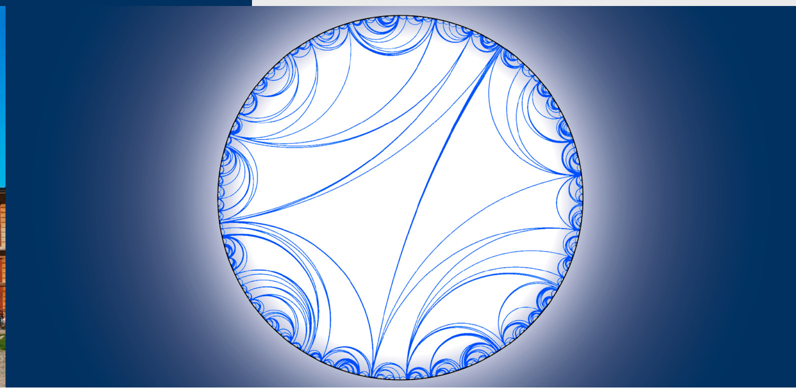
Universitätsverlag Potsdam



Sylvie Roelly | Pierre Vallois

Convoluted Brownian motion

A semimartingale approach

Preprints des Instituts für Mathematik der Universität Potsdam
5 (2016) 9

Preprints des Instituts für Mathematik der Universität Potsdam

Preprints des Instituts für Mathematik der Universität Potsdam
5 (2016) 9

Sylvie Roelly | Pierre Vallois

Convoluted Brownian motion

A semimartingale approach

Universitätsverlag Potsdam

Bibliografische Information der Deutschen Nationalbibliothek
Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über http://dnb.dnb.de abrufbar.

Universitätsverlag Potsdam 2016
http://verlag.ub.uni-potsdam.de/

Am Neuen Palais 10, 14469 Potsdam
Tel.: +49 (0)331 977 2533 / Fax: 2292
E-Mail: verlag@uni-potsdam.de

Die Schriftenreihe **Preprints des Instituts für Mathematik der Universität Potsdam** wird herausgegeben vom Institut für Mathematik der Universität Potsdam.

ISSN (online) 2193-6943

Kontakt:
Institut für Mathematik
Karl-Liebknecht-Straße 24/25
14476 Potsdam
Tel.: +49 (0)331 977 1499
WWW: http://www.math.uni-potsdam.de

Titelabbildungen:
1. Karla Fritze | Institutsgebäude auf dem Campus Neues Palais
2. Nicolas Curien, Wendelin Werner | Random hyperbolic triangulation
Published at: http://arxiv.org/abs/1105.5089

Online veröffentlicht auf dem Publikationsserver der Universität Potsdam
URL https://publishup.uni-potsdam.de/opus4-ubp/frontdoor/index/index/docId/9633
URN urn:nbn:de:kobv:517-opus4-96339
http://nbn-resolving.de/urn:nbn:de:kobv:517-opus4-96339

Convoluted Brownian motion: a semimartingale approach

Sylvie RŒLLY[(1)], **Pierre VALLOIS**[(2)]

September 21, 2016

(1) Universität Potsdam, Institut für Mathematik, Karl-Liebknecht-Str. 24-25, 14476 Potsdam OT Golm, Germany
(2) Université de Lorraine, Institut de Mathématiques Elie Cartan, INRIA-BIGS, CNRS UMR 7502, BP 239, 54506 Vandœuvre-lès-Nancy Cedex, France

Abstract.
In this paper we analyse semimartingale properties of a class of Gaussian periodic processes, called convoluted Brownian motions, obtained by convolution between a deterministic function and a Brownian motion. A classical example in this class is the periodic Ornstein-Uhlenbeck process.
We compute their characteristics and show that in general, they are never Markovian nor satisfy a time-Markov field property. Nevertheless, by enlargement of filtration and/or addition of a one-dimensional component, one can in some case recover the Markovianity. We treat exhaustively the case of the bidimensional trigonometric convoluted Brownian motion and the more dimensional monomial convoluted Brownian motion.

Key words and phrases : periodic Gaussian process, periodic Ornstein-Uhlenbeck process, Markov-field property, enlargement of filtration.
AMS 2000 subject classifications : 60 G 10, 60 G 15, 60 G 17, 60 H 10, 60 H 20.

1 Introduction

In this note we focus our attention on a class of processes constructed as convolution between a deterministic - possibly vector-valued - L^2-function and a Brownian motion. More precisely, for a (scalar) function φ in $L^2\big(0,1\big)$, we first define

$$X_t^{\varphi} := \int_0^t \varphi(t-s)\,dB_s + \int_t^1 \varphi(1+t-s)\,dB_s, \quad t \in [0,1], \tag{1.1}$$

where $(B_t, t \in [0,1])$ is a real-valued Brownian motion. We will call the process X^φ a (scalar) **convoluted Brownian motion**.

Some properties of this process are immediate: for a given φ, the process X^φ is a stationary, centered, Gaussian and belongs to the first chaos of B. It is periodic on the time interval $[0,1]$ and its law is also time reversal invariant. Less evident: in general a process of that type is neither Markovian, nor a time-Markov random field and nor a semimartingale in its natural filtration.

One key point of the paper is to study the linear map $\varphi \mapsto X^\varphi$. We propose in Proposition 3.5, for φ smooth enough, a decomposition of X^φ as a $d\varphi$-mixture of simple Gaussian processes $(Z(r,\cdot))_{r\in[0,1]}$ which satisfy interesting properties.

In particular, we prove that, for any r in $[0,1]$, $Z(r,\cdot)$ is itself a convoluted Brownian motion associated with the indicator function of a suitable interval. It also corresponds to the random concatenation of Brownian bridges, see Proposition 3.4.
Then, when φ is differentiable, the processes X^φ and $X^{\varphi'}$ are linked via Equation (3.22). This key identity will be useful. It first permits to interpret the particular case $X^{\exp(-\lambda\cdot)}$ as solution of the stochastic integral equation (2.1) and to identify it as the celebrated periodic Ornstein-Uhlenbeck process. When φ is a trigonometric function, due to the proportionality between φ'' and φ, one derives that the pair of processes $(X^{\cos}, X^{\sin})$ is solution of an (autonomous) bidimensional system of stochastic integral equations, see (3.27). We also consider the scalar process $X^{\sharp k} := X^\varphi$, when φ is the monomial function $x \mapsto x^k$. The process $X^{\sharp k}$ is not solution of an autonomous stochastic equation, but since the derivative of $x \mapsto x^k$ is a monomial of order $k-1$, it makes sense to consider $X^{\sharp k}$ as the first coordinate of the $(k+1)$-dimensional process $\mathbf{X}^{\sharp k}$ whose coordinates are $X^{\sharp k}, \cdots, X^{\sharp 1}, X^{\sharp 0}$. Indeed, $\mathbf{X}^{\sharp k}$ is a multidimensional convoluted process and solves the autonomous linear stochastic integral system (3.25).

Another central and difficult matter we are considering, is the Markovianity of X^φ. In general it fails, but we will present some partial results: the process $X^{\exp(-\lambda\cdot)}$ is not Markovian but a time-Markov random field, whose bridges coincide with the ones of the Ornstein-Uhlenbeck process. To recover its Markovianity one has to enlarge the natural filtration of the driving Brownian motion $(B_t, t \in [0,1])$ with the initial value of the process. In Section 4, we introduce a class of multidimensional convoluted processes $\mathbf{X}^{A,\phi}$ indexed by a matrix A and a vector ϕ. We compute in Proposition 4.3 the covariance function of this stationary Gaussian process. Under two additional assumptions $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$, $\mathbf{X}^{A,\phi}$ is a mixture of its bridges and solves a linear SDE, see Theorem 4.9. The bidimensional trigonometric convoluted Brownian motion is of the type $\mathbf{X}^{A,\phi}$ for suitable A and ϕ and Theorem 4.9 can be applied. In the case of the monomial convoluted Brownian motion, the process $\mathbf{X}^{\sharp k}$ is also of the type $\mathbf{X}^{A,\phi}$ but $(\mathfrak{H}_1)$ is not satisfied. Adding a one-dimensional component to the vector $\mathbf{X}^{\sharp k}$, we recover a Markov property, see Section 4.5.1 for details.

The originality of our contribution is based on various representations of convoluted Brownian motions and the use of initial enlargement of filtrations. This powerful tool of stochastic calculus permits to analyse them pathwise, to show their semimartingale decomposition and their (lack of) Markovianity.

2 A very special case: the periodic Ornstein-Ulhenbeck process (PerOU)

Let $(B_t, t \in [0,1])$ be a one-dimensional Brownian motion starting at 0 defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with its natural filtration $\mathcal{F}_t := \sigma(B_r, r \in [0,t]), 0 \le t \le 1$.
We are first interested in the process solution of the following stochastic integral equation with periodic boundary conditions:

$$\begin{cases} Y_t &= Y_0 + B_t - \lambda \int_0^t Y_s ds, \quad t \in [0,1], \\ Y_1 &= Y_0. \end{cases} \tag{2.1}$$

where $\lambda \in \mathbb{R}$.
Notice that a solution to equation (2.1), if it exists, could not be a classical strong one, since $(Y_t)_t$ is clearly *not adapted* to the filtration $(\mathcal{F}_t)_t$ due to the fact that the initial condition involves the final one.

Indeed this process is known in the literature as the periodic Ornstein-Uhlenbeck process, here called PerOU. It exhibits various interesting properties and was therefore studied by several authors with different motivations. E.g. its Gaussian aspect related to filtering problems is underlined by [8] or its construction as solution of a linear stochastic differential equation with time-boundary conditions appears in [13] or [4]. See also [14] for some generalisations including Lévy processes. The PerOU process is not Markov but satisfies a time-Markov field property, as proved in [1]. This time symmetric property, formalized by Jamison in [6] and also called *reciprocal property* states that, given the knowledge of the process at any pair of times s and u (with $s \le u$), the dynamics of the process *inside* $[s,u]$ and *outside* (s,u) are conditionally independent. See [10] for a recent review on the relationship between the Markov property and the reciprocal one.
Here we propose an alternative approach which is centered on the semimartingale decomposition with respect to an enlarged (*grossissement de*) filtration, to overcome the adaptibility problem we mentioned above. We will present this decomposition in Section 2.2, but before developing these arguments we review the Gaussian point of view in the next section.

2.1 PerOU as Gaussian process

Although the process Y solving (2.1) is not adapted to $(\mathcal{F}_t)_t$, it can be given explicitly.

Proposition 2.1 *1. The unique solution of* (2.1) *is given by*

$$Y_t = \frac{e^{-\lambda}}{1-e^{-\lambda}} \int_0^1 e^{-\lambda(t-s)} dB_s + \int_0^t e^{-\lambda(t-s)} dB_s, \quad t \in [0,1]. \tag{2.2}$$

In particular, its boundary conditions satisfy

$$Y_1 = Y_0 = \frac{1}{1-e^{-\lambda}} \int_0^1 e^{-\lambda(1-s)} dB_s. \tag{2.3}$$

2. *The process $(Y_t)_t$ admits the following representation, as the sum of a past-depending part and a future-depending one:*

$$Y_t = \frac{1}{1-e^{-\lambda}} \int_0^t e^{-\lambda(t-s)} dB_s + \frac{1}{1-e^{-\lambda}} \int_t^1 e^{-\lambda(1+t-s)} dB_s, \quad t \in [0,1]. \tag{2.4}$$

3. *The process $(Y_t)_t$ is a stationary Gaussian process whose covariance function*
$R(h) := \mathrm{Cov}(Y_s, Y_{s+h})$ satisfies

$$R(h) = \frac{1}{2\lambda(1-e^{-\lambda})} \left(e^{-\lambda(1-h)} + e^{-\lambda h} \right) = \frac{1}{2\lambda} \frac{\cosh\big(\lambda(h-1/2)\big)}{\sinh(\lambda/2)}. \tag{2.5}$$

The formulation (2.4) shows that the PerOU is a special element of the class of processes defined by (1.1): $Y = X^{\varphi}$ with $\varphi(s) := \frac{1}{1-e^{-\lambda}} e^{-\lambda s}$.

Proposition 2.1 is a particular case of a more general result, in which the final condition of the process is any given function of the initial condition Y_0, and not precisely equal to it. This will be treated in Proposition 2.5 below, and proved there.

2.2 PerOU as a semimartingale

One notices in the identity (2.3) that the random variable Y_0 is not $\mathcal{F}_0$-measurable. Consequently the process Y is not a $(\mathcal{F}_t)$-semimartingale. Anyway, by (2.1) and (2.3), the process $(Y_t)_t$ solves:

$$Y_t = \frac{e^{-\lambda}}{1-e^{-\lambda}} \int_0^1 e^{\lambda s} dB_s + B_t - \lambda \int_0^t Y_s ds, \quad t \in [0,1]. \tag{2.6}$$

An initial enlargement of filtration will therefore permit to consider (2.6) as the integral form of a usual SDE, whose solution is a semimartingale with respect to this new filtration. This is the subject of the present subsection.

Proposition 2.2 *Let $(\mathcal{G}_t)_t$ be the filtration obtained by an initial enlargement of $(\mathcal{F}_t)_t$ with the random variable $\int_0^1 e^{\lambda s} dB_s$. Then, there exists a $(\mathcal{G}_t)_t$-Brownian motion $\widetilde{B}$ independent of Y_0 such that the PerOU process $(Y_t)_t$ solves the SDE*

$$Y_t = Y_0 + \widetilde{B}_t - \lambda \int_0^t Y_s ds + \int_0^t \frac{\lambda}{\sinh\big(\lambda(1-s)\big)} \Big(Y_0 - e^{-\lambda(1-s)} Y_s\Big) ds. \quad (2.7)$$

Therefore equation (2.7) gives the $(\mathcal{G}_t)$-semimartingale decomposition of the process Y solution of (2.6).

