Geometric Electroelasticity

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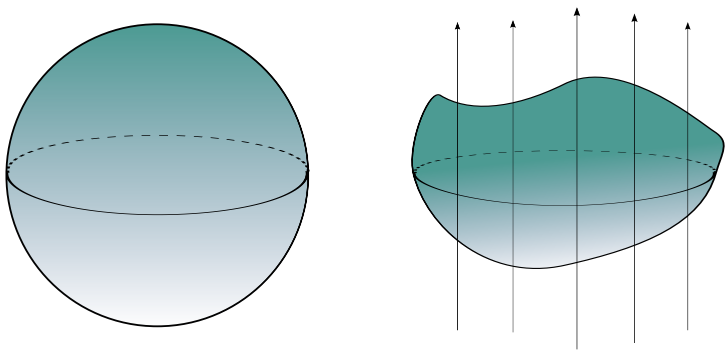
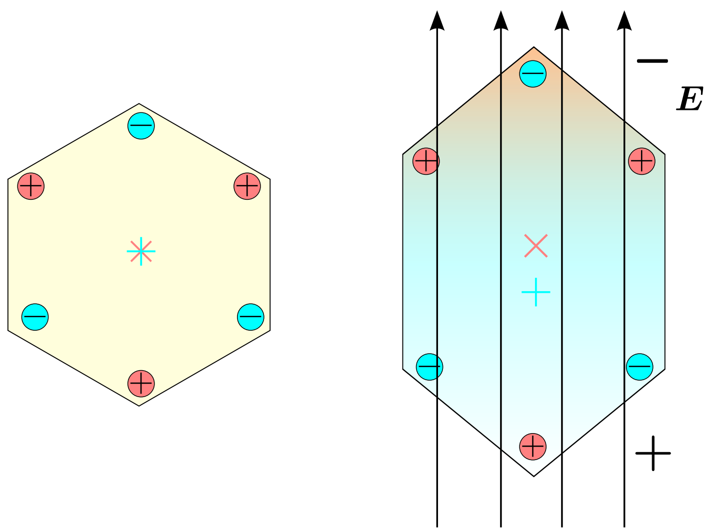
Contents

Introduction

The aim of this work is to describe the motion of bodies consisting of an elastic material that are deformed by the influence of an external electromagnetic field. In the picture below, such an electromagnetic field is illustrated by vertical arrows.



This process is called inverse piezo-electric effect.
In the body shown on the left hand side of the next picture, the centers of positive resp. negative charges coincide. Now suppose, the body is exposed to an external electrical field as indicated in the picture on the right hand side. Since the positive charges are attracted by the field's negative pole and the negative charges are attracted by the field's positive pole, the body expands.



For the description of bodies that are deformed by purely mechanical forces and undergo temperature changes, there is a well-established physical theory, called thermoelasticity. The deformation and the temperature development are governed by several balance laws (conservation of mass, balance of momentum, angular momentum, and energy).
For thermoelastic processes there is a broad agreement on the physically correct setup of these laws. The balance laws together with so-called constitutive relations, characterizing the material the body is made of, provide a complete description of the body's motion and temperature development. However, it seems that there is no complete differential geometric description of thermoelasticity. Even the rather geometric book by Marsden and Hughes [1983] mostly restricts to the case of simple bodies, that is, bodies that are open subsets of the Euclidean $\mathbb{R}^3$.

For the modeling of elastic bodies that are not only deformed by mechanical forces but also subjected to an electromagnetic field, there exist several approaches that are not compatible, most prominently the formulation by Ericksen [2008] in contrast to that by Kovetz [2000]. (Steigmann [2009] claims that these formulations are equivalent, but we will see that this is not true.)
Moreover, in the physics literature on electroelastic materials only simple bodies are treated. Shells, i.e. bodies that only consist of a very thin layer of material (and could thus be modeled by a hypersurface), are then approximated as simple bodies with a thickness that tends to zero. This situation is not very satisfactory from the differential geometric point of view.
We will base our considerations on the set of balance laws that Ericksen [2008] (see also Steigmann [2009]) provided for simple bodies. The form of balance of energy that Ericksen stated, can easily be generalized to bodies that are described by a Riemannian manifold. The remaining balance laws will then be obtained by means of our Theorem 3.5.1. It states that, if balance of energy is invariant under the action of arbitrary diffeomorphisms on the surrounding space, then this already implies the local forms of conservation of mass, balance of momentum and angular momentum, as well as the Doyle-Ericksen formula, which here provides a connection between the internal energy and the deformation. Thus, Theorem 3.5.1 provides a complete set of compatible balance laws that govern the deformation (and temperature development) of a body in a surrounding space. Our theorem generalizes a result that can already be found in the book by Marsden and Hughes [1983, ch. 2 Theorem 4.13] and has more recently been discussed by Kanso et al. [2007, sec.3]. Both these earlier results only pertain to bodies that have the same dimension as the surrounding space and do not allow the presence of electromagnetic fields.

Usually, in works on electroelasticity the entropy inequality is used to decide, which otherwise allowed deformations are physically admissible and which are not. It is also employed to derive the above mentioned Doyle-Ericksen formula and restrictions to the possible forms of constitutive relations describing the material. Unfortunately, the opinions on the physically correct statement of the entropy inequality diverge when electromagnetic fields are present [Ericksen, 2008; Kovetz, 2000; Hutter and Pao, 1974].

A further problem in the formulation of an electroelastic theory on manifolds is that the entropy inequality, as it relies on the entropy flux to be tangential to the deformed body, is only applicable to simple bodies. For general bodies, in particular if they are subjected to an electromagnetic field, this needs not to be the case.

In order to replace the entropy inequality from the outset, we demand that *for a given process*, balance of energy is invariant under the action of arbitrary diffeomorphisms on the surrounding space and under linear rescalings of the temperature (see Theorem 5.3.4). This generalizes a theorem of Marsden and Hughes [1983]. This time, our result is, like theirs, only valid for simple bodies, but it could possibly be generalized to arbitrary bodies. Moreover, in the physics literature one usually starts with quite strong assumptions on the form of the constitutive relations and deduces a number of further restrictions on its form by means of the entropy inequality. By means of Theorem 5.3.4 we are able to deduce the same restrictions using much weaker assumptions.

Finally, we shortly present the partial differential equation that governs the motion of a simple body that is made from a Neo-electroelastic material.

In the first chapter we reproduce some basic notions and concepts of classical elasticity theory. We explain what the deformation and the motion of a body are and give geometric definitions of the body's velocity and acceleration. For that purpose we have to define the substantial derivative that provides the time derivative of a moved continuous body.

Moreover, we recall the definition of certain tensors that describe, in what way the geometry (lengths, angles) of the body is deformed during its motion.

Furthermore we study important general aspects of balance laws.

In the second chapter we recall some basics of electrodynamics that will be needed later on. We state Maxwell's equations in terms of vector fields and also in terms of differential forms. Moreover we discuss Galilei transformations as well as Galilei invariants and introduce the notions of polarization and magnetization.

The third chapter comprises the formulation of the balance laws for electroelasticity. We introduce the Cauchy stress tensor that provides a measure for the forces that the deformation causes inside the body. Some of these laws are formulated at first only for the case that the surrounding space is the Euclidean $\mathbb{R}^3$.

The central theorem of this chapter is Theorem 3.5.1. It provides among other things local forms of balance of momentum and angular momentum that are also valid if the body and the surrounding space are arbitrary Riemannian manifolds, as well as the afore-mentioned Doyle-Ericksen formula. Since the motion of the body is governed by balance of momentum, Theorem 3.5.1 now provides the means to determine (in principle) the body's motion.

In the fourth chapter we reformulate the balance laws and the Doyle-Ericksen formula that we have obtained in the third chapter in terms of the coordinates on the undeformed body. This makes their study much easier.

The fifth chapter discusses certain properties of the material. These are encoded in the so-called constitutive relations providing a connection between the exterior influences (for example the electromagnetic field) and material quantities (for example the internal energy).
We recall the notion of thermoelastic materials, i.e. materials for which the material quantities only depend on the deformation and the temperature as well as their derivatives up to a certain order. Then we cite a result of Marsden and Hughes [1983] that provides some restrictions to the constitutive relations resulting from covariance assumptions and balance of energy. In contrast to section 3.5 we do not consider all processes, but rather all transformations of a given process. Next, material symmetries and in particular isotropic materials are discussed. As particular materials the Mooney-Rivlin and Neo-Hookean materials are considered.
Then we define thermoelectromagnetoelastic materials and deduce by means of Theorem 5.3.4 quite strong restrictions to the possible forms of constitutive relations for such materials. This generalizes a result of Marsden and Hughes [1983] that was given for thermoelastic materials.
After that we discuss constitutive relations for Neo-electroelastic materials, ie. for materials that can interact with an external electric field. They are defined as a generalization of the Neo-Hookean materials that are known from classical elasticity theory.

Finally, in the sixth chapter we shortly present the partial differential equation that governs the motion of a simple body that is made from a Neo-electroelastic material.

1 Geometric Setup and Basic Definitions

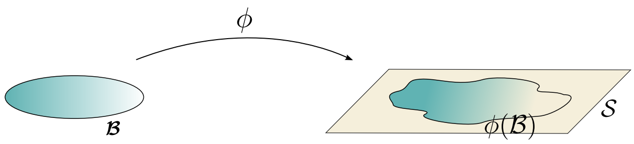
In this chapter we reproduce some basic notions and concepts of classical elasticity theory. We explain what the deformation and the motion of a body are and give geometric definitions of the body's velocity and acceleration. For that purpose we have to define the substantial derivative that provides the time derivative of a moved continuous body. Moreover, we recall the definition of certain tensors that describe in what way the geometry (length, arcs) of the body is deformed during its motion. Later on, in chapter 3, we will establish a number of balance laws, like for example balance of momentum, that govern the body's motion. Since all of these laws have essentially the same form, it is convenient to study them in general. This is done in section 1.3. In anticipation of chapter 3, we will already treat conservation of mass, for the treatment of all the other balance laws will become easier, if conservation of mass is valid. Most of the material for this introductory chapter is taken from Bär [2014] and Marsden and Hughes [1983].

1.1 Deformations

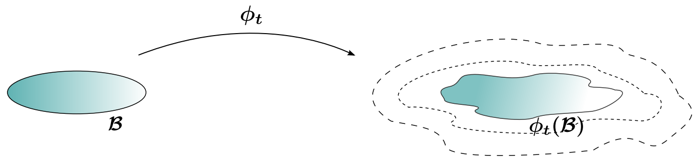
The body and the surrounding space are modeled by a Riemannian manifold $(\mathcal{B}, G)$ and a Riemannian manifold $(\mathcal{S}, g)$, respectively, where $\dim \mathcal{B} \leq \dim \mathcal{S}$. $\mathcal{B}$ respresents the abstract deformable body, $\mathcal{S}$ the surrounding space. The deformation of $\mathcal{B}$ is then described by an embedding:

Definition 1.1.1. A **deformation** of $\mathcal{B}$ is a C^2-embedding $\phi : \mathcal{B} \to \mathcal{S}$. $\mathcal{B}$ is called **reference configuration**.

$\phi(\mathcal{B})$ represents the final, deformed state of $\mathcal{B}$, without any regard of how this deformation took place.



Now imagine that the body is deformed continuously over a certain period of time. We can describe this situation by "glueing together" many single deformed states in a regular way, as is illustrated in the following picture:



More formally, we define:

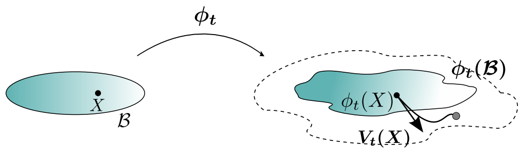
Definition 1.1.2. Denote by $\boldsymbol{\mathcal{D}(\mathcal{B})}$ the set of all deformations of $\mathcal{B}$ in $\mathcal{S}$, and let $I \subset \mathbb{R}$ be an open interval. A **motion** of a body $\mathcal{B}$ in $\mathcal{S}$ is a curve in $\mathcal{D}(\mathcal{B})$, that is, a C^2-map

$$\begin{array}{rcl} \phi : \mathcal{B} \times I & \to & \mathcal{D}(\mathcal{B}) \\ (X,t) & \mapsto & \phi(X,t), \end{array}$$

such that for each $t \in I$ the map $\phi_t := \phi(\cdot, t)$ is a deformation.

In particular, the set $\mathcal{B}_t := \phi_t(\mathcal{B})$ describes the state of the body at the time t.

If we keep a point $X \in \mathcal{B}$ fixed and let the time t vary, that is, defining $\phi_X(t) := \phi(X,t)$, we trace the motion of the single point X.



Definition 1.1.3. The **material velocity** is a vector field along ϕ, defined by

$$\boldsymbol{V}(X,t) := \frac{\partial \phi}{\partial t}(X,t).$$

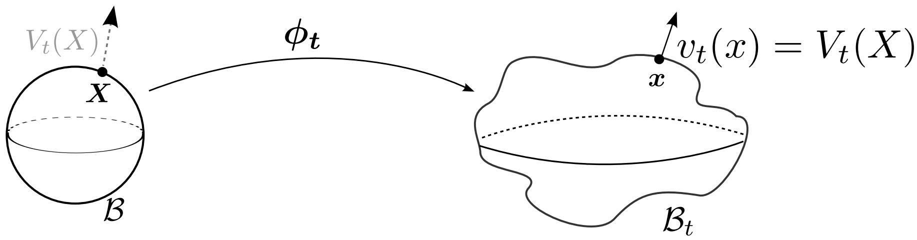
In general, the vector field $\boldsymbol{V}_t := \boldsymbol{V}(\cdot, t)$ is only a vector field along ϕ_t. If the dimensions of $\mathcal{B}$ and $\mathcal{S}$ coincide, however, then $\boldsymbol{V}_t$ is tangential to $\mathcal{B}_t$.

We now assign a velocity field $\boldsymbol{v}_t$ to the deformed body in the obvious way:

Definition 1.1.4. The **spatial velocity** of a motion ϕ is the map $\boldsymbol{v}_t : \mathcal{B}_t \to T\mathcal{S}$, defined by

$$\boldsymbol{v}_t := \boldsymbol{V}_t \circ \phi_t^{-1},$$

that is, if X and x are related by $\phi_t(X) = x$, then $\boldsymbol{v}_t(x) = \boldsymbol{V}_t(X)$.



If the dimensions of $\mathcal{B}$ and $\mathcal{S}$ coincide, then $\boldsymbol{v}_t$ is a vector field on $\mathcal{B}_t$. Otherwise it is just a vector field along the inclusion $\mathcal{B}_t \subset \mathcal{S}$.
To express physical laws like Newton's second law of motion we also need a notion of acceleration.

Definition 1.1.5. The **material acceleration** of a motion is a vector field along ϕ that is given by the covariant derivative of $\boldsymbol{V}$ along the map ϕ,

$$\boldsymbol{A}(X,t) := \frac{\nabla^{\mathcal{S}}}{\partial t}\boldsymbol{V}(X,t),$$

where $\nabla^{\mathcal{S}}$ denotes the connection on $(\mathcal{S}, g)$.
The **spatial acceleration** of a motion is the map $\boldsymbol{a}_t : \mathcal{B}_t \to TS$, defined by

$$\boldsymbol{a}_t := \boldsymbol{A}_t \circ \phi_t^{-1}.$$

Like the velocity $\boldsymbol{V}_t$, the acceleration $\boldsymbol{A}_t := \boldsymbol{A}(\cdot, t)$ is in general not tangential to $\mathcal{B}_t$, but only a vector field along ϕ_t. Moreover, $\boldsymbol{a}_t$ is a vector field along the inclusion $\mathcal{B}_t \subset \mathcal{S}$. Only if the dimensions of $\mathcal{B}$ and $\mathcal{S}$ coincide, $\boldsymbol{a}_t$ is tangential to $\mathcal{B}_t$.

The substantial derivative

We would like to express $\boldsymbol{a}_t$ directly by $\boldsymbol{v}_t$.
For that purpose we compute $\frac{\nabla}{\partial t}\boldsymbol{W}_t$ for an arbitrary vector field $\boldsymbol{W}_t$ along ϕ_t. We define

$$\widetilde{\mathcal{B}} := \bigcup_{t \in I}\Big(\mathcal{B}_t \times \{t\}\Big) \subset \mathcal{S} \times I =: \widetilde{\mathcal{S}}.$$

The tangent space of $\widetilde{\mathcal{S}}$ can be decomposed into

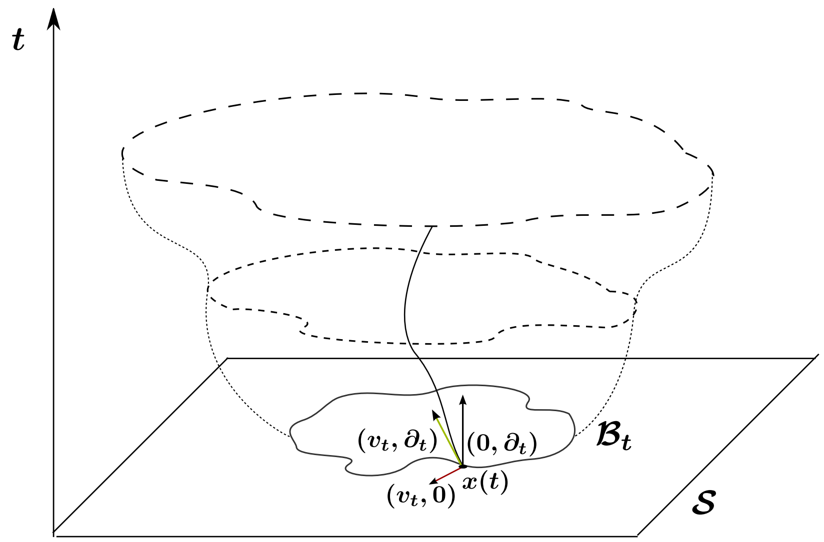
$$T_{(x,t)}\widetilde{\mathcal{S}} = T_x\mathcal{S} \oplus T_t I, \tag{1.1}$$

and we define a metric on $\widetilde{\mathcal{S}}$ by $g^{\widetilde{\mathcal{S}}} := g \oplus dt \otimes dt$.

For $X \in \mathcal{B}$ fix, the velocity field of the curve (see the picture below)

$$\begin{aligned} \gamma : I \quad &\to \quad \widetilde{\mathcal{B}} \\ t \quad &\mapsto \quad (\phi(X,t), t) \end{aligned}$$

is given by $(\boldsymbol{v}_t, \frac{\partial}{\partial t})$ and provides a vector field that is tangential to $\widetilde{\mathcal{B}}$. [1]



Theorem 1.1.6 (Bär [2014, sec. 1.1])
Let $\boldsymbol{W}_t$ be a vector field along ϕ_t. Define $\boldsymbol{w}_t := \boldsymbol{W}_t \circ \phi_t^{-1}$. Then

$$\frac{\nabla^{\mathcal{S}}}{\partial t} \boldsymbol{W}_t = \nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} \boldsymbol{w}_t,$$

where $\nabla^{\widetilde{\mathcal{S}}}$ denotes the connection on $(\widetilde{\mathcal{S}}, g^{\widetilde{\mathcal{S}}})$.

Proof. Let $\boldsymbol{w}_t := \boldsymbol{W}_t \circ \phi_t^{-1}$, assume $X \in \mathcal{B}$ and $x = \phi_t(X)$. Then the vector fields $(\boldsymbol{w}_t, \boldsymbol{\partial_t})$ and $(\boldsymbol{w}_t, \boldsymbol{0})$ are only defined along $\widetilde{\mathcal{B}}$, but so is $(\boldsymbol{v}_t, \boldsymbol{\partial_t})$, thus the following computation makes sense.

On the one hand, since $\boldsymbol{\partial_t}$ is constant,

$$\begin{aligned} \nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} (\boldsymbol{w}_t, \boldsymbol{\partial_t}) \Big|_{(x,t)} &= \nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} (\boldsymbol{w}_t, \boldsymbol{0}) \Big|_{(x,t)} + \nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} (\boldsymbol{0}, \boldsymbol{\partial_t}) \Big|_{(x,t)} \\ &= \left(\nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} \boldsymbol{w}_t, \boldsymbol{0} \right) \Big|_{(x,t)}. \end{aligned}$$

[1] By contrast, the vector field $(\boldsymbol{a}_t, \boldsymbol{\partial_t})$ is only for $\dim \mathcal{B} = \dim \mathcal{S}$ tangential to $\widetilde{\mathcal{B}}$. If $\dim \mathcal{B} < \dim \mathcal{S}$, then it is only a vector field along the inclusion $\widetilde{B} \subset \widetilde{S}$.

On the other hand, $(\boldsymbol{v}_t, \boldsymbol{\partial_t})$ is the velocity field of the curve γ and hence

$$
\begin{aligned}
\nabla^{\widetilde{S}}_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})} (\boldsymbol{w}_t, \boldsymbol{\partial_t}) \Big|_{(x,t)} &= \frac{\nabla^{\widetilde{S}}}{\partial t} (\boldsymbol{w}_t, \boldsymbol{\partial_t}) \Big|_{(x,t)} \\
&= \frac{\nabla^{\widetilde{S}}}{\partial t} (\boldsymbol{w}_t, \boldsymbol{0}) \Big|_{(x,t)} + \frac{\nabla^{\widetilde{S}}}{\partial t} (\boldsymbol{0}, \boldsymbol{\partial_t}) \Big|_{(x,t)} \\
&= \left(\frac{\nabla^{\mathcal{S}}}{\partial t} \boldsymbol{w}_t, \boldsymbol{0} \right) \Big|_{(x,t)} \\
&= \left(\frac{\nabla^{\mathcal{S}}}{\partial t} \boldsymbol{W}_t(X), \boldsymbol{0} \right),
\end{aligned}
$$

where we have used again that $\boldsymbol{\partial_t}$ is constant.
Comparing both expressions for $\nabla^{\widetilde{S}}_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})} (\boldsymbol{w}_t, \boldsymbol{\partial_t}) \Big|_{(x,t)}$, we conclude that

$$
\frac{\nabla^{\mathcal{S}}}{\partial t} \boldsymbol{W}_t = \nabla^{\widetilde{S}}_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})} \boldsymbol{w}_t. \qquad \square
$$

If the dimensions of the body and the surrounding space coincide, then not only $(\boldsymbol{v}_t, \boldsymbol{\partial_t})$ is tangential to $\widetilde{\mathcal{B}}$, but $(\boldsymbol{v}_t, \boldsymbol{0})$ and $(\boldsymbol{0}, \boldsymbol{\partial_t})$ are tangential to $\widetilde{\mathcal{B}}$, too. Then we are allowed to decompose $\nabla^{\widetilde{S}}_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})} \boldsymbol{w}_t$ into

$$
\nabla^{\widetilde{S}}_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})} \boldsymbol{w}_t = \nabla^{\widetilde{S}}_{(\boldsymbol{v}_t,\boldsymbol{0})} \boldsymbol{w}_t + \nabla^{\widetilde{S}}_{(\boldsymbol{0},\boldsymbol{\partial_t})} \boldsymbol{w}_t. \tag{1.2}
$$

Now we can use the decomposition (1.1) and obtain

$$
\begin{aligned}
\nabla^{\widetilde{S}}_{(\boldsymbol{v}_t,\boldsymbol{0})} \boldsymbol{w}_t &= \nabla^{\mathcal{S}}_{\boldsymbol{v}_t} \boldsymbol{w}_t \\
\nabla^{\widetilde{S}}_{(\boldsymbol{0},\boldsymbol{\partial_t})} \boldsymbol{w}_t &= \frac{\nabla \boldsymbol{w}_t}{\partial t}.
\end{aligned}
$$

Thus,

$$
\frac{\nabla^{\mathcal{S}}}{\partial t} \boldsymbol{W}_t = \frac{\nabla \boldsymbol{w}_t}{\partial t} + \nabla^{\mathcal{S}}_{\boldsymbol{v}_t} \boldsymbol{w}_t.
$$

If $\dim \mathcal{B} < \dim \mathcal{S}$, then $(\boldsymbol{v}_t, \boldsymbol{0})$ and $(\boldsymbol{0}, \boldsymbol{\partial_t})$ are not necessarily tangential to $\widetilde{\mathcal{B}}$. Thus the decomposition (1.2) does not make sense, since the single summands $\nabla^{\widetilde{S}}_{(\boldsymbol{v}_t,\boldsymbol{0})} \boldsymbol{w}_t$ and $\nabla^{\widetilde{S}}_{(\boldsymbol{0},\boldsymbol{\partial_t})} \boldsymbol{w}_t$ are not defined. In this case, we use the orthogonal projection $\tan : T_{(x,t)}\widetilde{S} \to T_{(x,t)}\widetilde{\mathcal{B}}$ onto the tangent space $T_{(x,t)}\widetilde{\mathcal{B}}$ and write

$$
\begin{aligned}
\frac{\nabla^{\mathcal{S}}}{\partial t} \boldsymbol{W}_t &= \nabla^{\widetilde{S}}_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})} \boldsymbol{w}_t \\
&= \nabla^{\widetilde{S}}_{\tan(\boldsymbol{v}_t,\boldsymbol{\partial_t})} \boldsymbol{w}_t \\
&= \nabla^{\widetilde{S}}_{\tan(\boldsymbol{v}_t,\boldsymbol{0})} \boldsymbol{w}_t + \nabla^{\widetilde{S}}_{\tan(\boldsymbol{0},\boldsymbol{\partial_t})} \boldsymbol{w}_t.
\end{aligned}
$$

Definition 1.1.7. The vector field

$$\dot{\boldsymbol{w}}_t := \nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})}\boldsymbol{w}_t = \nabla^{\widetilde{\mathcal{S}}}_{\tan(\mathbf{0},\boldsymbol{\partial_t})}\boldsymbol{w}_t + \nabla^{\widetilde{\mathcal{S}}}_{\tan(\boldsymbol{v}_t,\mathbf{0})}\boldsymbol{w}_t.$$

is called **substantial** (or **material**) **derivative** of $\boldsymbol{w}_t$. If the dimensions of $\mathcal{B}$ and $\mathcal{S}$ coincide, then the projections are redundant and

$$\dot{\boldsymbol{w}}_t = \frac{\nabla \boldsymbol{w}_t}{\partial t} + \nabla^{\mathcal{S}}_{\boldsymbol{v}_t}\boldsymbol{w}_t.$$

Thus, we have in particular

$$\boldsymbol{a}_t = \frac{\nabla^{\mathcal{S}}}{\partial t}\boldsymbol{V}_t \circ \phi_t^{-1} = \dot{\boldsymbol{v}}_t.$$

Let us consider the directional derivative $\partial_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})}f$ of a function $f \in C^\infty\big(\widetilde{B},\mathbb{R}\big)$. This derivative is well-defined, since $(\boldsymbol{v}_t,\boldsymbol{\partial_t})$ is tangential to $\widetilde{\mathcal{B}}$. If the dimensions of $\mathcal{B}$ and $\mathcal{S}$ coincide, then $(\boldsymbol{v}_t,\mathbf{0})$ and $(\mathbf{0},\boldsymbol{\partial_t})$ are tangential to $\widetilde{\mathcal{B}}$, too, and we are allowed to decompose $\partial_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})}f$ into

$$\partial_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})}f = \partial_{(\boldsymbol{v}_t,\mathbf{0})}f + \partial_{(\mathbf{0},\boldsymbol{\partial_t})}f. \tag{1.3}$$

Again, we can use the decomposition (1.1) and obtain

$$\begin{aligned}\partial_{(\boldsymbol{v}_t,\mathbf{0})}f &= g(\mathbf{grad}_x f,\boldsymbol{v}_t)\\ \partial_{(\mathbf{0},\boldsymbol{\partial_t})}f &= \frac{\partial f}{\partial t},\end{aligned}$$

where $\mathbf{grad}_x f$ denotes the spatial gradient of f, that is, $\mathbf{grad}_x f$ is tangential to $\mathcal{B}_t$. Thus,

$$\partial_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})}f = \frac{\partial f}{\partial t} + g(\mathbf{grad}_x f,\boldsymbol{v}_t). \tag{1.4}$$

If $\dim\mathcal{B} < \dim\mathcal{S}$, then the decomposition (1.3) does not make sense, since the single summands $\partial_{(\boldsymbol{v}_t,\mathbf{0})}f$ and $\partial_{(\mathbf{0},\boldsymbol{\partial_t})}f$ are not defined. Thus, we use again the orthogonal projection $\tan : T_{(x,t)}\widetilde{S} \to T_{(x,t)}\widetilde{\mathcal{B}}$ onto the tangent space $T_{(x,t)}\widetilde{\mathcal{B}}$ and write

$$\begin{aligned}\partial_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})}f &= \partial_{\tan(\boldsymbol{v}_t,\boldsymbol{\partial_t})}f\\ &= \partial_{\tan(\boldsymbol{v}_t,\mathbf{0})}f + \partial_{\tan(\mathbf{0},\boldsymbol{\partial_t})}f.\end{aligned}$$

Definition 1.1.8. For any function $f \in C^\infty\big(\tilde{B},\mathbb{R}\big)$,

$$\dot{f} := \partial_{(\boldsymbol{v}_t,\boldsymbol{\partial_t})}f = \partial_{\tan(\mathbf{0},\boldsymbol{\partial_t})}f + \partial_{\tan(\boldsymbol{v}_t,\mathbf{0})}f$$

is called substantial derivative of f. If the dimensions of $\mathcal{B}$ and $\mathcal{S}$ coincide, then this simplifies to

$$\dot{f} = \frac{\partial f}{\partial t} + g(\mathbf{grad}_x f, \boldsymbol{v}_t).$$

Lemma 1.1.9
The substantial derivative satisfies:

1) *For any functions $f, h \in C^\infty(\tilde{B}, \mathbb{R})$*

$$\dot{\widehat{f \cdot h}} = \dot{f}\, h + f\, \dot{h}.$$

2) *Let $\boldsymbol{W}_t$ and $\boldsymbol{Z}_t$ be vector fields along ϕ_t. Define $\boldsymbol{w}_t := \boldsymbol{W}_t \circ \phi_t^{-1}$ and $\boldsymbol{z}_t := \boldsymbol{Z}_t \circ \phi_t^{-1}$. Then*

$$\dot{\widehat{g(\boldsymbol{w}_t, \boldsymbol{z}_t)}} = g(\dot{\boldsymbol{w}}_t, \boldsymbol{z}_t) + g(\boldsymbol{w}_t, \dot{\boldsymbol{z}}_t).$$

Proof.

1) This follows immediately from the definition of the substantial derivative.

$$\begin{aligned}\dot{\widehat{f \cdot h}} &= \partial_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})}(f \cdot h) \\ &= (\partial_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} f) \cdot h + f \cdot (\partial_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} h) \\ &= \dot{f}\, h + f\, \dot{h}.\end{aligned}$$

2) A direct computation yields

$$\begin{aligned}\dot{\widehat{g(\boldsymbol{w}_t, \boldsymbol{z}_t)}} &= \partial_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} g(\boldsymbol{w}_t, \boldsymbol{z}_t) \\ &= \partial_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} g^{\widetilde{\mathcal{S}}}(\boldsymbol{w}_t \oplus \mathbf{0}, \boldsymbol{z}_t \oplus \mathbf{0}) \\ &= g^{\widetilde{\mathcal{S}}}\big(\nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})}(\boldsymbol{w}_t \oplus \mathbf{0}), \boldsymbol{z}_t \oplus \mathbf{0}\big) + g^{\widetilde{\mathcal{S}}}\big(\boldsymbol{w}_t \oplus \mathbf{0}, \nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})}(\boldsymbol{z}_t \oplus \mathbf{0})\big) \\ &= g^{\widetilde{\mathcal{S}}}\big((\nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} \boldsymbol{w}_t) \oplus \mathbf{0}, \boldsymbol{z}_t \oplus \mathbf{0}\big) + g^{\widetilde{\mathcal{S}}}\big(\boldsymbol{w}_t \oplus \mathbf{0}, (\nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} \boldsymbol{z}_t) \oplus \mathbf{0}\big) \\ &= g\big(\nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} \boldsymbol{w}_t, \boldsymbol{z}_t\big) + g\big(\boldsymbol{w}_t, \nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} \boldsymbol{z}_t\big) \\ &= g(\dot{\boldsymbol{w}}_t, \boldsymbol{z}_t) + g(\boldsymbol{w}_t, \dot{\boldsymbol{z}}_t).\end{aligned}$$

□

Remark 1.1.10. Assume that $f : \widetilde{\mathcal{B}} \to \mathbb{R}$ is differentiable. Let us define a corresponding function $F : \mathcal{B} \times I \to \mathbb{R}$ by

$$F(X,t) := f(\phi(X,t),t).$$

Let $\psi : \mathcal{B} \times I \to \widetilde{\mathcal{B}}$ be given by $(X,t) \overset{\psi}{\mapsto} (\phi(X,t),t)$. Then for $x = \phi_t(X)$,

$$\begin{aligned}
\dot{f}(x,t) &= \partial_{(\boldsymbol{v}_t, \boldsymbol{\partial_t})} f\Big|_{(x,t)} \\
&= df\Big|_{(x,t)} (\boldsymbol{v}_t, \boldsymbol{\partial_t}) \\
&= \left[dF\Big|_{\psi^{-1}(x,t)} \circ d\psi^{-1}\Big|_{(x,t)} \right] (\boldsymbol{v}_t, \boldsymbol{\partial_t}) \\
&= dF\Big|_{(X,t)} (\boldsymbol{\partial_t}) \\
&= \frac{\partial F}{\partial t}(X,t).
\end{aligned}$$

Thus, the time derivatives of F and f are related by

$$\frac{\partial F}{\partial t} = \dot{f} \circ \psi. \tag{1.5}$$

Notation 1.1.11. In accordance with standard notation of elasticity, points in $\mathcal{B}$ or other geometrical expressions concerning $\mathcal{B}$ are denoted by capital letters, whereas quantities concerning $\mathcal{S}$ are denoted by small letters. Bold letters are used for vector and tensors fields, calligraphic letters for tensor fields coming from elasticity theory. Note that a list of symbols can be found in the index at the end of this work.

Occasionally we will consider bodies for which the substantial derivatives and later the equations governing the motion of the body are especially simple:

Definition 1.1.12. A **simple body** is the closure $\bar{\Omega}$ of a bounded, open, connected subset $\Omega \subset \mathbb{R}^3$.

1.2 Deformation Tensors

The contents of the first part of this section are taken from Marsden and Hughes [1983].

Definition 1.2.1. Let $\phi : \mathcal{B} \to \mathcal{S}$ be a deformation. The **deformation gradient** $\boldsymbol{\mathcal{F}}$ of ϕ is given by the differential of ϕ:

$$\boldsymbol{\mathcal{F}}|_X := d\phi|_X.$$

Remark 1.2.2. Let $\{X^A\}$ and $\{x^a\}$ denote local coordinates on $\mathcal{B}$ and $\mathcal{S}$, respectively. Then in components,

$$\mathcal{F}^a{}_A(X) = \frac{\partial \phi^a}{\partial X^A}(X).$$

The coordinates of the transpose, $\boldsymbol{\mathcal{F}}^T$, are given by

$$\left(\boldsymbol{\mathcal{F}}^T\right)^A{}_a(x) = g_{ab}(x)\,\mathcal{F}^b{}_B(X)\,G^{AB}(X),$$

where $x = \phi(X)$.

For the deformation $\phi : \mathcal{B} \to \mathbb{R}^3$ of a simple body $\mathcal{B}$,

$$\phi(x+h) - \phi(x) = \boldsymbol{\mathcal{F}}(h) + o(h)$$

and hence

$$\begin{aligned}\|\phi(x+h) - \phi(x)\|^2 &= \|\boldsymbol{\mathcal{F}}(h)\|^2 + o\big(\|h\|^2\big)\\ &= \big\langle h, \boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}}(h)\big\rangle + o\big(\|h\|^2\big).\end{aligned}$$

Thus, we expect $\boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}}$ to be a measure for local changes of the body's form. Hence we make the following definition.

Definition 1.2.3. The tensor

$$\begin{aligned}\boldsymbol{\mathcal{C}}\Big|_x : T_X\mathcal{B} &\to T_X\mathcal{B}\\ \boldsymbol{V} &\mapsto \boldsymbol{\mathcal{C}}|_X(\boldsymbol{V}) := (\boldsymbol{\mathcal{F}}|_X)^T\boldsymbol{\mathcal{F}}|_X(\boldsymbol{V}).\end{aligned}$$

is called **deformation** or (right Cauchy-Green) **strain tensor**.

Theorem 1.2.4 (Marsden and Hughes [1983, ch.1, Prop. 3.6])
$\boldsymbol{\mathcal{C}}$ is symmetric and positive definite and in particular invertible.

Proof. For $\boldsymbol{V}, \boldsymbol{W} \in T_X\mathcal{B}$

$$G(\boldsymbol{\mathcal{C}}(\boldsymbol{V}), \boldsymbol{W}) = G\left(\boldsymbol{\mathcal{F}}^T\,\boldsymbol{\mathcal{F}}(\boldsymbol{V}), \boldsymbol{W}\right) = G\left(\boldsymbol{V}, \boldsymbol{\mathcal{F}}^T\,\boldsymbol{\mathcal{F}}(\boldsymbol{W})\right) = G(\boldsymbol{V}, \boldsymbol{\mathcal{C}}(\boldsymbol{W})),$$

so $\boldsymbol{\mathcal{C}}$ is symmetric. Since $G(\boldsymbol{\mathcal{C}}(\boldsymbol{V}), \boldsymbol{V}) = g(\boldsymbol{\mathcal{F}}(\boldsymbol{V}), \boldsymbol{\mathcal{F}}(\boldsymbol{V})) \geq 0$, $\boldsymbol{\mathcal{C}}$ is non-negative. Assume now $G(\boldsymbol{\mathcal{C}}(\boldsymbol{V}), \boldsymbol{V}) = 0$. Then $g(\boldsymbol{\mathcal{F}}(\boldsymbol{V}), \boldsymbol{\mathcal{F}}(\boldsymbol{V})) = 0$. Since ϕ is an embedding, $\boldsymbol{\mathcal{F}}$ is one-to-one, and the last equation implies that $\boldsymbol{V}$ must be zero. $\square$

Remark 1.2.5. $\boldsymbol{\mathcal{C}}$ can also be defined, if ϕ is not an embedding, but only smooth. Then $\boldsymbol{\mathcal{C}}$ will still be symmetric, but it will only be non-negative and thus not invertible in general.

Remark 1.2.6 (Marsden and Hughes [1983], ch.1, Prop. 4.13).
The deformation tensor is related to the metric g by $\boldsymbol{\mathcal{C}}^\flat = \phi^* g$:

$$\begin{aligned}\boldsymbol{\mathcal{C}}^\flat(\boldsymbol{W}_1, \boldsymbol{W}_2) &= G(\boldsymbol{\mathcal{C}}(\boldsymbol{W}_1), \boldsymbol{W}_2) = G(\boldsymbol{\mathcal{F}}^T \boldsymbol{\mathcal{F}}(\boldsymbol{W}_1), \boldsymbol{W}_2) = g(\boldsymbol{\mathcal{F}}(\boldsymbol{W}_1), \boldsymbol{\mathcal{F}}(\boldsymbol{W}_2)) \\ &= (\phi^* g)(\boldsymbol{W}_1, \boldsymbol{W}_2).\end{aligned}$$

If $\{X^A\}$ and $\{x^a\}$ denote local coordinate systems on $\mathcal{B}$ and $\mathcal{S}$, respectively, then in components,

$$(\boldsymbol{\mathcal{C}}^\flat|_X)_{AB} = g_{ab}|_x (\mathcal{F}|_X)^a{}_A \, (\mathcal{F}|_X)^b{}_B = g_{ab}|_x \frac{\partial \phi^a}{\partial X^A} \frac{\partial \phi^b}{\partial X^B}.$$

Recall that the length of a piecewise C^1-curve $\gamma : [a, b] \to \mathcal{B}$ is given by

$$l(\gamma) = \int_a^b \|\gamma'(s)\| \, ds.$$

Theorem 1.2.7 (Marsden and Hughes [1983, ch.1, Prop. 3.16])
Let γ be a C^1-curve in $\mathcal{B}$ and let ϕ be a deformation of $\mathcal{B}$ in $\mathcal{S}$. Let $\widetilde{\gamma} = \phi \circ \gamma$ be the image of γ under ϕ. Then

$$l(\widetilde{\gamma}) = \int_a^b \sqrt{G\big(\gamma'(s), \boldsymbol{\mathcal{C}}(\gamma'(s))\big)} \, ds.$$

In particular, the length of $\widetilde{\gamma}$ only depends on the deformation tensor $\boldsymbol{\mathcal{C}}$ and γ.

Proof. By the chain rule, $\widetilde{\gamma}'(s) = \boldsymbol{\mathcal{F}}|_{\gamma(s)}(\gamma'(s))$. Thus,

$$\begin{aligned}|\widetilde{\gamma}'(s)|^2 &= g\big(\boldsymbol{\mathcal{F}}(\gamma'(s)), \boldsymbol{\mathcal{F}}(\gamma'(s))\big) \\ &= G\big(\gamma'(s), \boldsymbol{\mathcal{F}}^T \boldsymbol{\mathcal{F}}(\gamma'(s))\big) \\ &= G\big(\gamma'(s), \boldsymbol{\mathcal{C}}(\gamma'(s))\big).\end{aligned}$$

This yields

$$l(\widetilde{\gamma}) = \int_a^b \sqrt{G\big(\gamma'(s), \boldsymbol{\mathcal{C}}(\gamma'(s))\big)} \, ds. \qquad \square$$

We have just seen that for some curve γ on $\mathcal{B}$ and some fixed deformation ϕ the deformation tensor $\boldsymbol{\mathcal{C}}$ relates the length of γ with the length of $\phi(\gamma)$. The following tensor describes how the angle between the images of tangent vectors on $\mathcal{B}$ changes during a motion $\phi : \mathcal{B} \times I \to \mathcal{S}$.

Definition 1.2.8. The tensor $\boldsymbol{\mathcal{D}}|_{(X,t)} : T_X\mathcal{B} \to T_X\mathcal{B}$, defined by

$$\boldsymbol{\mathcal{D}}|_{(X,t)} := \frac{1}{2}\frac{\partial \boldsymbol{\mathcal{C}}}{\partial t}|_{(X,t)}$$

is called **material rate of deformation tensor**.

Theorem 1.2.9 (Marsden and Hughes [1983, ch.1, Prop. 3.30])
$\boldsymbol{\mathcal{D}}$ is a measure for the geodesic deviation caused by the deformation ϕ_t:
Let $\boldsymbol{W}_1, \boldsymbol{W}_2 \in T_X\mathcal{B}$ and $\boldsymbol{w}_i(t) := \boldsymbol{\mathcal{F}}|_{(X,t)}(\boldsymbol{W}_i)$, $i = 1, 2$. Then

$$\frac{d}{dt}\, g\big(\boldsymbol{w}_1(t), \boldsymbol{w}_2(t)\big) = G\big(\boldsymbol{W}_1, 2\boldsymbol{\mathcal{D}}|_{(X,t)}(\boldsymbol{W}_2)\big).$$

That is, $\boldsymbol{\mathcal{D}}$ measures the change of the angle between $\boldsymbol{w}_1(t)$ and $\boldsymbol{w}_2(t)$ with time.

Proof. Let $\boldsymbol{W}_1, \boldsymbol{W}_2 \in T_X\mathcal{B}$ and $\boldsymbol{w}_i(t) := \boldsymbol{\mathcal{F}}|_{(X,t)}(\boldsymbol{W}_i)$, $i = 1, 2$. Then

$$\begin{aligned}
\frac{d}{dt}\, g\big(\boldsymbol{w}_1(t), \boldsymbol{w}_2(t)\big) &= \frac{d}{dt}\, g\big(\boldsymbol{\mathcal{F}}|_{(X,t)}(\boldsymbol{W}_1), \boldsymbol{\mathcal{F}}|_{(X,t)}(\boldsymbol{W}_2)\big) \\
&= \frac{d}{dt}\, G\big(\boldsymbol{W}_1, (\boldsymbol{\mathcal{F}}|_{(X,t)})^T \boldsymbol{\mathcal{F}}|_{(X,t)}(\boldsymbol{W}_2)\big) \\
&= \frac{d}{dt}\, G(\boldsymbol{W}_1, \boldsymbol{\mathcal{C}}|_{(X,t)}(\boldsymbol{W}_2)) \\
&= G\big(\boldsymbol{W}_1, 2\boldsymbol{\mathcal{D}}|_{(X,t)}(\boldsymbol{W}_2)\big).
\end{aligned}$$ □

We would like to express $\boldsymbol{\mathcal{D}}$ directly in terms of the motion.

