Relating Diameter and Mean Curvature for Varifolds

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Zusammenfassung

In dieser Arbeit geht es um *Varifaltigkeiten*, deren *erste Variation* lokal endlich ist. Eine m *dimensionale Varifaltigkeit* V in $\mathbf{R}^n$ ist ein Radonmaß auf dem Produkt $\mathbf{R}^n \times \mathbf{G}(n,m)$ mit der Grassmann-Mannigfaltigkeit $\mathbf{G}(n,m)$. Das zugehörige *Gewichtsmaß* $\|V\|$ auf $\mathbf{R}^n$ erhält man durch Projektion auf $\mathbf{R}^n$. Die Notation für Varifaltigkeiten richtet sich nach Allard [All72]. In den Kapiteln 3 und 4 werden die notwendigen Begriffe bereitgestellt.

In Kapitel 5 geht es um die *erste Variation* δV der Varifaltigkeit V. Das ist eine lineare Abbildung, welche jedem kompakt getragenem Vektorfeld g eine reelle Zahl, genauer gesagt das Integral über die relative Divergenz von g zuordnet. Es wird gezeigt, wie die erste Variation mittels Integration dargestellt werden kann. Korrespondiert die Varifaltigkeit zu einer glatten, eigentlich eingebetteten m dimensionalen Untermannigfaltigkeit M des $\mathbf{R}^n$ ohne Rand, das heißt V ist gegeben durch

$$V(A) = \mathscr{H}^m(M \cap \{x : (x, \operatorname{Tan}(M,x)) \in A\}) \quad \text{für } A \subset \mathbf{R}^n \times \mathbf{G}(n,m),$$

was $\|V\| = \mathscr{H}^m \llcorner M$ impliziert, so lässt sich die erste Variation ausdrücken durch die mittlere Krümmung von M. Dieser Zusammenhang motiviert die Definition der *verallgemeinerten mittleren Krümmung* $\mathbf{h}(V;\cdot)$ für *Varifaltigkeiten*. Das im Sinne des Rieszschen Darstellungssatzes zu δV resultierende Radonmaß auf $\mathbf{R}^n$ wird mit $\|\delta V\|$ bezeichnet und *totale Variation* genannt. Ferner wird definiert, was eine *unzerlegbare Varifaltigkeit* ist. Ist eine Varifaltigkeit unzerlegbar, so ist der Träger $\operatorname{spt}\|V\|$ des Gewichtsmaßes zusammenhängend. Andererseits ist jede Varifaltifkeit, die zu einer zusammenhängenden geschlossenen glatten Untermannigfaltigkeit des $\mathbf{R}^n$ korrespondiert unzerlegbar.

In Kapitel 6 wird die *isoperimetrische Ungleichung* bewiesen. Sie besagt, dass das Maß geeigneter Teilmengen des $\mathbf{R}^n$ beschränkt ist durch den Rand und durch die Krümmung dieser Menge. Gleichwohl ist dieser Zusammenhang nicht linear. Mathematisch ausgedrückt lautet die isoperimetrische Ungleichung wie folgt. Es gibt eine positive und endliche Konstante Γ, sodass

$$\|V\|\{x : 1 \le \mathbf{\Theta}^m(\|V\|, x)\} \le \Gamma \|V\|(\mathbf{R}^n)^{1/m} \|\delta V\|(\mathbf{R}^n)$$

falls $\|V\|(\mathbf{R}^n) < \infty$. Dabei bezeichnet $\mathbf{\Theta}^m(\|V\|,\cdot)$ die m dimensionale Dichte von $\|V\|$ und die Konstante Γ hängt nur von der Dimension m der Varifaltigkeit V ab. Gilt $\mathbf{\Theta}^m(\|V\|, x) \ge 1$ für $\|V\|$ fast alle x und $0 < \|V\|(\mathbf{R}^n) < \infty$, so lässt sich die isoperimetrische Ungleichung vereinfachen zu

$$\|V\|(\mathbf{R}^n)^{1-1/m} \le \Gamma \|\delta V\|(\mathbf{R}^n).$$

Die wichtigsten Hilfsmittel des Beweises stellen die klassische *Monotonie Identität* (in der Version von Menne [Men16a, 4.5]) und ein Lemma von Simon [Sim83, 18.7], welches erlaubt, den Überdeckungssatz von Vitali anzuwenden, dar.

In Kapitel 7 wird untersucht werden, wie sehr sich die Menge $\{x : 1 \le \mathbf{\Theta}^m(\|V\|, x)\}$ von der Menge $\operatorname{spt}\|V\|$ unterscheidet. Im folgenden sei angenommen, dass $\mathbf{\Theta}^m(\|V\|, x) \ge 1$ für $\|V\|$ fast alle x gilt. Mithilfe der isoperimetrischen Ungleichung wird gezeigt, dass der m dimensionale Dichtequotient von $\|V\|$ durch eine positive Zahl nach unten beschränkt ist, sofern der Ableitungsquotient von $\|\delta V\|$ nach $\|V\|^{1-1/m}$ klein ist. Ist V unzerlegbar, so kann die Voraussetzung an den Ableitungsquotienten reduziert werden zu der Annahme, dass der 1 dimensionale Dichtequotient

von $\|V\| \llcorner |\mathbf{h}(V;\cdot)|^{m-1}$ klein genug ist. Sei V eine m dimensionale Varifaltigkeit in $\mathbf{R}^n$, $\|\delta V\|$ ein Radonmaß, V unzerlegbar, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ für $\|V\|$ fast alle x, $\|\delta V\|$ absolut stetig bezüglich $\|V\|$ und $\mathbf{h}(V;\cdot)$ lokal zur Potenz $(m-1)$ summierbar. Es wird gezeigt, dass dann

$$\boldsymbol{\Theta}^m_*(\|V\|, x) \geq 1 \quad \text{für } \mathscr{H}^1 \text{ fast alle } x \in \operatorname{spt} \|V\|.$$

Auf die Unzerlegbarkeit von V kann nicht verzichtet werden.

In Kapitel 8 wird der geodätische Abstand in abgeschlossenen Teilmengen des $\mathbf{R}^n$, das heißt die kürzeste Länge von stetigen Verbindungskurven zwischen zwei Punkten eingeführt. Ist dieser Abstand zwischen zwei Punkten endlich, so gibt es eine minimale stetige Verbindungskurve. Diese kann Lipschitz-stetig und nach Bogenlänge parametrisiert gewählt werden. Sei V eine Varifaltigkeit wie im letzten Abschnitt der Beschreibung von Kapitel 7 und $\operatorname{spt} \|V\|$ kompakt. Es wird gezeigt, dass dann der Durchmesser d von $\operatorname{spt} \|V\|$ bezüglich dem geodätischen Abstand beschränkt ist durch

$$d \leq \Gamma \int |\mathbf{h}(V, x)|^{m-1} \, \mathrm{d}\|V\| x$$

wobei Γ eine positive und endliche Konstante ist, die nur von der Dimension m der Varifaltigkeit abhängt und der Ausdruck 0^0 als 1 interpretiert werden soll. Auf die Unzerlegbarkeit von V kann nicht verzichtet werden. Absolut Stetigkeit von $\|\delta V\|$ bezüglich $\|V\|$ wird nur für $m \geq 3$ benötigt. Den Hauptteil des Beweises liefern die Techniken aus Kapitel 7. Diese können verwendet werden, indem der geodätische Abstand approximiert wird durch eine Folge von Metriken auf $\operatorname{spt} \|V\|$, welche für Punkte, die im eigentlichem Sinn nahe beieinander liegen, dem euklidischen Abstand gleich sind.

Abstract

The main results of this thesis are formulated on the following set of hypotheses. Suppose $m \leq n$ are positive integers, V is an m dimensional rectifiable varifold in $\mathbf{R}^n$, the total variation $\|\delta V\|$ is a Radon measure, $\|\delta V\|$ is absolutely continuous with respect to the weight measure $\|V\|$, V is indecomposable, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x and the generalized mean curvature $\mathbf{h}(V;\cdot)$ is locally summable to the power $m-1$ with respect to $\|V\|$.

If m is at least 2, then it will be shown in this thesis that

$$\boldsymbol{\Theta}^m_*(\|V\|, x) \geq 1 \quad \text{for } \mathscr{H}^1 \text{ almost all } x \in \operatorname{spt} \|V\|.$$

One cannot drop the assumption that V is indecomposable.

If $\operatorname{spt} \|V\|$ is compact, then the intrinsic diameter d of $\operatorname{spt} \|V\|$ will be estimated in terms of the generalized mean curvature $\mathbf{h}(V;\cdot)$ by

$$d \leq \Gamma \int |\mathbf{h}(V; x)|^{m-1} \, \mathrm{d}\|V\| x$$

where Γ is a positive and finite number depending only on m. This generalizes the diameter control for closed and connected smooth submanifolds of $\mathbf{R}^n$ of Topping [Top08]. However, it was not known whether the present hypothesis implies that two points in $\operatorname{spt} \|V\|$ have finite geodesic distance in $\operatorname{spt} \|V\|$. The absolute continuity of $\|\delta V\|$ with respect to $\|V\|$ is only needed for $m \geq 3$. One cannot drop the assumption that V is indecomposable.

Keywords Varifold, rectifiable varifold, indecomposable varifold, first variation, mean curvature, isoperimetric inequality, density of a measure, geodesic distance, intrinsic diameter.

Contents

1 Introduction

In this thesis the objects of investigation are varifolds with locally finite first variation, first introduced by Almgren [Alm65]. The definition and notation for varifolds in this thesis follow Allard [All72]. As a consequence of the isoperimetric inequality, the main results of this thesis are an estimate of the set where the density quotient is small and an estimate of the intrinsic diameter with respect to the support of the weight measure in terms of the generalized mean curvature.

The isoperimetric inequality for varifolds was first published by Allard [All72]. Allard shows in [All72, 7.1] the existence of a constant γ such that

$$\|V\|(\mathbf{R}^n)^{1-1/m} \leq \gamma \|\delta V\|(\mathbf{R}^n)$$

whenever V is an m dimensional varifold in $\mathbf{R}^n$ satisfying $0 < \|V\|(\mathbf{R}^n) < \infty$ and $\mathbf{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x. The constant γ only depends on m and n. Loosely speaking, the isoperimetric inequality says that the measure of a set is bounded by its boundary and its mean curvature. The isoperimetric inequality has lots of applications. Some of them are part of this thesis. Michael and Simon improve in [MS73, 2.1], see also [Sim83, 18.6], the isoperimetric inequality for the class of rectifiable varifolds with finite weight for which the first variation is absolutely continuous with respect to the weight measure in this way that the constant γ only depends on the dimension of the varifold. Menne indicates in [Men09, 2.2] that the isoperimetric inequality holds true for all rectifiable varifolds with finite weight and a constant depending only on the dimension of the varifold. In this thesis it will be proved in Section 6 that the isoperimetric inequality holds true for all varifolds with finite weight and a constant depending only on the dimension of the varifold.

Suppose V is an m dimensional varifold in $\mathbf{R}^n$ and the total variation $\|\delta V\|$ is a Radon measure. Then the dimension of the set $\operatorname{spt}\|V\|$ might be strictly greater than m, see 5.16. Instead, the subset $\{x : 1 \leq \boldsymbol{\Theta}^m(\|V\|, x)\}$ of $\operatorname{spt}\|V\|$ is of geometric interest. In particular the question of how much the mentioned set differs from the support of the weight measure arises. This question is answered for one dimensional varifolds in Euclidean space by Menne [Men16a, 4.8]. To study this question for higher dimensions, suppose V is an m dimensional rectifiable varifold in $\mathbf{R}^n$, m is at least 2, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, the total variation $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ and the generalized mean curvature $\mathbf{h}(V; \cdot)$ is locally summable to the power $m-1$ with respect to $\|V\|$. In this case, Menne shows in [Men09, 2.9, 2.11] that there holds

$$\text{either } \boldsymbol{\Theta}^m_*(\|V\|, x) \geq 1 \quad \text{or } \boldsymbol{\Theta}^m(\|V\|, x) = 0$$

for $\mathscr{H}^1$ almost all $x \in \operatorname{spt}\|V\|$. Nevertheless, it may happen that the set of those $x \in \operatorname{spt}\|V\|$ for which $\boldsymbol{\Theta}^m(\|V\|, x) = 0$ has positive m dimensional Hausdorff measure. A sufficient condition to avoid this is the additional assumption that the varifold is indecomposable, see 5.10. This is a strong condition which implies amongst other things that the support of the weight measure is connected. In this thesis it will be proved in Section 7 that the hypothesis described above implies

$$\boldsymbol{\Theta}^m_*(\|V\|, x) \geq 1 \quad \text{for } \mathscr{H}^1 \text{ almost all } x \in \operatorname{spt}\|V\|.$$

In view of the isoperimetric inequality it seems natural that not only the measure of a set is bounded by its mean curvature but also the diameter of a set without boundary. This dependence is already described for immersed manifolds in the following way. Suppose m is a positive integer, $\mathcal{M}$ is a connected and closed m dimensional manifold smoothly immersed in $\mathbf{R}^n$ with mean curvature $\mathbf{H}$ and intrinsic diameter d_{int}. Topping shows in [Top08] the existence of a finite constant $C(m)$ satisfying

$$d_{\text{int}} \leq C(m) \int_{\mathcal{M}} |\mathbf{H}|^{m-1} \,\mathrm{d}\mu,$$

where μ is the measure on $\mathcal{M}$ induced by the ambient space and the constant $C(m)$ only depends on m. Menne rephrases this result in the unpublished notes [Men12b] in the context of indecomposable varifolds. That is if V is an m dimensional rectifiable varifold in $\mathbf{R}^n$, $\|\delta V\|$ is a Radon meausre, V is indecomposable, $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, $\operatorname{spt}\|V\|$ is compact, the generalized mean curvature $\mathbf{h}(V; \cdot)$ is locally summable to the power $(m-1)$ with respect to the weight measure $\|V\|$ and $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, then the extrinsic diameter d_{ext} of $\operatorname{spt}\|V\|$ is estimated in terms of the generalized mean curvature $\mathbf{h}(V; \cdot)$ by

$$d_{\text{ext}} \leq \Gamma \int |\mathbf{h}(V; x)|^{m-1} \,\mathrm{d}\|V\| x$$

where the constant Γ only depends on m and the expression 0^0 should be interpreted as 1. The absolute continuity of $\|\delta V\|$ with respect to $\|V\|$ is only needed if $m \geq 3$. In this thesis it will be proved in Section 8 that the inequality above can be sharpened by replacing the extrinsic diameter of $\operatorname{spt} \|V\|$ by its intrinsic diameter. It was not known that the present hypothesis implies that two points in $\operatorname{spt} \|V\|$ have finite geodesic distance in $\operatorname{spt} \|V\|$. One cannot drop the assumption that the varifold V is indecomposable.

A reference for the measure theoretic statements used in this thesis is [Fed69].

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2 Notation

The notation of Federer [Fed69] and Allard [All72] will be used throughout the thesis. Amongst other things, this means $\mathscr{P}$ denotes the set of positive integers and $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$ denotes the space of all real valued smooth functions with compact support in $\mathbf{R}^n$. Based on Menne [Men16a], the following modification and additional notation are employed. If f is a relation, then $f[A] = \{y : (x, y) \in f \text{ for some } x \in A\}$ whenever A is a set, see [Kel75, p. 8]. For each positive integer n, the number $\boldsymbol{\beta}(n)$ denotes the least positive integer in the Besicovitch Federer covering theorem [Fed69, 2.8.14] for $\mathbf{R}^n$. That is if G is a family of closed balls in $\mathbf{R}^n$ with $\sup\{\operatorname{diam} B : B \in G\} < \infty$, then there exist disjointed subfamilies $G_1, \ldots, G_{\boldsymbol{\beta}(n)}$ of G such that

$$\{x : \mathbf{B}(x, r) \in G \text{ for some } 0 < r < \infty\} \subset \bigcup \bigcup \{G_i : i = 1, \ldots, \boldsymbol{\beta}(n)\}.$$

A further constant $\boldsymbol{\gamma}(m)$ for the isoperimetric inequality, which depends on positive integers m will be defined in 6.9. Be also aware of the conventions described in 3.8 and 3.9.

3 Preliminaries

This section is a collection of definitions and elementary measure theoretic statements which are preparations for this thesis and will be needed later on. These include some notations concerning Radon measures and the notion of Grassmann manifold.

3.1 Definition (see [Fed69, 2.2.1])**.** Suppose ϕ is a measure over a topological space X.

Then the closed set

$$X \sim \bigcup \{U : U \text{ is open and } \phi(U) = 0\}$$

is called the *support of* ϕ and is denoted by $\operatorname{spt} \phi$.

3.2 Remark. Suppose the topology of X has a countable basis.

Then $\phi(X \sim \operatorname{spt} \phi) = 0$. In this case there holds

$$\phi = \phi \llcorner \operatorname{spt} \phi.$$

If X is a metric space, then it follows

$$\operatorname{spt}\phi = X \cap \{x : \phi(\mathbf{U}(x,r)) > 0 \text{ whenever } 0 < r < \infty\}.$$

3.3. Suppose ϕ is a Borel regular measure over a topological space X and f maps a subset of X into $\mathbf{R}$.

Then there holds

$$\lim_{s \to r-} \phi\{x : f(x) < s\} = \lim_{s \to r-} \phi\{x : f(x) \leq s\} = \phi\{x : f(x) < r\}$$

for all $r \in \mathbf{R}$. Moreover, if f is a Borel function and $r \in \mathbf{R}$ such that $\phi\{x : f(x) < t\} < \infty$ for some $t > r$, then

$$\lim_{s \to r+} \phi\{x : f(x) < s\} = \lim_{s \to r+} \phi\{x : f(x) \leq s\} = \phi\{x : f(x) \leq r\}.$$

These facts may be verified by [Fed69, 2.1.5 (1)] and [Fed69, 2.1.3 (5)].