The proof of Proposition 2.2 is based on results concerning initial enlargement of the Brownian filtration $(\mathcal{F}_t)_t$ by a one-dimensional random variable ξ which is $\mathcal{F}_1$-measurable and belongs to the first chaos of (B_t). This initial enlargement $(\mathcal{G}_t)_t$ is the smallest filtration which satisfies the usual conditions and such that:

- $\mathcal{F}_t$ is included in $\mathcal{G}_t$ for any $t \in [0,1]$
- $\mathcal{G}_0$ coincides with the σ-algebra generated by ξ.

Let us recall Théorème I.1.1 in [2].

Lemma 2.3 *Consider a random variable ξ belonging to the first chaos of $(B_t)_{0\le t\le 1}$, i.e. $\xi := \int_0^1 h(s) dB_s$ where $h \in L^2\big(0,1\big)$. Then*

$$B_t - \int_0^t \frac{h(s)}{\sigma^2(s)} \Big(\int_s^1 h(u) dB_u\Big) ds, \quad 0 \le t \le 1, \quad (2.8)$$

is a standard $(\mathcal{G}_t)_t$-Brownian motion independent of ξ, where the function σ is given by $\sigma^2(s) := \int_s^1 h^2(u) du$.

Remark that the stochastic integral between time s and time 1 appearing in the formula (2.8) is indeed $\mathcal{G}_s$-measurable:

$$\int_s^1 h(u) dB_u = \int_0^1 h(u) dB_u - \int_0^s h(u) dB_u = \xi - \int_0^s h(u) dB_u. \quad (2.9)$$

Proof of Proposition 2.2: Choose $h(t) := e^{\lambda t}$ in Lemma 2.3. One gets $Y_0 = \dfrac{e^{-\lambda}}{1-e^{-\lambda}} \xi$ and $\sigma^2(s) = \dfrac{1}{2\lambda}(e^{2\lambda} - e^{2\lambda s})$. Then the process defined by

$$\widetilde{B}_t := B_t - \int_0^t \frac{\lambda}{\sinh\big(\lambda(1-s)\big)} \Big(\int_s^1 e^{-\lambda(1-u)} dB_u\Big) ds, \quad (2.10)$$

is a $(\mathcal{G}_t)_t$-Brownian motion.
Set $Y'_t := Y_0 - e^{-\lambda(1-t)} Y_t$. Using (2.2) and (2.3), we have:

$$
\begin{aligned}
Y'_t &= \frac{1}{1-e^{-\lambda}} \int_0^1 e^{-\lambda(1-s)} dB_s - e^{-\lambda(1-t)} \Big(\frac{e^{-\lambda}}{1-e^{-\lambda}} \int_0^1 e^{-\lambda(t-s)} dB_s \\
&\qquad + \int_0^t e^{-\lambda(t-s)} dB_s \Big) \\
&= \int_t^1 e^{-\lambda(1-s)} dB_s.
\end{aligned}
$$

Thus, (2.7) is a direct consequence of (2.6), (2.3) and (2.10).

■

2.3 PerOU and its bridges

In this paragraph, we review the disintegration of the PerOU process along its initial (and final) time marginal.

Proposition 2.4 *Denote by ν the initial Gaussian law $\mathcal{L}(Y_0)$. Then, the PerOU process is a ν-mixture of its bridges, that is $\mathcal{L}(Y) = \int_{\mathbb{R}} \mathcal{L}(Y^{xx}) \nu(dx)$, where the $x \hookrightarrow x$ bridge, denoted by Y_t^{xx}, solves the SDE*

$$
\left\{
\begin{aligned}
dX_t &= d\widetilde{B}_t - \lambda X_t \, dt + \frac{\lambda}{\sinh\big(\lambda(1-t)\big)} \Big(x - e^{-\lambda(1-t)} X_t \Big) \, dt, \quad t \in [0,1[, \\
X_0 &= x.
\end{aligned}
\right.
\tag{2.11}
$$

Therefore the family of bridges $(Y^{xx})_x$ of the PerOU process coincides with those of an Ornstein-Ulhenbeck process.

Proof. (2.11) is a direct consequence of (2.7).
Now, consider the linear SDE with fixed initial condition x (but with free final condition),

$$
\left\{
\begin{aligned}
dX_t &= d\widetilde{B}_t - \lambda X_t \, dt, \quad t \in [0,1], \\
X_0 &= x.
\end{aligned}
\right.
\tag{2.12}
$$

Its unique Markov solution $X^{OU,x}$ is the celebrated Ornstein-Ulhenbeck process with initial deterministic condition x, given by

$$
X_t^{OU,x} = x e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} dB_s.
$$

One has

$$
X_1^{OU,x} - e^{-\lambda(1-t)} X_t^{OU,x} = x e^{-\lambda} + \int_0^1 e^{-\lambda(1-s)} dB_s - e^{-\lambda} \Big(x + \int_0^t e^{-\lambda s} dB_s \Big) = \int_t^1 e^{-\lambda(1-s)} dB_s.
$$

Using (2.12) and (2.10), we get

$$X_t^{OU,x} = x+\widetilde{B}_t-\lambda \int_0^t X_s^{OU,x} ds+\int_0^t \frac{\lambda}{\sinh\big(\lambda(1-s)\big)} \Big(X_1^{OU,x}-e^{-\lambda(1-s)}X_s^{OU,x}\Big)ds.$$

This leads to the identification of the process $Y_\cdot^{xx}$ as the $x \hookrightarrow x$ bridge of the Ornstein-Ulhenbeck process $X^{OU,x}$. ■

In other words, the PerOU process belongs to the *reciprocal class* of the Ornstein-Ulhenbeck process, which is defined as the set of any mixture of bridges of the Ornstein-Ulhenbeck process. This fact was already mentioned and proved in [15], via a completely different way. In the latter paper, the reciprocal class is characterized as the set of solutions of an integration by part formula on the path space. In particular the PerOU process satisfies the time-Markov field property.

2.4 Ornstein-Ulhenbeck process with prescripted time-boundaries

Let us now relax the periodic boundary conditions of the PerOU imposed in (2.1) and replace it by $Y_1 = f(Y_0)$ where f is a measurable real-valued map. Consider the process Y_t (if it exists) solution of the stochastic integral equation

$$\begin{cases} Y_t &= Y_0 + B_t - \lambda \int_0^t Y_s ds, \quad t \in [0,1], \\ Y_1 &= f(Y_0). \end{cases} \tag{2.13}$$

This class of pinned Ornstein-Ulhenbeck process was treated in [12] under the assumption called $(\mathfrak{H}_1)$ by the authors, which corresponds to the fact that

$$x \mapsto f(x) - e^{-\lambda} x \quad \text{ is a bijective map.} \tag{2.14}$$

Let us solve (2.13) under a weaker assumption than (2.14). First, since $t \mapsto \int_0^t Y_s ds$ is differentiable and B admits a finite quadratic variation, one can use the generalised stochastic calculus (see [17]) to get

$$d(Y_t e^{\lambda t}) = e^{\lambda t}\big(\lambda Y_t dt + dB_t - \lambda Y_t dt\big) = e^{\lambda t} dB_t.$$

Therefore

$$Y_t = Y_0 e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} dB_s.$$

Considering the boundary conditions, one obtains

$$Y_1 - e^{-\lambda} Y_0 = f(Y_0) - e^{-\lambda} Y_0 = \int_0^1 e^{-\lambda(1-s)} dB_s. \tag{2.15}$$

Suppose now that the map defined in (2.14) is surjective, then there exists a measurable function g such that $\big(f - e^{-\lambda} Id\big) \circ g = Id$. Therefore one solution to (2.13) is given by

$$Y_t = e^{-\lambda t} g\Big(\int_0^1 e^{-\lambda(1-s)} dB_s\Big) + \int_0^t e^{-\lambda(t-s)} dB_s. \tag{2.16}$$

Notice that, in general, it is no more a Gaussian process. Furthermore, the above representation of solutions of (2.13) implies their non-uniqueness as soon as the map (2.14) fails to be injective.

Take:

$$f(x) := e^{-\lambda} x + x \mathbb{1}_{]-\infty,1[}(x) + (2-x)\mathbb{1}_{[1,2]}(x) + (x-2)\mathbb{1}_{[2,+\infty[}(x).$$

Then, both functions g_1 and g_2 defined by

$$\begin{array}{rrcl} & g_1(y) & = & y\mathbb{1}_{]-\infty,1]}(y) + (2+y)\mathbb{1}_{]1,+\infty]}(y) \\ \text{and} & g_2(y) & = & y\mathbb{1}_{]-\infty,0[}(y) + (2-y)\mathbb{1}_{[0,1]}(y) + (2+y)\mathbb{1}_{]1,+\infty]}(y) \end{array}$$

solve the identity $\big(f - e^{-\lambda} Id\big) \circ g = Id$, which induces two non identical solutions for the equation (2.13). Moreover one can randomize the choice of the map g in the following way to obtain infinitely many solutions: take any random variable ϵ with values in $\{1, 2\}$ which is measurable with respect to $\mathcal{F}_1$. Then

$$e^{-\lambda t} g_\epsilon\Big(\int_0^1 e^{-\lambda(1-s)} dB_s\Big) + \int_0^t e^{-\lambda(t-s)} dB_s$$

is a also a solution to (2.13). Remark that, if ϵ is a.s. not a constant, the initial condition of this process is not $\sigma(\int_0^1 e^{-\lambda(1-s)} dB_s)$-measurable. Let us summarize these results in the following proposition.

Proposition 2.5 *Take any measurable function f. If the map defined by $x \mapsto f(x) - e^{-\lambda} x$ is surjective, there exists at least one pinned Ornstein-Uhlenbeck process solution to (2.13). It belongs to the reciprocal class of the Ornstein-Uhlenbeck process since, for all x and $y = f(x)$, its $x \hookrightarrow y$ bridge satisfies the SDE*

$$\left\{ \begin{array}{rcl} dX_t & = & d\widetilde{B}_t - \lambda X_t \, dt + \dfrac{\lambda}{\sinh\big(\lambda(1-t)\big)} \Big(y - e^{-\lambda(1-t)} X_t \Big) \, dt, \quad t \in [0,1[, \\ X_0 & = & x. \end{array} \right. \tag{2.17}$$

as the $x \hookrightarrow y$ bridge of the Ornstein-Uhlenbeck process does.

3 Convoluted Brownian motion

We now go back to the study of more general processes denoted by X^φ, admitting the representation (1.1) which is a kind of convolution between a square integrable determinist function φ - not necessarily of exponential type - and the Brownian motion.

Doing that, we consider processes which are no more in the reciprocal class of the Ornstein-Ulhenbeck process but still belong to the first Wiener chaos. Moreover we will see that the constructed processes - among other interesting properties - are stationary and periodic.

3.1 Definition of the process X^φ and first properties

For any fixed $\varphi \in L^2\big(0,1\big)$, let us recall the definition of the process X^φ.

$$X_t^\varphi := \int_0^t \varphi(t-s)\, dB_s + \int_t^1 \varphi(1+t-s)\, dB_s, \quad t \in [0,1]. \tag{3.1}$$

The following proposition extends the properties enounced in Proposition 2.1.

Proposition 3.1 *1. The process $(X_t^\varphi)_{0\le t\le 1}$ is stationary, centered and Gaussian with covariance function $R^\varphi(h) := \mathrm{Cov}\big(X_s^\varphi, X_{s+h}^\varphi\big)$ given by*

$$R^\varphi(h) = \int_0^h \varphi(1-u)\varphi(h-u)du + \int_0^{1-h} \varphi(u)\varphi(h+u)du. \tag{3.2}$$

More generally, the covariance between X_s^ψ and X_t^φ for $s \le t$, $\psi, \varphi \in L^2\big(0,1\big)$, is:

$$\int_0^s \varphi(t-u)\psi(s-u)du + \int_s^t \varphi(t-u)\psi(1+s-u)du + \int_t^1 \varphi(1+t-u)\psi(1+s-u)du. \tag{3.3}$$

2. $(X_t^\varphi)_{0\le t\le 1}$ is pathwise periodic and satisfies

$$X_0^\varphi = X_1^\varphi = \int_0^1 \varphi(1-s)\, dB_s. \tag{3.4}$$

3. $(X_t^\varphi)_{0\le t\le 1}$ is invariant under time reversal:

$$\big(X_{1-t}^\varphi, 0 \le t \le 1\big) \overset{(d)}{=} \big(X_t^\varphi, 0 \le t \le 1\big). \tag{3.5}$$

It also satisfies

$$\big(X_t^{\widehat{\varphi}}, 0 \le t \le 1\big) \overset{(d)}{=} \big(X_t^\varphi, 0 \le t \le 1\big), \tag{3.6}$$

where $\widehat{\varphi}(t) := \varphi(1-t)$ denotes the time reversal of the function φ.

4. *The linear map $\varphi \mapsto X_t^\varphi$ is an isometry from $L^2\big(0,1\big)$ in $L^2(\Omega)$ for any fixed $t \in [0,1]$. Moreover, the linear map $\varphi \mapsto \int_0^t X_u^\varphi \, du$ has a norm bounded by 1.*

Remark 3.2 *The reversibility* (3.5) *of the process X^φ holds not only in law but also pathwise, in the following sense. X^φ admits the symmetric path representation*

$$X_t^\varphi \quad = \quad I^\varphi(B)(t) + I^{\widehat{\varphi}}(\widehat{B})(1-t), \quad t \in [0,1], \tag{3.7}$$

where $I^\varphi(B)(t) := \int_0^t \varphi(t-s)\, dB_s$ $I^\varphi(B)$ is a stochastic convolution and $\big(\widehat{B}_t := B_{1-t} - B_1, t \in [0,1]\big)$ is the time reversal of the Brownian motion B. Indeed, for any $f \in L^2\big(0,1\big)$,

$$\int_0^1 f(s) dB_s = \int_0^1 \widehat{f}(s) d\widehat{B}_s = \int_0^1 f(1-s) d\widehat{B}_s. \tag{3.8}$$

Consequently,

$$\begin{aligned}\int_t^1 \varphi(1+t-s) dB_s \quad &= \quad \int_0^1 1_{\{s \geq t\}} \varphi(1+t-s) dB_s = \int_0^{1-t} \varphi(t+s) d\widehat{B}_s \\ &= \quad \int_0^{1-t} \widehat{\varphi}(1-t-s) d\widehat{B}_s = I^{\widehat{\varphi}}(\widehat{B})(1-t)\end{aligned}$$

which leads to (3.7).

