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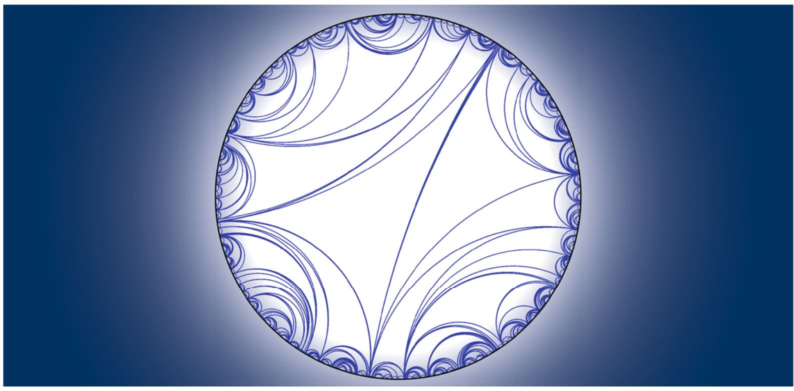

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ASYMPTOTIC EXPANSIONS AT NONSYMMETRIC CUSPIDAL POINTS

IBRAHIM LY AND NIKOLAI TARKHANOV

Abstract. We study asymptotics of solutions to the Dirichlet problem in a domain $\mathcal{X} \subset \mathbb{R}^3$ whose boundary contains a singular point O. In a small neighbourhood of this point the domain has the form $\{z > \sqrt{x^2 + y^4}\}$, i.e., the origin is a nonsymmetric conical point at the boundary. So far the behaviour of solutions to elliptic boundary value problems has not studied well in the case of nonsymmetric singular points. This problem was posed by V.A. Kondrat'ev in 2000. We establish a complete asymptotic expansion of solutions near the singular point.

Contents

Introduction

In 2000 V.A. Kondrat'ev called the attention of the second author to the problem of asymptotics of solutions of the Dirichlet problem in a bounded domain $\mathcal{X} \subset \mathbb{R}^3$ with a nonsymmetric singular point O at the boundary. By the local principle in elliptic theory the behaviour of any solution in a neighbourhood of O is completely determined by the nature of the singular point and the data of the problem nearby O. This allows one to restrict the study to a small neighbourhood U of O, in which the boundary surface does not contain any singular point different from O. One chooses a coordinate system in $\mathbb{R}^3$ with origin at O, and so the domain in question looks like $\{z > \sqrt{x^2 + y^4}\}$.

The behaviour of solutions to elliptic boundary value problems in domains with conical points at the boundary was completely described in the seminal paper [Kon67]. However, the singular point of $\{z > \sqrt{x^2 + y^4}\}$ goes beyond the class of singular points treated in [Kon67]. The plane $\{y = 0\}$ meets $\mathcal{X}$ over the corner $\{z > |x|\}$ while the intersection of $\mathcal{X}$ with any plane $\{y = c\}$, $|c| \ll 1$, is a smoothly

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bounded domain. Hence, on thinking of the intersection of $\partial\mathcal{X}$ with the plane $\{x = 0\}$ as artificial smooth edge at the boundary surface we don't achieve any smooth cone bundle structure of $\mathcal{X}$ close to the origin. This clarifies the surprise at this problem.

In [RST01] a simple change of variables is shown which reduces the Dirichlet problem in $\mathcal{X}$ to analysis of pseudodifferential operators with slowly varying symbols in domains with cuspidal points at the boundary. Within the framework of calculus of such operators elaborated in [RST00] it is possible to give certain criteria in terms of operator-valued symbols for the Fredholm property of the Dirichlet problem in appropriate weighted Sobolev spaces. However, the theory of [RST00] falls short of providing explicit asymptotic expansions of solutions in a neighbourhood of the singular point.

The present paper is aimed at constructing asymptotic expansions of solutions close to the singular point, thus completing [RST01]. To do this we first localise the problem to a neighbourhood of the singular point O to see that it degenerates at O to a nonelliptic problem. Hence it follows that the techniques of constructing formal solutions to the boundary value problem developed in [AKT14] do not apply to suggest any crude solution. Although the problem reduces to an ordinary differential equations $\dot{U} = A(t)U + F$ with operator-valued coefficients independent of t up to a separate interfering factor e^{2t}, the long-standing results of [Paz67] do not lead to a satisfactory solution, for the limit problem related to $A(-\infty)$ is quite sophisticated.

