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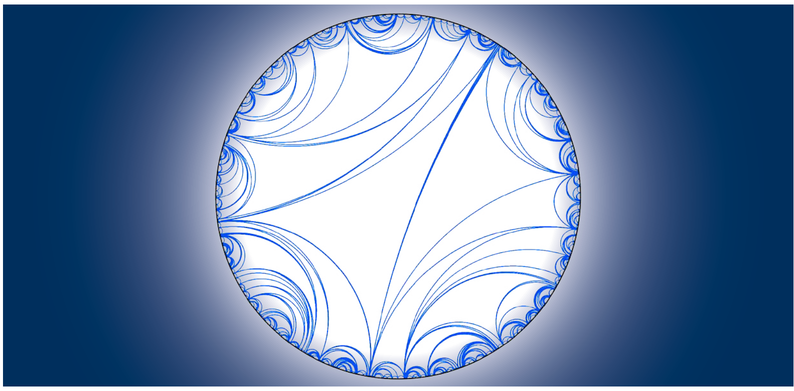

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Tel.: +49 (0)331 977 2533 / Fax: 2292
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Kontakt:
Institut für Mathematik
Am Neuen Palais 10
14469 Potsdam
Tel.: +49 (0)331 977 1028
WWW: http://www.math.uni-potsdam.de

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AN INDEX FORMULA FOR TOEPLITZ OPERATORS

D. FEDCHENKO AND N. TARKHANOV

Abstract. We prove a Fedosov index formula for the index of Toeplitz operators connected with the Hardy space of solutions to an elliptic system of first order partial differential equations in a bounded domain in $\mathbb{R}^n$ with smooth boundary.

Contents

Introduction

In [FT13] we initiated the problem of evaluating the index of Toeplitz type operators on Hardy spaces related to overdetermined elliptic systems. This problem is easily specified within the framework of analysis on manifolds. Namely, let $\mathcal{X}$ be a compact C^∞ manifold with boundary $\mathcal{S}$ and $A : C^\infty(\mathcal{X}, E) \to C^\infty(\mathcal{X}, F)$ a differential operator of order m between sections of smooth vector bundles E and F over $\mathcal{X}$. Assume the symbol of A is injective away from the zero section of $T^*\mathcal{X}$ and A satisfies the uniqueness condition for the local Cauchy problem in $\mathcal{X}$. Pick any Dirichlet system $\{B_j\}_{j=0,1,\ldots,m-1}$ of order $m-1$ in a neighbourhood of $\mathcal{S}$ which represents the Cauchy data of smooth solutions to $Au = 0$ at $\mathcal{S}$. If u is a solution of $Au = 0$ in the interior of $\mathcal{X}$ and has finite order of growth at $\mathcal{S}$, then the Cauchy data $\{B_j u\}$ of u has weak limit values at the boundary belonging to $\oplus \mathcal{D}'(\mathcal{S}, F_j)$, where F_j are smooth vector bundles over $\mathcal{S}$, see [Tar95, 9.4]. Each solution u of finite order of growth in the interior of $\mathcal{X}$ is uniquely determined through its Cauchy data at $\mathcal{S}$ by means of a Green formula. Denote by H the subspace of $L^2(\mathcal{S}, \oplus F_j)$ consisting of those $\oplus u_j \in L^2(\mathcal{S}, \oplus F_j)$ which are the Cauchy data of solutions to $Au = 0$ of finite order of growth at $\mathcal{S}$. By Theorem 10.3.14 of [Tar95], H is a closed

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subspace of $L^2(\mathcal{S}, \oplus F_j)$ and it can be thought of as natural generalisation of the Hardy space H^2 of holomorphic functions. Moreover, H is a Hilbert space with reproducing kernel.

The orthogonal projection Π of $L^2(\mathcal{S}, \oplus F_j)$ onto H is given by a so-called kernel function K_Π which is a C^∞ section of the bundle $E^* \otimes E$ over the interior of $\mathcal{X} \times \mathcal{X}$ satisfying $A(x, D) *_x^{-1} K_\Pi(x, y) = 0$. Here, $* : E \to E^*$ stands for the Hodge star operator induced by a Hermitean structure on E. For a deeper discussion of K_Π, see [Tar95, 11.2.3].

In general, Π fails to be a classical pseudodifferential operator on $\mathcal{S}$, rather it raises new classes of pseudodifferential operators. As but one example we mention the case A is the Cauchy-Riemann operator on a compact complex manifold $\mathcal{X}$ of dimension n whose boundary $\mathcal{S}$ is nonempty. If $n = 1$ then Π is the Szegö projection in $L^2(\mathcal{S})$ which is a classical pseudodifferential operator of order zero on $\mathcal{S}$. For $n > 1$, the nature of the Szegö projection is much more complicated and it well understood only in the case of strongly pseudoconvex manifolds, see [KS78]. Ii this case it is specified within pseudodifferential operators based on the Heisenberg group, cf. [NS79].

In [FT13] we prove that Π is a classical pseudodifferential operator of order zero on $\mathcal{S}$, provided that A is an elliptic differential operator on $\mathcal{X}$ in the classical sense, i.e. the principal homogeneous symbol of A is invertible away from the zero section of $T^*\mathcal{X}$. Thus, the generalised Hardy space H is rigged out with pseudodifferential projection, the structure being first studied in [BS82]. In particular, the latter paper introduces pseudodifferential operators on H which are readily recognised as generalised Toeplitz operators on $\mathcal{S}$ associated with the subspace H. By this are meant operators of the form $T_\Psi := \Pi\Psi$ in H, where Ψ is a classical pseudodifferential operator of order ≤ 0 which maps $L^2(\mathcal{S}, \oplus F_j)$ continuously into $L^2(\mathcal{S}, \oplus F_j)$.

Since $T_\Psi T_\Phi = T_{\Psi\Phi} + T_{\Psi[\Pi,\Phi]}$, the generalised Toeplitz operators form an operator algebra with symbols. Hence, they survive under parametrix construction. To wit, an operator is said to be elliptic if its symbol is invertible. It should be noted that the ellipticity refers to the subspace H through the projection Π. For an operator to be Fredholm it is necessary and sufficient that the operator is elliptic. The axioms of the symbol mapping allow one to accomplish a parametrix construction on the level of symbols, see [Wel73].

The purpose of this paper is to obtain an index formula à la Fedosov [Fed70] for elliptic Toeplitz operators. If the projection Π is pseudodifferential, any generalised Toeplitz operator T_Ψ is specified within the framework of pseudodifferential operators between sections of the vector bundle $\oplus F_j$ on $\mathcal{S}$. To evaluate the index of an elliptic operator T_Ψ we can therefore apply the Atiyah-Singer index formula [AS68]. This formula expresses the index by means of pairing the Chern character of the virtual bundle associated with the symbol of T_Ψ, and the Todd class of the tangent bundle of $\mathcal{S}$. In contrast to this cohomological formula [Fed70] gives an expression for the index in the form of the integral of a differential form over a cosphere bundle $S^*\mathcal{S}$ of $\mathcal{S}$. If the Todd class is equal to 1, then this form is explicitly written through the symbol of T_Ψ. In the general case the form is expressed through the symbol of T_Ψ and the curvature tensor of $\mathcal{S}$.