Theorem 1.2.10 (Marsden and Hughes [1983, ch. 1, Prop. 5.14])
The rate of deformation tensor satisfies

$$\boldsymbol{\mathcal{D}}^\flat = \frac{1}{2}\frac{\partial}{\partial t}\big(\phi_t^* g\big) = \frac{1}{2}\,\phi_t^*\left((\nabla^{\mathcal{S}}\boldsymbol{v})^\flat + \left[(\nabla^{\mathcal{S}}\boldsymbol{v})^\flat\right]^T\right).$$

To prove this theorem we need to compute the time derivative of $\boldsymbol{\mathcal{F}}_t := d\phi_t$.

Lemma 1.2.11
Let us define $\boldsymbol{\mathcal{F}}_t := d\phi_t$*. Then*

$$\frac{d}{dt}\boldsymbol{\mathcal{F}}_t = (\nabla^{\mathcal{S}}\boldsymbol{v}) \circ \boldsymbol{\mathcal{F}}_t.$$

Proof. Assume that $\boldsymbol{W} \in T_X\mathcal{B}$. Let $\gamma : (-a, a) \to \mathcal{B}$ be a curve on $\mathcal{B}$, such that $\dot{\gamma}(0) = \boldsymbol{W}$. Then

$$\begin{aligned}
\frac{d\boldsymbol{\mathcal{F}}_t}{dt}(\boldsymbol{W}) &= \frac{\nabla}{\partial t}\left(\boldsymbol{\mathcal{F}}_t(\boldsymbol{W})\right) \\
&= \frac{\nabla}{\partial t}\left(d\phi_t(\boldsymbol{W})\right) \\
&= \frac{\nabla}{\partial t}\left(\frac{d}{ds}(\phi_t \circ \gamma)|_{s=0}\right) \\
&= \frac{\nabla}{\partial s}\left(\frac{\partial}{\partial t}(\phi_t \circ \gamma)\right)|_{s=0} \\
&= \frac{\nabla}{\partial s}\left(\boldsymbol{V}_t \circ \gamma(s)\right)|_{s=0} \\
&= \nabla_{\boldsymbol{W}}\boldsymbol{V} \\
&= \nabla^{\mathcal{S}}_{d\phi_t(\boldsymbol{W})}\boldsymbol{v}.
\end{aligned}$$

Here, $\nabla_{\boldsymbol{W}}\boldsymbol{V}$ denotes the covariant derivative of $\boldsymbol{V}$ in the direction of $\boldsymbol{W}$ along the map ϕ_t. Thus, $\frac{d}{dt}\boldsymbol{\mathcal{F}}_t = (\nabla^{\mathcal{S}}\boldsymbol{v}) \circ \boldsymbol{\mathcal{F}}_t$. □

Proof of Theorem 1.2.10. By Remark 1.2.6,

$$2\,\boldsymbol{\mathcal{D}}^\flat = \frac{\partial}{\partial t}\boldsymbol{\mathcal{C}}^\flat = \frac{\partial}{\partial t}\left(\phi_t^* g\right).$$

Let $\boldsymbol{W}_1, \boldsymbol{W}_2 \in T\mathcal{B}$. Then

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\phi_t^* g\right)(\boldsymbol{W}_1, \boldsymbol{W}_2) &= \frac{\partial}{\partial t}\left(g(\boldsymbol{\mathcal{F}}_t(\boldsymbol{W}_1), \boldsymbol{\mathcal{F}}_t(\boldsymbol{W}_2))\right) \\
&= g\left(\frac{d\boldsymbol{\mathcal{F}}_t}{dt}(\boldsymbol{W}_1), \boldsymbol{\mathcal{F}}_t(\boldsymbol{W}_2)\right) + g\left(\boldsymbol{\mathcal{F}}(\boldsymbol{W}_1), \frac{d\boldsymbol{\mathcal{F}}_t}{dt}(\boldsymbol{W}_2)\right)
\end{aligned}$$

Employing Lemma 1.2.11 we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\phi_t^* g\right)(\boldsymbol{W}_1, \boldsymbol{W}_2) &= g\left(\nabla^{\mathcal{S}}_{d\phi_t(\boldsymbol{W}_1)}\boldsymbol{v}, d\phi_t(\boldsymbol{W}_2)\right) + g\left(d\phi_t(\boldsymbol{W}_1), \nabla^{\mathcal{S}}_{d\phi_t(\boldsymbol{W}_2)}\boldsymbol{v}\right) \\
&= \phi_t^*(\nabla^{\mathcal{S}}\boldsymbol{v})^\flat(\boldsymbol{W}_1, \boldsymbol{W}_2) + \phi_t^*\left[(\nabla^{\mathcal{S}}\boldsymbol{v})^\flat\right]^T(\boldsymbol{W}_1, \boldsymbol{W}_2).
\end{aligned}$$

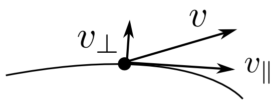
□

Definition 1.2.12. We define the **spatial rate of deformation tensor** $\boldsymbol{d} : T\widetilde{\mathcal{B}} \to T\widetilde{\mathcal{B}}$ by

$$\boldsymbol{d}^\flat := \phi_* \boldsymbol{\mathcal{D}}^\flat = \frac{1}{2}\, L_{\boldsymbol{v}} g.$$

Our next aim is to express $\boldsymbol{\mathcal{D}}$ directly in terms of the motion. To do this, we decompose the spatial velocity into its tangent and normal part (with respect to the metric g)

$$\begin{aligned} T_x\mathcal{S} &= T_x\mathcal{B}_t \oplus N_x\mathcal{B}_t \\ \boldsymbol{v}(x,t) &= \boldsymbol{v}_\|(x,t) + \boldsymbol{v}_\perp(x,t). \end{aligned}$$



The proof of the following theorem was originally given by Marsden and Hughes [1983], but recently a more elaborate proof was given by Grabs [2014].

Theorem 1.2.13 (see Marsden and Hughes [1983, ch.1, Box 5.1, p.92]) *The spatial rate of deformation tensor satisfies*

$$\boldsymbol{d}^\flat = \frac{1}{2}\left(\left(\nabla^{\mathcal{B}_t}\boldsymbol{v}_\|\right)^\flat + \left[\left(\nabla^{\mathcal{B}_t}\boldsymbol{v}_\|\right)^\flat\right]^T - 2\, g(\boldsymbol{v}_\perp, \mathbf{II}(\cdot,\cdot)) \right),$$

where $\mathbf{II}$ *denotes the second fundamental form of* $\mathcal{B}_t$ *in* $\mathcal{S}$*.*

Proof. By Theorem 1.2.10

$$\boldsymbol{\mathcal{D}}^\flat = \frac{1}{2}\,\phi_t^*\left((\nabla^{\mathcal{S}}\boldsymbol{v})^\flat + \left[(\nabla^{\mathcal{S}}\boldsymbol{v})^\flat\right]^T \right).$$

We decompose the velocity into its tangential and normal part, $\boldsymbol{v} = \boldsymbol{v}_\| + \boldsymbol{v}_\perp$. For all $\boldsymbol{W}, \boldsymbol{Z} \in T_X\mathcal{B}$,

$$\begin{aligned} \left(\phi_t^*(\nabla^{\mathcal{S}}\boldsymbol{v}_\|)^\flat\right)(\boldsymbol{W},\boldsymbol{Z}) &= \phi_t^*\, g(\nabla^{\mathcal{S}}_{(\cdot)}\boldsymbol{v}_\|, \cdot)(\boldsymbol{W},\boldsymbol{Z}) \\ &= g(\nabla^{\mathcal{S}}_{d\phi_t(\boldsymbol{W})}\boldsymbol{v}_\|, d\phi_t(\boldsymbol{Z})) \\ &= g\left(\nabla^{\mathcal{B}_t}_{d\phi_t(\boldsymbol{W})}\boldsymbol{v}_\| + \mathbf{II}(d\phi_t(\boldsymbol{W}), \boldsymbol{v}_\|), d\phi_t(\boldsymbol{Z})\right) \\ &= g\left(\nabla^{\mathcal{B}_t}_{d\phi_t(\boldsymbol{W})}\boldsymbol{v}_\|, d\phi_t(\boldsymbol{Z})\right), \end{aligned}$$

where we have used in the last step that the image of $\mathbf{II}$ is contained in $N_x\mathcal{B}_t$, while $d\phi_t(\boldsymbol{Z})$ is contained in $T_x\mathcal{B}_t$. Moreover, since $\boldsymbol{v}_\perp$ is perpendicular to $d\phi_t(\boldsymbol{Z})$,

$$\begin{aligned} \left(\phi_t^*(\nabla^{\mathcal{S}}\boldsymbol{v}_\perp)^\flat\right)(\boldsymbol{W},\boldsymbol{Z}) &= g\left(\nabla^{\mathcal{S}}_{d\phi_t(\boldsymbol{W})}\boldsymbol{v}_\perp, d\phi_t(\boldsymbol{Z})\right) \\ &= -g\left(\boldsymbol{v}_\perp, \nabla^{\mathcal{S}}_{d\phi_t(\boldsymbol{W})} d\phi_t(\boldsymbol{Z})\right) \\ &= -g\big(\boldsymbol{v}_\perp, \mathbf{II}(d\phi_t(\boldsymbol{W}), d\phi_t(\boldsymbol{Z}))\big). \end{aligned}$$

Thus, $\phi_t^*\left((\nabla^{\mathcal{S}}\boldsymbol{v})^\flat\right) : T_X\mathcal{B}\times T_X\mathcal{B}\to\mathbb{R}$ is given by

$$\phi_t^*\left((\nabla^{\mathcal{S}}\boldsymbol{v})^\flat\right) = \phi_t^*\left((\nabla^{\mathcal{B}_t}\boldsymbol{v}_\|)^\flat - g(\boldsymbol{v}_\perp, \mathbf{II}(\cdot,\cdot))\right)$$

Hence,

$$\begin{aligned}\boldsymbol{d}^\flat &= \frac{1}{2}\left((\nabla^{\mathcal{B}_t}\boldsymbol{v}_\|)^\flat - g(\boldsymbol{v}_\perp, \mathbf{II}(\cdot,\cdot)) + \left[(\nabla^{\mathcal{B}_t}\boldsymbol{v}_\|)^\flat - g(\boldsymbol{v}_\perp, \mathbf{II}(\cdot,\cdot))\right]^T\right)\\ &= \frac{1}{2}\left((\nabla^{\mathcal{B}_t}\boldsymbol{v}_\|)^\flat + \left[(\nabla^{\mathcal{B}_t}\boldsymbol{v}_\|)^\flat\right]^T - 2\,g(\boldsymbol{v}_\perp, \mathbf{II}(\cdot,\cdot))\right). \qquad \square\end{aligned}$$

Notation 1.2.14.
Let $\boldsymbol{S}$ be a (p,q) tensor and $\boldsymbol{T}$ be a (r,s) tensor, where $p+q=r+s$. Then we define the scalar product of $\boldsymbol{S}$ and $\boldsymbol{T}$ as the contraction of $\boldsymbol{S}^\sharp$ and $\boldsymbol{T}^\flat$ on all entries. That is, if $E_1,\dots,E_m$ is a basis of $T_X\mathcal{B}$ and $E^1,\dots,E^m$ is the corresponding dual basis, then

$$\langle\boldsymbol{S},\boldsymbol{T}\rangle := \boldsymbol{S}^\sharp(E^{A_1},\dots,E^{A_p},E^{B_1},\dots,E^{B_q})\boldsymbol{T}^\flat(E_{A_1},\dots,E_{A_p},E_{B_1},\dots,E_{B_q}).$$

In components,

$$\langle\boldsymbol{S},\boldsymbol{T}\rangle = S^{A_1\dots A_pB_1\dots B_q}T_{A_1\dots A_pB_1\dots B_q}.$$

This definition is independent of the choice of the basis $E_1,\dots,E_m$.
If $E_1,\dots,E_m$ is an orthonormal basis, then we may also write

$$\langle\boldsymbol{S},\boldsymbol{T}\rangle = \boldsymbol{S}^\flat(E_{A_1},\dots,E_{A_p},E_{B_1},\dots,E_{B_q})\boldsymbol{T}^\flat(E_{A_1},\dots,E_{A_p},E_{B_1},\dots,E_{B_q}),$$

or in components,

$$\langle\boldsymbol{S},\boldsymbol{T}\rangle = \sum_{A_1,\dots,B_s=1}^{m} S_{A_1\dots A_pB_1\dots B_q}T_{A_1\dots A_rB_1\dots B_s}.$$

Moreover, if $\boldsymbol{S}$ and $\boldsymbol{T}$ are $(1,1)$ tensors, then $\langle\boldsymbol{S},\boldsymbol{T}\rangle$ coincides with

$$\operatorname{tr}(\boldsymbol{S}\circ\boldsymbol{T}) = \sum_{A=1}^{m} g(\boldsymbol{S}\circ\boldsymbol{T}(E_A),E_A).$$

If $\boldsymbol{V}_t$ and $\boldsymbol{W}_t$ are vector fields along the deformation ϕ_t and $x=\phi_t(X)$, then

$$\langle\boldsymbol{V}_t,\boldsymbol{W}_t\rangle(X) = (V_t)^a\,(W_t)_a = g_{ab}(x)\,(V_t)^a(X)\,(W_t)^b(X) = g_x(\boldsymbol{V}_t(X),\boldsymbol{W}_t(X)).$$

For vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$ on $\mathcal{S}$

$$\langle\boldsymbol{v},\boldsymbol{w}\rangle = v^a\,w_a = g_{ab}\,v^a\,w^b = g(\boldsymbol{v},\boldsymbol{w}).$$

1.3 The Master Balance Law

1.3.1 The Spatial Master Balance Law

Let $\phi_t : \mathcal{B} \to \mathcal{S}$ be a time-dependent deformation. The metric g of $\mathcal{S}$ can be restricted to a metric g_t on $\mathcal{B}_t$, which induces the volume element $\mathbf{vol}_t$ on $\mathcal{B}_t$.
For each subset U of $\mathcal{B}$ we define $U_t := \phi_t(U)$. We will call a set $U \subset \mathcal{B}$ **nice** if it is open and relatively compact with piecewise C^1-boundary.

The motion of $\mathcal{B}$ is governed by a system of partial differential equations that consists of balance laws including the acting forces and exchanged energies. All of these balance laws are essentially of the following form:

Definition 1.3.1 (Marsden and Hughes [1983]). Let $a, b : \mathcal{B}_t \times I \to \mathbb{R}$ be scalar functions on $\mathcal{B}_t$ and c be a scalar function on the unit tangent bundle of $\mathcal{B}_t \times I$.
We say that a, b, and c satisfy the **(spatial) master balance law**, if for any nice set $U \subset \mathcal{B}$

i) the integrals in iii) exist,

ii) $\int\limits_{U_t} a\,\mathbf{vol}_t$ is differentiable in t,

iii) and
$$\frac{d}{dt}\int\limits_{U_t} a\,\mathbf{vol}_t = \int\limits_{U_t} b\,\mathbf{vol}_t + \int\limits_{\partial U_t} c\big(x,t,\boldsymbol{n}\big)\,\mathbf{vol}_{\partial U_t}, \tag{1.6}$$
where $\boldsymbol{n}$ is the unit outward normal field to ∂U_t.

Remark 1.3.2. Sloppily said, the master balance law states that the rate of increase of a over the volume U_t is equal to sources b inside of U_t inducing the growth of a and some inflow c through the boundary of U_t.
If the body is isolated (which means that $b = 0$ and that $c = 0$ on ∂U_t), then a is constant in time.

The following theorem is one of the most fundamental theorems of elasticity theory. It can be found, for example, in the book of [Marsden and Hughes, 1983, ch.2, Th. 1.9]. Here, we state it in a form that is similar to the one presented in Bär [2014].

Let $S_t \subset T\mathcal{B}_t$ be the unit tangent bundle. Similar to the construction of $\widetilde{\mathcal{B}}$ in section 1.1, we define the sphere bundle $S\widetilde{\mathcal{B}}$ over $\widetilde{\mathcal{B}}$ by

$$S\widetilde{\mathcal{B}} := \bigcup_{t\in I}\Big(S_t \times \{t\}\Big) \to \bigcup_{t\in I}\Big(\mathcal{B}_t \times \{t\}\Big) = \widetilde{\mathcal{B}}$$

Theorem 1.3.3 (Cauchy's Theorem, Bär [2014, Satz 2.2.1])
Let $a, b : \widetilde{\mathcal{B}} \to \mathbb{R}$ be C^2 functions and assume that $c : S\widetilde{\mathcal{B}} \to \mathbb{R}$ is C^2. Moreover, assume that a, b, and c satisfy the master balance law, which means for all nice sets $U \Subset \mathcal{B}$

$$\frac{d}{dt}\int\limits_{U_t} a(x,t)\,\mathbf{vol}_t = \int\limits_{U_t} b(x,t)\,\mathbf{vol}_t + \int\limits_{\partial U_t} c(x,t,\boldsymbol{n})\,\mathbf{vol}_{\partial U_t},$$

where $\boldsymbol{n}$ is the unit outward normal to ∂U_t. Then for every $(x,t) \in \widetilde{\mathcal{B}}$ the function c is on $T_x\mathcal{B}_t$ the restriction of a linear function. In particular, there is a unique vector field $\boldsymbol{c}$ on ∂U_t, such that

$$c(x,t,\boldsymbol{n}) = g_t(\boldsymbol{c}(x,t),\boldsymbol{n}).$$

Corollary 1.3.4
a, b, and $\boldsymbol{c}$ as defined in Theorem 1.3.3 satisfy the spatial master balance law if and only if

$$\frac{d}{dt}\int\limits_{U_t} a\,\mathbf{vol}_t = \int\limits_{U_t} b\,\mathbf{vol}_t + \int\limits_{\partial U_t} g_t(\boldsymbol{c}(x,t),\boldsymbol{n})\,\mathbf{vol}_{\partial U_t}. \tag{1.7}$$

To find a local form form of the master balance law, we have to pull the derivative $\frac{d}{dt}$ into the integral $\int_{U_t} a(x,t)\,\mathbf{vol}_t$, keeping in mind that the domain of integration also depends on t. The Transport Theorem below will explain how this can be done.

The Transport Theorem

Let $\mathbf{VOL}$ be the volume element induced on $\mathcal{B}$ by G. Let $\phi_t : \mathcal{B} \to \mathcal{S}$ be a time-dependent configuration, and for each $X \in \mathcal{B}$ define as usual $x \in \mathcal{B}_t$ by $x = \phi_t(X)$.

Definition 1.3.5. We define $\mathcal{J} : \mathcal{B} \times I \to \mathbb{R}$ by

$$\phi_t^*(\mathbf{vol}_t|_x) = \mathcal{J}(X,t)\,\mathbf{VOL}|_X.$$

Then, for every integrable function $f : \mathcal{B}_t \to \mathbb{R}$ and $U \subset \mathcal{B}$

$$\int\limits_{U_t} f\,\mathbf{vol}_t = \int\limits_{U} (f \circ \phi_t)\,\mathcal{J}(\cdot,t)\,\mathbf{VOL}. \tag{1.8}$$

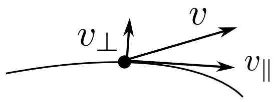
By the transformation formula from integration theory,

$$\mathcal{J}(X,t) = \det(\mathcal{I} \circ d\phi_t\big|_X),$$

where $\mathcal{I} : T_x\mathcal{B}_t \to T_X\mathcal{B}$ is an arbitrary isometry, such that $\mathcal{I} \circ d\phi_t\big|_X : T_X\mathcal{B} \to T_X\mathcal{B}$ is orientation-preserving.

Again, we decompose the spatial velocity into its tangential and normal part

$$\begin{aligned} T_xS &= T_x\mathcal{B}_t \oplus N_x\mathcal{B}_t \\ \boldsymbol{v}(x,t) &= \boldsymbol{v}_\parallel(x,t) + \boldsymbol{v}_\perp(x,t). \end{aligned}$$



Let $\mathbf{II}$ be the second fundamental form of $\mathcal{B}_t$ in $\mathcal{S}$ and $\boldsymbol{\mathcal{H}} = \frac{1}{m}\sum_{a=1}^m \mathbf{II}(e_a, e_a)$ be the mean curvature field, where $m := \dim \mathcal{B}_t$. Moreover, denote by div_t the divergence of vector fields on $\mathcal{B}_t$.

Theorem 1.3.6 (Bär [2014, Prop. 1.3.1])
In the situation above, we have

$$\frac{\partial \mathcal{J}}{\partial t} = \mathcal{J} \cdot \left(\mathrm{div}_t\left(\boldsymbol{v}_\parallel\right) - m \cdot g\left(\boldsymbol{v}_\perp, \boldsymbol{\mathcal{H}}\right)\right) \circ \phi_t.$$

Proof. We consider the vector bundle $\tan \psi^* T\mathcal{S}$ having at the point (X,t) the fiber $T_x\mathcal{B}_t$ and assign to it the connection $\widetilde{\nabla} := \tan \circ \psi^* \nabla^{\mathcal{S}}$, where as in Remark 1.1.10, $\psi : \mathcal{B} \times I \to \widetilde{\mathcal{B}}$ is given by $(X,t) \mapsto (\phi(X,t), t)$.
We fix a point $(X_0, t) \in \widetilde{\mathcal{B}}$ and choose an isometry $A : T_{\phi_t(X_0)}\mathcal{B}_{t_0} \to T_X\mathcal{B}$ with the correct orientation. Let $\mathcal{I}_t : T_{\phi_t(X)}\mathcal{B}_t \to T_X\mathcal{B}$ be the parallel transport along the curve $t \mapsto \phi_t(X)$ with respect to the connection $\widetilde{\nabla}$. Then at the point (X, t_0),

$$\begin{aligned} \frac{1}{\mathcal{J}}\frac{\partial \mathcal{J}}{\partial t} &= \det\left((\mathcal{I}_{t_0} \circ d\phi_{t_0})^{-1}\right) \cdot \frac{\partial}{\partial t}\Big|_{t=t_0} \det\left(\mathcal{I}_t \circ d\phi_t\right) \\ &= \mathrm{tr}\left((\mathcal{I}_{t_0} \circ d\phi_{t_0})^{-1} \circ \left(\frac{\partial}{\partial t}\Big|_{t=t_0} (\mathcal{I}_t \circ d\phi_t)\right)\right) \\ &= \mathrm{tr}\left((d\phi_{t_0})^{-1} \circ A^{-1} \circ \left(\frac{\partial}{\partial t}\Big|_{t=t_0} (\mathcal{I}_t \circ d\phi_t)\right)\right) \\ &= \mathrm{tr}\left((d\phi_{t_0})^{-1} \circ \left(\frac{\partial}{\partial t}\Big|_{t=t_0} (A^{-1} \circ \mathcal{I}_t \circ d\phi_t)\right)\right) \\ &= \mathrm{tr}\left((d\phi_{t_0})^{-1} \circ \frac{\widetilde{\nabla}}{\partial t}\Big|_{t=t_0} d\phi_t\right) \\ &= \mathrm{tr}\left((d\phi_{t_0})^{-1} \circ \widetilde{\nabla}\, \frac{\partial \phi_t}{\partial t}\Big|_{t=t_0}\right) \\ &= \mathrm{tr}\left((d\phi_{t_0})^{-1} \circ \widetilde{\nabla}\, \boldsymbol{V}\right) \\ &= \mathrm{tr}\left((d\phi_{t_0})^{-1} \circ \left(\nabla^{\mathcal{S}}\boldsymbol{v}\right)_\parallel \circ d\phi_{t_0}\right) \\ &= \mathrm{tr}\left(\left(\nabla^{\mathcal{S}}\boldsymbol{v}\right)_\parallel\right) \circ \phi_{t_0}. \qquad (1) \end{aligned}$$

Let e_a, $a = 1, \ldots, m$, be a local orthonormal frame for $T\mathcal{B}_t$. Then

$$\begin{aligned}
\operatorname{tr}\left(\left(\nabla^{\mathcal{S}}\boldsymbol{v}\right)_{\|}\right) \circ \phi_{t_0} &= \sum_{a=1}^{m} g\left(\nabla^{\mathcal{S}}_{e_a}\boldsymbol{v}, e_a\right) \circ \phi_{t_0} \\
&= \sum_{a=1}^{m} \left[g\left(\nabla^{\mathcal{S}}_{e_a}\boldsymbol{v}_{\|}, e_a\right) + g\left(\nabla^{\mathcal{S}}_{e_a}\boldsymbol{v}_{\perp}, e_a\right)\right] \circ \phi_{t_0} \qquad (2)
\end{aligned}$$

Now we compute

$$\begin{aligned}
\sum_{a=1}^{m} g\left(\nabla^{\mathcal{S}}_{e_a}\boldsymbol{v}_{\|}, e_a\right) &= \sum_{a=1}^{m} \left(g\left(\nabla^{\mathcal{B}_t}_{e_a}\boldsymbol{v}_{\|}, e_a\right) + g\left(\mathbf{II}(e_a, \boldsymbol{v}_{\|}), e_a\right)\right) \\
&= \operatorname{div}_t \boldsymbol{v}_{\|} \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{a=1}^{m} g\left(\nabla^{\mathcal{S}}_{e_a}\boldsymbol{v}_{\perp}, e_a\right) &= \sum_{a=1}^{m} \left(\partial_{e_a} g(\boldsymbol{v}_{\perp}, e_a) - g(\boldsymbol{v}_{\perp}, \nabla^{\mathcal{S}}_{e_a} e_a)\right) \\
&= -\sum_{a=1}^{m} g\left(\boldsymbol{v}_{\perp}, \nabla^{\mathcal{B}_t}_{e_a} e_a + \mathbf{II}(e_a, e_a)\right) \\
&= -m\, g(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}}). \qquad (4)
\end{aligned}$$

where we have used in the last step that $\sum_{a=1}^{m} \mathbf{II}(e_a, e_a) = m\boldsymbol{\mathcal{H}}$. Thus, by inserting (3) and (4) into (2) and by use of (1), we obtain

$$\frac{\partial \mathcal{J}}{\partial t} = \mathcal{J} \cdot \left(\operatorname{div}_t \boldsymbol{v}_{\|} - m \cdot g\left(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}}\right)\right) \circ \phi_{t_0}. \qquad \square$$

With the knowledge of $\frac{\partial \mathcal{J}}{\partial t}$ we immediately obtain the variation of the volume of the deformed body.

Corollary 1.3.7 (Bär [2014, Satz 1.3.2 and Bem. 1.3.3])
Assume that $\mathcal{B}$ is compact and that $\phi : \mathcal{B} \times I \to \mathcal{S}$ is a motion. Then the volume variation of $\mathcal{B}$ is given by

$$\frac{d}{dt}\mathbf{vol}(\mathcal{B}_t) = \int_{\mathcal{B}_t} \left(\operatorname{div}_t \boldsymbol{v}_{\|} - m \cdot g\left(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}}\right)\right) \mathbf{vol}_t.$$

Proof. Theorem 1.3.6 implies that for each nice subset U of $\mathcal{B}$ and its image U_t,

$$\begin{aligned}
\frac{d}{dt}\,\mathbf{vol}(\mathcal{B}_t) &= \frac{d}{dt}\int\limits_{\mathcal{B}_t} \mathbf{vol}_t \\
&= \frac{d}{dt}\int\limits_{\mathcal{B}} \mathcal{J}(\cdot,t)\,\mathbf{VOL} \\
&= \int\limits_{\mathcal{B}} \frac{\partial \mathcal{J}}{\partial t}\,\mathbf{VOL} \\
&= \int\limits_{\mathcal{B}} \left[\left(\mathrm{div}_t\left(\boldsymbol{v}_\parallel\right) - m\cdot g\left(\boldsymbol{v}_\perp, \boldsymbol{\mathcal{H}}\right)\right)\circ \phi_t\right] \mathcal{J}(\cdot,t)\,\mathbf{VOL} \\
&= \int\limits_{\mathcal{B}_t} \left(\mathrm{div}_t\left(\boldsymbol{v}_\parallel\right) - m\cdot g\left(\boldsymbol{v}_\perp, \boldsymbol{\mathcal{H}}\right)\right)\mathbf{vol}_t.
\end{aligned}$$

□

Theorem 1.3.8 (Transport Theorem, [see Bär, 2014, Kor. 1.3.4])
Let $f : \widetilde{\mathcal{B}} \to \mathbb{R}$ be smooth and assume that each $f(\cdot,t)$ is compactly supported. Then for the image $U_t := \phi_t(\mathcal{B})$ of each nice set $U \subset \mathcal{B}$,

$$\frac{d}{dt}\int\limits_{U_t} f(x,t)\,\mathbf{vol}_t = \int\limits_{U_t}\left(\dot{f} + f\cdot \mathrm{div}_t\,\boldsymbol{v}_\parallel - mf\cdot g\left(\boldsymbol{v}_\perp, \boldsymbol{\mathcal{H}}\right)\right)\mathbf{vol}_t.$$

In particular, if $\dim\mathcal{B} = \dim\mathcal{S}$, *then*

$$\begin{aligned}
\frac{d}{dt}\int\limits_{U_t} f(x,t)\,\mathbf{vol}_t &= \int\limits_{U_t}\left(\dot{f} + f\cdot \mathrm{div}_t\,\boldsymbol{v}\right)\mathbf{vol}_t \\
&= \int\limits_{U_t}\left(\frac{\partial f}{\partial t} + \mathrm{div}_t(f\boldsymbol{v})\right)\mathbf{vol}_t.
\end{aligned}$$

Proof.
1) By (1.8) and Theorem 1.3.6,

$$\begin{aligned}
\frac{d}{dt}\int\limits_{U_t} f(x,t)\,\mathbf{vol}_t(x) &= \frac{d}{dt}\int\limits_{U} f\left(\phi_t(X),t\right)\cdot\mathcal{J}(X,t)\,\mathbf{VOL}(X) \\
&= \int\limits_{U}\frac{\partial}{\partial t}\left(f\left(\phi_t(X),t\right)\cdot\mathcal{J}(X,t)\right)\mathbf{VOL}(X) \\
&= \int\limits_{U}\left(\dot{f} + f\cdot\mathrm{div}_t\left(\boldsymbol{v}_\parallel\right) - mf\cdot g\left(\boldsymbol{v}_\perp,\boldsymbol{\mathcal{H}}\right)\right)\left(\phi_t(X)\right)\cdot\mathcal{J}(X,t)\,\mathbf{VOL}(X)
\end{aligned}$$

$$= \int\limits_{U_t} \left(\dot{f} + f \cdot \operatorname{div}_t \left(\boldsymbol{v}_{\|} \right) - mf \cdot g \left(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}} \right) \right) (x) \, \mathbf{vol}_t(x).$$

2) If $\dim \mathcal{B} = \dim \mathcal{S}$, then $\boldsymbol{v} = \boldsymbol{v}_{\|}$ while $\boldsymbol{v}_{\perp} = 0$. In this case

$$\dot{f} + f \cdot \operatorname{div}_t(\boldsymbol{v}) = \frac{\partial f}{\partial t} + g(\mathbf{grad} f, \boldsymbol{v}) + f \cdot \operatorname{div}_t(\boldsymbol{v}) = \frac{\partial f}{\partial t} + \operatorname{div}_t(f \cdot \boldsymbol{v}). \qquad \square$$

Now we can use the Transport Theorem (Theorem 1.3.8) to obtain a local form of the Master Balance Law:

Theorem 1.3.9 (Spatial Localization Theorem)
Let $a, b : \widetilde{\mathcal{B}} \to \mathbb{R}$ be scalar functions and $\boldsymbol{c}$ a vector field on $\widetilde{\mathcal{B}}$. Assume that a and $\boldsymbol{c}$ are C^1 and b is C^0. Then a, b, and $\boldsymbol{c}$ satisfy the master balance law if and only if

$$\dot{a} + a \cdot \operatorname{div}_t(\boldsymbol{v}_{\|}) - m \, a \, g(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}}) = b + \operatorname{div}_t \boldsymbol{c}.$$

In particular, if $\dim \mathcal{B} = \dim \mathcal{S}$, then

$$\frac{\partial a}{\partial t} + \operatorname{div}_t(a\boldsymbol{v}) = b + \operatorname{div}_t \boldsymbol{c}.$$

Proof. By the Transport Theorem (1.3.8), the master balance law is equivalent to

$$\int\limits_{U_t} \left[\dot{a} + a \cdot \operatorname{div}_t(\boldsymbol{v}_{\|}) - ma \cdot g \left(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}} \right) \right] \mathbf{vol}_t = \int\limits_{U_t} b \, \mathbf{vol}_t + \int\limits_{U_t} \operatorname{div}_t \boldsymbol{c} \, \mathbf{vol}_t$$

for the image U_t of any nice $U \subset \mathcal{B}$. Thus, the first asserted formula follows immediately.
If the dimensions of $\mathcal{B}$ and $\mathcal{S}$ coincide, then $\boldsymbol{v} = \boldsymbol{v}_{\|}$ and $\boldsymbol{v}_{\perp}$ is zero. In this case

$$\begin{aligned} \dot{a} + a \cdot \operatorname{div}_t(\boldsymbol{v}_{\|}) - ma \cdot g \left(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}} \right) &= \frac{\partial a}{\partial t} + g(\mathbf{grad}\, a, \boldsymbol{v}) + a \cdot \operatorname{div}_t(\boldsymbol{v}) \\ &= \frac{\partial a}{\partial t} + \operatorname{div}_t(a \, \boldsymbol{v}). \qquad \square \end{aligned}$$

1.3.2 The Material Master Balance Law

As we have already seen in Remark 1.1.10, it is much easier to compute time derivatives on the undeformed body. Hence we would like to transform the spatial Master Balance Law to a balance law that is formulated in terms of functions and vector fields defined on $\mathcal{B} \times I$. That is, we would like to replace

$$\frac{d}{dt} \int\limits_{U_t} a \, \mathbf{vol}_t = \int\limits_{U_t} b \, \mathbf{vol}_t + \int\limits_{\partial U_t} g(\boldsymbol{c}, \boldsymbol{n}) \, \mathbf{vol}_t \tag{1.9}$$

by a balance law

$$\frac{d}{dt}\int\limits_U A\,\mathbf{VOL} = \int\limits_U B\,\mathbf{VOL} + \int\limits_{\partial U} G(\boldsymbol{C},\boldsymbol{N})\,\mathbf{VOL}_{\partial U}$$

on $\mathcal{B}$, where $\boldsymbol{N}$ is the unit normal vector field on ∂U, and A, B, $\boldsymbol{C}$ are somehow connected with a, b, $\boldsymbol{c}$, where we still have to determine how exactly they are related.
Because of the relation $\mathbf{vol}_t(x) = \mathcal{J}(X,t)\,\mathbf{VOL}(X)$ we can easily see that we could just define

$$A(X,t) := \mathcal{J}(X,t)\,a(x,t) \quad \text{and} \quad B(X,t) := J(X,t)\,b(x,t),$$

where $x = \phi_t(X)$. Then

$$\int\limits_{U_t} a(x,t)\,\mathbf{vol}_t(x) = \int\limits_U a(\phi(X,t),t)\,\mathcal{J}(X,t)\,\mathbf{VOL}(X) = \int\limits_U A(X,t)\,\mathbf{VOL}(X)$$

and analogously

$$\int\limits_{U_t} b(x,t)\,\mathbf{vol}_t(x) = \int\limits_U B(X,t)\,\mathbf{VOL}(X).$$

But how do we define $\boldsymbol{C}$ in such a way that the last term in (1.9) keeps its form ? We will see that $\boldsymbol{c}$ and $\boldsymbol{C}$ have to be related by a Piola transformation.

Recall that each $\phi_t : \mathcal{B} \to \mathcal{B}_t$ is a diffeomorphism and that the pull-back of a vector field w on $\mathcal{B}_t$ to a vector field on $\mathcal{B}$ is given by

$$\phi_t^* \boldsymbol{w}_t := d\phi_t^{-1} \circ \boldsymbol{w}_t \circ \phi_t.$$

Definition 1.3.10. The vector field

$$\boldsymbol{W}_t := \mathcal{J}(\cdot,t)\cdot(\phi_t^* \boldsymbol{w}_t)$$

is called the **Piola transformation** of $\boldsymbol{w}$.

Lemma 1.3.11 (Marsden and Hughes [1983, ch.1, Th. 7.19])
A vector field $\boldsymbol{W}_t$ on $\mathcal{B}$ is the Piola transformation of a vector field $\boldsymbol{w}_t$ on $\mathcal{B}_t$ if and only if

$$\phi_t^*\left(i_{\boldsymbol{w}_t}\mathbf{vol}_t\right) = i_{\boldsymbol{W}_t}\mathbf{VOL}.$$

Proof. For each vector field $\boldsymbol{w}_t$ that is tangential to $\mathcal{B}_t$,

$$\begin{aligned}\phi_t^*\left(i_{\boldsymbol{w}_t}\mathbf{vol}_t\right) &= i_{\phi_t^*\boldsymbol{w}_t}\left(\phi_t^*\mathbf{vol}_t\right)\\ &= i_{\phi_t^*\boldsymbol{w}_t}\left(\mathcal{J}(\cdot,t)\cdot\mathbf{VOL}\right)\\ &= i_{\mathcal{J}(\cdot,t)\cdot\phi_t^*\boldsymbol{w}_t}\mathbf{VOL}.\end{aligned}$$

Thus, $\boldsymbol{W}_t = \mathcal{J}(\cdot,t)\cdot(\phi_t^*\boldsymbol{w}_t)$ is equivalent to $\phi_t^*\left(i_{\boldsymbol{w}_t}\mathbf{vol}_t\right) = i_{\boldsymbol{W}_t}\mathbf{VOL}$. □

Theorem 1.3.12 (Piola Identity, [Marsden and Hughes, 1983, ch. 1, Th. 7.20])
If $\boldsymbol{W}_t$ is the Piola transformation of $\boldsymbol{w}_t$, then

$$\mathrm{DIV}(\boldsymbol{W}_t) = \mathcal{J}(\cdot,t)\cdot(\mathrm{div}_t(\boldsymbol{w}_t)\circ\phi_t).$$

Proof. For each nice $U\subset\mathcal{B}$,

$$\int_U \mathrm{DIV}(\boldsymbol{W}_t)\,\mathbf{VOL} = \int_{\partial U} i_{\boldsymbol{W}_t}\mathbf{VOL} \tag{1}$$

and

$$\begin{aligned}\int_U \mathcal{J}(\cdot,t)\cdot(\mathrm{div}_t(\boldsymbol{w}_t)\circ\phi_t)\,\mathbf{VOL} &= \int_{U_t}\mathrm{div}_t(\boldsymbol{w}_t)\,\mathbf{vol}_t\\ &= \int_{\partial U_t} i_{\boldsymbol{w}_t}\mathbf{vol}_t\\ &= \int_{\partial U}\phi_t^*\left(i_{\boldsymbol{w}_t}\mathbf{vol}_t\right) \qquad (2)\end{aligned}$$

If $\boldsymbol{W}_t$ is the Piola transformation of $\boldsymbol{w}_t$, then by Lemma 1.3.11 the right hand sides of (1) and (2) coincide. Since U is arbitrary, we conclude that

$$\mathrm{DIV}(\boldsymbol{W}_t) = \mathcal{J}(\cdot,t)\cdot(\mathrm{div}_t(\boldsymbol{w}_t)\circ\phi_t).$$ □

As an immediate consequence of Theorem 1.3.12 we obtain

Theorem 1.3.13
Let $a,b:\mathcal{B}_t\times I\to\mathbb{R}$ be scalar functions and let $\boldsymbol{c}_t$ be a vector field on $\mathcal{B}_t$.
Let $A,B:\mathcal{B}\times I\to\mathbb{R}$ be scalar functions and let $\boldsymbol{C}_t$ be a vector field on $\mathcal{B}$. Assume that A, B, $\boldsymbol{C}_t$ and a, b, $\boldsymbol{c}_t$ are related by

$$\begin{aligned}A(X,t) &= \mathcal{J}(X,t)\,a(x,t)\\ B(X,t) &= \mathcal{J}(X,t)\,b(x,t)\\ \boldsymbol{C}_t(X) &= \mathcal{J}(X,t)\boldsymbol{\mathcal{F}}^{-1}(X,t)\boldsymbol{c}_t(x), \quad \textit{i.e., } \boldsymbol{C}_t \textit{ is the Piola transformation of } \boldsymbol{c}_t.\end{aligned}$$

Then a, b, $\boldsymbol{c}_t$ satisfy the spatial master balance law if and only if A, B, $\boldsymbol{C}_t$ satisfy

$$\frac{d}{dt}\int\limits_U A\,\mathbf{VOL} = \int\limits_U B\,\mathbf{VOL} + \int\limits_{\partial U} \langle \boldsymbol{C}_t, \boldsymbol{N}\rangle\,\mathbf{VOL}_{\partial U}.$$

Remark 1.3.14. Marsden and Hughes [1983] define the scalar functions A and B without using the factor $\mathcal{J}$ and thus give a slightly different version of this theorem (see their Prop. 1.6 in ch.2).

Proof. As we have already seen,

$$\frac{d}{dt}\int\limits_{U_t} a\,\mathbf{vol}_t = \frac{d}{dt}\int\limits_U A\,\mathbf{VOL},$$
$$\int\limits_{U_t} b\,\mathbf{vol}_t = \int\limits_U B\,\mathbf{VOL}.$$

Moreover, by Theorem 1.3.12,

$$\begin{aligned}\int\limits_{\partial U} \langle \boldsymbol{C}_t, \boldsymbol{N}\rangle\,\mathbf{VOL}_{\partial U} &= \int\limits_U \mathrm{DIV}\boldsymbol{C}_t\,\mathbf{VOL} = \int\limits_U \mathcal{J}\cdot(\mathrm{div}_t\,\boldsymbol{c}_t)\circ\phi_t\,\mathbf{VOL}\\ &= \int\limits_{U_t} \mathrm{div}_t\,\boldsymbol{c}_t\,\mathbf{vol}_t = \int\limits_{\partial U_t} \langle \boldsymbol{c}_t, \boldsymbol{n}\rangle\,\mathbf{vol}_{\partial U_t}.\end{aligned}$$ $\square$

Thus we define the notion of a master balance law on $\mathcal{B}$ as follows:

Definition 1.3.15. Let $A, B : \mathcal{B}\times I \to \mathbb{R}$ be scalar functions and $\boldsymbol{C}_t$ a vector field on $\mathcal{B}$. We say that A, B, and $\boldsymbol{C}_t$ satisfy the **material master balance law**, if for any nice set $U \in \mathcal{B}$

i) the integrals in the following equation exist,

ii) $\int\limits_U A\,\mathbf{VOL}$ is differentiable in t,

iii) and
$$\frac{d}{dt}\int\limits_U A\,\mathbf{VOL} = \int\limits_U B\,\mathbf{VOL} + \int\limits_{\partial U} \langle \boldsymbol{C}_t, \boldsymbol{N}\rangle\,\mathbf{VOL}_{\partial U},$$
where $\boldsymbol{N}$ is the unit outward normal to ∂U.