The approach we take is an extension of [PT03]. It is based on a structure theorem for solutions with a compact set of singularities for elliptic equations with real analytic coefficients, see [Tar95, Ch. 3]. For an efficient use of this theorem one ought to have merely a fundamental solution of the differential equation, which is often the case. The approach can be extended to boundary value problems both for parabolic and hyperbolic equations. However, this topic exceeds the scope of this paper.

1. Reduction of the Dirichlet problem

Let $\mathcal{X}$ be a bounded domain in $\mathbb{R}^3$ whose boundary is smooth except for a singular point O which we identify with the origin of $\mathbb{R}^3$. In a neighbourhood U of O the domain $\mathcal{X}$ has the form $\{z > \sqrt{x^2 + y^4}\}$. It is easy to see that the boundary of $\mathcal{X}$ is a Lipschitz surface.

Given a distribution $f \in H^{-1}(\mathbb{R}^3)$ with support in the closure of $\mathcal{X}$ and a function $u_0 \in H^{1/2}(\partial\mathcal{X})$ at the boundary of $\mathcal{X}$, there is a unique function $u \in H^1(\mathcal{X})$ satisfying

$$\left\{ \begin{array}{rclcl} \Delta u & = & f & \text{in} & \mathcal{X}, \\ u & = & u_0 & \text{at} & \partial\mathcal{X}, \end{array} \right. \tag{1.1}$$

see for instance [Agr13]. By the equality $u = u_0$ at $\partial\mathcal{X}$ is meant that the trace of u at the boundary in the sense of Sobolev spaces coincides with u_0. However, this no longer holds if the data f and u_0 fail to be as regular as above. In particular, there are infinitely many linearly independent harmonic functions u in $\mathcal{X}$ of finite order of growth near the boundary whose weak limit values vanish at $\partial\mathcal{X} \setminus \{O\}$, cf. [PT03].

Pick an open cover U_1 and U_2 of the closure of $\mathcal{X}$ in $\mathbb{R}^3$, such that $U_1 = U$ and U_2 is bounded away from the origin. Let $\varphi_1 = \varphi$ and $\varphi_2 = 1 - \varphi$ be a C^∞

partition of unity on $U_1 \cup U_2$ subordinate to this cover. Then φ is equal to 1 in a neighbourhood of $U \setminus U_2$. Choose any subdomain $\mathcal{X}'$ of $\mathcal{X}$ with smooth boundary, such that $\overline{\mathcal{X}} \setminus \mathcal{X}'$ is a compact subset of $U \setminus \overline{U}_2$. Consider the Dirichlet problem in $\mathcal{X}'$ with data $\varphi_2 f$ and $\varphi_2 u_0$ in $\mathcal{X}'$ and at $\partial\mathcal{X}'$, respectively. This problem has a unique solution $u_2 = G(\varphi_2 f) + P(\varphi_2 u_0)$ in compliance with elliptic regularity in Sobolev spaces in $\mathcal{X}'$. Let ψ_2 be a smooth function in $U_1 \cup U_2$, such that ψ_2 vanishes in a neighbourhood of $\overline{\mathcal{X}} \setminus \mathcal{X}'$ and $\psi_2 \equiv 1$ on the support of φ_2. The product $\psi_2 u_2$ is well defined in all of $\mathcal{X}$ and it satisfies $\Delta(\psi_2 u_2) = f$ in a neighbourhood of $\mathcal{X} \setminus U$ and $\psi_2 u_2 = u_0$ in a neighbourhood of $\partial\mathcal{X} \setminus U$. If now u is any solution of problem (1.1), then the difference $u - \psi_2 u_2$ satisfies $\Delta(u - \psi_2 u_2) = 0$ in a neighbourhood of $\mathcal{X} \setminus U$ and $u - \psi_2 u_2 = 0$ in a neighbourhood of $\partial\mathcal{X} \setminus U$. By the local elliptic regularity we conclude that $u - \psi_2 u_2$ is smooth in $\mathcal{X} \setminus U$ up to the boundary. Hence, to describe the behaviour of solutions of the Dirichlet problem near the singular point O, it suffices to solve (1.1) in $\mathcal{X} \cap U$ with data f and u compactly supported in $\overline{\mathcal{X}} \cap U$ and $\partial\mathcal{X} \cap U$, respectively.

To this end we introduce new coordinates in the upper half-space $\{z > 0\}$ by the formulas

$$\left\{ \begin{array}{rcl} x & = & r^2 x_1, \\ y & = & r x_2, \\ z & = & r^2, \end{array} \right. \tag{1.2}$$

where $(x_1, x_2) \in \mathbb{R}^2$ and $r > 0$. The determinant of the Jacobi matrix of (1.2) is $2r^4$, hence the coordinates are singular at the hyperplane $\{r = 0\}$. Under (1.2) this hyperplane is blown-down to the origin O.