Proof of Proposition 3.1: Since

$$\int_0^t \varphi(t-s)^2 ds + \int_t^1 \varphi(1+t-s)^2 ds = \int_0^1 \varphi(s)^2 ds < +\infty$$

then (3.1) defines a centered Gaussian process.
Identity (3.4) is a direct consequence of (3.1).
Let us calculate $\mathrm{Cov}\big(X_t^\varphi, X_s^\psi\big)$ for $0 \leq s \leq t \leq 1$ and $\varphi, \psi \in L^2\big(0,1\big)$. Using (3.1), we easily get:

$$\begin{aligned}\mathrm{Cov}\big(X_t^\varphi, X_s^\psi\big) \quad &= \quad \int_0^s \psi(s-u)\varphi(t-u) du + \int_s^t \psi(1+s-u)\varphi(t-u) du \\ &\quad + \int_t^1 \psi(1+s-u)\varphi(1+t-u) du.\end{aligned}$$

We now take $\psi = \varphi$ and we use the change of variables $r := s-u$ in the first integral, $r := u-s$ in the second and $r := 1-u+s$ in the third one, and reformulate the covariance as

$$\begin{aligned}&\int_0^s \varphi(r)\varphi(t-s+r) dr + \int_0^{t-s} \varphi(1-r)\varphi(t-s-r) dr + \int_s^{1-t+s} \varphi(r)\varphi(t-s+r) dr \\ = \quad &\int_0^{1-t+s} \varphi(r)\varphi(t-s+r) dr + \int_0^{t-s} \varphi(1-r)\varphi(t-s-r) dr.\end{aligned}$$

Setting $h := t - s$, we get:

$$R^{\varphi}(h) = \int_0^{1-h} \varphi(r)\varphi(h+r)dr + \int_0^h \varphi(1-r)\varphi(h-r)dr.$$

The time reversibility of X^{φ} is a consequence of its Gaussianity and its stationarity:

$$\mathrm{Cov}(X^{\varphi}_{1-t}, X^{\varphi}_{1-s}) = R^{\varphi}\big(|t-s|\big) = \mathrm{Cov}(X^{\varphi}_t, X^{\varphi}_s), \quad 0 \le s,t \le 1.$$

The identity (3.6) is a consequence of

$$R^{\varphi}(1-h) = R^{\widehat{\varphi}}(h), \quad h \in [0,1]. \tag{3.9}$$

Last, let us prove the assertion 4.:

$$E\Big[\Big(\int_0^t X^{\varphi}_u du\Big)^2\Big] \le t\int_0^t E\Big[\big(X^{\varphi}_u\big)^2\Big]du = t^2\int_0^1 \varphi^2(s)ds.$$

Thus the continuity of $\varphi \mapsto \displaystyle\int_0^t X^{\varphi}_u\, du$ follows. ■

Examples 3.3 *1. For $\varphi(u) = u^k$, monomial of degree $k \in \mathbb{N}$, the corresponding convoluted process, denoted by $X^{\sharp k}$, satisfies*

$$X^{\sharp k}_t = \int_0^t (t-s)^k\, dB_s + \int_t^1 (1+t-s)^k\, dB_s. \tag{3.10}$$

In particular, for $\varphi \equiv 1$ (monomial of degree 0) one has $X^{\sharp 0}_t \equiv B_1$ which is a constant process.

2. When $\varphi = \mathbb{1}_{[a,1]}, a \in [0,1]$, the corresponding convoluted process denoted by $Z(a,\cdot)$ is given by

$$Z(a,t) = \begin{cases} B_{t+1-a} - B_t & \text{if } t \in [0,a] \\ B_{t-a} - B_t + B_1 & \text{if } t \in [a,1]. \end{cases} \tag{3.11}$$

Qualitative and quantitative analysis of the process $Z(a,\cdot)$.

For a fixed, $t \mapsto Z(a,t)$ is a stationary Gaussian process whose covariance function, which we denote by R^a, depends on the value of a.

$$\text{If } 0 \le a \le 1/2, \qquad R^a(h) := \begin{cases} 1-a-h & \text{if } h \in [0,a] \\ 1-2a & \text{if } h \in [a,1-a] \\ h-a & \text{if } h \in [1-a,1]. \end{cases} \tag{3.12}$$

$$\text{If } 1/2 \le a \le 1, \qquad R^a(h) := \begin{cases} 1-a-h & \text{if } h \in [0,1-a] \\ 0 & \text{if } h \in [1-a,a] \\ h-a & \text{if } h \in [a,1]. \end{cases} \tag{3.13}$$

Let us now study the time-Markov field property of this process, and the structure of its bridges.
We only analyze the case $a \le 1/2$ (that is $a \le 1-a$) since the study of the case $a \ge 1/2$ is similar (replace a by $1-a$).

Proposition 3.4 *Suppose $a \le 1/2$. The process $t \mapsto Z(a,t)$ considered on the time interval $[0,1]$ is not a time-Markov field, but a concatenation of such ones on each time intervals $[0,a]$, $[a,1-a]$ and $[1-a,1]$. Indeed the bridges of the process $\frac{1}{\sqrt{2}}Z(a,\cdot)$ between times 0 and a (resp. between a and $1-a$, resp. between $1-a$ and 1) are Brownian bridges. Therefore the conditional law of $(\frac{1}{\sqrt{2}}Z(a,t), t \in [0,1])$ given the four values $Z(a,0), Z(a,a), Z(a,1-a), Z(a,1)$ is equal to the law of a Brownian motion pinned at the four instants $0, a, 1-a, 1$. With other words, the process $(\frac{1}{\sqrt{2}}Z(a,t), t \in [0,1])$ is a mixture of concatenation of Brownian bridges.*

Proof. Consider a stationary Gaussian process with unit variance on the time interval $]0,T[$ and an affine covariance function R. Following [1] Théorème 2.2, (iii) (which improves and corrects a result presented by Jamison in [6]), one knows that this Gaussian process is a Markov field on the time interval $]0,T[$ if and only if, on this interval, R is of the form

$$R(h) = 1 - c\,h \quad \text{with } 0 \le c \le \frac{2}{T}\ . \tag{3.14}$$

- Thus, on the time interval $]0,a[$, the stationary Gaussian process $Z(a,\cdot)$ is a Markov field since the normed Gaussian process

  $$\tilde{Z}(t) := \frac{1}{\sqrt{1-a}}Z(a,t) = \frac{1}{\sqrt{1-a}}(B_{t+1-a} - B_t), t \in]0,a[,$$

  satisfies the condition (3.14): $c = \frac{1}{1-a} \le \frac{2}{a}$. See also the remark of Slepian in [18].
  Let us compute its bridge between x at time 0 and y at time a (the pinned values are then $x = B_{1-a}$ and $y = B_1 - B_{1-a}$). We decompose $Z(a,\cdot)$ as follows:

  $$Z(a,t) = Z^1(t) + Z^2(t) \text{ where } \begin{cases} Z^1(t) := B_{t+1-a} - B_{1-a} \\ Z^2(t) := B_{1-a} - B_t. \end{cases} \tag{3.15}$$

  Notice that the process $\big(Z^1(t), t \in [0,a]\big)$ is a Brownian motion which is independent from $\big(Z^2(t), t \in [0,a]\big)$.
  Let $\big(B^{xy}_{oa}(t), t \in [0,a]\big)$ be the $x \hookrightarrow y$ Brownian bridge which starts at x and ends at y at time a. Recall :

  $$\mathcal{L}\big(B^{xy}_{oa}(t), t \in [0,a]\big) = \mathcal{L}\big(x + \frac{t}{a}(y-x) + B^{oo}_{oa}(t), t \in [0,a]\big) \tag{3.16}$$

and

$$\mathcal{L}\big(B^{oo}_{oa}(t) + \widetilde{B}^{oo}_{oa}(t), t \in [0,a]\big) = \mathcal{L}\big(\sqrt{2}B^{oo}_{oa}(t), t \in [0,a]\big) \tag{3.17}$$

where B^{oo}_{oa} is a Brownian loop on the time interval $[0,a]$ and $\widetilde{B}^{oo}_{oa}$ is an independent copy of B^{oo}_{oa}.

By construction,

$$\begin{aligned}\mathcal{L}\big(Z^1(t), t \in [0,a] \,\big|\, Z^1(a) = z\big) &= \mathcal{L}\big(B_t, t \in [0,a] \,\big|\, B_0 = 0, B_a = z\big) \\ &= \mathcal{L}\big(\frac{t}{a}z + B^{oo}_{oa}(t), t \in [0,a]\big). \quad (3.18)\end{aligned}$$

On the other side, $\mathcal{L}\big(Z^2(t), t \in [0,a]\big) = \mathcal{L}\big(B_{1-a-t}, t \in [0,a]\big)$. Thus

$$\begin{aligned}&\mathcal{L}\Big(Z^2(t); t \in [0,a] \,\big|Z^2(0) = x, Z^2(a) = \mathfrak{z}\Big) \\ &\quad = \mathcal{L}\Big(B_{1-a-t}, t \in [0,a] \,\big|B_{1-a} = x, B_{1-2a} = \mathfrak{z}\Big) \\ &\quad = \mathcal{L}\Big(x + \frac{t}{a}(\mathfrak{z} - x) + B^{oo}_{oa}(t), t \in [0,a]\Big).\end{aligned}$$

Consequently,

$$\begin{aligned}&\mathcal{L}\Big(Z(t), t \in [0,a]\big|Z^2(0) = x, Z^1(a) = z, Z^2(a) = \mathfrak{z}\Big) \\ &= \mathcal{L}\Big(x + \frac{t}{a}(z + \mathfrak{z} - x) + \sqrt{2}B^a_{0\hookrightarrow 0}(t); t \in [0,a]\Big).\end{aligned}$$

Finally, since $Z(0) = Z^2(0)$ and $Z(a) = Z^1(a) + Z^2(a)$, one obtains:

$$\begin{aligned}&\mathcal{L}\Big(Z(t), t \in [0,a]\big|Z(0) = x, Z(a) = y\Big) \\ &\quad = \mathcal{L}\Big(x + \frac{t}{a}(y - x) + \sqrt{2}B^{oo}_{oa}(t); t \in [0,a]\Big) \\ &\quad = \mathcal{L}\Big(\sqrt{2}B^{\frac{x}{\sqrt{2}}\frac{y}{\sqrt{2}}}_{oa}(t), t \in [0,a]\Big).\end{aligned}$$

- On the time interval $]a, 1-a[$, the study of the process $Z(a,\cdot)$ can be reduced to the study of $\big(Z(a, s+a), s \in]0, 1-2a[\big)$; therefore it satisfies the Markov field property if and only if $1 - 2a < a \Leftrightarrow a > 1/3$. In that case, the condition (3.14) is always satisfied: $c = \dfrac{1}{1-a}$ is always smaller than $\dfrac{2}{1-2a}$.
  Furthermore, we are able to compute explicitly the bridge of $\big(Z(a, s+a), s \in]0, 1-2a[\big)$ thanks a decomposition as above. Indeed

  $$Z(a, a+s) = B_s + B_1 - B_{a+s} = B_s + W_{1-a-s} \tag{3.19}$$

where W is a Brownian motion, independent to B. By a similar argumentation as above, we conclude that the bridges of $Z(a, \cdot + a)$ on $]0, 1-2a[$ are Brownian bridges.

- On the time interval $]1-a, 1[$, the study of the process $Z(a, \cdot)$ can be reduced to the study of $\big(Z(a, s+1-a), s \in]0, a[\big)$, which disintegrates as

$$Z(a, 1-a+s) = B_{1-2a+s} + B_1 - B_{1-a+s} = B_{1-2a+s} + \widetilde{W}_{a-s} \quad (3.20)$$

where $\widetilde{W}$ is a Brownian motion, independent to B. By a similar argumentation as above, its bridges are Brownian bridges.

■

Recall that $\varphi \mapsto X^\varphi$ is a linear map. In case the function φ enjoys some mild regularity, one gets the following useful path representation of $X^\varphi(t)$ as $d\varphi$-mixture of the processes $Z(a, t)$.