The local version of the Material Master Balance Law is easily found:

Theorem 1.3.16 (Material Localization Theorem)
Let $A, B : \mathcal{B} \times I \to \mathbb{R}$ be scalar functions and $\boldsymbol{C}_t$ a vector field on $\mathcal{B}$. Assume that A and B are C^0, $\frac{\partial A}{\partial t}$ exists, and $\boldsymbol{C}_t$ is C^1. Then A, B, and $\boldsymbol{C}_t$ satisfy the master balance law if and only if

$$\frac{\partial A}{\partial t} = B + \mathrm{DIV}\boldsymbol{C}_t. \tag{1.10}$$

Proof. A direct differentiation under the integral sign and the Theorem of Stokes provide the stated equation. □

Remark 1.3.17. The form of the Material Localization Theorem, given by Theorem 1.3.16, slightly differs from the one given by Marsden and Hughes [1983] (see their Th. 1.5 in ch.2).

1.3.3 Consequences of Conservation of Mass

If conservation of mass is given, then the Transport Theorem and the Localization Theorem can be considerably simplified.

Assume we are given a smooth **mass density** $\rho : \widetilde{\mathcal{B}} \to \mathbb{R}$.

Definition 1.3.18. The mass $\mathcal{M}$ of a nice set $U_t \subset \mathcal{B}_t$ is given by the integral of the mass density

$$\mathcal{M}(U_t) = \int\limits_{U_t} \rho(x,t)\,\mathbf{vol}_t.$$

We say that ρ obeys conservation of mass, if for all nice $U_t \subset \mathcal{B}_t$,

$$\frac{d}{dt}\int\limits_{U_t} \rho(x,t)\,\mathbf{vol}_t = 0.$$

Conservation of mass states that the mass of any nice subset of $U \subset \mathcal{B}$ is constant in time.
We get at once from the spatial localization theorem (1.3.9) the local form of conservation of mass:

Theorem 1.3.19
Conservation of mass is equivalent to the ***continuity equation***

$$\dot{\rho} + \rho\,\mathrm{div}_t(\boldsymbol{v}_{\|}) - m\,\rho\,g(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}}) = 0,$$

where $\boldsymbol{v}$ denotes the body's spatial velocity, m is the body's dimension and $\mathcal{H}$ denotes the mean curvature field.
In particular, if $\dim \mathcal{B} = \dim \mathcal{S}$, then conservation of mass is equivalent to

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\, \boldsymbol{v}) = 0.$$

It is now convenient to refer the quantities a and b in the spatial master balance law (Corollary 1.3.4) to the mass density ρ, that is, to consider instead of

$$\frac{d}{dt}\int\limits_{U_t} a(x,t)\,\mathbf{vol}_t = \int\limits_{U_t} b(x,t)\,\mathbf{vol}_t + \int\limits_{\partial U_t} \langle \boldsymbol{c}_t(x,t), \boldsymbol{n}\rangle\,\mathbf{vol}_{\partial U_t}$$

the balance law

$$\frac{d}{dt}\int\limits_{U_t} a(x,t)\rho(x,t)\,\mathbf{vol}_t = \int\limits_{U_t} b(x,t)\rho(x,t)\,\mathbf{vol}_t + \int\limits_{\partial U_t} \langle \boldsymbol{c}_t(x,t), \boldsymbol{n}\rangle\,\mathbf{vol}_{\partial U_t}.$$

Then the assumption of conservation of mass simplifies the Transport and the Localization Theorem considerably. For simple bodies this has already been discussed by Marsden and Hughes [1983, p. 124].

Theorem 1.3.20 (Simplified Transport Theorem)
Let $f : \widetilde{\mathcal{B}} \to \mathbb{R}$ be smooth and assume that each $f(\cdot,t)$ is compactly supported. If conservation of mass holds, then

$$\frac{d}{dt}\int\limits_{U_t} f(x,t)\,\rho(x,t)\,\mathbf{vol}_t = \int\limits_{U_t} \dot{f}(x,t)\,\rho(x,t)\,\mathbf{vol}_t.$$

Proof. By the Transport Theorem (Theorem 1.3.8) and the product rule for the substantial derivative,

$$\begin{aligned}\frac{d}{dt}\int\limits_{U_t} f\,\rho\,\mathbf{vol}_t &= \int\limits_{U_t} \left(\dot{f}\,\rho + f\,\dot{\rho} + f\,\rho\left[\operatorname{div}_t(\boldsymbol{v}_{\|}) - m\,g\,(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}})\right]\right)\,\mathbf{vol}_t \\ &= \int\limits_{U_t} \dot{f}\,\rho\,\mathbf{vol}_t + \int\limits_{U_t} f\cdot\left[\dot{\rho} + \rho\operatorname{div}_t(\boldsymbol{v}_{\|}) - m\,\rho\,g\,(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}})\right]\,\mathbf{vol}_t.\end{aligned}$$

If conservation of mass is given, then the second integral in the last line is equal to zero, and we obtain,

$$\frac{d}{dt}\int\limits_{U_t} f\,\rho\,\mathbf{vol}_t = \int\limits_{U_t} \dot{f}\,\rho\,\mathbf{vol}_t.$$

□

Now we immediately obtain a simpler version of the Spatial Localization Theorem (1.3.9). A similar proof as the one given for Theorem 1.3.9 shows:

Theorem 1.3.21 (Simplified Spatial Localization Theorem)
Let $a, b : \mathcal{B}_t \times I \to \mathbb{R}$ be scalar functions and $\boldsymbol{c}_t$ a vector field on $\mathcal{B}_t$. Assume that a and $\boldsymbol{c}_t$ are C^1 and that b is C^0. Assume that conservation of mass holds.
Then a, b, and $\boldsymbol{c}_t$ satisfy the master balance law if and only if

$$\rho\,\dot{a} = \rho\,b + \mathrm{div}_t \boldsymbol{c}_t.$$

Definition 1.3.22. We say, that conservation of mass is valid for the subset U of the undeformed body $\mathcal{B}$ if

$$\frac{d}{dt}\int\limits_U \rho_{ref}(X,t)\,\mathbf{VOL} = 0,$$

where ρ_{ref} is the mass density on $\mathcal{B}$.

The local form of conservation of mass on the undeformed body $\mathcal{B}$ is simply given by

$$\rho_{ref} = \mathrm{const.in}\, t.$$

By equation (1.8), conservation of mass on U_t is equivalent to balance of mass on U, if and only if

$$\rho_{ref} = \mathcal{J}\,\rho. \tag{1.11}$$

In the Material Master Balance Law we also refer the quantities A and B to the reference mass density ρ_{ref}:

$$\frac{d}{dt}\int\limits_U A\,\rho_{ref}\mathbf{VOL} = \int\limits_U B\,\rho_{ref}\mathbf{VOL} + \int\limits_{\partial U} \langle \boldsymbol{C}_t, \boldsymbol{N}\rangle\,\mathbf{VOL}_{\partial U}. \tag{1.12}$$

If mass is conserved, then the localization of this equation is simply

$$\rho_{ref}\frac{\partial A}{\partial t} = \rho_{ref} B + \mathrm{DIV}(\boldsymbol{C}_t). \tag{1.13}$$

To connect the quantities a and b with the quantities A and B, respectively, we now define instead of the relations that were used in Theorem 1.3.13,

$$A(X,t) := a(x,t) \quad \text{and} \quad B(X,t) := b(x,t).$$

Then, using $\rho\,\mathcal{J} = \rho_{ref}$, we obtain

Theorem 1.3.23
Let $a, b : \mathcal{B}_t \times I \to \mathbb{R}$ be scalar functions and let $\boldsymbol{c}_t$ be a vector field on $\mathcal{B}_t$.
Let $A, B : \mathcal{B} \times I \to \mathbb{R}$ be scalar functions and let $\boldsymbol{C}_t$ be a vector field on $\mathcal{B}$. Assume that A, B, $\boldsymbol{C}_t$ and a, b, $\boldsymbol{c}_t$ are related by

$$\begin{aligned} A(X,t) &= a(x,t) \\ B(X,t) &= b(x,t) \\ \boldsymbol{C}_t(X) &= \mathcal{J}(X,t)\boldsymbol{\mathcal{F}}^{-1}(X,t)\boldsymbol{c}_t(x), \quad \textit{i.e., } \boldsymbol{C}_t \textit{ is the Piola transformation of } \boldsymbol{c}_t. \end{aligned}$$

Then a, b, $\boldsymbol{c}_t$ satisfy the spatial master balance law in the form

$$\frac{d}{dt}\int_{U_t} a(x,t)\rho(x,t)\,\mathbf{vol}_t = \int_{U_t} b(x,t)\rho(x,t)\,\mathbf{vol}_t + \int_{\partial U_t} \langle \boldsymbol{c}_t(x,t), \boldsymbol{n}\rangle\,\mathbf{vol}_{\partial U_t},$$

if and only if A, B, $\boldsymbol{C}_t$ satisfy

$$\frac{d}{dt}\int_{U} A\,\rho_{ref}\,\mathbf{VOL} = \int_{U} B\,\rho_{ref}\,\mathbf{VOL} + \int_{\partial U} \langle \boldsymbol{C}_t, \boldsymbol{N}\rangle\,\mathbf{VOL}_{\partial U}$$

Remark 1.3.24. Nevertheless, there are quantities, for example the density of electric charges, for which it does not make much sense to refer them to the mass density. If these occur in a balance law on $\mathcal{B}_t$, it is better to use the relations given in Theorem 1.3.13 to define the corresponding quantities on $\mathcal{B}$. Otherwise the law on $\mathcal{B}$ will have a slightly different form than the one on $\mathcal{B}_t$.

2 Electrodynamics

While in the previous chapter, we discussed the geometrical aspects required for a proper formulation of elasticity, in this chapter we state some basics of electrodynamics that we will need later on. We will adhere to the texts of Kovetz [2000], Hehl and Obukhov [2003, 2006] and Frankel [2006]. A more fundamental description of Maxwell's equations can be found in Fleisch [2008].

In the first two sections we state the usual form of Maxwell's equations in terms of vector fields and discuss Galilei transformations as well as Galilei invariants. In this context we recall the definitions of the conductive current density, the electromotive and the magnetomotive intensity, and the Lorentz force. We also introduce the flux derivative and state Maxwell's equations in terms of Galilei invariants.
In the third section Maxwell's equations are written in terms of differential forms. This formulation can also be used, if the surrounding space is not the Euclidean $\mathbb{R}^3$. In section 2.4 the notions of polarization and magnetization are introduced. This is done in the same way as they are treated in the book by Kovetz [2000], but here they are expressed in terms of forms. Kovetz' definition also works on arbitrary Riemannian manifolds; the usual definition using dipole and magnetic moments (see e.g. Jackson [2002]) only makes sense in the Euclidean $\mathbb{R}^3$. Moreover, we recall Poynting's Theorem.
Section 2.5 covers the definition of the electromagnetic quantities as vector fields and forms along some deformation map $\phi_t : \mathcal{B} \to \mathcal{B}_t \subset \mathcal{S}$.

2.1 Maxwell's equations

In this section we assume that $\mathcal{S}$ is the Euclidean $\mathbb{R}^3$. Nevertheless, for later purposes it will be beneficial, to distinguish $\mathcal{S}$ and its tangent space.
As in section 1.1, we define

$$\widetilde{\mathcal{B}} := \bigcup_{t\in I} \Big(\mathcal{B}_t \times \{t\}\Big) \subset \mathcal{S} \times I =: \widetilde{\mathcal{S}}.$$

Let $\rho_e : \widetilde{\mathcal{S}} \to \mathbb{R}$ be a charge density describing a 3-dimensional charge distribution and $\boldsymbol{u} : \widetilde{\mathcal{S}} \to T\mathcal{S}$ the velocity density of charge carriers. Then the current density is the vector field $\boldsymbol{j} : \widetilde{\mathcal{S}} \to T\mathcal{S}$, defined by $\boldsymbol{j} := \rho_e\, \boldsymbol{u}$. We say that balance of charge is satisfied, if for each compact region $\Omega \subset \mathbb{R}^3$

$$\frac{d}{dt}\int\limits_{\Omega} \rho_e(x,t)\, \mathbf{vol}_\Omega = -\int\limits_{\partial\Omega} g(\boldsymbol{j},\boldsymbol{n})\, \mathbf{vol}_{\partial\Omega}, \qquad \text{(Balance of charge)}$$

where $\boldsymbol{n}$ denotes the (outer) unit normal vector field on $\partial\Omega$.
Let $\boldsymbol{d} : \widetilde{\mathcal{S}} \to T\mathcal{S}$ be the electric flux density. Then by Gauss' Law, for any compact region $\Omega \subset \mathbb{R}^3$,

$$\int\limits_{\partial\Omega} \langle \boldsymbol{d}, \boldsymbol{n} \rangle \, \mathbf{vol}_{\partial\Omega} = \int\limits_{\Omega} \rho_e \, \mathbf{vol}_\Omega. \tag{Gauss' Law}$$

Let us denote by $\boldsymbol{b} : \widetilde{\mathcal{S}} \to T\mathcal{S}$ the magnetic flux density. Then the conservation law of magnetic flux states that for each compact oriented region $\Omega \subset \mathbb{R}^3$,

$$\int\limits_{\partial\Omega} \langle \boldsymbol{b}, \boldsymbol{n} \rangle \, \mathbf{vol}_{\partial\Omega} = 0. \tag{Conservation of magnetic flux}$$

Moreover, let $\boldsymbol{e} : \widetilde{\mathcal{S}} \to T\mathcal{S}$ be the electric field. By Faraday's Induction Law, for each compact oriented surface Σ,

$$\int\limits_{\partial\Sigma} \langle \boldsymbol{e}, \xi \rangle \, \mathbf{vol}_{\partial\Sigma} = -\int\limits_{\Sigma} \Big\langle \frac{\partial \boldsymbol{b}}{\partial t}, \boldsymbol{n} \Big\rangle \, \mathbf{vol}_\Sigma. \tag{Faraday's Induction Law}$$

Here, ξ denotes the unit tangent vector field to $\partial\Sigma$. Furthermore, let us denote by $\boldsymbol{h} : \widetilde{\mathcal{S}} \to T\mathcal{S}$ the magnetic field. Ampère's Law states that for each compact 2-sided surface Σ with prescribed normal $\boldsymbol{n}$,

$$\int\limits_{\partial\Sigma} \langle \boldsymbol{h}, \xi \rangle \, \mathbf{vol}_{\partial\Sigma} = \int\limits_{\Sigma} \langle \boldsymbol{j}, \boldsymbol{n} \rangle \, \mathbf{vol}_\Sigma + \int\limits_{\Sigma} \Big\langle \frac{\partial \boldsymbol{d}}{\partial t}, \boldsymbol{n} \Big\rangle \, \mathbf{vol}_\Sigma. \tag{Ampère's Law}$$

It is important to note that the balance of charge as well as Maxwell's equations are three-dimensional concepts and that the electromagnetic fields are in a sense a property of the surrounding space. If one wants to formulate balance of charge for a hypersurface or a one-dimensional body, one has to work with distributions. But here we will not need to do that. Our present aim is just the statement of Maxwell's equations for the surrounding space.
Later on, we are only interested in what happens *inside* the deformed body. Thus, we restrict the charge density and the electromagnetic fields to $\widetilde{\mathcal{B}}$ and demand that the maps $\rho_e : \widetilde{\mathcal{B}} \to \mathbb{R}$, $\boldsymbol{u} : \widetilde{\mathcal{B}} \to T\mathcal{S}$, $\boldsymbol{j} : \widetilde{\mathcal{B}} \to T\mathcal{S}$ as well as $\boldsymbol{d} : \widetilde{\mathcal{B}} \to T\mathcal{S}$, $\boldsymbol{b} : \widetilde{\mathcal{B}} \to T\mathcal{S}$, $\boldsymbol{e} : \widetilde{\mathcal{B}} \to T\mathcal{S}$ and $\boldsymbol{h} : \widetilde{\mathcal{B}} \to T\mathcal{S}$ are C^2. Note that the flux density $\boldsymbol{j}$ is in general not tangential to $\mathcal{B}_t$, we will discuss that later on.
Then the local form of Maxwell's equations states that on $\widetilde{\mathcal{B}}$

$$\operatorname{div} \boldsymbol{d} = \rho_e \tag{Gauss' Law}$$
$$\operatorname{div} \boldsymbol{b} = 0 \tag{Conservation of magnetic flux}$$
$$\mathbf{rot}\ \boldsymbol{e} = -\frac{\partial \boldsymbol{b}}{\partial t} \tag{Faraday's Induction Law}$$
$$\mathbf{rot}\ \boldsymbol{h} = \boldsymbol{j} + \frac{\partial \boldsymbol{d}}{\partial t}. \tag{Ampère's Law}$$

The local expression for the balance of charge is given by

$$\frac{\partial \rho_e}{\partial t} + \operatorname{div} \boldsymbol{j} = 0. \qquad \text{(Balance of charge)}$$

Remark 2.1.1. In Notation 1.1.11 we have stipulated to denote all quantities that are defined on $\mathcal{B}_t$ or $\widetilde{\mathcal{B}}$ by small letters, whereas quantities that are defined on $\mathcal{B}$ or $\mathcal{B} \times I$ are denoted by capital letters. $\boldsymbol{d}$, $\boldsymbol{b}$, $\boldsymbol{e}$ and $\boldsymbol{h}$ are vector fields on the surrounding space and can be evaluated along $\widetilde{\mathcal{B}}$. Thus, in contrast to the classical literature on electrodynamics, we denote all the electromagnetic quantities by small letters.

2.2 Galilei-invariants and the Lorentz force

In this section we recall some material on Galilei transformations. Maxwell's equations are Lorentz invariant. But in the following chapters we will use notions and concepts from thermodynamics. Unfortunately, it seems that a generally accepted theory of relativistic thermodynamics does not (yet) exist (see for instance Nakamura [2012] and Requardt [2008]).
Moreover, we will need in the proof of Theorem 3.5.1 the splitting of the energy density into the internal energy density and the (macroscopic) kinetic energy. But relativistically, this splitting is not covariant.
Thus, we will always restrict our considerations to motions with velocities that are small compared to the speed of light.

We continue to assume that $\mathcal{S}$ is the Euclidean $\mathbb{R}^3$. Let Σ' be a coordinate system that moves with some *constant small* (compared to the speed of light) velocity $\boldsymbol{w} : \mathcal{S} \to T\mathcal{S}$ with respect to a given coordinate system Σ. Then the coordinates $\boldsymbol{x}'$ and $\boldsymbol{x}$ of some point of $\mathcal{S}$ with respect to the systems Σ' and Σ, respectively, are related by a **Galilei transformation**, usually stated as

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{w}\, t. \tag{2.1}$$

Velocities $\boldsymbol{v}'$ and $\boldsymbol{v}$ that are measured in the primed and unprimed system are therefore connected by

$$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{w}. \tag{2.2}$$

On a deformed body $\mathcal{B}_t$ we can at each tangent space $T_x\mathcal{B}_t$ consider a different Galilei transformation with the velocity $\boldsymbol{w}_t(x)$, demanding that these Galilei transformations depend smoothly on the point x, in other words, demanding that $\boldsymbol{w}_t : \mathcal{B}_t \to T\mathcal{S}$ is C^∞. Then velocities in the primed and in the unprimed system are related by

$$\boldsymbol{v}'(x', t) = \boldsymbol{v}(x, t) - \boldsymbol{w}(x, t). \tag{2.3}$$

The charge density, the electric flux density, and the magnetic flux density are invariant under Galilei transformations, that is, $\rho_e'(x', t) = \rho_e(x, t)$, $\boldsymbol{d}'(x', t) = \boldsymbol{d}(x, t)$, and

$\boldsymbol{b}'(x',t) = \boldsymbol{b}(x,t)$, whereas the current density, the electric field $\boldsymbol{e}$, and the magnetic field $\boldsymbol{h}$ are not invariant. Recall that for charge carriers with the density ρ_e moving with the velocity $\boldsymbol{u} : \widetilde{\mathcal{B}} \to T\mathcal{S}$ with respect to Σ, the current density $\boldsymbol{j} : \widetilde{\mathcal{B}} \to T\mathcal{S}$ is given by

$$\boldsymbol{j} = \rho_e\, \boldsymbol{u}.$$

The current densities in the primed and the unprimed system are related by

$$\boldsymbol{j}' = \rho_e'\, \boldsymbol{u}' = \rho_e\, (\boldsymbol{u} - \boldsymbol{w}) = \boldsymbol{j} - \rho_e\, \boldsymbol{w}.$$

Assume now that the material moves with the spatial velocity field $\boldsymbol{v} : \widetilde{\mathcal{B}} \to T\mathcal{S}$. We would like to construct a current density that is Galilei-invariant. The obvious way to do is to take at each point the current density as it is seen in the system that moves along with the material in that point. That is, in the co-moved system Σ' the material has the velocity $\boldsymbol{v}' = \mathbf{0}$. Hence, the velocity of Σ' with respect to Σ is given by $\boldsymbol{w} = \boldsymbol{v}$. Thus, we define the **convective current density** $\bar{\boldsymbol{j}} : \widetilde{\mathcal{B}} \to T\mathcal{S}$ as

$$\bar{\boldsymbol{j}} := \boldsymbol{j} - \rho_e\, \boldsymbol{v}.$$

$\bar{\boldsymbol{j}}$ is Galilei-invariant. The convective current density can be seen as a more fundamental notion than the current density $\boldsymbol{j}$: It gives the current as it is seen from the material's point of view. Thus, $\bar{\boldsymbol{j}}(\cdot,t)$ is by definition tangential to $\mathcal{B}_t$, otherwise the charge carriers would leave the material. The current density $\boldsymbol{j}(\cdot,t) = \bar{\boldsymbol{j}}(\cdot,t) + \rho_e(\cdot,t)\, \boldsymbol{v}(\cdot,t)$, however, is in general not tangential to $\mathcal{B}_t$. It describes the motion of the material's charge carriers from the point of view of an external observer, and for this observer the velocity of their motion inside the material and the motion of the material itself superimpose.

Recall that the electric fields as seen in the primed and the unprimed system, resp., are connected by

$$\boldsymbol{e}' = \boldsymbol{e} + \boldsymbol{w} \times \boldsymbol{b}.$$

Again, we would like to define a quantity that is Galilei-invariant. If the material moves with the velocity field $\boldsymbol{v}$, then we take at each point of the material the electric field as it is seen in the co-moved frame, in which the material is at rest. Then $\boldsymbol{w} = \boldsymbol{v}$ and we define the **electromotive intensity** $\bar{\boldsymbol{e}} : \widetilde{\mathcal{S}} \to T\mathcal{S}$ as

$$\bar{\boldsymbol{e}} := \boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b}.$$

Similarly, the magnetic fields in the primed and the unprimed system, resp., are connected by

$$\boldsymbol{h}' = \boldsymbol{h} - \boldsymbol{w} \times \boldsymbol{d},$$

and we can construct a Galilei-invariant version $\bar{\boldsymbol{h}} : \widetilde{\mathcal{S}} \to T\mathcal{S}$ of $\boldsymbol{h}$, the magnetomotive intensity, by

$$\bar{\boldsymbol{h}} := \boldsymbol{h} - \boldsymbol{v} \times \boldsymbol{d}.$$

Up to now we have not specified how electromagnetic fields act upon charged particles moving with the velocity $\boldsymbol{u}$. This connection is provided by the Lorentz force. The **Lorentz force** $\boldsymbol{f}_L : \widetilde{\mathcal{B}} \to T\mathcal{S}$ is defined as

$$\boldsymbol{f}_L = \rho_e\, \boldsymbol{e}'',$$

where $\boldsymbol{e}'' : \widetilde{\mathcal{B}} \to T\widetilde{\mathcal{S}}$ is now the electric field as it is seen from *the charge carrier's* point of view, that is, in a frame Σ'' that moves with respect to Σ with the velocity $\boldsymbol{w} = \boldsymbol{u}$. Thus, $\boldsymbol{e}'' = \boldsymbol{e} + \boldsymbol{u} \times \boldsymbol{b}$. Using $\boldsymbol{j} = \rho_e\, \boldsymbol{u}$, we obtain

$$\boldsymbol{f}_L = \rho_e\, \boldsymbol{e} + \boldsymbol{j} \times \boldsymbol{b}.$$

This definition of the Lorentz force is Galilei-invariant. We might also express $\boldsymbol{f}_L$ completely in terms of Galilei-invariants (i.e. give the Lorentz force as it is seen from the material's point of view) and write

$$\boldsymbol{f}_L = \rho_e\, \bar{\boldsymbol{e}} + \bar{\boldsymbol{j}} \times \boldsymbol{b}.$$

Of course, one could also exchange the roles of the primed and the unprimed system and write

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{w}\, t. \tag{2.4}$$

Then the velocities, current densities and electric fields measured in the primed and the unprimed system are related by

$$\begin{aligned} \boldsymbol{v}' &= \boldsymbol{v} + \boldsymbol{w} \\ \boldsymbol{j}' &= \boldsymbol{j} + \rho_e\, \boldsymbol{w} \\ \boldsymbol{e}' &= \boldsymbol{e} - \boldsymbol{w} \times \boldsymbol{b}. \end{aligned} \tag{2.5}$$

Nevertheless, the definition of the Galilean invariants $\bar{\boldsymbol{j}}$, $\bar{\boldsymbol{e}}$ and $\boldsymbol{f}_L$ stays the same: If we observe the current density and the electric field from the co-moved system, in which the velocity of the material is $\boldsymbol{v}' = 0$, then the velocity of this system is $\boldsymbol{w} = -\boldsymbol{v}$, and we obtain again that $\bar{\boldsymbol{j}} = \boldsymbol{j} - \rho_e\, \boldsymbol{v}$ and $\bar{\boldsymbol{e}} = \boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b}$. If we observe the electric field from the co-moved system Σ'' of charge carriers moving with the velocity field $\boldsymbol{u}$, then the velocity of Σ'' with respect to Σ is given by $\boldsymbol{w} = -\boldsymbol{u}$, and we obtain once more that $\boldsymbol{f}_L = \rho_e\, \boldsymbol{e} + \boldsymbol{j} \times \boldsymbol{b}$.

It will turn out to be convenient to express the vectorial form of Maxwell's equations completely in terms of the Galilei-invariants $\boldsymbol{d}$, $\boldsymbol{b}$, $\bar{\boldsymbol{e}}$, $\bar{\boldsymbol{h}}$, as well as ρ_e and $\bar{\boldsymbol{j}}$:

$$\begin{aligned} \operatorname{div} \boldsymbol{d} &= \rho_e && \text{(Gauss's Law)} \\ \operatorname{div} \boldsymbol{b} &= 0 && \text{(Conservation of magnetic flux)} \\ \textbf{rot}\ \bar{\boldsymbol{e}} &= -\overset{*}{\boldsymbol{b}} && \text{(Faraday's Induction Law)} \\ \textbf{rot}\ \bar{\boldsymbol{h}} &= \bar{\boldsymbol{j}} + \overset{*}{\boldsymbol{d}}. && \text{(Ampère's Law)} \end{aligned}$$

Here, the star denotes the **flux derivative** [Kovetz, 2000, see sections 25 and 54]. For any vector field $\boldsymbol{b}$ on $\widetilde{\mathcal{S}}$, its flux derivative $\overset{*}{\boldsymbol{b}} : \widetilde{\mathcal{S}} \to T\mathcal{S}$ is defined by

$$\frac{d}{dt} \int_{\partial U_t} \langle \boldsymbol{b}, \boldsymbol{n} \rangle \, \mathbf{vol}_{\partial U_t} = \int_{\partial U_t} \langle \overset{*}{\boldsymbol{b}}, \boldsymbol{n} \rangle \, \mathbf{vol}_{\partial U_t}.$$

It can be shown that $\overset{*}{\boldsymbol{b}}$ is given by

$$\overset{*}{\boldsymbol{b}} = \frac{\partial \boldsymbol{b}}{\partial t} + (\operatorname{div} \boldsymbol{b})\, \boldsymbol{v} - \mathbf{rot}\, (\boldsymbol{v} \times \boldsymbol{b}),$$

where $\boldsymbol{v}$ denotes the spatial velocity of the deformation. Using the identity

$$\mathbf{rot}\, (\boldsymbol{v} \times \boldsymbol{b}) = (\operatorname{div} \boldsymbol{b})\boldsymbol{v} - (\operatorname{div} \boldsymbol{v})\boldsymbol{b} + \nabla^{\mathcal{S}}_{\boldsymbol{b}} \boldsymbol{v} - \nabla^{\mathcal{S}}_{\boldsymbol{v}} \boldsymbol{b},$$

we see that $\overset{*}{\boldsymbol{b}}$ and the material derivative $\dot{\boldsymbol{b}}$ are related by

$$\overset{*}{\boldsymbol{b}} = \dot{\boldsymbol{b}} + (\operatorname{div} \boldsymbol{v})\, \boldsymbol{b} - \nabla^{\mathcal{S}}_{\boldsymbol{b}} \boldsymbol{v}. \tag{2.6}$$

2.3 Maxwell's Equations in terms of differential forms

If the body and the surrounding space are arbitrary manifolds, then we regard the electromagnetic fields as differential forms.

Let us denote the volume form of $\mathcal{S}$ by $\mathbf{vol}$ and the induced volume form on $\mathcal{B}_t$ by $\mathbf{vol}_t$. We can think of the charge density as a 3-form σ_e on $\widetilde{\mathcal{B}}$, related to the scalar charge density $\rho_e : \widetilde{\mathcal{S}} \to \mathbb{R}$ by $\sigma_e(\cdot, t) = \rho_e(\cdot, t)\, \mathbf{vol}$.

The current density can be considered either as a C^1 map $\mathfrak{j} : \widetilde{\mathcal{S}} \to \Omega^2\mathcal{S}$, or as vector field $\boldsymbol{j} : \widetilde{\mathcal{S}} \to T\mathcal{S}$, where both notions are related by $\mathfrak{j} = i_{\boldsymbol{j}} \mathbf{vol}$.

Moreover, the electric flux density and the electric field are regarded as maps $\mathfrak{d} : \widetilde{\mathcal{S}} \to \Omega^2\mathcal{S}$ and $\mathfrak{e} : \widetilde{\mathcal{S}} \to \Omega^1\mathcal{S}$, respectively. The magnetic flux density and the magnetic field can be taken as maps $\mathfrak{b} : \widetilde{\mathcal{S}} \to \Omega^2\mathcal{S}$ and $\mathfrak{h} : \widetilde{\mathcal{S}} \to \Omega^1\mathcal{S}$, respectively. (This distinction in the mathematical structure is usually ignored in the standard vectorial formulation.)

As in section 2.1 we restrict the charge density and the electromagnetic fields to $\widetilde{\mathcal{B}}$ and demand that these restrictions are C^2.

Then Maxwell's equations state that on $\widetilde{\mathcal{B}}$,

$$d\mathfrak{d} = \sigma_e \qquad \text{(Gauss' Law)}$$

$$d\mathfrak{b} = 0 \qquad \text{(Conservation of magnetic flux)}$$

$$d\mathfrak{e} = -\frac{\partial \mathfrak{b}}{\partial t} \qquad \text{(Faraday's Induction Law)}$$

$$d\mathfrak{h} = \mathfrak{j} + \frac{\partial \mathfrak{d}}{\partial t}. \qquad \text{(Ampère's Law)}$$

The electric flux density and the magnetic flux density are related with the electric and the magnetic field via the so-called aether relations

$$\begin{aligned} \mathfrak{d} &= \varepsilon_0 \ast \mathfrak{e} \\ \mathfrak{h} &= \frac{1}{\mu_0} \ast \mathfrak{b}, \end{aligned} \qquad \text{(Aether relations)}$$

where $\ast$ denotes the Hodge star operator. ε_0 and μ_0 are constants, called the vacuum permittivity and the vacuum permeability, respectively. The local expression for the balance of charge is given by

$$\frac{\partial \sigma_e}{\partial t} + d\mathfrak{j} = 0. \qquad \text{(Balance of charge)}$$

In the usual vectorial formulation, the electric flux density and the magnetic flux density are related with the electric and the magnetic field via

$$\begin{aligned} \boldsymbol{d} &= \varepsilon_0 \, \boldsymbol{e} \\ \boldsymbol{h} &= \frac{1}{\mu_0} \boldsymbol{b}. \end{aligned} \qquad \text{(Aether relations)}$$

This is an extremely awkward situation, since on the level of forms it is obvious that $\boldsymbol{d}$ and $\boldsymbol{e}$ or $\boldsymbol{b}$ and $\boldsymbol{h}$, respectively, are completely different objects with a different transformation behavior. This blurring of their differences is the source of much distress in physical texts, in particular, when Maxwell's equations are to be transferred to the undeformed body.

Assume once more that the material moves with the velocity field $\boldsymbol{v}$. Then we define the electromotive intensity as the C^2 map $\mathfrak{e} : \widetilde{\mathcal{S}} \to \Omega^1 \mathcal{S}$ that is given by

$$\bar{\mathfrak{e}} := \mathfrak{e} - i_{\boldsymbol{v}} \mathfrak{b}.$$

while the magnetomotive intensity, seen as a C^2 map $\bar{\mathfrak{h}} : \widetilde{\mathcal{S}} \to \Omega^1 \mathcal{S}$ is defined as

$$\bar{\mathfrak{h}} := \mathfrak{h} + i_{\boldsymbol{v}} \mathfrak{d}.$$

Both these objects are Galilei-invariant. The **Lorentz force** $\mathfrak{f}_L : \widetilde{\mathcal{B}} \to \Omega^1 \mathcal{S}$ is defined by

$$\mathfrak{f}_L = \rho_e \, \mathfrak{e} - i_{\boldsymbol{j}} \mathfrak{b}.$$

or in terms of Galilei-invariants, by

$$\mathfrak{f}_L = \rho_e \, \bar{\mathfrak{e}} - i_{\bar{\boldsymbol{j}}} \mathfrak{b}.$$

2.4 Polarization and Magnetization

In most texts on electrodynamics, balance of charge is seen as a consequence of Gauss' Law and Ampère's Law. Alternatively, one might take it as a postulate and deduce *from it* the laws of Gauss and Ampère under the additional assumption of covariance. [Kovetz, 2000] Then the charge density σ_e is seen as the source of the electric flux density $\mathfrak{d}$, while the current density $\mathfrak{j}$ and the temporal change of the electric flux density generate the magnetic field $\mathfrak{h}$. We adopt the second point of view to define the polarization and the magnetization of the material:

Experiments have shown that many materials react to an external electromagnetic field by setting up charge and current distributions. These contributions are usually called **bound** charge and current distributions and are denoted by $(\sigma_e)_b$ and $\mathfrak{j}_b$, respectively. But the bound charges and currents not necessarily constitute the total charge that is contained in the material or the total current passing through it. We call the other charges and currents **free** and denote them by $(\sigma_e)_f$ and $\mathfrak{j}_f$. Thus the total charge density and the total current density are given by

$$\sigma_e = (\sigma_e)_b + (\sigma_e)_f \quad \text{and} \quad \mathfrak{j} = \mathfrak{j}_b + \mathfrak{j}_f,$$

respectively. We now assume that not only the total charges, but also the bound charges (and hence the free charges, too) are conserved (see [Hehl and Obukhov, 2003]):

$$\frac{\partial(\sigma_e)_b}{\partial t} + d\mathfrak{j}_b = 0. \tag{2.7}$$

The total charge σ_e gave rise to the potential $\mathfrak{d}$. Similarly, $(\sigma_e)_b$ generates the potential $\mathfrak{d}_b : \widetilde{\mathcal{B}} \to \Omega^2\mathcal{S}$, that is,

$$d\mathfrak{d}_b = (\sigma_e)_b. \qquad \text{(Gauss Law for bound charges)}$$

By Ampère's Law, currents and temporal changes of the electric flux density generate a magnetic field. The bound current density $\mathfrak{j}_b$ and temporal changes of the electric flux density generated by bound charges induce a magnetic field $\mathfrak{h}_b : \widetilde{\mathcal{B}} \to \Omega^1\mathcal{S}$, that is,

$$d\mathfrak{h}_b = \mathfrak{j}_b + \frac{\partial \mathfrak{d}_b}{\partial t} \qquad \text{(Ampère's Law for bound charges)}$$

The negative of the bound part of $\mathfrak{d}$ is called polarization and denoted by $\mathfrak{p}$, the bound part of $\mathfrak{h}$ is called magnetization and denoted by $\mathfrak{m}$,

$$\begin{aligned} \mathfrak{d}_b &=: -\mathfrak{p} \\ \mathfrak{h}_b &=: \mathfrak{m}. \end{aligned}$$

We may now define the parts of $\mathfrak{d}$ and $\mathfrak{h}$ that are generated by the free charges and currents. Using the aether relations, we obtain

$$\begin{aligned} \mathfrak{d}_f &:= \mathfrak{d} - \mathfrak{d}_b = \varepsilon_0 * \mathfrak{e} + \mathfrak{p}, \\ \mathfrak{h}_f &:= \mathfrak{h} - \mathfrak{h}_b = \frac{1}{\mu_0} * \mathfrak{b} - \mathfrak{m}. \end{aligned} \tag{2.8}$$

Then we can write Maxwell's equations in a way that only uses the free charges and currents

$$\begin{aligned} d\mathfrak{d}_f &= (\sigma_e)_f \\ d\mathfrak{b} &= 0 \\ d\mathfrak{e} &= -\frac{\partial \mathfrak{b}}{\partial t} \\ d\mathfrak{h}_f &= \mathfrak{j}_f + \frac{\partial \mathfrak{d}_f}{\partial t}. \end{aligned} \tag{2.9}$$

but we have to complement them by the relations (2.8).
It is common practice to omit the index f in the relations (2.9), but here we will not do so.

The polarization is a Galilei-invariant, the magnetization is not. If the body moves with the velocity $\boldsymbol{v}$, then we construct a Galilei-invariant version of $\mathfrak{m}$ by

$$\bar{\mathfrak{m}} := \mathfrak{m} - i_{\boldsymbol{v}}\mathfrak{p}.$$

If $\mathcal{S}$ is the Euclidean $\mathbb{R}^3$, then we can also give a vectorial formulation of (2.9): Expressed solely in terms of free charges and currents, Maxwell's equations take on the form

$$\begin{aligned} \operatorname{div} \boldsymbol{d}_f &= (\rho_e)_f && \text{(Gauss' Law)} \\ \operatorname{div} \boldsymbol{b} &= 0 && \text{(Conservation of magnetic flux)} \\ \mathbf{rot}\ \boldsymbol{e} &= -\frac{\partial \boldsymbol{b}}{\partial t} && \text{(Faraday's Induction Law)} \\ \mathbf{rot}\ \boldsymbol{h}_f &= \boldsymbol{j}_f + \frac{\partial \boldsymbol{d}_f}{\partial t}. && \text{(Ampère's Law)} \end{aligned}$$

The connections between the electric flux density and the polarization and between the magnetic flux density and the magnetization are then given by

$$\begin{aligned} \boldsymbol{d}_f &= \varepsilon_0\, \boldsymbol{e} + \boldsymbol{p} \\ \boldsymbol{h}_f &= \frac{1}{\mu_0}\, \boldsymbol{b} - \boldsymbol{m}. \end{aligned}$$

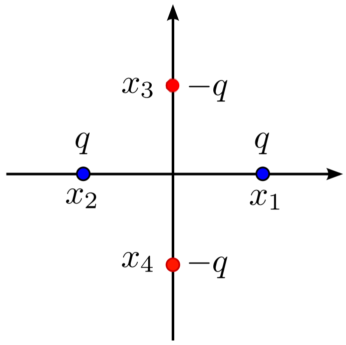
Again, this is an awkward situation, since $\boldsymbol{e}$ and $\boldsymbol{p}$ as well as $\boldsymbol{b}$ and $\boldsymbol{m}$ have completely different properties which can be easily seen if they are considered as differential forms. This constitutes an extremely annoying source of confusion in the literature. If, for example (in the case of a simple body), Maxwell's equations are pulled back to $\mathcal{B} \times I$, then some authors (e.g. Dorfmann and Ogden [2005]) wonder what the correct transformation of $\boldsymbol{p}$ might be. Shall it behave like $\boldsymbol{d}$, whose counterpart $\widetilde{\boldsymbol{D}}$ on $\mathcal{B} \times I$ is given by $\widetilde{\boldsymbol{D}} = \mathcal{J}\, \boldsymbol{\mathcal{F}}^{-1}\, \boldsymbol{d}$, or rather like $\boldsymbol{e}$, whose counterpart $\widetilde{\boldsymbol{E}}$ is given by $\widetilde{\boldsymbol{E}} = \boldsymbol{\mathcal{F}}^T(\boldsymbol{e})$? Of course, when seen as forms, it is clear that p must behave like $\boldsymbol{d}$, so its counterpart $\widetilde{\boldsymbol{P}}$ on $\mathcal{B} \times I$ must be defined by $\widetilde{\boldsymbol{P}} = \mathcal{J}\, \boldsymbol{\mathcal{F}}^{-1} \boldsymbol{p}$.

Remark 2.4.1. In most textbooks on electrodynamics (e.g. in Jackson [2002]) one finds another definition of the polarization and the magnetization:
Usually, the polarization is introduced as the dipole moment density: For a charge distribution consisting of point charges $q_1, \dots, q_n$ that are placed at the points $x_1, \dots, x_n$,

the dipole moment is defined as the vector

$$\sum_{i=1}^{n} q_i\, x_i.$$

For example, the charge distribution in the alongside picture has dipole moment 0.



In the case of a continuous charge distribution, the dipole moment is given by

$$\int \rho(\boldsymbol{x}, t)\, \boldsymbol{x}\, \mathbf{vol}.$$

The integrand of this expression is then called polarization.
In a similar way the magnetization is defined as the density of the magnetic moment, which is given by

$$\int \boldsymbol{x} \times \boldsymbol{j}\, \mathbf{vol}.$$

These definitions of $\boldsymbol{p}$ and $\boldsymbol{m}$ require the surrounding space to be the Euclidean $\mathbb{R}^3$. The definitions we gave in section 2.4, however, are also valid on arbitrary Riemannian manifolds. If $\mathcal{S}$ is the Euclidean $\mathbb{R}^3$, both definitions coincide (see Kovetz [2000], p.77).

