UNIVERSITÄT POTSDAM

WIRTSCHAFTS- UND SOZIALWISSENSCHAFTLICHE FAKULTÄT

Lehrstuhl für Finanzwissenschaft

Ralf E. Becker

General Classification of Social Choice Situations

Diskussionsbeitrag 46
Potsdam 2004

Ralf E. Becker
Magdeburg (Germany)*
Tel.: (+49) 0391 555 86-30; Fax: ...-29
E-mail: *ralf_e_becker@t-online.de*

Ralf E. Becker

General Classification of Social Choice Situations

July 2004

* JEL-class: D71. I thank for support implied by the remarks in Arrow (1953).

ISSN 0948 - 7549

General Classification of Social Choice Situations

Ralf E. Becker

Magdeburg (Germany) *

An exhaustive and disjoint decomposition of social choice situations is derived in a general set theoretical framework using the new tools of the Lifted Pareto relation on the power set of social states representing a pre-choice comparison of choice option sets. The main result is the classification of social choice situations which include three types of social choice problems. First, we usually observe the common incompleteness of the Pareto relation. Second, a kind of non-compactness problem of a choice set of social states can be generated. Finally, both can be combined. The first problem root can be regarded as natural everyday dilemma of social choice theory whereas the second may probably be much more due to modeling technique implications. The distinction is enabled at a very general set theoretical level. Hence, the derived classification of social choice situations is applicable on almost every relevant economic model.

* JEL-class: D71. I thank for support implied by the remarks in Arrow (1953).

1 Introduction

We want to classify social choice situations in a general framework of sets of social states. For our purpose there are two sources of trouble for social choice. First, the major problem in social choice theory is the incompleteness of the Pareto relation, i.e., the potential existence of winners *and* losers. This incompleteness problem is interesting especially on the Pareto-efficient border of the set of social states. That is what can be associated with the utility possibility frontier. But there is a second root of choice difficulty: The frontier or border itself in part may not be complete due to a potential for openness of intervals or decision fields and unlimited resources. For example, if all individuals prefer higher numbers and we regard an open or proper endless interval then in both cases a Pareto efficient social state does not exist. Hence, a rather fundamental help for social choice considerations is to classify social choice situations according to the character of the root of the choice problem on a general set theoretical ground. This is the task of this contribution. Below an exhaustive decomposition of social choice situations is derived.

As a tool we introduce what we call the *Lifted* Pareto relation on the power set of social states. It can be interpreted as a pre-choice comparison of sets of social states: The intuitive idea is that one set of social states as choice option is “at least as good as” another set if and only if for every state in the other set there is at least one corresponding state in the first set that is Pareto indifferent or Pareto superior to this state. Hence, the society can be sure to loose nothing if it neglects the other set of social states and limits his choice to the first choice option set. This relation is used for the definition of an compact economy of a (sub)set of social states which essentially means that the border is sufficient to *maximizes* all Pareto-inferior social states.[†] Afterwards we derive a list of criteria for this property which opens the floor for the general classification of social choice situation.

[†] “Maximized” means, that for a social state (which actually must be a Pareto-inferior element) we can find a suitable maximal (i.e., Pareto efficient) element that is Pareto-superior to *this* Pareto-inefficient social state.

2 Tools

2.1 Lifted Pareto relation

Let X be any set (interpreted as set of social states considered) and $i \in IN$ the index of one of the countable (finite or denumerable) number of individuals, then we define the *Pareto relation* (Pareto preference or indifference) in dependence of the individual preorders R_i as

$$R^{Pa} \; := \; \{(x, y) \in X \times X \mid x R_i y \;\; \forall i \} \text{ (interpreted as } \succ \text{ or } \sim) \qquad \text{“}\textbf{Pareto relation}\text{” } (\boldsymbol{R^{Pa}}) \quad [1]$$

Both properties of the individual preorders (eventually incomplete orderings) transmit to the Pareto relation and the vector of individual preorders $\underline{R} := (R_1, R_2, \ldots)$ can be used to abbreviate the elementary dependence of the Pareto relation on the individual preorders:‡

$$[R] \; \forall R_i \Rightarrow [R] \text{ for } R^{Pa} \tag{2}$$

$$[T] \; \forall R_i \Leftrightarrow [T] \text{ for } R^{Pa} \tag{3}$$

$$R^{Pa} = R^{Pa}(\underline{R}) \qquad \text{(with } \underline{R} \text{ as } (R_1, R_2, \ldots)) \tag{4}$$

If we apply the I-definition ($x \, I^{Pa} y \Leftrightarrow x \, R^{Pa} y \wedge y \, R^{Pa} x$), P-definition ($x \, P^{Pa} y \Leftrightarrow x \, R^{Pa} y \wedge \neg \, y \, R^{Pa} x$), and N-definition ($x \, N^{Pa} y \Leftrightarrow \neg \, x \, R^{Pa} y \wedge \neg \, y \, R^{Pa} x$) on [1] we immediately have the following translations:

$$I^{Pa} = \{(x, y) \in X \times X \mid \forall i : \; x \, I_i \, y \} \qquad \text{“}\textbf{\textit{Pareto indifference}}\text{”} \quad (\boldsymbol{I^{Pa}}) \tag{5}$$

$$P^{Pa} = \{(x, y) \in X \times X \mid \forall i : \; x \, R_i \, y \wedge \exists j : \neg \, y \, R_j \, x\} \qquad \text{“}\textbf{\textit{Pareto preference}}\text{”} \quad (\boldsymbol{P^{Pa}}) \tag{6}$$

$$N^{Pa} = \{(x, y) \in X \times X \mid \exists i : \neg \, y \, R_j \, x \wedge \exists j : \neg \, x \, R_j \, y\} \qquad \text{“}\textbf{\textit{Pareto incompleteness}}\text{”} \quad (\boldsymbol{N^{Pa}}) \tag{7}$$

Now we will define a relation that compares *any* pair of sets of social states. The intuitive idea is that one set X' ($\subset X$) is “at least as good as” another set Y iff (if and only if) for every state in Y there is at least one corresponding state in X' that is Pareto indifferent or Pareto superior to this state. Hence, corresponding to our simple example in the introduction of this section, the society can be sure to

‡ [R] and [T] are shorthands for refexivity and transitivity.

loose nothing if it neglects Y and concentrates on X'. The considered sets can arbitrarily be chosen. The comparison represents a pre-choice evaluation.

Let us introduce this more formally. The *power set* $P(X)$ of X is simply the set of all subsets of X inclusive the empty set $\emptyset$ and X itself. It is defined by

$$P(X) := \{X' \text{ set} \mid X' \subset X\} \qquad (\emptyset, X \in P(X); \text{ for any set } X) \qquad [8]$$

Let $\mathsf{X} \subset P(X)$ be any set of subsets of X, R_i individual relations on X with $[R,T,E2.1]$ and R^{Pa} the Pareto relation (depending on the R_i's). then we define a new relation $R^{LPa(\mathsf{X})}$ on X ($R^{LPa(\mathsf{X})} \subset \mathsf{X} \times \mathsf{X} \subset P(X) \times P(X)$) by

$$R^{LPa(\mathsf{X})} := \{(X',Y) \in \mathsf{X} \times \mathsf{X} \mid \forall\, y \in Y\ \exists\, x \in X' : x\, R^{Pa}\, y\}, \text{ for any } \mathsf{X} \subset P(X)$$

"Lifted Pareto relation (on X)" ($\boldsymbol{R^{LPa(\mathsf{X})}}$) [§] [9]

We call any element $X' \in \mathsf{X}$ a **choice *option* set** because in contrast to the choice set and the maximal set which are normally interpretated as after-choice sets these elements play the rule of *pre-choice sets*. Because we presume individual preorders the Lifted Pareto relation is also a preorder:

$$[R] \text{ for } R_i\ \forall\, i \ \Rightarrow\ [R] \text{ for } R^{LPa} \qquad (10)$$

$$[T] \text{ for } R_i\ \forall\, i \ \Rightarrow\ [T] \text{ for } R^{LPa} \qquad (11)$$