Proposition 3.5 *Suppose that φ is a right-continuous map with bounded variation over $[0, 1]$. Then*

$$X_t^\varphi = \varphi(0)B_1 + \int_0^1 Z(r, t)\, d\varphi(r), \quad \forall\, t \in [0, 1]. \quad (3.21)$$

Proof. Suppose first that φ is of class C^1, then $d\varphi(r) = \varphi'(r)dr$ and φ' is a continuous function. We have

$$\begin{aligned}
\int_0^t (B_{t-r} - B_t)\varphi'(r)dr &= \int_0^t \varphi'(t-u)B_u du - B_t\big(\varphi(t) - \varphi(0)\big) \\
&= -\varphi(0)B_t + \int_0^t \varphi(t-u)dB_u - B_t\big(\varphi(t) - \varphi(0)\big) \\
&= -\varphi(t)B_t + \int_0^t \varphi(t-u)dB_u.
\end{aligned}$$

We proceed similarly with the second integral:

$$\begin{aligned}
\int_t^1 (B_{1-r+t} - B_t)\varphi'(r)dr &= \int_t^1 \varphi'(1+t-u)B_u du - B_t\big(\varphi(1) - \varphi(t)\big) \\
&= -\varphi(t)B_1 + \varphi(1)B_t + \int_t^1 \varphi(1+t-u)dB_u \\
&\quad -B_t\big(\varphi(1) - \varphi(t)\big) \\
&= \varphi(t)B_t - \varphi(t)B_1 + \int_t^1 \varphi(1+t-u)dB_u.
\end{aligned}$$

By (3.1), we deduce

$$\int_0^t (B_{t-r} - B_t)\varphi'(r)dr + \int_t^1 (B_{1-r+t} - B_t)\varphi'(r)dr = -\varphi(t)B_1 + X_t^\varphi.$$

Since $\varphi(t) = \varphi(0) + \int_0^t \varphi'(r)dr$, we get(3.21).
When the function φ is no more of class C^1, the representation remains valid as long as φ is of bounded variation over $[0,1]$, via limiting procedure and the continuity of $\varphi \mapsto X_t^\varphi$. ■

With other words the map $\varphi \mapsto X^\varphi$ admits the following decomposition:

$$X^\varphi = X^{\varphi(0)} + X^{\varphi - \varphi(0)} =< B_1\delta_0 - (Z(\cdot,t))', \varphi >$$

where the derivative should be understood in the sense of distributions and $< \mu, \varphi >$ means that the distribution μ acts on the function φ.

3.2 Comparison between the processes X^φ and $X^{\varphi'}$

It is of interest to relate both processes X^φ and $X^{\varphi'}$ when φ is differentiable in $L^2\big(0,1\big)$.

Proposition 3.6 *Suppose that the function φ belongs to the Cameron-Martin space. Then*

$$X_t^\varphi = X_0^\varphi + \big(\varphi(0) - \varphi(1)\big)B_t + \int_0^t X_s^{\varphi'}\,ds, \quad 0 \le t \le 1. \tag{3.22}$$

Proof of Proposition 3.6.

- Suppose first that φ is of class C^2.

  Then we use as tool the following stochastic Fubini theorem (cf Exercise 5.17, Chapter IV in [16]). Let ψ be in $L^2\big([0,1]^2\big)$, then

$$\int_0^1 \Big(\int_0^1 \psi(u,s)dB_s\Big)du = \int_0^1 \Big(\int_0^1 \psi(u,s)du\Big)dB_s. \tag{3.23}$$

  Using (3.1), we have:

$$\int_0^t X_u^{\varphi'}du = A_1(t) + A_2(t)$$

  where

$$A_1(t) := \int_0^t \Big(\int_0^u \varphi'(u-s)dB_s\Big)du,\ A_2(t) := \int_0^t \Big(\int_u^1 \varphi'(1+u-s)dB_s\Big)du.$$

  Using (3.23), we get:

$$\begin{aligned} A_1(t) &= \int_0^t \Big(\int_s^t \varphi'(u-s)du\Big)dB_s \\ &= \int_0^t \big(\varphi(t-s) - \varphi(0)\big)dB_s \\ &= -\varphi(0)B_t + \int_0^t \varphi(t-s)dB_s. \end{aligned}$$

We proceed similarly with $A_2(t)$:

$$\begin{aligned}
A_2(t) &= \int_0^1 \Big(\int_0^{s\wedge t} \varphi'(1+u-s)du\Big)dB_s \\
&= \int_0^t \Big(\int_0^s \varphi'(1+u-s)du\Big)dB_s + \int_t^1 \Big(\int_0^t \varphi'(1+u-s)du\Big)dB_s \\
&= \int_0^t \big(\varphi(1)-\varphi(1-s)\big)dB_s + \int_t^1 \big(\varphi(1+t-s)-\varphi(1-s)\big)dB_s \\
&= \varphi(1)B_t - \int_0^1 \varphi(1-s)dB_s + \int_t^1 \varphi(1+t-s)dB_s.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_0^t X_u^{\varphi'} du &= \big(\varphi(1)-\varphi(0)\big)B_t - \int_0^1 \varphi(1-s)dB_s + \int_0^t \varphi(t-s)dB_s \\
&\quad + \int_t^1 \varphi(1+t-s)dB_s.
\end{aligned}$$

The result follows from (3.1) and (3.4).

- Suppose now that φ belongs to the Cameron-Martin space, that is φ is differentiable and the two functions φ and φ' are elements of $L^2\big(0,1\big)$. Let $(\psi_n)_{n\geq 1}$ be a sequence of functions of class C^1 defined on $[0,1]$ and converging to φ' in $L^2\big(0,1\big)$. Define $\varphi_n(x) := \varphi(0) + \int_0^x \psi_n(u)du, \quad \forall\ x \in [0,1]$. Since φ' is integrable, $\varphi(x) := \varphi(0) + \int_0^x \varphi'(u)du$. Consequently,

$$\sup_{0\leq x\leq 1} \big|\varphi_n(x)-\varphi(x)\big| \leq \int_0^1 |\psi_n(u)-\varphi'(u)|du.$$

Since φ_n is of class C^2, then

$$X_t^{\varphi_n} = X_0^{\varphi_n} + \big(\varphi_n(0)-\varphi_n(1)\big)B_t + \int_0^t X_s^{\varphi_n'} ds, \quad 0\leq t\leq 1.$$

By assertion 4. of Proposition 3.1, each term converges in $L^2(\Omega)$ as n grows, which implies (3.22). ■

Examples 3.7 *1. Take* $\varphi(u) = \dfrac{1}{1-e^{-\lambda}} e^{-\lambda u}$ *as in Section 2. Then* $\varphi' = -\lambda\varphi$. *According to Proposition 3.6, since* $\varphi(0)-\varphi(1) = 1$, *the process* $Y = X^\varphi$ *satisfies*

$$Y_t = Y_0 + B_t - \lambda \int_0^t Y_u\, du$$

and we recover the equation (2.1). *Reciprocally, suppose that the process* X^φ *satisfies* $X^{\varphi'} = -\lambda X^\varphi$ *for some regular function* φ. *This*

identity is equivalent to $X^{\varphi'+\lambda\varphi} = 0$. *Then, the isometry property proved in Proposition 3.1 implies that the function* φ *itself solves the differential equation* $\varphi' + \lambda\varphi = 0$, *which means that it is proportional to* $u \mapsto e^{-\lambda u}$.
This case is the unique one where the integral equation (3.22) *on* X^φ *is indeed autonomous, due to the proportionality between* φ *and* φ'.

2. *With the notation introduced in Examples 3.3, the convoluted process* $X^{\sharp k}$ *associated with the monomial of degree* k *satisfies the non-autonomous integral equation*

$$X_t^{\sharp k} = X_0^{\sharp k} - B_t + k\int_0^t X_s^{\sharp (k-1)}\, ds. \tag{3.24}$$

To obtain an autonomous equation, one has to consider the $\mathbb{R}^{k+1}$-*valued process* $\mathbf{X}^{\sharp k}$ *whose coordinates are* $X^{\sharp k}, \cdots, X^{\sharp 1}, X^{\sharp 0}$, *which then satisfies the linear integral system*

$$\mathbf{X}_t^{\sharp k} = \mathbf{X}_0^{\sharp k} - B_t \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix} + \int_0^t A\, \mathbf{X}_s^{\sharp k}\, ds, \tag{3.25}$$

where the $(k+1)\times(k+1)$ *matrix* A *is given by* $\begin{bmatrix} 0 & k & 0 & \dots & 0 \\ 0 & 0 & k-1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \dots & 0 & 1 \\ 0 & & \dots & & 0 \end{bmatrix}$.
This more general vector-valued framework will be studied in Section 4.

3. *The random variable* $(1-e^{-\lambda})\, Y_t$ *defined in the first example can be obtained as limit in* $L^2(\Omega)$ *of the sequence* $\displaystyle\sum_{k=0}^n \frac{(-\lambda)^k}{k!} X_t^{\sharp k}$ *when* n *tends to infinity. It is a consequence of Proposition 3.1 and the fact that* $\displaystyle\sum_{k=0}^n \frac{(-\lambda)^k}{k!} X^{\sharp k} = X^{\psi_n}$ *where* $\displaystyle\psi_n(x) := \sum_{k=0}^n \frac{(-\lambda x)^k}{k!}$.

3.3 The trigonometric convoluted Brownian motion

Take now for function φ a trigonometric one, either $x \mapsto \cos(\lambda x)$ or $x \mapsto \sin(\lambda x)$, where λ is a real number.

$$\begin{cases} X_t^{\cos} &= \displaystyle\int_0^t \cos\big(\lambda(t-s)\big)\, dB_s + \int_t^1 \cos\big(\lambda(1+t-s)\big)\, dB_s, \\ X_t^{\sin} &= \displaystyle\int_0^t \sin\big(\lambda(t-s)\big)\, dB_s + \int_t^1 \sin\big(\lambda(1+t-s)\big)\, dB_s. \end{cases} \tag{3.26}$$

In a more elegant way, one considers the complex-valued process $\mathbf{X}_t^\lambda := X_t^{\cos} + i\, X_t^{\sin}$ which satisfies

$$\begin{aligned}
\mathbf{X}_t^\lambda &:= \int_0^t \exp\big(i\lambda(t-s)\big)\, dB_s + \int_t^1 \exp\big(i\lambda(1+t-s)\big)\, dB_s \\
&= \int_0^1 \exp\big(i\lambda(t-s)\big)\, dB_s + \big(\exp(i\lambda) - 1\big) \int_t^1 \exp\big(i\lambda(t-s)\big)\, dB_s.
\end{aligned}$$

Let us first analyze one special case.

1- The periodic case, $\lambda \in 2\pi\mathbb{Z}$.

For $\lambda = 2k\pi, k \in \mathbb{Z}^*$, $\mathbf{X}^\lambda$ admits a simple representation.

$$\begin{aligned}
\mathbf{X}_t^{2k\pi} &= \int_0^1 \exp\big(i2k\pi(t-s)\big)\, dB_s \\
&= \exp(i2k\pi t) \int_0^1 \exp(-i2k\pi s)\, dB_s = \exp(i2k\pi t)\, \mathbf{X}_0^{2k\pi}.
\end{aligned}$$

This process is degenerated - as the product of a determinist time function by a fixed random variable - and it is $1/k$-periodic: $\mathbf{X}_{t+\frac{1}{k}}^{2k\pi} = \mathbf{X}_t^{2k\pi}$. Therefore the stationary centered Gaussian process $X^{\cos}$, real part of $\mathbf{X}^{2k\pi}$ (resp. $X^{\sin}$ the imaginary part of $\mathbf{X}^{2k\pi}$), disintegrates as a mixture of two Gaussian random variables:

$$X_t^{\cos} = \cos(2k\pi t) \int_0^1 \cos(2k\pi s)\, dB_s + \sin(2k\pi t) \int_0^1 \sin(2k\pi s)\, dB_s.$$

Moreover the above two stochastic integrals are independent. Thus following [5], p.524 and [6], Theorem p.1627, $X^{\cos}$ (resp. $X^{\sin}$) is a Markov field on the time interval $[0, \frac{1}{2k}[$. Nevertheless it is not a Markov field on the full time interval $[0, 1]$.

2- The general case, $\lambda \notin 2\pi\mathbb{Z}$.

When the function φ is trigonometric, there is no proportionality between φ and φ' but there is proportionality between φ and φ''. Indeed, following (3.22), the pair of processes $(X^{\cos}, X^{\sin})$ satisfies the autonomous system of equations:

$$\begin{cases}
X_t^{\cos} &= \int_0^1 \cos\big(\lambda(1-s)\big) dB_s + \big(1 - \cos\lambda\big) B_t - \lambda \int_0^t X_s^{\sin}\, ds, \\
X_t^{\sin} &= \int_0^1 \sin\big(\lambda(1-s)\big) dB_s - \sin\lambda\, B_t + \lambda \int_0^t X_s^{\cos}\, ds,
\end{cases} \tag{3.27}$$

or, equivalently, the complex-valued process $\mathbf{X}_t^\lambda$ satisfies the equation:

$$\mathbf{X}_t^\lambda = \int_0^1 e^{i\lambda(1-s)} dB_s + \big(1 - e^{i\lambda}\big) B_t + \lambda i \int_0^t \mathbf{X}_s^\lambda\, ds. \tag{3.28}$$

Notice that $\dfrac{1}{1-e^{i\lambda}}\mathbf{X}_t^\lambda$ satisfies a similar equation to (2.1), where the parameter λ is replaced by $-i\lambda$. We have proved the following.

