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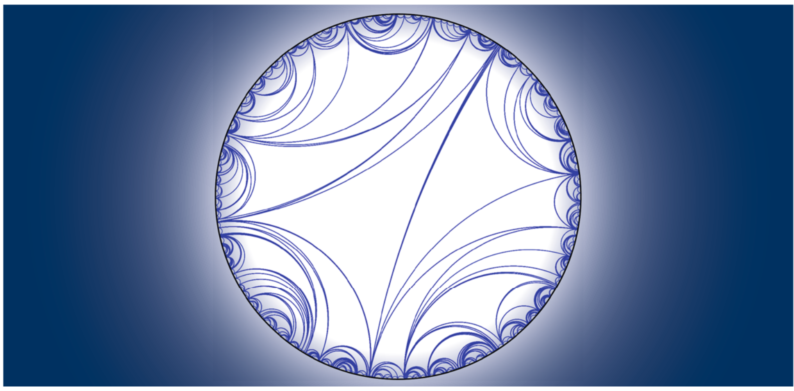

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A RADÓ THEOREM FOR p-HARMONIC FUNCTIONS

IBRAHIM LY AND NIKOLAI TARKHANOV

Abstract. Let A be a nonlinear differential operator on an open set $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{S}$ a closed subset of $\mathcal{X}$. Given a class $\mathcal{F}$ of functions in $\mathcal{X}$, the set $\mathcal{S}$ is said to be removable for $\mathcal{F}$ relative to A if any weak solution of $A(u) = 0$ in $\mathcal{X} \setminus \mathcal{S}$ of class $\mathcal{F}$ satisfies this equation weakly in all of $\mathcal{X}$. For the most extensively studied classes $\mathcal{F}$ we show conditions on $\mathcal{S}$ which guarantee that $\mathcal{S}$ is removable for $\mathcal{F}$ relative to A.

Contents

Introduction

The problem under study lies in the following. Suppose A is a nonlinear differential operator on an open set $\mathcal{X}$ in $\mathbb{R}^n$ and $\mathcal{S}$ is a closed subset of $\mathcal{X}$. For a given class $\mathcal{F}$ of functions on $\mathcal{X} \setminus \mathcal{S}$, the set $\mathcal{S}$ is said to be removable for $\mathcal{F}$ with respect to A if each function $u \in \mathcal{F}$ satisfying $A(u) = 0$ on $\mathcal{X} \setminus \mathcal{S}$ extends to a solution of this equation on the whole set $\mathcal{X}$. What balance between the growth of functions in $\mathcal{F}$ near $\mathcal{S}$ and the "smallness" of $\mathcal{S}$ is sufficient in order that $\mathcal{S}$ be removable for $\mathcal{F}$ relative to A?

The first result of this type is perhaps the Riemann theorem on the removability of one-point singularities for bounded holomorphic functions. For linear differential operators with C^∞ coefficients the problem was studied in [Boc56], [HP70], etc. The paper [HP70] is of special importance for it singles out the crucial step in the study of removable singularities. To wit, on assuming $\mathcal{F}$ to be a class of functions on all of $\mathcal{X}$ one asks if any weak solution u to $A(u) = 0$ in $\mathcal{X} \setminus \mathcal{S}$ satisfies this equation weakly in all of $\mathcal{X}$. This paper facilitated considerable progress in the

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study of removable sets for solutions of linear equations, see [Tar95, Ch. 1] and the references given there.

A starting point for nonlinear differential equations is the pioneering work on the local behaviour of solutions of quasilinear equations by Serrin [Ser64]. The comparatively recent book [Ver96] presents in a unified way the development of the theory of singularities for solutions of second order elliptic or parabolic quasilinear equations starting from the linear equations and the work [Ser64]. As but one motivation of the present paper we mention that the book [Ver96] does not contain any reference to [HP70] while the approach of the latter paper may be undoubtedly of use for nonlinear equations, too. For general nonlinear equations there is no reasonable concept of a weak solution, however, one gets it immediately by turning to a variational setting and relaxing the initial equation into the Euler-Lagrange equation.

