 3 *If $u \in C^1_{\mathrm{loc}}(\mathcal{X})$ satisfies $A(u) = 0$ in $\mathcal{X} \setminus u^{-1}(0)$, then $A(u) = 0$ away from the set of all $x \in \mathcal{X}$ satisfying $\partial^\beta u(x) = 0$ for each multi-index β with $|\beta| \le 1$.*

Proof Set $S = u^{-1}(0)$, and so S is a closed subset of $\mathcal{X}$. Denote by S_{reg} the subset of S consisting of those $x \in S$ which satisfy $u'(x) \neq 0$. Clearly, S_{reg} is an open set in S, and so the set $S^{(1)} := S \setminus S_{\mathrm{reg}}$, which consists of all $x \in S$ satisfying $u(x) = u'(x) = 0$, is closed in $\mathcal{X}$. Each point $x \in S_{\mathrm{reg}}$ has a neighbourhood U in $\mathcal{X}$, such that $S \cap U$ is a hypersurface in U. On applying Lemma 2 we see that $A(u) = 0$ holds weakly in U and hence everywhere in $\mathcal{X} \setminus S^{(1)}$, as desired.

We now elucidate the main analytical problem in studying the Radó theorem for solutions of the equation $A(u) = 0$. Let $u \in C^1_{\mathrm{loc}}(\mathcal{X})$ satisfy $A(u) = 0$ in $\mathcal{X} \setminus S$, where $S = u^{-1}(0)$. By Theorem 3, the function u satisfies $A(u) = 0$ away from the closed set $S^{(1)}$ in $\mathcal{X}$. In all interesting cases the Hausdorff dimension of the set $S^{(1)}$ is less than n, and so the hypothesis of Theorem 2 is satisfied. By this theorem, one gets $A(u) = 0$ in all of $\mathcal{X}$, showing the Radó theorem. Clearly, no conclusion on the size of $S^{(s-1)}$ can be drawn in the case of general operators A.

For a counterexample to a Radó theorem we refer the reader to Example 5 in [12]. It should be noted, however, that the equation of this example is not a diffusion equation.

7 A Radó theorem for solutions of the porous medium equation

The following theorem presents a class of diffusion equations for which the Radó theorem is valid in the proper formulation.

Theorem 4 *Assume that $D(u) = O(u)$ as $u \to 0$. If $u \in C^{0,1}_{\mathrm{loc}}(\mathcal{X})$ satisfies $A(u) = 0$ weakly in $\mathcal{X} \setminus u^{-1}(0)$, then it is a weak solution to this equation in all of $\mathcal{X}$.*

Proof Set $S = u^{-1}(0)$. Suppose $u \in C^{0,1}_{\mathrm{loc}}(\mathcal{X})$ satisfies $A(u) = 0$ weakly in $\mathcal{X} \setminus S$. We wish to show that $A(u) = 0$ in $\mathcal{X}$, i.e.,

$$-\langle u, \partial_t g\rangle + \langle D(u)u', g'\rangle = 0$$

for all test functions $g \in C^{\infty}_{\mathrm{comp}}(\mathcal{X})$.

Fix a function $g \in C^{\infty}_{\mathrm{comp}}(\mathcal{X})$ and denote by K the intersection of the support of g with S. By a familiar lemma of Bochner [1], for each $\varepsilon > 0$ there is a C^{∞} function χ_ε with compact support in the ε-neighbourhood K_ε of K, equal to 1 in a smaller neighbourhood of K and such that $|\partial^\alpha \chi_\varepsilon(x)| \le c_\alpha \varepsilon^{-|\alpha|}$ for all x, where the constant c_α is independent of ε. Since the support of $A(u)$ belongs to S, we get

$$-\langle u, \partial_t g\rangle + \langle D(u)u', g'\rangle = -\langle u, \partial_t(\chi_\varepsilon g)\rangle + \langle D(u)u', (\chi_\varepsilon g)'\rangle.$$

Hence,

$$|-\langle u, \partial_t g\rangle + \langle D(u)u', g'\rangle| \le C\,\varepsilon^{-1} \int_{K_\varepsilon} |u(y)|\,(1 + |u'(y)|)\,dy \qquad (7)$$

for all sufficiently small $\varepsilon > 0$, where C is a constant depending on g but not on ε.

For a point $x \in \mathcal{X}$, we denote by

$$m_x(u)\,(\delta) = \sup_{\substack{y \in X \\ |y-x|<\delta}} |u(y) - u(x)|$$

the modulus of continuity of u at x. Since, by hypothesis, $u \in C^{0,1}_{\mathrm{loc}}(\mathcal{X})$, it follows that

$$\sup_{x \in K} m_x(u)\,(\delta) \le c\,\delta$$

for all $\delta > 0$ small enough. Thus, (7) yields

$$\begin{aligned} | - \langle u, \partial_t g \rangle + \langle D(u) u', g' \rangle | &\leq C \, \varepsilon^{-1} \sup_{x \in K} m_x(u) \, (\varepsilon) \, \mathrm{meas} \, (K_\varepsilon \setminus K) \\ &\leq C c \, \mathrm{meas} \, (K_\varepsilon \setminus K) \end{aligned}$$

for sufficiently small $\varepsilon > 0$. However, the measure of $K_\varepsilon \setminus K$ tends to zero as $\varepsilon \to 0$ (see for instance [3, 3.2.34]). Thus, $- \langle u, \partial_t g \rangle + \langle D(u) u', g' \rangle = 0$, which completes the proof.

As but one consequence of Theorem 4 we mention a Radó theorem for the porous medium equation.

Example 4 For the equation $\partial_t u = \mathrm{div} \, (p |u|^{p-1} u')$ the diffusion coefficient is $D(u) = p |u|^{p-1}$. The Nemytskii operator N_D maps $C_{\mathrm{loc}}(\mathcal{X})$ continuously into itself and $D(u) = O(u)$ as $u \to 0$, provided that $p \geq 2$. Hence, if $u \in C^{0,1}_{\mathrm{loc}}(\mathcal{X})$ satisfies the equation weakly in $\mathcal{X} \setminus u^{-1}(0)$, then it is a weak solution to this equation in all of $\mathcal{X}$.

Note that this result does not hold for $p = 1$. For example, the function $u(x,t) = \max\{0, x_n\}$ in $\mathbb{R}^n \times \mathbb{R}$ belongs obviously to $C^{0,1}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and satisfies $\partial_t u - \Delta u = - \delta_{\{x_n = 0\}}$, where $\delta_{\{x_n = 0\}}$ is the distribution given by the integration over the hyperplane $\{x_n = 0\}$ in $\mathbb{R}^{n+1}$.

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