On using the chain rule we get

$$\begin{array}{rcl} \dfrac{\partial}{\partial x} & = & \dfrac{1}{r^2} \dfrac{\partial}{\partial x_1}, \\ \dfrac{\partial}{\partial y} & = & \dfrac{1}{r} \dfrac{\partial}{\partial x_2}, \\ \dfrac{\partial}{\partial z} & = & \dfrac{1}{2r^2} \Big(r \dfrac{\partial}{\partial r} - 2x_1 \dfrac{\partial}{\partial x_1} - x_2 \dfrac{\partial}{\partial x_2} \Big), \end{array}$$

and so the pullback of the Laplace operator $c^\sharp \Delta = c^* \Delta (c^*)^{-1}$ under the change of variables (1.2) takes the form

$$c^\sharp \Delta = \frac{1}{4r^4} \Big((r\partial_r)^2 - 2\,(E+1)\,(r\partial_r) + 4\partial_1^2 + 4r^2 \partial_2^2 + E\,(E+2) \Big),$$

where

$$E = 2x_1 \partial_1 + x_2 \partial_2.$$

Note that the Fuchs derivative $r\partial_r$ is typical for analysis on spaces with conic singularities.

By abuse of notation, we continue to write u for the pullback $c^* u$ under the change of variables (1.2), and similarly for the data f and u_0. Then problem (1.1) reduces to

$$\left\{ \begin{array}{rcll} c^\sharp \Delta u & = & f & \text{in} \quad D \times (0,1), \\ u & = & u_0 & \text{at} \quad \partial D \times (0,1), \end{array} \right.$$

where D is the domain in the plane of variables x_1 and x_2 consisting of those points (x_1, x_2) which satisfy $x_1^2 + x_2^4 < 1$. The data f and u_0 are assumed to vanish for r close to $r = 1$. Thus, after blowing up the singularity at O we arrive at a boundary value problem in the cylinder $D \times (0,1)$ with no data posed at the base $D \times \{0\}$ of the cylinder.

Our next goal is to transform the scalar equation $c^\sharp \Delta u = f$ to a system of two equations of the first order in the Fuchs derivative. To this end we introduce the function

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with values in $\mathbb{R}^2$, where $u_1 = u$ and $u_2 = (r\partial_r)u_1$. For U we obtain immediately the boundary value problem

$$\left\{ \begin{array}{rcll} (r\partial_r)U & = & A(r)\,U + F & \text{in} \quad D \times (0,1), \\ BU & = & U_0 & \text{at} \quad \partial D \times (0,1), \end{array} \right. \tag{1.3}$$

where

$$A(r) = \begin{pmatrix} 0 & 1 \\ -4\partial_1^2 - 4r^2\partial_2^2 - E(E{+}2) & 2(E{+}1) \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $F = \begin{pmatrix} 0 \\ 4r^4 f \end{pmatrix}$ and $U_0 = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}$.

2. Specification within edge calculus

Problem (1.1) can be also handled within the framework of analysis on spaces with edges. To do this, we think of the curve $z = y^2$ in the plane of variables (y, z) as artificial edge at the boundary of $\mathcal{X}$ lying in U, and give $\mathcal{X} \cap U$ the structure of (stretched) cone bundle over the edge. In our case the only singular point of the edge is the origin O.

To wit, we introduce the new coordinates

$$\begin{array}{rcl} x_1 & = & y, \\ x_2 & = & \dfrac{x}{\sqrt{z^2 - y^4}}, \\ r & = & \sqrt{z^2 - y^4}, \end{array}$$

so that x_1 is a coordinate along the edge, $x_2 \in (-1, 1)$ a coordinate along the base of the cylinder which is actually the fibre cone blown up at the vertex, and $r > 0$ is a coordinate along the axis of the cylinder which is the distance to the edge. The inverse transformation is

$$\begin{array}{rcl} x & = & rx_2, \\ y & = & x_1, \\ z & = & \sqrt{x_1^4 + r^2}, \end{array} \tag{2.1}$$

whence

$$\begin{array}{rcl} \dfrac{\partial}{\partial x} & = & \dfrac{1}{r} \dfrac{\partial}{\partial x_2}, \\ \dfrac{\partial}{\partial y} & = & \dfrac{\partial}{\partial x_1} + \dfrac{2x_1^3 x_2}{r^2} \dfrac{\partial}{\partial x_2} - \dfrac{2x_1^3}{r} \dfrac{\partial}{\partial r}, \\ \dfrac{\partial}{\partial z} & = & -\dfrac{\sqrt{x_1^4 + r^2}\, x_2}{r^2} \dfrac{\partial}{\partial x_2} + \dfrac{\sqrt{x_1^4 + r^2}}{r} \dfrac{\partial}{\partial r}. \end{array}$$