Specifically we discuss a Radó type theorem for p-harmonic functions on an open set $\mathcal{X}$ in $\mathbb{R}^n$ which are defined to be weak solutions $u \in W^{1,p}_{\mathrm{loc}}(\mathcal{X})$ of the quasilinear equation $\operatorname{div}(|u'|^{p-2}u') = 0$, where u' stands for the gradient of u and $p > 1$. The operator $\Delta_p u := \operatorname{div}(|u'|^{p-2}u')$ is called the p-Laplace operator, it is elliptic away from the critical points of u. The classical Radó theorem states that if u is a continuous function on an open set $\mathcal{X}$ in the complex plane which is holomorphic away from the set of zeroes then u is actually holomorphic on all of $\mathcal{X}$, see [Rad37]. By the very nature, this is a result on removable sets for the class of continuous functions with respect to the Cauchy-Riemann operator in the plane. In 1983 Král extended the Radó theorem to harmonic functions showing that each C^1 function on an open set $\mathcal{X}$ in $\mathbb{R}^n$, which is harmonic away from the set of its zeroes, is actually harmonic on all of $\mathcal{X}$, see [Kra83]. The paper [GZ12] contains a Radó type theorem for matrix factorisations of the Laplace operator in $\mathbb{R}^n$. For a deeper discussion see [Tar95, 1.3.4].

We now dwell on the contents of the paper. In Section 1 we remind of how variational formulations of nonlinear equations lead to quasilinear relaxations of these equations. This gives immediately rise to the natural concept of a weak solution. In Section 2 we specify the notion of a removable set for solutions of quasilinear partial differential equations. In Section 3 we adduce a fundamental lemma of [HP70] which is of key importance in the study of removable sets for solutions of linear equations. We show that the lemma is still useful in characterising removable singularities for solutions of quasilinear equtions. Section 4 deals with removable sets for functions of Sobolev classes while Section 5 does with removable sets for classes of C^s functions. The results of Section 5 apply to study removability of the zero sets in Section 6. In Section 7 we discuss shortly a Radó type theorem for p-harmonic functions.

The paper [HP72] rises immediately from [HP70] to introduce a notion of capacity which characterizes removable sets for solutions of linear equations. In [GP08], a concept of nonlinear capacity related to a nonlinear operator is applied to blow-up problems for diverse nonlinear partial differential equations including those with nonlocal nonlinearities.

1. Lagrangian problems

Suppose that $\mathcal{X}$ is an open set in $\mathbb{R}^n$. For a nonnegative integer s and $1 \le p \le \infty$, we denote by $W^{s,p}(\mathcal{X})$ the space of all distributions in $\mathcal{X}$ whose derivatives up to order s belong to the Lebesgue space $L^p(\mathcal{X})$. This space is given a standard Banach space structure.

When solving a partial differential equation for a function $u \in W^{m,p}(\mathcal{X})$, one can often relax this problem to that of minimising a nonlinear functional

$$F(u) := \int_{\mathcal{X}} L(x, (\partial^\beta u(x))_{|\beta| \le m})\, dx$$

over all $u \in W^{m,p}(\mathcal{X})$, where $L(x, (u_\beta)_{|\beta|\le m})$ is a real-valued function of its numerical variables in $\mathcal{X} \times \oplus_{|\beta|\le m}\mathbb{R}$. The function L is usually referred to as Lagrange function and the mapping $N_L(u) := L(x, (\partial^\beta u)_{|\beta|\le m})$ as Nemytskii operator. A standing assumption on L is that N_L is a differentiable mapping of $W^{m,p}(\mathcal{X})$ to $L^1(\mathcal{X})$.

If F takes on its local minimum at a function $u \in W^{m,p}(\mathcal{X})$, then, given any fixed $v \in W^{m,p}(\mathcal{X})$, the function $f(\varepsilon) := F(u + \varepsilon v)$ of $\varepsilon \in \mathbb{R}$ takes on a local minimum at $\varepsilon = 0$. Since f is differentiable at the origin, we conclude that $f'(0) = 0$. Hence it follows that

$$\int_{\mathcal{X}} \sum_{|\alpha|\le m} L'_{u_\alpha}(x, (\partial^\beta u(x))_{|\beta|\le m})\, \partial^\alpha v(x)\, dx = 0 \tag{1.1}$$

for all $v \in W^{m,p}(\mathcal{X})$.

Choosing arbitrary $v \in C^\infty_{\mathrm{comp}}(\mathcal{X})$ we deduce from (1.1) that u satisfies the equation

$$\sum_{|\alpha|\le m} (-\partial)^\alpha\, L'_{u_\alpha}(x, (\partial^\beta u(x))_{|\beta|\le m}) = 0$$

in the sense of distributions in the interior of $\mathcal{X}$. This is a nonlinear equation of generalised divergence form whose "coefficients" $L'_{u_\alpha}(x, (\partial^\beta u(x))_{|\beta|\le m})$ map $W^{m,p}(\mathcal{X})$ continuously into $L^{p'}(\mathcal{X})$, where $p' = p/(p-1)$ is the dual exponent of p. Moreover, if the "coefficients" are smooth enough then partial integration on the left-hand side of (1.1) yields