We use

$$\neg\, X'\, R^{LPa(\mathsf{X})}\, Y \ \Leftrightarrow\ \exists\ y \in Y :\ (y\, P^{Pa}\, x' \,\underline{\vee}\, y\, N^{Pa}\, x')\ \ \forall\, x' \in X' \qquad (12)$$

to calculate the corresponding element tests for the *IPN*-relations:

$$X'\, I^{LPa(\mathsf{X})}\, Y \ \Leftrightarrow\ [\forall\, y \in Y\ \exists\, x \in X' : x\, R^{Pa}\, y] \ \wedge\ [\forall\, x \in X'\ \exists\ y \in Y : y\, R^{Pa}\, x\,] \qquad (13)$$

$$X'\, P^{LPa(\mathsf{X})}\, Y \ \Leftrightarrow\ [\forall\, y \in Y\ \exists\, x \in X' : x\, R^{Pa}\, y] \ \wedge\ [\exists\, x \in X' : (x P^{Pa}\, y \,\underline{\vee}\, x\, N^{Pa}\, y)\ \ \forall\, y \in \mathrm{Y}] \qquad (14)$$

$$X'\, N^{LPa(\mathsf{X})}\, Y \ \Leftrightarrow\ [\exists y \in Y : (y P^{Pa} x \,\underline{\vee}\, y\, N^{Pa}\, x)\ \ \forall x \in X'] \wedge [\exists x \in X' : (x\, P^{Pa}\, y \,\underline{\vee}\, x\, N^{Pa}\, y)\ \ \forall y \in Y] \qquad (15)$$

Naturally, any set is pre-choice equivalent to itself, simply because we presume that every social state is self-equivalent for all individuals ($Y \sim^{LPa} Y\ \forall\ Y \subset X$):

$$Y\, I^{LPa}\, Y \qquad \forall\ Y \in P(X) \qquad \text{for } [R] \text{ for } R_i\ \forall\, i \quad (\text{incl. } Y = \emptyset) \qquad (16)$$

---

§ We abbreviate $R^{LPa} := R^{LPa(P(X))}$ (i.e., $\mathsf{X} = P(X)$) and naturally can use R^{LPa} whenever we do not want to specify X.

But the immediate results

$$\varnothing\, I^{LPa}\, \varnothing \tag{17}$$

$$Y\, P^{LPa}\, \varnothing \qquad \forall\; Y \subset X \text{ with } Y \neq \varnothing \tag{18}$$

raise the question how we have to interpret the empty set as choice option set? [**] A choice option set with only one element can be interpreted as option set with actually no choice option, i.e., as some kind of special case (degenerated) options set, or as *trivial choice-option set*. But what is the meaning of ∅? The first reaction could be to restrict the domain of the lifted Pareto relation (or lifted relations in general) to the comparisons of non-empty sets in its definition (i.e., to require that $\mathrm{X} \subset P(X)\backslash\{\varnothing\}$ in [9]). But despite the following remark this is nevertheless superfluous:

<u>REMARK 1</u>: (***omnipresent must-choice***). *Choice option sets (pre-choice sets) are not empty*.

Proof: Axiom: Something happens. □ { 19}

Hence, ∅ cannot represent a choice option set. Therefore, (18) expresses that generally any non-empty choice option set is formally enforced to be pre-choice preferred (here applied to Pareto) to the empty set which cannot be regarded as choice option set. The proper interpretation for the empty set is surprisingly simple. The preference (18) is the natural logical outcome of the must-choice interpretation of non-empty pre-choice set for pre-choice set comparisons. I.e., ∅ is simply impossible and if you can choose between (hypothetically) non-empty choice-option sets *Y* and non-existing other choice-option set then you actually must choose *Y*. (17) is just a formally useful completion: the choice set ∅ representing that nothing happens (which is excluded to be valid) is self-equivalent.[††]

Finally, it should be clear that the author was tempted to alternatively use the adjective *Generalized* for the *Lifted* Pareto relation. The reason is simply

$$\{x\}R^{LPa}\{y\} \;\Leftrightarrow\; x\, R^{Pa}\, y \qquad \forall\; x, y \in X \tag{20}$$

which so far gives us the mathematical equivalence of the relations, because up to pure formalism which is easily avoidable both are identical:[‡‡]

[**] This is an example that even convincing definitions may generate unexpected work for interpretational coherence.

[††] Another equivalent reinterpretation of ∅ for *Y* or *Z* in $Y\, R^{LPa}Z$-considerations is to think about the empty set as a symbol of a non-empty choice set with a single element that represents an outside option which is presumed to be inferior to all elements of *X*, and in addition is self-equivalent. But this would be a very uncommon, formally dangerous and avoidable interpretation of ∅.

[‡‡] One can even avoid this formal difference by a definition of the Pareto relation which refers to {*x*} instead of *x*.

$$R^{LPa(\mathsf{X}1)} \cong R^{Pa} \qquad \text{for} \quad \mathsf{X}_1 := \{\{x\} \subset P(X) \mid x \in X\} \tag{21}$$

Hence, the Pareto relation R^{Pa} can be regarded as a special case of R^{LPa} if the Lifted Pareto relation is applied to singletons. In addition, the concept of *lifted* relations can be generalized by itself on any binary relation and is interpretable as tool of a general pre-choice comparison of subsets of the original set on which the reference relation works.

2.2 Compact Economy

Naturally, in theory, there is the potential of economies which generate no Pareto-optimal social states at all. But the central problem is that even if the whole maximal set (the set of Pareto-efficient social states) is not empty there are regularly still empty maximal sets of certain samples of social states. There are two major intuitions for the potential of empty maximal sets of the Pareto relation within the set of social states or respective subsets: the openness problem and an imagination of endless resources. For example, if all individuals prefer higher numbers and Y is the open or endless interval (0,1) or (0,+∞) then in both cases the maximal set for the Pareto relation is empty, i.e., $M(Y, R^{Pa}) =_{def} \{x \in Y \mid \neg\ z\ P^{Pa}\ x\ \forall\ z \in Y\} = \varnothing$. Therefore, at a first glance one could be tempted to generally exclude empty maximal sets of any non-empty subset of social states (naturally, with the test for maximal property, i.e., Pareto-efficiency, always restricted to the respective subset) from further considerations. But this would both be a too strong (*any*) and a too weak (require "*maximizing*" maximal sets; see below) assumption:

Half closed welfare circle (too weak): For example, if for a circle including only half of its closure a greater distance from the origin is preferred but points on different radial lines are generally regarded as incomparable then circle-inner points on lines which include the intersections with the circle's closure are "maximized" by the point of intersection and all points on non-closed lines are not but simultaneously the maximal set of this half-closed circle is *not* empty. So what we actually want to ensure is that if Y is any subset then all (Y-) inefficient social states can be "maximized", i.e., for any Pareto-inferior element we can find a suitable maximal (Pareto efficient) element that is Pareto-superior to *this* Pareto-inefficient social state. The guarantee of the exclusion of our problems with endless resources and openness or half-openness for a certain subset is what we describe by the following notion:

<u>DEFINITION 1</u>: (*compact*; *maximized*): Let X be a set, $Y \subset X$, $\underline{R} \in (\Re_{R,T})^n$ vector of individual preorders on X and $R^{Pa} = R^{Pa}(\underline{R})$ the corresponding Pareto relation. Then we define Y to be *compact*:

$$Y \subset X \textit{ compact} \;:\Leftrightarrow\; \forall\, x \in Y \setminus M(Y, R^{Pa})\ \exists\, y \in M(Y, R^{Pa}) : y\ P^{Pa}\ x \quad (\forall\ Y \subset X) \tag{1}$$

We call $x \in Y$ (not $Y \setminus M(Y, R^{Pa})$ for purist definition) *maximized in Y* iff (if and only if) $\exists\ y \in M(Y, R^{Pa}) : y\ P^{Pa}\ x$. ♦