The index problem for Toeplitz operators goes back at least as far as the index formula for classical Toeplitz operators, cf. for instance [Dou73], [HH75]. An index formula for Toeplitz operators in strongly pseudoconvex domains in $\mathbb{C}^n$ was first obtained in [Ven72]. This paper motivated the study of Boutet de Monvel [BdM79] who developed an index theory for an algebra of Toeplitz operators in several complex variables. The monograph [Upm96] elaborates Fredholm theory for diverse classes of Toeplitz operators in complex domains which give rise to rather exotic index formulas. We contribute to this area by a very explicit index formula for a new class of Toeplitz operators in several variables. Our standing assumption throughout the whole paper is that the order of the operator A is equal to one. As is shown in [Cos91], this assumption does not contain any restriction of generality.

1. Subspaces admitting a pseudodifferential projector

In problems of spectral theory of differential and pseudodifferential operators one needs sometimes to consider an operator or a quadratic form on a proper subspace of its natural domain. The analysis of such a problem is especially substantial if the projection operator onto the subspace is pseudodifferential. The study of those subspaces which admit a pseudodifferential projector goes back at least as far as [BS82].

Pseudodifferential projectors onto subspaces give rise to pseudodifferential operators there which generalise classical Toeplitz operators, see for instance [Dou73] and elsewhere.

Let $\mathcal{X}$ be a compact closed C^∞ manifold of dimension n. We write $T^*\mathcal{X}$ for the cotangent bundle of $\mathcal{X}$ whose points are denoted by (x, ξ), where $x \in \mathcal{X}$ and $\xi \in T^*_x\mathcal{X}$. Furthermore, let E be a C^∞ vector bundle over $\mathcal{X}$ and E^* be the conjugate bundle. By π^*E is meant the vector bundle over $T^*\mathcal{X}$ induced by the projection π of E onto $\mathcal{X}$. The fibre of π^*E over a point $(x, \xi) \in T^*\mathcal{X}$ coincides with the fibre E_x of E over x.

Fix a smooth positive density dx on $\mathcal{X}$ and a Hermitean metric $(\cdot, \cdot)_x$ on the bundle E. Then $H^s(\mathcal{X}, E)$ stands for the scale of Sobolev spaces of smoothness $s \in \mathbb{R}$ based on the Lebesgue space $L^2(\mathcal{X}, E)$. By the Sobolev embedding theorem, the intersection of all spaces $H^s(\mathcal{X}, E)$ just amounts to $C^\infty(\mathcal{X}, E)$, and the union to $\mathcal{D}'(\mathcal{X}, E)$, the space of distributions on $\mathcal{X}$ with values in E, which is due to the compactness of $\mathcal{X}$.

The Hermitean metric on E determines a conjugate linear bundle isomorphism $* : E \to E^*$ which generalises the Hodge star operator in the bundles of exterior forms over $\mathcal{X}$. Namely, given any vector $f \in E_x$, we define a linear form $*f$ on E_x by the formula $\langle *f, g\rangle_x := (g, f)_x$ for all $g \in E_x$. Hence, $*f \in E^*_x$ and the family $* : E_x \to E^*_x$ parametrised by $x \in \mathcal{X}$ defines a conjugate linear bundle isomorphism $* : E \to E^*$. It acts also on sections of E in a natural way, which gives rise to conjugate linear isomorphisms $* : H^s(\mathcal{X}, E) \to H^s(\mathcal{X}, E^*)$. The use of $*$ makes the proofs more elegant.

Assume $A : C^\infty(\mathcal{X}, E) \to C^\infty(\mathcal{X}, F)$ is a classical pseudodifferential operator of order m between sections of vector bundles E and F over $\mathcal{X}$. It extends to a continuous linear mapping $H^s(\mathcal{X}, E) \to H^{s-m}(\mathcal{X}, F)$, for each $s \in \mathbb{R}$, and to $\mathcal{D}'(\mathcal{X}, E) \to \mathcal{D}'(\mathcal{X}, F)$. We write A' and A^* for the transposed and formal adjoint pseudodifferential operators on $\mathcal{X}$, respectively. Obviously, they are interrelated through $A^* = *_F^{-1} A' *_E$. The principal homogeneous symbol $\sigma^m(A)$ of A is a bundle morphism $\pi^*E \to \pi^*F$ away from the zero section of $T^*\mathcal{X}$. For fixed (x, ξ)

in $T^*\mathcal{X} \setminus \{0\}$, the value $\sigma^m(A)(x,\xi)$ is a linear map of E_x to F_x, and the principal homogeneous symbols of A' and A^* at (x,ξ) are actually the transpose and adjoint of $\sigma^m(A)(x,\xi)$.

Let Σ be a subspace of $C^\infty(\mathcal{X},E)$ of infinite dimension. We are interested in finding conditions on Σ which guarantee that there is a pseudodifferential operator Π projecting $C^\infty(\mathcal{X},E)$ onto Σ, i.e. $\Pi^2 = \Pi$ and $\Pi\, C^\infty(\mathcal{X},E) = \Sigma$. From the equality $\Pi^2 = \Pi$ it follows readily that the order of Π just amounts to zero. For $s \in \mathbb{R}$, we denote by $\Sigma^{(s)}$ the closure of Σ in $H^s(\mathcal{X},E)$. Since Π is a continuous self-mapping of $H^s(\mathcal{X},E)$, it follows immediately that $\Sigma^{(s)} = \Pi\, H^s(\mathcal{X},E)$. Obviously, $\Sigma^{(s)} \cap C^\infty(\mathcal{X},E) = \Sigma$.

The principal homogeneous symbol $\sigma^0(\Pi)(x,\xi)$ of Π projects the fibre E_x onto the subspace

$$\Sigma_{(x,\xi)} := \sigma^0(\Pi)(x,\xi)E_x. \tag{1.1}$$

Lemma 1.1. *Assume $\Sigma \subset C^\infty(\mathcal{X},E)$ admits a pseudodifferential projector Π. Then, the subspaces (1.1) are independent of the choice of the projector Π.*

Proof. Two pseudodifferential operators Π and Ψ project onto the same subspace if and only if $\Pi\Psi = \Psi$ and $\Psi\Pi = \Pi$. Since both Π and Ψ have order zero, their principal homogeneous symbols fulfill the same equations. Therefore, the ranges of $\sigma^0(\Pi)(x,\xi)$ and $\sigma^0(\Psi)(x,\xi)$ coincide, as desired. □

Thus, to each subspace $\Sigma \subset C^\infty(\mathcal{X},E)$ admitting a pseudodifferential projector one relates a unique family of finite dimensional spaces $\Sigma_{(x,\xi)}$ which depends smoothly on the point $(x,\xi) \in T^*\mathcal{X} \setminus \{0\}$. More precisely, for each connected component $\mathcal{C}$ of $T^*\mathcal{X} \setminus \{0\}$, the union of $\Sigma_{(x,\xi)}$ over all $(x,\xi) \in \mathcal{C}$ forms a smooth subbundle of $\pi^*E \restriction_{\mathcal{C}}$. The dimension of $\Sigma_{(x,\xi)}$ is locally constant, however, fails to be constant on all of $T^*\mathcal{X} \setminus \{0\}$ in general.