$$\int_{\mathcal{X}} \sum_{|\alpha|\le m} (-\partial)^\alpha\, L'_{u_\alpha}(x, (\partial^\beta u)_{|\beta|\le m})\, v\, dx - \int_{\partial\mathcal{X}} \sum_{j=0}^{m-1} B_j(u) \Big(\frac{\partial}{\partial\nu}\Big)^j v\, ds = 0 \tag{1.2}$$

for all $v \in W^{m,p}(\mathcal{X})$, where B_j are uniquely determined nonlinear differential operators of order $2m - j - 1$, ν the unit outward normal vector of the hypersurface $\partial\mathcal{X}$ and ds the area form on $\partial\mathcal{X}$. Since (1.1) is fulfilled for all $v \in W^{m,p}(\mathcal{X})$, we conclude readily that $B_j(u) = 0$ in a weak sense at the boundary of $\mathcal{X}$, for each $j = 0, 1, \ldots, m-1$.

We have thus arrived at the so-called Euler-Lagrange equations for the local minima u of the functional F, which actually constitute a Neumann type boundary value problem in $\mathcal{X}$.

Example 1.1. If $L(x, (\partial^\beta u)_{|\beta|\le 1}) := (1/p)\, |u'|^p$, then the Euler-Lagrange equations become

$$\begin{array}{rcll} \operatorname{div} L'_{u'} & = & 0 & \text{in} \quad \mathcal{X}, \\ \langle L'_{u'}, \nu\rangle_x & = & 0 & \text{at} \quad \partial\mathcal{X}, \end{array}$$

where $L'_{u'} = |u'|^{p-2} u'$.

Using the notation of differential forms, the problem of Example 1.1 just amounts to the Cauchy problem $d^*U = 0$ in $\mathcal{X}$ and $\nu(U) = 0$ at $\partial\mathcal{X}$ for the 1-form $U = L'_{u'}$ in $\mathcal{X}$.

2. Removable sets for solutions of quasilinear equations

The Euler-Lagrange equations give rise to a broad class of nonlinear operators in generalised divergence form

$$A(u)\,(x) := \sum_{|\alpha|\le m} (-\partial)^{\alpha}\, A_{\alpha}(x, (\partial^{\beta} u(x))_{|\beta|\le s-m}), \tag{2.1}$$

where s and $m \le s$ are nonnegative integers, and $A_{\alpha}(x, (u_{\beta})_{|\beta|\le s-m})$ complex-valued functions of its numerical variables in $\mathcal{X} \times \oplus_{|\beta|\le s-m}\mathbb{C}$. The number s can be thought of as the order of A. The following assumption on the “coefficients” A_{α} will be needed throughout the paper. For every multi-index α with $|\alpha| \le m$, the Nemytskii operator $N_{A_{\alpha}}$ is required to map $W^{s-m,p}_{\mathrm{loc}}(\mathcal{X})$ continuously into the space $L^{p'}_{\mathrm{loc}}(\mathcal{X})$. As usual, the designations “loc” and “comp” specify the “local” and “with compact support” versions of the corresponding global Sobolev spaces in $\mathcal{X}$.

Under this assumptions, the operator A is given the domain $W^{s-m,p}_{\mathrm{loc}}(\mathcal{X})$ to map it continuously into the dual of $W^{m,p}_{\mathrm{comp}}(\mathcal{X})$ (usually denoted by $W^{-m,p'}_{\mathrm{loc}}(\mathcal{X})$). More precisely, we set

$$\langle A(u), g\rangle := \sum_{|\alpha|\le m} \langle A_{\alpha}(x, (\partial^{\beta} u)_{|\beta|\le s-m}), \partial^{\alpha} g\rangle$$

for all $g \in W^{m,p}_{\mathrm{comp}}(\mathcal{X})$.

Given any $u \in W^{s-m,p}_{\mathrm{loc}}(\mathcal{X})$, the image $A(u)$ is specified within the framework of distributions in $\mathcal{X}$. In this way, a function $u \in W^{s-m,p}_{\mathrm{loc}}(\mathcal{X})$ is said to satisfy $A(u) = 0$ on an open set $U \subset \mathcal{X}$ if $A(u) = 0$ in the sense of distributions in U, i.e., $\langle A(u), g\rangle = 0$ for all $g \in C^{\infty}_{\mathrm{comp}}(U)$. Hence, by solutions of $A(u) = 0$ are meant weak solutions. This allows one to extend the definition of removable sets, introduced in [HP70] for linear differential operators A, to solutions of nonlinear differential equations.