Using the following shortcuts

$$M(Y, R^{Pa}) \quad = \quad \{x \in Y \mid \neg\ z\ P^{Pa}\ x\ \forall\ z \in Y\}$$

$$M^{\circ}(Y, R^{Pa}) \quad := \quad \{x \in Y \mid \exists\ z \in M(Y, R^{Pa}) : z\ P^{Pa}\ x\} \qquad \text{(“\textbf{maximized elements}”)} \qquad [2a]$$

$$M^{\circ\circ}(Y, R^{Pa}) \quad := \quad \{x \in Y \setminus M^{\circ}(Y, R^{Pa}) \mid \exists\ z \in Y : z\ P^{Pa}\ x\} \qquad \text{(“\textbf{non-maximized non-maximal el.}”)} \qquad [2b]$$

and

$$M(Y, R^{Pa})^{C} \quad := \quad Y \setminus M(Y, R^{Pa}) \qquad \text{(“\textbf{non-border elements}”)} \qquad [2c]$$

$$M^{\circ 1}(Y, R^{Pa}) \quad := \quad \{x \in M^{\circ}(Y, R^{Pa}) \mid \forall\ z \in M^{\circ\circ}(Y, R^{Pa}) : \neg\ z\ P^{Pa}\ x\} \qquad [2d]$$

$$M^{\circ 2}(Y, R^{Pa}) \quad := \quad \{x \in M^{\circ}(Y, R^{Pa}) \mid \exists\ z \in M^{\circ\circ}(Y, R^{Pa}) : \quad z\ P^{Pa}\ x\} \qquad [2e]$$

we decompose any set Y in its disjoint subsets of elements which are maximal, maximized, or neither maximized nor maximal:

$$Y \quad = \quad M(Y, R^{Pa}) \ \dot{\cup}\ M^{\circ}(Y, R^{Pa}) \ \dot{\cup}\ M^{\circ\circ}(Y, R^{Pa}) \qquad (3)$$

$$M(Y, R^{Pa})^{C} \quad = \quad M^{\circ}(Y, R^{Pa}) \ \dot{\cup}\ M^{\circ\circ}(Y, R^{Pa}) \qquad (4)$$

$$M^{\circ}(Y, R^{Pa}) \quad = \quad M^{\circ 1}(Y, R^{Pa}) \ \dot{\cup}\ M^{\circ 2}(Y, R^{Pa}) \qquad (5)$$

$$\begin{aligned} M(Y, R^{Pa}) = \varnothing \quad &\Leftrightarrow \quad M^{\circ\circ}(Y, R^{Pa}) = Y \quad \Leftrightarrow \quad M^{\circ}(Y, R^{Pa}) = \varnothing \ \wedge\ M^{\circ\circ}(Y, R^{Pa}) = Y \\ &\Leftrightarrow \quad (\forall\ w \in S(x)\ \exists\ z \in S(x) : z\ P^{Pa}\ w)\ \ \forall\ w \in Y \end{aligned} \qquad (6)$$

$$\begin{aligned} M^{\circ\circ}(Y, R^{Pa}) &= \{x \in Y \mid \exists\ z \in Y : z\ P^{Pa}\ x \wedge \neg\ z\ P^{Pa}\ x\ \forall\ z \in M(Y, R^{Pa})\} \\ &= \{x \in Y \mid \exists\ z \in Y : z\ P^{Pa}\ x \wedge \neg\ x'\ P^{Pa}\ x\ \forall\ x' \in X : \neg\ w\ P^{Pa}\ x'\ \forall\ w \in Y\} \end{aligned} \qquad (7)$$

A set Y is compact (relative to itself) if and only if any non-maximal element is maximized:

$$Y\ \textit{compact} \quad \Leftrightarrow \quad (x \notin M(Y,R^{Pa}) \Rightarrow x \in M^{\circ}(Y,R^{Pa}))\quad \forall\ x \in Y \qquad (8)$$

The criterion (8) is simply a rewriting of definition [1]. This definition of an compact set (or economy) looks very similar to the definition of the lifted Pareto relation R^{LPa} applied on maximal sets and respective complements and indeed we could equivalently have used R^{LPa} in this definition:

$$\forall\ x \in Y \setminus M(Y,R^{Pa})\ \exists\ y \in M(Y,R^{Pa}) : y\ P^{Pa}\ x \quad \Leftrightarrow \quad M(Y,R^{Pa})\ R^{LPa}\ Y \setminus M(Y,R^{Pa}) \qquad (9)$$ [§§]

[§§] Appendix A1 contains the short proof of (9).

But non-purist versions like the r.h.s. of [1] are directly more informative.[***] A set is not compact iff it contains a non-maximized inefficient element. The intention is similar to the concept of compactness defined for metric spaces. Obvious conclusions are:

$$Y \subset X \text{ compact} \;\Rightarrow\; M(Y, R^{Pa}) \neq \emptyset \;\underline{\vee}\; Y = \emptyset \tag{10}$$

$$\emptyset \text{ compact} \tag{11}$$

$$x \in Y \text{ maximized} \;\Rightarrow\; x \notin M(Y, R^{Pa}) \tag{12}$$

$$Y \text{ compact}, Y \supset Z, M(Z, R) \supset M(Y, R) \Rightarrow\; Z \text{ compact} \tag{13}$$

The last statement means that for smaller subsets the test by definition of compact sets simultaneously becomes less as well as more restrictive because even if the number of tested inefficient elements may decrease also the number of candidates which can maximize them does. But if you only drop inefficient elements then nothing can destroy the property.

The following lemma shows that for every non-compact set there must even exist a whole endless preference chain of non-maximized (i.e., no preferred maximal element exists) inefficient elements. Because the other direction of implication is trivial the exclusion of such chains is another potential alternative for our definition.

<u>LEMMA 1</u>: (*non-maximized inefficient preference chain*): *If for any transitive relation there exists an inefficient (non-maximal) element that is not inferior to any maximal element then (and only then) there even exists a whole preference chain of non-maximized inefficient elements (non-maximized inefficient preference chain), or formally:*

Let X be a set, [*T*] *for* $R \subset Y \times Y$. *Then:*

$$\exists\, x_0 \in Y \setminus M(Y, R) : \forall\, y \in M(Y, R) : \neg\, y\, P\, x_0 \qquad (Y \text{ not compact})^{\dagger\dagger\dagger}$$

$$\Leftrightarrow \quad \exists\, x_0, x_1, x_2, \ldots \in Y \setminus M(Y, R) \qquad (\text{“\textbf{\textit{inefficient}}”})$$

$$\textit{with} \quad \ldots x_3\, P\, x_2\, P\, x_1\, P\, x_0 \qquad (\text{“\textbf{\textit{preference chain}}”})$$

$$\wedge\;\; \neg\, y\, P\, x_i \;\; \forall\, y \in M(Y, R),\, i = 0, 1, 2, \ldots^{\ddagger\ddagger\ddagger} \qquad (\text{“\textbf{\textit{non-maximized}}”}) \tag{14}$$

<u>Proof</u>: see appendix A2.

[***] E. g., take puristic transitivity ($x\,R\,y \wedge y\,R\,z \Rightarrow x\,R\,z \;\forall\, x, y, z \in X$ with $x \neq y \wedge y \neq z$) =: [*pT*] $\Leftrightarrow$ [*T*].

[†††] We can express the l.h.s. of this equivalence als by $M(Y \setminus M(Y,R), M(Y,R), R) \neq \emptyset$.

[‡‡‡] We can express this second condition by $x_i \in M^{\circ\circ}(Y, R)$.

(Too strong). Naturally the requirement to be compact cannot be applied to any subset of the whole economy. In our example, (.25, .75) is not compact and, hence, this requirement for all subsets of [0, 1] produces a contradiction but any $[a, b] \subset [0, 1]$ does not. But normally resources are presumed to be finite and the openness problem is more or less a formal mathematical phenomenon because it guarantees an unlimited power of approximation and, therefore, has limited economic relevance. Hence, this could be the motivation for a focus on systems of compact sets of social states, i.e., on *compact subsystems* as subset of the power set of the set of social states with exclusively compact elements. This would be no restriction on X itself but on the subset system $\mathcal{X}$. At a first glance one could be mislead to presume the compactness property for example before starting the investigation of the social choice situations. But opponents could reject even this very elementary presumption and, i.e., we would not recommend to exclude non-compact redistribution sets at once. We will see that such a potential presumption is superfluous and would be a redundant weakening of our position. The character of maximal sets gives us the following strict translation of compactness and we use this together with (16) to illustrate the "pre-choice surprise", which is an important general element of understanding the pre-choice comparison of sets by lifted relations.

LEMMA 1: (*lifted Pareto-relation and compactness*): *To be compact for non-empty sets is equivalent with lifted Pareto preference of the maximal set over his relative complement: X set, $Y \subset X$, $[T]$ for $R \subset Y \times Y$:*

$$Y \text{ compact} \wedge Y \neq \varnothing \quad \Leftrightarrow \quad M(Y, R^{Pa})\, P^{LPa}\, Y \setminus M(Y, R^{Pa}) \qquad \forall\, Y \subset X \tag{15}$$

Proof: see appendix A3.

There is little surprise here except that one should keep in mind that even if we can always find Pareto *preferred* elements to any other elements of another set then this does not definitely imply a pre-choice preference. For a fitted preference case, $(0,1) \sim^{LPa} (0,1)$ is an example. Hence, keep in mind:

$$\forall\, y \in Y\, \exists\, y \in X' : x \succ^{Pa} y \quad \not\Rightarrow \quad X' \succ^{LPa} Y \ \ (\text{only} \ldots \Rightarrow X' \succsim^{LPa} Y\,) \ \ (\text{"\textbf{\textit{pre-choice surprise}}"}) \tag{16}$$

Note, that we also have almost by negation:

COROLLAY 1: **(*not compact*)** *We have for X set, $Y \subset X$, $[T]$ for $R^{Pa} \subset Y \times Y$:*

$$Y \text{ not compact} \quad \Leftrightarrow \quad Y \neq \varnothing \ \wedge\ \neg\, M(Y, R^{Pa})\, P^{LPa}\, Y \setminus M(Y, R^{Pa}) \qquad \forall\, Y \subset X \tag{17}$$

Proof: see appendix A4.