Theorem 1.2. *Suppose in each fibre of the bundle π^*E over $T^*\mathcal{X} \setminus \{0\}$ one chooses a subspace $V_{(x,\xi)}$ smoothly depending on (x,ξ), such that $V_{(x,t\xi)} = V_{(x,\xi)}$ for all $(x,\xi) \in T^*\mathcal{X} \setminus \{0\}$ and for all $t > 0$. Then there is a subspace Σ of $C^\infty(\mathcal{X},E)$ which admits a pseudodifferential projector and satisfies $\Sigma_{(x,\xi)} = V_{(x,\xi)}$ for all points $(x,\xi) \in T^*\mathcal{X} \setminus \{0\}$.*

Proof. Fix a Hermitean metric $(\cdot,\cdot)_x$ in E. Denote by $\pi(x,\xi$ the orthogonal projection onto $V_{(x,\xi)}$ in E_x and let P be a pseudodifferential operator of order zero in $C^\infty(\mathcal{X},E)$ whose principal homogeneous symbol coincides with $\pi(x,\xi)$. The spectrum of P in $L^2(\mathcal{X},E)$ away from the points $\lambda = 0$ and $\lambda = 1$ consists of isolated eigenvalues. Assume that the circle $|\lambda - 1| = r$, where $r < 1$, does not meet the spectrum of P. Consider the projector

$$\Pi = \int_{|\lambda-1|=r} \frac{-1}{2\pi\imath}\,(P - \lambda I)^{-1} d\lambda.$$

In each coordinate patch on $\mathcal{X}$ it is easy to calculate the full symbol of $(P - \lambda I)^{-1}$. It has asymptotic expansion

$$\sum_{j=0}^{\infty} a_{-j}(x,\xi,\lambda),$$

where $a_{-j}(x,\xi,\lambda)$ are homogeneous functions of degree $-j$ in $\xi \in \mathbb{R}^n \setminus \{0\}$, which are moreover rational functions of λ with poles at the points $\lambda = 0, 1$. In particular,

$a_0(x,\xi,\lambda) = \pi(x,\xi)(1-\lambda)^{-1} - (I_{E_x} - \pi(x,\xi))\lambda^{-1}$. Hence it follows that Π is a classical pseudodifferential operator of order zero whose principal homogeneous symbol just amounts to $\pi(x,\xi)$. Obviously, $\Sigma := \Pi C^\infty(\mathcal{X}, E)$ is the desired subspace of $C^\infty(\mathcal{X}, E)$. □

It is clear that the subspace Σ of $C^\infty(\mathcal{X}, E)$ satisfying the conditions of this theorem is not determined uniquely.

2. Pseudodifferential operators elliptic on a subspace

Let $A : C^\infty(\mathcal{X}, E) \to C^\infty(\mathcal{X}, F)$ be a pseudodifferential operator of order m between sections of vector bundles E and F over $\mathcal{X}$. Write $a_m(x,\xi) := \sigma^m(A)(x,\xi)$ for the principal homogeneous symbol of A. The operator A is said to be elliptic on a subspace Σ of $C^\infty(\mathcal{X}, E)$ which admits a pseudodifferential projector Π if the rank of $a_m(x,\xi) \restriction_{\Sigma_{(x,\xi)}}$ is equal to the dimension of $\Sigma_{(x,\xi)}$ for all (x,ξ) away from the zero section of $T^*\mathcal{X} \setminus \{0\}$.

If $\Sigma = C^\infty(\mathcal{X}, E)$, the notion of ellipticity on Σ coincides with that of overdetermined ellipticity. The basic results of elliptic theory remain still valid for pseudodifferential operators which are elliptic on a subspace with pseudodifferential projector.

Theorem 2.1. *Let $A : C^\infty(\mathcal{X}, E) \to C^\infty(\mathcal{X}, F)$ be a classical pseudodifferential operator of order m elliptic on a subspace Σ admitting a pseudodifferential projector Π. Then:*

1) The null-space of $A^{(s)} : \Sigma^{(s)} \to H^{s-m}(\mathcal{X}, F)$ is independent of $s \in \mathbb{R}$, has finite dimension and belongs to $C^\infty(\mathcal{X}, E)$.

2) The range B of $A : \Sigma \to C^\infty(\mathcal{X}, F)$ admits a pseudodifferential projector and $B_{(x,\xi)} = a_m(x,\xi)\Sigma_{(x,\xi)}$. For each $s \in \mathbb{R}$, the range of $A^{(s)} : \Sigma^{(s)} \to H^{s-m}(\mathcal{X}, F)$ coincides with $B^{(s-m)}$ (and so it is closed in $H^{s-m}(\mathcal{X}, F)$).

3) Let P and N be the orthogonal projections of $L^2(\mathcal{X}, E)$ onto $\Sigma^{(0)}$ and the null-space of $A \restriction_\Sigma$, respectively, and R the orthogonal projection of $L^2(\mathcal{X}, F)$ onto $B^{(0)}$. There is a pseudodifferential operator $G : C^\infty(\mathcal{X}, F) \to C^\infty(\mathcal{X}, E)$ of order $-m$, such that

$$\begin{aligned} GAP &= P - N, \\ APG &= R. \end{aligned}$$

Any operator G with the properties described in 3) is called a regulariser of A restricted to Σ.

Proof. We tacitly assume that the manifold $\mathcal{X}$ and the bundles E, F are endowed with Hermitean structures. Pick an elliptic pseudodifferential operator Ψ of order m in $C^\infty(\mathcal{X}, E)$ whose null-space is trivial. An easy calculation shows that the complex

$$C^\infty(\mathcal{X}, E) \stackrel{(I-P)\Psi^*}{\longrightarrow} C^\infty(\mathcal{X}, E) \stackrel{AP}{\longrightarrow} C^\infty(\mathcal{X}, F)$$

is elliptic, i.e. its Laplace operator

$$\Delta = (I-P)\Psi^*\Psi(I-P) + PA^*AP$$

is elliptic of order $2m$. Therefore, Theorem 2.1 follows from the Hodge theory for elliptic complexes on compact closed manifolds.

Since

$$\ker \Delta^{(s)} = \ker (AP)^{(s)} \cap \ker (\Psi(I-P))^{(s)} = \ker \left(A^{(s)} \restriction_{\Sigma^{(s)}} \right),$$

we get readily the assertion of 1).

Obviously,

$$\begin{array}{rcl} \operatorname{im} (A \restriction_{\Sigma}) & = & \operatorname{im} AP, \\ \operatorname{im} \left(A^{(s)} \restriction_{\Sigma^{(s)}} \right) & = & \operatorname{im} (AP)^{(s)} \end{array}$$

and $a_m(x,\xi)\Sigma_{(x,\xi)} = a_m(x,\xi)\sigma^0(P)(x,\xi)E_x$ for all (x,ξ) away from the zero section of $T^*\mathcal{X}$. This implies the first part of 2).