Definition 2.1. Let S be a closed subset of $\mathcal{X}$ and $\mathcal{F}$ a class of functions in $W^{s-m,p}_{\mathrm{loc}}(\mathcal{X})$. The set S is called removable for $\mathcal{F}$ relative to the differential operator A if any function $u \in \mathcal{F}$ satisfying $A(u) = 0$ in $\mathcal{X} \setminus S$ actually satisfies $A(u) = 0$ in all of $\mathcal{X}$.

One may ask what conditions on the “size” of S are sufficient for S to be a removable set for $\mathcal{F}$ relative to A. For a survey of results on removable singularities we refer the reader to [Tar95, Ch. 1] and [Ver96]. For the most extensively studied classes $\mathcal{F}$ and differential operators A there have been known sharp sufficient conditions on removable sets in terms of the Hausdorff measure of $\mathcal{S}$. For both necessary and sufficient conditions on removable sets one appeals to the so-called capacity, see [HP72].

Example 2.2. The extreme case $m = 0$ is vapid. Indeed, in this case A reduces to the Nemytskii operator $N_{A_0}(u) = A_0(x, (\partial^{\beta} u)_{|\beta|\le s})$ mapping $W^{s,p}_{\mathrm{loc}}(\mathcal{X})$ continuously into $L^{p'}_{\mathrm{loc}}(\mathcal{X})$. As the elements of $L^{p'}_{\mathrm{loc}}(\mathcal{X})$ are defined up to functions vanishing almost everywhere in $\mathcal{X}$, any closed set $S \subset \mathcal{X}$ of measure zero is removable for $\mathcal{F} := W^{s,p}_{\mathrm{loc}}(\mathcal{X})$ relative to A. Obviously, this condition is necessary provided that N_{A_0} is surjective.

3. A fundamental lemma

In order to characterize the removable sets in terms of the Hausdorff measure one uses a fundamental lemma of [HP70]. We first recall the definition of the Hausdorff measure.

For $0 \leq d \leq n$ we set

$$h_{d,\varepsilon}(S) := \inf \sum_{\nu} v_d\, r_\nu^d,$$

where the infimum is taken over all countable coverings $\{B_\nu\}$ of the set S by balls with radii $r_\nu \leq \varepsilon$, and v_d is the volume of the unit ball in $\mathbb{R}^d$. Obviously, $h_{d,\varepsilon}(S)$ is a monotone increasing function of $\varepsilon \to 0+$, and so it has a limit as $\varepsilon \to 0+$. The number

$$h_d(S) = \lim_{\varepsilon \to 0+} h_{d,\varepsilon}(S)$$

is called the d-dimensional Hausdorff measure of the set S.

Hausdorff measure is a regular metric outer measure on $\mathbb{R}^n$. Therefore, $h_d(S) = 0$ if and only if $h_d(K) = 0$ for each compact subset $K \subset S$. Note that h_n agrees with the standard Lebesgue measure in $\mathbb{R}^n$. In most cases one is interested only in whether the measure $h_d(S)$ is zero, finite, or infinite. From this point of view, instead of coverings by balls in the definition of h_d, we may use coverings by cubes or arbitrary (convex) sets of diameter $2r_v$, because all such coverings lead to equivalent measures.

Lemma 3.1. *Suppose K is a compact subset of $\mathbb{R}^n$. Then, for each $d = n-mp$ and $\varepsilon > 0$, there is a C^∞ function χ_ε with compact support in $\mathbb{R}^n$, such that the support of χ_ε belongs to the ε-neighbourhood of K, $\chi_\varepsilon \equiv 1$ in a smaller neighbourhood of K, and*

$$\|\partial^\alpha \chi_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq C_\alpha\, \varepsilon^{m-|\alpha|} \left(h_d(K) + \varepsilon\right)^{1/p}$$

for all α with $|\alpha| \leq m$, where the constant C_α is independent of ε.

Proof. This is a refinement of a well-known lemma of [Boc56]. For a proof, see [HP70] or [Tar95, 1.2.1]. □

It is worth pointing out that d is assumed to be nonnegative. Hence it follows that $n - mp \geq 0$, i.e., $p \leq n/m$.

4. Removable sets for Sobolev functions

In this section we characterize removable sets for the class $\mathcal{F} := W^{s-m,p}_{\mathrm{loc}}(\mathcal{X})$. As mentioned, p is bounded away from the point at infinity unless $m = 0$, in which case the problem is vapid.