Proposition 3.8 *The processes $X^{\cos}$ and $X^{\sin}$ are centered, stationary, periodic Gaussian processes. Their covariance functions satisfy*

$$\begin{cases} R^{\cos}(h) & = \frac{\sin(\lambda/2)}{2\lambda}\Big(\cos\big(\lambda(\frac{3}{2}-h)\big)+\cos\big(\lambda(\frac{1}{2}+h)\big)\Big)+h\sin\big(\frac{\lambda}{2}\big)\sin\big(\lambda(h-\frac{1}{2})\big) \\ & \qquad +\frac{\cos(\lambda h)}{2} \\ R^{\sin}(h) & = -\frac{\sin(\lambda/2)}{2\lambda}\Big(\cos\big(\lambda(\frac{3}{2}-h)\big)+\cos\big(\lambda(\frac{1}{2}+h)\big)\Big)+h\sin\big(\frac{\lambda}{2}\big)\sin\big(\lambda(h-\frac{1}{2})\big) \\ & \qquad +\frac{\cos(\lambda h)}{2}. \end{cases} \tag{3.29}$$

Proof. By (3.2)

$$\begin{aligned} R^{\cos}(h) &= \int_0^h \cos\big(\lambda(1-u)\big)\cos\big(\lambda(h-u)\big)du+\int_0^{1-h}\cos(\lambda u)\cos\big(\lambda(h+u)\big)du \\ &=: I_1(h)+I_2(h). \end{aligned}$$

Using the identity $\cos a\,\cos b=\dfrac{\cos(a+b)+\cos(a-b)}{2}$ one gets

$$\begin{aligned} I_1(h) &= \frac{1}{2}\int_0^h \cos\big(\lambda(1+h-2u)\big)\,du+\frac{1}{2}\int_0^h\cos\big(\lambda(1-h)\big)\,du \\ &= \frac{1}{4\lambda}\Big(\sin\big(\lambda(1+h)\big)-\sin\big(\lambda(1-h)\big)\Big)+\frac{h}{2}\cos\big(\lambda(1-h)\big). \end{aligned}$$

In a similar way

$$\begin{aligned} I_2(h) &= \frac{1}{2}\int_0^{1-h}\cos\big(\lambda(2u+h)\big)\,du+\frac{1}{2}\int_0^{1-h}\cos(\lambda h)\,du \\ &= \frac{1}{4\lambda}\Big(\sin\big(\lambda(2-h)\big)-\sin(\lambda h)\Big)+\frac{1-h}{2}\cos(\lambda h). \end{aligned}$$

Therefore

$$\begin{aligned} R^{\cos}(h) &= \frac{1}{4\lambda}\Big(\sin\big(\lambda(2-h)\big)-\sin\big(\lambda(1-h)\big)+\sin\big(\lambda(1+h)\big)-\sin(\lambda h)\Big) \\ &\quad +\frac{h}{2}\Big(\cos\big(\lambda(1-h)\big)-\cos(\lambda h)\Big)+\frac{\cos(\lambda h)}{2}. \end{aligned}$$

Using the identities

$$\sin a-\sin b=2\sin\frac{a-b}{2}\cos\frac{a+b}{2},\qquad \cos a-\cos b=-2\sin\frac{a-b}{2}\sin\frac{a+b}{2},$$

one obtains the first equality of (3.29). Let us prove the second equality of (3.29).

$$\begin{aligned} R^{\sin}(h) &= \int_0^h \sin\big(\lambda(1-u)\big)\sin\big(\lambda(h-u)\big)du+\int_0^{1-h}\sin(\lambda u)\sin\big(\lambda(h+u)\big)du \\ &=: J_1(h)+J_2(h) \end{aligned}$$

Using the identity $\sin a \, \sin b = \dfrac{\cos(a-b) - \cos(a+b)}{2}$ one gets

$$J_1(h) = \frac{h}{2} \cos\big(\lambda(1-h)\big) + \frac{1}{4\lambda}\Big(\sin\big(\lambda(1-h)\big) - \sin\big(\lambda(1+h)\big)\Big)$$

and

$$J_2(h) = \frac{1-h}{2} \cos(\lambda h) + \frac{1}{4\lambda}\Big(\sin(\lambda h) - \sin\big(\lambda(2-h)\big)\big)\Big).$$

Therefore

$$\begin{aligned} R^{\sin}(h) &= \frac{1}{4\lambda}\Big(\sin\big(\lambda(1-h)\big) - \sin\big(\lambda(2-h)\big) + \sin(\lambda h) - \sin\big(\lambda(1+h)\big)\Big) \\ &\quad + \frac{h}{2}\Big(\cos\big(\lambda(1-h)\big) - \cos(\lambda h)\Big) + \frac{\cos(\lambda h)}{2} \\ &= -\frac{\sin(\lambda/2)}{2\lambda}\Big(\cos\big(\lambda(\frac{3}{2}-h)\big) + \cos\big(\lambda(\frac{1}{2}+h)\big)\Big) \\ &\quad + h \, \sin\big(\frac{\lambda}{2}\big) \sin\big(\lambda(h-\frac{1}{2})\big) + \frac{\cos(\lambda h)}{2}. \end{aligned}$$

■

As in Section 2, the process $\mathbf{X}^\lambda$ admits a semimartingale decomposition in a filtration enlarged by two initial conditions. This is a particular case of a more general question we address in the next section.

4 Vector-valued convoluted processes

4.1 General properties

We extend here the definition (1.1) of convoluted Brownian motion to the multidimensional case. Let A be a $n \times n$ matrix and ϕ a vector in $\mathbb{R}^n$. We introduce the $\mathbb{R}^n$-valued process

$$\mathbf{X}_t^{A,\phi} := \int_0^t e^{(t-s)A} \phi \, dB_s + \int_t^1 e^{(1+t-s)A} \phi \, dB_s, \quad t \in [0,1], \tag{4.1}$$

where $\big(B_t, t \in [0,1]\big)$ is as before a standard real-valued Brownian motion. Remark that if $n = 1$, $A = -\lambda$ and $\phi = \dfrac{1}{1-e^{-\lambda}}$, then $\mathbf{X}^{A,\phi}$ is the PerOU process Y defined by (2.4).
As in (2.2), the process $\mathbf{X}^{A,\phi}$ admits indeed another representation and solves a stochastic linear integral system.

Proposition 4.1 *1. The process $\mathbf{X}_t^{A,\phi}, t \in [0,1]$, admits the following representation:*

$$\mathbf{X}_t^{A,\phi} = e^A \int_0^1 e^{(t-s)A} \phi \, dB_s + (Id - e^A) \int_0^t e^{(t-s)A} \phi \, dB_s. \tag{4.2}$$

In particular it is 1-periodic and

$$\mathbf{X}_0^{A,\phi} = \mathbf{X}_1^{A,\phi} = \int_0^1 e^{(1-s)A}\phi\, dB_s. \tag{4.3}$$

2. *Reciprocally, the unique solution of the integral system*

$$Z_t = \int_0^1 e^{(1-s)A}\phi\, dB_s + (Id - e^A)\phi B_t + \int_0^t AZ_s\, ds\ , \quad t \in [0,1]. \tag{4.4}$$

is the process $Z \equiv \mathbf{X}^{A,\phi}$.

Proof. Identities (4.2) and (4.3) are consequences of (4.1). The proof of the second assertion is omitted since it is a direct generalization of Section 2.4. ■

Remark 4.2 1. *Equation* (4.4) *is not a classical stochastic integral system since the r.v.* $\int_0^1 e^{(1+t-s)A}\phi\, dB_s$ *is not* $\mathcal{F}_0$*-measurable.*

2. *We recover the two-dimensional process* $(X^{\cos}, X^{\sin})^*$ *defined by* (3.26) *setting* $\phi = (1,0)^*$ *and* $A = \lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. *Indeed, since*

$$e^{tA} = \begin{pmatrix} \cos(\lambda t) & -\sin(\lambda t) \\ \sin(\lambda t) & \cos(\lambda t) \end{pmatrix} \tag{4.5}$$

then equation (4.1) *and equation* (3.26) *are identical.*

Clearly the vector-valued process $(\mathbf{X}_t^{A,\phi})_{t\in[0,1]}$ is centered and Gaussian. It is therefore characterized by its covariance matrix done in the proposition below, which will permit us to develop several examples in the next Section 4.2.
Recall first that if $\mathbf{X}_1$ and $\mathbf{X}_2$ are two $\mathbb{R}^n$-valued centered Gaussian vectors, their covariance is the $n \times n$-matrix defined by: $\mathbf{Cov}(\mathbf{X}_1, \mathbf{X}_2) := \mathbb{E}\big[\mathbf{X}_1\mathbf{X}_2^*\big]$. In particular, if ξ_1, ξ_2 are any vectors in $\mathbb{R}^n$, then

$$\xi_1^*\, \mathbf{Cov}(\mathbf{X}_1, \mathbf{X}_2)\, \xi_2 = \mathrm{Cov}\big(\xi_1^*\mathbf{X}_1, \xi_2^*\mathbf{X}_2\big). \tag{4.6}$$

To simplify the notations, we define the function ρ from $[0,1]$ into the set of $n \times n$-matrices by

$$\rho(t) := \int_0^t e^{uA}\phi\phi^* e^{uA^*} du, \quad t \in [0,1]. \tag{4.7}$$

Proposition 4.3 *The process* $(\mathbf{X}_t^{A,\phi})_{t\in[0,1]}$ *is Gaussian and stationary. Moreover, for any* $0 \le s \le s+h \le 1$, *we have:*

$$R^{A,\phi}(h) := \mathbf{Cov}\big(\mathbf{X}_s^{A,\phi}, \mathbf{X}_{s+h}^{A,\phi}\big) = e^{hA}\rho(1-h) + \rho(h)e^{(1-h)A^*}. \tag{4.8}$$

Proof: Let $\xi_i \in \mathbb{R}^n, i \in \{1,2\}$. We deduce from (4.1) that

$$\xi_i^* \mathbf{X}_t^{A,\phi} = \int_0^t \xi_i e^{(t-u)A}\phi \, dB_u + \int_t^1 \xi_i e^{(1+t-u)A}\phi \, dB_u, \quad 0 \le t \le 1.$$

Therefore

$$\xi_i^* \mathbf{X}^{A,\phi} = X^{\varphi_i}, \text{ where } \quad \varphi_i(t) := \xi_i^* e^{tA}\phi, \quad t \in [0,1]. \tag{4.9}$$

Relation (4.6) implies, for $0 \le s \le s+h \le 1$,

$$\xi_1^* \, \mathbf{Cov}\big(\mathbf{X}_s^{A,\phi}, \mathbf{X}_{s+h}^{A,\phi}\big) \, \xi_2 = \mathrm{Cov}\big(X_s^{\varphi_1}, X_{s+h}^{\varphi_2}\big).$$

We now apply (3.3) with $\varphi = \varphi_1$ and $\psi = \varphi_2$:

$$\begin{aligned}
\mathrm{Cov}\big(X_s^{\varphi_1}, X_{s+h}^{\varphi_2}\big) \quad &= \quad \int_0^s \varphi_2(s-u)\varphi_1(s+h-u)du \\
&\quad + \int_s^{s+h} \varphi_2(1+s-u)\varphi_1(s+h-u)du \\
&\quad + \int_{s+h}^1 \varphi_2(1+s-u)\varphi_1(1+s+h-u)du.
\end{aligned}$$

Proceeding as in the proof of Proposition 3.1, we get:

$$\mathrm{Cov}\big(X_s^{\varphi_1}, X_{s+h}^{\varphi_2}\big) = \int_0^{1-h} \varphi_2(r)\varphi_1(h+r)dr + \int_0^h \varphi_2(1-r)\varphi_1(h-r)dr.$$

Using (4.9) and (4.6) leads to:

$$\mathbf{Cov}\big(\mathbf{X}_s^{A,\phi}, \mathbf{X}_{s+h}^{A,\phi}\big) = \int_0^{1-h} e^{(h+r)A}\phi\phi^* e^{rA^*} dr + \int_0^h e^{(h-r)A}\phi\phi^* e^{(1-r)A^*} dr.$$

The change of variable $u := h - r$ in the second integral gives:

$$\begin{aligned}
\mathbf{Cov}\big(\mathbf{X}_s^{A,\phi}, \mathbf{X}_{s+h}^{A,\phi}\big) \quad &= e^{hA}\Big(\int_0^{1-h} e^{rA}\phi\phi^* e^{rA^*} dr\Big) \\
&\qquad + \Big(\int_0^h e^{rA}\phi\phi^* e^{rA^*} dr\Big) e^{(1-h)A^*}.
\end{aligned}$$

■

Remark 4.4 *1. In the case $n = 1$, $A = -\lambda$ and $\phi = \dfrac{1}{1-e^{-\lambda}}$, it is easy to check that Identity (4.8) corresponds to (2.5).*

2. In the particular case $h = 0$, then (4.8) leads to the covariance matrix $K^{A,\phi}$ (which does not depend on t) of the vector $\mathbf{X}_t^{A,\phi}$:

$$K^{A,\phi} = \int_0^1 e^{uA}\phi\phi^* e^{uA^*} du.$$

3. *The covariance function* $R^{A,\phi}$ *has the following structure*

$$R^{A,\phi}(h) = \sigma(h) + \sigma(1-h)^*, \tag{4.10}$$

where the matrix-valued map σ *is defined by*

$$\sigma(h) := e^{hA}\rho(1-h), \quad h \in [0,1]. \tag{4.11}$$

4.2 Some illustrating examples

4.2.1 The trigonometric convoluted Brownian motion

We begin with the two-dimensional convoluted process $(X^{\cos}, X^{\sin})^*$ defined in (3.26) through the trigonometric functions sin and cos. We already observed in Remark 4.2 that

$$(X^{\cos}, X^{\sin})^* = \mathbf{X}^{A,e_1} \text{ with } e_1 := (1,0)^* \text{ and } A := \lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We also computed in Proposition 3.8 the covariance terms $\mathrm{Cov}\big(X_s^{\cos}, X_t^{\cos}\big)$ and $\mathrm{Cov}\big(X_s^{\sin}, X_t^{\sin}\big)$. Anyway the formula (4.8) permits to go further computing the mixed covariance terms of the form $\mathrm{Cov}\big(X_s^{\cos}, X_t^{\sin}\big)$. Indeed, using (4.5), one obtains for the explicit computation of the matrix-valued map ρ defined in (4.7):

$$\rho(t) = \begin{pmatrix} \dfrac{t}{2} + \dfrac{\sin(2\lambda t)}{4\lambda} & \dfrac{1-\cos(2\lambda t)}{4\lambda} \\[2ex] \dfrac{1-\cos(2\lambda t)}{4\lambda} & \dfrac{t}{2} - \dfrac{\sin(2\lambda t)}{4\lambda} \end{pmatrix}.$$