3.4 Theorem (see [Fed69, 2.2.3])**.** *Suppose ϕ is a Borel regular measure over a topological space X and A is a countably ϕ measurable subset of X.*

Then there exist Borel subsets B and D of X such that

$$D \subset A \subset B \quad \textit{and} \quad \phi(B \sim D) = 0.$$

Proof. Choose ϕ measurable subsets $A_1, A_2, \ldots$ of X such that

$$A = \textstyle\bigcup_{i=1}^{\infty} A_i, \quad \phi(A_j) < \infty$$

for all positive integers j. The conclusion follows from [Fed69, 2.2.3] by taking $D = \bigcup_{i=1}^{\infty} D_i$ and $B = \bigcup_{i=1}^{\infty} B_i$ where the D_j and B_j are Borel subsets of X which correspond to $j \in \mathscr{P}$ such that $D_j \subset A_j \subset B_j$ and $\phi(B_j \sim D_j) = 0$. $\square$

3.5 Theorem (see [Fed69, 2.4.10])**.** *Suppose ϕ is a Borel regular measure over a topological space X, X is countably ϕ measurable, f is a nonnegative ϕ measurable function and*

$$\psi(A) = \textstyle\int_A^* f \,\mathrm{d}\phi \quad \textit{whenever } A \subset X.$$

Then ψ defines a Borel regular measure over X and all ϕ measurable sets are ψ measurable.

Proof. Assume A is any subset of X. By [Fed69, 2.10.4] and [Fed69, 2.2.3] it remains to show

$$\textstyle\int_A^* f \,\mathrm{d}\phi = \inf\{\int_B f \,\mathrm{d}\phi : B \text{ is a Borel subset of } X \text{ and } A \subset B\}.$$

Obviously the left hand side is less or equal to the right hand side. To prove the reverse inequality suppose $\int_A^* f \,\mathrm{d}\phi < \infty$ and let $\varepsilon > 0$. Choose a ϕ step function u such that $u(x) \geq 0$ for ϕ almost all x, $u(x) \geq f(x)$ for ϕ almost all x in A and

$$\varepsilon + \textstyle\int_A^* f \,\mathrm{d}\phi \geq \int u \,\mathrm{d}\phi.$$

Let

$$P = X \cap \{x : f(x) \leq u(x)\}.$$

Then P is ϕ measurable and 3.4 ensures the existence of a Borel subset D of X such that

$$D \subset P \quad \text{and} \quad \phi(P \sim D) = 0.$$

There holds

$$\phi(A \sim D) \le \phi(A \sim P) + \phi(A \cap P \sim D) = 0.$$

Choose a Borel subset C of X such that $A \sim D \subset C$ and $\phi(C) = 0$. For the Borel set $B = D \cup C$ it follows $A \subset B$ and

$$\textstyle\int u \, \mathrm{d}\phi \ge \int_B u \, \mathrm{d}\phi \ge \int_B f \, \mathrm{d}\phi$$

which completes the proof. □

3.6. Suppose X is a topological space and A is a subset of X.

Then the Borel family generated by the relative topology of A in X equals the set

$$\{A \cap B : B \text{ is a Borel subset of } X\}.$$

Hence, if ϕ is a Borel regular measure over X, then $\phi | \mathbf{2}^A$ is a Borel regular measure over A. This may be verified with help of [Fed69, 2.1.2].

3.7 Theorem (see [Fed69, 2.2.17])**.** *Suppose X and Y are locally compact and separable metric spaces, μ is a Radon measure over X and $f : X \to Y$ is a μ measurable function.*

Then $f_\# \mu$ is a Borel regular measure over Y.

Proof. Assume A is a μ measurable subset of X such that $\mu(A) < \infty$. Use Lusin's theorem [Fed69, 2.3.5] to construct a sequence $C_1, C_2, \ldots$ of compact subsets of X such that for each positive integer i there holds $f | C_i$ is continuous and

$$C_i \subset A \sim \textstyle\bigcup_{j=1}^{i-1} C_j, \quad \mu(A \sim \bigcup_{j=1}^{i} C_j) < 1/i.$$

Then $\mu(A \sim \bigcup_{j=1}^{\infty} C_j) = 0$ and $C_i \cap C_j = \varnothing$ whenever $i, j \in \mathscr{P}$ and $i \neq j$. Since X is countably μ measurable, this procedure provides a sequence $K_1, K_2, \ldots$ of pairwise disjointed compact subsets of X such that $f | K_i$ is continuous whenever $i \in \mathscr{P}$ and

$$\mu \left(X \sim \textstyle\bigcup_{j=1}^{\infty} K_j \right) = 0.$$

By 3.6 and [Fed69, 2.2.17], $(f | K_i)_\# \mu | \mathbf{2}^{K_i}$ is a Radon measure over Y whenever $i \in \mathscr{P}$. Since

$$f_\# \mu = \sum_{j=1}^{\infty} (f | K_j)_\# \mu | \mathbf{2}^{K_j},$$

the conclusion follows. □

3.8 (see [Fed69, 2.5.13, 14])**.** Suppose X is a locally compact Hausdorff space and μ is a Radon measure over X. Let

$$\mathscr{K}(X)$$

denote the space of continuous real valued functions on X with compact support. Sometimes the notation

$$\mu(f) \quad \text{for} \quad \textstyle\int f \, \mathrm{d}\mu \quad \text{when } f \in \mathscr{K}(X)$$

will be used. This is natural since the Riesz representation theorem asserts that Radon measures over X correspond in this way with the linear functionals on $\mathscr{K}(X)$ wich are nonnegative on the nonnegative members of $\mathscr{K}(X)$. The notation above also will be used to define Radon measures over X.

3.9 (Grassmann manifold, see [All72, 2.3])**.** Suppose n is a positive integer and k is a nonnegative integer not exceeding n.

Then the so-called Grassmann manifold

$$\mathbf{G}(n, k)$$

is defined to be the space of k dimensional subspaces of $\mathbf{R}^n$. Suppose $S \in \mathbf{G}(n, k)$. Then there exists exactly one member P of $\operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ such that

$$P \circ P = P, \quad P^* = P \quad \text{and} \quad \operatorname{im} P = S.$$

This induces an injective map $\mathbf{G}(n, k) \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$. A member of $\mathbf{G}(n, k)$ also will be considered as its evaluation under this injection. Moreover, $\mathbf{G}(n, k)$ will be endowed with a metric by the requirement that the injection above is distance preserving with respect to the operator norm over $\operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$. With this metric $\mathbf{G}(n, k)$ is compact.

The following definition will be needed to define varifolds in a smooth submanifold of some Euclidean space.

3.10 Definition (see [All72, 2.5])**.** Suppose $k \le m \le n$ are nonnegative integers, $1 \le n$ and M is a smooth m dimensional submanifold of $\mathbf{R}^n$.

Then define

$$\mathbf{G}_k(M) = (M \times \mathbf{G}(n, k)) \cap \{(x, P) : P \subset \operatorname{Tan}(M, x)\}.$$

3.11 Remark. Define a map $F : M \times \mathbf{G}(n, k) \to \mathbf{R}$ by letting

$$F(x, P) = \operatorname{Tan}(M, x) \bullet P = \operatorname{trace}(\operatorname{Tan}(M, x) \circ P)$$

whenever $x \in M$ and $P \in \mathbf{G}(n, k)$, where $\operatorname{Tan}(M, x)$ and P are considered as members of $\operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$, see 3.9. Then F is continuous and $\mathbf{G}_k(M) = F^{-1}\{k\}$. Therefore, $\mathbf{G}_k(M)$ is closed in $M \times \mathbf{G}(n, k)$ and $\mathbf{G}_k(M)$ itself is a locally compact and separable metric space. Moreover, the projection map of $\mathbf{G}_k(M)$ onto M is proper.

4 Varifolds

This section is a summary of [All72, Section 3], which is supposed to make reading the thesis easier. The notion of varifold and rectifiable varifold in a smooth submanifold of some Euclidean space will be recalled. Moreover, it will be shown how to apply a smooth mapping to a varifold. Notice that the following approach is more general than the approach of rectifiable varifolds in [Sim83, Section 4], which does not need the notion of Grassmann manifold. This will be indicated at the end of this section. According to [All72, 4.8 (2), 3.5 (1)], there exist varifolds with locally bounded first variation which are not rectifiable. Hence, in view of the isoperimetric inequality 6.5 the following more general approach makes sense for this thesis.

4.1. A useful set of hypotheses is gathered here for later reference.

Suppose $0 \le k \le m \le n$ *are nonnegative integers,* $1 \le n$*,* M *is a smooth* m *dimensional submanifold of* $\mathbf{R}^n$*,* U *is an open subset of* $\mathbf{R}^n$*,* $M \subset U$*,* $i : M \to U$ *is the inclusion map and* i *is proper.*

4.2 Definition (Varifold, see [All72, 3.1])**.** Suppose k, m, n and M are related as in 4.1.

Then V is said to be a k *dimensional varifold in* M if and only if V is a Radon measure over $\mathbf{G}_k(M)$. Let

$$\mathbf{V}_k(M)$$

be the weakly topologized space of k dimensional varifolds in M. Whenever $V \in \mathbf{V}_k(M)$, let

$$\|V\|(A) = V(\mathbf{G}_k(M) \cap \{(x,P) : x \in A\}) \quad \text{for } A \subset M.$$

In view of 3.7 and 3.11, $\|V\|$ is a Radon measure over M; it is called the *weight* of V.

4.3 Definition (Mapping a varifold, see [All72, 3.2])**.** Suppose k, m, n and M are related as in 4.1, $V \in \mathbf{V}_k(M)$, $\mu \le \nu$ are nonnegative integers, $1 \le \nu$, N is a smooth μ dimensional submanifold of $\mathbf{R}^\nu$ and $F : M \to N$ is smooth.

Then the Borel regular measure

$$F_\# V$$

over $\mathbf{G}_k(N)$ is characterized by the requirement that

$$F_\# V(B) = \int_{\{(x,P):(F(x),\mathrm{D}F(x)[P])\in B\}} |\Lambda_k(\mathrm{D}F(x) \circ P)| \,\mathrm{d}V(x,P)$$

whenever B is a Borel subset of $\mathbf{G}_k(N)$.

4.4 Remark. One may use 3.5 and 3.7 to see that such a Borel regular measure $F_\# V$ exists. The properness of the inclusion map i makes sure that $i_\# V$ is a varifold in U.

4.5 Theorem (see [All72, 3.5])**.** *Suppose* k*,* m*,* n*,* M *and* U *are related as in 4.1,* E *is an* $\mathscr{H}^k$ *measurable subset of* M *which meets every compact subset of* U *in an* $(\mathscr{H}^k, k)$ *rectifiable subset of* U *and* $\mathbf{v}(E)$ *is defined by*

$$\mathbf{v}(E)(A) = \mathscr{H}^k\{x : (x, \operatorname{Tan}^k(\mathscr{H}^k \llcorner E, x)) \in A\}$$

whenever $A \subset \mathbf{G}_k(M)$*.*

Then $\mathbf{v}(E) \in \mathbf{V}_k(M)$*. Moreover,*

$$\|\mathbf{v}(E)\| = \mathscr{H}^k \llcorner E.$$

Proof. Use [Fed69, 3.2.25] and 3.7 for a possible approach. □

4.6 Remark. This notation is ambiguous since $\mathbf{v}(E)$ could be considered a member of $\mathbf{V}_k(U)$. It will always be clear in which space $\mathbf{v}(E)$ shall lie.

4.7 Definition (Rectifiable varifolds, see [All72, 3.5])**.** Suppose k, m, n and M are related as in 4.1.

Then $V \in \mathbf{V}_k(M)$ is said to be a *k dimensional rectifiable varifold in M* if there are positive real numbers $c_1, c_2, \ldots$ and $\mathscr{H}^k$ measurable subsets $E_1, E_2, \ldots$ of M which meet every compact subset of U in an $(\mathscr{H}^k, k)$ rectifiable subset of U such that

$$V = \sum_{j=1}^{\infty} c_j \mathbf{v}(E_j).$$

Let

$$\mathbf{RV}_k(M)$$

denote the space of k dimensional rectifiable varifolds in M.

4.8 Remark. Suppose M is an $\mathscr{H}^m$ measurable and countably $(\mathscr{H}^m, m)$ rectifiable subset of $\mathbf{R}^n$, $\theta : \mathbf{R}^n \to \mathbf{R}$ is a nonnegative $\mathscr{H}^m$ measurable function, $M = \{x : \theta(x) > 0\}$ and the Borel regular measure μ over $\mathbf{R}^n$ defined by

$$\mu(A) = \textstyle\int_A^* \theta(x) \,\mathrm{d}\mathscr{H}^m x \quad \text{whenever } A \subset \mathbf{R}^n$$

is a Radon measure. In other words, $\mathbf{v}(M, \theta)$ is a rectifiable m-varifold in the sense of [Sim83, Section 4]. Its weight equals μ.

Let $r_1, r_2, \ldots$ be a sequence of positive real numbers such that

$$\lim_{j \to \infty} r_j = 0, \quad \textstyle\sum_{j=1}^{\infty} r_j = \infty$$

and let $g_1, g_2, \ldots$ are the characteristic functions of the sets $E_1, E_2, \ldots$ which are inductively defined by

$$E_i = \{x : \theta(x) \geq r_i + \textstyle\sum_{j=1}^{i-1} r_j g_j(x)\}$$

whenever $i \in \mathscr{P}$. If $i \in \mathscr{P}$ and K is a compact subset of $\mathbf{R}^n$, then $\mathscr{H}^m(K \cap E_i) \leq r_i^{-1} \mu(K \cap E_i) < \infty$. Therefore, E_i meets every compact subset of $\mathbf{R}^n$ in an $(\mathscr{H}^m, m)$ rectifiable subset of $\mathbf{R}^n$. By [Fed69, 2.3.3] there holds

$$\theta = \textstyle\sum_{j=1}^{\infty} r_j g_j.$$

It follows

$$\left(\textstyle\sum_{j=1}^{\infty} r_j \|\mathbf{v}(E_j)\|\right)(B) = \mu(B)$$

for all Borel subsets B of $\mathbf{R}^n$ by [Fed69, 2.4.8]. Hence,

$$V = \textstyle\sum_{j=1}^{\infty} r_j \mathbf{v}(E_j)$$

defines an m dimensional rectifiable varifold in $\mathbf{R}^n$ which satisfies $\|V\| = \mu$. There holds

$$\theta(x) = \boldsymbol{\Theta}^m(\mu, x) \quad \text{for } \mathscr{H}^m \text{ almost all } x \in \mathbf{R}^n,$$
$$V(\alpha) = \textstyle\int \alpha(x, \operatorname{Tan}^m(\mu, x)) \theta(x) \,\mathrm{d}\mathscr{H}^m x \quad \text{whenever } \alpha \in \mathscr{K}(\mathbf{G}_m(\mathbf{R}^n))$$

by [All72, 3.5 (1)(b)].

Now suppose V is any m dimensional rectifiable varifold in $\mathbf{R}^n$. Then there holds

$$\|V\|(A) = \textstyle\int_A^* \boldsymbol{\Theta}^m(\|V\|, x) \,\mathrm{d}\mathscr{H}^m x \quad \text{whenever } A \subset \mathbf{R}^n$$

and $\{x : \mathbf{\Theta}^m(\|V\|, x) > 0\}$ is an $\mathscr{H}^m$ measurable and countably $(\mathscr{H}^m, m)$ rectifiable subset of $\mathbf{R}^n$ by [All72, 3.5 (1)] and [All72, 2.8 (5)]. Hence,

$$\mathbf{v}\big(\{x : \mathbf{\Theta}^m(\|V\|, x) > 0\}, \mathbf{\Theta}^m(\|V\|, \cdot)\big)$$

is a rectifiable m-varifold in the sense of [Sim83, Section 4].

This shows that the notion of rectifiable varifold in Euclidean space given in this section leads to the same notion of rectifiable varifold in [Sim83, Section 4].

5 The First Variation of a Varifold

This section is based on [All72, Section 4]. Motivated by the variational formula [All72, 4.1], one associates with each varifold in a smooth submanifold M of some Euclidean space, a real valued linear map on $\mathscr{X}(M)$, called the first variation of the varifold. If this first variation is representable by integration, one obtains the generalized mean curvature vectorfield, which is the same as the mean curvature vector field [All72, 2.5 (2)] whenever the varifold corresponds to a smooth submanifold of Euclidean space. Given a varifold in a smooth manifold properly embedded in some open set U of an Euclidean space, it will be shown how to extend this varifold to a varifold in U. The first variations of these varifolds are related by a simple formula. This makes it possible to apply the isoperimetric inequality 6.7, which is stated for varifolds in some Euclidean space to varifolds in an appropriate submanifold of some Euclidean space. Moreover, the formula [All72, 4.10 (1)] which shows how to cut a varifold, motivates a definition of indecomposable varifolds. Varifolds which correspond to connected and compact submanifolds of some Euclidean space are indecomposable. A class of varifolds which are decomposable will be given. Finally, it will be shown how to construct examples of varifolds which will be useful later on.

5.1 Definition (First variation of a varifold, see [All72, 4.2]). Suppose k, m, n and M are related as in 4.1 and $V \in \mathbf{V}_k(M)$.