We summarize the different options to express the compactness of sets in the following proposition also already including the usage of a weak form of the lifted Pareto relation not introduced here:

PROPOSITION 2: (***compactness criteria***): *Let X be a set, $Y \subset X$, $\underline{R} \in (\Re_{R,T})^n$ vector of individual preoders on X and $R^{Pa} = R^{Pa}(\underline{R})$ the corresponding Pareto relation: Then the following statements are equivalent:*[§§§]

i)	*Y compact*	(*name*)
ii)	$\forall\, x \in Y \setminus M(Y, R^{Pa})\ \exists\, y \in M(Y, R^{Pa}) : y\, P^{Pa}\, x$	(*definition*)
iii)	$x \notin M(Y, R^{Pa}) \Rightarrow x \in M°(Y, R^{Pa})\ \forall\, x \in Y$	(*rewriting of definition*)
iv)	$M°°(Y, R^{Pa}) = \emptyset$	(“***state-absence rewriting***”)
v)	$M(Y, R^{Pa})\ R^{wLPa}\ Y \setminus M(Y, R^{Pa})$	(R^{wLPa}-*rewriting*)
vi)	$Y = \emptyset \,\underline{\vee}\, M(Y, R^{Pa})\ P^{LPa}\ Y \setminus M(Y, R^{Pa})$	(P^{LPa}-*criterion*)
vii)	$Y = \emptyset \,\underline{\vee}\, M(Y, R^{Pa})\ P^{wLPa}\ Y \setminus M(Y, R^{Pa})$	(P^{wLPa}-*criterion*)
viii)	$M(Y, R^{Pa})\ R^{LPa}\ Y \setminus M(Y, R^{Pa})$	(R^{LPa}-*criterion*)
ix)	$\neg(\exists\, x_0, x_1, x_2, \ldots \in M°°(Y, R^{Pa}) : \ldots x_3\, P^{Pa}\, x_2\, P^{Pa}\, x_1\, P^{Pa}\, x_0)$	(“***chain criterion***”)

Proof: *ix*) $\Leftrightarrow_{\text{LEMMA 1}}$ *iv*) $\Leftrightarrow_{[2b]\,(3)}$ *iii*) $\Leftrightarrow_{(8)}$ *i*) $\Leftrightarrow_{[1]}$ *ii*) $\Leftrightarrow$ (easy to show) *v*) and *i*) $\Leftrightarrow_{(17)\text{-negation}}$ *vi*) and *ii*) $\Leftrightarrow_{(9)}$ *viii*). Easy to show: *vii*) $\Leftrightarrow$ *vi*) (dropped here). □

It is also easily proven that:

$$[T] \text{ for } R^{Pa} \Rightarrow (x_0 \in M°°(Y, R^{Pa}) \Leftrightarrow \exists\, x_1, x_2, \ldots \in M°°(Y, R^{Pa}) : \ldots x_3\, P^{Pa} x_2\, P^{Pa} x_1\, P^{Pa} x_0) \tag{18}$$

$$[T] \text{ for } R^{Pa} \Rightarrow (x \in M°°(Y, R^{Pa}) \Leftrightarrow \neg\, z\, P^{Pa}\, x\ \ \forall\, z \in M(Y, R^{Pa})\ \dot{\cup}\ M°(Y, R^{Pa})) \tag{19}$$

Now we are ready to apply the concepts of compactness and of the Lifted Pareto relation:

[§§§] Practical hint: if you implicitly want to express with compactness that a certain set *Y* is not empty note simply $M(Y, R^{Pa})\ P^{LPa}\ Y \setminus M(Y, R^{Pa})$ which automatically excludes $Y = \emptyset$. If you want allow empty sets then take *iv*).

3 Social Choice Situations

DEFINITION 1: (*Pareto relation Cases*) Let X be a set, $\underline{R} \in (\Re_{R,T})^n$ vector of individual preorders and $R^{Pa} = R^{Pa}(\underline{R})$ the resulting Pareto relation [1]. We define the follow. *Pareto rel.* or ***R^Pa^-Cases***:

Pareto incompleteness (*P*1 – *P*4) ("***Incomplete*** **Pareto relations**") *redistribution existence*[****]

Incomplete Pareto relations, hence *potential social choice problem*

P*1**: $P^{Pa} \neq \varnothing \wedge N^{Pa} \neq \varnothing \wedge I^{Pa} \backslash M \neq \varnothing$ "Busy*** **Pareto relation**"

P*2**: $P^{Pa} \neq \varnothing \wedge N^{Pa} \neq \varnothing \wedge I^{Pa} \backslash M = \varnothing$ "Exclusively Strong*** **Pareto relation**"

P*3**: $P^{Pa} = \varnothing \wedge N^{Pa} \neq \varnothing \wedge I^{Pa} \backslash M \neq \varnothing$ "Exclusively Symmetric*** **Pareto relation**"

P*4**: $P^{Pa} = \varnothing \wedge N^{Pa} \neq \varnothing \wedge I^{Pa} \backslash M = \varnothing$ "Omnipresent redistribution*** **Pareto relation**"

Pareto completeness (*P*5 – *P*8) ("***Complete*** **Pareto relations**") *redistribution nonexistence*

Complete Pareto relations, hence *no incompleteness social choice problem* [††††]

P*5**: $P^{Pa} \neq \varnothing \wedge N^{Pa} = \varnothing \wedge I^{Pa} \backslash M \neq \varnothing$ "Exclusively Complete*** **Pareto relation**" (card(X) ≥ 3)

P*6**: $P^{Pa} \neq \varnothing \wedge N^{Pa} = \varnothing \wedge I^{Pa} \backslash M = \varnothing$ "Chain*** **Pareto relation**" (card(X) ≥ 2)

P*7**: $P^{Pa} = \varnothing \wedge N^{Pa} = \varnothing \wedge I^{Pa} \backslash M \neq \varnothing$ "Universal Indiff.*** **Pareto rel.**" $I^{Pa}=X \times X$, card(X)≥2 [‡‡‡‡]

P*8**: $P^{Pa} = \varnothing \wedge N^{Pa} = \varnothing \wedge I^{Pa} \backslash M = \varnothing$ "Trivial*** **Pareto relation**" card(X) ≤ 1

(*P*3, *P*4 & *P*7, *P*8: "***Symmetric*** **Pareto relations**") ♦

We define types of social choice situations:

DEFINITION 2: (*Social Choice Situation Types*) Let X be a set, $\underline{R} \in (\Re_{R,T})^n$ vector of individual preorders and R^{Pa} the resulting Pareto relation. Then we define the types of social choice situations (*SCS*) within table 1. ♦

[****] The translation "redistribution" for N^{Pa} presumes $N^{Pa} = N^{Pa(S)}$ witch is only guaranteed for [*C*] for R_i ∀ *i*.

[††††] To be precise we mean no social choice problems caused by social or individual Pareto incompleteness. There can still be a social choice problem due to an non-compactness (def. 1, p. 7) of the economy (cf. *SCP*2 in tab. 1, p. 13) possible for *P*5 or *P*6 which implies the existence of an endless non-maximized inefficient preference chain ($\ldots x_3 P^{Pa} x_2 P^{Pa} x_1 P^{Pa} x_0$) according to LEMMA 1 (p. 9). We only presume *welfarism* (cf., e.g., Boadway and Bruce 1984, p. 5), if we presume *individual welfarism* represented by the R_i's. Cf. Sen (1982, p. …) for different interpretations of the R_i's.

[‡‡‡‡] *P*7 represents *unimportant* social choice and *P*8 *absent* social choice.