Then, the matrix $\sigma(h)$ defined by (4.11) has the form

$$\sigma(h) = \begin{pmatrix} \sigma_{11}(h) & \sigma_{12}(h) \\ \sigma_{21}(h) & \sigma_{22}(h) \end{pmatrix}$$

where

$$\sigma_{12}(h) := -\frac{1-h}{2}\sin(\lambda h) - \frac{1}{4\lambda}\cos\big(\lambda(2-h)\big),$$

$$\sigma_{21}(h) := \frac{1-h}{2}\sin(\lambda h) - \frac{1}{4\lambda}\cos\big(\lambda(2-h)\big) + \frac{1}{4\lambda}\cos(\lambda h).$$

We thus deduce:

$$\mathrm{Cov}\big(X_s^{\cos}, X_{s+h}^{\sin}\big) = \begin{cases} \sigma_{12}(h) + \sigma_{21}(1-h) & \text{for } h \geq 0 \\ \sigma_{12}(1-h) + \sigma_{21}(h) & \text{for } h \leq 0. \end{cases} \tag{4.12}$$

4.2.2 The monomial convoluted Brownian motion

We now analyze in more detail the $(k+1)$-dimensional convoluted process $\mathbf{X}^{\sharp k} := (X^{\sharp k}, \cdots, X^{\sharp 1}, X^{\sharp 0})^*$ defined in (3.10) as convolution with monomials of degree lower than k (or, equivalently, defined by the linear system (3.25)). We set

$$e_{k+1} := (0, \cdots, 0, 1)^* \text{ and } A := \begin{bmatrix} 0 & k & 0 & \dots & 0 \\ 0 & 0 & k-1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \dots & 0 & 1 \\ 0 & & \dots & & 0 \end{bmatrix}. \tag{4.13}$$

Proposition 4.5 *The process $\mathbf{X}^{\sharp k}$ coincides with the vector-valued convoluted process $X^{A,e_{k+1}}$ where A and e_{k+1} are defined by (4.13).*

1. *Let $\rho(t)$ be the associated $(k+1)\times(k+1)$-matrix defined by (4.7). The entries of $\rho(t)$ are monomials in t and satisfy:*

$$\rho_{i,j}(t) = \frac{1}{2k+3-(i+j)}\, t^{2k+3-(i+j)}, \quad 1 \le i, j \le k+1,\ 0 \le t \le 1. \tag{4.14}$$

2. *The covariance matrix of $\mathbf{X}^{\sharp k}$, denoted by $R^{\sharp k}(h)$ and defined in (4.8), has as (i,j)-entry the following polynomial in h of degree $2k+3-(i+j)$:*

$$R^{\sharp k}_{ij}(h) = \int_0^h s^{k+1-i}(1-h+s)^{k+1-j} ds + \int_h^1 (h+1-s)^{k+1-i}(1-s)^{k+1-j} ds \tag{4.15}$$

where $1 \le i, j \le k+1$ and $0 \le h \le 1$.

Proof. 1) We first prove that $\mathbf{X}^{\sharp k}$ satisfies (4.4) with A and e_{k+1} as above. Then part 2 of Proposition 4.1 will imply $\mathbf{X}^{\sharp k} \equiv \mathbf{X}^{A,e_{k+1}}$.
According to (3.25), it remains to prove that

$$\mathbf{X}^{\sharp k}_1 = e^A \int_0^1 e^{(1-s)A} e_{k+1}\, dB_s \tag{4.16}$$

$$\text{and} \quad (e^A - Id)e_{k+1} = (1, \cdots, 1, 0)^*. \tag{4.17}$$

a) Let us first compute e^{tA} for any real number t. Noting that A is nilpotent with index of nilpotency $k+1$, then e^{tA} is a polynomial in t of degree k. For any $1 \le i, j \le k+1$, it is convenient to introduce the matrix $E^{i,j}$ whose entries vanish excepted the (i,j)-one which is equal to 1 ($E^{i,j}_{kl} = \mathbb{1}_{(i,j)}(k,l)$). We claim that:

$$e^{tA} = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{k-i}{j} t^j E^{i+1,i+j+1}. \tag{4.18}$$

$$\text{Indeed,}\ \ A = \sum_{l=0}^{k-1}(k-l)E^{l+1,l+2}. \tag{4.19}$$

Let $\Gamma(t)$ be equal to the right hand-side of (4.18).
It is clear that $\Gamma(0) = \sum_{i=1}^{k+1} E^{i,i} = Id$. Relations (4.19) and (4.18) imply:

$$A\Gamma(t) = \sum_{i=0}^{k}\sum_{j=0}^{k-i}\sum_{l=0}^{k-1}\binom{k-i}{j}(k-l)t^j E^{l+1,l+2}E^{i+1,i+j+1}.$$

But $E^{l+1,l+2}E^{i+1,i+j+1} = E^{l+1,i+j+1}\mathbb{1}_{\{l+2=i+1\}}$ and $l \geq 0 \Rightarrow i \geq 1$. Setting $i' := i-1$ $j' := j+1$, after some easy calculations we have:

$$A\Gamma(t) = \sum_{i'=0}^{k-1}\sum_{j'=1}^{k-i'}\binom{k-i'}{j'}j't^{j'-1}E^{i'+1,i'+j'+1}.$$

On the other side, taking the t-derivative of $\Gamma(t)$ leads to:

$$\begin{aligned}\frac{d\Gamma}{dt}(t) &= \sum_{i=0}^{k}\sum_{j=1}^{k-i}\binom{k-i}{j}jt^{j-1}E^{i+1,i+j+1}\\ &= \sum_{i=0}^{k-1}\sum_{j=1}^{k-i}\binom{k-i}{j}jt^{j-1}E^{i+1,i+j+1}.\end{aligned}$$

The second equality follows from : $k-i \geq 1 \Rightarrow i \leq k-1$. Finally, $\frac{d\Gamma}{dt}(t) = A\Gamma(t)$ and $\Gamma(0) = Id$, therefore (4.18) holds.
b) Let $e_i := (0,\cdots,0,1,0,\cdots,0)^*$ be the i-th basis vector of $\mathbb{R}^{k+1}$. Since $E^{i,l}e_{k+1} = \mathbb{1}_{\{l=k+1\}}e_i$, then

$$e^{tA}e_{k+1} = \sum_{i=1}^{k+1}t^{k+1-i}e_i. \tag{4.20}$$

In particular for $t=1$, $e^A e_{k+1} = \sum_{i=1}^{k+1} e_i = (1,\cdots,1,0)^* + e_{k+1}$. That implies (4.17).
c) We now prove that the entries of the matrix $\rho(t)$ are monomials in t and satisfy (4.14) for $1 \leq i,j \leq k+1$ and $0 \leq t \leq 1$. By (4.20),

$$e_{k+1}^* e^{uA^*} = \sum_{i=1}^{k+1}u^{k+1-i}e_i^*,$$

which implies

$$e^{uA}e_{k+1}e_{k+1}^*e^{uA^*} = \sum_{1\leq i,j\leq k+1}u^{2k+2-i-j}e_i^*e_j^*.$$

Since $e_i e_j^* = E^{i,j}$, then $e^{uA} e_{k+1} e_{k+1}^* e^{uA^*} = \sum_{1 \le i,j \le k+1} u^{2k+2-(i+j)} E^{i,j}$.
Integrating this identity in u over the interval $[0,t]$ and using (4.7) gives (4.14).
2) We now prove (4.15). Let $\sigma_{ij}(h)$ be the (i,j)-entry of $\sigma(h)$. Using (4.11), (4.18) and (4.14), we have:

$$\sigma_{ij}(h) = \sum_{l=i}^{k+1} \binom{k-i+1}{l-i} \frac{1}{2k+3-(l+j)} h^{l-i} (1-h)^{2k+3-(l+j)}.$$

For i, j, h fixed, we define the polynomial function g_h by

$$g_h(x) := \sum_{l=i}^{k+1} \binom{k-i+1}{l-i} \frac{1}{2k+3-l-j} h^{l-i} x^{2k+3-(l+j)}.$$

Then

$$\begin{aligned} g_h'(x) &= \sum_{l=i}^{k+1} \binom{k-i+1}{l-i} h^{l-i} x^{2k+2-(l+j)} \\ &= \left[\sum_{m=0}^{k+1-i} \binom{k-i+1}{m} h^m x^{k+1-i-m} \right] x^{k+1-j} \\ &= (h+x)^{k+1-i} x^{k+1-j}. \end{aligned}$$

Note that $g_h(0) = 0$, since, for any $j \le k+1$, $2k+3-(k+1+j) \ge 1$. Therefore $g_h(x) = \int_0^x (h+s)^{k+1-i} s^{k+1-j} ds$ and

$$\sigma_{ij}(h) = g_h(1-h) = \int_0^{1-h} (h+s)^{k+1-i} s^{k+1-j} ds.$$

Finally (4.15) follows from (4.10). ■

4.3 The semimartingale representation of $\mathbf{X}^{A,\phi}$

According to (4.3), the initial value of the process $\mathbf{X}^{A,\phi}$ is given by

$$\mathbf{X}_0^{A,\phi} = \int_0^1 h(s) dB(s) \quad \text{where} \quad h(s) := e^{(1-s)A}\phi. \tag{4.21}$$

Since this $\mathbb{R}^n$-valued random vector is not $\mathcal{F}_0$-measurable, generalizing the approach developed in Section 2.2, we propose to enlarge the filtration $(\mathcal{F}_t)_t$ with $\mathbf{X}_0^{A,\phi}$ to obtain a semimartingale representation of $X^{A,\phi}$. Therefore we need an n-dimensional version of Lemma 2.3. This is done in [2], Théorème II.1, which we now recall.

Lemma 4.6 *Consider a $\mathbb{R}^n$-valued random vector $\xi = (\xi_1, \cdots, \xi_n)^*$ whose coordinates belong to the first chaos of $(B_t)_{0\le t\le 1}$, each of them represented as $\xi_i := \int_0^1 h_i(s)\, dB_s$, where $h_i \in L^2(0,1)$. Denote by h the $\mathbb{R}^n$-valued map $t \mapsto h(t) = \big(h_1(t), \cdots, h_n(t)\big)^*$. Let $(\mathcal{G}_t)_t$ be the initial enlargement of the filtration $(\mathcal{F}_t)_t$ by ξ. Suppose that, for any $t \in [0,1[$, the matrix $H(t) := \int_t^1 h(u)h(u)^* du$ is invertible. Then, defining the matrix-valued map G by*

$$G(t,u) := h(t)^* H(t)^{-1} h(u) \mathbb{1}_{\{0\le t\le u\le 1\}}, \tag{4.22}$$

one gets that

$$W_t := -B_t + \int_0^t \int_s^1 G(s,u) dB_u\, ds, \quad 0 \le t \le 1, \tag{4.23}$$

is a $(\mathcal{G}_t)_t$-Brownian motion which is independent of ξ.

Proposition 4.7 *Suppose that the matrix A and the vector ϕ satisfy the following assumptions:*
$(\mathfrak{H}_1)$ *The matrix $e^A - Id$ is invertible*
$(\mathfrak{H}_2)$ $\mathrm{Span}(A^k\phi, k \in \mathbb{N}) = \mathbb{R}^n$.
Let $(\mathcal{G}_t)$ be the filtration obtained from the initial enlargement of the Brownian filtration by the random vector $\mathbf{X}_0^{A,\phi}$.

1. *Define a real-valued bounded variation process V by*

$$V_t := \int_0^t \phi^* e^{(1-s)A^*} H(s)^{-1} (e^A - Id)^{-1} \Big[e^{(1-s)A} \mathbf{X}_s^{A,\phi} - \mathbf{X}_0^{A,\phi} \Big] ds.$$

   Then the real-valued process

$$\tilde{B}_t := -B_t + V_t \tag{4.24}$$

   is a $(\mathcal{G}_t)_t$-Brownian motion independent from $\mathbf{X}_0^{A,\phi}$.

2. *The vector-valued process $\mathbf{X}_t^{A,\phi}$ admits the following semimartingale decomposition:*

$$\mathbf{X}_t^{A,\phi} = \mathbf{X}_0^{A,\phi} - (Id - e^A)\phi\, \tilde{B}_t + \int_0^t A\mathbf{X}_s^{A,\phi}\, ds + (Id - e^A)\phi\, V_t\,, \quad t \in [0,1]. \tag{4.25}$$

We begin with a preliminary result.

Lemma 4.8 *Under assumption* $(\mathfrak{H}_2)$*, the matrix*

$$H(t) := \int_t^1 e^{(1-s)A}\phi\phi^* e^{(1-s)A^*} ds, \quad t \in [0, 1[\tag{4.26}$$

is invertible for any $t \in [0, 1[$.

Proof of Lemma 4.8: We prove in fact that $H(t)$ is invertible if and only if $\mathrm{Span}(A^k\phi, k \in \mathbb{N}) = \mathbb{R}^n$. Let $u \in \mathbb{R}^n$. Then,

$$u^* H(t) u = \int_t^1 \left(u^* e^{(1-s)A}\phi\right)^2 ds.$$

Note that $s \mapsto e^{(1-s)A}\phi$ is a continuous function, therefore $u^*H(t)u = 0$ if and only if

$$u^* e^{sA}\phi = 0, \quad \forall\, s \in [0, 1-t[. \tag{4.27}$$

Since $u^* e^{sA}\phi = \sum_{k\geq 0} u^* A^k \phi \,\frac{s^k}{k!}$, then (4.27) is equivalent to

$$u^* A^k \phi = 0, \quad \forall\, k \in \mathbb{N}. \tag{4.28}$$

It is clear that (4.28) holds true if and only if $\mathrm{Span}(A^k\phi, k \geq 0) \subset u^\perp$, which completes the proof. ■

Proof of Proposition 4.7: First, according to Lemma 4.8, the matrix $H(t)$ is invertible.