Theorem 4.1. *Let $d = n - mp$ be nonnegative.*

1) If $1 < p < \infty$ and $h_{n-mp}(K) < \infty$ for each compact set $K \subset S$, then S is removable for $W^{s-m,p}_{\mathrm{loc}}(\mathcal{X})$ relative to A.

2) If $p = 1$ and $h_{n-m}(S) = 0$, then the set S is removable for $W^{s-m,1}_{\mathrm{loc}}(\mathcal{X})$ relative to A.

For general linear partial differential operators A, Theorem 4.1 is contained in [HP70].

Proof. Let $u \in W^{s-m,p}_{\mathrm{loc}}(\mathcal{X})$ satisfy $A(u) = 0$ in $\mathcal{X} \setminus \mathcal{S}$. Pick any $g \in C^{\infty}_{\mathrm{comp}}(\mathcal{X})$ and set $K = S \cap \operatorname{supp} g$. Then

$$\langle A(u), g\rangle = \langle A(u), \chi_\varepsilon g\rangle + \langle A(u), (1-\chi_\varepsilon)g\rangle$$

for all $\varepsilon > 0$, where χ_ε is the function of Lemma 3.1.

Since $A(u) = 0$ in $\mathcal{X} \setminus \mathcal{S}$ and the support of $(1-\chi_\varepsilon)g$ is a compact subset of $\mathcal{X} \setminus \mathcal{S}$, it follows that

$$\begin{aligned}
\langle A(u), g\rangle &= \langle A(u), \chi_\varepsilon g\rangle \\
&= \sum_{|\alpha|\le m} \langle N_{A_\alpha}(u), \partial^\alpha(\chi_\varepsilon g)\rangle \\
&= \sum_{|\alpha|\le m} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \langle N_{A_\alpha}(u), \partial^{\alpha-\beta} g\, \partial^\beta \chi_\varepsilon\rangle
\end{aligned}$$

where, for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, by $\beta \le \alpha$ is meant that $\beta_j \le \alpha_j$ for all $j = 1, \ldots, n$. Consequently, by the Hölder inequality and Lemma 3.1, we get

$$\begin{aligned}
|\langle A(u), g\rangle| &\le C \sum_{|\alpha|\le m} \sum_{\beta\le\alpha} \|N_{A_\alpha}(u)\|_{L^{p'}(K_\varepsilon)} \|\partial^\beta \chi_\varepsilon\|_{L^p(\mathcal{X})} \\
&\le C' \left(h_d(K) + \varepsilon\right)^{1/p} \sum_{|\alpha|\le m} \varepsilon^{m-|\alpha|} \|N_{A_\alpha}(u)\|_{L^{p'}(K_\varepsilon)}
\end{aligned} \tag{4.1}$$

for all sufficiently small $\varepsilon > 0$, where K_ε stands for the ε-neighbourhood of K and C, C' are constant independent of ε.

If $p > 1$, then $p' < \infty$ and so $\|N_{A_\alpha}(u)\|_{L^{p'}(K_\varepsilon)} \to 0$ as $\varepsilon \to 0$, for each multi-index α. On the other hand, if $p = 1$ and $h_d(K) = 0$, then $(h_d(K) + \varepsilon)^{1/p}$ tends to zero as $\varepsilon \to 0$.

By the hypotheses of 1) or of 2), the right-hand side of (4.1) tends to zero as $\varepsilon \to 0+$. Thus, $\langle A(u), g\rangle = 0$, as desired. □

The arguments of [Tar95, 1.2.2] show that the assumptios on S in Theorem 4.1 cannot be improved in terms of the Hausdorff measure.

Example 4.2. Assume that $1 < p < \infty$ and $h_{n-p}(K)$ is finite for each compact set $K \subset S$. Then the set S is removable for $W^{1,p}_{\mathrm{loc}}(\mathcal{X})$ relative to the p-Laplace operator.

5. Removable sets for smooth functions

In this section we will be concerned with removable sets for $C^{s-m}_{\mathrm{loc}}(\mathcal{X})$ relative to A. As usual, for $k \ge 0$ an integer, we denote by $C^k_{\mathrm{loc}}(\mathcal{X})$ the space of k times continuously differentiable functions on $\mathcal{X}$. In order to get substantial results, it is necessary to put some restrictions on the coefficients $A_\alpha(x, (u_\beta)_{|\beta|\le s-m})$ of A. Since this work is intended as an attempt at motivating the Radó type theorem for p-harmonic functions, we choose the abstract setting of the p-Laplace equation. To wit, assume that the Nemytskii operator N_{A_α} maps $C^{s-m}_{\mathrm{loc}}(\mathcal{X})$ continuously into $C_{\mathrm{loc}}(\mathcal{X})$, for $|\alpha| \le m$.