Table 1: Types of social choice situations

Social choice situations				***Potential Pareto cases***
	Social choice problems (SCP)			
		SCP1	***border-incompleteness* problem (exclusive)**	***P1 – P4***
			$N^{Pa} \cap M(X, R^{Pa})^2 \neq \varnothing \wedge M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa})$	
		SCP2	***non-compactness* problem (exclusive)**	***P1, P2, P5, P6***
			$N^{Pa} \cap M(X, R^{Pa})^2 = \varnothing \wedge \neg M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa}) \wedge X \neq \varnothing$	
		SCP3	***combined* problem (border-incomplete and not compact)**	***P1, P2***
			$N^{Pa} \cap M(X, R^{Pa})^2 \neq \varnothing \wedge \neg M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa})$	
	Pareto-bests existence (SCB) ***(no SCP)***			(*cf. def. 1*)
		SCB	***Pareto-best* social state(s) existence** [§§§§]	***P1 – P8*** **(*all*)**
			$N^{Pa} \cap M(X, R^{Pa})^2 = \varnothing \wedge M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa})$	
Remarks				
	1.	[*T*] for R^{Pa} $\Rightarrow$ (SCB $\Leftrightarrow_{(25)}$ $C(X, R^{Pa}) \neq \varnothing$)		
	2.	[*C*] for $R_i\ \forall\ i \Rightarrow N^{Pa} = N^{Pa(Social)}$, then you can interpret "border-incompleteness problem" as "***border-redistribution* problem**" within SCP1–3-definitions		
	3.	The ***non-compactness problem*** is defined as X not compact (cf. [1]) $\Leftrightarrow_{(17)} \neg M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa}) \wedge X \neq \varnothing$		
	4.	$M(X, R^{Pa})^2 := M(X, R^{Pa}) \times M(X, R^{Pa})$		

The notion "non-compactness problem" can be motivated by the negation of the economic restriction [20] that the whole set of social states is compact with respect to a given Pareto relation R^{Pa}:

Restriction *FX* :⇔ X compact ("***compact economy***") [20]

We can translate in our main result:

Corollary 2: **(*SCS-descriptions*)** (to Proposition 2). *Let X be a set,* $\underline{R} \in (\mathfrak{R}_{R,T})^n$ *vector of individual preoders on X and* $R^{Pa} = R^{Pa}(\underline{R})$ *the corresponding Pareto relation:*

$$FX \Leftrightarrow \textbf{SCB} \veebar \textbf{SCP1} \veebar X = \varnothing \Leftrightarrow M^{\circ\circ}(X, R^{Pa}) = \varnothing \quad (21)$$ [*****]

$$\neg FX \Leftrightarrow \textbf{SCP2} \veebar \textbf{SCP3} \Leftrightarrow M^{\circ\circ}(X, R^{Pa}) \neq \varnothing \quad (22)$$

$$\textbf{SCB} \Leftrightarrow C(X, R^{Pa}) \neq \varnothing \quad (23)$$

[§§§§] This verbal translation of the following logical condition presumes [*T*] for R^{Pa} according to (25). The axiom of choice from set theory may be used if we have more than one Pareto-best (i.e., "at least as good") social state (i.e., card($C(X, R^{Pa})$) ≥ 1) to ensure that this unessential choice can actually be made.

[*****] Hence, $FX \Leftrightarrow M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa}) \veebar X = \varnothing$ (cf. Proposition 2).

In order to note this also explicitly:

$$\mathbf{SCB} \;\underline{\vee}\; \mathbf{SCP1} \quad \Leftrightarrow\; M(X, R^{Pa})\; P^{LPa}\; X \setminus M(X, R^{Pa}) \qquad (\Leftrightarrow\; M^{\circ\circ}(X, R^{Pa}) = \emptyset \wedge X \neq \emptyset)^{\dagger\dagger\dagger\dagger\dagger} \quad (24)$$

<u>Proof</u>: see appendix A5.

If and only if there exists an element in X that is neither maximal nor maximized (cf. (22): $M^{\circ\circ}(X, R^{Pa}) \neq \emptyset$) then we have also (SCP3) or exclusively (SCP2) an non-compactness problem of the economy. This is equivalent with the case where we already have a whole chain of such elements according to LEMMA 1 (or PROPOSITION 2). If not (cf. (21)) then the economy is compact, either trivially ($X = \emptyset$) or not (SCB $\underline{\vee}$ SCP1). The natural translation of the absence of a social choice problem is the existence of one of many (Pareto-indifferent) social states for $X \neq \emptyset$ (SCB).

††††† Here, $M(X, R^{Pa})\; P^{LPa}\; X \setminus M(X, R^{Pa})$ implies $X \neq \emptyset$.

4 Appendix

A1 Proof of *compact*-criteria with R^{LPa} (9)

$$\forall\, x \in Y \setminus M(Y,R^{Pa})\ \exists\, y \in M(Y,R^{Pa}) : y\, P^{Pa}\, x \quad \Leftrightarrow \quad M(Y,R^{Pa})\ R^{LPa}\ Y \setminus M(Y,R^{Pa}) \qquad \text{“(9)”}$$

Proof of (9): to “$\Rightarrow$”: obvious: to “$\Leftarrow$”: Presume: $M(Y,R^{Pa})\ R^{LPa}\ Y \setminus M(Y,R^{Pa})\ {}_{def}\Leftrightarrow \forall\, y \in Y \setminus M(Y,R^{Pa})$ $\exists\, x \in M(Y,R^{Pa}) : x\, R^{Pa}\, y$ (*). Ass.: $\exists\, y_0 \in Y \setminus M(Y,R^{Pa}) : \neg\, x\, P^{Pa}\, y_0\ \forall\, x \in M(Y,R^{Pa})$ (**). By *) $\Rightarrow \exists\, x_0 \in M(Y,R^{Pa}) : x_0\, R^{Pa}\, y_0\ {}_{**)}\Rightarrow x_0\, I^{Pa}\, y_0 \Rightarrow y_0 \in M(Y,R^{Pa}) \Rightarrow$ 7 to $y_0 \in Y \setminus M(Y,R^{Pa})$. $\Rightarrow \forall\, y \in Y\setminus M(Y,R^{Pa})$ $\exists\, x \in M(Y,R^{Pa})\ x\, P^{Pa} y$. □

A2 Proof of *not-compact* criteria (14) of LEMMA 1 (chain-criteria)

Proof of (14): “$\Leftarrow$”: obvious inclusion. “$\Rightarrow$” (a) [T] for $R \Rightarrow$ [T] for P. Presume [$\exists\, x_0 \in Y \setminus M(Y,R)$ (*) : $\forall\, y \in M(Y,R) : \neg\, y\, P\, x_0$] (**). Complete Induction: $i = 1$ (x_1 exists): to show: [$\exists\, x_1 \in Y \setminus M(Y,R)$: $x_1 P x_0\ \wedge \neg\, y\, P\, x_1\ \forall\, y \in M(Y,R)$]. Accord. to *) $x_0 \notin M(Y,R) \Rightarrow \exists\, w \in Y : w\, P\, x_0$. Ass.: $w \in M(Y,R)$: ${}_{def}\Rightarrow \neg\, x\, P\, w\ \forall\, x \in Y \Rightarrow \neg\, x_0\, P\, w \Rightarrow$ 7 to **). $\Rightarrow w \in Y \setminus M(Y,R)$: Then define $x_1 := w \Rightarrow x_1\, P\, x_0 \wedge x_1 \in Y\setminus M(Y,R)$. Ass.: $\exists\, y \in M(Y,R)$ with $y\, P\, x_1 \Rightarrow y\, P\, x_1 \wedge x_1\, P\, x_0\ {}_{(a)}\Rightarrow y\, P\, x_0$ 7$\Rightarrow$ to **) $\Rightarrow \exists\ x_1 \in Y \setminus M(Y,R)$ with $x_1\, P\, x_0 \wedge\ \neg\, y\, P\, x_1\ \forall\, y \in M(Y,R)$ $i \Rightarrow i + 1$ (if x_i exists then also $x_{i+1}\ \forall\, i \geq 1$): Presume: [$\exists\, x_1, x_2, \ldots, x_i \in Y \setminus M(Y,R) : x_1 P x_0, x_2 P x_1, \ldots, x_i P x_{i-1} \wedge \neg\, y\, P\, x_i\ \ \forall\, y \in M(Y,R)$ for $i = 1, 2, \ldots, i$] (***). to show [$\exists\, x_{i+1} \in Y \setminus M(Y,R) : x_{i+1} P\, x_i \wedge \neg\, y\, P\, x_{i+1}\ \ \forall\, y \in M(Y,R)$]. Accord. to ***) $x_i \notin M(Y,R) \Rightarrow \exists\, w \in Y : w\, P\, x_i$ (****) Ass.: $w \in M(Y,R)$: ${}_{def}\Rightarrow \neg\, x_i\, P\, w\ \forall\, w \in Y \Rightarrow \neg\, x_i\, P\, w \Rightarrow$ 7 to ***) $\Rightarrow w \in Y \setminus M(Y,R)$. Then define $x_{i+1} := w\ {}_{****)}\Rightarrow x_{i+1} P\, x_i \wedge x_{i+1} \in Y \setminus M(Y,R)$. Ass.: $\exists\, y \in M(Y,R)$ with $y\, P\, x_{i+1} \Rightarrow y\, P\, x_{i+1} \wedge x_{i+1} P\, x_i\ {}_{(a)}\Rightarrow y\, P\, x_i \Rightarrow$ 7 to ***) $\Rightarrow \exists\ x_{i+1} \in Y \setminus M(Y,R)$ with $x_{i+1}\, P\, x_i\ \wedge \neg\, y\, P\, x_1\ \forall\, y \in M(Y,R)$. □