Then, since the random variable $\mathbf{X}_0^{A,\phi}$ satisfies (4.3), it has coordinates which belong to the first chaos of B, and we can apply Lemma 4.6: the process $W_t := -B_t + \int_0^t v_s \, ds$ is a $(\mathcal{G}_t)$-Brownian motion which is independent of $\mathbf{X}_0^{A,\phi}$, where $v_s := \int_s^1 G(s,u)\, dB_u,\ 0 \leq s \leq 1$. According to (4.22),

$$G(s,u) = \phi^* e^{(1-s)A^*} H(s)^{-1} e^{(1-u)A}\phi$$

and then

$$v_s = \phi^* e^{(1-s)A^*} H(s)^{-1} \int_s^1 e^{(1-u)A}\phi \, dB_u. \tag{4.29}$$

We decompose the above stochastic integral as:

$$\int_s^1 e^{(1-u)A}\phi \, dB_u = \mathbf{X}_0^{A,\phi} - \int_0^s e^{(1-u)A}\phi \, dB_u, \tag{4.30}$$

so that identity (4.2) can be re-written as:

$$\mathbf{X}_s^{A,\phi} = e^{sA}\mathbf{X}_0^{A,\phi} + (Id - e^A)e^{(s-1)A}\int_0^s e^{(1-u)A}\phi\, dB_u. \tag{4.31}$$

Under the assumption $(\mathfrak{H}_1)$,

$$\int_0^s e^{(1-u)A}\phi\, dB_u = \left(e^A - Id\right)^{-1}\Big[e^A\mathbf{X}_0^{A,\phi} - e^{(1-s)A}\mathbf{X}_s^{A,\phi}\Big],$$

which implies:

$$\int_s^1 e^{(1-u)A}\phi\, dB_u = (e^A - Id)^{-1}\Big(e^{(1-s)A}\mathbf{X}_s^{A,\phi} - \mathbf{X}_0^{A,\phi}\Big)$$

and completes, with (4.29), the proof of item 1.
Back to (4.4), replacing the Brownian motion B by $\tilde{B} + V$ leads to (4.25). ■

4.4 The bridges of the process $\mathbf{X}^{A,\phi}$

As in the one-dimensional case, we are interested in the disintegration of $\mathbf{X}^{A,\phi}$ along its initial (and final) time marginal, which leads to its time-Markov field property.
To that aim, we prove that conditionally on $\mathbf{X}_0^{A,\phi} = \mathbf{x}$, the process $\big(\mathbf{X}_t^{A,\phi}, t \in [0,1]\big)$ is Markov. More precisely, let us define, for any $t \in [0,1]$, the $n \times n$ matrices

$$\begin{aligned}
\Lambda_t^0 &:= (e^A - Id)\phi\phi^* e^{(1-t)A^*} H(t)^{-1}(e^A - Id)^{-1} &(4.32)\\
\Lambda_t^1 &:= A - \Lambda_t^0 e^{(1-t)A}. &(4.33)
\end{aligned}$$

In the next theorem we identify the $\mathbf{x} \hookrightarrow \mathbf{x}$ bridge of $\mathbf{X}^{A,\phi}$ as a Markov process solution of an explicit linear stochastic differential system.

Theorem 4.9 *Suppose Assumptions $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$ are satisfied and denote by ν the Gaussian law of the random vector $\mathbf{X}_0^{A,\phi}$. Then, $\mathbf{X}^{A,\phi}$ is a ν-mixture of its bridges, where the $\mathbf{x} \hookrightarrow \mathbf{x}$ bridge solves the affine SDE in $\mathbb{R}^n$*

$$\begin{cases} dZ_t &= (e^A - Id)\phi\, d\tilde{B}_t + (\Lambda_t^0\, \mathbf{x} + \Lambda_t^1 Z_t)\, dt, \quad t \in [0,1[, \\ Z_0 &= \mathbf{x}. \end{cases} \tag{4.34}$$

Proof. It is a consequence of Definitions (4.32) and (4.33) and identity (4.25). ■

We thus obtained, under some additional assumptions, a multidimensional generalization of Proposition 2.4.

Application to the trigonometric convoluted Brownian motion

According to item 2 of Remark 4.2, the trigonometric convoluted case corresponds to $n = 2$, $\phi = (1,0)^*$ and $A = \lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
We now verify that Assumptions $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$ are satisfied. Indeed

$$\begin{aligned}(Id - e^A)^{-1} &= \frac{1}{2(1-\cos\lambda)} \begin{pmatrix} 1-\cos\lambda & -\sin\lambda \\ \sin\lambda & 1-\cos\lambda \end{pmatrix} \\ &= 1/2 \begin{pmatrix} 1 & -\cot(\lambda/2) \\ \cot(\lambda/2) & 1 \end{pmatrix}\end{aligned}$$

and $\mathrm{Span}(\phi, A\phi) = \mathrm{Span}((1,0)^*, \lambda(0,1)^*) = \mathbb{R}^2$. Consequently Theorem 4.9 applies.
Let us compute forward some matrices. First one obtains that

$$H(t) = \frac{1}{2} \begin{pmatrix} 1 - t + \frac{\sin(2\lambda(1-t))}{2\lambda} & \frac{1-\cos(2\lambda(1-t))}{2\lambda} \\ \frac{1-\cos(2\lambda(1-t))}{2\lambda} & 1 - t - \frac{\sin(2\lambda(1-t))}{2\lambda} \end{pmatrix}$$

and

$$H(t)^{-1} = \frac{2}{(1-t)^2 - \frac{1-\cos(2\lambda(1-t))}{2\lambda^2}} \begin{pmatrix} 1 - t - \frac{\sin(2\lambda(1-t))}{2\lambda} & \frac{\cos(2\lambda(1-t))-1}{2\lambda} \\ \frac{\cos(2\lambda(1-t))-1}{2\lambda} & 1 - t + \frac{\sin(2\lambda(1-t))}{2\lambda} \end{pmatrix}.$$

Thus, to obtain Λ_t^0, one has to multiply

$$(e^A - Id)\phi\phi^* e^{(1-t)A^*} = \begin{pmatrix} (\cos\lambda - 1)\cos(\lambda(1-t)) & (\cos\lambda - 1)\sin(\lambda(1-t)) \\ \sin\lambda\cos(\lambda(1-t)) & \sin\lambda\sin(\lambda(1-t)) \end{pmatrix}$$

with

$$\frac{1}{2(1-\cos\lambda)} H(t)^{-1} \begin{pmatrix} 1-\cos\lambda & -\sin\lambda \\ \sin\lambda & 1-\cos\lambda \end{pmatrix}.$$

All the entries of Λ_t^0 are thus calculable but we do not go further because the explicit formulas are complicated.

4.5 More on the monomial convoluted Brownian motion $\mathbf{X}^{\sharp k}$

The $(k+1)$-dimensional monomial convoluted Brownian motion $\mathbf{X}^{\sharp k}$, whose covariance was calculated in Section 4.2.2, does not satisfy Assumption $(\mathfrak{H}_1)$: $Id - e^A$ is not invertible, when the matrix A is given by (4.13). Therefore one can not derive its semimartingale representation (resp. the structure of its bridges) as a direct application of Section 4.3 (resp. Section 4.4). Nevertheless we recover some Markovianity considering this process jointly with an additional coordinate $\bar{X}$, constructed as a weighted primitive.

4.5.1 A Markovian enhancement of $\mathbf{X}^{\sharp k}$

In all this section, A denotes the $(k+1)\times(k+1)$ matrix given by (4.13).

Proposition 4.10 *Define the real-valued process $\bar{X}$ by $\bar{X}_t = \int_0^t (1-u)^{k-1} X_u^{\sharp 1}\, du$ and consider the $\mathbb{R}^{k+2}$-valued process $\mathbf{Z}_t =: \begin{pmatrix} \mathbf{X}_t^{\sharp k} \\ \bar{X}_t \end{pmatrix}, t \in [0,1]$.*

Conditionally on $\mathbf{Z}_0 = \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix}$, $\mathbf{Z}$ is a Markov process which solves the affine SDE:

$$\left\{ \begin{array}{rl} d\mathbf{X}_t^{\sharp k} & = (e^A - Id)e_{k+1}\, d\tilde{B}_t + \left(A\mathbf{X}_t^{\sharp k} + \tilde{\Lambda}_t^0\, \mathbf{x} + \tilde{\Lambda}_t^1 \mathbf{Z}_t\right) dt, \\ d\bar{X}_t & = (1-t)^{k-1} X_t^{\sharp 1}\, dt, \\ \mathbf{Z}_0 & = \ (\mathbf{x}, 0) \end{array} \right. \tag{4.35}$$

where $\tilde{\Lambda}_t^0$ and $\tilde{\Lambda}_t^1$ are deterministic matrices defined by (4.42)*.*

Before proving Proposition 4.10, we begin with three preliminary results, Lemmas 4.11- 4.13. In the first one, we prove that $Id - \exp A$ can be, in some sense, "weakly" inverted.

Lemma 4.11 *1. The $k \times k$ matrix C with entries*

$$C_{ij} := \binom{k-(i-1)}{j-(i-1)} \mathbb{1}_{\{j \geq i\}}, \quad 1 \leq i, j \leq k \tag{4.36}$$

is invertible.

2. Let $\mathbf{y} = (y_1, \cdots, y_{k+1})^* \in \mathbb{R}^{k+1}$. The equation

$$(e^A - Id)\ \mathbf{x} = \mathbf{y}$$

admits a solution in $\mathbb{R}^{k+1}$ if and only if $y_{k+1} = 0$. In that case the set of solutions is the 1-dimensional vector subspace $\mathbb{R} \times C^{-1}(y_1, \cdots, y_k)^*$.

Proof of Lemma 4.11: C is a triangular matrix whose diagonal entries are $k-i+1, 1 \leq i \leq k$. They do not vanish, therefore C is invertible.
We keep the notations introduced in the proof of Proposition 4.5. Using (4.18) we have:

$$e^{tA} = \sum_{i=0}^{k} \sum_{j'=i}^{k} \binom{k-i}{j'-i} t^{j'-i} E^{i+1, j'+1} \ . \tag{4.37}$$

Then we deduce:

$$\begin{aligned}
e^A - Id \quad &= \sum_{i=0}^{k-1} \sum_{j'=i+1}^{k} \binom{k-i}{j'-i} E^{i+1,j'+1} \\
&= \sum_{i'=1}^{k} \sum_{j'=i'}^{k} \binom{k-(i'-1)}{j'-(i'-1)} E^{i',j'+1} \\
\Rightarrow (e^A - Id)\ \mathbf{x} \quad &= \sum_{i'=1}^{k} \Big(\sum_{j'=i'}^{k} C_{i'j'} x_{j'+1} \Big) e_{i'}
\end{aligned}$$

where $\mathbf{x} = (x_1, \cdots, x_{k+1})^*$. Item 2 follows immediately.

∎

Lemma 4.12 *The first component of the vector* $\int_0^t e^{(1-u)A} e_{k+1}\ dB_u$ *is given by the scalar stochastic integral* $\int_0^t (1-u)^k dB_u$.

Proof of Lemma 4.12: Identity (4.37) implies:

$$e^{tA} e_{k+1} = \sum_{i=1}^{k+1} t^{k-(i-1)} e_i.$$

Consequently, the first component of $e^{(1-u)A} e_{k+1}$ is equal to $(1-u)^k$ and the first component of $\int_0^t e^{(1-u)A} e_{k+1}\ dB_u$ is $\int_0^t (1-u)^k dB_u$. ∎

Lemma 4.13 *For any t fixed, the stochastic integral* $\int_0^t (1-u)^k dB_u$ *is a linear combination of the r.v.* $X_t^{\sharp 1}, \bar{X}_t, X_0^{\sharp 1}$ *and* $X_0^{\sharp 0}$. *More precisely, we have:*

$$\int_0^t (1-u)^k dB_u = -(1-t)^k X_t^{\sharp 1} - k \bar{X}_t + X_0^{\sharp 1} + \frac{1}{k+1} \big(1 - (1-t)^{k+1}\big) X_0^{\sharp 0}.$$

Proof of Lemma 4.13: First we integrate by part the stochastic integral:

$$\int_0^t (1-u)^k dB_u = (1-t)^k B_t + k \int_0^t (1-u)^{k-1} B_u du. \tag{4.38}$$

Then, using item 1 of Example 3.3, and (3.24) with $k = 1$, we write B_t as a linear combination of $X_t^{\sharp 1}, X_0^{\sharp 1}$ and $X_0^{\sharp 0}$:

$$B_t = X_0^{\sharp 1} + t X_0^{\sharp 0} - X_t^{\sharp 1}, \quad t \in [0,1]. \tag{4.39}$$

Equality (4.38) becomes

$$\int_0^t (1-u)^k dB_u = -(1-t)^k X_t^{\sharp 1} - k\int_0^t (1-u)^{k-1} X_u^{\sharp 1} du + a_1(t) X_0^{\sharp 1} + a_0(t) X_0^{\sharp 0}$$

where

$$\begin{aligned} a_0(t) &:= (1-t)^k t + k\int_0^t (1-u)^{k-1} u \; du = \frac{1}{k+1}\big(1-(1-t)^{k+1}\big),\\ a_1(t) &:= (1-t)^k + k\int_0^t (1-u)^{k-1} \; du = 1. \end{aligned}$$

■

Proof of Proposition 4.10: We revisit the proof of Proposition 4.7, using now Lemma 4.11 instead of Assumption $(\mathfrak{H}_1)$. Relation (4.31) reads in our framework:

$$(e^A - Id)\int_0^s e^{(1-u)A} e_{k+1} \; dB_u = e^A \, \mathbf{X}_0^{\sharp k} - e^{(1-s)A} \, \mathbf{X}_s^{\sharp k}. \tag{4.40}$$