Then the linear function

$$\delta V : \mathscr{X}(M) \to \mathbf{R}$$

defined by

$$\delta V(g) = \textstyle\int (\mathrm{D}g(x) \circ P) \bullet P \, \mathrm{d}V(x, P) \quad \text{for } g \in \mathscr{X}(M)$$

is called the *first variation of* V, where $\mathscr{X}(M)$ denotes the space of smooth vector fields with compact support in M. The *total variation of* δV

$$\|\delta V\|$$

is defined by letting

$$\|\delta V\|(G) = \sup\{\delta V(g) : g \in \mathscr{X}(M), \operatorname{spt} g \subset G \text{ and } |g| \leq 1\}$$

whenever G is an open subset of M and

$$\|\delta V\|(A) = \inf\{\|\delta V\|(G) : G \text{ is open in } M, \, A \subset G\}$$

whenever A is any subset of M.

5.2 Remark. An analogous approach of the Riesz representation theorem [Sim83, 4.1] shows that $\|\delta V\|$ defines a Borel regular measure over M. Clearly there holds

$$\operatorname{spt}\|\delta V\| \subset \operatorname{spt}\|V\|.$$

Suppose U is an open subset of $\mathbf{R}^n$, $M = U$ and $g \in \mathscr{X}(U)$. Then there holds

$$(\mathrm{D}g(x) \circ P) \bullet P = \mathrm{D}g(x) \bullet P = \operatorname{div}(P \circ g)(x)$$

whenever $(x, P) \in \mathbf{G}_k(U)$.

5.3 (Functions of bounded variation)**.** Suppose n is a positive integer, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{RV}_n(U)$ and $\|\delta V\|$ is a Radon measure.

Then $\boldsymbol{\Theta}^n(\|V\|, \cdot)$ has locally bounded variation in U and if $\|\delta V\|(U) < \infty$, then $\boldsymbol{\Theta}^n(\|V\|, \cdot)$ has bounded variation in U in the sense of [EG92, 5.1].

On the other hand, for each nonnegative function f which has bounded variation in U one obtains a varifold $V \in \mathbf{RV}_n(U)$ such that $\|\delta V\|(U) < \infty$ by letting

$$V(\alpha) = \textstyle\int \alpha(x, \mathbf{1}_{\mathbf{R}^n}) f(x)\, \mathrm{d}\mathscr{L}^n x \quad \text{whenever } \alpha \in \mathscr{K}(\mathbf{G}_n(U)).$$

Actually $\|\delta V\|(U) = \|\mathrm{D}f\|(U)$. In particular the isoperimetric inequality for bounded sets with finite perimeter in $\mathbf{R}^n$ [EG92, 5.6.2 (i)] is a special case of 6.7.

5.4 (Representing the first variation by integration, see [All72, 4.3])**.** Suppose k, m, n, M and i are related as in 4.1, $V \in \mathbf{V}_k(M)$ and $\|\delta V\|$ is a Radon measure.

Then well known representation theorems [Fed69, 2.5.12] assert the existence of a $\|\delta V\|$ measurable function $\boldsymbol{\eta}(V; \cdot)$ with values in $\mathbf{S}^{n-1}$ such that

$$\boldsymbol{\eta}(V; x) \in \operatorname{Tan}(M, x) \quad \text{for } \|\delta V\| \text{ almost all } x$$

and

$$\delta V(g) = \textstyle\int g(x) \bullet \boldsymbol{\eta}(V; x)\, \mathrm{d}\|\delta V\| x \quad \text{for } g \in \mathscr{X}(M).$$

Notice that $\boldsymbol{\eta}(V; \cdot)$ is $\|\delta V\|$ almost unique.

Moreover, the theory of symmetrical derivation [Fed69, 2.8.18, 2.9] implies the following. The formula

$$(\|\delta V\| / \|V\|)(x) = \lim_{r \to 0+} i_\# \|\delta V\| \mathbf{B}(x, r) / i_\# \|V\| \mathbf{B}(x, r) \quad \text{for } \|V\| \text{ almost all } x$$

defines a real valued $\|V\|$ measurable function,

$$\mathbf{h}(V; x) = -(\|\delta V\| / \|V\|)(x) \boldsymbol{\eta}(V; x) \quad \text{for } \|V\| \text{ almost all } x$$

defines a $\|V\|$ measurable function with values in $\mathbf{R}^n$ such that

$$\mathbf{h}(V; x) \in \operatorname{Tan}(M, x) \quad \text{for } \|V\| \text{ almost all } x$$

and such that if

$$\|\delta V\|_{\mathrm{sing}} = \|\delta V\| \llcorner \{x : (\|\delta V\| / \|V\|)(x) = \infty\},$$

then

$$\delta V(g) = - \textstyle\int g(x) \bullet \mathbf{h}(V; x)\, \mathrm{d}\|V\| x + \int g(x) \bullet \boldsymbol{\eta}(V; x)\, \mathrm{d}\|\delta V\|_{\mathrm{sing}} x$$

whenever g is a Borel function with values in $\mathbf{R}^n$ such that $g(x) \in \operatorname{Tan}(M, x)$ for $x \in M$ and $\int |g| \,\mathrm{d}\|\delta V\| < \infty$, where $\delta V(g)$ is defined to be the value of the unique $\|\delta V\|_{(1)}$ continuous extension of δV to

$$\mathbf{L}_1(\|\delta V\|, \mathbf{R}^n) \cap \{h : h(x) \in \operatorname{Tan}(M, x) \text{ for } \|\delta V\| \text{ almost all } x\}.$$

The function $\mathbf{h}(V; \cdot)$ is called the *generalized mean curvature vector of* V.

The following theorem is a generalization of [All72, 4.8 (3)].

5.5 Theorem. *Suppose $m \leq n$ are positive integers, U is an open subset of $\mathbf{R}^n$, $a \in U$, $V \in \mathbf{V}_m(U)$ and $\|V\|(\{a\}) > 0$.*

Then there holds $\|\delta V\|(\{a\}) = \infty$.

Proof. Choose $S \in \mathbf{G}(n, m)$ and $v \in \mathbf{S}^{n-1}$ such that

$$V(\{a\} \times W) > 0, \quad |S(v)| > 0$$

for all neighbourhoods W of S in $\mathbf{G}(n, m)$. Then, for

$$l = \textstyle\int_{\{a\} \times \mathbf{G}(n,m)} |P(v)|^2 \,\mathrm{d}V(x, P)$$

there holds $0 < l < \infty$. Let $0 < \varepsilon < 1$ such that $\mathbf{U}(a, \varepsilon) \subset U$ and let $4\varepsilon^{-1} < \lambda < \infty$. It will be shown that

$$l\lambda - m\|V\|\mathbf{U}(a, \varepsilon) \leq \|\delta V\|\mathbf{U}(a, \varepsilon).$$

For this purpose choose a smooth function $\varphi : \mathbf{R} \to \mathbf{R}$ such that

$$\begin{gathered} 0 \leq \varphi \leq 1, \quad \varphi' \leq 0, \\ \varphi(t) = 1 \quad \text{for } -\infty < t \leq 2^{-1}\lambda^{-1}, \\ \varphi'(t) \leq -\lambda^2 \quad \text{for } t = \lambda^{-1}, \\ \varphi(t) = 0 \quad \text{for } 2\lambda^{-1} \leq t < \infty. \end{gathered}$$

Let $b = a - \lambda^{-1}v$ and define $g \in \mathscr{X}(U)$ by

$$g(x) = -\varphi(|x - b|)(x - b)$$

for $x \in U$. There holds $|x - a| - \lambda^{-1} \leq |x - b| \leq |x - a| + \lambda^{-1}$ whenever $x \in U$. Hence, $\operatorname{spt} g \subset \mathbf{B}(a, 3\lambda^{-1}) \subset \mathbf{U}(a, \varepsilon)$ and $|g| \leq 1$. By [All72, 2.3 (2)(3)(4)] one calculates

$$\begin{gathered} \mathrm{D}g(b) \bullet P = 0, \\ \mathrm{D}g(x) \bullet P = -\varphi'(|x - b|)|x - b|^{-1}|P(x - b)|^2 - \varphi(|x - b|)m \end{gathered}$$

for $b \neq x \in U$ and $P \in \mathbf{G}(n, m)$. Therefore,

$$\begin{aligned} &\textstyle\int \mathrm{D}g(x) \bullet P \,\mathrm{d}V(x, P) \\ &\quad \geq \textstyle\int_{\{a\} \times \mathbf{G}(n,m)} -\varphi'(\lambda^{-1})\lambda|P(\lambda^{-1}v)|^2 \,\mathrm{d}V(x, P) - m\|V\|\mathbf{U}(a, \varepsilon) \\ &\quad \geq l\lambda - m\|V\|\mathbf{U}(a, \varepsilon) \end{aligned}$$

and the conclusion follows. □

5.6 (see [All72, 4.4]). Suppose $1 \le k \le m < n$, M, U and i are related as in 4.1 and $V \in \mathbf{V}_k(M)$.

Then, if $k = m$, [All72, 2.5(2)] yields

$$\delta(i_\# V)(g) = \delta V(\operatorname{Tan}(M,g)) - \textstyle\int \operatorname{Nor}(M,g)(x) \bullet \mathbf{h}(M,x)\, \mathrm{d}\|V\|x$$

whenever $g \in \mathscr{X}(U)$ and if $k < m$, [All72, 2.5(3)] yields

$$\delta(i_\# V)(g) = \delta V(\operatorname{Tan}(M,g)) - \textstyle\int \operatorname{Nor}(M,g)(x) \bullet \mathbf{h}(M;x,P)\, \mathrm{d}V(x,P)$$

whenever $g \in \mathscr{X}(U)$.

5.7 Theorem. *Suppose $m \le n$ are positive integers, U is an open subset of $\mathbf{R}^n$, $j : U \to \mathbf{R}^n$ is the inclusion map, $V \in \mathbf{V}_m(U)$ and $j[\operatorname{spt}\|V\|]$ is closed in $\mathbf{R}^n$.*

Then $j_\# V \in \mathbf{V}_m(\mathbf{R}^n)$, $\|j_\# V\| = j_\# \|V\|$ and $\|\delta(j_\# V)\| = j_\# \|\delta V\|$.

Proof. Clearly there holds $(j_\# V)(B) = V(\mathbf{G}_m(U) \cap B)$ for all Borel subsets B of $\mathbf{G}_m(\mathbf{R}^n)$. Hence, $\|j_\# V\| = j_\# \|V\|$. By 3.2, this leads to

$$\operatorname{spt}\|j_\# V\| = j[\operatorname{spt}\|V\|]$$

as $j[\operatorname{spt}\|V\|]$ is closed in $\mathbf{R}^n$. Let G be an open subset of U, $g \in \mathscr{X}(U)$ such that $\operatorname{spt} g \subset G$ and define $\bar{g} = g \cup ((\mathbf{R}^n \sim U) \times \{0\})$. Then $\operatorname{spt} \bar{g} = j[\operatorname{spt} g]$, $\bar{g} \in \mathscr{X}(\mathbf{R}^n)$ and $\delta(j_\# V)(\bar{g}) = \delta V(g)$. Therefore, $\|\delta(j_\# V)\| \ge j_\# \|\delta V\|$. Now let G be an open subset of $\mathbf{R}^n$ and $g \in \mathscr{X}(\mathbf{R}^n)$ such that $\operatorname{spt} g \subset G$. Choose a smooth function $\varphi : \mathbf{R}^n \to \mathbf{R}$ such that $0 \le \varphi \le 1$, $\operatorname{spt}\varphi \subset U \cap G$ and $\operatorname{spt} g \cap j[\operatorname{spt}\|V\|] \subset \operatorname{Int}\{x : \varphi(x) = 1\}$. Then $\operatorname{spt}(\varphi g)|U \subset U \cap G$, $(\varphi g)|U \in \mathscr{X}(U)$ and $\delta(j_\# V)(g) = \delta V((\varphi g)|U)$. Hence, $\|\delta(j_\# V)\| \le j_\# \|\delta V\|$. ☐

5.8 Lemma (see [All72, 4.5]). *Suppose m, n, M and U are related as in 4.1, W is an open and connected subset of U, Y is an open subset of $\mathbf{R}^m$, $\psi : W \to Y$ and $\varphi : Y \to W$ are smooth, $\psi \circ \varphi = \mathbf{1}_Y$ and $W \cap \operatorname{im}\varphi = W \cap M$.*

Then, for $V = \mathbf{v}(M) \in \mathbf{V}_m(M)$ there holds

$$\delta V(g) = 0 \quad \text{whenever } g \in \mathscr{X}(M) \text{ and } \operatorname{spt} g \subset W \cap M.$$

Proof. See the first part of the proof [All72, 4.5]. ☐

5.9 Theorem. *Suppose $1 \le m < n$, M, U and i are related as in 4.1 and $V = \mathbf{v}(M) \in \mathbf{V}_m(M)$.*

Then there holds $i_\# V = \mathbf{v}(M) \in \mathbf{V}_m(U)$ and

$$\delta(i_\# V)(g) = - \textstyle\int_M g(x) \bullet \mathbf{h}(M,x)\, \mathrm{d}\mathscr{H}^m x$$

whenever $g \in \mathscr{X}(U)$.

Proof. The first statement follows as $\operatorname{Tan}^m(\mathscr{H}^m \llcorner M, x) = \varnothing$ for $\mathscr{H}^m$ almost all $x \in U \sim M$ by [Fed69, 2.10.19 (4)]. Let $g \in \mathscr{X}(U)$. Notice that $M \cap \operatorname{spt} g$ is compact and choose a finite covering of $M \cap \operatorname{spt} g$ which consists of open sets W as in 5.8. Take a partition of unity subordinate to this open covering. Then 5.8 leads to $\delta V(\operatorname{Tan}(M,g)) = 0$. The conclusion follows by 5.6. ☐

5.10 Definition (see [Men16a, 6.2])**.** Suppose $m \leq n$ are positive integers, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$ and $\|\delta V\|$ is a Radon measure.

Then V is called *indecomposable* if and only if there exists no Borel subset E of U such that

$$\|V\|(E) > 0, \quad \|V\|(U \sim E) > 0, \quad (\delta V) \llcorner E = \delta(V \llcorner E \times \mathbf{G}(n,m))$$

where $((\delta V) \llcorner E)(g) = \delta V(\chi_E g)$ whenever $g \in \mathscr{X}(U)$ and χ_E is the characteristic function of E, see [Men16a, 2.20].

5.11 Theorem (see [Men16a, 6.5])**.** *Suppose $m \leq n$ are positive integers, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure and V is indecomposable.*

Then $\operatorname{spt} \|V\|$ *is connected.*

Proof. See the proof of [Men16a, 6.5]. □

5.12 Theorem. *Suppose $1 \leq m < n$, M and U, are related as in 4.1, M is connected, $V = \mathbf{v}(M) \in \mathbf{V}_m(U)$ and $\|\delta V\|$ is a Radon measure.*

Then V is indecomposable.

Proof. Assume E is a Borel subset of U such that $\mathscr{H}^m(M \cap E) > 0$ and $(\delta V) \llcorner E = \delta(V \llcorner E \times \mathbf{G}(n,m))$. Let $W = V \llcorner E \times \mathbf{G}(n,m)$. Then $W \in \mathbf{V}_m(U)$, $\operatorname{spt} \|W\| \subset \operatorname{spt} \|V\| \subset M$ and $\|\delta W\|$ is a Radon measure. One calculates by 5.9 and [All72, 2.5 (2), 3.5 (1)(b)]

$$\begin{aligned}
0 &= ((\delta V) \llcorner E)(g) - \delta(V \llcorner E \times \mathbf{G}(n,m))(g) \\
&= - \textstyle\int_E g(x) \bullet \mathbf{h}(M,x) \,\mathrm{d}(\mathscr{H}^m \llcorner M)x - \int_{E \times \mathbf{G}(n,m)} \mathrm{D} g(x) \bullet P \,\mathrm{d}V(x,P) \\
&= - \textstyle\int_{E \times \mathbf{G}(n,m)} (\mathrm{D}\operatorname{Tan}(M,g)(x) \circ P) \bullet P \,\mathrm{d}V(x,P) \\
&= - \delta W(\operatorname{Tan}(M,g))
\end{aligned}$$

whenever $g \in \mathscr{X}(U)$. Hence, [All72, 4.6 (3)] implies $W = \mathbf{v}(M)$. This means $\mathscr{H}^m \llcorner M = \mathscr{H}^m \llcorner (M \cap E)$, $\mathscr{H}^m(M \sim E) = 0$. □

5.13 Remark. The previous proof is based on the proof [Men16a, 5.9 (1)].

5.14 Lemma. *Suppose $m < n$ are positive integers, G is a nonempty countable family of compact and smooth m dimensional submanifolds of $\mathbf{R}^n$, $E = \bigcup G$, $\mathscr{H}^m(E) < \infty$, $V = \mathbf{v}(E) \in \mathbf{RV}_m(\mathbf{R}^n)$ and $\mathscr{H}^m(M \cap N) = 0$ whenever $M, N \in G$ and $M \neq N$.*

Then there holds

$$\begin{gathered}
\boldsymbol{\Theta}^m(\|V\|, x) = 1 \quad \text{for } \|V\| \text{ almost all } x, \\
\delta V(g) = - \sum_{M \in G} \textstyle\int_M g(x) \bullet \mathbf{h}(M,x) \,\mathrm{d}\mathscr{H}^m x \quad \text{whenever } g \in \mathscr{X}(\mathbf{R}^n)
\end{gathered}$$

and if $\sum_{M \in G} \int_M |\mathbf{h}(M,x)| \,\mathrm{d}\mathscr{H}^m x < \infty$, then $\|\delta V\|$ is a Radon measure, $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ and for all $M \in G$ there holds

$$\mathbf{h}(V;x) = \mathbf{h}(M,x) \quad \text{for } \mathscr{H}^m \text{ almost all } x \in M.$$