A3 Proof of *compact*-criteria (15) with P^{LPa} in LEMMA 1

Proof of (15): $^{(a)}$ [T] for $R \Rightarrow$ [PI, PP]. We locally abbrev. $M := M(Y, R^{Pa})$:

“$\Rightarrow$”: Presume: Y compact $\wedge\, Y \neq \varnothing$ (*). [to show: $M\, P^{LPa}\, Y \setminus M$

${}_{(14)}\Leftrightarrow$ [$\forall\, x \in Y \setminus M\ \ \exists\, y \in M$: $y\, R^{Pa}\, x$] (°) $\wedge$ [$\exists\, y \in M : \forall\, x \in Y \setminus M : (y\, P^{Pa}\, x \veebar y\, N^{Pa}\, x)$] (°°)]

to (°): $Y \subset X$ *compact* $\Leftrightarrow_{def}\ \forall\ x \in Y \setminus M\ \ \exists\, y \in M : y\, P^{Pa}\, x$ (**) $\Rightarrow_{P\text{-}def}$ (°)

to (°°): Ass ($\neg$°°).: $\forall\, y \in M\ \ \exists\, x \in Y \setminus M : \neg\, (y\, P^{Pa}\, x \veebar\ y\, N^{Pa}\, x) \Leftrightarrow\ \ \forall\, y \in M\ \ \exists\, x \in Y \setminus M : x\, R^{Pa}\, y$ (***). Accord. to *) Y compact $\wedge\, Y \neq \varnothing\ {}_{(10)}\Rightarrow M \neq \varnothing \Rightarrow_{M\text{-def}} \exists\, y_0 \in M$ ***) $\Rightarrow \exists\, x_0 \in Y \setminus M : x_0\, R^{Pa}$ y_0 (****) $\Rightarrow_{M\text{-def}}\ \exists\, z_0 \in Y\ : z_0\, P^{Pa}\, x_0$ ****) $\Rightarrow z_0\, P^{Pa}\, x_0 \wedge x_0\, R^{Pa}\, y_0 \Rightarrow$

$(z_0 P^{Pa} x_0 \wedge x_0 I^{Pa} y_0) \veebar (z_0 P^{Pa} x_0 \wedge x_0 P^{Pa} y_0)$ $_{(a)}\Rightarrow z_0 P^{Pa} y_0 \Rightarrow y_0 \notin M \Rightarrow 7$ to ***) $\Rightarrow$ (°°).

<u>"$\Leftarrow$"</u>: <u>Presume</u>: $M\ P^{LPa}\ Y \setminus M$ (i.e., °/°°). [<u>to show</u>: a) $\forall\ x \in Y \setminus M\ \ \exists\, y \in M : y\ P^{Pa}\ x \wedge$ b) $Y \neq \varnothing$]

to a): Ass.: $\exists\, x \in Y \setminus M : \neg\, y\ P^{Pa}\ x\ \forall\ y \in M \ \Leftrightarrow \exists\, x \in Y \setminus M\ :\ x\ P^{Pa}\ y \veebar\ x\ N^{Pa}\ y \veebar x\ I^{Pa}\ y\ \forall\ y \in M$

°) $\Rightarrow \exists\, y \in M$ (***) : $x\ I^{Pa}\ y$ (****). $x \in Y \setminus M \Rightarrow_{M\text{-}def} \exists\, z \in Y : z\ P^{Pa}\ x$ ****) $\Rightarrow z\ P^{Pa}\ x \wedge x\ I^{Pa}\ y$ $_{(a)}\Rightarrow$ $z\ P^{Pa}\ x$ for $y \in Y \Rightarrow_{M\text{-}def}\ x \notin M \Rightarrow 7$ to ***) $\Rightarrow Y$ compact. to b): °°) $\Rightarrow \exists\, y \in M \Rightarrow_{M\subset Y} Y \neq \varnothing$. □

A4 Proof of *not-compact* translation (17) in COROLLAY 1

<u>Proof of</u> <u>(17)</u>: Presumed: [T] for R^{Pa}. <u>to "$\Rightarrow$"</u>: presume: Y not compact (*) $_{(11)} \Rightarrow Y \neq \varnothing$ (**). Ass.: $M(Y, R^{Pa})\ P^{LPa}\ Y \setminus M(Y, R^{Pa})$ $_{P\text{-}def}\Rightarrow$ $M(Y, R^{Pa})\ R^{LPa}\ Y \setminus M(Y,R^{Pa})$ $_{(9)} \Leftrightarrow \forall\ x \in Y \setminus M(Y,R^{Pa})\ \exists\ y \in M(Y,R^{Pa}) : y\ P^{Pa}\ x$ $_{def}\Leftrightarrow$ *Y compact* $\Rightarrow 7$ to *) $\Rightarrow \neg\ M(Y, R^{Pa})\ R^{LPa}\ Y \setminus M(Y,R^{Pa})$ $_{**)}\Rightarrow Y \neq \varnothing \wedge \neg\, M(Y, R^{Pa})\ R^{LPa}\ Y \setminus M(Y,R^{Pa})$. <u>to "$\Leftarrow$"</u>: By negation: Y not compact $\vee\ Y = \varnothing$ $_{(15)}\Leftrightarrow \neg\, M(Y, R^{Pa})\ P^{LPa}\ Y \setminus M(Y, R^{Pa})$ (*). presume: $Y \neq \varnothing \wedge np$ $_{*)}\Rightarrow nf$. □

A5 Proof of COROLLARY 2 (*SCS*-descriptions; p. 4)

<u>to (21)</u>: $FX \Leftrightarrow_{def} X$ compact $\Leftrightarrow M°°(X, R^{Pa}) = \varnothing \Leftrightarrow (M°°(X, R^{Pa}) = \varnothing \wedge X \neq \varnothing) \veebar X = \varnothing \Leftrightarrow$ $M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa}) \veebar X = \varnothing \Leftrightarrow$ SCB $\veebar$ SCP1 $\veebar$ $X = \varnothing$

<u>to (22)</u>: SCP2 $\vee$ SCP3 $\Leftrightarrow \neg\, M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa}) \wedge X \neq \varnothing$

$\Leftrightarrow \neg[M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa}) \vee X = \varnothing]$

$\Leftrightarrow \quad M°°(X, R^{Pa}) \neq \varnothing$ $(\Leftrightarrow_{(7)}\ \exists\, x \in X : \exists\, z \in X : z\ P^{Pa}\ x \wedge \neg\, z\ P^{Pa}\ x\ \forall\ z \in M(X, R^{Pa}))$

$\Leftrightarrow\ \neg\, X$ compact

$\Leftrightarrow_{def} \neg\, FX$

<u>to (23)</u>: Presumed [T] for $R_i\ \forall\ i \Rightarrow$ [T] for R^{Pa}

SCB $\Leftrightarrow N^{Pa} \cap M(X, R^{Pa})^2 = \varnothing \wedge M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa})$. Hence to show:

$$C(X, R^{Pa}) \neq \varnothing \ \Leftrightarrow\ N^{Pa} \cap M(X, R^{Pa})^2 = \varnothing \wedge M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa}),\ \text{ for } [T] \text{ for } R^{Pa} \qquad (25)$$

<u>Proof of (25)</u>: Generally presumed: [T] for $R^{Pa} \Rightarrow$ [PI,II] for R^{Pa} (*)

<u>to "$\Rightarrow$"</u>: Presumed: $\varnothing \neq C(X, R^{Pa}) = \{x \in X \mid x\ R^{Pa}\ y\ \forall y \in X\} \Rightarrow \exists\, x_0 \in X : x_0\ R^{Pa}\ y\ \forall\ y \in X$

$_{P\text{-}def}\Rightarrow \exists\, x_0 \in X : x_0\ R^{Pa}\ y \wedge \neg\, y\ P^{Pa}\ x_0\ \forall\ y \in X \ \Rightarrow \exists\, x_0 \in M(X, R^{Pa}) : x_0\ R^{Pa}\ y\ \forall\ y \in X$ (1*)