The operator N is a smoothing pseudodifferential operator of finite rank on $\mathcal{X}$. The operator $\Delta + N$ is invertible and its inverse $(\Delta + N)^{-1}$ is a pseudodifferential operator of order $-2m$. Since the pairwise products of the operators $(\Psi(I-P))^*(\Psi(I-P))$, $(AP)^*(AP)$ and N vanish, these operators commute with $\Delta + N$, and so with $(\Delta + N)^{-1}$. It follows that

$$\begin{array}{rcl} AP & = & (AP)(\Delta + N)(\Delta + N)^{-1} \\ & = & (AP)(AP)^*(AP)(\Delta + N)^{-1} \\ & = & (AP)(\Delta + N)^{-1}(AP)^*(AP). \end{array}$$

Using this equality we deduce immediately that the (formally) self-adjoint pseudodifferential operator $R = AP(\Delta + N)^{-1}PA^*$ gives the orthogonal projection of $L^2(\mathcal{X}, F)$ onto $B^{(0)}$.

We get $AP = R(AP)$, hence the image of $\Sigma^{(s)}$ by $A^{(s)}$ belongs to the image of $H^{s-m}(\mathcal{X}, F)$ by $R^{(s-m)}$. The inverse inclusion is clear, which gives the second part of 2).

The pseudodifferential operator $P - N$ is an orthogonal projection onto the range of $(AP)^*$ whence $P - N = (\Delta + N)^{-1}(AP)^*(AP)$. Hence it follows that $G := (\Delta + N)^{-1}(AP)^*$ is a regulariser of $A \restriction_{\Sigma}$. □

Theorem 2.1 goes back at least as far as [BS82]. We just specify it within the Hodge theory for elliptic complexes of pseudodifferential operators on compact closed manifolds.

3. Generalised Szegö projection

Assume $\mathcal{X}$ is a bounded closed domain with smooth boundary in $\mathbb{R}^n$. Its boundary $\mathcal{S}$ is a compact closed submanifold of $\mathbb{R}^n$ of dimension $n-1$ with induced orientation.

Let A be an $(l \times k)$-matrix of scalar first order partial differential operators in a neighbourhood U of $\mathcal{X}$. We assume that $l \geq k$ and the principal homogeneous symbol $\sigma^1(A)(x,\xi)$ of A has rank k for all $(x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})$. Such differential operators A are usually referred to as overdetermined elliptic operators in U. As but one example we mention the Cauchy-Riemann operator in several complex variables.

Our basic assumption is that A satisfies the uniqueness condition for the local Cauchy problem in U, see [Tar95]. This is the case, in particular, if the coefficients of A are real analytic functions on U. By Theorem 4.4.3 of [Tar95], the operator A has a left fundamental solution Φ in U which is a $(k \times l)$-matrix of pseudodifferential operators of order -1 in U. By the very definition, this is an operator $\Phi : C^\infty_{\mathrm{comp}}(U, \mathbb{C}^l) \to C^\infty(U, \mathbb{C}^k)$ satisfying $\Phi A = I$ on $C^\infty_{\mathrm{comp}}(U, \mathbb{C}^k)$. We often use the kernel theorem to identify Φ with its Schwartz kernel for which we write $\Phi(x,y)$ by abuse of notation.

In the sequel we write $\nu(y)$ for the unit outward normal vector of $\mathcal{S}$ at a point $y \in \mathcal{S}$. Denote by $\sigma(y)$ the principal homogeneous symbol of the operator A evaluated at the point $(y, \nu(y)/\imath)$ of the complexified cotangent bundle of U, where $y \in \mathcal{S}$.

Theorem 3.1. *Let u_0 be an arbitrary square integrable function on $\mathcal{S}$ with values in $\mathbb{C}^k$. In order that there be a solution u to $Au = 0$ in the interior of $\mathcal{X}$, which has finite order of growth at the boundary and coincides with u_0 at $\mathcal{S}$, it is necessary and sufficient that*

$$\int_{\mathcal{S}} (\sigma(y)u_0, g)_y \, ds = 0 \tag{3.1}$$

for all solutions of the formal adjoint equation $A^* g = 0$ near $\mathcal{X}$, where ds is the surface measure on $\mathcal{S}$.

Proof. See Theorem 10.3.14 in [Tar95]. □

We denote by H the (closed) subspace of $L^2(\mathcal{S}, \mathbb{R}^k)$, consisting of all functions u_0 satisfying the orthogonality conditions (3.1). The elements of H can be actually specified as solutions to $Au = 0$ of Hardy class H^2 in the interior of $\mathcal{X}$, see [Tar95, 11.2.2]. The orthogonal projection Π of $L^2(\mathcal{S}, \mathbb{R}^k)$ onto H is therefore an analogue of the Szegö projection.

Actually H is a Hilbert space with reproducing kernel, the concept going back at least as far as Aronszajn (1950). But we will not develop this point here.

Definition 3.2. Let M be a $(k \times k)$-matrix whose entries are bounded functions on $\mathcal{S}$. By a Toeplitz operator T_M with multiplier M is meant the operator $u \mapsto \Pi(Mu)$ in H.

More generally, if Ψ is a $(k \times k)$-matrix of pseudodifferential operators of order 0 on $\mathcal{S}$, then Ψ maps $L^2(\mathcal{S}, \mathbb{C}^k)$ continuously into $L^2(\mathcal{S}, \mathbb{C}^k)$. Therefore, the composition $T_\Psi = \Pi \Psi$ is a continuous self-mapping of H which we call a generalised Toeplitz operator.

If the projection Π is a classical pseudodifferential operator on $\mathcal{S}$, then from the equality $\Pi^2 = \Pi$ it follows readily that the order of Π just amounts to zero. Hence, the generalised Toeplitz operators on $\mathcal{S}$ form a subalgebra of the C^*-algebra of all zero order classical pseudodifferential operators on $C^\infty(\mathcal{S}, \mathbb{C}^k)$. When restricted to $L^2(\mathcal{S}, \mathbb{C}^k)$, the generalised Toeplitz operators are specified within $\mathcal{L}(H)$ and their closure is a unital operator subalgebra of $\mathcal{L}(H)$. The construction of Toeplitz operators still survives in any abstract context where both Π and Ψ belong to a unital operator algebra in $\mathcal{L}(H)$.

4. The generalised Cauchy type integral

To clarify the nature of the generalised Szegö projection Π we introduce a singular Cauchy type integral

$$\mathcal{C}u\,(x) = -\text{p.v.} \int_{\mathcal{S}} \Phi(x, \cdot)\sigma u \, ds \tag{4.1}$$

for $x \in \mathcal{S}$, where $u \in L^2(\mathcal{S}, \mathbb{C}^k)$ is a given function. The principal value of the integral on the right hand side is known to exist for almost all $x \in \mathcal{S}$ and it induces a bounded linear operator in $L^2(\mathcal{S}, \mathbb{C}^k)$, see for instance Theorem 2.3.12 of [Tar95] and elsewhere.