Lemma 2.4.2 (Poynting's Theorem)
Let $\mathcal{S}$ be the Euclidean $\mathbb{R}^3$. Then all solutions $(\rho_e, \boldsymbol{j}, \boldsymbol{d}, \boldsymbol{b}, \boldsymbol{e}, \boldsymbol{h})$ of Maxwell's equations satisfy

$$\langle \bar{\boldsymbol{j}}, \bar{\boldsymbol{e}} \rangle = -\langle \bar{\boldsymbol{e}}, \overset{*}{\boldsymbol{d}} \rangle - \langle \bar{\boldsymbol{h}}, \overset{*}{\boldsymbol{b}} \rangle - \operatorname{div}(\bar{\boldsymbol{e}} \times \bar{\boldsymbol{h}}).$$

For the bound charges and currents alone,

$$\langle \bar{\boldsymbol{j}}_b, \bar{\boldsymbol{e}} \rangle = \langle \bar{\boldsymbol{e}}, \overset{*}{\boldsymbol{p}} \rangle - \langle \bar{\boldsymbol{m}}, \overset{*}{\boldsymbol{b}} \rangle - \operatorname{div}(\bar{\boldsymbol{e}} \times \bar{\boldsymbol{m}}).$$

Proof. By inserting the laws of Faraday and Ampère, $\mathbf{rot}\, \bar{\boldsymbol{e}} = -\overset{*}{\boldsymbol{b}}$ and $\mathbf{rot}\, \bar{\boldsymbol{h}} = \bar{\boldsymbol{j}} + \overset{*}{\boldsymbol{d}}$, into the identity

$$-\operatorname{div}(\bar{\boldsymbol{e}} \times \bar{\boldsymbol{h}}) = -\langle \mathbf{rot}\, \bar{\boldsymbol{e}}, \bar{\boldsymbol{h}} \rangle + \langle \bar{\boldsymbol{e}}, \mathbf{rot}\, \bar{\boldsymbol{h}} \rangle,$$

we immediately obtain the first of the asserted formulas. By inserting the laws of Faraday and Ampère for bound charges instead, $\mathbf{rot}\,\bar{\boldsymbol{e}} = -\overset{*}{\boldsymbol{b}}$ and $\mathbf{rot}\,\bar{\boldsymbol{m}} = \bar{\boldsymbol{j}}_b - \overset{*}{\boldsymbol{p}}$, we obtain the second one. □

2.5 The electromagnetic fields along the map ϕ_t

In chapter 3, we will establish balance laws (like for instance balance of momentum and energy) that govern the motion of $\mathcal{B}$. Of course, these laws contain electromagnetic quantities. For example, balance of momentum contains the Lorentz force $\mathfrak{f}_L$, and balance of energy contains the term $(\bar{\boldsymbol{j}}, \bar{\mathfrak{e}})$, where $(\cdot,\cdot)$ denotes the dual pairing of vector fields and 1-forms. As we have already seen in section 1.3.2, it is advisable to express these laws in terms of the coordinates on the undeformed body. Thus, we will, starting in chapter 4, regard the electromagnetic quantities as vector fields or forms along the map $\phi_t : \mathcal{B} \to \mathcal{B}_t \subset \mathcal{S}$, defining

$$\begin{aligned}
\widetilde{\rho}_e(X,t) &:= \mathcal{J}(X,t)\,\rho_e(x,t)\\
\boldsymbol{J}(X,t) &:= \mathcal{J}(X,t)\,\boldsymbol{j}(x,t)\\
\mathfrak{E}(X,t) &:= \mathfrak{e}(x,t)\\
\mathfrak{B}(X,t) &:= \mathfrak{b}(x,t)\\
\mathfrak{P}(X,t) &:= \mathcal{J}(X,t)\,\mathfrak{p}(x,t).
\end{aligned}$$

Then we obtain

$$\mathcal{J}\,(\boldsymbol{j}, \mathfrak{e}) = (\boldsymbol{J}, \mathfrak{E}), \tag{2.10}$$

which will be helpful in chapter 4 for expressing balance of energy in terms of the coordinates on $\mathcal{B}$. Moreover, the vector fields or forms along the map $\phi_t : \mathcal{B} \to \mathcal{B}_t \subset \mathcal{S}$, defined by

$$\begin{aligned}
\bar{\boldsymbol{J}} &:= \boldsymbol{J} - \widetilde{\rho}_e\,\boldsymbol{V}\\
\bar{\mathfrak{E}} &:= \mathfrak{E} - i_{\boldsymbol{V}}\mathfrak{B}\\
\mathfrak{F}_L &:= \widetilde{\rho}_e\,\mathfrak{E} + i_{\boldsymbol{J}}\mathfrak{B}
\end{aligned}$$

satisfy the following relations, that we will need in chapter 4:

$$\bar{\boldsymbol{J}}(X,t) = \mathcal{J}(X,t)\,\bar{\boldsymbol{j}}(x,t) \tag{2.11}$$
$$\bar{\mathfrak{E}}(X,t) = \bar{\mathfrak{e}}(x,t) \tag{2.12}$$
$$\mathfrak{F}_L(X,t) = \mathcal{J}(X,t)\,\mathfrak{f}_L(x,t), \tag{2.13}$$

and

$$\mathcal{J}\,(\bar{\boldsymbol{j}}, \bar{\mathfrak{e}}) = (\bar{\boldsymbol{J}}, \bar{\mathfrak{E}}). \tag{2.14}$$

3 The Balance Laws on $\mathcal{B}_t$

The motion of $\mathcal{B}$ is governed by a system of partial differential equations that consists of the conservation of mass and the balance laws for momentum, angular momentum, and energy.

For simple elastic bodies that are not exposed to an electromagnetic field, there is a broad agreement on how these balance laws must be formulated (see for example Liu [2002] or Chadwick [1999]). Marsden and Hughes [1983] even discuss the balance laws for the case that $\mathcal{B}$ and $\mathcal{S}$ are manifolds, but they mostly restrict to the case, where the body has the same dimension as the surrounding space.
For electromagnetic fields alone, that is, without any matter that is moved or deformed under the influence of these fields, there are also statements on balance of momentum and energy. [Griffiths, 2008]
But for the modeling of elastic bodies that are subjected to an electromagnetic field, there exist several approaches that are not compatible, most prominently the formulation by Ericksen [2008] in contrast to that by Kovetz [2000]. Steigmann [2009] claims that these formulations are equivalent, but we will see that this is not true (see Remark 3.5.3). Other formulations that also involve so-called surface couples, can be found in Eringen and Maugin [1990], Hutter et al. [2006], Hutter and Pao [1974], and Maugin [1988].
Moreover, in the physics literature on electroelastic materials only simple bodies are treated. Shells, i.e. bodies that only consist of a very thin layer of material (and could thus be modeled by a hypersurface), are then approximated as simple bodies with a thickness that tends to zero.
Usually, in works on electroelasticity the entropy inequality is used to decide, which otherwise allowed deformations are physically admissible and which are not. It is also employed to derive the Doyle-Ericksen formula that provides an important connection between the free energy density of the material and the deformation. Unfortunately, the opinions on the physically correct statement of the entropy inequality diverge when electromagnetic fields are present [Ericksen, 2008; Kovetz, 2000; Hutter and Pao, 1974]. A further problem in the formulation of an electroelastic theory on manifolds is that the entropy inequality, as it relies on the entropy flux to be tangential to the deformed body, is only applicable to simple bodies. For general bodies, in particular if they are subjected to an electromagnetic field, this needs not to be the case.

If the balance laws are to be formulated for general bodies, one has to decide upon a setup that is physically acceptable and at the same time can be carried over to a manifold. We will base our considerations on the set of balance laws that Ericksen [2008](see also Steigmann [2009]) provided for simple bodies. The form of balance of

energy that Ericksen stated, can be easily generalized to bodies that are described by a Riemannian manifold. The other balance laws will then be obtained by means of our Theorem 3.5.1. It states that, if balance of energy is invariant under the action of arbitrary diffeomorphisms on the surrounding space, then this *already implies* the local forms of conservation of mass, balance of momentum and angular momentum, as well as the Doyle-Ericksen formula which here provides a connection between the internal energy and the deformation. Theorem 3.5.1 generalizes a result that can already be found in the book by Marsden and Hughes [1983, ch. 2, Th. 4.13], and has more recently been discussed by Kanso et al. [2007, sec.3]. Both earlier statements of this result only pertain to bodies that have the same dimension as the surrounding space and do not allow the presence of electromagnetic fields.
Of course, it is also desirable from the physical point of view to have this invariance of balance of energy; it meets the demand that physics is independent of the choice of coordinate system. The proof of Theorem 3.5.1, however, does not work nicely for other setups of the balance laws that were suggested in the literature.
The formulation by Hutter and Pao [1974] is not suitable, since it uses transformations of the electromagnetic fields that are neither Galilean nor Lorentzian, but something in between.
The formulation by Kovetz [2000] is quite elegant, but it does not tell how the single contributions to the energy density depend on the velocity. But this knowledge is vital to the proof of Theorem 3.5.1. Furthermore, Kovetz uses the Poynting vector in his statement of balance of energy. Unfortunately, this vector is in general not tangential to the deformed body and thus hardly usable if one wants to find a description of the balance laws that works on a manifold.

As before, we will denote by $\mathcal{B}_t$ the state of $\mathcal{B}$ at the time t, i.e., $\mathcal{B}_t := \phi_t(\mathcal{B})$, and for some subset $U \subset \mathcal{B}$ of the undeformed body, $U_t := \phi_t(U)$ denotes the state of U at the time t.

3.1 Conservation of Mass

Recall from section 1.3.3 that the mass $\mathcal{M}$ of a nice set $U_t \subset \mathcal{B}_t$ is given by the integral of the (smooth) mass density $\rho : \widetilde{\mathcal{B}} \to \mathbb{R}$,

$$\mathcal{M}(U_t) = \int\limits_{U_t} \rho(x,t)\, \mathbf{vol}_t,$$

and that ρ obeys conservation of mass, if for all nice $U_t \subset \mathcal{B}_t$,

$$\frac{d}{dt} \int\limits_{U_t} \rho(x,t)\, \mathbf{vol}_t = 0.$$

By Theorem 1.3.19, the local form of conservation of mass is

$$\dot{\rho} + \rho \operatorname{div}_t \left(\boldsymbol{v}_{\|}\right) - m\, \rho\, g(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}}) = 0,$$

where $\boldsymbol{v}$ denotes the body's spatial velocity, m is the body's dimension and $\mathcal{H}$ denotes the mean curvature field. If the dimension of $\mathcal{B}$ coincides with the dimension of the surrounding space (and so in particular for simple bodies), then this simplifies to

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\, \boldsymbol{v}) = 0.$$

3.2 Balance of Momentum

By Newton's second law, the change of momentum $\mathscr{P}$ of a point mass is equal to the applied force $\mathscr{F}$:

$$\frac{d\mathscr{P}}{dt} = \mathscr{F}.$$

To set up balance of momentum in electroelasticity, we have to define, what momentum and force are in the case of a continuum.
The formulations of balance of momentum and angular momentum that are used in this and the following section only make sense if the surrounding space is the Euclidean $\mathbb{R}^3$. Later on in section 3.5 we will justify why the local forms of the laws that we obtain here are also valid on arbitrary Riemannian manifolds.

Definition 3.2.1. The **momentum** $\mathscr{P}$ of a part $U_t \subset \mathcal{B}_t$ in the motion ϕ is given by the integral of the **momentum density** $\rho\, \boldsymbol{v}$,

$$\mathscr{P}(U_t) := \int\limits_{U_t} \rho(x,t)\, \boldsymbol{v}\, \mathbf{vol}_t,$$

where $\rho : \widetilde{\mathcal{B}} \to \mathbb{R}$ denotes the mass density and $\boldsymbol{v} : \widetilde{\mathcal{B}} \to \mathcal{S}$ the velocity.

Forces

There are two kinds of forces acting on matter. On the one hand there are **long-range forces** (for example gravity and external electromagnetic forces) which decrease slowly when the distance between the interacting particles grows. If such a force acts from the outside on a part U of $\mathcal{B}$, it is capable of passing through the interior of U. The total force is proportional to the size of the concerned volume. Thus, long-range forces are also called **volume** or **body forces**.

On the other hand, there are forces that have only a **short range** and that decrease extremely rapidly when the distance between the interacting parts grows. They are called surface forces or contact forces and are negligible unless there is direct mechanical contact between the interacting parts. If a part U of $\mathcal{B}$ is acted on by short-range forces arising from reactions with the material outside U, these forces can act only on a thin layer adjacent to the boundary of U, of thickness corresponding to the penetration depth of the forces. The total of the short-range forces acting on U is thus determined by the surface area of U while the volume of U is not directly relevant. That is why they are also called **surface forces** or **contact forces** [Batchelor, 2000; Liu, 2002].

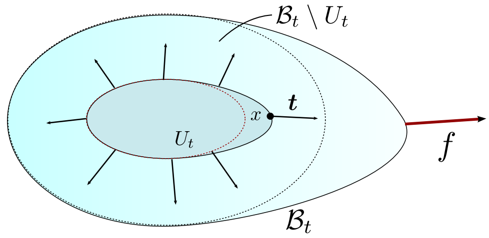
The only long-range forces that we will take into account are a purely mechanical force with the force density $\boldsymbol{f}$ (referred to the mass density ρ) and the Lorentz force with the force density $\boldsymbol{f}_L = \rho_e\,\boldsymbol{e} + \boldsymbol{j} \times \boldsymbol{b}$. Here $\rho_e : \widetilde{\mathcal{B}} \to \mathbb{R}$ denotes the charge density, $\boldsymbol{j} : \widetilde{\mathcal{B}} \to \mathcal{S}$ the current density, $\boldsymbol{e} : \widetilde{\mathcal{B}} \to \mathcal{S}$ the electric field and $\boldsymbol{b} : \widetilde{\mathcal{B}} \to \mathcal{S}$ the magnetic field (see chapter 2).

Definition 3.2.2. The **exterior forces** acting on a subset $U_t \subset \mathcal{B}_t$ are given by

$$\int_{U_t} \boldsymbol{f}\,\rho\,\mathbf{vol}_t + \int_{U_t} \boldsymbol{f}_L\,\mathbf{vol}_t,$$

where $\boldsymbol{f} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is a mechanical force density (referred to the mass density ρ) and $\boldsymbol{f}_L = \rho_e\,\boldsymbol{e} + \boldsymbol{j} \times \boldsymbol{b} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is the Lorentz force.

Let us consider a subset $U_t \subset \mathcal{B}_t$ of the deformed body. If its complement $U_t^c := \mathcal{B}_t \setminus U_t$ is deformed by an exterior force, then U_t will also be moved or deformed, since it is connected more or less fix with U_t^c. *How exactly* U_t reacts to the deformation of U_t^c, depends on the material the body is made of.



By the Euler-Cauchy stress principle, the action of U_t^c on U_t is equivalent to the interior forces that act on the boundary ∂U_t of U_t. These forces are characterized by a vector field $\boldsymbol{t}$ on ∂U_t that represents the force per area element.

Definition 3.2.3. The vector field $\boldsymbol{t} : \partial U_t \to TU_t$ representing the forces inside U_t is called **Cauchy stress vector field**.

The force characterized by the Cauchy stress vector is a short range force.

Axiom 1 (Cauchy's Postulate, Liu [2002]). For all surfaces passing through the point $x \in \partial U_t$ having the same tangent space at x (and thus the same normal vector at x), $\boldsymbol{t}|_x$ is the same. More precisely: $\boldsymbol{t}|_x$ only depends on the surfaces' (outward pointing) unit normal $\boldsymbol{n}$ at x.
In particular, the stress acting on $x \in \partial U_t$ is independent of the curvature of ∂U_t in $\mathcal{B}_t$.

Thus, the interior force or **the stress** that is caused inside U_t by the deformation of the body is given by

$$\int_{\partial U_t} \boldsymbol{t}(x, t, \boldsymbol{n})\, \mathbf{vol}_{\partial U_t}.$$

Definition 3.2.4. The **total force** acting on a part $U_t \subset \mathcal{B}_t$ is given by

$$\mathscr{F} := \int_{U_t} (\boldsymbol{f}\, \rho + \boldsymbol{f}_L)\, \mathbf{vol}_t + \int_{\partial U_t} \boldsymbol{t}\, \mathbf{vol}_{\partial U_t},$$

where the Cauchy stress $\boldsymbol{t}$ depends on the unit normal field $\boldsymbol{n}$ of ∂U_t.

Now we can formulate the balance of momentum:

Definition 3.2.5. We say that **balance of momentum** is satisfied, if for every nice set $U \subset \mathcal{B}$,

$$\frac{d}{dt} \int_{U_t} \rho\, \boldsymbol{v}\, \mathbf{vol}_t = \int_{U_t} (\boldsymbol{f}\, \rho + \boldsymbol{f}_L)\, \mathbf{vol}_t + \int_{\partial U_t} \boldsymbol{t}\, \mathbf{vol}_{\partial U_t}, \tag{3.1}$$

where $U_t = \phi_t(U)$, $\boldsymbol{f} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is a mechanical force density (referred to the mass density ρ), $\boldsymbol{f}_L = \rho_e\, \boldsymbol{e} + \boldsymbol{j} \times \boldsymbol{b} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is the Lorentz force and where the Cauchy stress $\boldsymbol{t}$ depends on the (outward) unit normal field $\boldsymbol{n}$ of ∂U_t.

Theorem 3.2.6
*Let $\boldsymbol{t}(x, t, \boldsymbol{n})$ be a continuous function of its arguments. Moreover, assume that balance of momentum holds. Then there is a unique $(1, 1)$-tensor field $\boldsymbol{\sigma}$ on $\mathcal{B}_t$, called the **Cauchy stress tensor field**, such that*

$$\boldsymbol{t}(x, t, \boldsymbol{n}) = \boldsymbol{\sigma}(\boldsymbol{n})(x, t).$$

That is, in local coordinates on $\mathcal{B}_t$, the components of the vector $\boldsymbol{t}$ are given by

$$t^a = \boldsymbol{\sigma}(\boldsymbol{n})^a = \sigma^a{}_b\, n^b.$$

Proof. Let e_a, $a = 1, \ldots, m$, be a local frame for the tangent space of $\mathcal{B}_t$, and let E_i, $i = 1, 2, 3$, be the Euclidean base of the surrounding $\mathbb{R}^3$. Moreover, we denote by $g^{\mathcal{B}_t}$ the metric that is induced on $\mathcal{B}_t$ by the restriction of g to $\mathcal{B}_t$. The i-th component of (3.1),

$$\frac{d}{dt} \int_{U_t} v^i\, \rho\, \mathbf{vol}_t = \int_{U_t} [f^i\, \rho + f^i_L] \mathbf{vol}_t + \int_{\partial U_t} t^i\, \mathbf{vol}_{\partial U_t},$$

is a conservation law for the i-th component of the momentum. By Cauchy's Theorem (1.3.3) it follows that there is a vector field $\boldsymbol{\sigma^i}$, such that $t^i = g^{\mathcal{B}_t}(\boldsymbol{\sigma^i}, \boldsymbol{n})$, and hence

$$\boldsymbol{t} = t^i\, E_i = g^{\mathcal{B}_t}(\boldsymbol{\sigma^i}, \boldsymbol{n})\, E_i. \tag{1}$$

Thus, $\boldsymbol{t}$ depends linearly on $\boldsymbol{n}$. Hence, there must be a $(1,1)$-tensor field $\boldsymbol{\sigma}$ on $\mathcal{B}_t$ such that $\boldsymbol{t} = \boldsymbol{\sigma}(\boldsymbol{n})$. By (1)

$$\boldsymbol{\sigma}(\boldsymbol{n}) = g^{\mathcal{B}_t}(\boldsymbol{\sigma^i}, \boldsymbol{n})\, E_i.$$

Consider a fixed point $x \in \mathcal{B}_t$. For each vector field $\boldsymbol{w}_t \in T_x\mathcal{B}_t$ we can find a subset $U_t \subset \mathcal{B}_t$, such that $\boldsymbol{w}_t$ is a multiple of the outward unit normal vector $\boldsymbol{n}$ of ∂U_t in x. Thus, we define $\boldsymbol{\sigma}$ by

$$\boldsymbol{\sigma}(\boldsymbol{w}_t) = g^{\mathcal{B}_t}(\boldsymbol{\sigma^i}, \boldsymbol{w}_t)\, E_i \quad \text{for all } \boldsymbol{w}_t \in T\mathcal{B}_t. \tag{2}$$

□

Thus, balance of momentum can also be expressed by

$$\frac{d}{dt}\int_{U_t} \rho\, \boldsymbol{v}\, \mathbf{vol}_t = \int_{U_t} (\boldsymbol{f}\, \rho + \boldsymbol{f}_L)\, \mathbf{vol}_t + \int_{\partial U_t} \boldsymbol{\sigma}(\boldsymbol{n})\, \mathbf{vol}_{\partial U_t}. \tag{3.2}$$

Theorem 3.2.7 (Equations of Motion)
Assume that conservation of mass and balance of momentum hold. Then

$$\rho\, \boldsymbol{a} = \rho\, \boldsymbol{f} + \boldsymbol{f}_L + \operatorname{div}_t \boldsymbol{\sigma} + \operatorname{tr}_{\mathbf{II}} \boldsymbol{\sigma}, \tag{3.3}$$

with $\boldsymbol{a} = \dot{\boldsymbol{v}}$ *and* $\boldsymbol{f}_L = \rho_e\, \boldsymbol{e} + \boldsymbol{j} \times \boldsymbol{b}$. $\operatorname{tr}_{\mathbf{II}}\boldsymbol{\sigma}$ *denotes the trace of* $\boldsymbol{\sigma}$ *with respect to* $\mathbf{II}$, *i.e., if* $e_1, \ldots, e_m$ *is a local orthonormal basis of* $T_x\mathcal{B}_t$, *then*

$$\operatorname{tr}_{\mathbf{II}} \boldsymbol{\sigma} = \sum_{a=1}^{m} \mathbf{II}(e_a, \boldsymbol{\sigma}(e_a)).$$

If $\mathcal{B}_t$ *is a hypersurface in* $\mathcal{S}$, *then*

$$\rho\, \boldsymbol{a} = \rho\, \boldsymbol{f} + \boldsymbol{f}_L + \operatorname{div}_t \boldsymbol{\sigma} + \operatorname{tr}(\boldsymbol{S_\nu} \circ \boldsymbol{\sigma})\, \boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ *denotes the (outer) unit normal vector field of* $\mathcal{B}_t$ *and* $\boldsymbol{S_\nu}$ *is the corresponding Weingarten map.*

Some parts of the following proof are taken from Grabs [2014].

Proof. We consider some arbitrary point $x \in U_t$. Let $e_1(x), \ldots, e_m(x)$ be an orthonormal basis of $T_x\mathcal{B}_t$, and extend it to a synchronous frame $e_1, \ldots, e_m$ around x. Let E_1, E_2, E_3 be the standard basis of the surrounding $\mathbb{R}^3$. For reasons of clarity we denote by $g^{\mathcal{S}}$ the metric on $\mathcal{S}$ and by $g^{\mathcal{B}_t}$ the metric that is induced on $\mathcal{B}_t$ by the restriction of $g^{\mathcal{S}}$ to $\mathcal{B}_t$. Once more, we consider the i-th component of (3.1),

$$\frac{d}{dt}\int\limits_{U_t} v^i\,\rho\,\mathbf{vol}_t = \int\limits_{U_t}[f^i\,\rho + f_L^i]\mathbf{vol}_t + \int\limits_{\partial U_t} t^i\,\mathbf{vol}_{\partial U_t}. \tag{1}$$

In the course of the proof of Theorem 3.2.6 we have seen that if balance of momentum is given, then the i-th component of $\boldsymbol{t}$ can be expressed by

$$t^i = g^{\mathcal{B}_t}(\boldsymbol{\sigma^i}, \boldsymbol{n}).$$

Hence, according to Theorem 1.3.21, the local form of (1) is given by

$$\rho\,\dot{\widehat{v^i}} = \rho\,f^i + f_L^i + \mathrm{div}_t\,\boldsymbol{\sigma^i}, \tag{2}$$

where $\dot{\widehat{v^i}}$ denotes the substantial derivative of v^i. Here the surrounding space is the Euclidean $\mathbb{R}^3$, thus the substantial derivative of any vector field $\boldsymbol{w}_t = w_t^i\,E_i$ that is defined along ϕ_t satisfies

$$\dot{\widehat{\boldsymbol{w}_t}} = \dot{\widehat{w_t^i}}\,E_i.$$

(In other words, $\dot{\widehat{w_t^i}} = \dot{\boldsymbol{w}}_t{}^i$, i.e., the substantial derivative of the i-th component of $\boldsymbol{w}_t$ is equal to the i-th component of the substantial derivative of $\boldsymbol{w}_t$.) This can be seen as follows:
Since the E_i are constant,

$$\begin{aligned}\dot{\boldsymbol{w}}_t &= \nabla^{\widetilde{\mathcal{S}}}_{(\boldsymbol{v}_t,\partial_t)}(w_t^i\,E_i)\\ &= \left[\partial_{(\boldsymbol{v}_t,\partial_t)} w_t^i\right] E_i\\ &= \dot{\widehat{w_t^i}}\,E_i.\end{aligned}$$

(Note that this step really requires $\mathcal{S}$ to be Euclidean.) Thus multiplying (2) by E_i and summing over i yields

$$\rho\,\dot{\boldsymbol{v}} = \rho\,\boldsymbol{f} + \boldsymbol{f}_L + (\mathrm{div}_t\,\boldsymbol{\sigma^i})\,E_i. \tag{3}$$

Using the synchronicity of $e_1, \ldots, e_m$, we obtain at the point x

$$\begin{aligned}\mathrm{div}_t\boldsymbol{\sigma^i} &= \sum_{a=1}^{m} g^{\mathcal{B}_t}\left(\nabla^{\mathcal{B}_t}_{e_a}\boldsymbol{\sigma^i}, e_a\right)\\ &= \sum_{a=1}^{m} \partial_{e_a}\,g^{\mathcal{B}_t}\left(\boldsymbol{\sigma^i}, e_a\right).\end{aligned}$$

Using (2) from the proof of Theorem 3.2.6, we obtain

$$\begin{aligned}\mathrm{div}_t\boldsymbol{\sigma}^{\boldsymbol{i}} &= \sum_{a=1}^{m} \partial_{e_a}\, g^{\mathcal{S}}(\boldsymbol{\sigma}(e_a), E_i)\\ &= \sum_{a=1}^{m} g^{\mathcal{S}}\Big(\nabla^{\mathcal{S}}_{e_a}\big(\boldsymbol{\sigma}(e_a)\big), E_i\Big),\end{aligned}$$

where we have used that the E_i are constant. In particular, $\nabla^{\mathcal{S}}_{e_a} E_i = 0$ for all $a = 1, \ldots, m$ and all $i = 1, \ldots, 3$. Now we can decompose $\nabla^{\mathcal{S}}$ into its tangent and normal part with respect to $\mathcal{B}_t$ and obtain

$$\begin{aligned}\mathrm{div}_t\boldsymbol{\sigma}^{\boldsymbol{i}} &= \sum_{a=1}^{m} g^{\mathcal{S}}\Big(\nabla^{\mathcal{B}_t}_{e_a}\big(\boldsymbol{\sigma}(e_a)\big) + \mathbf{II}\big(e_a, \boldsymbol{\sigma}(e_a)\big), E_i\Big)\\ &= \sum_{a=1}^{m} g^{\mathcal{S}}\Big(\big(\nabla^{\mathcal{B}_t}_{e_a}\boldsymbol{\sigma}\big)(e_a) + \mathbf{II}\big(e_a, \boldsymbol{\sigma}(e_a)\big), E_i\Big),\end{aligned}$$

where we have used again that the e_a are synchronous with respect to x. Thus,

$$\mathrm{div}_t(\boldsymbol{\sigma}^{\boldsymbol{i}})\, E_i = \mathrm{div}_t\, \boldsymbol{\sigma} + \sum_{a=1}^{m} \mathbf{II}(e_a, \boldsymbol{\sigma}(e_a)).$$

Inserting this into (3) yields

$$\rho\, \dot{\boldsymbol{v}} = \rho\, \boldsymbol{f} + \boldsymbol{f}_L + \mathrm{div}_t\boldsymbol{\sigma} + \sum_{a=1}^{m} \mathbf{II}(e_a, \boldsymbol{\sigma}(e_a)).$$

If $\mathcal{B}_t$ is a hypersurface in $\mathcal{S}$, then we may write $\mathbf{II}(\boldsymbol{\xi}, \boldsymbol{\eta}) = g(\boldsymbol{S_\nu}(\boldsymbol{\xi}), \boldsymbol{\eta})\, \boldsymbol{\nu}$, where $\boldsymbol{\xi}, \boldsymbol{\eta} \in T\mathcal{B}_t$, and $\boldsymbol{\nu}$ denotes the (outer) unit normal vector field of $\mathcal{B}_t \subset \mathcal{S}$. Then, using the symmetry of the Weingarten map $\boldsymbol{S_\nu}$, we obtain

$$\begin{aligned}\sum_{a=1}^{m} \mathbf{II}\big(e_a, \boldsymbol{\sigma}(e_a)\big) &= \sum_{a=1}^{m} g\big(\boldsymbol{S_\nu}(e_a), \boldsymbol{\sigma}(e_a)\big)\, \boldsymbol{\nu}\\ &= \sum_{a=1}^{m} g\big((\boldsymbol{S_\nu} \circ \boldsymbol{\sigma})(e_a), e_a\big)\, \boldsymbol{\nu}\\ &= \mathrm{tr}(\boldsymbol{S_\nu} \circ \boldsymbol{\sigma})\, \boldsymbol{\nu}.\end{aligned}$$

This completes the proof. □

3.3 Balance of Angular Momentum

By Newton's Law for a rigid body the change of angular momentum $\mathscr{L}_0$ is equal to the applied torque $\mathscr{M}_0$:

$$\frac{d\mathscr{L}_0}{dt} = \mathscr{M}_0.$$

In the case of a point particle the angular momentum (with respect to a point $\boldsymbol{x_0}$) is given by $(\boldsymbol{x} - \boldsymbol{x_0}) \times \boldsymbol{p}$, where $\boldsymbol{x}$ denotes the position and $\boldsymbol{p}$ the momentum of the particle. The torque that is caused by a force $\boldsymbol{f}$ is given by $(\boldsymbol{x} - \boldsymbol{x_0}) \times \boldsymbol{f}$. For a continuous body this is replaced by the following definition.

Definition 3.3.1. We define the **angular momentum** with respect to some point $x_0 \in U_t$ of a part $U_t \subset \mathcal{B}_t$ in the motion ϕ by

$$\mathscr{L}_0(U_t, \boldsymbol{x_0}) := \int\limits_{U_t} (\boldsymbol{x} - \boldsymbol{x_0}) \times \boldsymbol{v}\, \rho \,\mathbf{vol}_t.$$

and the **torque** acting on a part U_t by

$$\mathscr{M}_0(U_t, \boldsymbol{x_0}) := \int\limits_{U_t} (\boldsymbol{x} - \boldsymbol{x_0}) \times \boldsymbol{f}\, \rho \,\mathbf{vol}_t + \int\limits_{U_t} (\boldsymbol{x} - \boldsymbol{x_0}) \times \boldsymbol{f}_L \,\mathbf{vol}_t + \int\limits_{\partial U_t} (\boldsymbol{x} - \boldsymbol{x_0}) \times \boldsymbol{t}\,\mathbf{vol}_{\partial U_t}.$$

Here, as usual, $\boldsymbol{f} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is a mechanical force density (referred to the mass density ρ), $\boldsymbol{f}_L = \rho_e\, \boldsymbol{e} + \boldsymbol{j} \times \boldsymbol{b} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is the Lorentz force density, and $\boldsymbol{t}$ is the Cauchy stress vector.

Thus, for a continuous body, balance of angular momentum has the following form.

Definition 3.3.2. We say that **balance of angular momentum** is satisfied, if for every nice $U \subset \mathcal{B}$,

$$\begin{aligned} \frac{d}{dt} \int\limits_{U_t} (\boldsymbol{x} - \boldsymbol{x_0}) \times \boldsymbol{v}\, \rho \,\mathbf{vol}_t = \int\limits_{U_t} (\boldsymbol{x} - \boldsymbol{x_0}) \times \boldsymbol{f}\, \rho \,\mathbf{vol}_t + \int\limits_{U_t} (\boldsymbol{x} - \boldsymbol{x_0}) \times \boldsymbol{f}_L \,\mathbf{vol}_t \\ + \int\limits_{\partial U_t} (\boldsymbol{x} - \boldsymbol{x_0}) \times \boldsymbol{t}\,\mathbf{vol}_{\partial U_t}, \end{aligned} \tag{3.4}$$

where $U_t = \phi_t(U)$, $\boldsymbol{f} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is a mechanical force density (referred to the mass density ρ), $\boldsymbol{f}_L = \rho_e\, \boldsymbol{e} + \boldsymbol{j} \times \boldsymbol{b} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is the Lorentz force density, and $\boldsymbol{t}$ is the Cauchy stress vector.

In the following we set $\boldsymbol{x_0}$ to $\mathbf{0}$.

Theorem 3.3.3
Assume that conservation of mass and balance of momentum hold. Then balance of angular momentum holds if and only if

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T.$$

Proof. Let E_1, E_2, E_3 be the Euclidean base of the surrounding $\mathbb{R}^3$. Again, for reasons of clarity, we denote by $g^{\mathcal{S}}$ the (Euclidean) metric on $\mathcal{S}$ and by $g^{\mathcal{B}_t}$ the metric that is induced on $\mathcal{B}_t$ by the restriction of $g^{\mathcal{S}}$ to $\mathcal{B}_t$.
We consider the i-th component of (3.4):

$$\frac{d}{dt}\int\limits_{U_t} (\boldsymbol{x}\times\boldsymbol{v})^i\,\rho\,\mathbf{vol}_t = \int\limits_{U_t} (\boldsymbol{x}\times\boldsymbol{f})^i\,\rho\,\mathbf{vol}_t + \int\limits_{U_t} (\boldsymbol{x}\times\boldsymbol{f}_L)^i\,\mathbf{vol}_t + \int\limits_{\partial U_t} (\boldsymbol{x}\times\boldsymbol{t})^i\,\mathbf{vol}_{\partial U_t}. \tag{1}$$

If balance of momentum is valid, then by Theorem 3.2.6 there is a vector field $\boldsymbol{\sigma^k}$, tangential to $\mathcal{B}_t$, such that $t^k = g^{\mathcal{B}_t}(\boldsymbol{\sigma^k}, \boldsymbol{n})$. Then

$$(\boldsymbol{x}\times\boldsymbol{t})^i = \varepsilon^i_{jk}\,x^j\,t^k = g^{\mathcal{B}_t}(\varepsilon^i_{jk}\,x^j\,\boldsymbol{\sigma^k}, \boldsymbol{n}),$$

where ε denotes the Levi-Civita symbol. Thus, using conservation of mass and

$$\mathrm{div}_t(x^j\,\boldsymbol{\sigma^k}) = g^{\mathcal{B}_t}(\mathbf{grad}\,x^j, \boldsymbol{\sigma^k}) + x^j\,\mathrm{div}_t\,\boldsymbol{\sigma^k},$$

by Theorem 1.3.21 the localization of (1) is

$$\rho\,\dot{\overbrace{(\boldsymbol{x}\times\boldsymbol{v})^i}} = \rho\,(\boldsymbol{x}\times\boldsymbol{f})^i + (\boldsymbol{x}\times\boldsymbol{f}_L)^i + \varepsilon^i_{jk}\,\left[g^{\mathcal{B}_t}(\mathbf{grad}\,x^j, \boldsymbol{\sigma^k}) + x^j\,\mathrm{div}_t\,\boldsymbol{\sigma^k}\right]. \tag{2}$$

For the term on the left hand side of (2) we compute

$$\begin{aligned}\rho\,\dot{\overbrace{(\boldsymbol{x}\times\boldsymbol{v})^i}} &= \rho\,\dot{\overbrace{\varepsilon^i_{jk}\,x^j\,v^k}}\\ &= \rho\,\varepsilon^i_{jk}\left[\dot{\widehat{x^j}}\,v^k + x^j\,\dot{\widehat{v^k}}\right].\end{aligned}$$

x^j is a function on $\widetilde{\mathcal{B}}$. Let us define as in Remark 1.1.10 the function $\psi : \mathcal{B}\times I \to \widetilde{\mathcal{B}}$ by $(X,t) \overset{\psi}{\mapsto} (\phi(X,t),t)$. Then

$$\dot{\widehat{x^j}}\circ\psi = \frac{\partial\phi^j}{\partial t} = V^j = v^j\circ\psi.$$

Moreover, in the course of the proof of Theorem 3.2.7, we have seen that $\dot{\widehat{v^k}} = \dot{\boldsymbol{v}}^k$, i.e., the substantial derivative of the k-th component of $\boldsymbol{v}$ is equal to the k-th component of the substantial derivative of $\boldsymbol{v}$. (Of course, this works only with components with respect to the standard Euclidean basis.) Thus,

$$\begin{aligned}\rho\,\dot{\overbrace{(\boldsymbol{x}\times\boldsymbol{v})^i}} &= \rho\,[\boldsymbol{v}\times\boldsymbol{v} + \boldsymbol{x}\times\dot{\boldsymbol{v}}]^i\\ &= \rho\,[\boldsymbol{x}\times\dot{\boldsymbol{v}}]^i\,.\end{aligned} \tag{3}$$

We have also seen in the same proof that balance of momentum can be formulated as $\rho\,\dot{\boldsymbol{v}} = \rho\,\boldsymbol{f} + \boldsymbol{f}_L + \mathrm{div}_t\,\boldsymbol{\sigma}^{\boldsymbol{l}} E_l$. Inserting this into (3) gives

$$\begin{aligned}\rho\,\dot{\widehat{(\boldsymbol{x}\times\boldsymbol{v})^i}} &= \rho\,(\boldsymbol{x}\times\boldsymbol{f})^i + (\boldsymbol{x}\times\boldsymbol{f}_L)^i + \mathrm{div}_t\boldsymbol{\sigma}^{\boldsymbol{l}}\,[\boldsymbol{x}\times E_l]^i \\ &= \rho\,(\boldsymbol{x}\times\boldsymbol{f})^i + (\boldsymbol{x}\times\boldsymbol{f}_L)^i + \varepsilon^i_{jk}\,x^j\,\mathrm{div}_t\boldsymbol{\sigma}^{\boldsymbol{k}}.\end{aligned} \tag{4}$$

Combining (2) and (4), we obtain

$$\varepsilon^i_{jk}\,g^{\mathcal{B}_t}(\mathbf{grad}\,x^j, \boldsymbol{\sigma}^{\boldsymbol{k}}) = 0. \tag{5}$$

Let e_a, $a = 1, \dots, m$, be a local frame for the tangent space of $\mathcal{B}_t$. Then

$$\begin{aligned}\mathbf{grad}\,x^j &= g^{ab}\,\partial_{e_a}(x^j)\,e_b \\ &= g^{ab}\,(e_a)^i\,E_i(x^j)\,e_b \\ &= g^{ab}\,(e_a)^i\,\delta_i^{\,j}\,e_b \\ &= g^{ab}\,(e_a)^j\,e_b.\end{aligned}$$

We have derived in the proof of Theorem 3.2.6 that $g^{\mathcal{B}_t}(\boldsymbol{\sigma}^{\boldsymbol{k}}, \boldsymbol{w}) = g^{\mathcal{S}}(\boldsymbol{\sigma}(\boldsymbol{w}), E_k)$ for each vector $\boldsymbol{w}$ that is tangential to $\mathcal{B}_t$. Thus,

$$\begin{aligned}g^{\mathcal{B}_t}(\mathbf{grad}\,x^j, \boldsymbol{\sigma}^{\boldsymbol{k}}) &= g^{ab}\,(e_a)^j\,g^{\mathcal{B}_t}(\boldsymbol{\sigma}^{\boldsymbol{k}}, e_b) \\ &= g^{ab}\,(e_a)^j\,g^{\mathcal{S}}(\boldsymbol{\sigma}(e_b), E_k) \\ &= g^{ab}\,(e_a)^j\,\sigma^c{}_b\,(e_c)^k \\ &= \sigma^{ba}\,(e_a)^j\,(e_b)^k.\end{aligned}$$

Hence, (5) is equivalent to

$$\sigma^{ba}\,e_a \times e_b = \mathbf{0}.$$

Using the anti-symmetry of the vector product, we can rewrite this to

$$\frac{1}{2}(\sigma^{ba} - \sigma^{ab})\,e_a \times e_b = \mathbf{0},$$

from which we conclude that $\boldsymbol{\sigma}$ is symmetric. □

3.4 Balance of Energy

The next balance law, balance of energy, is a scalar law and can be formulated, if $\mathcal{B}$ and $\mathcal{S}$ are Riemannian manifolds. But for the computation of its local form we will need balance of momentum and angular momentum, and up to now we have only established them for the case that $\mathcal{S}$ is the Euclidean $\mathbb{R}^3$. Thus, we will at first demand that $\mathcal{S}$ is the Euclidean $\mathbb{R}^3$. At the end of this section, we will formulate balance of energy for the general setting and derive balance of momentum and angular momentum from the assumption that balance of energy is invariant under arbitrary spatial diffeomorphisms.

The first law of thermodynamics states that the change of energy $\mathscr{E}$ equals the power Π of the work that is done by body and surface forces and the exchanged amount of heat $\mathscr{Q}$:

$$\frac{d\mathscr{E}}{dt} = \Pi + \mathscr{Q} + \mathscr{W}_{\text{diss}},$$

where $\mathscr{E}$ denotes the energy, Π the power, $\mathscr{Q}$ the exchanged amount of heat, and $\mathscr{W}_{\text{diss}}$ represents some dissipative work. The energy is the sum of internal energy and (macroscopic) kinetic energy. Thus we define in the continuous case:

Definition 3.4.1. The **energy** $\mathscr{E}$ of a part $U_t \subset \mathcal{B}_t$ in the motion ϕ is given by

$$\mathscr{E}(U_t) := \int_{U_t} \Big(u + \frac{1}{2}\, g(\boldsymbol{v}, \boldsymbol{v})\Big)\, \rho\, \mathbf{vol}_t.$$

Here, $u : \widetilde{\mathcal{B}} \to \mathbb{R}$ is the internal energy density including among other things the energy which is stored microscopically inside the body in form of binding energy and the energy of molecular translations, rotations, and vibrations.

3.4.1 Special case: $\mathcal{S} = \mathbb{R}^3$ with the Euclidean metric

The power is equal to the scalar product of the applied forces with the velocity field. Here, as before, the applied forces are a purely mechanical force with the force density $\boldsymbol{f}$, stress forces inside the body, encoded in the stress vector $\boldsymbol{t}$, and the Lorentz force density $\boldsymbol{f}_L$.

Definition 3.4.2. The power Π of the forces acting on a part $U_t \subset \mathcal{B}_t$ in the motion ϕ is given by

$$\Pi(U_t) := \int_{U_t} g(\rho\, \boldsymbol{f} + \boldsymbol{f}_L, \boldsymbol{v})\, \mathbf{vol}_t + \int_{\partial U_t} g(\boldsymbol{t}, \boldsymbol{v})\, \mathbf{vol}_{\partial U_t},$$

where as usual, $\boldsymbol{f} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is a mechanical force density (referred to the mass density ρ), $\boldsymbol{f}_L = \rho_e\, \boldsymbol{e} + \boldsymbol{j} \times \boldsymbol{b} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is the Lorentz force, and $\boldsymbol{t}$ is the Cauchy stress vector.

Heat can be exchanged by an external **heat supply** with the density $r_\theta : \widetilde{\mathcal{B}} \to \mathbb{R}$, and by some **gain or loss of heat through the boundary** of U_t, encoded in some function h, depending on the outward unit normal vector field $\boldsymbol{n}$ of ∂U_t.

Definition 3.4.3. We define the **exchanged amount of heat** $\mathscr{Q}$ of a part $U_t \subset \mathcal{B}_t$ in the motion ϕ by

$$\mathscr{Q}(U_t) := \int\limits_{U_t} r_\theta \, \rho \, \mathbf{vol}_t + \int\limits_{\partial U_t} h(x, t, \boldsymbol{n}) \, \mathbf{vol}_{\partial U_t},$$

with the heat supply density $r_\theta : \widetilde{\mathcal{B}} \to \mathbb{R}$ and the gain or loss of heat through the boundary h.

If $\mathcal{B}$ does not have the same dimension as the surrounding space, then we could collect a possible heat flux between the body and its surrounding space into the heat supply density r_θ.

The only dissipative work that we take into account is the **Joule heating**. It is defined as $g(\bar{\boldsymbol{j}}, \bar{\boldsymbol{e}})$ where $\bar{\boldsymbol{j}} := \boldsymbol{j} - \rho_e \, \boldsymbol{v}$ is the conduction current density and $\bar{\boldsymbol{e}} := \boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b}$ is the electromotive intensity (see chapter 2). The Joule heating represents the amount of energy that charge carriers inside the material transfer during collisions to the rest of the material. For this process only the velocity of the charge carriers with respect to the material and the electric field as seen from the material's point of view are relevant. Hence the expression for the Joule heating involves the Galilei invariants $\bar{\boldsymbol{j}}$ and $\bar{\boldsymbol{e}}$, i.e. the current density and the electric field as they are seen from the material's point of view and not the corresponding fields that might be perceived by another observer.

Definition 3.4.4. The **dissipative work** that occurs inside some part $U_t \subset \mathcal{B}_t$ during the motion ϕ is given by

$$\mathscr{W}_{\mathrm{diss}}(U_t) := \int\limits_{U_t} g(\bar{\boldsymbol{j}}, \bar{\boldsymbol{e}}) \, \mathbf{vol}_t,$$

where $\bar{\boldsymbol{j}} : \widetilde{\mathcal{B}} \to \mathcal{S}$ denotes the convective current density and $\bar{\boldsymbol{e}} : \widetilde{\mathcal{B}} \to \mathcal{S}$ the electromotive intensity.