<u>Ass.</u>: $N^{Pa} \cap M(X, R^{Pa})^2 = \varnothing \Rightarrow \exists\, y_0, z_0 \in X : y_0\ N^{Pa}\ z_0$ (2*) $\wedge\ (\neg\, z\ P^{Pa}\ y_0 \wedge \neg\, z\ P^{Pa}\ z_0\ \forall\ z \in X)$ (3*)

$_{1*)}\Rightarrow x_0\ R^{Pa}\ y_0 \wedge x_0\ R^{Pa}\ z_0$ $_{3*)}\Rightarrow x_0\ I^{Pa}\ y_0 \wedge x_0\ I^{Pa}\ z_0$ $_{I\text{-}def}\Leftrightarrow\ y_0\ I^{Pa}\ x_0 \wedge x_0\ I^{Pa}\ z_0$ $_{*)}\Rightarrow y_0\ I^{Pa}\ z_0$

$\Rightarrow \neg\, y_0\ N^{Pa}\ z_0 \Rightarrow 7$ to 2*). $\Rightarrow N^{Pa} \cap M(X, R^{Pa})^2 \neq \varnothing$

Presumed $\varnothing \neq C(X, R^{Pa}) \subset X \Rightarrow X \neq \varnothing$ (4*)

<u>Ass.</u>: $\neg\, M(X, R^{Pa})\ P^{LPa}\ X \setminus M(X, R^{Pa})$ $_{4*)\,(15)} \Rightarrow X$ not compact $_{[1]} \Rightarrow$

$\exists\, w_0 \in X \setminus M(X, R^{Pa})$ (5*) : $\neg\, z\ P^{Pa}\ w_0\ \forall\ z \in M(X, R^{Pa})$ $_{1*)} \Rightarrow x_0\ I^{Pa}\ w_0 \Leftrightarrow w_0\ I^{Pa}\ x_0$ (5*)

Sub-Ass.: $\exists\, z \in X : z\, P^{Pa}\, w_0 \;_{5*)} \Rightarrow z\, P^{Pa}\, w_0 \wedge w_0\, I^{Pa}\, x_0 \;_{*)} \Rightarrow z\, P^{Pa}\, x_0 \;_{P\text{-}def} \Rightarrow \neg\, x_0\, R^{Pa}\, z \Rightarrow$ 7 to 1*).

$\Rightarrow \neg\, z\, P^{Pa}\, w_0\ \forall\, z \in X \Rightarrow w_0 \in M(X, R^{Pa}) \Rightarrow$ 7 to 5*). $\Rightarrow M(X, R^{Pa})\, P^{LPa}\, X \setminus M(X, R^{Pa})$

to "$\Leftarrow$": $N^{Pa} \cap M(X, R^{Pa})^2 = \varnothing$ (1*) $\wedge\, M(X, R^{Pa})\, P^{LPa}\, X \setminus M(X, R^{Pa})$ (2*)

Ass.: $X = \varnothing \;_{2*)} \Rightarrow \varnothing\, P^{LPa}\, \varnothing \Rightarrow$ 7 to (17). $\Rightarrow X \neq \varnothing$ (3*)

Ass.: $M(X, R^{Pa}) = \varnothing \;_{2*)} \Rightarrow \varnothing\, P^{LPa}\, X \;_{P\text{-}def}$ (4*). But $X \;_{3*)} \neq \varnothing \;_{\mathbf{(18)}} \Rightarrow X\, P^{LPa}\, \varnothing \Rightarrow \neg\, X\, P^{LPa}\, \varnothing$

$\Rightarrow$ 7 to 4*). $\Rightarrow M(X, R^{Pa}) \neq \varnothing \Rightarrow \exists\, x_0 \in M(X, R^{Pa})$ (5*) $_{1*)} \Rightarrow \neg\, x_0\, N^{Pa}\, y\ \forall\, y \in X$

$\Leftrightarrow (x_0\, P^{Pa}\, y \veebar x_0\, I^{Pa}\, y \veebar y\, P^{Pa}\, x_0)\ \forall\, y \in X \;_{5*)} \Leftrightarrow (x_0\, P^{Pa}\, y \veebar x_0\, I^{Pa}\, y)\ \forall\, y \in X$

$\Leftrightarrow x_0\, R^{Pa}\, y\ \forall\, y \in X \Leftrightarrow x_0 \in C(X, R^{Pa}) \Rightarrow C(X, R^{Pa}) \neq \varnothing$

to (24): SCB $\veebar$ SCP1 $\Leftrightarrow$ $\neg$ (SCP2 $\vee$ SCP3) $\wedge\, X \neq \varnothing$

$\Leftrightarrow$ $\neg\, [\neg\, M(X, R^{Pa})\, P^{LPa}\, X \setminus M(X, R^{Pa}) \wedge X \neq \varnothing] \wedge X \neq \varnothing$

$\Leftrightarrow$ $[M(X, R^{Pa})\, P^{LPa}\, X \setminus M(X, R^{Pa}) \vee X = \varnothing] \wedge X \neq \varnothing$

$\Leftrightarrow$ $M^{\circ\circ}(X, R^{Pa}) = \varnothing \wedge X \neq \varnothing$

$\Leftrightarrow_{(3)}$ $X = M(X, R^{Pa}) \,\dot\cup\, M^{\circ}(X, R^{Pa}) \wedge X \neq \varnothing$

$\Leftrightarrow_{[2a]}$ $X = \{x \in X \mid \neg\ z\, P^{Pa}\, x\ \forall\, z \in X\} \,\dot\cup\, \{x \in X \mid \exists\, z \in M(X, R^{Pa}) : z\, P^{Pa}\, x\} \wedge X \neq \varnothing$

$/\!\!\Leftrightarrow$ $M(X, R^{Pa})\, P^{LPa}\, X \setminus M(X, R^{Pa})$ □

5 References

Arrow, Kenneth Joseph (1963), Social Choice and Individual Values, 2. ed., New Haven, Conn., Yale University Press

Boadway, Robin W. / Bruce, Neil (1984), Welfare Economics, Oxford, England, Basil Blackwell

Feldman, Allan M. (1994), Welfare Economics and Social Choice Theory, 9. printing (1980), Boston et al., Kluwer Nijhoff

Fishburn, Peter C. (1970), Utility Theory for Decision Making, New York et al., John Wiley & Sons

Hausdorff, Felix (1914), Grundzüge der Mengenlehre, Leibzig, Von Veit & Comp.

Also published in this series:

Nr. 1	7/95	H.-G. Petersen	Economic Aspects of Agricultural Areas Management and Land/Water Ecotones Conservation
Nr. 2	7/95	H.-G. Petersen	Pros and Cons of a Negative Income Tax
Nr. 3	7/95	C. Sowada	Haushaltspolitische Konsequenzen steigender Staatsverschuldung in Polen
Nr. 4	8/95	C. Bork	Die Elektrizitätswirtschaft in der Bundesrepublik Deutschland - Das Tarifpreisgenehmigungsverfahren und seine Auswirkungen auf eine potentielle Netzübernahme nach Ablauf von Konzessions-verträgen
Nr. 5	10/95	H.-G. Petersen	Transformation Process After Five Years: Behavioral Adaptation and Institutional Change - The Polish Case
Nr. 6	11/95	C. Bork K. Müller H.-G. Petersen S. Wirths	Wider den Sachzeitwert - Untersuchung zur Frage des angemessenen Übernahmepreises von Elektrizitätsversorgungsnetzen
Nr. 7	1/96	C. Sowada	Sozialpolitik im Transformationsprozess am Beispiel Polens
Nr. 8	4/96	K. Müller T. Nagel H.-G. Petersen	Ökosteuerreform und Senkung der direkten Abgaben: Zu einer Neugestaltung des deutschen Steuer- und Transfersystems
Nr. 9	6/96	H.-P. Weikard	The Rawlsian Principles of Justice Reconsidered
Nr. 10	9/96	H.-G. Petersen	Effizienz, Gerechtigkeit und der Standort Deutschland
Nr. 11	10/96	H.-P. Weikard	Sustainable Freedom of Choice - A New Concept
Nr. 12	2/97	C. Bork K. Müller	Aufkommens- und Verteilungswirkungen einer Reform der Renten-besteuerung mit einem Kommentar von H.-P. Weikard zu Renten-besteuerung und Korrespondenzprinzip
Nr. 13	2/97	C. Bork	Ein einfaches mikroökonomisches Gruppensimulationsmodell zur Einkommensbesteuerung
Nr. 14	3/97	H.-G. Petersen	Das Neuseeland Experiment: Ist das die zukünftige Entwicklung des deutschen Sozialstaats?
Nr. 15	4/97	H.-P. Weikard	Contractarian Approaches to Intergenerational Justice
Nr. 16	8/97	H.-G. Petersen C. Bork	Schriftliche Stellungnahme zum Entwurf eines Steuerreform gesetzes (StRG) 1999 der Fraktionen CDU/CSU und F.D.P.
Nr. 17	10/97	H.-P. Weikard	Property Rights and Resource Allocation in an Overlapping Generations Modell
Nr. 18	10/97	C. Sowada	Wieviel Staat braucht der Markt und wieviel Staat braucht die Gerechtigkeit? Transformation des polnischen Sozialversiche-rungssystems im Lichte der deutschen Erfahrungen
Nr. 19	12/97	C. Bork K. Müller	Effekte der Verrechnungsmöglichkeit negativer Einkünfte im deutschen Einkommensteuerrecht
Nr. 20	01/98	C. Bork H.-G. Petersen	Ein Vergleich möglicher Datensätze zur Eignung für steuerpolitische Simulationsrechnungen
Nr. 21	02/98	S. Gabbert H.-P. Weikard	Food Deficits, Food Security and Food Aid: Concepts and Measurement
Nr. 22	01/99	H.-G. Petersen C. Bork	Finanzpolitischer Reformbedarf jenseits der Besteuerung Konsequenzen für die Aufgabenseite
Nr. 23	02/99	C. Sowada	Soziale Reformen in Polen. Zwischen Bewahrung und Neuanfang.