Since the kernel $\Phi(x,y)$ is smooth away from the diagonal of $U \times U$, it follows that the integral

$$-\int_{\mathcal{S}} \Phi(x,\cdot)\sigma u\, ds$$

is a C^∞ function of $x \in U \setminus \mathcal{S}$ with values in $\mathbb{C}^k$. Let $\mathcal{C}^{\pm}u$ stand for the restrictions of the integral to the interior of $\mathcal{X}$ and $U \setminus \mathcal{X}$, respectively. If u satisfies a Hölder condition at a point $x^0 \in \mathcal{S}$, then both $\mathcal{C}^+u\,(x)$ and $\mathcal{C}^-u\,(x)$ has limit values at x^0, as x tends to x^0 along a transversal way to $\mathcal{S}$ lying in the interior of $\mathcal{X}$ or $U \setminus \mathcal{X}$, respectively.

Lemma 4.1. *If u satisfies a Hölder condition at a point $x^0 \in \mathcal{S}$, then*

$$\mathcal{C}^{\pm}u\,(x^0) = \pm\frac{1}{2}\,u(x^0) + \mathcal{C}u\,(x^0).$$

Proof. These are classical jump formulas first proved for the Cauchy integral by Sokhotskii and Plemelj, see Theorem 3.2.6 in [Tar90]. □

By the definition of the generalised Hardy space H, it is the operator $\mathcal{C}^+$ that might be a projection of $L^2(\mathcal{S}, \mathbb{C}^k)$ onto H. A necessary condition for this is $A\Phi = 0$, i.e. the columns of $\Phi(x,y)$ satisfy $Au = 0$ for all x in the interior of $\mathcal{X}$ whenever $y \in \mathcal{S}$.

Lemma 4.1 provides another possibility to verify if $\mathcal{C}^+$ is really a projection. To wit,

$$\begin{aligned}(\mathcal{C}^+)^2 &= \Big(\frac{1}{2}\,I + \mathcal{C}\Big)^2 \\ &= \frac{1}{4}\,I + \mathcal{C} + \mathcal{C}^2\end{aligned}$$

coincides with $\mathcal{C}^+$ if and only if $\mathcal{C}^2 = \dfrac{1}{4}\,I$.

Remark 4.2. If A is an elliptic (i.e. $l = k$) partial differential operator in U, then $\mathcal{C}^2 = \dfrac{1}{4}\,I$, see [Tar90, 3.2.8].

The generalised Cauchy type integral (4.1) is a classical pseudodifferential operator of order 0 in $C^\infty(\mathcal{S}, \mathbb{C}^k)$. We finish this section by evaluating its principal homogeneous symbol. To this end, we identify the cotangent space $T^*_x\mathcal{S}$ of $\mathcal{S}$ at a point $x \in \mathcal{S}$ with all linear forms on $T^*_x\mathbb{R}^n$ which vanish on the one-dimensional subspace of $T^*_x\mathbb{R}^n$ spanned by the vector $\nu(x)$. Since $T^*_x\mathbb{R}^n \cong \mathbb{R}^n$, one can actually specify $T^*_x\mathcal{S}$ as the hyperplane through the origin in $\mathbb{R}^n$ which is orthogonal to the vector $\nu(x)$.

Theorem 4.3. *For all $x \in \mathcal{S}$ and $\xi \in T^*_x\mathcal{S}$, the symbol of order 0 of the operator $\mathcal{C}$ is equal to*

$$\sigma^0(\mathcal{C})(x,\xi) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \Big(\sigma^2(A^*A)(x, t\nu(x){+}\xi)\Big)^{-1} dt\, \sigma^1(A^*)(x,\xi)\, \sigma^1(A)\Big(x, \frac{1}{\imath}\nu(x)\Big).$$

Set

$$a(x,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big(\sigma^2(A^*A)(x, t\nu(x) + \xi)\Big)^{-1} dt$$

for $(x,\xi) \in T^*\mathcal{S}$. Then

$$\begin{aligned} a(x,\lambda\xi) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\sigma^2(A^*A)(x,t\nu(x)+\lambda\xi)\Big)^{-1}dt \\ &= \lambda^{-1}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\sigma^2(A^*A)\Big(x,\frac{t}{\lambda}\,\nu(x)+\xi\Big)\Big)^{-1}d\,\frac{t}{\lambda} \\ &= \lambda^{-1}\,a(x,\xi) \end{aligned}$$

for all $\lambda > 0$, showing that $a(x,\xi)$ is a homogeneous symbol of order -1 on $T^*\mathcal{S}$ indeed.

Proof. We first recall the construction of the left fundamental solution Φ. By assumption, the Laplacian A^*A is a second order elliptic differential operator on $C^\infty(U,\mathbb{C}^k)$. It has a parametrix G which is a $(k\times k)$-matrix of scalar pseudodifferential operators of order -2 on U. The operator Φ differs from GA^* by a smoothing operator, and so it has the principal homogeneous symbol $(\sigma^2(A^*A))^{-1}\sigma^1(A^*)$ which is a left inverse for $\sigma^1(A)$.

Formula (4.1) just amounts to saying that

$$\mathcal{C}u\,(x) = -\Phi\Big(\sigma^1(A)\Big(y,\frac{1}{\imath}\nu(y)\Big)u(y)\ell_{\mathcal{S}}\Big),$$

where $\ell_{\mathcal{S}}$ is the surface layer on $\mathcal{S}$. We thus see that the pseudodifferential $\mathcal{C}$ on $\mathcal{S}$ is the restriction to $\mathcal{S}$ of the pseudodifferential operator

$$\Psi = -\Phi\circ\Big(\sigma^1(A)\Big(x,\frac{1}{\imath}\nu(x)\Big)\Big)$$

defined in a neighbourhood of $\mathcal{S}$. This latter is of order -1 and its principal symbol is easily evaluated, namely

$$\begin{aligned} \sigma^{-1}(\Psi)(x,\xi) &= -\sigma^{-1}(\Phi)(x,\xi)\,\sigma^1(A)\Big(x,\frac{1}{\imath}\nu(x)\Big) \\ &= -\big(\sigma^2(A^*A)(x,\xi)\big)^{-1}\,\sigma^1(A^*)(x,\xi)\,\sigma^1(A)\Big(x,\frac{1}{\imath}\nu(x)\Big) \end{aligned}$$

for x in a neighbourhood of $\mathcal{S}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. A familiar argument now shows that the principal symbol of $\mathcal{C}$ is given by the formula

$$\begin{aligned} \sigma^0(\mathcal{C})(x,\xi) &= -\frac{1}{2\pi}\,\text{p.v.}\int_{-\infty}^{\infty}\Big(\sigma^2(A^*A)(x,t\nu(x)+\xi)\Big)^{-1}\sigma^1(A^*)(x,t\nu(x)+\xi)dt \\ &\times\ \sigma^1(A)\Big(x,\frac{1}{\imath}\nu(x)\Big) \end{aligned}$$

for all $x \in S$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ orthogonal to $\nu(x)$. Note that the integral on the right-hand side diverges, however, its Cauchy principal value exists, which is due to the condition $\langle\nu(x),\xi\rangle = 0$. To wit,

$$\sigma^0(\mathcal{C})(x,\xi) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\sigma^2(A^*A)(x,t\nu(x)+\xi)\Big)^{-1}\sigma^1(A^*)(x,\xi)dt\,\sigma^1(A)\Big(x,\frac{1}{\imath}\nu(x)\Big),$$

which establishes the theorem. □

If A is a Dirac operator, then Theorem 4.3 can be sharpened. By a Dirac operator is meant any $(l\times k)$-matrix A of scalar partial differential operators of order 1 on U, such that $A^*A = -\Delta E_k$ modulo first order operators on U, where Δ is the usual (nonpositive) Laplace operator on $\mathbb{R}^n$ and E_k the unit $(k\times k)$-matrix. In this case we get $\Phi(x,y) = -A^*(y,D)G_n(x-y)$ modulo pseudodifferential operators of order less than -1 on U, where

$$G_n(x) = \frac{1}{\sigma_n(2-n)}\,\frac{1}{|x|^{n-2}}$$

is the standard fundamental solution of convolution type of the Laplace operator in $\mathbb{R}^n$ and σ_n the area of the $(n-1)$-dimensional sphere in $\mathbb{R}^n$.