A routine computation now shows that the pullback of the Laplace operator under the change of variables (2.1) is

$$\begin{array}{rcl} c^\sharp \Delta & = & \Big(\dfrac{\partial}{\partial x_1}\Big)^2 + \dfrac{4x_1^3 x_2}{r^2}\,\dfrac{\partial}{\partial x_1}\dfrac{\partial}{\partial x_2} + \Big(\dfrac{1}{r^2} + \dfrac{(4x_1^6 + x_1^4 + r^2)x_2^2}{r^4}\Big)\Big(\dfrac{\partial}{\partial x_2}\Big)^2 \\ & - & \dfrac{4x_1^3}{r}\,\dfrac{\partial}{\partial x_1}\dfrac{\partial}{\partial r} - \dfrac{2(4x_1^6 + x_1^4 + r^2)x_2}{r^3}\,\dfrac{\partial}{\partial x_2}\dfrac{\partial}{\partial r} + \dfrac{4x_1^6 + x_1^4 + r^2}{r^2}\,\Big(\dfrac{\partial}{\partial r}\Big)^2 \end{array}$$

up to terms containing the derivatives of order less than or equal to one in x_1, x_2 and r.

On comparing the pullbacks of the Laplace operator under transformations (1.2) and (2.1) we deduce that the first of the two suits better for analysis than the second one. Explicit results seem thus to depend on the good fortune of researcher in discovering proper cordinates in a neighbourhood of the singular point 0 to interpret the singularity.

3. Formal solutions

In order to set up an adequate functional theoretical approach to problem (1.3) we specify the formal solutions to this problem. Since (1.3) is of Fuchs type, analysis on manifolds with conical singularities suggests an Ansatz for solutions. To wit, this is

$$U(x,r) = \sum_{j=1}^{\infty} \sum_{k=0}^{m_j-1} U_{j,k}(x)\, r^{\lambda_j} (\log r)^k, \tag{3.1}$$

where λ_j are complex numbers satisfying $\Re\lambda_1 \leq \Re\lambda_2 \leq \ldots$, and m_j are natural numbers to be determined. The coefficients $U_{j,k}$ are assumed to be sufficiently smooth functions in the closure of D. For definiteness, we assume that the coefficient U_{1,m_0-1} is different from zero. The advantage of using this Ansatz lies in the fact that

$$(r\partial_r)\left(r^{\lambda_j} (\log r)^k\right) = r^{\lambda_j}\left(\lambda_j (\log r)^k + k(\log r)^{k-1}\right).$$

A routine computation shows that

$$\begin{array}{rcl} (r\partial_r)U & = & \displaystyle\sum_{j=1}^{\infty}\sum_{k=0}^{m_j-1} (\lambda_j U_{j,k} + (k+1)U_{j,k+1})\, r^{\lambda_j} (\log r)^k, \\ A(r)\,U & = & \displaystyle\sum_{j=1}^{\infty}\sum_{k=0}^{m_j-1} A(0)U_{j,k}\, r^{\lambda_j} (\log r)^k + \sum_{j=1}^{\infty}\sum_{k=0}^{m_j-1} (A(1)-A(0))U_{j,k}\, r^{\lambda_j+2} (\log r)^k, \end{array}$$

for the matrix $A(r) = A(0) + r^2(A(1)-A(0))$ depends polynomially on r. On substituting the series for $U(x,r)$ into the boundary value problem (1.3) and equating the coefficients of the same terms $r^{\lambda_1}(\log r)^k$ we obtain a collection of successive equations

$$\left\{ \begin{array}{rcll} kU_{1,k} & = & (A(0)-\lambda_1 I)\, U_{1,k-1} + F_{1,k-1} & \text{in} \quad D, \\ BU_{1,k-1} & = & U_{0;1,k-1} & \text{at} \quad \partial D \end{array} \right. \tag{3.2}$$

for $k = 1, \ldots, m_1$, where $V_{1,m_1} := 0$.

By $F_{1,k}$ and $U_{0;1,k}$ are meant the coefficients of $r^{\lambda_1}(\log r)^k$ in the expansions of F and U_0, respectively.

We restrict our attention to formal solutions of the homogeneous boundary value problems, for the data f and u_0 may initiate their own "frequencies" in the solution.