Nr. 24	06/99	G. Leßmann	Zur Theorie der Einstellungen zur Staatstätigkeit - Möglichkeiten und Grenzen der Erfassung -
Nr. 25	07/99	H.-G. Petersen	The German Tax and Transfer System: A Problem Oriented Overview
Nr. 26	07/99	C. Bork H.-G. Petersen	Revenue and Distributional Effects of the Current Tax Reform Proposals in Germany – An Evaluation by Microsimulation
Nr. 27	11/99	H.-G. Petersen	Arbeit organisieren – Sozialstaat erneuern
Nr. 28	11/99	U. Paschen	Die Regionalisierte Ökologische Gesamtrechnung: Mittel zur Darstellung regionaler umweltökonomischer Tatbestände -Hintergrund, Konzeption und Anwendungsmöglichkeiten im Rahmen des interdisziplinären GRANO-Projektes
Nr. 29 a	04/00	H.-G. Petersen S. Anton C. Bork C. Sowada	Gutachten im Auftrag des Ministeriums der Finanzen des Landes Brandenburg: Modernisierung der bundesstaatlichen Ordnung: Mischfinanzierungen und Gemeinschaftsaufgaben (Teil a, Text des Gutachtens)
Nr. 29 b	04/00	s. Nr. 29 a	Titel s. Nr. 29 a (Teil b, tabellarischer Anhang)
Nr. 30	04/00	H.-G. Petersen B. Raffelhüschen	Die gesetzliche und freiwillige Altersvorsorge als Element eines konsumorientierten Steuer- und Sozialsystems
Nr. 31	07/02	S. Anton M. Brehe H.-G. Petersen	Das Konzept der Einfachsteuer im empirischen Text
Nr. 32	08/02	H.-G. Petersen	The Polit-economic Situation in Germany: Chances for Changes in Resource and Energy Economics
Nr. 33	12/02	H.- G. Petersen	Fiskalischer Föderalismus als Mittel friedlicher Integration – das Beispiel Bosnien und Herzegowina
Nr. 34	01/03	H.- G. Petersen M. Rose	Zu einer Fundamentalreform der deutschen Einkommensteuer: Die Einfachsteuer des „Heidelberger Kreises“
Nr. 35	02/03	H.-G. Petersen	Soziale Gerechtigkeit und Leistungsfähigkeit in dynamischer Perspektive
Nr. 36	07/03	H.-G. Petersen	Globalisierung und soziale Gerechtigkeit
Nr. 37	08/03	A. Keser	Staatliche Belastung fabrikneuer PKW im europäischen Vergleich und Preisdiskriminierung auf dem Automobilmarkt
Nr. 38	08/03	J. Ehrke	Die Strukturfonds der EU. Eine ökonomische Einschätzung vor dem Hintergrund ihrer historischen Entwicklung
Nr. 39	12/03	H.-G. Petersen A. Fischer J. Flach	Wirkungen der Einfachsteuer auf die Steuerbelastung von Haushalten und Unternehmen
Nr. 40	12/03	J. Flach	Die Auswirkungen der Unternehmenssteuerreform auf ausgewählte Unternehmen
Nr. 41	02/04	H.-G. Petersen	Capital Flight and Capital Income Taxation
Nr. 42	03/04	H.-G. Petersen	Redistribution and the Efficiency-Justice Trade-off
Nr. 43	06/04	H.-G. Petersen	Vom Nehmen vor dem Geben: Ist der Staat als Wohltäter ein starker Staat?
Nr. 44	07/04	H.-G. Petersen	International Experience with alternative Forms of Social Protection: Lessons for the Reforms Process in Russia
Nr. 45	07/04	H.-G. Petersen	Systematic Change Instead of Curing Symptoms: Coordinating Social and Private Health Insurance in Germany and Beyond

Specials Series:

Industrial and Social Policies in Countries in Transition

No. S-1	12/97	H.-P. Weikard	Industrial Policies and Social Security: Investigating the Links
No. S-2	06/98	H.-G. Petersen C. Sowada	On the Integration of Industrial and Social Policy in the Transition Process
No. S-3	06/98	B. Czasch A. Balmann M. Odening T. Sobczak M. Switlyk	Die Umstrukturierung landwirtschaftlicher Unternehmen beim Übergang zur Marktwirtschaft unter besonderer Berücksichtigung des Faktors Arbeit
No. S-4	06/98	R. Bakardjieva C. Sowada	Soziale Sicherung in Bulgarien 1991-1997. Entwicklung - Stand - Perspektiven
No. S-5	06/98	R. Bakardjieva	Der Privatisierungsprozeß in Bulgarien - Strategien, Widersprüche und Schlußfolgerungen
No. S-6	06/98	M. Bednarski	Privatisation Policy and Industrial Policy in Poland in the Period of Transformation
No. S-7	06/98	G. D. Demopoulos E. K. Fratzeskos	Macroeconomic Developments and Problems in the Transition Process of the Bulgarian Economy
No. S-8	10/98	P. Kurowski	Scope and Forms of State Support to Enterprises in Poland in Period of Transition
No. S-9	11/98	S. Golinowska	Public Social Expenditures in Poland in the Period of Transition
No. S-10	03/99	M. Switlyk	The Economic Standing of the Partnership Companies which Lease Agricultural Real Estate from the Agricultural Property Agency of the State Treasury in Gorzów Voivodeship in 1996 and 1997
No. S-11	05/99	B. Czasch A. Balmann M. Odening	Organisation und Effizienz landwirtschaftlicher Unternehmen während der Umstrukturierung des Agrarsektors - Eine empirische Analyse für Brandenburg -
No. S-12	06/99	M. Bednarski P. Kurowski	Industrial Policy and Social Strategy at the Corporate Level in Poland: Qestionnaire Results
No. S-13	06/99	H.-G. Petersen A. Naydenov	The Tax and Social Contribution System in Bulgaria: Formal Structure and Possible Impacts
No. S-14	07/99	R. Bakardjieva C. Sowada	The Employment Crisis, Pensions and Poverty in Bulgaria 1990-1998. Trends Consequences – Preventative measures
No. S-15	07/99	R. Rusielik T. Sobczak M. Switlyk	Organisation and Efficiency of Agricultural Enterprises in Transformation: An Empirical Analysis of the Gorzów Voivodeship
No. S-16	07/99	R. Bakardjieva C. Sowada	Privatisation in Bulgaria. Strategies, Methods, Results and Conclusions
No. S-17	07/99	A. Christev H.-G. Petersen	Privatisation and Ownership: The Impact on Firms in Transition Survey Evidence from Bulgaria
No. S-18	07/99	A. Christev H.-P. Weikard	Social Benefits and the Enterprise: Some Recent Evidence from Bulgaria and Poland
No. S-19	07/99	A. Christev F. FitzRoy	Employment and Wages in Transition: Panel Evidence from Poland
No. S-20	07/99	H.-G. Petersen C. Sowada	The Polish an Bulgarian Questionnaires