Definition 3.4.5 (Balance of Energy). We say that **balance of energy** holds if for every nice open set $U \subset \mathcal{B}$,

$$\begin{aligned} &\frac{d}{dt} \int\limits_{U_t} \left(u + \frac{1}{2} g(\boldsymbol{v}, \boldsymbol{v}) \right) \rho \, \mathbf{vol}_t \\ &= \int\limits_{U_t} (g(\boldsymbol{f}, \boldsymbol{v}) + r_\theta) \rho \, \mathbf{vol}_t + \int\limits_{U_t} g(\boldsymbol{f}_L, \boldsymbol{v}) \, \mathbf{vol}_t + \int\limits_{U_t} g(\bar{\boldsymbol{j}}, \bar{\boldsymbol{e}}) \, \mathbf{vol}_t + \int\limits_{\partial U_t} \Big(g(\boldsymbol{t}, \boldsymbol{v}) + h \Big) \, \mathbf{vol}_{\partial U_t}, \end{aligned}$$

where as usual $U_t = \phi_t(U)$, $\boldsymbol{f} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is a mechanical force density (referred to the mass density ρ), $\boldsymbol{f}_L = \rho_e\, \boldsymbol{e} + \boldsymbol{j} \times \boldsymbol{b} : \widetilde{\mathcal{B}} \to \mathcal{S}$ is the Lorentz force, and $\boldsymbol{t}$ is the Cauchy stress vector. Moreover, $u : \widetilde{\mathcal{B}} \to \mathbb{R}$ denotes the internal energy, $r_\theta : \widetilde{\mathcal{B}} \to \mathbb{R}$ the heat supply, h the gain/loss of heat through the boundary, and $\bar{\boldsymbol{j}} : \widetilde{\mathcal{B}} \to \mathcal{S}$ and $\bar{\boldsymbol{e}} : \widetilde{\mathcal{B}} \to \mathcal{S}$ denote the convective current density and the electromotive intensity, respectively. This is equivalent to

$$\frac{d}{dt} \int\limits_{U_t} \Big(u + \frac{1}{2}\, g(\boldsymbol{v}, \boldsymbol{v}) \Big)\, \rho\, \mathbf{vol}_t$$
$$= \int\limits_{U_t} \big(g(\boldsymbol{f}, \boldsymbol{v}) + r_\theta \big) \rho\, \mathbf{vol}_t + \int\limits_{U_t} g(\boldsymbol{j}, \boldsymbol{e}) \mathbf{vol}_t + \int\limits_{\partial U_t} \big(g(\boldsymbol{t}, \boldsymbol{v}) + h \big) \mathbf{vol}_{\partial U_t},$$

with the current density $\boldsymbol{j} : \widetilde{\mathcal{B}} \to \mathcal{S}$ and the electric field $\boldsymbol{e} : \widetilde{\mathcal{B}} \to \mathcal{S}$.

Remark 3.4.6. One part of the gained energy, the kinetic energy, is used for the deformation of the body, while the other part increases the internal energy of the body.

Theorem 3.4.7 (Marsden and Hughes [1983])
Assume that $\boldsymbol{t}(x, t, \boldsymbol{n}) = \boldsymbol{\sigma}_{(x,t)}(\boldsymbol{n})$ for a $(1, 1)$ tensor field $\boldsymbol{\sigma}$ on $\mathcal{B}_t$, where $\boldsymbol{n}$ denotes the outward unit normal vector field along ∂U_t. (This is given, if balance of momentum holds.)
Then balance of energy implies the existence of a unique vector field $\boldsymbol{q_\theta}$ on $\mathcal{B}_t$, such that for all $\boldsymbol{n}$,

$$h(x, t, \boldsymbol{n}) = -g(\boldsymbol{q_\theta}(x, t), \boldsymbol{n}).$$

We call $\boldsymbol{q_\theta}$ the ***heat flux vector****.*

Proof. We define $c := \langle \boldsymbol{t}, \boldsymbol{v} \rangle + h$. By Cauchy's Theorem (1.3.3) there exists a vector field $\boldsymbol{c}$ on U_t, such that $c = g(\boldsymbol{c}, \boldsymbol{n})$. Hence,

$$h = g(\boldsymbol{c}, \boldsymbol{n}) - g(\boldsymbol{t}, \boldsymbol{v}).$$

If $\boldsymbol{t} = \boldsymbol{\sigma}(\boldsymbol{n})$ for a $(1, 1)$ tensor field $\boldsymbol{\sigma}$ on $\mathcal{B}_t$, then

$$g(\boldsymbol{t}, \boldsymbol{v}) = g^{\mathcal{B}_t}(\boldsymbol{\sigma}(\boldsymbol{n}), \boldsymbol{v}_\parallel) = g^{\mathcal{B}_t}(\boldsymbol{\sigma}^T(\boldsymbol{v}_\parallel), \boldsymbol{n})$$

and thus

$$h = g(\boldsymbol{c} - \boldsymbol{\sigma}^T(\boldsymbol{v}_\parallel), \boldsymbol{n}).$$

Now we can define $\boldsymbol{q_\theta}$ by $-\boldsymbol{q_\theta} = \boldsymbol{c} - \boldsymbol{\sigma}^T(\boldsymbol{v}_\parallel)$. $\square$

Remark 3.4.8. The minus sign in the definition of $\boldsymbol{q_\theta}$ is just a matter of convention: It leads to $h = -g(\boldsymbol{q_\theta}, \boldsymbol{n})$ where $\boldsymbol{n}$ is the outward unit normal field to ∂U_t. Thus, if $\boldsymbol{q_\theta}$ points into the same direction as $-\boldsymbol{n}$, then the contribution of the heat flux is positive.

An immediate consequence of Theorem 3.4.7 is

Corollary 3.4.9
If balance of momentum holds, then balance of energy is equivalent to

$$\frac{d}{dt}\int\limits_{U_t}\Big(u + \frac{1}{2}\,g(\boldsymbol{v},\boldsymbol{v})\Big)\,\rho\,\mathbf{vol}_t = \int\limits_{U_t}\big(g(\boldsymbol{f},\boldsymbol{v}) + r_\theta\big)\rho\,\mathbf{vol}_t + \int\limits_{U_t} g(\boldsymbol{f}_L,\boldsymbol{v})\,\mathbf{vol}_t$$
$$+ \int\limits_{U_t} g(\bar{\boldsymbol{j}},\bar{\boldsymbol{e}})\,\mathbf{vol}_t + \int\limits_{\partial U_t} g\Big(\boldsymbol{\sigma}^T(\boldsymbol{v}_{\|}) - \boldsymbol{q_\theta},\boldsymbol{n}\Big)\,\mathbf{vol}_{\partial U_t}.$$

Theorem 3.4.10 (Local spatial form of balance of energy)
Assume that conservation of mass and balance of momentum hold.
Then the local form of balance of energy is given by

$$\rho\,\dot{u} = \langle\boldsymbol{\sigma},\nabla^{\mathcal{B}_t}\boldsymbol{v}_{\|}\rangle - g\left(\mathrm{tr}_{\mathbf{II}}\boldsymbol{\sigma},\boldsymbol{v}\right) + \rho\,r_\theta + g(\bar{\boldsymbol{j}},\bar{\boldsymbol{e}}) - \mathrm{div}_t\,\boldsymbol{q_\theta}. \tag{3.5}$$

Here, $\nabla^{\mathcal{B}_t}$ denotes the Levi-Civita connection on $(\mathcal{B}_t, g^{\mathcal{B}_t})$ and $\langle\boldsymbol{\sigma},\nabla^{\mathcal{B}_t}\boldsymbol{v}_{\|}\rangle$ the scalar product of the $(1,1)$ tensor fields $\boldsymbol{\sigma}$ and $\nabla^{\mathcal{B}_t}\boldsymbol{v}_{\|}$ on $\mathcal{B}_t$ (see Notation 1.2.14). If moreover, balance of angular momentum holds, then this simplifies to

$$\rho\,\dot{u} = \langle\boldsymbol{\sigma},\boldsymbol{d}\rangle + \rho\,r_\theta + g(\bar{\boldsymbol{j}},\bar{\boldsymbol{e}}) - \mathrm{div}_t\,\boldsymbol{q_\theta}, \tag{3.6}$$

where $\boldsymbol{d}$ is the rate of deformation tensor field we defined in section 1.2 (see Definition 1.2.12).

Proof. Since by assumption balance of momentum holds, it is sufficient to localize the equation from corollary 3.4.9.
Since conservation of mass holds, we can apply Theorem 1.3.21. Thus, balance of energy is equivalent to

$$\rho\,\dot{u} + \frac{1}{2}\,\rho\,\dot{\overline{g(\boldsymbol{v},\boldsymbol{v})}} = \rho\,g(\boldsymbol{f},\boldsymbol{v}) + \rho\,r_\theta + g(\boldsymbol{f}_L,\boldsymbol{v}) + g(\bar{\boldsymbol{j}},\bar{\boldsymbol{e}}) + \mathrm{div}_t\big(\boldsymbol{\sigma}^T(\boldsymbol{v}_{\|})\big) - \mathrm{div}_t\,\boldsymbol{q_\theta}. \tag{1}$$

By Lemma 1.1.9,

$$\frac{1}{2}\,\rho\,\dot{\overline{g(\boldsymbol{v},\boldsymbol{v})}} = \rho\,g(\boldsymbol{v},\dot{\boldsymbol{v}}) = \rho\,g(\boldsymbol{v},\boldsymbol{a}).$$

Inserting this and balance of momentum (see Theorem 3.2.7),

$$\rho\,\boldsymbol{a} = \rho\,\boldsymbol{f} + \boldsymbol{f}_L + \mathrm{div}_t\boldsymbol{\sigma} + \mathrm{tr}_{\mathbf{II}}\boldsymbol{\sigma}$$

into (1) while using that $\mathrm{div}_t\,\boldsymbol{\sigma}$ is tangential, gives

$$\rho\,\dot{u} + g(\mathrm{div}_t\boldsymbol{\sigma}, \boldsymbol{v}_\parallel) + g\left(\mathrm{tr}_{\mathbf{II}}\boldsymbol{\sigma}, \boldsymbol{v}\right) = \rho\, r_\theta + g(\bar{\boldsymbol{j}}, \bar{\boldsymbol{e}}) + \mathrm{div}_t\big(\boldsymbol{\sigma}^T(\boldsymbol{v}_\parallel)\big) - \mathrm{div}_t\,\boldsymbol{q_\theta}. \tag{1'}$$

Consider some arbitrary point $x \in \mathcal{B}_t$. Let $e_1(x), \ldots, e_m(x)$ be an orthonormal basis of $T_x\mathcal{B}_t$ and extend it to a synchronous frame $e_1, \ldots, e_m$ around x. Then in x

$$\begin{aligned}
\mathrm{div}_t\big(\boldsymbol{\sigma}^T(\boldsymbol{v}_\parallel)\big) &= g\left(\nabla^{\mathcal{B}_t}_{e_a}\big(\boldsymbol{\sigma}^T\,\boldsymbol{v}_\parallel\big), e_a\right) \\
&= \partial_{e_a} g\left(\boldsymbol{\sigma}^T\,\boldsymbol{v}_\parallel, e_a\right) \\
&= \partial_{e_a} g\left(\boldsymbol{\sigma}(e_a), \boldsymbol{v}_\parallel\right) \\
&= g\left(\big(\nabla^{\mathcal{B}_t}_{e_a}\boldsymbol{\sigma}\big)(e_a), \boldsymbol{v}_\parallel\right) + g\left(\boldsymbol{\sigma}(e_a), \nabla^{\mathcal{B}_t}_{e_a}\boldsymbol{v}_\parallel\right) \\
&= g(\mathrm{div}_t\boldsymbol{\sigma}, \boldsymbol{v}_\parallel) + g\left(\boldsymbol{\sigma}^T \circ \nabla^{\mathcal{B}_t}\boldsymbol{v}_\parallel(e_a), e_a\right) \\
&= g(\mathrm{div}_t\boldsymbol{\sigma}, \boldsymbol{v}_\parallel) + \mathrm{tr}_g(\boldsymbol{\sigma}^T \circ \nabla^{\mathcal{B}_t}\boldsymbol{v}_\parallel) \\
&= g(\mathrm{div}_t\boldsymbol{\sigma}, \boldsymbol{v}_\parallel) + \langle \boldsymbol{\sigma}, \nabla^{\mathcal{B}_t}\boldsymbol{v}_\parallel\rangle.
\end{aligned}$$

Thus, (1') becomes

$$\rho\,\dot{u} = \langle \boldsymbol{\sigma}, \nabla^{\mathcal{B}_t}\boldsymbol{v}_\parallel\rangle - g\left(\mathrm{tr}_{\mathbf{II}}\boldsymbol{\sigma}, \boldsymbol{v}\right) + \rho\, r_\theta + g(\bar{\boldsymbol{j}}, \bar{\boldsymbol{e}}) - \mathrm{div}_t\,\boldsymbol{q_\theta}. \tag{1''}$$

If balance of angular momentum is given, then $\boldsymbol{\sigma}$ is symmetric and hence,

$$\begin{aligned}
\langle \boldsymbol{\sigma}, \nabla^{\mathcal{B}_t}\boldsymbol{v}_\parallel\rangle &= \frac{1}{2}\left[\langle \boldsymbol{\sigma}, \nabla^{\mathcal{B}_t}\boldsymbol{v}_\parallel\rangle + \langle \boldsymbol{\sigma}^T, (\nabla^{\mathcal{B}_t}\boldsymbol{v}_\parallel)^T\rangle\right] \\
&= \frac{1}{2}\,\langle \boldsymbol{\sigma}, \nabla^{\mathcal{B}_t}\boldsymbol{v}_\parallel + (\nabla^{\mathcal{B}_t}\boldsymbol{v}_\parallel)^T\rangle \\
&= \frac{1}{2}\,\langle \boldsymbol{\sigma}, (\nabla^{\mathcal{B}_t}\boldsymbol{v}_\parallel)^\flat + \left[(\nabla^{\mathcal{B}_t}\boldsymbol{v}_\parallel)^\flat\right]^T\rangle.
\end{aligned}$$

Moreover, the symmetry of $\boldsymbol{\sigma}$ implies

$$\begin{aligned}
g(\mathrm{tr}_{\mathbf{II}}\boldsymbol{\sigma}, \boldsymbol{v}) &= \sum_{a=1}^{m} g\left(\mathbf{II}(e_a, \boldsymbol{\sigma}(e_a)), \boldsymbol{v}\right) \\
&= \sum_{a=1}^{m} g(\boldsymbol{\sigma}(e_a), e_b)\, g\left(\mathbf{II}(e_a, e_b), \boldsymbol{v}_\perp\right) \\
&= \sum_{a=1}^{m} \boldsymbol{\sigma}^\flat(e_a, e_b)\, g\left(\mathbf{II}(e_a, e_b), \boldsymbol{v}_\perp\right) \\
&= \langle \boldsymbol{\sigma}, g(\mathbf{II}(\cdot,\cdot), \boldsymbol{v}_\perp)\rangle.
\end{aligned}$$

Thus, by the definition of the rate of deformation tensor $\boldsymbol{d}$ (see Definition 1.2.12) and Theorem 1.2.13,

$$\begin{aligned}\langle \boldsymbol{\sigma}, \nabla^{\mathcal{B}_t} \boldsymbol{v}_\parallel \rangle - g(\mathrm{tr}_{\mathbf{II}} \boldsymbol{\sigma}, \boldsymbol{v}) &= \Big\langle \boldsymbol{\sigma}, \frac{1}{2} \left[(\nabla^{\mathcal{B}_t} \boldsymbol{v}_\parallel)^\flat + ((\nabla^{\mathcal{B}_t} \boldsymbol{v}_\parallel)^\flat)^T \right] - g(\mathbf{II}(\cdot,\cdot), \boldsymbol{v}_\perp) \Big\rangle \\ &= \langle \boldsymbol{\sigma}, \boldsymbol{d} \rangle,\end{aligned}$$

and (1”) becomes

$$\rho\, \dot{u} = \langle \boldsymbol{\sigma}, \boldsymbol{d} \rangle + \rho\, r_\theta + g(\bar{\boldsymbol{j}}, \bar{\boldsymbol{e}}) - \mathrm{div}_t\, \boldsymbol{q}_{\boldsymbol{\theta}}. \qquad \square$$

3.4.2 The general case

On general manifolds we regard the mechanical force and the Lorentz force as 1-forms $\mathfrak{f}$ and $\mathfrak{f}_L$ and write $\mathfrak{f}_L = \rho_e\, \mathfrak{e} - i_{\boldsymbol{j}} \mathfrak{b}$, where $\mathfrak{e}$ and $\mathfrak{b}$ denote the electric field 1-form and the magnetic flux 2-form, respectively.

Then the power Π of the forces acting on a part $U_t \subset \mathcal{B}_t$ in the motion ϕ can be expressed by

$$\Pi(U_t) = \int_{U_t} (\rho\, \mathfrak{f} + \mathfrak{f}_L, \boldsymbol{v})\, \mathbf{vol}_t + \int_{\partial U_t} g(\boldsymbol{t}, \boldsymbol{v})\, \mathbf{vol}_{\partial U_t},$$

where $(\mathfrak{f}, \boldsymbol{v})$ denotes the dual pairing of the vector field $\boldsymbol{v}$ and the 1-form $\mathfrak{f}$.

The scalar product $g(\bar{\boldsymbol{j}}, \bar{\boldsymbol{e}})$ can be replaced by the pairing $(\bar{\boldsymbol{j}}, \bar{\mathfrak{e}})$ with $\bar{\mathfrak{e}} = \mathfrak{e} - i_{\boldsymbol{v}} \mathfrak{b}$. Then the dissipative work for a part $U_t \subset \mathcal{B}_t$ in the motion ϕ is given by

$$\mathscr{W}_{\mathrm{diss}}(U_t) := \int_{U_t} (\bar{\boldsymbol{j}}, \bar{\mathfrak{e}})\, \mathbf{vol}_t.$$

Altogether we formulate balance of energy as follows:

Definition 3.4.11 (Balance of Energy in the general setting). We say that **balance of energy** holds if for every nice open set $U \subset \mathcal{B}$,

$$\begin{aligned}&\frac{d}{dt} \int_{U_t} \Big(u + \frac{1}{2} g(\boldsymbol{v}, \boldsymbol{v}) \Big)\, \rho\, \mathbf{vol}_t \\ &= \int_{U_t} \big((\mathfrak{f}, \boldsymbol{v}) + r_\theta \big) \rho\, \mathbf{vol}_t + \int_{U_t} (\mathfrak{f}_L, \boldsymbol{v})\, \mathbf{vol}_t + \int_{U_t} (\bar{\boldsymbol{j}}, \bar{\mathfrak{e}})\, \mathbf{vol}_t + \int_{\partial U_t} \Big(g(\boldsymbol{t}, \boldsymbol{v}) + h \Big)\, \mathbf{vol}_{\partial U_t},\end{aligned}$$

with $\boldsymbol{f}_L = \rho_e\, \mathfrak{e} - i_{\boldsymbol{j}} \mathfrak{b}$. This is equivalent to

$$\begin{aligned}&\frac{d}{dt} \int_{U_t} \Big(u + \frac{1}{2} g(\boldsymbol{v}, \boldsymbol{v}) \Big)\, \rho\, \mathbf{vol}_t \\ &= \int_{U_t} \big((\mathfrak{f}, \boldsymbol{v}) + r_\theta \big) \rho\, \mathbf{vol}_t + \int_{U_t} (\boldsymbol{j}, \mathfrak{e})\, \mathbf{vol}_t + \int_{\partial U_t} \big(g(\boldsymbol{t}, \boldsymbol{v}) + h \big) \mathbf{vol}_{\partial U_t},\end{aligned}$$

with notation as in Def. 3.4.5 but adapted to forms.

3.5 The balance laws for the general case

Conservation of mass and balance of energy are scalar laws and can be easily formulated even if the body and the surrounding space are Riemannian manifolds: Conservation of mass is given if (see section 3.1)

$$\frac{d}{dt}\int_{U_t} \rho\,\mathbf{vol}_t = 0,$$

or locally,

$$\dot{\rho} + \rho\,\mathrm{div}_t(\boldsymbol{v}_{\|}) - m\,\rho\,g(\boldsymbol{v}_{\perp}, \boldsymbol{\mathcal{H}}) = 0. \tag{3.7}$$

Balance of energy is given, if (see Def. 3.4.11)

$$\frac{d}{dt}\int_{U_t} \Big(u + \frac{1}{2}\,g(\boldsymbol{v},\boldsymbol{v})\Big)\rho\,\mathbf{vol}_t = \int_{U_t} ((\mathfrak{f},\boldsymbol{v}) + r_\theta)\,\rho\,\mathbf{vol}_t + \int_{U_t} (\boldsymbol{j},\mathfrak{e})\,\mathbf{vol}_t + \int_{\partial U_t} g(\boldsymbol{t},\boldsymbol{v})\,\mathbf{vol}_{\partial U_t} + \int_{\partial U_t} h\,\mathbf{vol}_{\partial U_t}. \tag{3.8}$$

For the balance laws of momentum and angular momentum there is a problem: For the simple case, where $\mathcal{S}$ is the Euclidean $\mathbb{R}^3$, we have derived local formulations of these laws in sections 3.2 and 3.3:

$$\rho\,\boldsymbol{a} = \rho\,\boldsymbol{f} + \boldsymbol{f}_L + \mathrm{div}_t\boldsymbol{\sigma} + \mathrm{tr}_{\mathbf{II}}\boldsymbol{\sigma} \tag{3.9}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{3.10}$$

(where we have already used conservation of mass) and with their help a local form of balance of energy. But to derive these laws we explicitly used the Euclidian structure of the surrounding space. So why should these laws also be valid if $\mathcal{B}$ and $\mathcal{S}$ are arbitrary Riemannian manifolds ?

In this section we will show that the demand that balance of energy is invariant under the action of arbitrary diffeomorphisms on the surrounding space together with some physical assumptions already implies (3.9), (3.10), and also (3.7). Then, by use of (3.7), (3.9), and (3.10) we also obtain the local form of balance of energy as we know it from Theorem 3.4.10. As a nice by-product we obtain the Doyle-Ericksen formula that provides a connection between the stress tensor, the internal energy, and the metric.

By Theorem 1.3.8 and Lemma 1.1.9, equation (3.8) is equivalent to

$$\int\limits_{U_t} [\dot{u} + g(\boldsymbol{v}, \boldsymbol{a})]\, \rho\, \mathbf{vol}_t + \int\limits_{U_t} \left[u + \frac{1}{2}\, g(\boldsymbol{v}, \boldsymbol{v})\right] \dot{\rho}\, \mathbf{vol}_t$$
$$+ \int\limits_{U_t} \left[u + \frac{1}{2}\, g(\boldsymbol{v}, \boldsymbol{v})\right] \rho \left[\operatorname{div}_t(\boldsymbol{v}_\|) - m\, g(\boldsymbol{v}_\perp, \mathcal{H})\right] \mathbf{vol}_t$$
$$= \int\limits_{U_t} [(\mathfrak{f}, \boldsymbol{v}) + r_\theta]\, \rho\, \mathbf{vol}_t + \int\limits_{U_t} (\boldsymbol{j}, \mathfrak{e})\, \mathbf{vol}_t + \int\limits_{\partial U_t} g(\boldsymbol{t}, \boldsymbol{v})\, \mathbf{vol}_{\partial U_t} + \int\limits_{\partial U_t} h\, \mathbf{vol}_{\partial U_t}. \quad (1)$$

Axiom 2. Balance of energy is invariant under the action of spatial diffeomorphisms $\xi_t : \mathcal{S} \to \mathcal{S}$. That is, on $U_t' = \xi_t(U_t) = \phi_t'(U)$, where $\phi_t' := \xi_t \circ \phi_t$, balance of energy is given by

$$\int\limits_{U_t'} \left[\dot{u}' + g'(\boldsymbol{v}', \boldsymbol{a}')\right] \rho'\, \mathbf{vol}_t' + \int\limits_{U_t'} \left[u' + \frac{1}{2}\, g'(\boldsymbol{v}', \boldsymbol{v}')\right] \dot{\rho}'\, \mathbf{vol}_t'$$
$$+ \int\limits_{U_t'} \left[u' + \frac{1}{2}\, g'(\boldsymbol{v}', \boldsymbol{v}')\right] \rho' \left[\operatorname{div}_t'(\boldsymbol{v}_\|') - m\, g'(\boldsymbol{v}_\perp', \mathcal{H}')\right] \mathbf{vol}_t'$$
$$= \int\limits_{U_t'} [(\mathfrak{f}', \boldsymbol{v}') + r_\theta']\, \rho'\, \mathbf{vol}_t' + \int\limits_{U_t'} (\boldsymbol{j}', \mathfrak{e}')\, \mathbf{vol}_t' + \int\limits_{\partial U_t'} g'(\boldsymbol{t}', \boldsymbol{v}')\, \mathbf{vol}_{\partial U_t'}' + \int\limits_{\partial U_t'} h'\, \mathbf{vol}_{\partial U_t'}'. \quad (2)$$

The coordinates in the primed and the unprimed systems are related by $x' = \xi_t(x)$. The corresponding expressions for the metric satisfy

$$g = \xi_t^* g',$$

the volume forms of U_t and U_t' are related by

$$\mathbf{vol}_t = \xi_t^*\, \mathbf{vol}_t'.$$

Let $\boldsymbol{w}$ be the velocity of ξ_t. Then differentiating $\phi_t' = \xi_t \circ \phi_t$ gives

$$\boldsymbol{v}'(x', t) = (\xi_t)_* \boldsymbol{v}(x, t) + \boldsymbol{w}(x, t).$$

Let p be an arbitrary point on $\mathcal{B}_t$. Let $e_1', \ldots, e_m'$ be an orthonormal basis of $T_p\mathcal{B}_t$ with respect to the coordinates x'. Then $e_1, \ldots, e_m$, defined by $e_a' = \xi_{t*} e_a$, $a = 1, \ldots, m$, is a basis of $T_p\mathcal{B}_t$ with respect to the coordinates x. Thus, in p,

$$\begin{aligned} \operatorname{div}_t'(\boldsymbol{v}_\|') &= \sum_{a=1}^m g'(\nabla^{\mathcal{B}_t}_{e_a'} \boldsymbol{v}_\|', e_a') \\ &= \sum_{a=1}^m ((\xi_t^{-1})^* g) \left(\nabla^{\mathcal{B}_t}_{(\xi_{t*} e_a)} ((\xi_t)_* \boldsymbol{v}_\| + \boldsymbol{w}_\|), \xi_{t*} e_a\right). \end{aligned}$$

We make the following **assumptions:**

1) The mass density, the heat supply density and the heat flow through the boundary satisfy

$$\begin{aligned} \rho'(x',t) &= \rho(x,t) \\ r'_\theta(x',t) &= r_\theta(x,t) \\ h'(x',t,\boldsymbol{n}') &= h(x,t,\boldsymbol{n}). \end{aligned}$$

2) The forces are transformed classically, i.e., they are transformed in the same way as the accelerations. Thus, $\boldsymbol{f}' - \boldsymbol{a}' = (\xi_t)_*(\boldsymbol{f} - \boldsymbol{a})$, or equivalently,

$$\mathfrak{f} - \boldsymbol{a}^\flat = \xi_t^* \left(\mathfrak{f}' - \boldsymbol{a}'^\flat \right).$$

3) The stress vectors that act along the boundaries of U_t and U'_t, respectively, are related by

$$\boldsymbol{t}'(x',t,\boldsymbol{n}') = (\xi_t)_* \boldsymbol{t}(x,t,\boldsymbol{n}).$$

4) The internal energy contains among other contributions the energy that is stored inside the material in the form of molecular rotations and vibrations. These energies depend on the distance of the molecules and thus on the metric of the surrounding space. Hence, the internal energy density u should depend on the metric g. We *assume* the simplest possible transformation

$$u'(x',t,g) = u(x,t,\xi_t^* g).$$

This assumption is often called assumption of "minimal coupling".

5) If $t = t_0$, the velocities satisfy $\boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{w}$. Thus, for $t = t_0$,

$$\begin{aligned} \boldsymbol{j}' &= \boldsymbol{j} + \rho_e \boldsymbol{w} \\ \mathfrak{e}' &= \mathfrak{e} + i_{\boldsymbol{w}} \mathfrak{b}, \end{aligned}$$

(see chapter 2).

We collect the relations between the primed and the unprimed quantitites at $t = t_0$:

$$\begin{aligned} \mathbf{vol}'_t(x') &= \mathbf{vol}_t(x), \\ \mathcal{H}(x') &= \mathcal{H}(x), \\ h'(x',t,\boldsymbol{n}') &= h(x,t,\boldsymbol{n}), \\ \boldsymbol{t}'(x',t,\boldsymbol{n}') &= \boldsymbol{t}(x,t,\boldsymbol{n}). \end{aligned}$$

Moreover,

$$
\begin{aligned}
\boldsymbol{v}' &= \boldsymbol{v} + \boldsymbol{w},\\
\mathrm{div}'_t(\boldsymbol{v}'_{\|}) &= \mathrm{div}_t(\boldsymbol{v}_{\|}) + \mathrm{div}_t(\boldsymbol{w}_{\|}),\\
\rho' &= \rho,\\
r'_\theta &= r_\theta,\\
\mathfrak{f}' - \boldsymbol{a}'^{\flat} &= \mathfrak{f} - \boldsymbol{a}^{\flat},\\
u' &= u,\\
\boldsymbol{j}' &= \boldsymbol{j} + \rho_e\, \boldsymbol{w},\\
\mathfrak{e}' &= \mathfrak{e} + i_{\boldsymbol{w}}\mathfrak{b},
\end{aligned}
$$

and

$$
\begin{aligned}
\langle \boldsymbol{v}', \boldsymbol{a}' \rangle &= \langle \boldsymbol{v} + \boldsymbol{w}, \boldsymbol{a}' \rangle,\\
\dot{u}'|_{t=t_0} &= \dot{u} + \Big\langle \frac{\partial u}{\partial(\xi_t^* g)}, \frac{d}{dt}(\xi_t^* g)|_{t=t_0} \Big\rangle = \dot{u} + \Big\langle \frac{\partial u}{\partial g}, L_{\boldsymbol{w}} g \Big\rangle,
\end{aligned} \tag{3.11}
$$

where the primed quantities are evaluated at (x', t) while the unprimed quantities are evaluated at (x, t).

Now we are ready to state one of the central theorems in this work:

Theorem 3.5.1
Let the body $(\mathcal{B}, G)$ and the surrounding space $(\mathcal{S}, g)$ be arbitrary Riemannian manifolds. Suppose, balance of energy is satisfied and invariant under the action of spatial diffeomorphisms $\xi_t : \mathcal{S} \to \mathcal{S}$ (Axiom 2) with the transformations that were stated above. Then there exists a $(1,1)$-tensor field $\boldsymbol{\sigma}$ on $\mathcal{B}_t$, such that $\boldsymbol{t} = \boldsymbol{\sigma}(\boldsymbol{n})$ and a vector field $\boldsymbol{q_\theta}$ on $\mathcal{B}_t$, such that $h(x, t, \boldsymbol{n}) = -g(\boldsymbol{q_\theta}, \boldsymbol{n})$.
Moreover, conservation of mass is satisfied,

$$
\dot{\rho} + \rho\, \mathrm{div}_t(\boldsymbol{v}_{\|}) - m\, \rho\, g(\boldsymbol{v}_\perp, \mathcal{H}) = 0,
$$

as well as

$$
\rho\, \boldsymbol{a}^{\flat} = \rho\, \mathfrak{f} + \rho_e\, \mathfrak{e} - (i_{\boldsymbol{j}}\mathfrak{b}) + (\mathrm{div}_t \boldsymbol{\sigma})^{\flat} + \mathrm{tr}_{\mathbf{II}} \boldsymbol{\sigma}^{\flat}, \tag{3.12}
$$

$$
\boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \tag{3.13}
$$

and the Doyle-Ericksen formula,

$$
\boldsymbol{\sigma}^{\sharp} = 2\, \rho\, \frac{\partial u}{\partial g}.
$$

Furthermore, the local form of balance of energy is given by

$$
\rho\, \dot{u} = \langle \boldsymbol{\sigma}, \boldsymbol{d} \rangle + \rho\, r_\theta + (\bar{\boldsymbol{j}}, \bar{\mathfrak{e}}) - \mathrm{div}_t\, \boldsymbol{q_\theta}.
$$

We identify (3.12) and (3.13) as balance of momentum and angular momentum, since they coincide with the local expressions (3.9) and (3.10) we had derived for the case that $\mathcal{S}$ is the Euclidean $\mathbb{R}^3$.

Remark 3.5.2. Theorem 3.5.1 provides a complete set of compatible balance laws that govern the deformation (and temperature development) of a body in a surrounding space. These balance laws are also valid, if the body and the surrounding space are arbitrary manifolds. Moreover, the Doyle-Ericksen formula states that the internal energy serves as a potential function for the stress tensor. In classical elasticity theory materials for which such a potential function exists are called *hyperelastic*. Thus, we have seen that the demand that the balance of energy is invariant under arbitrary spatial diffeomorphisms (together with some reasonable physical assumptions), already implies that the material the body consists of, must be hyperelastic. This is remarkable since we did not specify the kind of material yet.

As we have already mentioned, 3.5.1 generalizes a result by Marsden and Hughes [1983, ch. 2 Theorem 4.13] and Kanso et al. [2007, sec.3]. Both these earlier results only pertain to bodies that have the same dimension as the surrounding space and do not allow the presence of electromagnetic fields.

Proof. If $t = t_0$, then by use of the relations (3.11), balance of energy in the primed coordinates (2) becomes

$$\begin{aligned}
&\int_{U_t} \Big(\dot{u} + \big\langle \frac{\partial u}{\partial g}, L_{\boldsymbol{w}} g\big\rangle\Big)\,\rho\,\mathbf{vol}_t + \int_{U_t} \Big[u + \frac{1}{2}\,g(\boldsymbol{v}+\boldsymbol{w},\boldsymbol{v}+\boldsymbol{w})\Big]\,\big[\dot{\rho} + \partial_{(\boldsymbol{w}_t,\partial_t)}\rho\big]\,\mathbf{vol}_t \\
&\qquad + \int_{U_t} \Big[u + \frac{1}{2}\,g(\boldsymbol{v}+\boldsymbol{w},\boldsymbol{v}+\boldsymbol{w})\Big]\,\rho\,\big[\operatorname{div}_t(\boldsymbol{v}_{\|}+\boldsymbol{w}_{\|}) - m\,g(\boldsymbol{v}_{\perp}+\boldsymbol{w}_{\perp},\mathcal{H})\big]\,\mathbf{vol}_t \\
&\quad = \int_{U_t} \big((\mathfrak{f}-\boldsymbol{a}^{\flat},\boldsymbol{v}+\boldsymbol{w}) + r_\theta\big)\,\rho\,\mathbf{vol}_t + \int_{U_t} [(\boldsymbol{j},\mathfrak{e}) + (\rho_e\,\mathfrak{e} - i_{\boldsymbol{j}}\mathfrak{b},\boldsymbol{w})]\;\mathbf{vol}_t \\
&\qquad + \int_{\partial U_t} g(\boldsymbol{t},\boldsymbol{v}+\boldsymbol{w})\,\mathbf{vol}_{\partial U_t} + \int_{\partial U_t} h\,\mathbf{vol}_{\partial U_t},
\end{aligned} \tag{3}$$

where we also used that

$$\begin{aligned}
(\boldsymbol{j}',\mathfrak{e}') &= (\boldsymbol{j} + \rho_e\,\boldsymbol{w}, \mathfrak{e} + i_{\boldsymbol{w}}\mathfrak{b}) \\
&= (\boldsymbol{j},\mathfrak{e}) + (\boldsymbol{j}, i_{\boldsymbol{w}}\mathfrak{b}) + (\rho_e\,\boldsymbol{w},\mathfrak{e}) + \underbrace{(\rho_e\,\boldsymbol{w}, i_{\boldsymbol{w}}\mathfrak{b})}_{=0} \\
&= (\boldsymbol{j},\mathfrak{e}) + (\rho_e\,\mathfrak{e} - i_{\boldsymbol{j}}\mathfrak{b},\boldsymbol{w}).
\end{aligned}$$

Substracting (1) from (3) gives

$$\begin{aligned}
&\int\limits_{U_t} \Big\langle \frac{\partial u}{\partial g}, L_{\boldsymbol{w}} g \Big\rangle \rho \,\mathbf{vol}_t + \int\limits_{U_t} \frac{1}{2}\, g(\boldsymbol{w}, \boldsymbol{w}) \left[\dot{\rho} + \rho \operatorname{div}_t(\boldsymbol{v}_{\|}) - m\, \rho\, g(\boldsymbol{v}_{\perp}, \mathcal{H})\right] \,\mathbf{vol}_t \\
&\quad + \int\limits_{U_t} g(\boldsymbol{v}, \boldsymbol{w}) \left[\dot{\rho} + \rho \operatorname{div}_t(\boldsymbol{v}_{\|}) - m\, \rho\, g(\boldsymbol{v}_{\perp}, \mathcal{H})\right] \,\mathbf{vol}_t \\
&\quad + \int\limits_{U_t} \frac{1}{2}\, g(\boldsymbol{w}, \boldsymbol{w}) \left[\partial_{(\boldsymbol{w}_t, \partial_t)} \rho + \rho \operatorname{div}_t(\boldsymbol{w}_{\|}) - m\, \rho\, g(\boldsymbol{w}_{\perp}, \mathcal{H})\right] \mathbf{vol}_t \\
&\quad + \int\limits_{U_t} \Big(u + \frac{1}{2}\, g(\boldsymbol{v}, \boldsymbol{v})\Big) \left[\partial_{(\boldsymbol{w}_t, \partial_t)} \rho + \rho \operatorname{div}_t(\boldsymbol{w}_{\|}) - m\, \rho\, g(\boldsymbol{w}_{\perp}, \mathcal{H})\right] \mathbf{vol}_t \\
&\quad + \int\limits_{U_t} \frac{1}{2}\, g(\boldsymbol{v}, \boldsymbol{v}) \left[\partial_{(\boldsymbol{w}_t, \partial_t)} \rho + \rho \operatorname{div}_t(\boldsymbol{w}_{\|}) - m\, \rho\, g(\boldsymbol{w}_{\perp}, \mathcal{H})\right] \mathbf{vol}_t \\
&\quad = \int\limits_{U_t} (\mathfrak{f} - \boldsymbol{a}^{\flat}, \boldsymbol{w})\, \rho \,\mathbf{vol}_t + \int\limits_{U_t} (\rho_e\, \mathfrak{e} - i_{\boldsymbol{j}} \mathfrak{b}, \boldsymbol{w}) \,\mathbf{vol}_t + \int\limits_{\partial U_t} g(\boldsymbol{t}, \boldsymbol{w}) \,\mathbf{vol}_{\partial U_t}.
\end{aligned} \tag{4}$$

The only summand in this equation, in which $\boldsymbol{w}$ occurs in the third order, is the summand in the third line. Thus, a rescaling argument provides

$$\partial_{(\boldsymbol{w}_t, \partial_t)} \rho + \rho \operatorname{div}_t(\boldsymbol{w}_{\|}) - m\, \rho\, g(\boldsymbol{w}_{\perp}, \mathcal{H}) = 0.$$

Hence, (4) simplifies to

$$\begin{aligned}
&\int\limits_{U_t} \Big\langle \frac{\partial u}{\partial g}, L_{\boldsymbol{w}} g \Big\rangle \rho \,\mathbf{vol}_t + \int\limits_{U_t} \frac{1}{2}\, g(\boldsymbol{w}, \boldsymbol{w}) \left[\dot{\rho} + \rho \operatorname{div}_t(\boldsymbol{v}_{\|}) - m\, \rho\, g(\boldsymbol{v}_{\perp}, \mathcal{H})\right] \,\mathbf{vol}_t \\
&\quad + \int\limits_{U_t} g(\boldsymbol{v}, \boldsymbol{w}) \left[\dot{\rho} + \rho \operatorname{div}_t(\boldsymbol{v}_{\|}) - m\, \rho\, g(\boldsymbol{v}_{\perp}, \mathcal{H})\right] \,\mathbf{vol}_t \\
&\quad = \int\limits_{U_t} (\mathfrak{f} - \boldsymbol{a}^{\flat}, \boldsymbol{w})\, \rho \,\mathbf{vol}_t + \int\limits_{U_t} (\rho_e\, \mathfrak{e} - i_{\boldsymbol{j}} \mathfrak{b}, \boldsymbol{w}) \,\mathbf{vol}_t + \int\limits_{\partial U_t} g(\boldsymbol{t}, \boldsymbol{w}) \,\mathbf{vol}_{\partial U_t}.
\end{aligned} \tag{4'}$$

The only summand in (4'), in which $\boldsymbol{w}$ occurs quadratically, is the second summand in the first line. Thus, we conclude that

$$\dot{\rho} + \rho \operatorname{div}_t(\boldsymbol{v}_{\|}) - m\, \rho\, g(\boldsymbol{v}_{\perp}, \mathcal{H}) = 0.$$

This is the local form of conservation of mass. Hence, (4') simplifies to

$$\int\limits_{U_t} \Big\langle \frac{\partial u}{\partial g}, L_{\boldsymbol{w}} g \Big\rangle \rho \,\mathbf{vol}_t = \int\limits_{U_t} (\rho\, \mathfrak{f} - \rho\, \boldsymbol{a}^{\flat} + \rho_e\, \mathfrak{e} - i_{\boldsymbol{j}} \mathfrak{b}, \boldsymbol{w}) \,\mathbf{vol}_t + \int\limits_{\partial U_t} g(\boldsymbol{t}, \boldsymbol{w}) \,\mathbf{vol}_{\partial U_t}. \tag{4''}$$

In a similar way as in the proof of Theorem 3.2.6 we can apply Cauchy's Theorem to (4'') and deduce the existence of a vector field $\boldsymbol{\zeta}$ that is tangential to $\mathcal{B}_t$ and satisfies

$$g(\boldsymbol{t}, \boldsymbol{w}) = g(\boldsymbol{\zeta}, \boldsymbol{n}).$$

The left hand side of this equation depends linearly on $\boldsymbol{w}_\|$, so the $\boldsymbol{\zeta}$ on the right hand side must also depend linearly on $\boldsymbol{w}_\|$. Thus, there is a $(1,1)$-tensor field $\widehat{\boldsymbol{\sigma}}$ on $\mathcal{B}_t$, such that $\boldsymbol{\zeta} = \widehat{\boldsymbol{\sigma}}(\boldsymbol{w}_\|)$ and hence

$$g(\boldsymbol{t}, \boldsymbol{w}) = g(\boldsymbol{t}, \boldsymbol{w}_\|) = g(\widehat{\boldsymbol{\sigma}}(\boldsymbol{w}_\|), \boldsymbol{n}) = g(\widehat{\boldsymbol{\sigma}}^T(\boldsymbol{n}), \boldsymbol{w}_\|).$$

We now define $\boldsymbol{\sigma} := \widehat{\boldsymbol{\sigma}}^T$ and obtain $\boldsymbol{t} = \boldsymbol{\sigma}(\boldsymbol{n})$. As in the proof of Theorem 3.4.7 it follows from (1) that $h = -g(\boldsymbol{q_\theta}, \boldsymbol{n})$ for a vector field $\boldsymbol{q_\theta}$ on U_t. Moreover, (4") can be replaced by

$$\int_{U_t} \Big\langle \frac{\partial u}{\partial g}, L_{\boldsymbol{w}} g \Big\rangle \rho\, \mathbf{vol}_t = \int_{U_t} (\rho\, \mathfrak{f} - \rho\, \boldsymbol{a}^\flat + \rho_e\, \mathfrak{e} - i_{\boldsymbol{j}} \mathfrak{b}, \boldsymbol{w})\, \mathbf{vol}_t + \int_{U_t} \mathrm{div}_t(\boldsymbol{\sigma}^T \boldsymbol{w}_\|)\, \mathbf{vol}_t,$$

or, locally,

$$\rho \Big\langle \frac{\partial u}{\partial g}, L_{\boldsymbol{w}} g \Big\rangle = (\rho\, \mathfrak{f} - \rho\, \boldsymbol{a}^\flat + \rho_e\, \mathfrak{e} - i_{\boldsymbol{j}} \mathfrak{b}, \boldsymbol{w}) + \mathrm{div}_t(\boldsymbol{\sigma}^T \boldsymbol{w}_\|). \tag{5}$$

We reformulate the last term of (5). We have already computed in the proof of Theorem 3.4.10 that

$$\mathrm{div}_t(\boldsymbol{\sigma}^T \boldsymbol{w}_\|) = g(\mathrm{div}_t \boldsymbol{\sigma}, \boldsymbol{w}_\|) + \langle \boldsymbol{\sigma}^\sharp, (\nabla^{\mathcal{B}_t} \boldsymbol{w}_\|)^\flat \rangle.$$

We can also write this as

$$\mathrm{div}_t(\boldsymbol{\sigma}^T \boldsymbol{w}_\|) = g(\mathrm{div}_t \boldsymbol{\sigma}, \boldsymbol{w}_\|) + \frac{1}{2} \Big\langle \boldsymbol{\sigma}^\sharp, (\nabla^{\mathcal{B}_t} \boldsymbol{w}_\|)^\flat + \Big[(\nabla^{\mathcal{B}_t} \boldsymbol{w}_\|)^\flat\Big]^T \Big\rangle + \langle \boldsymbol{\sigma}^\sharp, \boldsymbol{\alpha} \rangle$$

with

$$\boldsymbol{\alpha} := \frac{1}{2} \Big[(\nabla^{\mathcal{B}_t} \boldsymbol{w}_\|)^\flat - \Big[(\nabla^{\mathcal{B}_t} \boldsymbol{w}_\|)^\flat\Big]^T \Big].$$

By Definition 1.2.12 and Theorem 1.2.13,

$$(\nabla^{\mathcal{B}_t} \boldsymbol{w}_\|)^\flat + \Big[(\nabla^{\mathcal{B}_t} \boldsymbol{w}_\|)^\flat\Big]^T = L_{\boldsymbol{w}} g + 2\, g(\boldsymbol{w}_\perp, \mathbf{II}(\cdot,\cdot)),$$

where $L_{\boldsymbol{w}}$ denotes the Lie derivative with respect to $\boldsymbol{w}$. Moreover, $\mathrm{div}_t \boldsymbol{\sigma}$ is tangential to $\mathcal{B}_t$. Hence,

$$\mathrm{div}_t(\boldsymbol{\sigma}^T \boldsymbol{w}_\|) = g(\mathrm{div}_t \boldsymbol{\sigma}, \boldsymbol{w}) + \frac{1}{2} \langle \boldsymbol{\sigma}^\sharp, L_{\boldsymbol{w}} g \rangle + \Big\langle \boldsymbol{\sigma}^\sharp, g(\boldsymbol{w}_\perp, \mathbf{II}(\cdot,\cdot)) \Big\rangle + \langle \boldsymbol{\sigma}^\sharp, \boldsymbol{\alpha} \rangle.$$

Consider an arbitrary point $x \in U_t$. Let $e_1(x), \ldots, e_m(x)$ be an orthonormal basis of $T_x U_t$ and extend it to a local frame $e_1, \ldots, e_m$ around x. Then at x

$$\Big\langle \boldsymbol{\sigma}^\sharp, g(\boldsymbol{w}_\perp, \mathbf{II}(\cdot,\cdot)) \Big\rangle = \sum_{a,b=1}^m \boldsymbol{\sigma}^\flat(e_a, e_b)\, g(\boldsymbol{w}_\perp, \mathbf{II}(e_a, e_b)) = \sum_{a,b=1}^m g(\boldsymbol{\sigma}^\flat(e_a, e_b)\, \mathbf{II}(e_a, e_b), \boldsymbol{w}),$$

since each $\mathbf{II}(e_a, e_b)$ is normal to $\mathcal{B}_t$. Thus,

$$\mathrm{div}_t(\boldsymbol{\sigma}^T \boldsymbol{w}_\parallel) = \sum_{a,b=1}^{m} g\Big(\mathrm{div}_t\boldsymbol{\sigma} + \boldsymbol{\sigma}^\flat(e_a, e_b)\,\mathbf{II}(e_a, e_b), \boldsymbol{w}\Big) + \frac{1}{2}\left\langle \boldsymbol{\sigma}^\sharp, L_{\boldsymbol{w}} g\right\rangle + \langle \boldsymbol{\sigma}^\sharp, \boldsymbol{\alpha}\rangle$$

and (5) becomes

$$\begin{aligned}\left\langle \rho\,\frac{\partial u}{\partial g} - \frac{1}{2}\,\boldsymbol{\sigma}^\sharp, L_{\boldsymbol{w}} g\right\rangle &= (\rho\,\mathfrak{f} - \rho\,\boldsymbol{a}^\flat + \rho_e\,\mathfrak{e} - i_{\boldsymbol{j}}\,\mathfrak{b} + (\mathrm{div}_t\boldsymbol{\sigma})^\flat \\ &\quad + \sum_{a,b=1}^{m} \boldsymbol{\sigma}^\flat(e_a, e_b)\,(\mathbf{II}(e_a, e_b))^\flat, \boldsymbol{w}) + \langle \boldsymbol{\sigma}^\sharp, \boldsymbol{\alpha}\rangle. \end{aligned} \tag{6}$$

This equation shall be valid for each diffeomorphism $\xi_t : \mathcal{S} \to \mathcal{S}$ and hence for each velocity $\boldsymbol{w}$.