In this case equations (3.2) just amount to saying that U_{1,m_1-1} is an eigenfunction of the operator $A(0)$ corresponding to the eigenvalue λ_1 of algebraic multiplicity m_1, and

$$\frac{k!}{(m_1-1)!}\,U_{1,k}$$

for $k = 0, 1, \ldots, m_1-1$ is the corresponding chain of associated functions. Moreover, each $U_{1,k}$ satisfies the boundary condition $BU_{1,k} = 0$.

Pick any eigenvalue λ_1 of algebraic multiplicity m_1 for the differential operator $A(0)$ in D subject to the boundary condition $BU = 0$. This enables us to find smooth functions $U_{1,k}$ for $k = 0, \ldots, m_1 - 1$, such that U is a solution to (1.3) up to terms of order

$$O(r^{\lambda_1+2}(\log r)^{m_1-1}).$$

Our next objective is to improve this solution U by introducing amendments into formula (3.1). If there exist different eigenvalues λ_j of $A(0)$ subject to B, which satisfy $\Re\lambda_1 \leq \Re\lambda_j \leq \Re\lambda_1 + 2$, then arguing as above we evaluate several new terms in expansion (3.1). However, the discrepancy of the formal solution obtained in this way still remains to be of order $O(r^{\lambda_1+2}(\log r)^{m_1-1})$. To reduce it, we go beyond the spectrum of $A(0)$.

Set $\lambda_2 := \lambda_1 + 2$ and $m_2 := m_1$. On substituting the series for $U(x, r)$ into the boundary value problem (1.3) and equating the coefficients of the same terms $r^{\lambda_2}(\log r)^k$ we obtain a collection of successive equations

$$\left\{ \begin{array}{rcll} kU_{2,k} & = & (A(0)-\lambda_2 I)\, U_{2,k-1} + (A(1)-A(0))\, U_{2,k-1} + F_{2,k-1} & \text{in} \quad D, \\ BU_{2,k-1} & = & U_{0;2,k-1} & \text{at} \quad \partial D \end{array} \right. \tag{3.3}$$

for $k = 1, \ldots, m_2$, where $U_{2,m_2} := 0$.

Even if the data f and u_0 are zero, the equation for U_{2,m_2-1} is inhomogeneous and so it possesses a nonzero solution, provided that λ_2 is away from the spectrum of the differential operator $A(0)$ subject to B. Under this choice of λ_2 the formal series U satisfies (1.3) up to terms of order $O(r^{\lambda_2+2}(\log r)^{m_2-1})$. If λ_2 is an eigenvalue of $A(0)$ subject to B, the question of solvability of equations (3.3) requires a careful examination, cf. [AKT14].

The analysis of formal solutions displays strikingly the main difficulty which occurs in the study of asymptotic expansions at nonsymmetric singularities. On passing to appropriate coordinates one obtains a problem with small parameter $r > 0$ which degenerates at $r = 0$ to a nonelliptic problem. Another way of stating this is to say that the degeneration is nonregular, i.e., it can no longer be treated within elliptic theory. To wit,

$$A(0) = \begin{pmatrix} 0 & 1 \\ -4\partial_1^2 - (E+1)^2 + 1 & 2(E+1) \end{pmatrix}$$

and (1.3) reduces to the boundary value problem

$$\left\{ \begin{array}{rcll} \big(4\partial_1^2 + E(E+2)\big)\, u & = & f & \text{in} \quad D, \\ u & = & u_0 & \text{at} \quad \partial D, \end{array} \right. \tag{3.4}$$

in the domain $D \subset \mathbb{R}^2$ consisting of those (x_1, x_2) which satisfy $x_1^2 + x_2^4 < 1$. This is the Dirichlet problem in D for a second order partial differential equation whose symbol vanishes with multiplicity 2 on the submanifold $\{x_2 = 0,\ \xi_1 = 0\}$ of the cotangent bundle $T^*\mathbb{R}^2 \cong \mathbb{R}^4$. This fits well into the analysis of the problem in the domain $\mathcal{X}$ containing the artificial edge $\{x = 0,\ z = y^2\}$ at the boundary, cf. Section 2. Perhaps the nonstandard problem (3.4) can be treated explicitly in detail, however, it gives rather an evidence to the inadequacy of the techniques chosen for studying asymptotics at nonsymmetric conical points, let alone singular points of more involved nature. This is just a good motivation for our approach which exploits a sophisticated structure formula for harmonic functions with arbitrary compact set of singularities, see Theorem 3.2.1 in [Tar95]. The approach applies not only to harmonic functions but also to solutions of arbitrary elliptic equations with real analytic coefficients in a neighbourhood of the closure of $\mathcal{X}$. In order to get asymptotic result, it is necessary to get a fundamental solution of the partial differential equation in an explicit form, which is obviously the case for the Laplace equation.