Theorem 5.1. *Suppose $h_{n-m}(K) < \infty$ for each compact set $K \subset S$. Then S is removable for $C^{s-m}_{\mathrm{loc}}(\mathcal{X})$ relative to A.*

For $m = 0$, the set S is removable for $C^{s}_{\mathrm{loc}}(\mathcal{X})$ relative to $A = N_{A_0}$ provided that the interior of S is empty. If moreover A is onto $C_{\mathrm{loc}}(\mathcal{X})$, then this condition is also necessary.

Proof. Assume $u \in C^{s-m}_{\mathrm{loc}}(\mathcal{X})$ satisfies $A(u) = 0$ on $\mathcal{X} \setminus \mathcal{S}$. Let $g \in C^{\infty}_{\mathrm{comp}}(\mathcal{X})$, and let $K = S \cap \operatorname{supp} g$.

Since the support of $A(u)$ belongs to S, we obtain for the function χ_ε from Lemma 3.1 that

$$\begin{aligned}
\langle A(u), g\rangle &= \sum_{|\alpha|\le m} \langle N_{A_\alpha}(u), \partial^\alpha(\chi_\varepsilon g)\rangle \\
&= \sum_{|\alpha|\le m} \Big(\langle \partial^\alpha \chi_\varepsilon N_{A_\alpha}(u), g\rangle + \sum_{\substack{\beta\le\alpha \\ \beta\ne\alpha}} \binom{\alpha}{\beta} \langle N_{A_\alpha}(u), \partial^{\alpha-\beta} g\, \partial^\beta \chi_\varepsilon\rangle \Big),
\end{aligned}$$

with ε an arbitrary positive number. Using the Hölder inequality and Lemma 3.1 we obtain

$$\begin{aligned}
|\langle N_{A_\alpha}(u), \partial^{\alpha-\beta} g\, \partial^\beta \chi_\varepsilon\rangle| &\le \|\partial^{\alpha-\beta} g N_{A_\alpha}(u)\|_{L^\infty(\mathcal{X})}\, \|\partial^\beta \chi_\varepsilon\|_{L^1(\mathcal{X})} \\
&\le C_{\alpha,\beta}\, \varepsilon^{m-|\beta|}\, (h_d(K) + \varepsilon)\,,
\end{aligned}$$

where the constants $C_{\alpha,\beta}$ are independent of ε. Consequently, $A(u)$ is the limit of the net of continuous functions

$$\sum_{|\alpha|\le m} N_{A_\alpha}(u)\, \partial^\alpha \chi_\varepsilon \tag{5.1}$$

in the space of distributions on $\mathcal{X}$.

By Lemma 3.1, we have $\|\partial^\alpha \chi_\varepsilon\|_{L^1(\mathcal{X})} \le C_\alpha\,(h_d(K) + \varepsilon)$ for all positive $\varepsilon \le 1$, provided that $|\alpha| \le m$. Since $h_d(K) < \infty$, we can assert that the net $\partial^\alpha \chi_\varepsilon$ is bounded in $L^1(\mathcal{X})$. Hence it follows that the net has a subsequence which converges in the weak* topology of $C_{\mathrm{loc}}(\mathcal{X})'$. The limit of this subsequence is necessarily zero, for the net χ_ε, and so also the net $\partial^\alpha \chi_\varepsilon$ converges to zero in the sense of distributions on $\mathcal{X}$. Multiplication by $N_{A_\alpha}(u)$, where $|\alpha| \le m$, defines a continuous operator in $C_{\mathrm{loc}}(\mathcal{X})'$. Therefore, some subsequence of (5.1) converges to zero in the weak* topology of $C_{\mathrm{loc}}(\mathcal{X})'$. Since, however, the net itself converges to $A(u)$ in the space of distributions on $\mathcal{X}$, it follows that $A(u) = 0$ on $\mathcal{X}$, as desired. □

Example 5.2. Suppose $h_{n-1}(K) < \infty$ for each compact set $K \subset S$. Then S is removable for continuously differentiable p-harmonic functions in $\mathcal{X}$, with any $p > 1$.

6. A Radó theorem

A Radó type theorem for solutions of linear differential equations was first formulated in the monograph [Tar95, 1.3.4] whose original Russian edition was published in 1991.