Example 4.4. Assume that A is a Dirac operator in U and $\mathcal{C}$ the corresponding Cauchy type integral given by (4.1). Then

$$\sigma^0(\mathcal{C})(x,\xi) = \frac{1}{2}\,\sigma^1(A^*)\Big(x,\frac{\xi}{|\xi|}\Big)\sigma^1(A)\Big(x,\frac{1}{\imath}\nu(x)\Big)$$

for all $(x,\xi) \in T^*\mathcal{S}$. Since

$$\sigma^2(A^*A)(x,\xi) = |\xi|^2\,E_k,$$

we shall have established the desired equality, by Theorem 4.3, if we prove that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|t\nu(x)+\xi|^2}\,dt = \frac{1}{2}\,\frac{1}{|\xi|}$$

for all $x \in S$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ orthogonal to the vector $\nu(x)$. A trivial verification shows that

$$|t\nu(x)+\xi|^2 = t^2 + |\xi|^2$$

whence

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|t\nu(x)+\xi|^2}\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+|\xi|^2}\,dt\\ &= \frac{1}{2}\,\frac{1}{|\xi|},\end{aligned}$$

as desired.

5. Index of Toeplitz operators

We now return to generalised Toeplitz operators $T_\Psi = \Pi\Psi$ in H introduced in Section 3, see Definition 3.2 and below. Here, Π is a projection of $L^2(\mathcal{S},\mathbb{C}^k)$ onto H. We restrict ourselves to the case where Π is a classical pseudodifferential operator of order zero in $C^\infty(\mathcal{S},\mathbb{C}^k)$. We are interested in characterising those Toeplitz operators in H which possess the Fredholm property. To this end we extend T_Ψ from H to all of $L^2(\mathcal{S},\mathbb{C}^k)$ in a special manner and use Fredholm criteria for operator algebras with symbols.

Lemma 5.1. *A Toeplitz operator T_Ψ in H is Fredholm if and only if so is the operator*

$$E_\Psi := T_\Psi\Pi + (I-\Pi)$$

in $L^2(\mathcal{S},\mathbb{C}^k)$.

Proof. Necessity. Suppose $T_\Psi : H \to H$ is Fredholm. Let $u \in L^2(\mathcal{S},\mathbb{C}^k)$ satisfy $E_\Psi u = 0$. Then $(I-\Pi)u = 0$ whence $u \in H$. It follows that $u \in \ker T_\Psi$. Therefore, $\ker E_\Psi \subset \ker T_\Psi$ is finite dimensional. Let now $f \in L^2(\mathcal{S},\mathbb{C}^k)$ be an arbitrary element. Write $f = \Pi f + (I-\Pi)f$. Since the cokernel of $T_\Psi : H \to H$ is finite dimensional and $\Pi f \in H$, there are a finite number of continuous linear functionals $g_1,\dots,g_\beta$ on H, such that the conditions $\langle g_1,\Pi f\rangle = 0,\dots,\langle g_\beta,\Pi f\rangle = 0$

are necessary and sufficient for the existence of $h \in H$ satisfying $T_\Psi h = \Pi f$. Set $u = h + (I - \Pi)f$, then $E_\Psi u = f$, showing that the cokernel of E_Ψ is finite dimensional.

Sufficiency. Since the restriction of E_Ψ to H coincides with T_Ψ, the sufficiency is obvious. □

As a byproduct of the proof of Lemma 5.1 we deduce that $\ker E_\Psi = \ker T_\Psi$ and $\operatorname{coker} E_\Psi = \operatorname{coker} T_\Psi$, hence

$$\operatorname{ind} E_\Psi = \operatorname{ind} T_\Psi. \tag{5.1}$$

The operator E_Ψ on $L^2(\mathcal{S}, \mathbb{C}^k)$ is pseudodifferential of order zero. Its principal homogeneous symbol just amounts to $\sigma^0(E_\Psi) = \sigma^0(\Pi)\sigma^0(\Psi)\sigma^0(\Pi) + (I - \sigma^0(\Pi))$ away from the zero section of $T^*\mathcal{S}$. Given any $s \in \mathbb{R}$, the operator E_Ψ in $H^s(\mathcal{S}, \mathbb{C}^k)$ is known to be Fredholm if and only if it is elliptic, i.e. $\sigma^0(E_\Psi)(x, \xi)$ is invertible for all $(x, \xi) \in T^*\mathcal{S}$ with $\xi \neq 0$. Moreover, the index of this operator is actually independent of the particular choice of s and it can be evaluated by the familiar Atiyah-Singer formula [AS68].

Remark 5.2. As is defined in Section 2, the ellipticity of the Toeplitz operator T_Ψ in H just amounts to the ellipticity of the operator E_Ψ in $H^s(\mathcal{S}, \mathbb{C}^k)$ for any one $s \in \mathbb{R}$.

If the operator E_Ψ is elliptic, then its principal homogeneous symbol $\sigma^0(E_\Psi)$ provides an isomorphism of the induced bundle $\pi^*(\mathcal{S} \times \mathbb{C}^k)$ over $T^*\mathcal{S} \setminus \{0\}$, where $\pi : T^*\mathcal{S} \to \mathcal{S}$ is the canonical projection. In this way E_Ψ gives rise to an element $d(E_\Psi)$ of the functor with compact support $K^{\mathrm{comp}}(T^*\mathcal{S})$. More precisely, $d(E_\Psi)$ is defined by the virtual bundle with compact support $\{\mathcal{S} \times \mathbb{C}^k, \mathcal{S} \times \mathbb{C}^k, \sigma^0(E_\Psi)\}$. This is called the difference bundle of the elliptic operator E_Ψ. The Chern character extends to virtual bundles by the formula $\mathrm{ch}\,(d(E_\Psi)) := \mathrm{ch}\,(\mathcal{S} \times \mathbb{C}^k) - \mathrm{ch}\,(\mathcal{S} \times \mathbb{C}^k)$, see [Pal65, Ch. 2, § 3].