4. A structure theorem

In order to apply the results of [Tar95, Ch. 3] to problem (1.1) we intend to reduce it to a boundary value problem for solutions of a homogeneous elliptic equation, to wit, to the Dirichlet problem for harmonic functions. We do this in a standard manner.

Let $f \in H^{-1}(\mathcal{X})$. Since

$$H^{-1}(\mathcal{X}) = (H^1(\mathcal{X}))' \cong \Big(\frac{H^1(\mathbb{R}^n)}{\overline{C^\infty_{\mathrm{comp}}(\mathbb{R}^n \setminus \overline{\mathcal{X}})}} \Big)' \cong \big(C^\infty_{\mathrm{comp}}(\mathbb{R}^n \setminus \overline{\mathcal{X}}) \big)^o = H^{-1}_{\overline{\mathcal{X}}}(\mathbb{R}^n),$$

the subspace of $H^{-1}(\mathbb{R}^n)$ consisting of all distributions with support in $\overline{\mathcal{X}}$, we can think of f as a distribution in $H^{-1}(\mathbb{R}^n)$ supported in $\overline{\mathcal{X}}$. Write Φ for the standard fundamental solution of convolution type of the Laplace operator in $\mathbb{R}^n$. For $n \neq 2$, we get

$$\Phi(x) = \frac{1}{\sigma_n} \frac{|x|^{2-n}}{2-n},$$

where σ_n is the surface area of the unit sphere in $\mathbb{R}^n$, and $\Phi(x) = (2\pi)^{-1} \log |x|$, if $n = 2$. The convolution $\Phi * f$ is known to belong to $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfy $\Delta(\Phi * f) = f$ in $\mathbb{R}^n$. Hence, if one looks for a solution of problem (1.1) of the form $u = v + \Phi * f$ in $\mathcal{X}$, then the problem reduces to

$$\left\{ \begin{array}{rcll} \Delta v & = & 0 & \text{in} \quad \mathcal{X}, \\ v & = & u_0 - \Phi * f & \text{at} \quad \partial\mathcal{X}, \end{array} \right.$$

the limit values of the solution at the boundary are required to constitute a function in $H^{1/2}(\partial\mathcal{X})$.

Thus, the contribution of the right hand side f into the solution u is localised in the term $\Phi * f$. Pick any C^∞ function φ with compact support in a small neighbourhood U of the singular point O, which is equal to one close to O. Then, the potential $\Phi * ((1-\varphi)f)$ is a harmonic function in a neighbourhood of O and the singular behaviour of $\Phi * f$ near O is encoded in the summand $\Phi * (\varphi f)$ to be handled independently.

We restrict our attention to the study of singularities of v near O. To this end, we note that $\mathcal{X}$ is a Lipschitz domain and so its boundary is a rectifiable surface. The surface $\partial\mathcal{X}$ is a regular compact subset of $\mathbb{R}^3$ and the induced Lebesgue measure ds on $\partial\mathcal{X}$ is massive, see [Tar95, S. 3.1]. The function V in $\mathbb{R}^3 \setminus \partial\mathcal{X}$, which is equal to v in $\mathcal{X}$ and to 0 in the complement of $\overline{\mathcal{X}}$, is harmonic away from the boundary of $\mathcal{X}$. On applying Corollary 3.2.11 of [Tar95] to the function V one arrives at the formula

$$v(x) = \sum_{j=0}^{\infty} \int_{\partial \mathcal{X}} \frac{h_j(y, x-y)}{|x-y|^{n+2(j-1)}}\, ds \tag{4.1}$$

for all $x \in \mathcal{X}$, where $h_j(y, \xi)$ are homogeneous harmonic polynomials of degree j in ξ with coefficients of class $L^2(\partial \mathcal{X})$ in y. The coefficients are shown to satisfy the growth condition

$$\lim_{j \to \infty} \Big(\frac{1}{j!} \int_{\partial \mathcal{X}} |h_j(y, \partial_\xi)^* h_j(y, \xi)|\, ds \Big)^{1/2j} = 0,$$

which is typical for hyperfunction theory. Under this condition, the series in (4.1) converges uniformly on compact subsets of $\mathcal{X}$.

Remark 4.1. Conversely, each function v of the form (4.1) is harmonic in the domain $\mathcal{X}$.