In order to formulate a Radó theorem in the context of nonlinear differential equations, we ought to rearrange the operator A. Suppose that $m = 1$, i.e., A is of the form

$$A(u) = -\sum_{j=1}^{n} \partial_j\, A_j(x, (\partial^\beta u)_{|\beta|\le s-1}) + A_0(x, (\partial^\beta u)_{|\beta|\le s-1}),$$

where the Nemytskii operators $N_{A_1}, \ldots, N_{A_n}$ and N_{A_0} are assumed to map $C^{s-1}_{\mathrm{loc}}(\mathcal{X})$ continuously into $C_{\mathrm{loc}}(\mathcal{X})$.

By Theorem 5.1, if S is a closed subset of $\mathcal{X}$ with the property that $h_{n-1}(K) < \infty$ for each compact set $K \subset S$, then S is removable for $C^{s-1}_{\mathrm{loc}}(\mathcal{X})$ relative to A.

Lemma 6.1. *Assume that S is a smooth hypersurface in $\mathcal{X}$. Then S is removable for $C^{s-1}_{\mathrm{loc}}(\mathcal{X})$ relative to A.*

Proof. Indeed, when restricted to subsets of a smooth submanifold S of $\mathcal{X}$ of dimension d, the Hausdorff measure h_d is commensurable with the corresponding surface measure on S induced by the Lebesgue measure in $\mathbb{R}^n$. Hence it follows immediately that $h_{n-1}(K) < \infty$ for each compact set $K \subset S$, showing the desired assertion. □

While Theorem 5.1 characterises those $S \subset \mathcal{X}$ which are removable for all solutions $u \in C^{s-1}_{\mathrm{loc}}(\mathcal{X})$ to $A(u) = 0$ in $\mathcal{X} \setminus S$, the Radó type theorems deal with individual solutions $u \in C^{s-1}_{\mathrm{loc}}(\mathcal{X})$ of this equation. As S one takes the preimage of a point by u, e.g., $S = u^{-1}(0)$ which is the set of all $x \in \mathcal{X}$ satisfying $u(x) = 0$. Then, a Radó theorem for solutions of the nonlinear equation $A(u) = 0$ states that if $u \in C^{s-1}_{\mathrm{loc}}(\mathcal{X})$ satisfies $A(u) = 0$ in $\mathcal{X} \setminus u^{-1}(0)$ then $A(u) = 0$ is actually fulfilled in all of $\mathcal{X}$.

Theorem 6.2. *If $u \in C^{s-1}_{\mathrm{loc}}(\mathcal{X})$ satisfies $A(u) = 0$ in $\mathcal{X} \setminus u^{-1}(0)$, then $A(u) = 0$ away from the set of all $x \in \mathcal{X}$ satisfying $\partial^\beta u(x) = 0$ for each multi-index β with $|\beta| \le s - 1$.*

Proof. We can certainly assume that $s \ge 2$, since otherwise the assertion is obvious. Set $S = u^{-1}(0)$, and so S is a closed subset of $\mathcal{X}$. Denote by S_{reg} the subset of S consisting of those $x \in S$ which satisfy $u'(x) \ne 0$. Clearly, S_{reg} is an open set in S, and so the set $S^{(1)} := S \setminus S_{\mathrm{reg}}$, which consists of all $x \in S$ satisfying $u(x) = u'(x) = 0$, is closed in $\mathcal{X}$. Each point $x \in S_{\mathrm{reg}}$ has a neighbourhood U in $\mathcal{X}$, such that $S \cap U$ is a hypersurface in U. By Lemma 6.1, $A(u) = 0$ in U and hence everywhere in $\mathcal{X} \setminus S^{(1)}$.

Further, denote by $S^{(1)}_{\mathrm{reg}}$ the subset of $S^{(1)}$ consisting of those $x \in S^{(1)}$ such that $u''(x) \ne 0$. (By $u''(x)$ is meant the tuple $(\partial^\beta u(x))_{|\beta|=2}$, and similarly for higher order total derivatives.) Then, $S^{(1)}_{\mathrm{reg}}$ is an open set in $S^{(1)}$, and hence the set $S^{(2)} := S^{(1)} \setminus S^{(1)}_{\mathrm{reg}}$ is closed in $\mathcal{X}$. Each point $x \in S^{(1)}_{\mathrm{reg}}$ has a neighbourhood U in $\mathcal{X}$ with the property that $S^{(1)} \cap U$ lies in some hypersurface $\{x \in U : \partial_j u(x) = 0\}$, where j is one of the numbers $1, \ldots, n$. On using Lemma 6.1 we see that $A(u) = 0$ in U, and hence in all of $\mathcal{X} \setminus S^{(2)}$.