The surface $\mathcal{S}$ is given the orientation induced by that of $\mathcal{X}$. The topological index of the operator E_Ψ is defined by

$$\operatorname{ind}_{\mathrm{top}}(E_\Psi) = \int_{T^*\mathcal{S}} \mathrm{ch}\,(d(E_\Psi))\, \mathcal{T}(T\mathcal{S}),$$

where $\mathcal{T}(T\mathcal{S})$ is the Todd class of the tangent bundle of $\mathcal{S}$ and the orientation of $T^*\mathcal{S}$ is defined by the differential form $d\xi_1 \wedge dx^1 \wedge \ldots \wedge d\xi_{n-1} \wedge dx^{n-1}$, see [Pal65, Ch. 3, § 5].

Theorem 5.3. *If the symbol $\sigma^0(E_\Psi)$ is invertible away from the zero section of $T^*\mathcal{S}$, then the analytical index of E_Ψ just amounts to its topological index, i.e.* $\operatorname{ind}(E_\Psi) = \operatorname{ind}_{\mathrm{top}}(E_\Psi)$.

Proof. See [Pal65, Ch. 19, § 1]. □

Corollary 5.4. *Suppose that Ψ is an elliptic pseudodifferential operator of order* 0 *on the subspace H. Then the index of the Toeplitz operator T_Ψ on H is evaluated by*

$$\operatorname{ind} T_\Psi = \int_{T^*\mathcal{S}} \mathrm{ch}\,(d(E_\Psi))\, \mathcal{T}(T\mathcal{S}).$$

In [Fed70] Fedosov derived from the formula of [AS68] an expression for the index of an elliptic operator on a compact closed manifold in the form of integral of a differential form. In the case where the Todd class of the manifold is equal to

1, this form is written explicitly through the symbol of the operator. In the general case the differential form is expressed through the symbol of the operator and the curvature tensor of the manifold.

To apply the index formula of [Fed70] to generalised Toeplitz operators, we denote by σ the principal homogeneous symbol of the operator $\Pi\Psi\Pi + (I - \Pi)$ of order 0 in $L^2(\mathcal{S}, \mathbb{C}^k)$. This is a smooth function on $S^*\mathcal{S}$ with values in invertible matrices of type $k \times k$. Consider the differential form ω_{2n-3} of degree $2n-3$ on $S^*\mathcal{S}$ given by

$$\omega_{2n-3} = \frac{1}{(2\pi\imath)^{n-1}} \frac{(n-2)!}{(2n-3)!} \operatorname{tr} (\sigma^{-1} d\sigma)^{2n-3},$$

where $(\sigma^{-1} d\sigma)^{2n-3}$ is evaluated by the usual rules of matrix multiplication but the entries of the matrices are multiplied by means of exterior product, and tr stands for the matrix trace.

Corollary 5.5. *If the Todd class of $T\mathcal{S}$ is equal to* 1*, then, under suitable orientation of $S^*\mathcal{S}$,*

$$\operatorname{ind} T_\Psi = - \int_{S^*\mathcal{S}} \omega_{2n-3}.$$

Since the Pontryagin classes lie actually in cohomology groups whose degree is a multiple of four, we conclude that $\mathcal{T}(T\mathcal{S}) = 1$ provided that $n = 2$. In this case we arrive at the well-known formula for the index of classical Toeplitz operators, namely

$$\begin{aligned} \operatorname{ind} T_\Psi &= - \int_{S^*\mathcal{S}} \frac{1}{2\pi\imath} \operatorname{tr} \sigma^{-1} d\sigma \\ &= - \int_{S^*\mathcal{S}} \frac{1}{2\pi\imath} \frac{d \det \sigma}{\det \sigma}. \end{aligned}$$

In the rest of the paper we clarify the results of [Fed70] in order to prove the formula of Corollary 5.5.

6. Curvature matrix

Let $\mathcal{S}$ be a compact smooth manifold and P an $(N \times N)$-matrix of smooth functions on $\mathcal{S}$, such that $P^2 = P$ and the rank of P is equal to r. Then, P gives rise to a vector bundle of rank r over $\mathcal{S}$, which we denote also by P. Indeed, the matrix P is a projection matrix and so for each $x \in \mathcal{S}$ we get a subspace of vectors $v \in \mathbb{C}^N$ satisfying $P(x)v = v$, which is thought of as the fibre of the bundle P at the point x. The space of the bundle P consists of all pairs (x, v) with $x \in \mathcal{X}$ and $v \in \mathbb{C}^N$, such that $P(x)v = v$.

The matrix $P(x)$ may be real or complex. The bundle P is called real or complex, respectively.

On bordering the matrix P with zeros one can define our bundle P by means of a projector in $\mathbb{C}^{N'}$, where $N' \geq N$. Two vector bundles of rank r are said to be equivalent if the corresponding projectors are stationary homotopic, i.e. they are homotopic in $\mathbb{C}^{N'}$ for N' large enough.

Operations with vector bundles are defined as the corresponding operators with matrices. For instance, the direct sum $P_1 \oplus P_2$ of two vector bundles is the vector bundle with projector

$$\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

Let U be a domain in $\mathcal{S}$. Assume that e is an $(N \times r)$-matrix and g a $(r \times N)$-matrix of smooth functions defined in U, such that

$$\begin{array}{rcl} Pe & = & e, \\ gP & = & g \end{array} \tag{6.1}$$

and $ge = E_r$, the identity $(r \times r)$-matrix. The matrix e is called a frame and g a coframe of the bundle P. From (6.1) it follows that the columns of the matrix $e(x)$ regarded as vectors of $\mathbb{C}^N$ belong to the fibre of P over x. Since the rank of $e(x)$ just amounts to r, these vectors form a basis in P_x. The rows of the matrix $g(x)$ determine a dual basis. It is easy to see that $P = eg$ in U.

If one can define a bundle with the help of frame and coframe over all of $\mathcal{S}$, i.e. if U coincides with $\mathcal{S}$, then the bundle is trivial. In the general case it holds merely locally, i.e., for each point $x \in U$, there is a neighbourhood U over which the bundle admits a frame a coframe. Indeed, in some neighbourhood of the point x one can fix r basis columns in the matrix P and take the matrix of this columns as e. As a coframe one can take the matrix $g = (\tilde{g}e)^{-1}\tilde{g}$, where $\tilde{g}$ is a matrix of basis rows of P.

As $\mathcal{S}$ is compact, there is a finite open covering $\{U_i\}$ of $\mathcal{S}$ with the property that in each U_i the bundle possesses a frame e_i and a coframe g_i. In the intersection of any two neighbourhoods $U_i \cap U_j$ there are invertible $(r \times r)$-matrices $f_{i,j} = g_i e_j$ which are transition matrices from e_i to e_j, i.e. $e_j = e_i f_{i,j}$ and $g_i = f_{i,j} g_j$. Obviously, the equality $f_{i,j} = f_{j,i}^{-1}$ holds. The matrices $f_{i,j}$ are called the transition functions of the bundle.

Conversely, if an open covering $\{U_i\}$ is given and in each set U_i there are matrices of smooth functions e_i and g_i, such that $g_i e_i = E_r$, then these matrices are frames and coframes of some vector bundle over $\mathcal{S}$, provided that in the intersections $U_i \cap U_j$ there are transitions matrices $f_{i,j}$ satisfying $f_{i,j} = f_{j,i}^{-1}$. To see this, we define a projector P_i over U_i by setting $P_i = e_i g_i$. Then, in the intersection $U_i \cap U_j$ we obtain

$$P_i = e_i g_i = e_j f_{j,i} f_{i,j} g_j = e_j g_j = P_j,$$

hence, the local projectors P_i define actually a global projector P over all of $\mathcal{S}$.