It is worth pointing out that if v is a harmonic function in $\mathcal{X}$ smooth up to the boundary of $\mathcal{X}$ then

$$v(x) = \int_{\partial \mathcal{X}} \Big(- \frac{1}{\sigma_n} \frac{1}{2-n} \frac{1}{|x-y|^{n-2}} \frac{\partial v}{\partial \nu}(y) - \frac{1}{\sigma_n} \frac{(\nu(y), x-y)}{|x-y|^n} v(y) \Big)\, ds$$

for all $x \in \mathcal{X}$, which is due to the Green formula. Here, $\nu(y)$ is the outward unit normal vector to the boundary at a point $y \in \partial \mathcal{X}$. Comparing this formula with (4.1) yields

$$\begin{array}{rcl} h_0(y, \xi) & = & - \dfrac{1}{\sigma_n} \dfrac{1}{2-n} \dfrac{\partial v}{\partial \nu}(y), \\ h_1(y, \xi) & = & - \dfrac{1}{\sigma_n} (\nu(y), \xi)\, v(y). \end{array}$$

For general $v \in H^1(\mathcal{X})$ the restriction of the normal derivative to the boundary is no longer defined in the sense of Sobolev spaces. It has mere weak limit values at $\partial \mathcal{X}$ belonging to $H^{-1/2}(\partial \mathcal{X})$. Hence, the first boundary integrand in the Green formula should be rearranged to fit well into representation (4.1). This shows that (4.1) is much more refined than the Green formula for harmonic functions. The harmonic polynomials $h_j(y, \xi)$ can be thought of as generalised boundary values of the harmonic function v. However, they fail to be uniquely determined by v in general.

A harmonic function v on $\mathcal{X}$ is said to be of finite order of growth near the boundary of $\mathcal{X}$ if there are a natural number Q and a constant $C > 0$ with the property that $|v(x)| \leq C\, \mathrm{dist}(x, \partial \mathcal{X})^{-Q}$ holds for all $x \in \mathcal{X}$. It is obvious that the summands of (4.1) are of finite order of growth near $\partial \mathcal{X}$. Hence, if the series breaks up, then the function v is of finite order of growth near $\partial \mathcal{X}$, too. Conversely, if v is a harmonic function in $\mathcal{X}$ of finite order of growth near the boundary, then it admits a representation (4.1) with finitely many terms. More precisely, denote by $\mathcal{S}(\partial \mathcal{X})$ the vector space of functions harmonic on neighbourhoods of $\partial \mathcal{X}$. Since each harmonic function is real analytic, the family of seminorms $\|\partial^\alpha g\|_{L^2(\partial \mathcal{X})}$ parametrised by $\alpha \in \mathbb{Z}^n_{\geq 0}$ makes $\mathcal{S}(\partial \mathcal{X})$ to a metrisable locally convex space. (Clearly, this space is not complete.) Any harmonic function v in $\mathcal{X}$ defines a linear functional $\mathcal{F}_v$ on $\mathcal{S}(\partial \mathcal{X})$ by the formula

$$\mathcal{F}_v(g) = \int_{\partial U_g} \Big(g \frac{\partial v}{\partial \nu} - \frac{\partial g}{\partial \nu} v \Big)\, ds$$

for $g \in \mathcal{S}(\partial\mathcal{X})$, where U_g is a relatively compact domain in $\mathcal{X}$ with piecewise smooth boundary, such that g is still harmonic near the closure of $\mathcal{X} \setminus U_g$. From the Green formula it follows that $\mathcal{F}_v$ is well defined, i.e., it does not depend on the particular choice of U_g.

Theorem 4.2. *In order that a harmonic function v in $\mathcal{X}$ have a representation of the form*

$$v(x) = \sum_{j=0}^{N} \int_{\partial\mathcal{X}} \frac{h_j(y, x-y)}{|x-y|^{n+2(j-1)}}\, ds,$$

where $h_j(y,\xi)$ are homogeneous harmonic polynomials of degree j in ξ with coefficients of class $L^2(\partial\mathcal{X})$ in y, it is necessary and sufficient that the functional $\mathcal{F}_v$ on $\mathcal{S}(\partial\mathcal{X})$ be continuous.