Proof. First notice that E is $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable. By [Fed69, 3.2.19] there holds

$$\boldsymbol{\Theta}^m(\|V\|, \cdot) = \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner E, \cdot)$$

is $\mathscr{H}^m$ almost equal the characteristic function of E. The definition of the approximate tangent space [Fed69, 3.2.16] implies

$$\operatorname{Tan}^m(\mathscr{H}^m \llcorner M, x) \subset \operatorname{Tan}^m(\mathscr{H}^m \llcorner E, x) \quad \text{whenever } M \in G \text{ and } x \in \mathbf{R}^n.$$

Therefore, if $M \in G$

$$\operatorname{Tan}^m(\mathscr{H}^m \llcorner M, x) = \operatorname{Tan}^m(\mathscr{H}^m \llcorner E, x) \quad \text{for } \mathscr{H}^m \text{ almost all } x \in M$$

by [Fed69, 3.2.19]. One calculates with help of [All72, 3.5 (1)(b)], Lebesgue's bounded convergence theorem and 5.9

$$\begin{aligned} \delta V(g) &= \textstyle\int \mathrm{D}g(x) \bullet \operatorname{Tan}^m(\mathscr{H}^m \llcorner E, x) \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner E, x) \,\mathrm{d}\mathscr{H}^m x \\ &= \textstyle\sum_{M \in G} \int_M \mathrm{D}g(x) \bullet \operatorname{Tan}^m(\mathscr{H}^m \llcorner M, x) \,\mathrm{d}\mathscr{H}^m x \\ &= -\textstyle\sum_{M \in G} \int_M g(x) \bullet \mathbf{h}(M, x) \,\mathscr{H}^m x \end{aligned}$$

whenever $g \in \mathscr{X}(\mathbf{R}^n)$. Now assume

$$\textstyle\sum_{M \in G} \int_M |\mathbf{h}(M, x)| \,\mathrm{d}\mathscr{H}^m x < \infty.$$

Whenever $M \in G$ let χ_M be the characteristic function of M in $\mathbf{R}^n$. Lebesgue's bounded convergence theorem implies

$$\delta V(g) = -\textstyle\int g(x) \bullet \big(\sum_{M \in G} \mathbf{h}(M, x) \chi_M(x) \big) \,\mathrm{d}\mathscr{H}^m x$$

whenever $g \in \mathscr{X}(\mathbf{R}^n)$. Therefore, $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ and

$$\delta V(g) = -\textstyle\int g(x) \bullet \mathbf{h}(V; x) \,\mathrm{d}(\mathscr{H}^m \llcorner E)x$$

whenever $g \in \mathscr{X}(\mathbf{R}^n)$. This implies the conclusion. $\square$

5.15 Remark. The previous lemma gives a quantity of examples for varifolds which are decomposable.

5.16 Theorem (see [Men16a, 14.1])**.** *Suppose $m < n$ are positive integers and U is an open and bounded subset of $\mathbf{R}^n$.*

Then there exists $V \in \mathbf{RV}_m(\mathbf{R}^n)$ such that

$$\begin{gathered} \|V\|(\mathbf{R}^n) < \infty, \quad \operatorname{spt} \|V\| = \operatorname{Clos} U, \\ \boldsymbol{\Theta}^m(\|V\|, x) = 1 \quad \textit{for } \|V\| \textit{ almost all } x \end{gathered}$$

and if $m \geq 2$, then $\|\delta V\|$ is a Radon measure and

$$\begin{gathered} \delta V(g) = -\textstyle\int g(x) \bullet \mathbf{h}(V; x) \,\mathrm{d}\|V\| x \quad \textit{for } g \in \mathscr{X}(\mathbf{R}^n), \\ \textstyle\int |\mathbf{h}(V; x)|^{m-1} \,\mathrm{d}\|V\| x < \infty. \end{gathered}$$

Proof. Define

$$S = \mathbf{R}^n \cap \{(x_1, \dots, x_n) : \textstyle\sum_{i=1}^{m+1} x_i^2 = 1,\, x_{m+2} = \dots = x_n = 0\}$$

which is an embedding of the m dimensional sphere and

$$Q(x,r) = \mathbf{R}^n \cap \{(y_1, \dots, y_n) : |x_i - y_i| < r \text{ for } i = 1, \dots, n\},$$
$$S(x,r) = rS + x$$

whenever $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ and $0 < r < \infty$. Define furthermore

$$F_i = \{Q(x, 2^{-i}) : 2^{i-1}x \in \mathbf{Z}^n\}, \quad G_i = F_i \cap \{Q : Q \subset U\},$$
$$H_i = \{S(x, 2^{-i(n+1)}) : 2^{i-1}x \in \mathbf{Z}^n,\, Q(x, 2^{-i}) \subset U\}$$

for all positive integers i. Choose an integer k such that $U \subset Q(0, 2^k)$. Then for each positive integer i, the set G_i consists of at most $2^{n(k+i)}$ elements. For $E = \bigcup_{i=1}^\infty \bigcup H_i$ it follows

$$\begin{aligned}\mathscr{H}^m(E) &\leq \textstyle\sum_{i=1}^\infty 2^{n(k+i)}(m+1)\boldsymbol{\alpha}(m+1)(2^{-i(n+1)})^m \\ &\leq 2^{nk}(m+1)\boldsymbol{\alpha}(m+1)(1-2^{-m})^{-1} < \infty.\end{aligned}$$

Hence, one may define $V = \mathbf{v}(E) \in \mathbf{RV}_m(\mathbf{R}^n)$. Notice that each F_i is a disjointed family and $\mathbf{R}^n = \bigcup\{\operatorname{Clos} Q : Q \in F_i\}$ whenever i is a positive integer. So it is easy to see that $\operatorname{spt}\|V\| = \operatorname{Clos} U$. By 5.14, the case $m = 1$ should now be clear. Hence, suppose $2 \leq m$. Notice that $|\mathbf{h}(M,x)| = m(2^{-i(n+1)})^{-1}$ whenever i is a positive integer, $M \in H_i$ and $x \in M$. Therefore, one calculates

$$\begin{aligned}&\textstyle\sum\{\int_M |\mathbf{h}(M,x)|\,\mathrm{d}\mathscr{H}^m x : M \in \bigcup_{i=1}^\infty \bigcup H_i\} \\ &\quad\leq \textstyle\sum_{i=1}^\infty 2^{n(k+i)}(m+1)\boldsymbol{\alpha}(m+1)(2^{-i(n+1)})^m m(2^{-i(n+1)})^{-1} \\ &\quad\leq 2^{nk}(m+1)\boldsymbol{\alpha}(m+1)m < \infty.\end{aligned}$$

Remember that the intersection of two different m dimensional spheres in $\mathbf{R}^{m+1}$ always has $\mathscr{H}^m$ measure zero. According to 5.14, it follows that $\|\delta V\|$ is a Radon measure,

$$\boldsymbol{\Theta}^m(\|V\|, x) = 1 \quad \text{for } \|V\| \text{ almost all } x,$$
$$\delta V(g) = \textstyle\int_E g(x) \bullet \mathbf{h}(V;x)\,\mathrm{d}\mathscr{H}^m x \quad \text{for } g \in \mathscr{X}(\mathbf{R}^n)$$

and for all $M \in \bigcup_{i=1}^\infty H_i$ there holds

$$\mathbf{h}(V;x) = \mathbf{h}(M,x) \quad \text{for } \mathscr{H}^m \text{ almost all } x \in M.$$

One calculates

$$\begin{aligned}&\textstyle\int |\mathbf{h}(V;x)|^{m-1}\,\mathrm{d}\|V\|x \\ &\quad\leq \textstyle\sum_{i=1}^\infty 2^{n(k+i)}(m+1)\boldsymbol{\alpha}(m+1)(2^{-i(n+1)})^m(m(2^{-i(n+1)})^{-1})^{m-1} \\ &\quad\leq 2^{nk}(m+1)\boldsymbol{\alpha}(m+1)m^{m-1} < \infty\end{aligned}$$

which completes the proof. □

5.17 Remark. The previous proof is based on [Men09, 1.2].

5.18 Example. Suppose $m \le n$ are positive integers, $T \in \mathbf{G}(n,m)$, $f : \mathbf{R}^m \to \mathbf{R}^n$ is an orthogonal injection, $\operatorname{im} f = T$, $\theta : \mathbf{R}^m \to \mathbf{R}$ is a nonnegative Borel function, θ is locally $\mathscr{L}^m$ summable and the Borel regular measure μ over $\mathbf{R}^n$ is defined by

$$\mu(A) = \textstyle\int^*_{A \cap T} \theta(f^*(y)) \,\mathrm{d}\mathscr{H}^m y \quad \text{whenever } A \subset \mathbf{R}^n.$$

The area formula [Fed69, 3.2.5] yields

$$\begin{aligned} \mu(B) &= \textstyle\int \theta(f^*(y)) N(f|f^*[B \cap T], y) \,\mathrm{d}\mathscr{H}^m y \\ &= \textstyle\int_{f^*[B \cap T]} \theta(x) \,\mathrm{d}\mathscr{L}^m x \end{aligned}$$

for all Borel subsets B of $\mathbf{R}^n$. In particular μ is a Radon measure. For each positive integer i define

$$M_i = T \cap \{y : i^{-1} \le \theta(f^*(y)) \le i\}.$$

Then there holds

$$\begin{gathered} i^{-1}\mathscr{H}^m(M_i \cap S) \le \mu(M_i \cap S) \le i\mathscr{H}^m(M_i \cap S) \quad \text{whenever } S \subset \mathbf{R}^n, \\ \operatorname{Tan}^m(\mu \llcorner M_i, a) = \operatorname{Tan}^m(\mathscr{H}^m \llcorner M_i, a) = T \quad \text{for } \mathscr{H}^m \text{ almost all } a \in M_i, \\ T \subset \operatorname{Tan}^m(\mu, a) \quad \text{for } \mathscr{H}^m \text{ almost all } a \in M_i, \\ T \subset \operatorname{Tan}^m(\mu, a) \quad \text{for } \mathscr{H}^m \text{ almost all } a \in T \cap \{y : \theta(f^*(y)) > 0\}. \end{gathered}$$

In view of 4.8,

$$V(\alpha) = \textstyle\int_T \alpha(y, T)\theta(f^*(y)) \,\mathrm{d}\mathscr{H}^m y \quad \text{whenever } \alpha \in \mathscr{K}(\mathbf{G}_m(\mathbf{R}^n))$$

defines an m dimensional varifold V in $\mathbf{R}^n$ with $\|V\| = \mu$. Assume $g \in \mathscr{X}(\mathbf{R}^n)$. One calculates

$$\begin{aligned} \delta V(g) &= \textstyle\int_T \mathrm{D}g(y) \bullet T\, \theta(f^*(y)) \,\mathrm{d}\mathscr{H}^m y = \int \mathrm{D}g(f(x)) \bullet T\, \theta(x) \,\mathrm{d}\mathscr{L}^m x \\ &= \textstyle\int \operatorname{div}(f^* \circ g \circ f)(x)\theta(x) \,\mathrm{d}\mathscr{L}^m x \end{aligned}$$

and if θ is a Lipschitzian function, it follows

$$\begin{aligned} \delta V(g) &= -\textstyle\int (f^* \circ g \circ f)(x) \bullet \operatorname{grad} \theta(x) \,\mathrm{d}\mathscr{L}^m x \\ &= -\textstyle\int (g \circ f)(x) \bullet (f \circ \operatorname{grad} \theta)(x) \,\mathrm{d}\mathscr{L}^m x \\ &= -\textstyle\int_T g(y) \bullet (f \circ \operatorname{grad} \theta \circ f^*)(y) \,\mathrm{d}\mathscr{H}^m y. \end{aligned}$$

Hence, if $|g| \le 1$, then there holds

$$\delta V(g) \le \textstyle\int_{f^*[\operatorname{spt} g]} |\operatorname{grad} \theta(x)| \,\mathrm{d}\mathscr{L}^m x.$$

6 Isoperimetric Inequality

In this section the isoperimetric inequality for varifolds in some Euclidean space with locally finite first variation will be proved. An important tool for the proof is the monotonicity identity in the version of Menne [Men16a]. In contrary to Simon's monotonicity identity [Sim83, 17.3, 17.4], this allows to neglect the assumptions that the varifold is rectifiable and that its total variation is absolutely continuous with respect to the weight. Moreover, a lemma of Simon [Sim83] will be employed, see 6.3. This is designed to apply the Vitali covering theorem. In contrary to the Besicovitch Federer covering theorem, which is used in the proof of Allard's isoperimetric inequality [All72, 7.1], the Vitali covering theorem does not involve a constant depending on the ambient space. Once the isoperimetric inequality for varifolds in some Euclidean space is proved, an isoperimetric inequality for varifolds in an appropriate submanifold of some Euclidean space follows instantly by 5.6.

6.1 Theorem (Monotonicity identity, see [Men16a, 4.5, 4.6])**.** *Suppose $m \leq n$ are positive integers, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$ and $\|\delta V\|$ is a Radon measure.*

Then there holds

$$\begin{aligned} &t^{-m}\|V\|\mathbf{B}(a,t) + \textstyle\int_{(\mathbf{B}(a,s)\sim\mathbf{B}(a,t))\times\mathbf{G}(n,m)} |x-a|^{-m-2}|P^\perp(x-a)|^2 \,\mathrm{d}V(x,P) \\ &\quad = s^{-m}\|V\|\mathbf{B}(a,s) + \textstyle\int_t^s u^{-m-1}\int_{\mathbf{B}(a,u)} (x-a)\bullet \boldsymbol{\eta}(V,x)\,\mathrm{d}\|\delta V\|x \,\mathrm{d}\mathscr{L}^1 u \end{aligned}$$

whenever $a \in \mathbf{R}^n$, $0 < t \leq s < r$ and $\mathbf{B}(a,r) \subset U$.

Proof. See [Men16a, 4.2, 5, 6]. □

6.2 Remark. The preceding theorem is a slight generalization of [Sim83, 17.3, 17.4].

6.3 Lemma (see [Sim83, 18.7])**.** *Suppose m is a positive integer, f and g are real valued functions on $\{t : 0 < t < \infty\}$, f is bounded and non-decreasing and*

$$1 \leq \limsup_{t\to 0+} t^{-m} f(t),$$

$$s^{-m} f(s) \leq r^{-m} f(r) + \int_s^r t^{-m} g(t)\,\mathrm{d}\mathscr{L}^1 t$$

whenever $0 < s < r < \infty$.

Then there exists r such that $0 < r \leq \mu = (2^{m+1}\lim_{t\to\infty} f(t))^{1/m}$ and

$$f(5r) < 2^{-1}5^m \mu g(r).$$

Proof. First notice that the hypothesis $1 \leq \limsup_{t\to 0+} t^{-m} f(t)$ yields $0 < f$ and $0 < \mu < \infty$. Assume the conclusion were false. That is

$$2^{-1}5^m \mu g(r) \leq f(5r) \quad \text{for all } 0 < r \leq \mu.$$

If $m \geq 2$, then for $0 < s < \mu/5$ it would follow by the Transformation formula [Fed69, 3.2.6]

$$\begin{aligned} \sup_{s\leq t\leq\mu} t^{-m} f(t) &\leq \mu^{-m} f(\mu) + \frac{2}{\mu}\int_s^\mu 5^{-m} t^{-m} f(5t)\,\mathrm{d}\mathscr{L}^1 t \\ &= \mu^{-m} f(\mu) + \frac{2}{5\mu}\int_{5s}^{5\mu} t^{-m} f(t)\,\mathrm{d}\mathscr{L}^1 t \\ &= \mu^{-m} f(\mu) + \frac{2}{5\mu}\left(\int_{5s}^{\mu} t^{-m} f(t)\,\mathrm{d}\mathscr{L}^1 t + \int_{\mu}^{5\mu} t^{-m} f(t)\,\mathrm{d}\mathscr{L}^1 t\right) \\ &\leq \mu^{-m} f(\mu) + \frac{2}{5}\sup_{s\leq t\leq\mu} t^{-m} f(t) + \frac{2}{5\mu}\lim_{t\to\infty} f(t)\int_\mu^{5\mu} t^{-m}\,\mathrm{d}\mathscr{L}^1 t \\ &\leq \mu^{-m}\lim_{t\to\infty} f(t) + \frac{2}{5}\sup_{s\leq t\leq\mu} t^{-m} f(t) + \frac{2}{5(m-1)}\mu^{-m}\lim_{t\to\infty} f(t) \end{aligned}$$

and therefore,

$$\frac{1}{2} \leq \frac{1}{2} \sup_{0<t\leq\mu} t^{-m} f(t) = \lim_{s\to 0+} \frac{1}{2} \sup_{s\leq t\leq\mu} t^{-m} f(t) < 2\mu^{-m} \lim_{t\to\infty} f(t) = 2^{-m},$$

a contradiction.

If $m = 1$, then

$$5 \leq 2\exp(1) \leq \exp(2),$$
$$\frac{2}{5} \int_{\mu}^{5\mu} t^{-m} \, \mathrm{d}\mathscr{L}^1 t = \frac{2\log(5)}{5} \leq \frac{4}{5} < 1$$

in the calculation above leads to the same result. □

6.4 Remark. The previous lemma and its proof are adaptations of [Sim83, 18.7]. The conclusion of this lemma holds true for the following hypothesis.