Continuing this process, after $s-1$ steps, we conclude that the distribution $A(u)$ vanishes in $\mathcal{X} \setminus S^{(s-1)}$, where $S^{(s-1)}$ is the closed subset of $\mathcal{X}$ consisting of all $x \in \mathcal{X}$ such that $u(x) = u'(x) = \ldots = u^{(s-1)}(x) = 0$. □

We now elucidate the main analytical problem in studying the Radó theorem for solutions of the equation $A(u) = 0$. Let $u \in C^{s-1}_{\mathrm{loc}}(\mathcal{X})$ satisfy $A(u) = 0$ in $\mathcal{X} \setminus S$, where $S = u^{-1}(0)$. By Theorem 6.2, the function u satisfies $A(u) = 0$ away from the closed set $S^{(s-1)}$ in $\mathcal{X}$. In all interesting cases already studied the Hausdorff dimension of the set $S^{(s-1)}$ is less than $n-1$, and so the hypothesis of Theorem 5.1 is satisfied. By this theorem, one gets $A(u) = 0$ in all of $\mathcal{X}$, showing the Radó theorem. Clearly, no conclusion on the size of $S^{(s-1)}$ can be drawn in the case of general operators A.

Example 6.3. Let $u(x)$ be a nonzero C^∞ function on $\mathcal{X}$ vanishing in a neighbourhood of a point $x_0 \in \mathcal{X}$. Choose any continuous function $A_0(x,u)$ on $\mathcal{X} \times \mathbb{R}$ which vanishes away from $u^{-1}(0) \times \mathbb{R}$ and is different from zero at $(x_0, 0)$. For the Nemytskii operator $A = N_{A_0}$, we obviously have $A(u) = 0$ in $\mathcal{X} \setminus u^{-1}(0)$ but not in all of $\mathcal{X}$.

This example demonstrates rather strikingly that strong restrictions on the type of A are necessary in order that the Radó type theorem might hold for solutions of $A(u) = 0$.

7. A Radó theorem for p-harmonic functions

Throughout this section we assume that $\mathcal{X}$ is an open set in $\mathbb{R}^n$ and $1 < p < \infty$ a fixed number.

The divergence operator $\Delta_p u := \operatorname{div}(|u'|^{p-2}u')$ is called the p-Laplacian. The p-Laplace equation $\Delta_p u = 0$ just amounts to the Euler-Lagrange equation for the variational problem

$$\int_{\mathcal{X}} \frac{1}{p}\,|u'(x)|^p\,dx \mapsto \min$$

over all functions $u \in W^{1,p}(\mathcal{X})$.

The p-Laplace operator is given the domain $W^{1,p}_{\mathrm{loc}}(\mathcal{X})$ and it maps $W^{1,p}_{\mathrm{loc}}(\mathcal{X})$ continuously to $W^{-1,p'}_{\mathrm{loc}}(\mathcal{X})$, the latter space being the dual space of $W^{1,p}_{\mathrm{comp}}(\mathcal{X})$, where $1/p + 1/p' = 1$. Namely, we set $< \Delta_p u, v >:= - < |u'|^{p-2}u', v' >$ for all $v \in W^{1,p}_{\mathrm{comp}}(\mathcal{X})$.

By a p-harmonic function in $\mathcal{X}$ is meant any solution $u \in W^{1,p}_{\mathrm{loc}}(\mathcal{X})$ to $\Delta_p u = 0$ in $\mathcal{X}$. Thus, the p-harmonic functions are weak solutions to $\Delta_p u = 0$ in $\mathcal{X}$ by the very definition.

A Radó type theorem for p-harmonic functions reads as follows. If u is a C^1 function on an open set $\mathcal{X} \subset \mathbb{R}^n$ satisfying $\Delta_p u = 0$ in $\mathcal{X} \setminus u^{-1}(0)$, then u is p-harmonic in all of $\mathcal{X}$.

For $n = 2$, a proof can be found in [Kil94]. It relies on the intimate connection between quasiregular mappings and planar p-harmonic functions. Therefore, the proof can be generalised neither for higher dimensions nor for more general equations in the plane.

As is noted in [Kil94], the conclusion of the theorem fails to hold if one assumes merely that u is Lipschitz. For let u be the nonnegative part of the first coordinate x_1 of $x \in \mathbb{R}^n$, then u is a Lipschitz continuous function in $\mathbb{R}^n$ satisfying $\Delta_p u = 0$ weakly for $x_1 \neq 0$ but not in all of $\mathbb{R}^n$.

If $n \geq 3$ and $p \neq 2$, then it is not known whether the set $\{x \in \mathcal{X} : u'(x) = 0\}$ can contain interior points, even if we knew a priori that u is nonconstant and p-harmonic in $\mathcal{X}$, see *ibid.*

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