Proof. For the proof it suffices to apply Theorem 3.2.9 of [Tar95] to the harmonic function V in $\mathbb{R}^n \setminus \partial\mathcal{X}$ which is equal to v in $\mathcal{X}$ and 0 away from the closure of $\mathcal{X}$. □

On returning back to the solution of the Dirichlet problem (1.1) we get

$$u(x) = \sum_{j=0}^{N} \int_{\partial\mathcal{X}} \varphi(y)\, \frac{h_j(y, x-y)}{|x-y|^{n+2(j-1)}}\, ds + \Phi * (\varphi f)(x),$$

for $x \in \mathcal{X}$ close to O, up to a harmonic function in some neighbourhood of the singular point O. This representation allows one to specify the asymptotics of $u(x)$, when $x \in \mathcal{X}$ converges to O, as long as f belongs to $H^{-s}(\mathcal{X})$ with a finite integer $s > 0$. One may conjecture that N can be taken to be $s+1$ but we will not develop this point here.

5. Return to classical potential theory

The approach developed in the last section extends actually to boundary value problems for solutions of elliptic partial differential equations of order two with real analytic coefficients. More precisely, assume that $\mathcal{X}$ is a bounded domain with rectifiable boundary in $\mathbb{R}^n$. The boundary of $\mathcal{X}$ may contain, e.g., a finite number of cuspidal points or edges, etc. The induced Lebesgue measure ds on $\partial\mathcal{X}$ is obviously massive, i.e., any subset of $\partial\mathcal{X}$ of measure zero has no interior points. Let A be a second order partial differential operator with real analytic coefficients in a neighbourhood of $\overline{\mathcal{X}}$. Our basic assumption is that A is elliptic. Then A possesses a two-sided fundamental solution Φ in a neighbourhood of $\overline{\mathcal{X}}$ which is a pseudodifferential operator of order -2. As usual, we write $\Phi(x,y)$ for the Schwartz kernel of A. The approach is efficient provided one is in a position to construct $\Phi(x,y)$ in an explicit form. Consider the problem of finding a function u in $\mathcal{X}$ satisfying the equation $Au = f$ in $\mathcal{X}$ and a boundary condition $Bu = u_0$ at $\partial\mathcal{X}$. In order to get asymptotic results, it is necessary to put some restrictions on the right hand side f. We require f to belong to $H^{-s}(\mathcal{X})$ for some integer $s > 0$ to not go beyond the class of solutions u of finite order of growth near the boundary of $\mathcal{X}$. Then f can be thought of as a distribution in $H^{-s}(\mathbb{R}^n)$ with support in the closure of $\mathcal{X}$. The potential $U := \Phi(f)$ restricted to $\mathcal{X}$ solves the differential equation $AU = f$ in $\mathcal{X}$. Hence, on setting $u = v + U$ we reduce our boundary value problem to $Av = 0$ in $\mathcal{X}$ and $Bv = v_0$ at $\partial\mathcal{X}$, where $v_0 = u_0 - BU$. This reduction initiates the problem of specifying BU within distributions on the surface

$\partial\mathcal{X}$. Since $AU = f$ in $\mathcal{X}$, under reasonable conditions the restriction of BU to the boundary can be defined with the help of the Green formula for A. The study of the behaviour of $\Phi(f)$ near the surface $\partial\mathcal{X}$ makes a part of the theory of classical pseudodifferential operators. Hence, we restrict our discussion to the behaviour of v close to $\partial\mathcal{X}$ in $\mathcal{X}$. Any Poisson type formula $v = \wp(v_0)$ assumes the knowledge of the singularities of the Poisson kernel $\wp(x, y)$ in $x \in \mathcal{X}$ and $y \in \partial\mathcal{X}$. Hence, it is an issue rather than a tool for studying asymptotic behaviour of v at the boundary. With this as our starting point we exploit the structure theorem for solutions of $Av = 0$ away from the compact set $\partial\mathcal{X}$, see Theorem 3.2.9 in [Tar95]. Assuming v to be of finite order of growth at the boundary and applying Theorem 3.2.9 of [Tar95] to the solution V given by v in $\mathcal{X}$ and to zero away from the closure of $\mathcal{X}$, we get

$$v(x) = \sum_{|\alpha|\le N} \int_{\partial\mathcal{X}} \partial_y^\alpha \Phi(x,y)\, c_{-\alpha}(y)\, ds \tag{5.1}$$

for all $x \in \mathcal{X}$, where $c_{-\alpha} \in L^2(\partial\mathcal{X})$ for $|\alpha| \le N$. Formula (5.1) is more general than the representation of Theorem 4.2. Using a suitable cut-off function φ we may localise this formula to any singular part of $\partial\mathcal{X}$, thus displaying typical asymptotics of the solution.

Remark 5.1. An easy calculation shows that in this way we recover, in particular, the so-called Fuchs type asymptotics which are typical for conical points at the boundary.

Formula (5.1) shows that the classical approach to boundary value problems for elliptic equations based on potential theory (see for instance [Lop53]) gains in importance for analysis on singular spaces.

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