Suppose m is a positive integer, f and g are bounded and non-decreasing functions on $\{t : 0 < t < \infty\}$ and

$$1 \leq \limsup_{t\to 0+} t^{-m} f(t) < \infty,$$
$$s^{-m} f(s) \leq r^{-m} f(r) + \textstyle\int_0^r t^{-m} g(t) \, \mathrm{d}\mathscr{L}^1 t$$

for all $0 < s < r < \infty$.

With regard to the proof [Sim83, 18.7] and the application [Sim83, 18.6], this is a necessary correction of the hypothesis [Sim83, 18.7]. According to [All72, 5.5 (1)], the hypothesis above would have been enough for the purpose of this thesis. See the proof 6.5.

6.5 Theorem (Isoperimetric inequality)**.** *Suppose m is a positive integer. Then there exists a positive and finite number Γ with the following property. If $n \geq m$ is an integer, $V \in \mathbf{V}_m(\mathbf{R}^n)$ and $\|\delta V\|$ is a Radon measure, then*

$$\begin{aligned} &\textstyle\int_{\{x : 1 \leq \varphi(x) \boldsymbol{\Theta}^{*m}(\|V\|, x)\}} \varphi \, \mathrm{d}\|V\| \\ &\qquad \leq \Gamma \left(\textstyle\int \varphi \, \mathrm{d}\|V\|\right)^{1/m} \left(\textstyle\int \varphi \, \mathrm{d}\|\delta V\| + \int |P(\operatorname{grad}\varphi(x))| \, \mathrm{d}V(x,P)\right). \end{aligned}$$

whenever $\varphi \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ with $0 \leq \varphi$. Here $\Gamma = 5^m 2^{1/m} \boldsymbol{\alpha}(m)^{-1/m}$.

Proof. Let $V \in \mathbf{V}_m(\mathbf{R}^n)$ such that $\|\delta V\|$ is a Radon measure, $\varphi \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ such that $0 \leq \varphi$ and define

$$V_\varphi(\alpha) = \textstyle\int \alpha(x,P)\varphi(x) \, \mathrm{d}V(x,P) \quad \text{whenever } \alpha \in \mathscr{K}(\mathbf{G}_m(\mathbf{R}^n)).$$

Then $V_\varphi \in \mathbf{V}_m(\mathbf{R}^n)$ and

$$\|V_\varphi\|(B) = \textstyle\int_B \varphi \, \mathrm{d}\|V\|$$

whenever B is a Borel subset of $\mathbf{R}^n$. Let U be an open subset of $\mathbf{R}^n$, $g \in \mathscr{X}(\mathbf{R}^n)$ such that $\operatorname{spt} g \subset U$ and $|g| \leq 1$. Then, by [All72, 2.3 (4)] and 5.4

$$\begin{aligned} \delta V_\varphi(g) &= \textstyle\int \mathrm{D}g(x) \bullet P \, \mathrm{d}V_\varphi(x,P) = \int (Dg(x) \bullet P)\varphi(x) \, \mathrm{d}V(x,P) \\ &= \textstyle\int \mathrm{D}(\varphi g)(x) \bullet P \, \mathrm{d}V(x,P) - \int (\mathrm{D}\varphi(x) \, g(x)) \bullet P \, \mathrm{d}V(x,P) \\ &= \textstyle\int \varphi(x) g(x) \bullet \boldsymbol{\eta}(V,x) \, \mathrm{d}\|\delta V\| x - \int P(g(x)) \bullet \operatorname{grad}\varphi(x) \, \mathrm{d}V(x,P) \\ &= \textstyle\int \varphi(x) g(x) \bullet \boldsymbol{\eta}(V,x) \, \mathrm{d}\|\delta V\| x - \int g(x) \bullet P(\operatorname{grad}\varphi(x)) \, \mathrm{d}V(x,P) \\ &\leq \textstyle\int_U \varphi \, \mathrm{d}\|\delta V\| + \int_{U \times \mathbf{G}(n,m)} |P(\operatorname{grad}\varphi(x))| \, \mathrm{d}V(x,P) \end{aligned}$$

and hence

$$\|\delta V_\varphi\|(U) \le \textstyle\int_U \varphi \,\mathrm{d}\|\delta V\| + \int_{U\times \mathbf{G}(n,m)} |P(\operatorname{grad}\varphi(x))| \,\mathrm{d}V(x,P).$$

Approximating closed sets by open sets from above like in 3.3 yields

$$\|\delta V_\varphi\|(A) \le \textstyle\int_A \varphi \,\mathrm{d}\|\delta V\| + \int_{A\times \mathbf{G}(n,m)} |P(\operatorname{grad}\varphi(x))| \,\mathrm{d}V(x,P)$$

for all closed subsets A of $\mathbf{R}^n$. In particular $\|\delta V_\varphi\|$ is a Radon measure. Let $a \in \mathbf{R}^n$ such that

$$1 \le \varphi(a)\mathbf{\Theta}^{*m}(\|V\|, a).$$

The continuity of φ leads to

$$\mathbf{\Theta}^{*m}(\|V_\varphi\|, a) = \varphi(a)\mathbf{\Theta}^{*m}(\|V\|, a).$$

Define a function $h : \{t : 0 < t < \infty\} \to \mathbf{R}$ by letting

$$h(t) = \textstyle\int_{\mathbf{B}(a,t)} \varphi \,\mathrm{d}\|\delta V\| + \int_{\mathbf{B}(a,t)\times \mathbf{G}(n,m)} |P(\operatorname{grad}\varphi(x))| \,\mathrm{d}V(x,P)$$

for $0 < t < \infty$. By 6.1 it follows

$$\begin{aligned} s^{-m}\|V_\varphi\|\mathbf{B}(a,s) &\le r^{-m}\|V_\varphi\|\mathbf{B}(a,r) + \textstyle\int_s^r t^{-m}\|\delta V_\varphi\|\mathbf{B}(a,t)\,\mathrm{d}\mathscr{L}^1 t \\ &\le r^{-m}\|V_\varphi\|\mathbf{B}(a,r) + \textstyle\int_s^r t^{-m} h(t)\,\mathrm{d}\mathscr{L}^1 t \end{aligned}$$

whenever $0 < s \le r < \infty$. Applying 6.3 with $f = \boldsymbol{\alpha}(m)^{-1}\|V_\varphi\|\mathbf{B}(a,\cdot)$ and $g = \boldsymbol{\alpha}(m)^{-1}h$ provides $0 < r \le (2^{m+1}\boldsymbol{\alpha}(m)^{-1}\|V_\varphi\|(\mathbf{R}^n))^{1/m}$ such that

$$\begin{aligned} &\textstyle\int_{\mathbf{B}(a,5r)} \varphi \,\mathrm{d}\|V\| \le \Gamma(\int \varphi \,\mathrm{d}\|V\|)^{1/m} \int_{\mathbf{B}(a,r)} \varphi \,\mathrm{d}\|\delta V\| \\ &\qquad + \Gamma(\textstyle\int \varphi \,\mathrm{d}\|V\|)^{1/m} \int_{\mathbf{B}(a,r)\times \mathbf{G}(n,m)} |P(\operatorname{grad}\varphi(x))| \,\mathrm{d}V(x,P), \end{aligned}$$

where $\Gamma = 5^m 2^{1/m}\boldsymbol{\alpha}(m)^{-1/m}$. Doing so for all $a \in \{x : 1 \le \varphi(x)\mathbf{\Theta}^{*m}(\|V\|, x)\}$ and applying Vitali's covering theorem [Fed69, 2.8.5, 6, 8] to the resulting family of closed balls, it follows

$$\begin{aligned} &\textstyle\int_{\{x : 1 \le \varphi(x)\mathbf{\Theta}^{*m}(\|V\|,x)\}} \varphi \,\mathrm{d}\|V\| \\ &\qquad \le \Gamma\left(\textstyle\int \varphi \,\mathrm{d}\|V\|\right)^{1/m} \left(\textstyle\int \varphi \,\mathrm{d}\|\delta V\| + \int |P(\operatorname{grad}\varphi(x))| \,\mathrm{d}V(x,P)\right), \end{aligned}$$

which completes the proof. □

6.6 Remark. The preceding theorem is a generalization of [All72, 7.1] since in [All72, 7.1] the constant Γ depends on m and n. Moreover, the preceding theorem is a slight generalization of [Sim83, 18.6] since in [Sim83, 18.6] the hypothesis includes that V is rectifiable and $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$. Finally, the preceding theorem is a slight generalization of [Men09, 2.2] since in [Men09, 2.2] the varifold V is assumed to be rectifiable.

The proof of the previous theorem is an adaptation of the proof [Sim83, 18.6].

6.7 Corollary. *Suppose $m \le n$ are positive integers, $V \in \mathbf{V}_m(\mathbf{R}^n)$ and either $\|V\|(\mathbf{R}^n) < \infty$ or $0 < \|\delta V\|(\mathbf{R}^n)$.*

Then there holds

$$\|V\|\{x : 1 \le \mathbf{\Theta}^{*m}(\|V\|, x)\} \le \Gamma \|V\|(\mathbf{R}^n)^{1/m}\|\delta V\|(\mathbf{R}^n)$$

for $\Gamma = 5^m 2^{1/m}\boldsymbol{\alpha}(m)^{-1/m}$.

Proof. This directly follows from 6.5 with help of Lebesgue's increasing convergence theorem [Fed69, 2.4.7] and Lebesgue's bounded convergence theorem [Fed69, 2.4.9]. ☐

6.8 Remark. There exists no positive and finite number Δ satisfying

$$\|V\|\{x : 1 \leq \boldsymbol{\Theta}^{*m}(\|V\|, x)\} \leq \Delta \|V\|(\mathbf{R}^n)^{1/m} \|\delta V\|\{x : 1 \leq \boldsymbol{\Theta}^{*m}(\|V\|, x)\}$$

whenever $V \in \mathbf{V}_m(\mathbf{R}^n)$ and $\|V\|(\mathbf{R}^n) < \infty$.

To see this, define functions $\varphi : \{t : 0 \leq t < \infty\} \to \mathbf{R}$ and $\theta : \mathbf{R}^m \to \mathbf{R}$ by letting

$$\varphi(t) = \sup\{0, \inf\{1, (2-t)\}\} \quad \text{whenever } 0 \leq t < \infty,$$
$$\theta(x) = \varphi(|x|) \quad \text{whenever } x \in \mathbf{R}^m.$$

Let V be the corresponding varifold as in 5.18 for some $T \in \mathbf{G}(n, m)$ and θ as above. In view of 5.18 there holds $\|\delta V\|\{x : 1 \leq \boldsymbol{\Theta}^{*m}(\|V\|, x)\} = 0$ but $\|V\|\{x : 1 \leq \boldsymbol{\Theta}^{*m}(\|V\|, x)\} = \boldsymbol{\alpha}(m)$ and $\|V\|(\mathbf{R}^n) < \infty$.

6.9 Definition (see [Men09, 2.3])**.** Suppose m is a positive integer.

Then the smallest real number Γ having the property described in 6.7 is denoted by $\boldsymbol{\gamma}(m)$.

6.10 Remark (see [Men09, 2.4]). It is not known to the author whether or not $\boldsymbol{\gamma}(m)$ is the same as the constant in [Men09, 2.3]. It could be larger but not smaller. There holds

$$m^{-1}\boldsymbol{\alpha}(m)^{-1/m} \leq \boldsymbol{\gamma}(m) \leq 5^m 2^{1/m} \boldsymbol{\alpha}(m)^{-1/m}$$

as one could take $V \in \mathbf{V}_m(\mathbf{R}^m)$ such that $V(\alpha) = \int_{\mathbf{U}(0,1)} \alpha(x, \mathbf{1}_{\mathbf{R}^m}) \, \mathrm{d}\mathscr{L}^m x$ whenever $\alpha \in \mathscr{K}(\mathbf{G}_m(\mathbf{R}^m))$.

6.11 Corollary. *Suppose $m \leq n$ are positive integers, $0 < d < \infty$, $V \in \mathbf{V}_m(\mathbf{R}^n)$ and either $\|V\|(\mathbf{R}^n) < \infty$ or $0 < \|\delta V\|(\mathbf{R}^n)$.*

Then there holds

$$\|V\|\{x : d \leq \boldsymbol{\Theta}^{*m}(\|V\|, x)\} \leq \boldsymbol{\gamma}(m) d^{-1/m} \|V\|(\mathbf{R}^n)^{1/m} \|\delta V\|(\mathbf{R}^n).$$

Proof. This is easy to calculate by 6.7. ☐

6.12 Remark. All of the following corollaries can be analogously formulated like above. But there is no positive and finite number Δ such that

$$\|V\|\{x : 0 < \boldsymbol{\Theta}^{*m}(\|V\|, x)\} \leq \Delta \|V\|(\mathbf{R}^n)^{1/m} \|\delta V\|(\mathbf{R}^n)$$

for all $V \in \mathbf{V}_m(\mathbf{R}^n)$ with $\|V\|(\mathbf{R}^n) < \infty$ and $\|\delta V\|(\mathbf{R}^n) < \infty$.

To see this, let W be the varifold which corresponds to the m dimensional sphere, $\varepsilon > 0$ and $V = \varepsilon W$. Then $\|V\| = \varepsilon \|W\|$, $\boldsymbol{\Theta}^{*m}(\|V\|, \cdot) = \varepsilon \boldsymbol{\Theta}^{*m}(\|W\|, \cdot)$ and $\|\delta V\| = \varepsilon \|\delta W\|$. Let ε tend to zero to see that there exists no such Δ.

6.13 Corollary. *Suppose $m \leq n$ are positive integers, U is an open subset of $\mathbf{R}^n$, $j : U \to \mathbf{R}^n$ is the inclusion map, $V \in \mathbf{V}_m(U)$, $j[\operatorname{spt} \|V\|]$ is closed in $\mathbf{R}^n$ and either $\|V\|(U) < \infty$ or $0 < \|\delta V\|(U)$.*

Then there holds

$$\|V\|\{x : 1 \leq \boldsymbol{\Theta}^{*m}(\|V\|, x)\} \leq \boldsymbol{\gamma}(m) \|V\|(U)^{1/m} \|\delta V\|(U).$$

Proof. The corollary above is a slight generalization of 6.7. Its proof follows from 5.7. □

6.14 Remark. One cannot drop the assumption that $j[\operatorname{spt}\|V\|]$ is closed in $\mathbf{R}^n$. Otherwise one could take $U = \mathbf{R}^n \cap \{x : |x| < 1\}$ and $V = \mathbf{v}(U \cap T) \in \mathbf{V}_m(U)$ for some $T \in \mathbf{G}(n,m)$. Then $\|V\|(U) = \boldsymbol{\alpha}(m)$ and $\|V\|\{x : 1 \le \boldsymbol{\Theta}^{*m}(\|V\|, x)\} = \boldsymbol{\alpha}(m)$ but $\delta V = 0$.

6.15 Corollary. *Suppose $k \le m < n$ are positive integers, M is a smooth m dimensional submanifold of $\mathbf{R}^n$, the inclusion map $i : M \to \mathbf{R}^n$ is proper, $V \in \mathbf{V}_k(M)$ and $\|V\|(M) < \infty$.*

Then, if $k = m$ there holds

$$\begin{aligned}&\|V\|\{x : 1 \le \boldsymbol{\Theta}^{*m}(i_\#\|V\|, x)\}\\ &\quad\le \boldsymbol{\gamma}(m)\|V\|(M)^{1/m}\left(\|\delta V\|(M) + \textstyle\int |\mathbf{h}(M,x)|\,\mathrm{d}\|V\|x\right)\end{aligned}$$

and if $k < m$ there holds

$$\begin{aligned}&\|V\|\{x : 1 \le \boldsymbol{\Theta}^{*m}(i_\#\|V\|, x)\}\\ &\quad\le \boldsymbol{\gamma}(m)\|V\|(M)^{1/m}\left(\|\delta V\|(M) + \textstyle\int |\mathbf{h}(M; x, P)|\,\mathrm{d}V(x,P)\right).\end{aligned}$$

Proof. This is a consequence of 5.6. □

6.16 Corollary (see [All72, 7.2])**.** *Suppose $m < n$ are positive integers, M is a smooth m dimensional submanifold of $\mathbf{R}^n$, $\mathscr{H}^m(M) < \infty$ and*

$$B = \liminf_{r \to 0+} \mathscr{H}^{m-1}(M \cap \{x : \operatorname{dist}(x, (\operatorname{Clos} M) \sim M) = r\}).$$

Then there holds

$$\mathscr{H}^m(M)^{1-1/m} \le \boldsymbol{\gamma}(m)(B + \textstyle\int_M |\mathbf{h}(M,x)|\,\mathrm{d}\mathscr{H}^m x).$$

Proof. See the proof [All72, 7.2]. □

7 Density Bounds

Suppose $m \le n$ are positive integers and V is an m dimensional rectifiable varifold in $\mathbf{R}^n$ such that the m dimensional density of $\|V\|$ is at least one outside a set of $\|V\|$ measure zero. In this section the isoperimetric inequality is used to study the set of points in $\operatorname{spt}\|V\|$ where the lower m dimensional density of $\|V\|$ is strictly less than one. This set will be estimated in terms of the one dimensional Hausdorff measure. If $m < n$, then this set might be far away from having finite m dimensional Hausdorff measure, see 5.16. Therefore, an additional assumption is required. Here this will be the assumption that V is indecomposable. First a lemma originally of Allard [All72] is established, see 7.2. This lemma is naturally related to the notion of indecomposable varifold. Actually, it shows how to cut a varifold along a level set of a real valued Lipschitzian function f. The resulting formula allows to make use of the isoperimetric inequality for the varifold $V \llcorner \{x : f(x) < r\} \times \mathbf{G}(n,m)$, where r is a positive and appropriate small number. This will be indicated in 7.5 and provides a lower bound of the weight's m dimensional density quotient presupposed the derivative quotient of

$\|\delta V\|$ by $\|V\|^{1-1/m}$ is small. The main part of the final proof is to reduce the mentioned assumption on the derivative quotient to the assumption that the one dimensional density quotient of $\|V\| \llcorner |\mathbf{h}(V;\cdot)|^{m-1}$ is small. This will be done in Lemma 7.10. Most of the statements are more general than necessary for the purpose of this section. This is designed to make use of 7.10 in Section 8.