We now consider (6) in an arbitrary point $x \in \mathcal{B}_t$. In the coordinates of the surrounding space, the Lie derivative of g is given by

$$(L_{\boldsymbol{w}} g)_{ij} = \frac{\partial g_{ij}}{\partial t} + g_{ij;k}\, w^k + w_{i;j} + w_{j;i}.$$

g does not depend on t. From now on we use Riemannian normal coordinates around x. Then in x all $g_{ij;k}$ are 0. In these coordinates,

$$L_{\boldsymbol{w}} g = (\nabla^{\mathcal{S}} \boldsymbol{w})^\flat + \left[(\nabla^{\mathcal{S}} \boldsymbol{w})^\flat\right]^T.$$

We now choose a ξ_t, such that $\boldsymbol{w}_\parallel = \mathbf{0}$ and $\nabla^{\mathcal{S}} \boldsymbol{w}_\perp = \mathbf{0}$. Then $\nabla^{\mathcal{B}_t} \boldsymbol{w}_\parallel = \mathbf{0}$ and $\nabla^{\mathcal{S}} \boldsymbol{w} = \nabla^{\mathcal{S}} \boldsymbol{w}_\perp = \mathbf{0}$. Thus, (6) implies

$$(\rho\,\mathfrak{f}_\perp - \rho\,\boldsymbol{a}_\perp^\flat + \rho_e\,\mathfrak{e}_\perp - (i_{\boldsymbol{j}}\mathfrak{b})_\perp + \sum_{a,b=1}^{m} \boldsymbol{\sigma}^\flat(e_a, e_b)\,(\mathbf{II}(e_a, e_b))^\flat, \boldsymbol{w}_\perp) = 0$$

and hence

$$\rho\,\boldsymbol{a}_\perp^\flat = \rho\,\mathfrak{f}_\perp + \rho_e\,\mathfrak{e}_\perp - (i_{\boldsymbol{j}}\mathfrak{b})_\perp + \sum_{a,b=1}^{m} \boldsymbol{\sigma}^\flat(e_a, e_b)\,(\mathbf{II}(e_a, e_b))^\flat. \tag{7}$$

On the other hand, if we choose a ξ_t, such that $\boldsymbol{w}_\perp = \mathbf{0}$ and $\nabla^{\mathcal{S}} \boldsymbol{w}_\parallel = \mathbf{0}$, then this implies $\nabla^{\mathcal{S}} \boldsymbol{w} = \mathbf{0}$ and $\nabla^{\mathcal{S}}_\eta \boldsymbol{w}_\parallel = \nabla^{\mathcal{B}_t}_\eta \boldsymbol{w}_\parallel + \mathbf{II}(\eta, \boldsymbol{w}_\parallel) = \mathbf{0}$ for each η that is tangential to $\mathcal{B}_t$. Since $\nabla^{\mathcal{B}_t}_\eta \boldsymbol{w}_\parallel$ is tangential to $\mathcal{B}_t$, while $\mathbf{II}(\eta, \boldsymbol{w}_\parallel)$ is normal, both must be equal to 0. Thus, $\nabla^{\mathcal{B}_t} \boldsymbol{w}_\parallel = \mathbf{0}$. Hence, (6) implies,

$$(\rho\,\mathfrak{f}_\parallel - \rho\,\boldsymbol{a}_\parallel^\flat + \rho_e\,\mathfrak{e}_\parallel - (i_{\boldsymbol{j}}\mathfrak{b})_\parallel + (\mathrm{div}_t\boldsymbol{\sigma})^\flat, \boldsymbol{w}_\parallel) = 0,$$

and thus, since $\boldsymbol{w}_\parallel \neq 0$,

$$\rho\,\boldsymbol{a}_\parallel^\flat = \rho\,\mathfrak{f}_\parallel + \rho_e\,\mathfrak{e}_\parallel - (i_{\boldsymbol{j}}\mathfrak{b})_\parallel + (\mathrm{div}_t\boldsymbol{\sigma})^\flat. \tag{8}$$

Adding (7) and (8) yields

$$\rho\, \boldsymbol{a}^\flat = \rho\, \mathfrak{f} + \rho_e\, \mathfrak{e} - (i_{\boldsymbol{j}} \mathfrak{b}) + (\mathrm{div}_t \boldsymbol{\sigma})^\flat + \sum_{a,b=1}^{m} \boldsymbol{\sigma}^\flat(e_a, e_b)\, (\mathbf{II}(e_a, e_b))^\flat. \tag{9}$$

Thus, (6) simplifies to

$$\Big\langle \rho\, \frac{\partial u}{\partial g} - \frac{1}{2}\, \boldsymbol{\sigma}^\sharp, L_{\boldsymbol{w}} g \Big\rangle = \langle \boldsymbol{\sigma}^\sharp, \boldsymbol{\alpha} \rangle, \tag{10}$$

Now we choose ξ_t in such a way that $\boldsymbol{w}_\| = \boldsymbol{0}$ while the normal velocity $\boldsymbol{w}_\perp$ is arbitrary. Then (10) becomes

$$\Big\langle \rho\, \frac{\partial u}{\partial g} - \frac{1}{2}\, \boldsymbol{\sigma}^\sharp, L_{\boldsymbol{w}_\perp} g \Big\rangle = 0.$$

By the arbitrariness of $\boldsymbol{w}_\perp$ we deduce that

$$\boldsymbol{\sigma}^\sharp = 2\, \rho\, \frac{\partial u}{\partial g}. \tag{11}$$

The only remnant of (6) is $\langle \boldsymbol{\sigma}^\sharp, \boldsymbol{\alpha} \rangle = 0$. Since α is skew-symmetric, this implies that $\boldsymbol{\sigma}$ is symmetric,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{12}$$

Thus, we can rewrite the last term of (9) by

$$\begin{aligned} \sum_{a,b=1}^{m} \boldsymbol{\sigma}^\flat(e_a, e_b)\, \mathbf{II}(e_a, e_b) &= \sum_{a,b=1}^{m} g(\boldsymbol{\sigma}(e_a), e_b)\, \mathbf{II}(e_a, e_b) \\ &= \sum_{a,b=1}^{m} \mathbf{II}(e_a, \boldsymbol{\sigma}(e_a)) \\ &= \mathrm{tr}_{\mathbf{II}} \boldsymbol{\sigma}. \end{aligned}$$

Hence, (9) becomes

$$\rho\, \boldsymbol{a}^\flat = \rho\, \mathfrak{f} + \rho_e\, \mathfrak{e} - (i_{\boldsymbol{j}} \mathfrak{b}) + (\mathrm{div}_t \boldsymbol{\sigma})^\flat + \mathrm{tr}_{\mathbf{II}} \boldsymbol{\sigma}^\flat. \tag{13}$$

Using conservation of mass, (13), (12), as well as the relations $\boldsymbol{t} = \boldsymbol{\sigma}(\boldsymbol{n})$ and $h = -g(\boldsymbol{q_\theta}, \boldsymbol{n})$, we can compute the local form of (1) as in the proof of Theorem 3.4.10. We obtain

$$\rho\, \dot{u} = \langle \boldsymbol{\sigma}, \boldsymbol{d} \rangle + \rho\, r_\theta + (\bar{\boldsymbol{j}}, \bar{\mathfrak{e}}) - \mathrm{div}_t\, \boldsymbol{q_\theta}. \qquad \square$$

Remark 3.5.3. We have based our formulations of the balance laws on the ansatz by Ericksen [2008], but we used the reformulation that was provided by Steigmann [2009] and carried it over to manifolds.

Steigmann [2009] claims that Kovetz' ansatz for the balance laws can be rephrased to give the formulation by Ericksen. But this is not true. Their formulations of balance of energy are different. Ericksen's ansatz contains an additional term $\mathrm{div}_t(\bar{\boldsymbol{e}} \times \bar{\boldsymbol{m}})$, where $\bar{\boldsymbol{e}}$ denotes the electromotive intensity and $\bar{\boldsymbol{m}}$ the magnetization that we introduced in chapter 2. It might be possible that Steigmann has overlooked that Kovetz [2000] starting from p. 81 solely uses Maxwell's equations in a form that contains only free charges and currents, without indicating this by use of the index f.

4 The Balance Laws on $\mathcal{B}$

In chapter 3 we have found a set of balance laws that govern the deformation of the body in the surrounding space. All of these laws were formulated on the deformed body, i.e. they were expressed by functions, vector and tensor fields (or forms) defined on $\mathcal{B}_t$. In section 1.3.1 we have called such laws spatial balance laws. In section 1.3.2, we have already seen that it is convenient to transform such spatial balance laws to material balance laws, i.e. to laws that are formulated in terms of quantities that are defined on $\mathcal{B} \times I$. It is the aim of this chapter to provide a material version of conservation of mass, balance of momentum, angular momentum, and energy.
For simple bodies that are not exposed to an external electromagnetic field the following material balance laws were already given by Marsden and Hughes [1983].

4.1 Conservation of mass

Recall from section 1.3.3 that conservation of mass on some subset U of the undeformed body is given by

$$\frac{d}{dt} \int_U \rho_{ref}(X,t)\,\mathbf{VOL} = 0,$$

where ρ_{ref} is the mass density on $\mathcal{B}$. The local form of conservation of mass is simply

$$\rho_{ref} = \text{const. in}\, t. \tag{4.1}$$

Conservation of mass on U_t is equivalent to conservation of mass on U, if and only if

$$\rho_{ref} = \mathcal{J}\,\rho, \tag{4.2}$$

where $\mathcal{J} : \mathcal{B} \times I \to \mathbb{R}$ denotes the factor of volume deformation (see Def. 1.3.5) and $\rho : \widetilde{\mathcal{B}} \to \mathbb{R}$ the mass density.

4.2 Balance of Momentum

For simple bodies the integral form of balance of momentum was found to read

$$\frac{d}{dt} \int_{U_t} \rho\,\boldsymbol{v}\,\mathbf{vol}_t = \int_{U_t} (\boldsymbol{f}\,\rho + \boldsymbol{f}_L)\,\mathbf{vol}_t + \int_{\partial U_t} \boldsymbol{\sigma}(\boldsymbol{n})\,\mathbf{vol}_{\partial U_t} \tag{4.3}$$

for the image $U_t = \phi_t(U)$ of any nice subset U of $\mathcal{B}$ (see section 3.2).

In section 3.5 we have seen that the local form of balance of momentum on any body, simple or not, is given by

$$\rho\, \boldsymbol{a}^\flat = \rho\, \mathfrak{f} + \mathfrak{f}_L + (\mathrm{div}_t \boldsymbol{\sigma})^\flat + \mathrm{tr}_{\mathbf{II}} \boldsymbol{\sigma}^\flat, \tag{4.4}$$

where $\mathfrak{f}_L = \rho_e\, \mathfrak{e} - i_{\boldsymbol{j}} \mathfrak{b}$.
The expression $\boldsymbol{\sigma}_{(x,t)}(\boldsymbol{n})$ in (4.3) gives the force per unit of deformed area. Now we use the Piola transformation to define a stress tensor field $\boldsymbol{\mathcal{P}}$ on $\mathcal{B}$ such that $\boldsymbol{\mathcal{P}}(\boldsymbol{N})$ gives the force per unit of undeformed area, where $\boldsymbol{N}$ denotes the outer unit normal vector field on $\mathcal{B}$.

Definition 4.2.1. The **first Piola-Kirchhoff stress tensor** $\boldsymbol{\mathcal{P}} : T_X\mathcal{B} \to T_x\mathcal{B}_t$, where $x = \phi_t(X)$, is defined by

$$\boldsymbol{\mathcal{P}} = \mathcal{J}\, \boldsymbol{\sigma} \circ (\boldsymbol{\mathcal{F}}^{-1})^T. \tag{4.5}$$

In components, this means

$$\mathcal{P}^a{}_A = \mathcal{J}\, \sigma^{ab}\, (\boldsymbol{\mathcal{F}}^{-1})_{Ab}.$$

The components of the corresponding tensor field $\boldsymbol{\mathcal{P}}^\sharp : T_x^*\mathcal{B}_t \times T_X^*\mathcal{B} \to \mathbb{R}$ are given by

$$\mathcal{P}^{aA} = \mathcal{J}\, (\boldsymbol{\mathcal{F}}^{-1})^A{}_b\, \sigma^{ab}.$$

Hence, some authors define the first Piola-Kirchhoff tensor by "applying a Piola transformation (see Definition 1.3.10) on the second index of $\boldsymbol{\sigma}$".

By use of the Piola Identity (Theorem 1.3.12), we see that

$$\mathcal{J}\, \mathrm{div}_t \boldsymbol{\sigma} \circ \phi_t = \mathrm{DIV} \boldsymbol{\mathcal{P}}. \tag{4.6}$$

Using the first Piola-Kirchhoff tensor we can rewrite (4.3) to arrive at

$$\frac{d}{dt} \int_U \boldsymbol{V}\, \rho_{ref}\, \mathbf{VOL} = \int_U \boldsymbol{F}\, \rho_{ref}\, \mathbf{VOL} + \int_U \boldsymbol{F}_L \mathbf{VOL} + \int_{\partial U} \boldsymbol{\mathcal{P}}(\boldsymbol{N})\, \mathbf{VOL}_{\partial U}$$

for any nice open set, where $\boldsymbol{V}(X,t) = \boldsymbol{v}(x,t)$, $\boldsymbol{F}(X,t) = \boldsymbol{f}(x,t)$, and $\boldsymbol{F}_L(X,t) = \mathcal{J}(X,t)\, \boldsymbol{f}_L(x,t)$.
For simple bodies, we could now obtain the local form of this equation with an argumentation similar to the proof of Theorem 3.2.7.
But we would like to have a local form of this law that is valid for arbitrary Riemannian manifolds. Thus, we multiply the local spatial form (4.4) that we had derived for any body (under assumption of conservation of mass) by $\mathcal{J}$ and express all quantities in

terms of the coordinates on the undeformed body: Using

$$\begin{aligned} \mathcal{J}(X,t)\,\rho(x,t) &= \rho_{ref}(X,t) && \text{(see eq. (4.2))} \\ \mathfrak{F}_L(X,t) &= \mathcal{J}(X,t)\,\mathfrak{f}_L(x,t), && \text{(see eq. (2.13))} \\ \mathcal{J}\,\mathrm{div}_t\boldsymbol{\sigma}\circ\phi_t &= \mathrm{DIV}\boldsymbol{\mathcal{P}} && \text{(see eq. (4.6))} \\ \mathcal{J}\,\mathrm{tr}_{\mathbf{II}}\boldsymbol{\sigma} &= \mathrm{tr}_{\mathbf{II}}(\boldsymbol{\mathcal{P}}\circ\boldsymbol{\mathcal{F}}^T) && \text{(see eq. (4.5))} \end{aligned}$$

as well as $\boldsymbol{V}(X,t) = \boldsymbol{v}(x,t)$ and $\mathfrak{F}(X,t) = \mathfrak{f}(x,t)$, we obtain

$$\rho_{ref}\,\boldsymbol{A}^\flat = \rho_{ref}\,\mathfrak{F} + \mathfrak{F}_L + (\mathrm{DIV}\boldsymbol{\mathcal{P}})^\flat + \left[\mathrm{tr}_{\mathbf{II}}(\boldsymbol{\mathcal{P}}\circ\boldsymbol{\mathcal{F}}^T)\right]^\flat\circ\phi,$$

where $\boldsymbol{A}$ denotes the material acceleration that we defined in section 1.1 (see Definition 1.1.5). If $\mathcal{B}_t$ is a hypersurface with normal unit field $\boldsymbol{\nu}$, then $\mathcal{J}\,\mathrm{tr}_{\mathbf{II}}\boldsymbol{\sigma} = \mathrm{tr}_g(\boldsymbol{S_\nu}\circ\boldsymbol{\mathcal{P}}\circ\boldsymbol{\mathcal{F}}^T)$ and balance of momentum can be expressed by

$$\rho_{ref}\,\boldsymbol{A}^\flat = \rho_{ref}\,\mathfrak{F} + \mathfrak{F}_L + (\mathrm{DIV}\boldsymbol{\mathcal{P}})^\flat + \left[\mathrm{tr}_g(\boldsymbol{S_\nu}\circ\boldsymbol{\mathcal{P}}\circ\boldsymbol{\mathcal{F}}^T)\right]^\flat\,\boldsymbol{\nu}\circ\phi,$$

where $\boldsymbol{S_\nu}$ denotes the Weingarten map.

Thus, we have shown

Theorem 4.2.2
The material form of balance of momentum, equivalent to (4.4) is

$$\rho_{ref}\,\boldsymbol{A}^\flat = \rho_{ref}\,\mathfrak{F} + \mathfrak{F}_L + (\mathrm{DIV}\boldsymbol{\mathcal{P}})^\flat + \left[\mathrm{tr}_{\mathbf{II}}(\boldsymbol{\mathcal{P}}\circ\boldsymbol{\mathcal{F}}^T)\right]^\flat\circ\phi.$$

For hypersurfaces this becomes

$$\rho_{ref}\,\boldsymbol{A}^\flat = \rho_{ref}\,\mathfrak{F} + \mathfrak{F}_L + (\mathrm{DIV}\boldsymbol{\mathcal{P}})^\flat + \left[\mathrm{tr}_g(\boldsymbol{S_\nu}\circ\boldsymbol{\mathcal{P}}\circ\boldsymbol{\mathcal{F}}^T)\right]^\flat\,\boldsymbol{\nu}\circ\phi.$$

4.3 Balance of Angular Momentum

We can now give a material version of balance of angular momentum. According to Theorem 1.3.23, for simple bodies it is given by

Theorem 4.3.1 (Material Balance of Angular Momentum)
In material form, the integral formulation of balance of angular momentum for simple bodies reads

$$\frac{d}{dt}\int_U \boldsymbol{X}\times\boldsymbol{V}\,\rho_{ref}\,\mathbf{VOL} = \int_U \boldsymbol{X}\times\boldsymbol{F}\,\rho_{ref}\,\mathbf{VOL} + \int_U \boldsymbol{X}\times\boldsymbol{F}_L\,\mathbf{VOL} + \int_{\partial U}\boldsymbol{X}\times(\boldsymbol{\mathcal{P}}(\boldsymbol{N}))\mathbf{VOL}_{\partial U}$$

for any nice $U\subset\mathcal{B}$, where $\boldsymbol{F}(X,t) := \boldsymbol{f}(x,t)$, $\boldsymbol{F}_L(X,t) := \mathcal{J}\,\boldsymbol{f}_L(x,t)$, and $\boldsymbol{\mathcal{P}}$ is the first Piola-Kirchhoff tensor.

Definition 4.3.2. The **second Piola-Kirchhoff tensor** $\boldsymbol{\mathcal{S}} : T_X\mathcal{B} \to T_X\mathcal{B}$ is defined by

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{F}}^{-1} \circ \boldsymbol{\mathcal{P}} = \mathcal{J}\, \boldsymbol{\mathcal{F}}^{-1} \circ \boldsymbol{\sigma} \circ (\boldsymbol{\mathcal{F}}^{-1})^T.$$

In components,

$$\mathcal{S}^A{}_B = \mathcal{J}\, (\boldsymbol{\mathcal{F}}^{-1})^A{}_a\, \sigma^{ab}\, (\boldsymbol{\mathcal{F}}^{-1})_{Bb}.$$

The components of the corresponding $(2,0)$ tensor field $\boldsymbol{\mathcal{S}}^\sharp : T^*_X\mathcal{B} \times T^*_X\mathcal{B} \to \mathbb{R}$ are given by

$$\mathcal{S}^{AB} = \mathcal{J}\, (\boldsymbol{\mathcal{F}}^{-1})^A{}_a\, \sigma^{ab}\, (\boldsymbol{\mathcal{F}}^{-1})^B{}_b.$$

In other words, $\boldsymbol{\mathcal{S}}^\sharp = \mathcal{J}\, \phi_t^* \boldsymbol{\sigma}^\sharp$.

Theorem 4.3.3 (Material Balance of Angular Momentum, Local Form)
Assume that Axiom 2) holds with the transformations that were stated above Theorem 3.5.1. Then balance of angular momentum holds if and only if

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}^T.$$

Proof. Under the given assumptions by Theorem 3.5.1 the local form of balance of angular momentum is given by the symmetry of $\boldsymbol{\sigma}$. By the definition of $\boldsymbol{\mathcal{S}}$, $\boldsymbol{\mathcal{S}}$ is symmetric if and only if $\boldsymbol{\sigma}$ is symmetric. $\square$

4.4 Balance of Energy

Under the assumption that the spatial form of balance of energy is invariant under the action of arbitrary spatial diffeomorphisms, we have balance of momentum and angular momentum at our disposal (see Theorem 3.5.1) and can express the spatial form of balance of energy by

$$\begin{aligned} &\frac{d}{dt} \int_{U_t} \Big(u + \frac{1}{2} g(\boldsymbol{v}, \boldsymbol{v}) \Big)\, \rho\, \mathbf{vol}_t \\ &= \int_{U_t} \big(g(\boldsymbol{f}, \boldsymbol{v}) + r_\theta \big) \rho\, \mathbf{vol}_t + \int_{U_t} (\boldsymbol{j}, \mathfrak{e})\, \mathbf{vol}_t + \int_{\partial U_t} g\big(\boldsymbol{\sigma}^T(\boldsymbol{v}_\parallel) - \boldsymbol{q_\theta}, \boldsymbol{n} \big)\, \mathbf{vol}_{\partial U_t}. \end{aligned} \tag{4.7}$$

Theorem 4.4.1 (Material Balance of Energy)
Define $U(X,t) := u(x,t)$, $\boldsymbol{F}(X,t) := \boldsymbol{f}(x,t)$, $R_\theta(X,t) := r_\theta(x,t)$, $\boldsymbol{Q_\theta}(X,t) := \mathcal{J}(X,t) \cdot \boldsymbol{\mathcal{F}}^{-1}(\boldsymbol{q_\theta}(x,t))$, *and as in section 2.5,* $\boldsymbol{J}(X,t) := \mathcal{J}(X,t) \cdot \boldsymbol{j}(x,t)$, $\boldsymbol{E}(X,t) = \boldsymbol{e}(x,t)$.

Then the material form of balance of energy, equivalent to (4.7), is

$$\frac{d}{dt}\int_U \left(U + \frac{1}{2}\langle \boldsymbol{V}, \boldsymbol{V}\rangle\right)\rho_{ref}\,\mathbf{VOL}$$
$$= \int_U \big(\langle \boldsymbol{F}, \boldsymbol{V}\rangle + R_\theta\big)\rho_{ref}\,\mathbf{VOL} + \int_U (\boldsymbol{J}, \mathfrak{E})\mathbf{VOL} + \int_{\partial U} G(\boldsymbol{\mathcal{P}}^T(\boldsymbol{V}_\parallel) - \boldsymbol{Q_\theta}, \boldsymbol{N})\,\mathbf{VOL}_{\partial U}.$$

Proof. We just apply Theorem 1.3.13 to (4.7). To do this, we need to determine the Piola transformation of $\boldsymbol{\sigma}^T(\boldsymbol{v}_\parallel)$. It follows immediately from the definition of the first Piola-Kirchhoff tensor $\boldsymbol{\mathcal{P}}$ that

$$\mathcal{J}\,\boldsymbol{\mathcal{F}}^{-1}\boldsymbol{\sigma}^T(\boldsymbol{v}_\parallel) = \boldsymbol{\mathcal{P}}^T(\boldsymbol{v}_\parallel).$$

Writing $\boldsymbol{v}_\parallel(x,t) = \boldsymbol{V}_\parallel(X,t)$, we obtain that the Piola transformation of $\boldsymbol{\sigma}^T(\boldsymbol{v}_\parallel)$ is $\boldsymbol{\mathcal{P}}^T(\boldsymbol{V}_\parallel)$. □

Theorem 4.4.2 (Balance of Energy - local material form)
Assume that conservation of mass, balance of momentum and balance of angular momentum hold. Then the local form of balance of energy is given by

$$\rho_{ref}\frac{\partial U}{\partial t} = \langle \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{D}}\rangle + \rho_{ref}R_\theta + (\bar{\boldsymbol{J}}, \bar{\mathfrak{E}}) - \mathrm{DIV}\boldsymbol{Q_\theta},$$

where as in section 2.5, $\bar{\boldsymbol{J}}(X,t) := \mathcal{J}(X,t)\,\bar{\boldsymbol{j}}(x,t)$ *and* $\bar{\mathfrak{E}}(X,t) := \bar{\mathfrak{e}}(x,t)$.

Proof. We immediately obtain from $\boldsymbol{\mathcal{S}}^\sharp = \mathcal{J}\,\phi_t^*\boldsymbol{\sigma}^\sharp$ and $\boldsymbol{\mathcal{D}}^\flat = \phi_t^*\boldsymbol{d}^\flat$ that

$$\mathcal{J}\,\langle \boldsymbol{\sigma}, \boldsymbol{d}\rangle = \langle \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{D}}\rangle.$$

We multiply the local form of balance of energy

$$\rho\,\dot{u} = \langle \boldsymbol{\sigma}, \boldsymbol{d}\rangle + \rho\,r_\theta + (\bar{\boldsymbol{j}}, \bar{\mathfrak{e}}) - \mathrm{div}_t\,\boldsymbol{q_\theta},$$

from Theorem 3.5.1 by $\mathcal{J}$ and express all terms in dependence on X and t: In Theorem 4.4.1 we have defined U and u by $U(X,t) = u(x,t)$ and $R_\theta(X,t) = r_\theta(x,t)$, respectively. Moreover, $\boldsymbol{Q_\theta}$ was defined as the Piola transformation of $\boldsymbol{q_\theta}$, thus by the Piola identity (Theorem 1.3.12) $\mathcal{J}\mathrm{div}_t\,\boldsymbol{q_\theta} = \mathrm{DIV}\,\boldsymbol{Q_\theta}$. Furthermore, as in section 2.5 we define $\bar{\boldsymbol{J}}(X,t) := \mathcal{J}(X,t)\cdot\bar{\boldsymbol{j}}(x,t)$ and $\bar{\mathfrak{E}}(X,t) = \bar{\mathfrak{e}}(x,t)$. By (4.2), $\mathcal{J}\,\rho = \rho_{ref}$. Thus we immediately obtain

$$\rho_{ref}\frac{\partial U}{\partial t} = \langle \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{D}}\rangle + \rho_{ref}R_\theta + (\bar{\boldsymbol{J}}, \bar{\mathfrak{E}}) - \mathrm{DIV}\boldsymbol{Q_\theta}.$$ □

4.5 The Equation of motion and the Doyle-Ericksen formula

We can also provide material versions of the Doyle-Ericksen formula $\boldsymbol{\sigma}^\sharp = 2\,\rho\,\frac{\partial u}{\partial g}$ (see Theorem 3.5.1) that links the internal energy u and the metric g of the surrounding space to the stress tensor $\boldsymbol{\sigma}$.

Theorem 4.5.1
The Doyle-Ericksen formula $\boldsymbol{\sigma}^\sharp = 2\,\rho\,\frac{\partial u}{\partial g}$ is equivalent to each of the following relations

$$\boldsymbol{\mathcal{S}}^\sharp = 2\,\rho_{ref}\,\frac{\partial U}{\partial \boldsymbol{\mathcal{C}}^\flat}, \qquad \textit{i.e.,} \qquad \mathcal{S}^{AB} = 2\rho_{ref}\,\frac{\partial U}{\partial \mathcal{C}_{AB}}, \tag{4.8}$$

$$\boldsymbol{\mathcal{P}}^\sharp = 2\,\rho_{ref}\,\boldsymbol{\mathcal{F}}\,\frac{\partial U}{\partial \boldsymbol{\mathcal{C}}^\flat} \qquad \mathcal{P}^{aB} = 2\rho_{ref}\,\mathcal{F}^a{}_A\,\frac{\partial U}{\partial \mathcal{C}_{AB}} \tag{4.9}$$

$$\boldsymbol{\mathcal{P}}^\sharp = g^\sharp\,\rho_{ref}\,\frac{\partial U}{\partial \boldsymbol{\mathcal{F}}}, \qquad \mathcal{P}^{aA} = g^{ab}\,\rho_{ref}\,\frac{\partial U}{\partial \mathcal{F}^b{}_A}. \tag{4.10}$$

Here, U is the material expression for the internal energy. $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{S}}$ denote the first and the second Piola-Kirchhoff tensor, respectively. Moreover, $\boldsymbol{\mathcal{F}}$ is the deformation gradient (see Def. 1.2.1) and $\boldsymbol{\mathcal{C}}$ the deformation tensor (see Def. 1.2.3).

Parts of the proof of this theorem are taken from the proofs of Proposition 2.11 and Proposition 2.12 in Marsden and Hughes [1983, see ch. 3].

Proof. By use of the chain rule and $\boldsymbol{\mathcal{C}}^\flat = \phi^* g$ (Remark 1.2.6), we see that

$$\left(\frac{\partial u}{\partial g}\right)^{ab} = \frac{\partial u}{\partial g_{ab}} = \frac{\partial U}{\partial \boldsymbol{\mathcal{C}}_{AB}}\,\frac{\partial \boldsymbol{\mathcal{C}}_{AB}}{\partial g_{ab}} = \frac{\partial U}{\partial \boldsymbol{\mathcal{C}}_{AB}}\,\boldsymbol{\mathcal{F}}^a{}_A\,\boldsymbol{\mathcal{F}}^b{}_B = \left(\phi_*\,\frac{\partial U}{\partial \boldsymbol{\mathcal{C}}^\flat}\right)^{ab},$$

or short, $\frac{\partial u}{\partial g} = \phi_*\,\frac{\partial U}{\partial \boldsymbol{\mathcal{C}}^\flat}$. Thus, the Doyle-Ericksen formula is equivalent to

$$\boldsymbol{\sigma}^\sharp = 2\,\rho\,\phi_*\,\frac{\partial U}{\partial \boldsymbol{\mathcal{C}}^\flat}.$$

We multiplicate this equation by $\mathcal{J}$ and apply ϕ^*. Since $\mathcal{J}\,\phi^*\boldsymbol{\sigma}^\sharp = \boldsymbol{\mathcal{S}}^\sharp$ and $\mathcal{J}\,\rho = \rho_{ref}$, (see p. 76 and eq. 4.2) we obtain equation (4.8),

$$\boldsymbol{\mathcal{S}}^\sharp = 2\,\rho_{ref}\,\frac{\partial U}{\partial \boldsymbol{\mathcal{C}}^\flat}.$$

Applying now the deformation gradient $\boldsymbol{\mathcal{F}}$ to both sides using the definition of $\boldsymbol{\mathcal{S}}$, immediately provides equation (4.9),

$$\boldsymbol{\mathcal{P}}^\sharp = 2\,\rho_{ref}\,\boldsymbol{\mathcal{F}}\,\frac{\partial U}{\partial \boldsymbol{\mathcal{C}}^\flat}.$$

To show the equivalence of (4.9) and (4.10), we compute $g^\sharp \frac{\partial U}{\partial \boldsymbol{\mathcal{F}}}$: In components,

$$\left(g^\sharp \frac{\partial U}{\partial \boldsymbol{\mathcal{F}}}\right)^{aA} = g^{ab} \frac{\partial U}{\partial \mathcal{C}_{BC}} \frac{\partial \mathcal{C}_{BC}}{\partial \mathcal{F}^b{}_A}.$$

By use of $\frac{\partial \mathcal{C}_{BC}}{\partial \mathcal{F}^b{}_A} = g_{bc} \left(\delta^A{}_B \, \mathcal{F}^c{}_C + \mathcal{F}^c{}_B \, \delta^A{}_C\right)$ and the symmetry of $\boldsymbol{\mathcal{C}}$ we obtain

$$\begin{aligned}\left(g^\sharp \frac{\partial U}{\partial \boldsymbol{\mathcal{F}}}\right)^{aA} &= \mathcal{F}^a{}_B \frac{\partial U}{\partial \mathcal{C}_{AB}} + \mathcal{F}^a{}_B \frac{\partial U}{\partial \mathcal{C}_{BA}} \\ &= 2\, \mathcal{F}^a{}_B \left(\frac{\partial U}{\partial \boldsymbol{\mathcal{C}}}\right)^{BA},\end{aligned}$$

or in short, $g^\sharp \frac{\partial U}{\partial \boldsymbol{\mathcal{F}}} = 2\, \boldsymbol{\mathcal{F}} \frac{\partial U}{\partial \boldsymbol{\mathcal{C}}^\flat}$. Thus,

$$2\, \rho_{\mathit{ref}} \boldsymbol{\mathcal{F}} \frac{\partial U}{\partial \boldsymbol{\mathcal{C}}^\flat} = g^\sharp \rho_{\mathit{ref}} \frac{\partial U}{\partial \boldsymbol{\mathcal{F}}},$$

whence the equivalence of (4.9) and (4.10) is proved. □

By use of (4.10) and $\boldsymbol{A} = \frac{\nabla}{\partial t} \frac{\partial \phi}{\partial t}$ we can rewrite the material form of balance of momentum (see Theorem 4.2.2) to

$$\rho_{\mathit{ref}} \frac{\nabla}{\partial t} \frac{\partial \phi}{\partial t} = \rho_{\mathit{ref}} \boldsymbol{F} + \boldsymbol{F}_L + \mathrm{DIV}\left(\rho_{\mathit{ref}} \frac{\partial U}{\partial \boldsymbol{\mathcal{F}}}\right) + \rho_{\mathit{ref}} \mathrm{tr}_{\mathbf{II}} \left(\frac{\partial U}{\partial \boldsymbol{\mathcal{F}}} \circ \boldsymbol{\mathcal{F}}^T\right) \circ \phi.$$

If we knew, how the internal energy U depends on the deformation ϕ and thus on the deformation gradient $\boldsymbol{\mathcal{F}}$, we could start immediately with the study of the equation of motion as a partial differential equation. We could make now an ad hoc assumption on this dependence and get started. But it is advisable, to reflect on how such a connection between U und $\boldsymbol{\mathcal{F}}$ could look like at all and whether it might be restricted by some reasonable physical demands. This will be the topic of the next chapter.

5 The Properties of the material: Constitutive relations

From here on, we always assume that balance of energy is invariant under diffeomorphisms acting on the surrounding space (see Axiom 2, section 3.5). Only then we have the local forms of balance of momentum, angular momentum, and energy at our disposal.

5.1 Constitutive relations

In material coordinates, the local forms of conservation of mass, balance of angular momentum, and balance of energy were given by

$$\rho_{ref} = \text{const. in } t, \qquad \text{(see eq. (4.1))}$$

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}^T, \qquad \text{(see Th. 4.3.3)}$$

$$\rho_{ref}\frac{\partial U}{\partial t} = \langle \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{D}} \rangle + \rho_{ref} R_\theta + (\bar{\boldsymbol{J}}, \bar{\mathfrak{E}}) - \mathrm{DIV}\boldsymbol{Q_\theta}. \qquad \text{(see Th. 4.4.2)}$$

The body's motion is governed by balance of momentum,

$$\rho_{ref}\,\boldsymbol{A}^\flat = \rho_{ref}\,\mathfrak{F} + \mathfrak{F}_L + (\mathrm{DIV}\boldsymbol{\mathcal{P}})^\flat + \left[\mathrm{tr_{II}}(\boldsymbol{\mathcal{P}} \circ \boldsymbol{\mathcal{F}}^T)\right]^\flat \circ \phi, \qquad \text{(see Th. 4.2.2)}$$

or in spatial coordinates,

$$\rho\,\boldsymbol{a}^\flat = \rho\,\mathfrak{f} + \mathfrak{f}_L + (\mathrm{div}_t\boldsymbol{\sigma})^\flat + \mathrm{tr_{II}}\boldsymbol{\sigma}^\flat. \qquad \text{(see Th. 3.5.1)}$$

Assume we are given the initial values of the mass density, the deformation, and the velocity

$$\rho_{ref}(X,0), \phi(X,0), \quad \text{and } \boldsymbol{V}(X,0),$$

as well as the external force $\boldsymbol{F}$ and the electromagnetic fields $\bar{\mathfrak{E}}$ and $\mathfrak{B}$. Can we determine the motion of $\mathcal{B}$?

Indeed, the equations that we have summarized above, are not sufficient to determine the motion of $\mathcal{B}$. There are the following problems:

1) We have more unknowns than relations. According to Theorem 3.5.1 and Theorem 4.5.1, the first Piola-Kirchhoff tensor is related to the motion ϕ and the internal energy U by the Doyle-Ericksen formula

$$\boldsymbol{\mathcal{P}}^\sharp = g^\sharp\,\rho_{ref}\frac{\partial U}{\partial \boldsymbol{\mathcal{F}}}.$$

But we do not know yet, how the internal energy depends on $\boldsymbol{\mathcal{F}}$ and the external influences.

2) Up to now, we have not taken the material the body is made of into account. But we should certainly do so, since we would expect the body to behave differently according to how elastic or stiff it is.

To solve these problems we introduce constitutive relations that encode the substance constituting $\mathcal{B}$ and its inner structure, and hence specify the materials' properties. Of course, only the quantities that actually depend on the kind of material will be given by a constitutive relation. These are

- the internal energy U, since it encodes the molecular structure of the material,
- the first Piola Kirchhoff tensor $\boldsymbol{\mathcal{P}}$, since it describes how one part of $\mathcal{B}$ reacts if another part of $\mathcal{B}$ is moved. This reaction clearly depends on the material,
- and the heat flux vector $\boldsymbol{Q_\theta}$ which states, how well $\mathcal{B}$ can conduct heat.

If we are given a constitutive relation for the internal energy, then by use of the Doyle-Ericksen formula we can compute the first Piola-Kirchhoff tensor.
The external mechanical force $\boldsymbol{F}$, the heat supply R_θ and the electromagnetic fields $\bar{\mathfrak{E}}$ and $\mathfrak{B}$, however, are purely external influences.

Remark 5.1.1.

1) It is easier to formulate constitutive laws on $\mathcal{B}$ and not on $\mathcal{B}_t$, since then the domain of the quantities is fixed.

2) Here we are not interested in the development of the body's temperature. It is governed by balance of energy and depends on a constitutive relation for the heat flux vector $\boldsymbol{Q_\theta}$.

Recall that $\mathcal{D}(\mathcal{B})$ denotes the set of all deformations of $\mathcal{B}$ in $\mathcal{S}$.