Applying Lemma 4.11, the i-th component of $\int_0^s e^{(1-u)A} e_{k+1} \; dB_u$ is given by

$$\Big(\int_0^s e^{(1-u)A} e_{k+1} \; dB_u\Big)_i = \Big(C^{-1}\big(e^A \, \mathbf{X}_0^{\sharp k} - e^{(1-s)A} \, \mathbf{X}_s^{\sharp k}\big)\Big)_{i-1}, \quad i = 2, \cdots, k+1.$$

Note that (4.40) does not determine the first component of $\int_0^s e^{(1-u)A} e_{k+1} \; dB_u$. But, by Lemmas 4.12 and 4.13,

$$\begin{aligned} \Big(\int_0^s e^{(1-u)A} e_{k+1} \; dB_u\Big)_1 &= -(1-s)^k X_s^{\sharp 1} - k\bar{X}_s + X_0^{\sharp 1} \\ &\quad + \frac{1}{k+1}\big(1-(1-s)^{k+1}\big) X_0^{\sharp 0}. \end{aligned}$$

Both identities imply:

$$\int_0^s e^{(1-u)A} e_{k+1} \; dB_u = \Gamma_s^0 \, \mathbf{X}_0^{\sharp k} + \Gamma_s^1 \, \mathbf{Z}_s$$

where Γ_s^0 (resp. Γ_s^1) is a deterministic suitable $(k+1)\times(k+1)$ matrix (resp. $(k+1)\times(k+2)$ matrix).
Now, remark that Assumption $(\mathfrak{H}_2)$ is satisfied since

$$\mathrm{Span}(A^i e_{k+1}, i = 0, \cdots, k) = \mathrm{Span}(e_{k+1}, e_k, \cdots, e_1) = \mathbb{R}^{k+1}.$$

Therefore Lemma 4.8 implies that the matrix $H(s)$ is invertible, and the strategy used in the proof of Proposition 4.7 can be developed in our context. Using (4.29) and (4.30), we deduce that

$$B_t = -\tilde{B}_t - \int_0^t e_{k+1}^* e^{(1-s)A^*} H(s)^{-1}\Big[(-Id + \Gamma_s^0)\mathbf{X}_0^{\sharp k} + \Gamma_s^1 \mathbf{Z}_s\Big] ds \tag{4.41}$$

where the process $\tilde{B}$ is a $(\mathcal{G}_t)$-Brownian motion independent from $\mathbf{X}_0^{\sharp k}$ and $(\mathcal{G}_t)_t$ is the filtration obtained with the initial enlargement of the Brownian filtration with the random vector $\mathbf{X}_0^{\sharp k}$.
Back to (4.25), replacing B by the right hand side of (4.41), one obtains for all $t \in [0, 1]$,

$$\begin{aligned}\mathbf{X}_t^{\sharp k} &= \mathbf{X}_0^{\sharp k} + (e^A - Id)e_{k+1}\,\tilde{B}_t + \int_0^t A\mathbf{X}_s^{\sharp k}\,ds \\ &+ \int_0^t (e^A - Id)e_{k+1}e_{k+1}^* e^{(1-s)A^*} H(s)^{-1}(-Id + \Gamma_s^0)\,\mathbf{X}_0^{\sharp k}\,ds \\ &+ \int_0^t (e^A - Id)e_{k+1}e_{k+1}^* e^{(1-s)A^*} H(s)^{-1}\Gamma_s^1\,\mathbf{Z}_s\,ds.\end{aligned}$$

Therefore, defining three matrices by

$$\begin{aligned}\tilde{\Gamma}_t &:= (e^A - Id)e_{k+1}e_{k+1}^* e^{(1-s)A^*} H(s)^{-1}, \\ \tilde{\Lambda}_t^0 &:= \tilde{\Gamma}_t(-Id + \Gamma_t^0), \\ \tilde{\Lambda}_t^1 &:= \tilde{\Gamma}_t\Gamma_t^1,\end{aligned} \tag{4.42}$$

one gets the affine stochastic differential system (4.35) satisfied by the process $\mathbf{Z} = (\mathbf{X}^{\sharp k}, \bar{X})$ pinned at time 0 in $(\mathbf{x}, 0)$.

■

4.5.2 The special case of $\mathbf{X}^{\sharp 1}$

In this section, we treat the case $k = 1$ explicitly. The $\mathbb{R}^2$-valued process $\mathbf{X}^{\sharp 1} = (X^{\sharp 1}, X^{\sharp 0})$ is associated with the 2×2-matrix $A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Since its second component is not time-dependent but is equal to the constant r.v. B_1 (see Examples 3.3), we are principally interested in the dynamics of $X^{\sharp 1}$. Indeed, the process $X^{\sharp 1}$ admits the following representation:

$$X_t^{\sharp 1} = \int_0^1 B_s\,ds + tB_1 - B_t, \quad t \in [0, 1]. \tag{4.43}$$

Thus it is the (non independent) sum of its initial condition $X_0^{\sharp 1} = \int_0^1 B_s\,ds$ and a $0 \hookrightarrow 0$-Brownian bridge. Indeed, identity (3.21) applies with $\varphi(x) \equiv x$ and (3.11) gives:

$$\begin{aligned}X_t^{\sharp 1} &= \int_0^1 Z(r, t)\,dr \\ &= \int_0^t \big(B_{t-r} - B_t + B_1\big)dr + \int_t^1 \big(B_{1-r+t} - B_t\big)dr \\ &= \int_0^t B_r dr + (B_1 - B_t)t + \int_t^1 B_r dr - (1-t)B_t \\ &= \int_0^1 B_r dr + tB_1 - B_t.\end{aligned}$$

Therefore $X^{\sharp 1}$ is not Markovian. Enlarging the filtration with its initial condition $\int_0^1 B_r dr$ - as we did in Section 2.2 for the PerOU process - is nevertheless not enough to recover the Markovianity. However, as seen in the latter section 4.5.1, the right enlargement is obtained with the 2-dimensional initial random vector $\mathbf{X}_0^{\sharp 1} = (X_0^{\sharp 1}, B_1) = (\int_0^1 B_r dr, B_1)$. Proposition 4.10 shows that once we add the primitive process $\bar{X}$ of $X^{\sharp 1}$ to $X^{\sharp 1}$ we recover the Markov property. We determine explicitly the SDE satisfied by the enhancement of $\mathbf{X}^{\sharp 1}$, that is by the (pinned) $\mathbb{R}^3$-valued process $(\mathbf{X}^{\sharp 1}, \bar{X})$. Remark that (4.43) implies that $X_0^{\sharp 0} = 2\bar{X}_1 = B_1$.

Proposition 4.14 *The process $(X^{\sharp 1}, X^{\sharp 0}, \bar{X})$, pinned at time 0 in $(x, y, 0)$, solves the following affine stochastic differential system:*

$$\begin{cases} dX_t^{\sharp 1} &= d\tilde{B}_t + \beta(t, X_t^{\sharp 1})\, dt + \gamma(t, \bar{X}_t)\, dt, \\ d\bar{X}_t &= X_t^{\sharp 1}\, dt, \quad t \in [0, 1], \end{cases} \tag{4.44}$$

where $\beta(t, z) := -\frac{2x}{1-t} + \frac{3y}{(1-t)^2} - \frac{4}{1-t}\, z$ *and* $\gamma(t, z) := -\frac{6}{(1-t)^2}\, z.$
Here is $\tilde{B}$ a $(\mathcal{G}_t)$-Brownian motion independent of $\mathbf{X}_0^{\sharp 1}$ where $(\mathcal{G}_t)_t$ is the filtration obtained by the initial enlargement of $(\mathcal{F}_t)_t$ with the random vector $\mathbf{X}_0^{\sharp 1}$.

Proof of Proposition 4.14: We apply Lemma 4.6 to $(\xi_1, \xi_2)^* = (X_0^{\sharp 0}, X_0^{\sharp 1})^*$ that is with $h_1(t) \equiv 1$ and $h_2(t) := 1 - t$. Computing the different terms of the matrix H:

$$H_{11}(t) = 1 - t, \quad H_{12}(t) = H_{21}(t) = \frac{(1-t)^2}{2}, \quad H_{22}(t) = \frac{(1-t)^3}{3},$$

which leads to

$$\det H(t) = \frac{(1-t)^4}{12} \text{ and } H^{-1}(t) = \frac{12}{(1-t)^4} \begin{pmatrix} \frac{(1-t)^3}{3} & -\frac{(1-t)^2}{2} \\ -\frac{(1-t)^2}{2} & 1-t \end{pmatrix}.$$

We are now able to calculate the matrix G:

$$\begin{aligned} G(t, u) &= \frac{12}{(1-t)^4} \begin{pmatrix} 1 & 1-t \end{pmatrix} \begin{pmatrix} \frac{(1-t)^3}{3} & -\frac{(1-t)^2}{2} \\ -\frac{(1-t)^2}{2} & 1-t \end{pmatrix} \begin{pmatrix} 1 \\ 1-u \end{pmatrix} \\ &= \frac{1}{(1-t)^2} \big(6(1-u) - 2(1-t)\big). \end{aligned}$$

Therefore

$$\int_s^1 G(s, u)\, dB_u = \frac{1}{(1-s)^2} \Big[6 \int_s^1 (1-u) dB_u - 2(1-s)(B_1 - B_s)\Big].$$

We reformulate it using first the integration by parts:

$$\int_s^1 (1-u)dB_u = -(1-s)B_s + \int_0^1 B_u du - \int_0^s B_u du,$$

and then the boundary conditions:

$$X_0^{\sharp 1} = \int_0^1 B_s ds = x, \quad X_0^{\sharp 0} = B_1 = y.$$

We obtain

$$\int_s^1 G(s,u)\, dB_u \quad = \quad \frac{-2(y+2B_s)}{1-s} + \frac{6x}{(1-s)^2} - \frac{6}{(1-s)^2}\int_0^s B_u du.$$

Using (4.43), we get $B_t = yt + x - X_t^{\sharp 1}$. Integrating this identity over $[0,s]$ leads to $\int_0^s B_t dt = s^2 y/2 + sx - \bar{X}_s$. Therefore

$$\int_s^1 G(s,u)\, dB_u \quad = \quad \frac{2x}{1-s} + \Big(1 - \frac{3}{(1-s)^2}\Big)y + \frac{4X_s^{\sharp 1}}{1-s} + \frac{6}{(1-s)^2}\bar{X}_s.$$

We can now conclude, using (4.23):

$$\begin{aligned} X_t^{\sharp 1} &= x - B_t + yt \\ &= x + \tilde{B}_t + \int_0^t \big(y - \int_s^1 G(s,u)\, dB_u\big) ds \\ &= x + \tilde{B}_t + \int_0^t \Big[-\frac{2x}{1-s} + \frac{3y}{(1-s)^2} - \frac{4X_s^{\sharp 1}}{1-s} - \frac{6}{(1-s)^2}\bar{X}_s\Big] ds. \end{aligned}$$

References

[1] J-P. Carmichael, J-C. Masse and R. Theodorescu, Processus gaussiens stationnaires réciproques sur un intervalle, C.R. Acad. Sc. Paris, Série I t.295 (1982) 291-293

[2] M. Chaleyat-Maurel and T. Jeulin, Grossissement gaussien de la filtration brownienne, C.R. Acad. Sc. Paris, Série I t.296-15 (1983) 699-702

[3] C. Dellacherie and P.-A. Meyer, Probabilités et Potentiel, Vol II. Théorie des Martingales, Hermann, Paris (1980)

[4] C. Donati-Martin, Equations différentielles stochastiques dans $\mathbb{R}$ avec conditions aux bords, Stochastics and Stochastics Reports 35 (1988) 143-173

[5] J. Doob, Stochastic processes, Wiley (1953)

[6] B. Jamison, Reciprocal processes: The stationary gaussian case, Ann. Math. Statist. 41 (1970) 1624-1630

[7] B. Jamison, Reciprocal processes, Z. Warsch. Verw. Geb. 30 (1974) 65-86

[8] H. Kwakernaak, Periodic linear differential stochastic processes, SIAM J. Control Optimization 13 (1975) 400-413

[9] A.J. Krener, R. Frezza and C.B. Levy, Gaussian reciprocal processes and self-adjoint stochastic differential equations of second order, Stochastics and Stoch. Reports 34 (1991) 29-56.

[10] C. Léonard, S. Rœlly and J.-C. Zambrini, Reciprocal processes. A measure-theoretical point of view, Probab. Surveys 11 (2014) 237-269.

[11] P.-A. Meyer, Probabilités et Potentiel, Hermann (1966)

[12] D. Nualart and E. Pardoux, Boundary value problems for Stochastic Differential Equations, Annals of Probab. 19-3 (1991) 1118-1144

[13] D. Ocone and E. Pardoux, Linear Stochastic Differential Equations with Boundary Conditions, Probab. Th. Rel. Fields 82 (1989) 489-526

[14] J. Pedersen, Periodic Ornstein-Uhlenbeck processes driven by Levy processes, Journal of applied probability 39-4 (2002) 748-763.

[15] S. Rœlly and M. Thieullen, A characterization of reciprocal processes via an integration by parts formula on the path space, Probab. Th. Rel. Fields 123 (2002) 97-120.

[16] D. Revuz and M. Yor, Continuous Martingales and Brownian Motion, Springer-Verlag (1991)

[17] F. Russo and P. Vallois, Elements of stochastic calculus via regularization, Séminaire de Probabilités XL, LNM 1899, Springer-Verlag (2007) 147-186

[18] D. Slepian, First Passage Time for a Particular Gaussian Process, Ann. Math. Statist. 32-2 (1961) 610-612