7.1. A useful set of hypotheses is gathered here for later reference.

Suppose $m \le n$ are positive integers, $V \in \mathbf{RV}_m(\mathbf{R}^n)$, $\|\delta V\|$ is a Radon measure, $f : \mathbf{R}^n \to \mathbf{R}$ is a Lipschitzian function and

$$F(x) = (\|V\|, m) \operatorname{ap} \mathrm{D} f(x) \circ \operatorname{Tan}^m(\|V\|, x)$$

for $\|V\|$ almost all x.

Notice that f is $(\|V\|, m)$ approximately differentiable at $\|V\|$ almost all x and F defines a $\|V\|$ measurable function by [Men12b, 4.5]. Moreover, f is generalized V weakly differentiable in the sense of [Men16a, 8.3], by [Men16a, 8.7]. Actually F equals the generalized V weak derivative of f.

7.2 Lemma (see [All72, 4.10 (1)], [Men16a, 8.29]). *Suppose m, n, V, f and F are related as in 7.1 and $g \in \mathscr{X}(\mathbf{R}^n)$.*

Then, for $r \in \mathbf{R}$ there holds

$$\begin{aligned} \delta V(\chi_{f,r} g) = \delta(V \llcorner \{x : f(x) < r\} \times \mathbf{G}(n,m))(g) \\ - \lim_{h \to 0+} h^{-1} \int_{\{x : r-h \le f(x) < r\}} \langle g(x), F(x) \rangle \, \mathrm{d}\|V\| x, \end{aligned}$$

where $\chi_{f,r}$ denotes the characteristic function of $\{x : f(x) < r\}$.

Proof. By [Men16a, 8.7], the same method of proof [All72, 4.10 (1)] applies. See also [Men16a, 8.29] for a more general statement. □

Besides the fact that the weight of an indecomposable varifold has connected support, which also follows from 7.2, see [Men16a, 6.5], the only additional property of indecomposable varifolds used in this thesis will be the following theorem. Actually, the theorem shows in which sense the connectedness of the weight's support persists if one subtracts sets of weight measure zero.

7.3 Theorem. *Suppose m, n, V, f and F are related as in 7.1, V is indecomposable, $S \subset \operatorname{spt} \|V\|$, $\|V\|(\mathbf{R}^n \sim S) = 0$, $\|F(x)\| \le 1$ for $\|V\|$ almost all x, $\inf f[\operatorname{spt} \|V\|] < r < s < \sup f[\operatorname{spt} \|V\|]$ and $(\operatorname{spt} \|V\|) \cap \{x : r \le f(x) \le s\}$ is compact.*

Then there holds

$$\mathscr{L}^1(\{y : r \le y \le s\} \sim f[S]) = 0.$$

Proof. Define $\mu = f_\#(\|V\| \llcorner \{x : r \le f(x) \le s\})$. Then μ is a Radon measure over $\mathbf{R}$ by 3.7. There holds $\mu(\mathbf{R}) < \infty$ and

$$\mu(\mathbf{R} \sim f[S]) \le \|V\|(\mathbf{R}^n \sim S) = 0.$$

Hence, $f[S]$ is μ measurable and one may apply [Fed69, 2.10.19 (4)] to obtain

$$\boldsymbol{\Theta}^1(\mu \llcorner f[S], y) = 0 \quad \text{for } \mathscr{L}^1 \text{ almost all } y \in \mathbf{R} \sim f[S].$$

If

$$0 < \mathscr{L}^1(\{y : r \le y \le s\} \sim f[S])$$

were true, there would exist $r < y < s$ such that

$$\boldsymbol{\Theta}^1(\mu, y) \le \boldsymbol{\Theta}^{*1}(\mu \llcorner f[S], y) + \boldsymbol{\Theta}^{*1}(\mu \llcorner \mathbf{R} \sim f[S], y) = 0.$$

Then 7.2 would imply

$$(\delta V) \llcorner \{x : f(x) < y\} = \delta(V \llcorner \{x : f(x) < y\} \times \mathbf{G}(n, m)),$$

which would be incompatible with the indecomposability of V as one could find $x_1, x_2 \in \operatorname{spt} \|V\|$ such that $f(x_1) < y < f(x_2)$. $\square$

7.4 Remark. A slightly weaker statement and the same method of proof have been used by Menne in [Men12a, 2.2].

7.5 Theorem. *Suppose m, n, V, f and F are related as in 7.1, $\|F(x)\| \le 1$ for $\|V\|$ almost all x, $0 < r < \infty$ and*

$$\begin{gathered} \|V\|\{x : f(x) \le 0\} = 0, \\ 0 < \|V\|\{x : f(x) < s\} < \infty \quad \textit{for all } 0 < s \le r, \\ \boldsymbol{\Theta}^m(\|V\|, a) \ge 1 \quad \textit{for } \|V\| \textit{ almost all } a \in \{x : f(x) < r\}. \end{gathered}$$

Then there holds

$$\begin{aligned} r &\le \boldsymbol{\gamma}(m) m \|V\|(\{x : f(x) < r\})^{1/m} \\ &\quad + \boldsymbol{\gamma}(m) \int_0^r \|\delta V\|\{x : f(x) < s\} \|V\|(\{x : f(x) < s\})^{(1/m)-1} \, \mathrm{d}\mathscr{L}^1 s. \end{aligned}$$

Proof. Define a function $\mu : \mathbf{R} \to \mathbf{R}$ by letting

$$\mu(t) = \|V\|\{x : f(x) < \inf\{t, r\}\} \quad \text{for } t \in \mathbf{R}.$$

Then μ is non-decreasing and $\mathscr{L}^1$ almost everywhere differentiable by [Fed69, 2.9.19]. Notice that

$$\boldsymbol{\Theta}^{*m}(\|V \llcorner \{x : f(x) < s\} \times \mathbf{G}(n, m)\|, a) = \boldsymbol{\Theta}^{*m}(\|V\|, a)$$

whenever $0 < s < r$ and $a \in \{x : f(x) < s\}$ as f is continuous. Hence, 6.7 and 7.2 yield

$$\begin{aligned} \mu(s)^{1-1/m} &\le \boldsymbol{\gamma}(m) \|\delta(V \llcorner \{x : f(x) < s\} \times \mathbf{G}(n, m))\|(\mathbf{R}^n) \\ &\le \boldsymbol{\gamma}(m) \left(\|\delta V\|\{x : f(x) < s\} + \lim_{h \to 0+} h^{-1} \|V\|\{x : s - h \le f(x) < s\} \right) \\ &= \boldsymbol{\gamma}(m) \left(\|\delta V\|\{x : f(x) < s\} + \mu'(s) \right) \end{aligned}$$

for $\mathscr{L}^1$ almost all $0 < s < r$. Define $p : \mathbf{R} \to \mathbf{R}$ by letting $p(t) = |t|^{1/m}$ whenever $t \in \mathbf{R}$. Dividing the inequality above by $\mu(s)^{1-(1/m)}$ leads to

$$\begin{aligned} 1 &\le \boldsymbol{\gamma}(m) \left(\mu(s)^{(1/m)-1} \|\delta V\|\{x : f(x) < s\} + \mu(s)^{(1/m)-1} \mu'(s) \right) \\ &= \boldsymbol{\gamma}(m) \left(\mu(s)^{(1/m)-1} \|\delta V\|\{x : f(x) < s\} + m (p \circ \mu)'(s) \right) \end{aligned}$$

for $\mathscr{L}^1$ almost all $0 < s < r$. The conclusion follows by integrating this inequality as

$$\textstyle\int_0^r (p \circ \mu)' \,\mathrm{d}\mathscr{L}^1 \leq \mathbf{V}_0^r(p \circ \mu) = \mu(r)^{1/m}$$

by [Fed69, 2.9.19]. □

7.6 Corollary (see [All72, 8.3], [Men09, 2.5])**.** *Suppose*

$$\|\delta V\|\{x : f(x) < s\} \leq (2\boldsymbol{\gamma}(m))^{-1}\|V\|(\{x : f(x) < s\})^{1-1/m}$$

for $\mathscr{L}^1$ almost all $0 < s < r$.

Then there holds

$$r^m \leq (2m\boldsymbol{\gamma}(m))^m\|V\|\{x : f(x) < r\}.$$

7.7 Remark. The proof of the previous theorem is an adaptation of [All72, 8.3]. Its corollary is a generalization of [Men09, 2.5]. The hypothesis in the previous corollary is equivalent to the following.

Suppose Q is dense in $\{t : 0 < t < r\}$ and

$$\|\delta V\|\{x : f(x) \leq s\} \leq (2\boldsymbol{\gamma}(m))^{-1}\|V\|(\{x : f(x) \leq s\})^{1-1/m}$$

for all $s \in Q$.

This is a consequence of 3.3.

7.8. Suppose $m \leq n$ are positive integers, $V \in \mathbf{RV}_m(\mathbf{R}^n)$, $\operatorname{spt}\|V\|$ is compact, $0 < \delta \leq \infty$ and σ is a metric over $\operatorname{spt}\|V\|$ such that

$$\sigma(x, y) \leq |x - y| \quad \text{whenever } x, y \in \operatorname{spt}\|V\| \text{ and } |x - y| \leq \delta.$$

Let $a \in \operatorname{spt}\|V\|$. Then $\sigma(a, \cdot)$ is a Lipschitzian function as $\operatorname{spt}\|V\|$ is compact. By [Fed69, 2.10.44] there exists a Lipschitzian extension $f : \mathbf{R}^n \to \mathbf{R}$ of $\sigma(a, \cdot)$. As already mentioned in 7.1, f is $(\|V\|, m)$ approximately differentiable at $\|V\|$ almost all x. By the definition of the $(\|V\|, m)$ approximative differential [Fed69, 3.2.16] and 3.2, the $(\|V\|, m)$ approximative differential does not depend on the Lipschitzian extension f of $\sigma(a, \cdot)$. Moreover, according to [Men16b, 6.2] and [Fed69, 3.2.16], there holds

$$\|(\|V\|, m) \operatorname{ap} \mathrm{D} f(x)\| \leq 1$$

for $\|V\|$ almost all x.

7.9. A useful set of hypotheses is gathered here for later reference.

Suppose $m \leq n$ are positive integers, $V \in \mathbf{RV}_m(\mathbf{R}^n)$, $\|\delta V\|$ is a Radon measure, V is indecomposable,

$$\boldsymbol{\Theta}^m(\|V\|, x) \geq 1 \quad \textit{for } \|V\| \textit{ almost all } x$$

and

1. *either $m = 1$ and $\psi = \|V\|$,*
2. *or $m = 2$ and $\psi = \|\delta V\|$,*

3. or $m \geq 3$, $\|\delta V\|$ *is absolutely continuous with respect to* $\|V\|$ *and*

$$\psi(A) = \textstyle\int_A^* |\mathbf{h}(V;x)|^{m-1} \,\mathrm{d}\|V\|x \quad \textit{whenever } A \subset \mathbf{R}^n.$$

Notice that ψ defines a Borel regular measure over $\mathbf{R}^n$, see 3.5.

7.10 Lemma. *Suppose* m, n, V *and* ψ *are related as in 7.9, the Borel regular measure* ψ *is a Radon measure,* $0 < \delta \leq \infty$, *either* $\delta = \infty$ *or* $\operatorname{spt}\|V\|$ *is compact,* σ *is a metric over* $\operatorname{spt}\|V\|$,

$$\begin{gathered} |x-y| \leq \sigma(x,y) \quad \textit{whenever } x, y \in \operatorname{spt}\|V\|, \\ \sigma(x,y) = |x-y| \quad \textit{whenever } x,y \in \operatorname{spt}\|V\| \textit{ and } |x-y| \leq \delta, \end{gathered}$$

$a \in \operatorname{spt}\|V\|$, $0 < r < \sup\{\sigma(a,x) : x \in \operatorname{spt}\|V\|\}$ *and*

$$\psi\{x : \sigma(a,x) \leq r\} < \eta r,$$

where $\eta = (m 2^{m+2} \boldsymbol{\beta}(1) \boldsymbol{\gamma}(m)^m)^{-1}$.
Then there holds

$$r^m \leq (m\boldsymbol{\gamma}(m))^m 2^{2m} \|V\|\{x : \sigma(a,x) \leq r\}.$$

Proof. Define balls with respect to σ by

$$\begin{aligned} U(x,s,\sigma) &= \operatorname{spt}\|V\| \cap \{y : \sigma(x,y) < s\}, \\ B(x,s,\sigma) &= \operatorname{spt}\|V\| \cap \{y : \sigma(x,y) \leq s\} \end{aligned}$$

whenever $x \in \operatorname{spt}\|V\|$ and $0 < s < \infty$. Let S be the set which consists exactly of those $x \in \operatorname{spt}\|V\|$ for which

$$\limsup_{s \to 0+} \|V\|(B(x,s,\sigma))^{(1/m)-1} \|\delta V\| B(x,s,\sigma) < (2\boldsymbol{\gamma}(m))^{-1}$$

and let P be the set which consists exactly of those $x \in U(a,r,\sigma) \cap S$ for which

$$s^m \leq (2m\boldsymbol{\gamma}(m))^m \|V\| B(x,s,\sigma) \quad \text{whenever } 0 < s < r - \sigma(a,x).$$

Let $f : \mathbf{R}^n \to \mathbf{R}$ be any Lipschitzan extension of $\sigma(a,\cdot)$. It will be shown that

$$\mathscr{L}^1(f[B(a,r/2,\sigma) \cap S \sim P]) < r/2.$$

For this purpose assume y is any point in $f[B(a,r/2,\sigma) \cap S \sim P]$, choose $x \in B(a,r/2,\sigma) \cap S \sim P$ such that $f(x) = y$ and let

$$s = \inf\{t : \|V\|(B(x,t,\sigma))^{1-1/m} < 2\boldsymbol{\gamma}(m) \|\delta V\| B(x,t,\sigma)\}.$$

The definition of S and P and 7.6 in conjunction with 7.8 and 3.2 yield

$$0 < s < r - \sigma(a,x), \quad B(x,s,\sigma) \subset B(a,r,\sigma).$$

Notice that the hypothesis for $m \geq 3$ implies

$$\|\delta V\|(B) = \textstyle\int_B |\mathbf{h}(V;\cdot)| \,\mathrm{d}\|V\|$$

whenever B is a Borel subset of $\mathbf{R}^n$. By 3.3 and Hölder's inequality [Fed69, 2.4.14] one calculates for $m \geq 2$

$$\begin{aligned}(2\boldsymbol{\gamma}(m))^{-1}\|V\|(B(x,s,\sigma))^{1-1/m} &\leq \|\delta V\|B(x,s,\sigma)\\ &\leq \|V\|(B(x,s,\sigma))^{1-1/(m-1)}\psi(B(x,s,\sigma))^{1/(m-1)}\end{aligned}$$

and therefore, by 7.6

$$\begin{gathered}(2m\boldsymbol{\gamma}(m))^{-1}s \leq \|V\|(B(x,s,\sigma))^{1/m} \leq (2\boldsymbol{\gamma}(m))^{m-1}\psi B(x,s,\sigma),\\ s \leq m(2\boldsymbol{\gamma}(m))^m\psi B(x,s,\sigma).\end{gathered}$$

If $m=1$, the last inequality directly follows from 7.6. Hence, there holds

$$m(2\boldsymbol{\gamma}(m))^m\big(f_\#(\psi \llcorner B(a,r,\sigma))\big)\mathbf{B}(y,s) \geq m(2\boldsymbol{\gamma}(m))^m\psi B(x,s,\sigma) \geq s,$$

as $B(x,s,\sigma) \subset B(a,r,\sigma) \cap f^{-1}[\mathbf{B}(y,s)]$. The Besicovitch Federer covering theorem leads to

$$\mathscr{L}^1(f[B(a,r/2,\sigma) \cap S \sim P]) \leq 2\boldsymbol{\beta}(1)m(2\boldsymbol{\gamma}(m))^m\psi B(a,r,\sigma) < r/2.$$

The hypothesis on σ yields that S consists exactly of those $x \in \operatorname{spt}\|V\|$ for which

$$\limsup_{s\to 0+}\|V\|(\mathbf{B}(x,s))^{(1/m)-1}\|\delta V\|\mathbf{B}(x,s) < (2\boldsymbol{\gamma}(m))^{-1}.$$

Hence, $\|V\|(\mathbf{R}^n \sim S) = 0$ by [Fed69, 2.8.18, 2.9.5] and the fact that $\|V\|\{x\} = 0$ for $\|V\|$ almost all x as V is rectifiable, see also 5.5. In view of 7.8, one may apply 7.3 to obtain

$$\begin{aligned}r/2 &= \mathscr{L}^1(\{y : 0 \leq y \leq r/2\} \cap f[S]) = \mathscr{L}^1(f[B(a,r/2,\sigma) \cap S])\\ &\leq \mathscr{L}^1(f[B(a,r/2,\sigma) \cap S \sim P]) + \mathscr{L}^1(f[B(a,r/2,\sigma) \cap P])\\ &< r/2 + \mathscr{L}^1(f[B(a,r/2,\sigma) \cap P]).\end{aligned}$$

This means

$$0 < \mathscr{L}^1(f[B(a,r/2,\sigma) \cap P]), \quad B(a,r/2,\sigma) \cap P \neq \varnothing.$$

Choose any $x \in B(a,r/2,\sigma) \cap P$. Then there holds $r - \sigma(x,a) \geq r/2$ and by the definition of P

$$\begin{gathered}s^m(2m\boldsymbol{\gamma}(m))^{-m} \leq \|V\|B(x,s,\sigma) \leq \|V\|B(a,r,\sigma) \quad \text{whenever } 0 < s < r/2,\\ r^m \leq (2^2m\boldsymbol{\gamma}(m))^m\|V\|B(a,r,\sigma)\end{gathered}$$

which completes the proof. □

7.11 Remark. The conclusion of the previous lemma is a weaker version of that in [Top08, 1.2]. Instead of the Michael-Simon Sobolev inequality [Top08, 2.1] the isoperimetric inequality, or more precisely 7.6, has been used in the proof above.