Definition 5.1.2 (see Marsden and Hughes [1983, ch.3, Def. 1.3]).
Let $F \in C^r(\mathcal{B})$ be a scalar function describing a material property (for instance, the internal energy density). A **thermoelectromagnetoelastic (TEM-elastic) constitutive relation** for F is a map

$$\widehat{F} : \mathcal{D}(\mathcal{B}) \times C^r(\mathcal{B}, \mathbb{R}^+) \times \Gamma^k(\mathcal{B}, T^*\mathcal{S}) \times \Gamma^k(\mathcal{B}, T^*\mathcal{S}) \to C^r(\widetilde{\mathcal{B}}).$$

The function F associated to $\widehat{F}$, the motion ϕ, the temperature field Θ, the electric field $\bar{\mathfrak{E}}$, and the magnetic flux $\mathfrak{B}$, is then given by

$$F(X,t) = \widehat{F}\big(\phi_t, \Theta_t, \bar{\mathfrak{E}}_t, \mathfrak{B}_t\big)(X), \tag{5.1}$$

where $\phi_t(X) = \phi(X,t)$, $\Theta_t(X) = \Theta(X,t)$, $\bar{\mathfrak{E}}_t(X) = \bar{\mathfrak{E}}(X,t)$ and $\mathfrak{B}_t(X) = \mathfrak{B}(X,t)$. Materials with such a kind of constitutive relation are called **thermoelectromagnetoelastic**.

The relation (5.1) means that F depends on the functions and vector fields ϕ_t, Θ_t, $\bar{\mathfrak{E}}_t$, and $\mathfrak{B}_t$ as a whole. For instance, F could depend on spatial or temporal derivatives of ϕ_t, Θ_t, $\bar{\mathfrak{E}}_t$, and $\mathfrak{B}_t$.

Similar definitions can be given for the TEM-elastic constitutive relations of material properties characterized by vector fields or tensor fields (e.g. the heat flux $\boldsymbol{Q_\theta}$ and the second Piola-Kirchhoff tensor $\boldsymbol{\mathcal{S}}$).

It could be possible that the value of some material property F at the time t not only depends on the values of ϕ, Θ, $\bar{\mathfrak{E}}$, and $\mathfrak{B}$ at the time t, but on the values these quantities had at earlier times. In such a case one says that there are memory effects. Here we have excluded such a behavior and only consider the simplest possible case: In the constitutive relations all quantities are evaluated at the same time t. That is, to define the current value of F, only the current values of the deformation ϕ_t, the temperature θ_t and the electromagnetic fields $\bar{\mathfrak{E}}_t$ and $\mathfrak{B}_t$ are necessary, their earlier values are irrelevant.

Remark 5.1.3. In particular the stress tensor $\boldsymbol{\sigma}$ (and thus $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{S}}$) at each point $x \in \mathcal{B}_t$ is determined only by the current state of deformation. The stress does not depend on the path of deformation that was taken to achieve the current state $\mathcal{B}_t$. Materials with such a simple dependence of the stress on the deformation are called **Cauchy elastic materials** [Ogden, 1984, p.175 ff.]. Nevertheless the *work* done by the interior forces could still depend on the past states of deformation, i.e. even for these materials, $\boldsymbol{\sigma}$ can in general not be derived from a scalar potential function. If, however, the stress can be computed as the derivative of some potential function with respect to the deformation, then the material is called **hyperelastic**. The potential function is then called **strain-energy function**, and the corresponding relation between the stress and the deformation is called **stress-deformation relation**.

We only want to consider materials for which all balance laws are available on a manifold. But then, according to Theorem 3.5.1, the Doyle-Ericksen formula $\boldsymbol{\sigma}^\sharp = 2\rho \frac{\partial u}{\partial g}$ or $\boldsymbol{\mathcal{P}} = \rho_{ref} \frac{\partial U}{\partial \boldsymbol{\mathcal{F}}}$, resp., holds, so the internal energy is a potential function for the stress. Thus, to have a theory of electroelasticity on a manifold, the material under consideration must be hyperelastic.

In section 5.3 we will set up a particular constitutive relation describing a certain electroelastic material that we want to study. But before we do that we will consider constitutive relations in general. We will make some (physically) reasonable demands to these relations and will see that they already restrict the form of the dependence of the material quantities on the deformation and the external influences.

5.2 Thermoelasticity

In this section, we recollect some results on thermoelastic materials.

Again we assume that we can always assign a well-defined **(absolute) temperature field** $\theta_t : \mathcal{B}_t \to \mathbb{R}^+$ to $\mathcal{B}_t$. Moreover, we define the temperature $\Theta : \mathcal{B} \to \mathbb{R}^+$ on the undeformed body by

$$\Theta(X,t) := \theta(x,t).$$

Definition 5.2.1 (see Marsden and Hughes [1983, ch.3, Def. 1.3]).
Let $F \in C^r(\mathcal{B})$ be a scalar function describing a material property (for instance, the internal energy density). A **thermoelastic constitutive relation** for F is a map

$$\widehat{F} : \mathcal{D}(\mathcal{B}) \times C^r(\mathcal{B}, \mathbb{R}^+) \to C^r(\mathcal{B}).$$

The function F associated to $\widehat{F}$, the motion ϕ and the temperature field Θ is then given by

$$F(X,t) = \widehat{F}\big(\phi_t, \Theta_t\big)(X), \tag{5.2}$$

where $\phi_t(X) = \phi(X,t)$ and $\Theta_t(X) = \Theta(X,t)$.
Materials with this kind of constitutive relation are called **thermoelastic**.

Definition 5.2.2 (Locality, Marsden and Hughes [1983], ch.3, Def. 2.1).
Let $\widehat{F} : \mathcal{D}(\mathcal{B}) \times C^r(\mathcal{B}, \mathbb{R}^+) \to \mathscr{F}(\mathcal{B})$ be a constitutive function for thermoelasticity, where $\mathscr{F}$ stands for the scalar functions, the smooth vector fields or the $(0,2)$ tensor fields on $\mathcal{B}$, respectively. Let $\phi_1, \phi_2 \in \mathcal{D}(\mathcal{B})$ and $\Theta_1, \Theta_2 \in C^r(\mathcal{B}, \mathbb{R}^+)$.
$\widehat{F}$ is called **local**, if for any open set $U \subset \mathcal{B}$

$$\begin{aligned} \phi_1 &= \phi_2 \quad \text{on } U \\ \Theta_1 &= \Theta_2 \quad \text{on } U \end{aligned}$$

already implies that

$$\widehat{F}(\phi_1, \Theta_1) = \widehat{F}(\phi_2, \Theta_2) \quad \text{on } U.$$

Definition 5.2.3. Let $\widehat{F} : \mathcal{D}(\mathcal{B}) \times C^r(\mathcal{B}, \mathbb{R}^+) \to \mathscr{F}(\mathcal{B})$ be a constitutive function for thermoelasticity, where $\mathscr{F}$ stands for the scalar functions, the smooth vector fields or the $(0,2)$ tensor fields on $\mathcal{B}$, respectively. We say that $\widehat{F}$ is **strongly local**, if the value of $\widehat{F}$ at X only depends on the values of ϕ and Θ and their derivatives up to a certain order k at X.

In particular, each strongly local constitutive relation is also local.

Axiom 3 (see Marsden and Hughes [1983, ch.3, p. 202]). For thermoelastic materials all constitutive functions are strongly local.

Definition 5.2.4. We assume that the **entropy** $\mathscr{S}$ on U_t is given by the integral of some **entropy density** $s : \mathcal{B}_t \to \mathbb{R}$ that is referred to the mass density ρ:

$$\mathscr{S}(U_t) = \int_{U_t} s\,\rho\,\mathbf{vol}_t.$$

We define the **free energy density** ψ on $\mathcal{B}_t$ by

$$\psi := u - \theta\, s.$$

As before u is the internal energy, θ the temperature, and ρ the mass density.
We also define a corresponding free energy density on the undeformed body: In section 4.4 we have already defined the internal energy on $\mathcal{B}$, by $U(X,t) := u(x,t)$. Analogously we define the entropy density on the undeformed body by

$$S(X,t) := s(x,t)$$

Thus, the free energy density Ψ on $\mathcal{B}$, defined by $\Psi(X,t) := \psi(x,t)$, satisfies

$$\Psi = U - \Theta\, S. \tag{5.3}$$

$\widehat{\Psi}$ is a material property and thus characterized by a constitutive relation.

Definition 5.2.5. A (thermoelastic) **process** is a tuple (ϕ, Θ), where $\phi : \mathcal{B} \times I \to \mathcal{S}$ is a motion of $\mathcal{B}$ in $\mathcal{S}$ and $\Theta : \mathcal{B} \times I \to \mathbb{R}$ a temperature distribution.

Let us denote by $D_\phi \widehat{\Psi}$ the partial derivative of $\widehat{\Psi}$ with respect to ϕ in the Fréchet sense, see Marsden and Hughes [1983, ch.3, Box 1.1].

We will now deduce some restrictions to the constitutive relations that result from covariance assumptions and balance of energy. In contrast to section 3.5, we do not consider all processes, but rather all transformations of a given process.
As before, we consider coordinate changes $\xi : \mathcal{S} \to \mathcal{S}$ of the surrounding space, but here we additionally use linear temperature rescalings.

Definition 5.2.6. A linear rescaling of the temperature is a monotone increasing linear diffeomorphism $r : \mathbb{R}^+ \to \mathbb{R}^+$. Moreover, we set

$$\mathcal{R} = \{r : \mathbb{R}^+ \to \mathbb{R}^+ \mid r \text{ is a temperature rescaling}\}.$$
$$\mathcal{C} = \{\xi : \mathcal{S} \to \mathcal{S} \mid \xi \text{ is a diffeomorphism}\}.$$

The following axiom is a reformulation of an axiom that has been given by Marsden and Hughes [1983].

Axiom 4 (see Marsden and Hughes [1983], ch.3, p. 202).
Let $(\phi,\ \Theta)$ be a process, where we assume that ϕ and Θ are as often continuously differentiable, as is needed to have the partial derivatives $D_\phi \widehat{\Psi}$ and $D_\Theta \widehat{\Psi}$ well-defined.

1) For each time t, the tuple $(\phi_t,\ \Theta_t)$ satisfies balance of energy (see Th. 4.4.2)

$$\rho_{ref} \frac{\partial U}{\partial t} = \langle \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{D}} \rangle + \rho_{ref} R_\theta - \mathrm{DIV} \boldsymbol{Q_\theta}.$$

2) There is a map $\hat{\hat{\Psi}} : \mathcal{D}(\mathcal{B}) \times \mathcal{O}_g \times C^r(\mathcal{B}) \times \mathbb{R}^+ \to C^r(\mathcal{B})$, such that for any diffeomorphism $\xi : \mathcal{S} \to \mathcal{S}$ and any temperature rescaling $r : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\widehat{\Psi}(\phi_t, \Theta_t) = \hat{\hat{\Psi}}(\xi \circ \phi_t, \xi_* g, r\Theta_t, r).$$

3) For curves in $\mathcal{C}$ and in $\mathcal{R}$, given by

$$I \to \mathcal{C}, \quad t \mapsto \xi_t \qquad \text{and} \qquad I \to \mathcal{R}, \quad t \mapsto r_t,$$

respectively, we assume that $\phi'_t := \xi_t \circ \phi_t$ and $\Theta'_t := r_t \Theta_t$ satisfy balance of energy. We demand that the primed and unprimed quantities are related by

$$\begin{aligned} S' &= S \\ \Psi' &= \Psi. \\ (R_\theta)'_t - \Theta'_t \frac{\partial S'_t}{\partial t} &= (R_\theta)_t - \Theta_t \frac{\partial S_t}{\partial t} \\ (\boldsymbol{Q_\theta})'_t &= r_t \cdot (\xi_t)_* (\boldsymbol{Q_\theta})_t \\ \boldsymbol{t}' &= (\xi_t)_* \boldsymbol{t}. \end{aligned} \tag{5.4}$$

Formula (5.4) accounts for the apparent heat supply due to some additional entropy production.
The following theorem was provided by Marsden and Hughes [1983], but we restrict its assertion to simple bodies to ensure the well-definedness of the partial derivatives $D_\phi \widehat{\Psi}$ and $D_\Theta \widehat{\Psi}$ that are needed for its proof.

Theorem 5.2.7 (see Marsden and Hughes [1983, ch.3, Theorem 3.6])
Let $\mathcal{B}$ be a simple body. Then under the assumptions of axiom 3 and axiom 4 the constitutive function $\widehat{\Psi}$ for the free energy density depends only on the point values of the deformation tensor $\boldsymbol{\mathcal{C}}$ and the temperature Θ. Moreover, the entropy S and the first Piola-Kirchhoff tensor $\boldsymbol{\mathcal{P}}$ can be obtained from $\widehat{\Psi}$ by

$$S = -\frac{\partial \widehat{\Psi}}{\partial \Theta}$$

$$\boldsymbol{\mathcal{P}}^{\sharp} = g^{\sharp}\,\rho_{ref}\,\frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{F}}} = 2\,\rho_{ref}\,\boldsymbol{\mathcal{F}}\,\frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{C}}^{\flat}}.$$

In section 5.3 we will give a version of this theorem that includes electromagnetic fields.

Remark 5.2.8. Observe that Theorem 5.2.7 provides no information on the dependence of $\boldsymbol{Q_\theta}$ on ϕ and Θ. In the literature on thermoelasticity it is often just assumed that $\boldsymbol{Q_\theta}$ depends only on the point values of $\boldsymbol{\mathcal{C}}$, Θ and **GRAD** Θ at X (Marsden and Hughes [1983, p. 193]; Kovetz [2000, p. 229]).
A material with such a simple constitutive relation is called grade $(1,1)$ material.

If the material possesses symmetries, then the constitutive relations can be further simplified.

Definition 5.2.9 (Marsden and Hughes [1983], ch.3, Def. 5.1).
A **material symmetry** for the free energy density Ψ at a point $X_0 \in \mathcal{B}$ is a linear isometry $\lambda : T_{X_0}\mathcal{B} \to T_{X_0}\mathcal{B}$, such that

$$\widehat{\Psi}(X_0, \boldsymbol{\mathcal{C}}, \Theta) = \widehat{\Psi}(X_0, \lambda^{*}\boldsymbol{\mathcal{C}}, \Theta),$$

where $\boldsymbol{\mathcal{C}}$ is an arbitrary symmetric positive-definite $(0,2)$ tensor field on $\mathcal{B}$.
The **material symmetry group** of $\widehat{\Psi}$ in X_0 consists of all material symmetries of $\widehat{\Psi}$ in X_0 and is denoted by $\boldsymbol{\mathscr{S}_{X_0}(\widehat{\Psi})}$.

Theorem 5.2.10 (Marsden and Hughes [1983], ch.3, Prop. 5.4)
Assume that $\widehat{\boldsymbol{\mathcal{S}}} = 2\,\rho_{ref}\,\frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{C}}}$. Then for each material symmetry $\lambda \in \mathscr{S}_{X_0}(\widehat{\Psi})$,

$$\widehat{\boldsymbol{\mathcal{S}}}(X_0, \lambda^{*}\boldsymbol{\mathcal{C}}, \Theta) = \lambda^{*}\widehat{\boldsymbol{\mathcal{S}}}(X_0, \boldsymbol{\mathcal{C}}, \Theta).$$

Definition 5.2.11 (Marsden and Hughes [1983], ch.3, Def. 5.6). Let $\mathcal{B}$ be a simple body. Then the material $\mathcal{B}$ is made of, is called **isotropic at a point** $X_0 \in \mathcal{B}$, if its

constitutive relation for the free energy density in X_0 is invariant under the action of $SO(3)$, that is, if $SO(T_{X_0}\mathcal{B}) = SO(3) \subset \mathscr{S}_{X_0}(\widehat{\Psi})$.
A material is called **isotropic**, if it is isotropic at every point.

Theorem 5.2.12 (Marsden and Hughes [1983], ch.3, Prop. 5.10)
Let $\boldsymbol{\mathcal{C}}$ be an arbitrary symmetric $m \times m$ matrix.
Then a scalar function f of $\boldsymbol{\mathcal{C}}$ is invariant under orthogonal transformations if and only if f depends on $\boldsymbol{\mathcal{C}}$ only by its elementary symmetric functions I_a, i.e. by

$$I_a(\boldsymbol{\mathcal{C}}) = \sum_{1 \leq i_1 < i_2 < \ldots < i_a \leq m} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_a} \quad a = 1, \ldots, m = \dim \mathcal{B},$$

where $\lambda_1, \ldots, \lambda_m$ denote the eigenvalues of $\boldsymbol{\mathcal{C}}$.

Remark 5.2.13. Recall that $I_1(\boldsymbol{\mathcal{C}}) = \operatorname{tr}(\boldsymbol{\mathcal{C}})$ and $I_m(\boldsymbol{\mathcal{C}}) = \det(\boldsymbol{\mathcal{C}})$.

Definition 5.2.14 (Marsden and Hughes [1983], ch.3, p.220). The thermoelastic constitutive function $\widehat{\Psi}$ for the free energy density is called **materially covariant**, if for all diffeomorphisms $\Xi : \mathcal{B} \to \mathcal{B}$,

$$\Xi^*(\widehat{\Psi}(G, \boldsymbol{\mathcal{C}}, \Theta)) = \widehat{\Psi}(\Xi^* G, \Xi^* \boldsymbol{\mathcal{C}}, \Xi^* \Theta),$$

where G denotes the Riemannian metric on $\mathcal{B}$.

Theorem 5.2.15 (Marsden and Hughes [1983], ch.3, Prop. 5.7)
Assume that $\widehat{\Psi}$ is materially covariant. Then for each $X_0 \in \mathcal{B}$, $SO(T_{X_0}\mathcal{B}) \subset \mathscr{S}_{X_0}(\widehat{\Psi})$.
In particular, any materially covariant body in $\mathbb{R}^3$ is isotropic.

Let $\widehat{\Psi}$ be an thermoelastic constitutive function for the free energy density. If $\widehat{\Psi}$ is materially covariant, then, motivated by Theorems 5.2.15 and 5.2.12, we regard it as a function of $I_1(\boldsymbol{\mathcal{C}}), \ldots, I_m(\boldsymbol{\mathcal{C}})$ and Θ.

There are several definitions concerning the constitutive laws of specified elastic materials. In the following we essentially adhere to the definitions as they are given by Ogden [1984, ch. 4.3.5], but we rewrite the definitions given there in terms of the elementary symmetric functions of $\boldsymbol{\mathcal{C}}$ and also provide a version of these relations for shells.

In the following we disregard temperature dependencies and consider the purely elastic case. Assume for the moment that the dimension of the body is three. Since for a materially covariant material the free energy density only depends on the elementary symmetric functions of $\boldsymbol{\mathcal{C}}$, we assume that Ψ is smooth with respect to $I_1(\boldsymbol{\mathcal{C}})$, $I_2(\boldsymbol{\mathcal{C}})$, and

$I_3(\boldsymbol{\mathcal{C}})$, and that we can expand it to an infinite power series

$$\Psi\big[I_1(\boldsymbol{\mathcal{C}}), I_2(\boldsymbol{\mathcal{C}}), I_3(\boldsymbol{\mathcal{C}})\big] = \sum_{p,q,r=0}^{\infty} c_{pqr}\big(I_1(\boldsymbol{\mathcal{C}}) - 3\big)^p \big(I_2(\boldsymbol{\mathcal{C}}) - 3\big)^q \big(I_3(\boldsymbol{\mathcal{C}}) - 1\big)^r. \tag{5.5}$$

The coefficients c_{pqr} are assumed to be independent of the deformation.
If the stress inside the body is zero, then $\boldsymbol{\mathcal{C}} = \mathbf{Id}$ and thus $I_1(\boldsymbol{\mathcal{C}}) = I_2(\boldsymbol{\mathcal{C}}) = 3$ while $I_3(\boldsymbol{\mathcal{C}}) = 1$. In this case $\Psi[I_1(\boldsymbol{\mathcal{C}}), I_2(\boldsymbol{\mathcal{C}}), I_3(\boldsymbol{\mathcal{C}})] = c_{000}$.

Remark 5.2.16. Ogden [1984] considers not explicitly the free energy, but an abstract function W, called "stored energy function" that serves as a potential for the first Piola-Kirchhoff tensor. To Ogden, W is just "a measure of the energy stored in the material as a result of deformation". That is why he demands that $c_{000} = 0$, which implies that W vanishes if there is no deformation.
In our setting this demand does not make sense: If we as above disregard the temperature, then Ψ coincides with the internal energy. But a vanishing internal energy is only possible if there is no material at all.
However, keeping a non-vanishing c_{000} has no impact in the following anyway; in the following deductions we will only consider the derivative of Ψ with respect to $\boldsymbol{\mathcal{C}}$.
Ogden's stored energy function W might be characterized as the *change of internal energy* due to deformation.

Definition 5.2.17. A material is called **incompressible**, if the factor of volume deformation $\mathcal{J}$ is constant and equal to 1.

For an incompressible material $I_3(\boldsymbol{\mathcal{C}}) = \mathcal{J}^2 = 1$, and the power series (5.5) simplifies to

$$\Psi\big[I_1(\boldsymbol{\mathcal{C}}), I_2(\boldsymbol{\mathcal{C}}), I_3(\boldsymbol{\mathcal{C}})\big] = \sum_{p,q=0}^{\infty} c_{pq}\big(I_1(\boldsymbol{\mathcal{C}}) - 3\big)^p \big(I_2(\boldsymbol{\mathcal{C}}) - 3\big)^q. \tag{5.6}$$

The simplest examples of isotropic materials are the Mooney-Rivlin and the Neo-Hookean materials. Their free energy density is obtained by using only some of the first terms of the power series (5.6).

Definition 5.2.18. Assume that the body is three-dimensional.
A **Mooney-Rivlin material** is a material with a constitutive law of the form

$$\Psi = \frac{\mu_1}{2}\left(I_1(\boldsymbol{\mathcal{C}}) - 3\right) + \frac{\mu_2}{2}\left(I_2(\boldsymbol{\mathcal{C}}) - 3\right) + \mu_3,$$

where $I_1(\boldsymbol{\mathcal{C}})$ and $I_2(\boldsymbol{\mathcal{C}})$ denote the first and the second elementary symmetric functions of $\boldsymbol{\mathcal{C}}$. μ_1, μ_2, μ_3 are constants, and we demand that $\mu_1 > 0$.

A **Neo-Hookean material** is a material with the constitutive law

$$\Psi = \frac{\mu}{2}\left(I_1(\boldsymbol{\mathcal{C}}) - 3\right) + \tau,$$

where $\mu > 0$ and τ are constants.

Remark 5.2.19.

1) By definition, the Mooney-Rivlin and the Neo-Hookean materials are incompressible.
2) The isotropy of the material is reflected in the fact that μ_1 and μ_2 are (constant) functions and not (non-trivial) tensors.
3) μ_1 and μ_2 (or μ) can be determined experimentally. μ_1 is called **shear modulus**. and is always positive. Materials with a high shear modulus are harder to deform than materials with a low shear modulus.

In two dimensions the power series (5.5) has to be replaced by

$$\Psi\left[I_1(\boldsymbol{\mathcal{C}}), I_2(\boldsymbol{\mathcal{C}})\right] = \sum_{p,q=0}^{\infty} c_{pq}\left(I_1(\boldsymbol{\mathcal{C}}) - 2\right)^p \left(I_2(\boldsymbol{\mathcal{C}}) - 1\right)^q. \tag{5.7}$$

If the material is incompressible, then this simplifies to

$$\Psi\left[I_1(\boldsymbol{\mathcal{C}}), I_2(\boldsymbol{\mathcal{C}})\right] = \sum_{p=0}^{\infty} c_p\left(I_1(\boldsymbol{\mathcal{C}}) - 2\right)^p. \tag{5.8}$$

In three dimensions we have defined Mooney-Rivlin and Neo-Hookean materials by setting in (5.6) all coefficients with the exception of c_{00}, c_{10} and c_{01} to zero. The analogy here is to set all coefficients c_p equal to zero with the exception of c_0 and c_1. But then, for two-dimensional bodies Mooney-Rivlin and Neo-Hookean materials are the same:

Definition 5.2.20. Assume that the body is two-dimensional.
A **Mooney-Rivlin material/Neo-Hookean material** is a material with a constitutive law of the form

$$\Psi = \frac{\mu}{2}\left(I_1(\boldsymbol{\mathcal{C}}) - 2\right) + \tau,$$

where $I_1(\boldsymbol{\mathcal{C}})$ denotes the first elementary symmetric functions of $\boldsymbol{\mathcal{C}}$, that is, $I_1(\boldsymbol{\mathcal{C}}) = \operatorname{tr}(\boldsymbol{\mathcal{C}})$. $\mu > 0$ and ν are constants.

For one-dimensional bodies $\boldsymbol{\mathcal{C}}$ is a scalar, and we write

$$\Psi[\mathcal{C}] = \sum_{p=0}^{\infty} c_p\left(\mathcal{C} - 1\right)^p. \tag{5.9}$$

But now it makes no sense to assume that the material is incompressible. In particular there are no one-dimensional Mooney-Rivlin or Neo-Hookean materials. To have a simple model for a one-dimensional body we can just take the first two summands of the power series (5.9) and work with

$$\Psi[\mathcal{C}] = \frac{\mu}{2}\big(\mathcal{C} - 1\big) + \tau$$

where $\mu > 0$ and τ are constants.

Remark 5.2.21. If the material is incompressible, then the Transport Theorem (Th. 1.3.8), the Spatial Localization Theorem (1.3.9), and the conservation of mass become much simpler. The first equation in the Transport Theorem (Theorem 1.3.8) is replaced by

$$\frac{d}{dt}\int_{U_t} f(x,t)\,\mathbf{vol}_t = \int_{U_t} \dot{f}(x,t)\,\mathbf{vol}_t(x),$$

the local form of

$$\frac{d}{dt}\int_{U_t} a(x,t)\,\mathbf{vol}_t = \int_{U_t} b(x,t)\,\mathbf{vol}_t + \int_{\partial U_t} g(\boldsymbol{c}(x,t),\boldsymbol{n})\,\mathbf{vol}_{\partial U_t}$$

becomes (see Theorem 1.3.9)

$$\dot{a} = b + \mathrm{div}_t \boldsymbol{c},$$

and hence conservation of mass (see Th. 1.3.19) simplifies to $\dot{\rho} = 0$.

5.3 Thermoelectromagnetoelasticity

We repeat the definition of TEM-elastic materials that was already stated in section 5.1.

Definition 5.3.1 (see Marsden and Hughes [1983, ch. 3, Def. 1.3]).
Let $F \in C^r(\mathcal{B})$ be a scalar function describing a material property (for instance, the internal energy density). A **thermoelectromagnetoelastic (TEM-elastic) constitutive relation** for F is a map

$$\widehat{F} : \mathcal{D}(\mathcal{B}) \times C^r(\mathcal{B}, \mathbb{R}^+) \times \Gamma^k(\mathcal{B}, T^*\mathcal{S}) \times \Gamma^k(\mathcal{B}, T^*\mathcal{S}) \to C^r(\mathcal{B}).$$

The function F associated to $\widehat{F}$, a motion ϕ, a temperature field Θ, an electric field $\bar{\mathfrak{E}}$ and a magnetic field $\mathfrak{B}$, is then given by

$$F(X,t) = \widehat{F}\big(\phi_t, \Theta_t, \bar{\mathfrak{E}}_t, \mathfrak{B}_t\big)(X), \tag{5.10}$$

where $\phi_t(X) = \phi(X,t)$, $\Theta_t(X) = \Theta(X,t)$, $\bar{\mathfrak{E}}_t(X) = \bar{\mathfrak{E}}(X,t)$ and $\mathfrak{B}_t(X) = \mathfrak{B}(X,t)$. Materials with such a kind of constitutive relation are called **thermoelectromagnetoelastic** (TEM-elastic).

We extend the notion of locality as stated in definition 5.2.2 to TEM-elastic materials.

Axiom 5. We assume that for TEM-elastic materials all constitutive functions are strongly local.

Definition 5.3.2. We define the **free energy density** ψ on $\mathcal{B}_t$ by

$$\psi := u - \theta\, s - \frac{1}{\rho}\,(\bar{\mathfrak{e}}, \boldsymbol{p}).$$

$\bar{\mathfrak{e}}$ is the electromotive intensity, $\boldsymbol{p}$ denotes the polarization that we introduced in chapter 2.
Again, we define a corresponding free energy density on the undeformed body: In section 4.4 we have already defined the electromotive intensity on $\mathcal{B}$, by $\bar{\mathfrak{E}}(X,t) := \bar{\mathfrak{e}}(x,t)$. Analogously we define the polarization on the undeformed body by

$$\boldsymbol{P}(X,t) := \mathcal{J}(X,t)\,\boldsymbol{p}(x,t).$$

Thus, the free energy density Ψ on $\mathcal{B}$, defined by $\Psi(X,t) := \psi(x,t)$, satisfies

$$\Psi = U - \Theta\, S - \frac{1}{\rho_{ref}}\,\langle \bar{\mathfrak{E}}, \boldsymbol{P}\rangle. \tag{5.11}$$

Definition 5.3.3. A (TEM-elastic) **process** is a tuple $(\phi, \Theta, \bar{\mathfrak{E}}, \mathfrak{B})$, where $\phi : \mathcal{B}\times I \to \mathcal{S}$ is a motion in $\mathcal{S}$, $\Theta : \mathcal{B}\times I \to \mathbb{R}$ a temperature distribution, $\bar{\mathfrak{E}} : \mathcal{B}\times I \to T^*\mathcal{S}$ an electromotive intensity and $\mathfrak{B} : \mathcal{B}\times I \to T^*\mathcal{S}$ a magnetic flux density.

Recall that $\mathcal{C}$ denotes the set of all diffeomorphisms on $\mathcal{S}$ and that $\mathcal{R}$ denotes the set of all linear temperature rescalings $r : \mathbb{R}^+ \to \mathbb{R}^+$.

We now generalize axiom 4 to arrive at an axiom that also includes the presence of electromagnetic fields.

Axiom 6.
Let $(\phi,\ \Theta, \bar{\mathfrak{E}}, \mathfrak{B})$ be a given process, where we assume that ϕ, Θ, $\bar{\mathfrak{E}}$, $\mathfrak{B}$ are as often continuously differentiable, as is needed to have the partial derivatives $D_\phi\Psi$, $D_\Theta\Psi$, $D_{\bar{\mathfrak{E}}}\Psi$ and $D_{\mathfrak{B}}\Psi$ well-defined.

1) For each time t, the tuple $(\phi_t, \Theta_t, \bar{\mathfrak{E}}_t, \mathfrak{B}_t)$ satisfies balance of energy (see Th. 4.4.2)
$$\rho_{ref}\frac{\partial U}{\partial t} = \langle \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{D}} \rangle + \rho_{ref} R_\theta + (\bar{\boldsymbol{J}}, \bar{\mathfrak{E}}) - \mathrm{DIV}\boldsymbol{Q_\theta}.$$

2) There is a map $\hat{\hat{\Psi}} : \mathcal{D}(\mathcal{B}) \times \mathcal{O}_g \times C^r(\mathcal{B}) \times \mathbb{R}^+ \times \Gamma(\mathcal{B}, T^*\mathcal{S}) \times \Gamma(\mathcal{B}, T^*\mathcal{S}) \to C^r(\mathcal{B})$, such that for any diffeomorphism $\xi : \mathcal{S} \to \mathcal{S}$ and any temperature rescaling $r : \mathbb{R}^+ \to \mathbb{R}^+$,
$$\widehat{\Psi}(\phi_t, \Theta_t, \bar{\mathfrak{E}}_t, \mathfrak{B}_t) = \hat{\hat{\Psi}}(\xi \circ \phi_t, \xi_* g, r\Theta_t, r, \xi_*\bar{\mathfrak{E}}_t, \xi_*\mathfrak{B}_t).$$

3) For curves in $\mathcal{C}$ and in $\mathcal{R}$, given by
$$I \to \mathcal{C}, \quad t \mapsto \xi_t \qquad \text{and} \qquad I \to \mathcal{R}, \quad t \mapsto r_t,$$
respectively, we assume that $\phi'_t := \xi_t \circ \phi_t$ and $\Theta'_t := r_t \Theta_t$ satisfy balance of energy. We demand that the primed and unprimed quantities are related by
$$\begin{aligned} S' &= S \\ \Psi' &= \Psi. \\ (R_\theta)'_t - \Theta'_t \frac{\partial S'_t}{\partial t} &= (R_\theta)_t - \Theta_t \frac{\partial S_t}{\partial t} \\ (\boldsymbol{Q_\theta})'_t &= r_t \cdot (\xi_t)_* (\boldsymbol{Q_\theta})_t \\ \boldsymbol{t}' &= (\xi_t)_* \boldsymbol{t}. \end{aligned}$$
Let us assume that at the time $t = t_0$, ξ_t and r are the identity. Then for $t = t_0$
$$\begin{aligned} \bar{\boldsymbol{J}}' &= \bar{\boldsymbol{J}}, \\ \bar{\mathfrak{E}}' &= \bar{\mathfrak{E}}, \\ \boldsymbol{P}' &= \boldsymbol{P}, \end{aligned}$$
(see chapter 2).

The following central theorem generalizes Theorem 5.2.7 to TEM-materials.

Theorem 5.3.4
Assume that $\mathcal{B}$ is a simple body and suppose that the axioms 5 and 6 hold.
Then the constitutive function $\widehat{\Psi}$ for the free energy density depends only on the point values of $\boldsymbol{\mathcal{F}}$, Θ, $\bar{\boldsymbol{E}}$ and $\boldsymbol{B}$. Moreover, the entropy, the temperature, the polarization, and the magnetization can be obtained as partial derivatives of $\widehat{\Psi}$:
$$S = -\frac{\partial \Psi}{\partial \Theta} \tag{5.12}$$
$$\boldsymbol{P} = -\rho_{ref}\frac{\partial \Psi}{\partial \bar{\boldsymbol{E}}} \tag{5.13}$$
$$\bar{\boldsymbol{M}} = -\rho_{ref}\frac{\partial \Psi}{\partial \boldsymbol{B}} \tag{5.14}$$

and

$$\boldsymbol{\mathcal{P}} = \rho_{ref} \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{F}}} + \left[\langle \bar{\boldsymbol{M}}, \boldsymbol{B} \rangle \, \boldsymbol{Id} - \bar{\boldsymbol{M}} \otimes \boldsymbol{B} + \bar{\boldsymbol{E}} \otimes \boldsymbol{P} \right] \left(\boldsymbol{\mathcal{F}}^T \right)^{-1}, \tag{5.15}$$

that is,

$$\boldsymbol{\mathcal{P}} = \rho_{ref} \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{F}}} + \langle \bar{\boldsymbol{M}}, \boldsymbol{B} \rangle \left(\boldsymbol{\mathcal{F}}^T \right)^{-1} - \langle \boldsymbol{B}, (\boldsymbol{\mathcal{F}}^T)^{-1}(\,\cdot\,) \rangle \, \bar{\boldsymbol{M}} + \langle \boldsymbol{P}, (\boldsymbol{\mathcal{F}}^T)^{-1}(\,\cdot\,) \rangle \, \bar{\boldsymbol{E}}. \tag{5.16}$$

Here we have identified vector fields and 1*-forms along* ϕ_t*.*
If $\widehat{\Psi}$ *depends on* $\bar{\boldsymbol{E}}$ *and* $\boldsymbol{B}$ *only by* $|\boldsymbol{E}|^2$ *and* $|\boldsymbol{B}|^2$*, then* $\widehat{\Psi}$ *depends on* ϕ *only be the deformation tensor* $\boldsymbol{\mathcal{C}}$*.*

Remark 5.3.5. By means of the entropy inequality Kovetz [2000, see p.230] provides a similar result, but he uses much stronger assumptions: Concerning the electromagnetic fields he already assumes that $\widehat{\Psi}$ depends only on the point values of $\bar{\boldsymbol{E}}$ and $\boldsymbol{B}$. Moreover he already assumes that the dependence on the deformation ϕ is at most given by the point values of the deformation gradient $\boldsymbol{\mathcal{F}}$ and the velocity. For the temperature dependence he assumes that $\widehat{\Psi}$ only depends on the point values of the temperature itself and on the temperature gradient. Then he shows that under these assumptions the free energy density is indeed independent from the velocity and the temperature gradient. Moreover he derives the same relations for the entropy, the polarization, and the magnetization that we obtained in Theorem 5.3.4 and also provides a connection for the Cauchy stress tensor that is similar to eq. (5.15).

Proof of Theorem 5.3.4. We have already seen in the course of the proof of Theorem 4.4.2 that $\langle \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{D}} \rangle = \mathcal{J} \, \langle \boldsymbol{\sigma}, \boldsymbol{d} \rangle$. For simple bodies, the spatial rate of deformation tensor $\boldsymbol{d}$ is given by

$$\boldsymbol{d}^\flat = \frac{1}{2} \left(\left(\nabla^{\mathcal{B}_t} \boldsymbol{v} \right)^\flat + \left[\left(\nabla^{\mathcal{B}_t} \boldsymbol{v} \right)^\flat \right]^T \right).$$

(see Theorem 1.2.13) and thus, $\langle \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{D}} \rangle = \mathcal{J} \, \langle \boldsymbol{\sigma}, \nabla^{\mathcal{B}_t} \boldsymbol{v} \rangle$. The definition of the first Piola-Kirchhoff tensor (see 4.2.1) now provides $\mathcal{J} \, \boldsymbol{\sigma} = \boldsymbol{\mathcal{P}} \circ \boldsymbol{\mathcal{F}}^T$. Moreover, for simple bodies $\nabla^{\mathcal{B}_t} = \nabla^{\mathcal{S}}$. Thus,

$$\langle \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{D}} \rangle = \langle \boldsymbol{\mathcal{P}} \circ \boldsymbol{\mathcal{F}}^T, \nabla^{\mathcal{S}} \boldsymbol{v} \rangle$$

Using this and the definition of Ψ, (see (5.11)), we express balance of energy in terms of Ψ:

$$\begin{aligned} \rho_{ref} \left(\frac{d\psi}{dt} + \frac{\partial \Theta}{\partial t} S + \Theta \frac{\partial S}{\partial t} \right) + \left\langle \frac{\partial \bar{\boldsymbol{E}}}{\partial t}, \boldsymbol{P} \right\rangle + \left\langle \bar{\boldsymbol{E}}, \frac{\partial \boldsymbol{P}}{\partial t} \right\rangle \\ = \rho_{ref} R_\Theta + \langle \boldsymbol{\mathcal{P}} \circ \boldsymbol{\mathcal{F}}^T, \nabla^{\mathcal{S}} \boldsymbol{v} \rangle + \langle \bar{\boldsymbol{J}}, \bar{\boldsymbol{E}} \rangle - \mathrm{DIV}\, \boldsymbol{Q_\theta}. \end{aligned}$$

Assuming $\Psi = \widehat{\Psi}(\phi, \theta, \bar{\boldsymbol{E}}, \boldsymbol{B})$ gives

$$\rho_{ref}\Big((D_\phi \widehat{\Psi})\, \boldsymbol{V} + (D_\Theta \widehat{\Psi})\, \frac{\partial \Theta}{\partial t} + (D_{\bar{\boldsymbol{E}}} \widehat{\Psi})\, \frac{\partial \bar{\boldsymbol{E}}}{\partial t} + (D_{\boldsymbol{B}} \widehat{\Psi})\, \frac{\partial \boldsymbol{B}}{\partial t} + S\, \frac{\partial \Theta}{\partial t} + \Theta\, \frac{\partial S}{\partial t} \Big)$$
$$+ \Big\langle \boldsymbol{P}, \frac{\partial \bar{\boldsymbol{E}}}{\partial t} \Big\rangle + \Big\langle \bar{\boldsymbol{E}}, \frac{\partial \boldsymbol{P}}{\partial t} \Big\rangle = \rho_{ref} R_\Theta + \langle \boldsymbol{\mathcal{P}} \circ \boldsymbol{\mathcal{F}}^T, \nabla^{\mathcal{S}} \boldsymbol{v} \rangle + \langle \bar{\boldsymbol{J}}, \bar{\boldsymbol{E}} \rangle - \mathrm{DIV}\, \boldsymbol{Q}_{\boldsymbol{\theta}}. \qquad (1)$$

We rewrite $\langle \bar{\boldsymbol{E}}, \frac{\partial \boldsymbol{P}}{\partial t} \rangle$. We had defined $\boldsymbol{P}(X,t) = \mathcal{J}(X,t)\, \boldsymbol{p}(x,t)$ and $\bar{\boldsymbol{E}}(X,t) = \bar{\boldsymbol{e}}(x,t)$. Using that for simple bodies $\frac{\partial \mathcal{J}}{\partial t} = \mathcal{J}\, \mathrm{div}\, \boldsymbol{v}$ (see Th. 1.3.6), the connection (2.6) between the flux derivative and the substantial derivative as well as Poynting's Theorem for bound charges (see Lemma 2.4.2), we obtain

$$\begin{aligned} \Big\langle \bar{\boldsymbol{E}}, \frac{\partial \boldsymbol{P}}{\partial t} \Big\rangle &= \frac{\partial \mathcal{J}}{\partial t}\, \langle \bar{\boldsymbol{e}}, \boldsymbol{p} \rangle + \mathcal{J}\, \langle \bar{\boldsymbol{e}}, \dot{\boldsymbol{p}} \rangle \\ &= \Big\langle \bar{\boldsymbol{M}}, \frac{\partial \boldsymbol{B}}{\partial t} \Big\rangle + \langle \bar{\boldsymbol{J}}_b, \bar{\boldsymbol{E}} \rangle + \mathcal{J}\, \mathrm{div}(\bar{\boldsymbol{e}} \times \bar{\boldsymbol{m}}) + \mathcal{J}\, \mathrm{div}\, \boldsymbol{v}\, \langle \bar{\boldsymbol{m}}, \boldsymbol{b} \rangle - \mathcal{J}\, \langle \bar{\boldsymbol{m}}, \nabla^{\mathcal{S}}_{\boldsymbol{b}} \boldsymbol{v} \rangle \\ &\quad + \mathcal{J}\, \langle \bar{\boldsymbol{e}}, \nabla^{\mathcal{S}}_{\boldsymbol{p}} \boldsymbol{v} \rangle, \end{aligned}$$

Here we used that $\boldsymbol{B}(X,t) = \boldsymbol{b}(x,t)$ (*without* the factor $\mathcal{J}$). Employing $\mathrm{div}\, \boldsymbol{v} = \langle \mathbf{Id}, \nabla^{\mathcal{S}} \boldsymbol{v} \rangle$ as well as $\langle \bar{\boldsymbol{e}}, \nabla^{\mathcal{S}}_{\boldsymbol{p}} \boldsymbol{v} \rangle = \langle \bar{\boldsymbol{e}} \otimes \boldsymbol{p}, \nabla^{\mathcal{S}} \boldsymbol{v} \rangle$ and $\langle \bar{\boldsymbol{m}}, \nabla^{\mathcal{S}}_{\boldsymbol{b}} \boldsymbol{v} \rangle = \langle \bar{\boldsymbol{m}} \otimes \boldsymbol{b}, \nabla^{\mathcal{S}} \boldsymbol{v} \rangle$ gives

$$\begin{aligned} \Big\langle \bar{\boldsymbol{E}}, \frac{\partial \boldsymbol{P}}{\partial t} \Big\rangle &= \Big\langle \bar{\boldsymbol{M}}, \frac{\partial \boldsymbol{B}}{\partial t} \Big\rangle + \langle \bar{\boldsymbol{J}}_b, \bar{\boldsymbol{E}} \rangle + \mathcal{J}\, \mathrm{div}(\bar{\boldsymbol{e}} \times \bar{\boldsymbol{m}}) \\ &\quad + \mathcal{J}\, \Big\langle \langle \bar{\boldsymbol{m}}, \boldsymbol{b} \rangle\, \mathbf{Id} - \bar{\boldsymbol{m}} \otimes \boldsymbol{b} + \bar{\boldsymbol{e}} \otimes \boldsymbol{p}, \nabla^{\mathcal{S}} \boldsymbol{v} \Big\rangle. \end{aligned}$$