7.12 Lemma. *Suppose n is a positive integer, $1 < d < \infty$, μ is a Radon measure over $\mathbf{R}^n$, $\boldsymbol{\Theta}^d(\mu, x) = 0$ for μ almost all x, f is a nonnegative μ measurable function and the Borel regular measure ψ over $\mathbf{R}^n$ defined by*

$$\psi(A) = \textstyle\int_A^* f \,\mathrm{d}\mu \quad \textit{whenever } A \subset \mathbf{R}^n$$

is a Radon measure.

Then there holds

$$\boldsymbol{\Theta}^1(\psi, x) = 0 \quad \textit{for } \mathscr{H}^1 \textit{ almost all } x.$$

Proof. First notice that μ is absolutely continuous with respect to $\mathscr{H}^d$ by [Fed69, 2.10.19 (1)]. For each positive integer i define

$$B_i = \{x : |x| \leq i, \boldsymbol{\Theta}^{*1}(\psi, x) > i^{-1}\}$$

and infer from [Fed69, 2.10.19 (3)] that

$$i^{-1}\mathscr{H}^1(B_i) \leq \psi(B_i).$$

Since B_i is bounded it follows $\psi(B_i) < \infty$, $\mathscr{H}^1(B_i) < \infty$, $\mathscr{H}^d(B_i) = 0$, $\mu(B_i) = 0$, $\psi(B_i) = 0$, $\mathscr{H}^1(B_i) = 0$. The conclusion follows as $\{x : \boldsymbol{\Theta}^{*1}(\psi, x) > 0\} = \bigcup_{j=1}^{\infty} B_j$. □

7.13 Remark. The previous proof is an adaptation of [FZ73, p. 152, l. 9–16].

7.14 Theorem. *Suppose $2 \leq m$, n, V and ψ are related as in 7.9, $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ and the Borel regular measure ψ is a Radon measure.*

Then there holds

$$\boldsymbol{\Theta}^m_*(\|V\|, x) \geq 1 \quad \textit{for } \mathscr{H}^1 \textit{ almost all } x \in \operatorname{spt} \|V\|.$$

Proof. Let $1 < d < 2$. By [All72, 3.5 (1)(a)] it follows that $\boldsymbol{\Theta}^d(\|V\|, x) = 0$ for $\|V\|$ almost all x. Apply 7.12 to infer that $\boldsymbol{\Theta}^1(\psi, x) = 0$ for $\mathscr{H}^1$ almost all $x \in \mathbf{R}^n$. Now 7.10 implies

$$\boldsymbol{\Theta}^m_*(\|V\|, x) \geq (m\boldsymbol{\gamma}(m))^{-m} 2^{-2m} \boldsymbol{\alpha}(m)^{-1} \quad \text{for } \mathscr{H}^1 \text{ almost all } x \in \operatorname{spt} \|V\|.$$

The conclusion follows by [Men09, 2.11]. □

7.15 Remark. Suppose $m < n$. Then one cannot replace the assumption that V is indecomposable by assuming $\operatorname{spt} \|V\|$ to be path-connected instead. To see this, let $V \in \mathbf{RV}_m(\mathbf{R}^n)$ be a varifold which is related to $U = \mathbf{R}^n \cap \{x : |x| < 1\}$ as described in 5.16. If there held $\boldsymbol{\Theta}^m_*(\|V\|, x) \geq 1$ for $\mathscr{H}^1$ almost all $x \in \mathbf{R}^n \cap \{x : |x| \leq 1\}$, then in particular $\boldsymbol{\Theta}^m_*(\|V\|, x) \geq 1$ for $\mathscr{H}^m$ almost all $x \in \mathbf{R}^n \cap \{x : |x| \leq 1\}$ and [Fed69, 2.10.19 (3)] would imply

$$\|V\|(\mathbf{R}^n \cap \{x : |x| \leq 1\}) \geq \mathscr{H}^m(\mathbf{R}^n \cap \{x : |x| \leq 1\}) = \infty.$$

A contradiction.

7.16 Remark. One cannot drop the assumption that $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$. To see this, let $T \in \mathbf{G}(n, m)$, and $V = \mathbf{v}(T \cap \{x : |x| \leq 1\}) \in \mathbf{RV}_m(\mathbf{R}^n)$. Then $\boldsymbol{\Theta}^m(\|V\|, a) \leq 1/2$ whenever $a \in T \cap \{x : |x| = 1\}$ and

$$0 < \mathscr{H}^{m-1}(\mathbf{S}^{m-1}) = \mathscr{H}^{m-1}(T \cap \{x : |x| = 1\}) \leq \mathscr{H}^1(T \cap \{x : |x| = 1\}).$$

7.17 Theorem (see [Men16a, 4.8 (4)]). *Suppose n is a positive integer, $V \in \mathbf{RV}_1(\mathbf{R}^n)$, $\|\delta V\|$ is a Radon measure, V is indecomposable and*

$$\boldsymbol{\Theta}^1(\|V\|, x) \geq 1 \quad \textit{for } \|V\| \textit{ almost all } x.$$

Then then there holds

$$\boldsymbol{\Theta}^1_*(\|V\|, x) \geq \Delta \quad \textit{whenever } x \in \operatorname{spt}\|V\|,$$

where $\Delta = \inf\{(2^3\boldsymbol{\beta}(1)\boldsymbol{\gamma}(1)\boldsymbol{\alpha}(1))^{-1}, (\boldsymbol{\gamma}(1)2^2\boldsymbol{\alpha}(1))^{-1}\}$.

Proof. This follows from 7.10. Actually, the theorem is a weaker version of [Men16a, 4.8 (4)]. □

8 Geodesic Distance and Diameter Control

Suppose $m \leq n$ are positive integers, V is an m dimensional rectifiable varifold in $\mathbf{R}^n$, $\|\delta V\|$ is a Radon measure, V is indecomposable, $\|V\|(\mathbf{R}^n) < \infty$, the m dimensional density of $\|V\|$ is bounded below by one, outside a set of $\|V\|$ measure zero and if $m \geq 3$, then $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$. In this section the intrinsic diameter d of $\operatorname{spt}\|V\|$ will be estimated in terms of the generalized mean curvature by

$$d \leq \Gamma \textstyle\int |\mathbf{h}(V;x)|^{m-1} \,\mathrm{d}\|V\|x$$

where the positive and finite number Γ only depends on m. The main part of the proof is already done in 7.10. First, the notion of geodesic distance over a closed subset of $\mathbf{R}^n$ will be established. If two points have finite geodesic distance, a shortest path connection of these points exists. Actually, this path may be chosen to be Lipschitzian and to be parametrized by arclength. It will be shown how to approximate the geodesic distance by a sequence of metrics, which are locally Lipschitzian with respect to the Euclidean distance. This allows to make use of 7.10. Finally, a local property concerning the geodesic distance over $\operatorname{spt}\|V\|$ will be established.

8.1 Definition (Pseudometric, see [Men16b, 2.2]). Suppose X is a set.

Then a map $\rho : X \times X \to \{t : 0 \leq t \leq \infty\}$ is called *pseudometric over X* if and only if the following three conditions are satisfied.

1. If $x \in X$, then $\rho(x,x) = 0$.
2. If $x, y \in X$, then $\rho(x,y) = \rho(y,x)$.
3. If $x, y, z \in X$, then $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

8.2 (Continuity of a metric). Suppose n is a positive integer, X is a subset of $\mathbf{R}^n$ and ρ is a metric over X. One calculates

$$\begin{aligned}|\rho(x_0,y_0) - \rho(x,y)| &\leq |\rho(x_0,y_0) - \rho(y_0,x)| + |\rho(y_0,x) - \rho(y,x)| \\ &\leq \rho(x_0,x) + \rho(y_0,y)\end{aligned}$$

whenever $(x_0, y_0), (x, y) \in X \times X$. Now assume $x_0 \in X$ and $\rho(x_0, \cdot)$ is continuous at x_0 with respect to the Euclidean distance over X. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\rho(x_0, x) = |\rho(x_0, x_0) - \rho(x_0, x)| \le \varepsilon \quad \text{whenever } x \in X \text{ and } |x_0 - x| \le \delta.$$

Hence, the map $\rho : X \times X \to \mathbf{R}$ is continuous with respect to the Euclidean distance over $X \times X$ if and only if for each $a \in X$, the map $\rho(a, \cdot)$ is continuous at a with respect to the Euclidean distance over X.

8.3 (Geodesic distance, see [Men16b, 6.3])**.** Suppose n is a positive integer and X is a closed subset of $\mathbf{R}^n$. Whenever $0 < \delta \le \infty$, one may define a pseudometric

$$\sigma_\delta : X \times X \to \overline{\mathbf{R}}$$

over X by letting $\sigma_\delta(a, x)$ for $a, x \in X$ denote the infimum of the set of numbers

$$\sum_{i=1}^{j} |x_i - x_{i-1}|$$

corresponding to all finite sequences $x_0, x_1, \ldots, x_j$ in X with $x_0 = a$, $x_j = x$ and $|x_i - x_{i-1}| \le \delta$ for $i = 1, \ldots, j$ and $j \in \mathscr{P}$.

Define a further pseudometric

$$\rho : X \times X \to \overline{\mathbf{R}}$$

over X by letting $\rho(a, x)$ for $a, x \in X$ denote the infimum of the set of numbers

$$\mathbf{V}_{\inf I}^{\sup I} g$$

corresponding to continuous maps $g : \mathbf{R} \to X$ such that $g(\inf I) = a$ and $g(\sup I) = x$ for some compact subinterval I of $\mathbf{R}$, where the length of g is computed with respect to the Euclidean distance.

8.4 Lemma (see [Fed69, 2.5.16])**.** *Suppose n is a positive integer, X is a closed subset of $\mathbf{R}^n$, $g : \mathbf{R} \to X$ is continuous, $-\infty < s \le t < \infty$ and $\mathbf{V}_s^t g < \infty$.*

Then there exists a map $G : \mathbf{R} \to X$ satisfying

$$\mathbf{V}_s^t g = \mathbf{V}_0^{\mathbf{V}_s^t g} G, \quad G(0) = g(s), \quad G(\mathbf{V}_s^t g) = g(t), \quad \operatorname{Lip}(G) \le 1.$$

Proof. See the proof [Fed69, 2.5.16]. □

8.5 Lemma (see [Men16b, 6.3])**.** *Suppose n, X, σ_δ and ρ are related as in 8.3 whenever $0 < \delta \le \infty$ and suppose $a, x \in X$.*

Then there holds

$$\rho(a, x) = \lim_{\delta \to 0+} \sigma_\delta(a, x).$$

Proof. First notice that $\lim_{\delta \to 0+} \sigma_\delta(a, x)$ exists in $\overline{\mathbf{R}}$ since $\sigma_\delta(a, x) \le \sigma_\varepsilon(a, x)$ whenever $0 < \varepsilon \le \delta \le \infty$. Denote this limit by $\sigma(a, x)$. First it will be shown that $\sigma(a, x) \le \rho(a, x)$. For this purpose assume $\rho(a, x) < \infty$ and let $\varepsilon > 0$. By 8.4, there exists $0 \le u \le \rho(a, x) + \varepsilon$ and a map $g : \mathbf{R} \to X$ satisfying

$$\mathbf{V}_0^u g \le \rho(a, x) + \varepsilon, \quad g(0) = a, \quad g(u) = x, \quad \operatorname{Lip}(g) \le 1.$$

Let $0 < \delta \leq \infty$, $m \in \mathscr{P}$ and $0 = t_0 \leq t_1 \leq \ldots \leq t_m = u$, such that $|t_j - t_{j-1}| \leq \delta$ for $j = 1, \ldots, m$. Then there holds $|g(t_j) - g(t_{j-1})| \leq \delta$ for $j = 1, \ldots, m$ and

$$\sigma_\delta(a,x) \leq \textstyle\sum_{j=1}^m |g(t_j) - g(t_{j-1})| \leq \sum_{j=1}^m \mathbf{V}_{t_{j-1}}^{t_j} g = \mathbf{V}_0^u g \leq \rho(a,x) + \varepsilon.$$

Hence, $\sigma(a,x) \leq \rho(a,x)$.

To prove the reverse inequality, assume $\sigma(a,x) < \infty$. For each positive integer i choose $0 < \delta_i \leq 1/i$, $m_i \in \mathscr{P}$ and a finite sequence $a = x_{i,0}, x_{i,1}, \ldots, x_{i,m_i} = x$ in X such that

$$\begin{gathered} |x_{i,j} - x_{i,j-1}| \leq \delta_i \quad \text{for } j = 1, \ldots, m_i, \\ \sigma(a,x) - (1/i) \leq \sigma_{\delta_i}(a,x) \leq \textstyle\sum_{j=1}^{m_i} |x_{i,j} - x_{i,j-1}| \leq \sigma_{\delta_i}(a,x) + (1/i). \end{gathered}$$

Choose a Lipschitzian function $f_i : \mathbf{R} \to \mathbf{R}^n$ satisfying $\operatorname{Lip}(f_i) \leq 1$ and

$$f_i \left(\textstyle\sum_{j=1}^k |x_{i,j} - x_{i,j-1}| \right) = x_{i,k} \quad \text{for } k = 0, \ldots, m_i.$$

There holds

$$f_i(0) = a, \quad f_i(t) = x \quad \text{for some } \sigma(a,x) - (1/i) \leq t \leq \sigma(a,x) + (1/i).$$

The Arzélà Ascoli theorem for Lipschitzian functions [Fed69, 2.10.21] provides a sequence $k_1 < k_2 < \ldots$ of positive integers and a Lipschitzian function $g : \mathbf{R} \to \mathbf{R}^n$ such that the sequence $f_{k_1}, f_{k_2}, \ldots$ converges locally uniformly to g. Then $\operatorname{Lip}(g) \leq 1$, $g(0) = a$ and $g(\sigma(a,x)) = x$. Next it will be shown that $\operatorname{im} g | \{t : 0 \leq t \leq \sigma(a,x)\} \subset X$. For this purpose let $0 \leq t \leq \sigma(a,x)$. The construction above provides a sequence $t_1, t_2, \ldots$ of real numbers in $\{s : 0 \leq s \leq \sigma(a,x) + 1\}$ such that $t_j \to t$ as $j \to \infty$ and such that for each $j \in \mathscr{P}$ there exists $k \in \{0, \ldots, m_{k_j}\}$ satisfying $f_{k_j}(t_j) = x_{k_j,k} \in X$. It follows

$$\begin{aligned} |g(t) - f_{k_j}(t_j)| &\leq |g(t) - f_{k_j}(t)| + |f_{k_j}(t) - f_{k_j}(t_j)| \\ &\leq |g(t) - f_{k_j}(t)| + |t - t_j| \to 0 \text{ as } j \to \infty. \end{aligned}$$

The closedness of X implies $g(t) \in X$. Hence,

$$\rho(a,x) \leq \mathbf{V}_0^{\sigma(a,x)} g \leq \sigma(a,x)$$

which completes the proof. $\square$

8.6 Lemma (Existence of geodesics, see [Men16b, 6.3])**.** *Suppose n, X and ρ are related as in 8.3, $a, x \in X$ and $\rho(a,x) < \infty$.*

Then there exists a map $g : \mathbf{R} \to X$ such that

$$\begin{gathered} g(0) = a, \quad g(\rho(a,x)) = x, \quad \operatorname{Lip}(g) \leq 1, \\ \mathbf{V}_0^{\rho(a,x)} g = \mathscr{H}^1 \{g(t) : 0 \leq t \leq \rho(a,x)\} = \rho(a,x), \end{gathered}$$

$g | \{t : 0 \leq t \leq \rho(a,x)\}$ is injective, g is a Lipschitzian function with respect to ρ, $W = \{g(t) : 0 \leq t \leq \rho(a,x)\}$ is compact with respect to ρ and $\rho | (W \times W)$ is continuous with respect to the Euclidean distance.

Proof. The second part of the proof 8.5 shows how to construct a map $g : \mathbf{R} \to X$ satisfying

$$g(0) = a, \quad g(\rho(a,x)) = x, \quad \operatorname{Lip}(g) \le 1, \quad \mathbf{V}_0^{\rho(a,x)} g = \rho(a,x).$$

Assume $0 \le t_1 \le t_2 \le \rho(a,x)$ such that $g(t_1) = g(t_2)$. Then there holds

$$\rho(a,x) \le \rho(a, g(t_1)) + \rho(g(t_2), x) \le \mathbf{V}_0^{t_1} g + \mathbf{V}_{t_2}^{\rho(a,x)} g \le t_1 + \rho(a,x) - t_2.$$

Hence, $t_1 = t_2$ and $g|\{t : 0 \le t \le \rho(a,x)\}$ is injective. This implies

$$\mathbf{V}_0^{\rho(a,x)} g = \mathscr{H}^1 \{g(t) : 0 \le t \le \rho(a,x)\}$$

by [Fed69, 2.10.13]. Whenever $-\infty < t_1 \le t_2 < \infty$, there holds

$$\rho(g(t_1), g(t_2)) \le \mathbf{V}_{t_1}^{t_2} g \le |t_2 - t_1|.$$

Therefore, g is a Lipschitzian function with respect to ρ and $W = \{g(t) : 0 \le t \le \rho(a,x)\}$ is compact with respect to ρ. The continuity of $\rho|(W \times W)$ follows from the inequality above, since $(g|\{t : 0 \le t \le \rho(a,x)\})^{-1}$ is continuous. $\square$

8.7 Example. For each positive integer i define

$$S_i = \mathbf{R}^2 \cap \{z : |z| = i^{-1}\},$$
$$T_i = \mathbf{R}^2 \cap \{(x,y) : (i+1)^{-1} \le (-1)^i x \le i^{-1},\ y = 0\}$$

and define

$$X = \operatorname{Clos} \textstyle\bigcup_{i=1}^{\infty} (S_i \cup T_i).$$

Then X is compact and path-connected but has infinite diameter with respect to ρ as in 8.3.

8.8 Example (The comb). For each positive integer i define

$$T_i = \mathbf{R}^2 \cap \{(x,y) : x = i^{-1},\ 0 \le y \le 1\}$$

and define

$$X = \mathbf{R}^2 \cap \{(x,y) : 0 \le x \le 1,\ y = 0\} \cup \operatorname{Clos} \textstyle\bigcup_{i=1}^{\infty} T_i.$$

Then X has finite diameter with respect to ρ as in 8.3, but ρ is not continuous with respect to the Euclidean distance.

The essential reason for the non-continuity of the geodesic distance over the comb, is that the comb has infinite one dimensional Hausdorff measure. This clarifies the following theorem.

8.9 Theorem. *Suppose n, X and ρ are related as is 8.3, X is connected and $\mathscr{H}^1(X) < \infty$.*

Then ρ is a metric over X, X is compact with respect to ρ and ρ is continuous with respect to the Euclidean distance.

Proof. Assume X consists of at least two elements. The connectedness of X implies

$$\operatorname{diam} X = \sup\{\mathscr{H}^1(\lambda[X]) : \lambda = |x - \cdot| \text{ for some } x \in X\} \leq \mathscr{H}^1(X).$$

Therefore, X is compact. According to [EH43, Theorem 2], there exists a continuous map $f : \{t : 0 \leq t \leq 1\} \to X$ satisfying

$$\operatorname{im} f = X, \quad \mathbf{V}_0^1 f \leq 2\mathscr{H}^1(X) - \operatorname{diam} X.$$

Hence, ρ is a metric over X. Define a map $s : \{t : 0 \leq t \leq 1\} \to \mathbf{R}$ by

$$s(t) = \mathbf{V}_0^t f \quad \text{whenever } 0 \leq t \leq 1.$$

Then s is continuous, see [Fed69, 2.5.16], and

$$\rho(f(t_1), f(t_2)) \leq |\mathbf{V}_{t_1}^{t_2} f| = |s(t_1) - s(t_2)|$$

whenever $t_1, t_2 \in \{t : 0 \leq t \leq 1\}$. Therefore, f is continuous with respect to ρ and X is compact with respect to ρ. Next the continuity of ρ will be shown. For this purpose let $a \in X$ and $x_1, x_2, \ldots$ be a sequence in X such that $|a - x_i| \to 0$ as $i \to \infty$. In view of 8.2, it is enough to show that there exists a sequence $k_1 < k_2 < \ldots$ of positive integers such that

$$\rho(a, x_{k_i}) \to 0 \quad \text{as } i \to \infty.$$

The compactness of X with respect to ρ provides a sequence $k_1 < k_2 < \ldots$ of positive integers and $b \in X$ satisfying $\rho(b, x_{k_i}) \to 0$ as $i \to \infty$. It follows $b = a$ as $|x - y| \leq \rho(x, y)$ whenever $x, y \in X$. □

8.10 Lemma (see [Men16b, 6.4])**.** *Suppose n, X, $0 < \delta \leq \infty$ and σ_δ are related as in 8.3 and X is connected.*

Then the following three statements hold true.

1. *The pseudometric σ_δ is a metric over X.*
2. *Whenever $x, y \in X$, there holds $|x - y| \leq \sigma_\delta(x, y)$.*
3. *Whenever $x, y \in X$ and $|x - y| \leq \delta$, there holds $\sigma_\delta(x, y) = |x - y|$.*

Proof. The statements (2) and (3) directly follow from the definition of σ_δ. They imply

$$\sigma_\delta(x, y) > 0 \quad \text{whenever } x, y \in X \text{ and } x \neq y.$$

To prove (1), it remains to show that $\sigma_\delta(x, y) < \infty$ whenever $x, y \in X$. For this purpose let a be any point in X and define

$$U(a) = X \cap \{x : \sigma_\delta(a, x) < \infty\}.$$

Then $U(a) \neq \varnothing$ as $a \in U(a)$. Moreover, $X \cap \{y : |x - y| < \delta\} \subset U(a)$ whenever $x \in U(a)$. Hence, $U(a)$ is open in X. Let $x_1, x_2, \ldots$ be a sequence in $U(a)$ which converges in X to some $x \in X$. Then there exists an integer j such that $|x_j - x| \leq \delta$. Therefore,

$$\sigma_\delta(a, x) \leq \sigma_\delta(a, x_j) + \delta < \infty$$

and $x \in U(a)$. This means that $U(a)$ is closed. The connectedness of X yields $U(a) = X$, which completes the proof. □

8.11 Theorem. *Suppose m, n, V and ψ are related as in 7.9, $\|V\|(\mathbf{R}^n) < \infty$ and d is the intrinsic diameter of $\operatorname{spt}\|V\|$, that is $d = \sup \rho[X \times X]$, where ρ is the pseudometric over $X = \operatorname{spt}\|V\|$ as in 8.3.*

Then there holds

$$d \leq \Gamma \psi(\mathbf{R}^n),$$

for some positive and finite number Γ which depends only on m. In particular $\operatorname{spt}\|V\|$ is compact if $\psi(\mathbf{R}^n) < \infty$.

Proof. Assume $\psi(\mathbf{R}^n) < \infty$ and $V \neq 0$. Define constants depending on m by

$$\Gamma_1(m) = m\boldsymbol{\gamma}(m)2^2, \quad \Gamma_2(m) = m\boldsymbol{\gamma}(m)^m 2^2,$$
$$\Delta(m) = m\boldsymbol{\beta}(1)\boldsymbol{\gamma}(m)^m 2^{m+2}.$$

Apply 7.10 with $\delta = \infty$ to obtain

$$\operatorname{diam} \operatorname{spt}\|V\| \leq \sup\{\Gamma_1(m), \Delta(m)\}\big(\psi(\mathbf{R}^n) + \|V\|(\mathbf{R}^n)^{1/m}\big).$$

Hence, $\operatorname{spt}\|V\|$ is compact. Moreover, $\operatorname{spt}\|V\|$ is connected by 5.11 as V is indecomposable. If $m = 1$, then 7.10 in conjunction with 8.10 and 8.5 implies

$$d \leq (\Gamma_1(1) + \Delta(1))\psi(\mathbf{R}^n).$$

Now assume $m \geq 2$. Let $0 < \delta < \infty$ and σ_δ be the pseudometric over $X = \operatorname{spt}\|V\|$ as in 8.3. Suppose a is any point in $\operatorname{spt}\|V\|$ and define

$$r = \sup\{\sigma_\delta(a, x) : x \in \operatorname{spt}\|V\|\}.$$

In the case

$$r \leq m\boldsymbol{\gamma}(m)2^2\|V\|(\mathbf{R}^n)^{1/m}$$

one calculates by the isoperimetric inequality and Hölder's inequality

$$\|V\|(\mathbf{R}^n)^{1-1/m} \leq \boldsymbol{\gamma}(m)\|\delta V\|(\mathbf{R}^n) \leq \boldsymbol{\gamma}(m)\|V\|(\mathbf{R}^n)^{1-1/(m-1)}\psi(\mathbf{R}^n)^{1/(m-1)},$$
$$r \leq m2^2\boldsymbol{\gamma}(m)^m\psi(\mathbf{R}^n) = \Gamma_2(m)\psi(\mathbf{R}^n).$$

In the case

$$m\boldsymbol{\gamma}(m)2^2\|V\|(\mathbf{R}^n)^{1/m} < r,$$

7.10 in conjunction with 8.10 implies

$$r \leq m2^{m+2}\boldsymbol{\beta}(1)\boldsymbol{\gamma}(m)^m\psi(\mathbf{R}^n) = \Delta(m)\psi(\mathbf{R}^n).$$

In both cases there holds

$$r \leq \sup\{\Gamma_2(m), \Delta(m)\}\psi(\mathbf{R}^n).$$

The conclusion follows for $\Gamma = \sup\{\Gamma_1(m) + \Delta(m), \Gamma_2(m), \Delta(m)\}$ by 8.5. $\square$

8.12 Remark. Obviously one cannot drop the assumption that $\|V\|(\mathbf{R}^n) < \infty$ as one could take $V = \mathbf{v}(T) \in \mathbf{RV}_m(\mathbf{R}^n)$ for some $T \in \mathbf{G}(n, m)$.

Moreover, if $m \geq 3$, one cannot drop the assumption that $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$. To see this, let T be as above and $V = \mathbf{v}(T \cap \{x : |x| \leq 1\}) \in \mathbf{RV}_m(\mathbf{R}^n)$. Then $\mathbf{h}(V; x) = 0$ for $\|V\|$ almost all x but $\operatorname{spt}\|V\| = T \cap \{x : |x| \leq 1\}$.

Finally, one cannot replace the assumption that the m dimensional density of $\|V\|$ is bounded below by a positive number $\|V\|$ almost everywhere by the assumption that $\operatorname{spt}\|V\|$ is compact. This follows analogous as in 6.12.

8.13 Remark. Suppose $2 \leq m < n$. Then one cannot replace the indecomposability condition of V by assuming $\operatorname{spt} \|V\|$ to be path-connected. To see this, it is enough by 5.16 to construct an open and bounded set U such that $\operatorname{Clos} U$ is path-connected but has infinite intrinsic diameter. For this purpose, define

$$\iota : \mathbf{R} \to \mathbf{R}^n$$

by letting

$$\iota(t) = (t, 0, \ldots, 0) \quad \text{whenever } t \in \mathbf{R}$$

and define

$$U(t, r) = \mathbf{U}(\iota(t), r) \quad \text{whenever } t \in \mathbf{R} \text{ and } 0 < r < \infty,$$
$$T = \mathbf{R}^n \cap \{x : (2i)^{-1} < |x| < (2i-1)^{-1} \text{ for some } i \in \mathscr{P}\},$$
$$U = T \cup \bigcup \{U((-1)^i a, (2i)^{-1} - a) : a = 2^{-1}[(2i)^{-1} + (2i+1)^{-1}],\, i \in \mathscr{P}\}.$$

Then $X = \operatorname{Clos} U$ is an analogous example as 8.7 for a compact and path-connected set which has infinite intrinsic diameter.

Suppose $m = 1$. Then one can replace the indecomposability condition of V by assuming $\operatorname{spt} \|V\|$ to be connected. This will be shown in 8.16.

8.14 Remark. The preceding theorem is not a generalization of Topping's diameter control [Top08] for two reasons. First, the corresponding varifold of a closed m dimensional manifold smoothly immersed in $\mathbf{R}^n$ does not need to be indecomposable. For example, the corresponding varifold of two spheres which meet exactly in one point is decomposable, see 5.14. Second, the geodesic distance over an m dimensional manifold smoothly immersed in $\mathbf{R}^n$ might be larger than the corresponding geodesic distance as in 8.3. The distance in 8.3 allows a shortcut whenever the immersion is not injective. However, the conclusion of the preceding theorem holds true for varifolds which correspond to m dimensional smooth connected and closed submanifolds of $\mathbf{R}^n$, see 5.12.

The preceding theorem is a generalization of Menne's diameter control [Men12a].

The preceding theorem is an extension of [Men16a, 14.2], see [Men16a, 14.4].

8.15 Theorem. *Suppose m, n, V and ψ are related as in 7.9, the Borel regular measure ψ is a Radon measure, $\operatorname{spt} \|V\|$ is compact, ρ is related to $X = \operatorname{spt} \|V\|$ as in 8.3, $a \in \operatorname{spt} \|V\|$ and $0 < r \leq \sup\{\rho(a, x) : x \in \operatorname{spt} \|V\|\}$.*

Then there holds

$$r \leq \Gamma \left(\psi\{x : \rho(a, x) \leq r\} + \|V\|(\{x : \rho(a, x) \leq r\})^{1/m} \right)$$

for $\Gamma = \sup\{m\boldsymbol{\gamma}(m)2^2, m\boldsymbol{\beta}(1)\boldsymbol{\gamma}(m)^m 2^{m+2}\}$.

Proof. First assume $r < \sup\{\rho(a, x) : x \in \operatorname{spt} \|V\|\}$. Then 8.5 provides $0 < \eta < \infty$ such that $r < \sup\{\sigma_\delta(a, x) : x \in \operatorname{spt} \|V\|\}$ whenever $0 < \delta < \eta$. Therefore, 7.10 in conjunction with 8.10 leads to

$$r \leq \Gamma \left(\psi\{x : \sigma_\delta(a, x) \leq r\} + \|V\|(\{x : \sigma_\delta(a, x) \leq r\})^{1/m} \right) < \infty$$

whenever $0 < \delta < \eta$. This implies

$$r \leq \Gamma \left(\psi\{x : \rho(a, x) \leq r\} + \|V\|(\{x : \rho(a, x) \leq r\})^{1/m} \right)$$

by 8.5 and [Fed69, 2.1.3 (5)]. Now the conclusion for $r = \sup\{\rho(a, x) : x \in \operatorname{spt} \|V\|\}$ follows. $\square$

8.16 Theorem. *Suppose n is a positive integer, $V \in \mathbf{V}_1(\mathbf{R}^n)$, $\|V\|(\mathbf{R}^n) < \infty$, $\|\delta V\|$ is a Radon measure,*

$$\boldsymbol{\Theta}^1(\|V\|, x) \geq 1 \quad \text{for } \|V\| \text{ almost all } x,$$

$\operatorname{spt}\|V\|$ *is connected, ρ is related to $X = \operatorname{spt}\|V\|$ as in 8.3 and d is the diameter of* $\operatorname{spt}\|V\|$ *with respect to ρ.*

Then there holds

$$d \leq \|V\|(\mathbf{R}^n),$$

ρ is a metric over $\operatorname{spt}\|V\|$, $\operatorname{spt}\|V\|$ *is compact with respect to ρ, ρ is continuous with respect to the Euclidean distance and the estimate of d is sharp.*

Proof. Define

$$Y = \{a : \|\delta V\|\{a\} > 0\}.$$

Then Y is countable as $\|\delta V\|$ is a Radon measure and [Fed69, 2.10.19 (3)] in conjunction with [Men16a, 4.8 (4)] implies

$$\mathscr{H}^1(\operatorname{spt}\|V\|) = \mathscr{H}^1(\operatorname{spt}\|V\| \sim Y) \leq \|V\|(\operatorname{spt}\|V\| \sim Y) \leq \|V\|(\mathbf{R}^n).$$

The conclusion follows by 8.9 and 8.6. □

A Outlook

This section presents questions based on the results of this thesis.

Isoperimetric inequality, see 6.7. One may investigate whether the isoperimetric inequality can be sharpened in the following way.

Suppose $m \leq n$ are positive integers $V \in \mathbf{V}_m(\mathbf{R}^n)$ and $\|V\|(\mathbf{R}^n) < \infty$.

Does there hold

$$\|V\|\{x : 1 \leq \boldsymbol{\Theta}^{*m}(\|V\|, x)\} \leq \Gamma \|V\|(\{x : 1 \leq \boldsymbol{\Theta}^{*m}(\|V\|, x)\})^{1/m} \|\delta V\|(\mathbf{R}^n)$$

for some positive and finite number Γ which only depends on m?

Indecomposability condition. The only property of indecomposable varifolds used in this thesis, is that one cannot decompose an indecomposable varifold by cutting the varifold along a level set of a Lipschitzian function in the way of 7.2. One may investigate whether the indecomposability condition is stronger than the property described above.

The set where the density is small, see 7.14. One may investigate whether the conclusion of 7.14 is sharp in the following way.

Suppose $0 < s < 1$.

Does there exists a varifold V which satisfies the hypothesis of 7.14 such that

$$\mathscr{H}^s(\operatorname{spt}\|V\| \cap \{x : \boldsymbol{\Theta}^m_*(\|V\|, x) < 1\}) > 0\,?$$

Diameter control, see 8.11. In view of the case $m = 2$ and the paper of Paeng [Pae14], one may investigate whether 8.11 can be generalized to an inequality which involves the singular part $\|\delta V\|_{\mathrm{sing}}$ of the total variation but does not need the absolute continuity of $\|\delta V\|$ with respect to $\|V\|$.

Continuity of the geodesic distance. In view of 8.16 one may investigate whether the following statement holds true.

Suppose V is a varifold which satisfies the hypothesis of 8.11 and $\psi(\mathbf{R}^n) < \infty$. Is the geodesic distance over $\operatorname{spt} \|V\|$ continuous with respect to the Euclidean distance?

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Selbstständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Masterarbeit eigenständig und nur unter Zuhilfenahme der angegeben Quellen und Hilfsmittel verfasst habe.

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