Here, $\bar{\boldsymbol{e}} \otimes \boldsymbol{p}$ denotes the $(1,1)$-tensor field that is defined by $(\bar{\boldsymbol{e}} \otimes \boldsymbol{p})(\boldsymbol{v}) = \langle \boldsymbol{p}, \boldsymbol{v} \rangle\, \bar{\boldsymbol{e}}$ for all $\boldsymbol{v} \in T\mathcal{B}_t$ and $\langle \bar{\boldsymbol{e}} \otimes \boldsymbol{p}, \nabla^{\mathcal{S}} \boldsymbol{v} \rangle$ denotes the scalar product of the $(1,1)$ tensor fields as it was introduced in Notation 1.2.14. Thus, (1) becomes

$$\begin{aligned} &\rho_{ref}\Big((D_\phi \widehat{\Psi})\, \boldsymbol{V} + (D_\Theta \widehat{\Psi})\, \frac{\partial \Theta}{\partial t} + (D_{\bar{\boldsymbol{E}}} \widehat{\Psi})\, \frac{\partial \bar{\boldsymbol{E}}}{\partial t} + (D_{\boldsymbol{B}} \widehat{\Psi})\, \frac{\partial \boldsymbol{B}}{\partial t} + S\, \frac{\partial \Theta}{\partial t} + \Theta\, \frac{\partial S}{\partial t} \Big) \\ &\quad + \Big\langle \boldsymbol{P}, \frac{\partial \bar{\boldsymbol{E}}}{\partial t} \Big\rangle + \Big\langle \bar{\boldsymbol{M}}, \frac{\partial \boldsymbol{B}}{\partial t} \Big\rangle + \mathcal{J}\, \mathrm{div}(\bar{\boldsymbol{e}} \times \bar{\boldsymbol{m}}) \\ &\quad = \rho_{ref} R_\Theta + \langle \boldsymbol{\mathcal{T}}\, \boldsymbol{\mathcal{F}}^T, \nabla^{\mathcal{S}} \boldsymbol{v} \rangle + \langle \bar{\boldsymbol{J}}_f, \bar{\boldsymbol{E}} \rangle - \mathrm{DIV}\, \boldsymbol{Q}_{\boldsymbol{\theta}} \end{aligned} \qquad (1')$$

with the electromagnetic stress tensor

$$\boldsymbol{\mathcal{T}} := \boldsymbol{\mathcal{P}} + \mathcal{J}\, \big[- \langle \bar{\boldsymbol{m}}, \boldsymbol{b} \rangle\, \mathbf{Id} + \bar{\boldsymbol{m}} \otimes \boldsymbol{b} - \bar{\boldsymbol{e}} \otimes \boldsymbol{p} \big] \circ \big(\boldsymbol{\mathcal{F}}^T \big)^{-1}, \qquad (5.17)$$

or, equivalently,

$$\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{P}} - \langle \bar{\boldsymbol{M}}, \boldsymbol{B} \rangle\, \big(\boldsymbol{\mathcal{F}}^T \big)^{-1} + \langle \boldsymbol{B}, (\boldsymbol{\mathcal{F}}^T)^{-1}(\,\cdot\,) \rangle\, \bar{\boldsymbol{M}} - \langle \boldsymbol{P}, (\boldsymbol{\mathcal{F}}^T)^{-1}(\,\cdot\,) \rangle\, \bar{\boldsymbol{E}}.$$

After the change of coordinates ξ_t and the temperature rescaling r_t, (1') becomes

$$\begin{aligned}
&\rho_{ref}\Big((D_{\phi'}\widehat{\Psi}')\,\boldsymbol{V}' + (D_{\Theta'}\widehat{\Psi}')\,\frac{\partial\Theta'}{\partial t} + (D_{\bar{\boldsymbol{E}}'}\widehat{\Psi}')\,\frac{\partial\bar{\boldsymbol{E}}'}{\partial t} + (D_{\boldsymbol{B}'}\widehat{\Psi}')\,\frac{\partial\boldsymbol{B}'}{\partial t} + S'\,\frac{\partial\Theta'}{\partial t} + \Theta'\,\frac{\partial S'}{\partial t}\Big)\\
&\qquad + \Big\langle\boldsymbol{P}', \frac{\partial\bar{\boldsymbol{E}}'}{\partial t}\Big\rangle + \Big\langle\bar{\boldsymbol{M}}', \frac{\partial\boldsymbol{B}'}{\partial t}\Big\rangle + \mathcal{J}'\,\mathrm{div}'(\bar{\boldsymbol{e}}'\times\bar{\boldsymbol{m}}')\\
&\quad = \rho_{ref}R'_{\Theta} + \langle\boldsymbol{\mathcal{T}}'\boldsymbol{\mathcal{F}}'^{T}, \nabla^{\mathcal{S}}\boldsymbol{v}'\rangle + \langle\bar{\boldsymbol{J}}'_f, \bar{\boldsymbol{E}}'\rangle - \mathrm{DIV}'\,\boldsymbol{Q}'_{\boldsymbol{\theta}}. \qquad (2)
\end{aligned}$$

Let $\boldsymbol{w}$ be the velocity of ξ_t. If at the time $t = t_0$, ξ_t is the identity, then the primed and the unprimed velocities are related by (see the discussion above Theorem 3.5.1)

$$\begin{aligned}
\boldsymbol{v}'(x', t) &= \boldsymbol{v}(x, t) + \boldsymbol{w}(x, t) \quad \text{or}\\
\boldsymbol{V}'(X, t) &= \boldsymbol{V}(X, t) + \boldsymbol{W}(X, t),
\end{aligned}$$

where $\boldsymbol{W}(X, t) := \boldsymbol{w}(x, t)$. (Since x and x' are the images under ϕ_t and ϕ'_t of the same point $X \in \mathcal{B}$, both the velocities $\boldsymbol{V}$ and $\boldsymbol{V}'$ depend on X.) $\bar{\boldsymbol{E}}$, $\boldsymbol{P}$, $\boldsymbol{B}$ and $\bar{\boldsymbol{J}}$ are Galilei invariants. Thus, for $t = t_0$,

$$\begin{aligned}
\bar{\boldsymbol{E}}'(X, t) &= \bar{\boldsymbol{E}}(X, t),\\
\boldsymbol{P}'(X, t) &= \boldsymbol{P}(X, t),\\
\boldsymbol{B}'(X, t) &= \boldsymbol{B}(X, t)\\
\bar{\boldsymbol{J}}'(X, t) &= \bar{\boldsymbol{J}}(X, t).
\end{aligned}$$

But the derivatives of these fields depend on the velocity $\boldsymbol{w}$. Consider for example the magnetic field $\boldsymbol{B}$. Its time derivative along the deformed body is given by $\dot{\boldsymbol{b}} = \frac{\partial\boldsymbol{b}}{\partial t} + \nabla^{\mathcal{S}}_{\boldsymbol{v}}\boldsymbol{b}$. In the primed system, at $t = t_0$,

$$\begin{aligned}
\dot{\boldsymbol{b}}'(x', t) &= \frac{\partial\boldsymbol{b}'}{\partial t}(x', t) + \nabla^{\mathcal{S}}_{\boldsymbol{v}'}\boldsymbol{b}'\,(x', t)\\
&= \frac{\partial\boldsymbol{b}}{\partial t}(x, t) + \left[\nabla^{\mathcal{S}}_{\boldsymbol{v}}\boldsymbol{b} + \nabla^{\mathcal{S}}_{\boldsymbol{w}}\boldsymbol{b}\right](x, t)\\
&= \dot{\boldsymbol{b}}(x, t) + \nabla^{\mathcal{S}}_{\boldsymbol{w}}\boldsymbol{b}\,(x, t).
\end{aligned}$$

In the coordinates of the undeformed body, this can be expressed by

$$\frac{\partial\boldsymbol{B}'}{\partial t}(X, t) = \frac{\partial\boldsymbol{B}}{\partial t}(X, t) + \nabla^{\mathcal{S}}_{\boldsymbol{w}}\boldsymbol{b}\,(x, t).$$

Next, we discuss the transformation behavior of the term $\boldsymbol{\mathcal{T}}\,\boldsymbol{\mathcal{F}}^T$:
By Def. 4.2.1, $\boldsymbol{\mathcal{P}} \circ \boldsymbol{\mathcal{F}}^T = \mathcal{J}\,\boldsymbol{\sigma}$. If $t = t_0$, then by Axiom 6 $\boldsymbol{t}' = \boldsymbol{t}$ and $\boldsymbol{\sigma}' = \boldsymbol{\sigma}$. Thus if $t = t_0$, then

$$\boldsymbol{\mathcal{P}}' \circ (\boldsymbol{\mathcal{F}}')^T(x', t) = \boldsymbol{\mathcal{P}}' \circ (\boldsymbol{\mathcal{F}}')^T(x, t).$$

Moreover, the second part of $\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{F}}^T$, given by

$$\mathcal{J}\left[-\langle\bar{\boldsymbol{m}},\boldsymbol{b}\rangle\,\mathbf{Id}+\bar{\boldsymbol{m}}\otimes\boldsymbol{b}-\bar{\boldsymbol{e}}\otimes\boldsymbol{p}\right],$$

involves only Galilei invariants. Thus, if $t=t_0$, then

$$\boldsymbol{\mathcal{T}'}\circ(\boldsymbol{\mathcal{F}}')^T(x,t)=\boldsymbol{\mathcal{T}}\circ\boldsymbol{\mathcal{F}}^T(x,t).$$

Let $u=\frac{dr_t}{dt}$ be the velocity of the temperature rescaling r_t. If for $t=t_0$ r_t is the identity, then

$$\frac{\partial\Theta'}{\partial t}=u\cdot\Theta+\frac{\partial\Theta}{\partial t}.$$

In summary, the above considerations provide that for $t=t_0$, equation (2) becomes

$$\begin{aligned}
\rho_{ref}\Bigg(&(D_\phi\widehat{\Psi})\,(\boldsymbol{V}+\boldsymbol{w})+(D_\Theta\widehat{\Psi})\left(u\Theta+\frac{\partial\Theta}{\partial t}\right)+(D_{\bar{\boldsymbol{E}}}\widehat{\Psi})\left(\frac{\partial\bar{\boldsymbol{E}}}{\partial t}+\nabla^{\mathcal{S}}_{\boldsymbol{w}}\bar{\boldsymbol{e}}\right)\\
&+(D_{\boldsymbol{B}}\widehat{\Psi})\left(\frac{\partial\boldsymbol{B}}{\partial t}+\nabla^{\mathcal{S}}_{\boldsymbol{w}}\bar{\boldsymbol{b}}\right)+S\left(u\Theta+\frac{\partial\Theta}{\partial t}\right)+\Theta\,\frac{\partial S}{\partial t}\Bigg)\\
&+\left\langle\boldsymbol{P},\frac{\partial\bar{\boldsymbol{E}}}{\partial t}\right\rangle+\langle\boldsymbol{p},\nabla^{\mathcal{S}}_{\boldsymbol{w}}\bar{\boldsymbol{e}}\rangle+\left\langle\bar{\boldsymbol{M}},\frac{\partial\boldsymbol{B}}{\partial t}\right\rangle+\langle\bar{\boldsymbol{m}},\nabla^{\mathcal{S}}_{\boldsymbol{w}}\boldsymbol{b}\rangle+\mathcal{J}\,\mathrm{div}(\bar{\boldsymbol{e}}\times\bar{\boldsymbol{m}})\\
=\;&\rho_{ref}R_\Theta+\langle\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{F}}^T,\nabla^{\mathcal{S}}(\boldsymbol{v}+\boldsymbol{w})\rangle+\langle\bar{\boldsymbol{J}}_f,\bar{\boldsymbol{E}}\rangle-\mathrm{DIV}\,\boldsymbol{Q_\theta}.
\end{aligned}\tag{3}$$

Subtracting (1’) from (3) gives

$$\begin{aligned}
\rho_{ref}\Big(&(D_\phi\widehat{\Psi})\,\boldsymbol{w}+(D_\Theta\widehat{\Psi})\,u\Theta+(D_{\bar{\boldsymbol{E}}}\widehat{\Psi})\,\nabla^{\mathcal{S}}_{\boldsymbol{w}}\bar{\boldsymbol{e}}+(D_{\boldsymbol{B}}\widehat{\Psi})\,\nabla^{\mathcal{S}}_{\boldsymbol{w}}\boldsymbol{b}+S\,u\Theta\Big)\\
&+\langle\boldsymbol{p},\nabla^{\mathcal{S}}_{\boldsymbol{w}}\bar{\boldsymbol{e}}\rangle+\langle\bar{\boldsymbol{m}},\nabla^{\mathcal{S}}_{\boldsymbol{w}}\boldsymbol{b}\rangle\\
=\;&\rho_{ref}R_\Theta+\langle\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{F}}^T,\nabla^{\mathcal{S}}\boldsymbol{w}\rangle.
\end{aligned}$$

This equation is valid for all diffeomorphisms $\xi_t:\mathcal{S}\to\mathcal{S}$ and all rescalings $r_t:\mathbb{R}^+\to\mathbb{R}^+$. Since ξ_t and r_t are arbitrary and do not depend on each other, we deduce that independently

$$\begin{aligned}
\rho_{ref}\Big(&(D_\phi\widehat{\Psi})\,\boldsymbol{w}+(D_{\bar{\boldsymbol{E}}}\widehat{\Psi})\,\nabla^{\mathcal{S}}_{\boldsymbol{w}}\bar{\boldsymbol{e}}+(D_{\boldsymbol{B}}\widehat{\Psi})\,\nabla^{\mathcal{S}}_{\boldsymbol{w}}\boldsymbol{b}\Big)+\langle\boldsymbol{p},\nabla^{\mathcal{S}}_{\boldsymbol{w}}\bar{\boldsymbol{e}}\rangle+\langle\bar{\boldsymbol{m}},\nabla^{\mathcal{S}}_{\boldsymbol{w}}\boldsymbol{b}\rangle\\
&=\langle\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{F}}^T,\nabla^{\mathcal{S}}\boldsymbol{w}\rangle
\end{aligned}\tag{4}$$

and

$$\rho_{ref}\Big((D_\Theta\widehat{\Psi})u\,\Theta+S\,u\,\Theta\Big)=0.\tag{5}$$

Keep in mind that $\boldsymbol{\mathcal{T}}$ also contains the point values of $\bar{\boldsymbol{e}}$ and $\boldsymbol{b}$.
Now consider (4) at an arbitrary point $x_0\in\mathcal{S}$. We can change $\bar{\boldsymbol{e}}$ and $\boldsymbol{b}$ in such a way that their values in x_0 stay the same, but their derivatives $\nabla^{\mathcal{S}}_{\boldsymbol{w}}\bar{\boldsymbol{e}}$ and $\nabla^{\mathcal{S}}_{\boldsymbol{w}}\boldsymbol{b}$ in the point x_0 are changed independently of each other. Thus, (4) can only hold, if in x_0

$$\rho_{ref}(D_\phi\widehat{\Psi})\,\boldsymbol{w}=\langle\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{F}}^T,\nabla^{\mathcal{S}}\boldsymbol{w}\rangle\tag{4a}$$

$$\rho_{ref}(D_{\bar{\boldsymbol{E}}}\widehat{\Psi})\,\nabla^{\mathcal{S}}_{\boldsymbol{w}}\bar{\boldsymbol{e}}+\langle\boldsymbol{p},\nabla^{\mathcal{S}}_{\boldsymbol{w}}\bar{\boldsymbol{e}}\rangle=0\tag{4b}$$

$$\rho_{ref}(D_{\boldsymbol{B}}\widehat{\Psi})\,\nabla^{\mathcal{S}}_{\boldsymbol{w}}\boldsymbol{b}+\langle\bar{\boldsymbol{m}},\nabla^{\mathcal{S}}_{\boldsymbol{w}}\boldsymbol{b}\rangle=0\tag{4c}$$

Since x_0 and the diffeomorphism ξ_t were arbitrarily chosen, the relations (4a)-(4c) must be valid in each point and for each $\xi_t : \mathcal{S} \to \mathcal{S}$.

1) Consider a fix point $X_0 \in \mathcal{B}$ and fix Θ, $\bar{\boldsymbol{E}}$, and $\boldsymbol{B}$. Let ϕ_0 and ϕ_1 be two configurations, such that

$$\begin{aligned} \phi_0(X_0) &= \phi_1(X_0) \\ \boldsymbol{\mathcal{F}}_0(X_0) &= \boldsymbol{\mathcal{F}}_1(X_0). \end{aligned}$$

We define a motion ϕ by

$$\phi(X,t) := \phi_0(X) + t(\phi_1(X) - \phi_0(X))$$

for a sufficiently small neighborhood $\mathcal{U}$ of X_0 and small t. Outside this neighborhood we continuate ϕ in a regular but otherwise arbitrary way. By the locality of $\widehat{\Psi}$ the choice of continuation does not influence the values of $\widehat{\Psi}$ inside the neighborhood.

The velocity and deformation gradient of the motion ϕ satisfy

$$\begin{aligned} \boldsymbol{V}_t(X_0) &= \phi_1(X_0) - \phi_0(X_0) = 0, \\ \frac{\partial \boldsymbol{\mathcal{F}}}{\partial t}(X_0) &= d\phi_1(X_0) - d\phi_0(X_0) = \mathbf{0}. \end{aligned}$$

Now we consider a particular coordinate transformation $\xi_t : \mathcal{S} \to \mathcal{S}$ that is given on $\mathcal{U}$ by

$$\xi_t(x) = \phi\left(\phi_1^{-1}(x), t\right) = \phi_0\left(\phi_1^{-1}(x)\right) + t\left(x - \phi_0(\phi_1^{-1}(x))\right)$$

and continuated regularly. Then

$$\boldsymbol{w}_t(x_0) = 0 = \boldsymbol{V}_t(X_0)$$

and (see Lemma 1.2.11) $\nabla^{\mathcal{S}} \boldsymbol{w}_t(x_0) = \frac{d}{dt} d\xi_t(x_0) = \mathbf{0}$. Hence, (4a) implies that at the point X_0,

$$\rho_{ref}(D_\phi \widehat{\Psi})\boldsymbol{V}(X_0, t) = 0.$$

Thus,

$$\frac{d}{dt}\widehat{\Psi}[\phi_t, \Theta, \bar{\boldsymbol{E}}, \boldsymbol{B}](X_0, t) = 0$$

and

$$\widehat{\Psi}[\phi_0, \Theta, \bar{\boldsymbol{E}}, \boldsymbol{B}](X_0, t) = \widehat{\Psi}[\phi_1, \Theta, \bar{\boldsymbol{E}}, \boldsymbol{B}](X_0, t).$$

Thus, we have shown: If Θ, $\bar{\boldsymbol{E}}$, and $\boldsymbol{B}$ are kept fixed and the deformation gradients of two deformations coincide in X_0, then the corresponding free energy densities coincide in X_0, too. In other words, $\widehat{\Psi}$ depends on ϕ only by the point values of $\boldsymbol{\mathcal{F}}$.

Hence, using Lemma 1.2.11 we can replace the term $(D_\phi \widehat{\Psi})\boldsymbol{V}$ in (1) by

$$\langle \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{F}}}, \frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} \rangle = \langle \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{F}}}, \nabla^{\mathcal{S}} \boldsymbol{v} \circ \boldsymbol{\mathcal{F}} \rangle.$$

Retracing the steps that led from (1) to (4a), we see that we can replace (4a) by

$$\left\langle \rho_{ref} \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{F}}} \boldsymbol{\mathcal{F}}^T - \boldsymbol{\mathcal{T}} \boldsymbol{\mathcal{F}}^T, \nabla^{\mathcal{S}} \boldsymbol{w} \right\rangle = 0$$

and conclude that $\boldsymbol{\mathcal{T}} = \rho_{ref} \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{F}}}$. Thus, by the definition of $\boldsymbol{\mathcal{T}}$,

$$\boldsymbol{\mathcal{P}} = \rho_{ref} \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{F}}} - \mathcal{J} \left[- \langle \bar{\boldsymbol{m}}, \boldsymbol{b} \rangle \, \mathbf{Id} + \bar{\boldsymbol{m}} \otimes \boldsymbol{b} - \bar{\boldsymbol{e}} \otimes \boldsymbol{p} \right] \left(\boldsymbol{\mathcal{F}}^T \right)^{-1}$$

or, in terms of vector fields along ϕ_t,

$$\boldsymbol{\mathcal{P}} = \rho_{ref} \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{F}}} + \left[\langle \bar{\boldsymbol{M}}, \boldsymbol{B} \rangle \, \mathbf{Id} - \bar{\boldsymbol{M}} \otimes \boldsymbol{B} + \bar{\boldsymbol{E}} \otimes \boldsymbol{P} \right] \left(\boldsymbol{\mathcal{F}}^T \right)^{-1}.$$

2) Let r_t be a rescaling of the temperature, such that in $t = t_0$ the derivative $u = 0$. (That is, the rescaling is the identity in a small neighborhood around $t = t_0$. This is admissible; we had only demanded that for each fix t the rescaling $r_t : \mathbb{R}^+ \to \mathbb{R}^+$ is a diffeomorphism.) Then in each point of $\mathcal{B}$, $S \, \boldsymbol{u} \, \Theta = 0$. Hence, (5) implies that $\widehat{\Psi}$ depends on Θ only by the point values of Θ, and not by higher derivatives.

   Thus, we can replace (5) by

$$\rho_{ref} \left(\frac{\partial \widehat{\Psi}}{\partial \Theta} + S \right) u \Theta = 0.$$

   Using $\rho_{ref} \neq 0$, we conclude that $S = -\frac{\partial \widehat{\Psi}}{\partial \Theta}$.

3) Let $x_0 \in \mathcal{S}$ be fix. We choose a diffeomorphism $\xi_t : \mathcal{S} \to \mathcal{S}$, such that $\nabla^{\mathcal{S}}_{\boldsymbol{w}} \bar{\boldsymbol{e}} \, (x_0) = 0$. Then (4b) implies that $\widehat{\Psi}$ depends on $\bar{\boldsymbol{E}}$ only by the point values of $\bar{\boldsymbol{E}}$.

   Thus, we can replace (4b) by

$$\left\langle \rho_{ref} \frac{\partial \widehat{\Psi}}{\partial \bar{\boldsymbol{E}}} + \boldsymbol{P}, \nabla^{\mathcal{S}}_{\boldsymbol{w}} \bar{\boldsymbol{e}} \right\rangle$$

   and conclude that $\boldsymbol{P} = -\rho_{ref} \frac{\partial \widehat{\Psi}}{\partial \bar{\boldsymbol{E}}}$.

   The statements concerning $\boldsymbol{B}$ can be deduced in exactly the same way from (4c).

4) By the items 1)-3) we have established that $\widehat{\Psi}$ depends only on X, the point values of Θ, $\bar{\boldsymbol{E}}$, and $\boldsymbol{B}$ and it depends on ϕ only by the deformation gradient $\boldsymbol{\mathcal{F}}$. Let us now assume that $\widehat{\Psi}$ depends on $\bar{\boldsymbol{E}}$ and $\boldsymbol{B}$ only by $|\bar{\boldsymbol{E}}|^2$ and $|\boldsymbol{B}|^2$. Then the second assumption in Axiom 6 implies the following:

Let $\xi : \mathcal{S} \to \mathcal{S}$ be an invertible, orientation-preserving map taking x to x', such that $d\xi|_x$ is an isometry from $T_x\mathcal{S}$ to $T_{x'}\mathcal{S}$. Then

$$\widehat{\Psi}(X, \boldsymbol{\mathcal{F}}, \Theta, |\bar{\boldsymbol{E}}|^2, |\boldsymbol{B}|^2) = \widehat{\Psi}(X, \boldsymbol{\mathcal{F}}', \Theta, |\bar{\boldsymbol{E}}|^2, |\boldsymbol{B}|^2), \tag{5.18}$$

where $\boldsymbol{\mathcal{F}} : T_X\mathcal{B} \to T_x\mathcal{S}$, $\boldsymbol{\mathcal{F}}' : T_X\mathcal{B} \to T_{x'}\mathcal{S}$ and $\boldsymbol{\mathcal{F}}' = \xi_*\boldsymbol{\mathcal{F}} = d\xi \circ \boldsymbol{\mathcal{F}}$.

x and x' can be either regarded as different coordinates for one and the same point of $\phi_t(\mathcal{B})$, or x can be regarded as the image of X by a deformation ϕ while x' is the image of X by a deformation ϕ'.

Let $\boldsymbol{\mathcal{F}} : T_X\mathcal{B} \to T_x\mathcal{S}$ and $\boldsymbol{\mathcal{F}}' : T_X\mathcal{B} \to T_{x'}\mathcal{S}$ be deformation gradients such that the resulting deformation tensors $\boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}}$ and $(\boldsymbol{\mathcal{F}}')^T\boldsymbol{\mathcal{F}}'$ coincide:

$$\boldsymbol{\mathcal{C}} := \boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}} = (\boldsymbol{\mathcal{F}}')^T\boldsymbol{\mathcal{F}}'.$$

If we can show that this implies $\widehat{\Psi}(X, \boldsymbol{\mathcal{F}}, \Theta, \bar{\boldsymbol{E}}, \boldsymbol{B}) = \widehat{\Psi}(X, \boldsymbol{\mathcal{F}}', \Theta, \bar{\boldsymbol{E}}, \boldsymbol{B})$, then $\widehat{\Psi}$ follows to depend only on $\boldsymbol{\mathcal{C}}$, but not on the values of $\boldsymbol{\mathcal{F}}$ resp. $\boldsymbol{\mathcal{F}}'$ themselves.

We had assumed our deformations to be diffeomorphisms, so $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{F}}'$ are invertible. Thus, there is a regular map $\xi : \mathcal{S} \to \mathcal{S}$, with $\xi(x) = x'$, such that $d\xi|_x\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}'$. Now we show that $d\xi|_x$ is an isometry: For all tangent vectors $V, W \in T_X\mathcal{B}$,

$$\begin{aligned}\left\langle d\xi|_x \cdot \boldsymbol{\mathcal{F}}|_X(V), d\xi|_x \cdot \boldsymbol{\mathcal{F}}|_X(W)\right\rangle &= \left\langle \boldsymbol{\mathcal{F}}'|_X(V), \boldsymbol{\mathcal{F}}'|_X(W)\right\rangle \\ &= \left\langle \boldsymbol{\mathcal{F}}'|_X^T\boldsymbol{\mathcal{F}}'|_X(V), W\right\rangle\end{aligned}$$

Since we had assumed that $\boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}} = (\boldsymbol{\mathcal{F}}')^T\boldsymbol{\mathcal{F}}'$, it follows that

$$\begin{aligned}\left\langle d\xi|_x \cdot \boldsymbol{\mathcal{F}}|_X(V), d\xi|_x \cdot \boldsymbol{\mathcal{F}}|_X(W)\right\rangle &= \left\langle \boldsymbol{\mathcal{F}}|_X^T\boldsymbol{\mathcal{F}}|_X(V), W\right\rangle \\ &= \left\langle \boldsymbol{\mathcal{F}}|_X(V), \boldsymbol{\mathcal{F}}|_X(W)\right\rangle.\end{aligned}$$

Thus, ξ is an isometry, so by (5.18), $\widehat{\Psi}(X, \boldsymbol{\mathcal{F}}, \Theta, |\bar{\boldsymbol{E}}|^2, |\boldsymbol{B}|^2) = \widehat{\Psi}(X, \boldsymbol{\mathcal{F}}', \Theta, |\bar{\boldsymbol{E}}|^2, |\boldsymbol{B}|^2)$.
□

Remark 5.3.6. The electromagnetic stress tensor $\boldsymbol{\mathcal{T}}$ that we defined in (5.17) also occurs in other texts on electroelasticity, e.g. in Kovetz [2000, p.221]. Ericksen [2007, see page 95-96] criticizes the use of this tensor, claiming that it does not vanish outside matter. $\boldsymbol{\mathcal{T}}$ indeed contains the electromotive intensity $\bar{\boldsymbol{e}}$ and the magnetic flux density $\boldsymbol{b}$, both of which also exist outside matter. But outside matter, the Piola-Kirchhoff tensor vanishes as do the polarization and the magnetization (at least in classical electrodynamics), so in vacuum, the electromagnetic stress tensor does vanish.

Remark 5.3.7. In physics it is a fundamental demand that at no instant during the motion ϕ the entropy inequality is violated. That is, by use of the entropy inequality one can rule out motions that are not physically possible. Kovetz [2000] and Ericksen [2008] use disagreeing entropy inequalities, and so obtain different classes of possible motions. Again Steigmann [2009] claims that the formulations of Kovetz and Ericksen are equivalent, but again this is not true. Kovetz assumes that

$$\rho\,\theta\,\dot{s} \geq \rho\,r_\theta - \theta\,\mathrm{div}_t\left(\frac{\boldsymbol{q_\theta}}{\theta}\right),$$

where s denotes the entropy density, while Ericksen makes the ansatz

$$\rho\,\theta\,\dot{s} \geq \rho\,r_\theta - \theta\,\mathrm{div}_t\left(\frac{\boldsymbol{q_\theta} + \bar{\boldsymbol{e}} \times \bar{\boldsymbol{m}}}{\theta}\right).$$

5.4 Electroelastic materials

Finally, we now consider a setting in which we neglect temperature dependencies and where the body is only exposed to a pure electric field. We assume that the experimental setup ensures that all magnetic influences are negligible. Then in particular $\bar{\mathfrak{E}} = \mathfrak{E}$.

Definition 5.4.1. Let $F \in C^r(\mathcal{B})$ be a scalar function describing a material property. An **electroelastic constitutive relation** for F is a map

$$\widehat{F} : \mathcal{D}(\mathcal{B}) \times \Gamma^k(\mathcal{B}, T^*\mathcal{S}) \to C^r(\mathcal{B}).$$

The function F associated to $\widehat{F}$, the motion ϕ and the electric field $\mathfrak{E}$, is then given by

$$F(X,t) = \widehat{F}\big(\phi_t, \mathfrak{E}_t\big)(X), \tag{5.19}$$

where $\phi_t(X) = \phi(X,t)$ and $\mathfrak{E}_t(X) = \mathfrak{E}(X,t)$.
Materials with such a kind of constitutive relation are called **electroelastic**.

We would now like to build a constitutive law for electroelastic materials that generalizes the Neo-Hookean law we formulated for purely elastic materials. Thus, if the electric field vanishes, the law to be constructed should simplify to

$$\Psi = \frac{\mu}{2}\,(I_1(\boldsymbol{\mathcal{C}}) - \dim\,\mathcal{B}) + \tau, \tag{5.20}$$

with the shear modulus $\mu = \text{const.} > 0$ and some $\tau = \text{const.}$

We assume that the free energy density Ψ depends on the electric field only by the trace of a symmetric $(0,2)$-tensor that can be constructed from $\mathfrak{E}$. A convenient choice is

$\mathrm{tr}(\mathfrak{E} \otimes \mathfrak{E}) = |\boldsymbol{E}|_g^2$. Note that for simple bodies, Theorem 5.3.4 then already implies that $\widehat{\Psi}$ depends on ϕ only by the point values of $\boldsymbol{\mathcal{C}}$.
To obtain a law that simplifies for $\boldsymbol{E} = \boldsymbol{0}$ to the Neo-Hookean law we make the shear modulus μ dependent on $|\boldsymbol{E}|_g^2$ and define

Definition 5.4.2. A Neo-electroelastic material is a material with a constitutive law of the form

$$\Psi = \frac{\mu(|\boldsymbol{E}|_g^2)}{2} \left(\mathrm{tr}(\boldsymbol{\mathcal{C}}) - \dim \mathcal{B}\right) + \nu(|\boldsymbol{E}|_g^2) + \tau, \tag{5.21}$$

where $\mu : C^0(\mathcal{B}_t) \to \mathbb{R}^+$, $\nu : C^0(\mathcal{B}_t) \to \mathbb{R}_0^+$, and $\tau \in \mathbb{R}^+$ is constant. If $\boldsymbol{E} = 0$, then μ is supposed to coincide with the constant shear modulus of a classical Neo-Hookean material and $\nu \equiv 0$.

Remark 5.4.3. For simple bodies the constitutive law of definition 5.4.2 is a simplification of a law that also occurs in the physics literature, for instance in the work of Dorfmann and Ogden [2005, p.177, eq. (76)].

6 The equation of motion for Neo-electroelastic materials

In section 4.2 we have obtained as the equation of motion (see Theorem 4.2.2)

$$\rho_{ref}\, \boldsymbol{A}^\flat = \rho_{ref}\, \mathfrak{F} + \mathfrak{F}_L + (\mathrm{DIV}\boldsymbol{\mathcal{P}})^\flat + \left[\mathrm{tr}_{\mathbf{II}}(\boldsymbol{\mathcal{P}} \circ \boldsymbol{\mathcal{F}}^T)\right]^\flat \circ \phi.$$

For purely electroelastic problems $\mathfrak{B} = 0$ and thus $\bar{\mathfrak{E}} = \mathfrak{E}$ and $\mathfrak{F}_L = \widetilde{\rho}_e\, \mathfrak{E}$. If we moreover express the acceleration by the motion ϕ, then this equation becomes

$$\rho_{ref} \left(\frac{\nabla}{\partial t} \frac{\partial \phi}{\partial t}\right)^\flat = \rho_{ref}\, \mathfrak{F} + \widetilde{\rho}_e\, \mathfrak{E} + (\mathrm{DIV}\boldsymbol{\mathcal{P}})^\flat + \left[\mathrm{tr}_{\mathbf{II}}(\boldsymbol{\mathcal{P}} \circ \boldsymbol{\mathcal{F}}^T)\right]^\flat \circ \phi.$$

The first Piola-Kirchhoff tensor $\boldsymbol{\mathcal{P}}$ is characterized by the material the body is made of. We would now like to set up the equation of motion for Neo-electroelastic materials. According to Def. 5.4.2 they are characterized by

$$\widehat{\Psi} = \frac{\mu(|\boldsymbol{E}|_g^2)}{2} \,(\mathrm{tr}(\boldsymbol{\mathcal{C}}) - \dim \mathcal{B}) + \nu(|\boldsymbol{E}|_g^2) + \tau. \tag{6.1}$$

To compute the divergence of $\boldsymbol{\mathcal{P}}$ and $\mathrm{tr}_{\mathbf{II}}(\boldsymbol{\mathcal{P}} \circ \boldsymbol{\mathcal{F}}^T)$ with the help of (6.1), we need to know how the first Piola-Kirchhoff tensor can be derived from the free energy density. Unfortunately, our Theorem 5.3.4 provides this knowledge only for simple bodies.

6.1 The initial boundary value problem for simple bodies

For simple bodies, Theorem 5.3.4 provides that

$$\boldsymbol{\mathcal{P}}^\sharp = g^\sharp\, \rho_{ref} \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{F}}} + \boldsymbol{E} \otimes \boldsymbol{P} \left(\boldsymbol{\mathcal{F}}^T\right)^{-1} = \rho_{ref} \left(2\, \boldsymbol{\mathcal{F}} \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{C}}^\flat} - \boldsymbol{E} \otimes \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{E}} \left(\boldsymbol{\mathcal{F}}^T\right)^{-1}\right),$$

where we have also used that $g^\sharp \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{F}}} = 2\, \boldsymbol{\mathcal{F}} \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{C}}^\flat}$. (We have already seen an analogous statement for the internal energy density in Theorem 4.5.1). Using $\frac{\partial \mathrm{tr}(\boldsymbol{\mathcal{C}})}{\partial \boldsymbol{\mathcal{C}}} = \mathbf{Id}$ and (6.1), we compute that for Neo-electroelastic materials

$$\mathrm{DIV} \left(2\, \rho_{ref} \boldsymbol{\mathcal{F}} \frac{\partial \widehat{\Psi}}{\partial \boldsymbol{\mathcal{C}}^\flat}\right) = \mathrm{DIV}\Big(\rho_{ref}\, \mu\big(|\boldsymbol{E}|_g^2\big)\, \boldsymbol{\mathcal{F}}\Big).$$

Moreover, for Neo-electroelastic materials

$$\begin{aligned}\frac{\partial\widehat{\Psi}}{\partial \boldsymbol{E}} &= \frac{1}{2}\left(\operatorname{tr}(\boldsymbol{\mathcal{C}}) - \dim\mathcal{B}\right)\frac{\partial\mu}{\partial|\boldsymbol{E}|^2}\,2|\boldsymbol{E}|\,\langle\boldsymbol{E},\,\cdot\,\rangle + 2\,\frac{\partial\nu}{\partial|\boldsymbol{E}|^2}\,|\boldsymbol{E}|\,\langle\boldsymbol{E},\,\cdot\,\rangle \\ &= 2\,|\boldsymbol{E}|\left(\frac{1}{2}\left(\operatorname{tr}(\boldsymbol{\mathcal{C}}) - \dim\mathcal{B}\right)\frac{\partial\mu}{\partial|\boldsymbol{E}|^2} + \frac{\partial\nu}{\partial|\boldsymbol{E}|^2}\right)\langle\boldsymbol{E},\,\cdot\,\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}&\operatorname{DIV}\left(\rho_{ref}\,\boldsymbol{E}\otimes\frac{\partial\widehat{\Psi}}{\partial\boldsymbol{E}}\left(\boldsymbol{\mathcal{F}}^T\right)^{-1}\right) \\ &= 2\operatorname{DIV}\left(\rho_{ref}|\boldsymbol{E}|\left(\frac{1}{2}\left(\operatorname{tr}(\boldsymbol{\mathcal{C}}) - \dim\mathcal{B}\right)\frac{\partial\mu}{\partial|\boldsymbol{E}|^2} + \frac{\partial\nu}{\partial|\boldsymbol{E}|^2}\right)\left\langle\boldsymbol{E},\left(\boldsymbol{\mathcal{F}}^T\right)^{-1}(\,\cdot\,)\right\rangle\boldsymbol{E}\right).\end{aligned}$$

Thus, for simple bodies made of a Neo-electroelastic material, the equation of motion reads

$$\begin{aligned}\rho_{ref}\frac{\partial^2\phi}{\partial t^2} &= \rho_{ref}\boldsymbol{F} + \widetilde{\rho}_e\,\boldsymbol{E} + \operatorname{DIV}\left(\rho_{ref}\,\mu\big(|\boldsymbol{E}|_g^2\big)\,\boldsymbol{\mathcal{F}}\right) \\ &\quad - 2\operatorname{DIV}\left(\rho_{ref}|\boldsymbol{E}|\left(\frac{1}{2}\left(\operatorname{tr}(\boldsymbol{\mathcal{C}}) - \dim\mathcal{B}\right)\frac{\partial\mu}{\partial|\boldsymbol{E}|^2} + \frac{\partial\nu}{\partial|\boldsymbol{E}|^2}\right)\left\langle\boldsymbol{E},\left(\boldsymbol{\mathcal{F}}^T\right)^{-1}(\,\cdot\,)\right\rangle\boldsymbol{E}\right),\end{aligned} \tag{6.2}$$

where $\boldsymbol{\mathcal{F}} = d\phi$ is the deformation gradient and $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}}$ the deformation tensor.

This equation is a system of non-linear inhomogeneous partial differential equations of second order. Observe that here, for simple bodies, all the single equations for the components of ϕ decouple. The mass density ρ_{ref} and the shear modulus μ are always positive, so if it were not for the last term, each of these partial differential equations would be hyperbolic.

Eq. (6.2) has to be supplemented by initial and boundary conditions. As initial conditions we might prescribe the values of ϕ and $\boldsymbol{V} = \frac{\partial\phi}{\partial t}$ for some initial time $t = t_0$. Suitable boundary conditions can be [Marsden and Hughes, 1983, ch.3, Def. 4.9]

1) displacement - ϕ is prescribed on $\partial\mathcal{B}$, the boundary of $\mathcal{B}$,

2) traction - the stress vector field $\boldsymbol{T} = \boldsymbol{\mathcal{P}}(\boldsymbol{N})$ is prescribed on $\partial\mathcal{B}$

3) mixed - ϕ is prescribed on a part $\partial\mathcal{B}_{\mathrm{D}}$ of $\partial\mathcal{B}$ and the stress vector field $\boldsymbol{T} = \boldsymbol{\mathcal{P}}(\boldsymbol{N})$ is prescribed on the remaining part of $\partial\mathcal{B}$, where $\partial\mathcal{B}_{\mathrm{D}}\cup\partial\mathcal{B}_T = \partial\mathcal{B}$ and $\partial\mathcal{B}_{\mathrm{D}}\cap\partial\mathcal{B}_T = \emptyset$.

The condition 2) is non-linear: $\widehat{\boldsymbol{\mathcal{P}}}$ depends on $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}}$ and is thus a non-linear function of ϕ.

Definition 6.1.1. Assume that the exterior mechanical force $\boldsymbol{F}$, the electric field $\boldsymbol{E}$ and one of the boundary conditions 1)-3) above are given. Moreover, prescribe the values of ϕ and $\boldsymbol{V} = \frac{\partial \phi}{\partial t}$ for some initial time $t = t_0$.
Then the **initial boundary value problem of Neo-electroelasticity** consists in finding ϕ, such that it satisfies (6.2) and also the boundary and initial conditions.

Index

Bibliography

Bär, C. (2010/2014). *Geometrische Elastizitätstheorie.* unpublished. University of Potsdam.

Batchelor, G. (2000). *An Introduction to Fluid Dynamics.* Cambridge University Press.

Chadwick, P. (1999). *Continuum Mechanics: Concise Theory and Problems.* Dover Publications.

Dorfmann, A. and Ogden, R. (2005). Nonlinear electroelasticity. *Acta Mechanica*, 174:167–183.

Ericksen, J. (2007). On formulating and assessing continuum theories of electromagnetic fields in elastic materials. *Journal of Elasticity*, 87:95–108.

Ericksen, J. (2008). Magnetizable and polarizable elastic materials. *Mathematics and Mechanics of Solids*, 13:38–54.

Eringen, A. C. and Maugin, G. A. (1990). *Electrodynamics of Continua.* Springer.

Fleisch, D. (2008). *A Student's Guide to Maxwell's Equations.* Cambridge University Press.

Frankel, T. (2006). *The geometry of physics: an introduction.* Cambridge University Press.

Grabs, C. (2014). *Geometrische Behandlung eines Problems aus der Elastostatik.* Diploma Thesis, University of Potsdam.

Griffiths, D. J. (2008). *Introduction to Electrodynamics.* Pearson.

Hehl, F. W. and Obukhov, Y. N. (2003). *Foundations of Classical Electrodynamics - Charge, Flux and Metric.* Birkhäuser.

Hehl, F. W. and Obukhov, Y. N. (2006). Spacetime metric from local and linear electrodynamics: A new axiomatic scheme. *Lecture Notes in Physics*, 702:163–187.

Hutter, K. and Pao, Y.-H. (1974). A dynamic theory for magnetizable elastic solids with thermal and electrical conduction. *Journal of Elasticity*, 4(2):89–114.

Hutter, K., van de Ven, Alfons, A., and Ursescu, A. (2006). *Electromagnetic Field Interactions in Thermoelastic Solids and Viscous Fluids.* Springer.

Jackson, J. D. (2002). *Klassische Elektrodynamik.* Walter de Gruyter.

Kanso, E., Arroyo, M., Tong, Y., Yavari, A., Marsden, J. E., and Desbrun, M. (2007). On the geometric character of stress in continuum mechanics. *Zeitschrift für angewandte Mathematik und Physik*, 58:1–14.

Kovetz, A. (2000). *Electromagnetic theory.* Oxford University Press.

Liu, I.-S. (2002). *Continuum Mechanics.* Springer.

Marsden, J. E. and Hughes, T. J. (1983). *Mathematical Foundations of Elasticity.* Dover Publications, New York.

Maugin, G. A. (1988). *Continuum Mechanics of Electromagnetic Solids*, volume 33 of *Applied Mathematics and Mechanics.* North Holland, Elsevier.

Nakamura, T. K. (2012). Three views of a Secret in relativistic thermodynamics. *Progress of Theoretical Physics*, 128(3):463–475.

Ogden, R. (1984). *Non-linear elastic deformations.* Ellis Harwood, Ltd. and Halsted Press/John Wiley and Sons.

Requardt, M. (2008). *Thermodynamics meets Special Relativity - or what is real in Physics ?* arXiv:0801.2639v1 [gr-qc].

Steigmann, D. J. (2009). On the formulation of balance laws for electromagnetic continua. *Mathematics and Mechanics of Solids*, 14:390–402.

Hiermit erkläre ich, dass diese Arbeit an keiner anderen Hochschule eingereicht sowie selbständig von mir und nur mit den angegebenen Mitteln angefertigt wurde.

Potsdam, den 29. April 2014