If P is an orthogonal projector, i.e. the matrix P is Hermitean, then, given a frame e_i, the coframe g_i is defined uniquely, to wit $g_i = (e_i^* e_i)^{-1} e_i^*$, where $*$ means the Hermitean conjugation. Note that each projector is homotopic to an orthogonal one. To show this, let P be a projector over $\mathcal{S}$ and let e_i and g_i be frames and coframes of P for some open covering $\{U_i\}$. Consider the orthogonal projector P' over $\mathcal{S}$ with frames e_i which is given by $P'_i = e_i(e_i^* e_i)^{-1} e_i^*$. It is easy to check that $PP' = P'$ and $P'P = P$. Hence it follows that, for each t with $0 \leq t \leq 1$, the matrix $P(t) = tP' + (1-t)P$ is a projector. In this way we arrive at a homotopy connecting P and P'.

To introduce a curvature matrix, we will consider matrices whose elements are differential forms on $\mathcal{S}$. If $A = (a_{i,j})$ is a matrix of p-forms and $B = (b_{j,k})$ a matrix of q-forms, then by $AB = (c_{i,k})$ is meant the matrix of $(p+q)$-forms obtained by multiplication of A and B according to the usual rules but the elements $a_{i,j}$ and $b_{j,k}$ are multiplied by exterior product rules, i.e. $c_{i,k}$ just amounts to the sum of $a_{i,j} \wedge b_{j,k}$ over all indices j. In the sequel we omit the symbol of exterior multiplication. By dA we mean the matrix with entries $da_{i,j}$. It is easy to verify that

$d(AB) = dA\,B + (-1)^p A\,dB$, $\mathrm{tr}\,(AB) = (-1)^{pq}\mathrm{tr}\,(BA)$ and $(AB)^* = (-1)^{pq}B^*A^*$, tr being the matrix trace.

Definition 6.1. Let P be a projector over the manifold $\mathcal{S}$. The matrix of 2-forms on $\mathcal{S}$ given by $\Omega = P\,dP\,dP$ is called the curvature matrix of the corresponding vector bundle P.

To clarify the geometric sense of the curvature matrix we construct a connection in the bundle P by means of the projector, i.e. we define the covariant differential of sections of P. The curvature form of this connection will just coincide with our matrix Ω.

Lemma 6.2. *If P is a projector over $\mathcal{S}$, then $P\,dP\,P = 0$. The matrices P and $dP\,dP$ commute.*

Proof. Differentiating the identity $P^2 = P$ yields $P\,dP + dP\,P = dP$. On multiplying this equality by P from the left and from the right we get $P\,dP\,P = 0$. Differentiating this identity we obtain immediately $dP\,dP\,P - P\,dP\,dP = 0$, as desired. □

We now consider an N-column f of differential forms on $\mathcal{S}$, such that $Pf = f$. This is precisely what is meant by the invariance of f relative to P, namely, the value of f at any point $x \in \mathcal{S}$ belongs to the fibre of P over x.

Definition 6.3. The covariant differential of a vector-valued form f invariant relative to P is defined to be $\partial f = P\,df$.

If g is a scalar-valued p-form, then $\partial(gf) = dg\,f + (-1)^p g\,\partial f$, as is easy to check. This equality serves as a basis for the axiomatic definition of connection, see for instance [Wel73]. Another designation for the vector-valued forms f invariant relative to P is sections of the vector bundle $P \otimes \Lambda^p T^*\mathcal{S}$, the space of smooth sections being $\Omega^p(\mathcal{S}, P)$. The connection ∂ maps $\Omega^p(\mathcal{S}, P)$ to $\Omega^{p+1}(\mathcal{S}, P)$ for each $p = 0, 1, \ldots, n-1$.

Consider the second covariant differential $\partial^2 f$. Using the invariance of f we get $df = dP\,f + P\,df$ whence

$$\partial^2 f = P\,dP\,df = P\,dP\,dP\,f + P\,dP\,P\,df = \Omega f,$$

which is due to Lemma 6.2. The last equality just amounts to saying that Ω is the curvature form of the connection ∂.

Our next objective is to establish some properties of the curvature matrix. To this end we extend the definition of the covariant differential to matrix-valued forms invariant relative to P. A matrix-valued form A is called invariant relative to P if $PA = AP = A$. By the covariant differential of A is meant the matrix $\partial A = P\,dA\,P$.

It follows from Lemma 6.3 that the curvature matrix Ω and all the powers of Ω are invariant relative to P. Indeed, since $dP\,dP$ and P commute, one can write Ω in the form $\Omega = P\,dP\,dP = P^2\,dP\,dP = P\,dP\,dP\,P$ which implies immediately that $P\Omega = \Omega P = \Omega$.

Lemma 6.4. *The covariant differential of Ω^k is equal to zero.*

Proof. It suffices to show the Bianchi identity $\partial\Omega = 0$. To to this, write

$$d\Omega = d(P\,dP\,dP\,P) = dP\,\Omega + \Omega\,dP,$$

and so, by Lemma 6.2, we get

$$\partial\Omega = P\,dP\,\Omega\,P + P\,\Omega\,dP\,P = P\,dP\,P\,\Omega + \Omega\,P\,dP\,P = 0.$$

□

Lemma 6.5. *Assume that A be a matrix-valued form invariant with respect to P. Then*

$$d\,\mathrm{tr}\,A = \mathrm{tr}\,\partial A.$$

Proof. Using the equality $P\,dP\,P = 0$, we get

$$\begin{array}{rcl} d\,\mathrm{tr}\,A & = & d\,\mathrm{tr}\,PAP \\ & = & \mathrm{tr}\,dP\,A\,P + \mathrm{tr}\,\partial A + (-1)^p\,\mathrm{tr}\,P\,A\,dP \\ & = & \mathrm{tr}\,P\,dP\,P\,A + \mathrm{tr}\,\partial A + (-1)^p\,\mathrm{tr}\,A\,P\,dP\,P \\ & = & \mathrm{tr}\,\partial A. \end{array}$$

□

Consider scalar-valued $2k$-forms

$$\varphi_{2k} = \mathrm{tr}\left(-\frac{\Omega}{2\pi\imath}\right)^k. \tag{6.2}$$

From Lemmata 6.4 and 6.5 it follows immediately that these differential forms are closed and so they determine some cohomology classes with complex coefficients of the manifold $\mathcal{S}$. Our next lemma shows that these cohomology classes are actually independent of the bundle P and they are uniquely determined by the equivalence class of P.

Lemma 6.6. *If P_0 and P_1 are equivalent bundles then the corresponding forms (6.2) differ by an exact form.*

Proof. Let P_t be the homotopy connecting P_0 and P_1, and let Ω_t stand for the curvature matrix of P_t. Then

$$\mathrm{tr}\,\Omega_1^k - \mathrm{tr}\,\Omega_0^k = \int_0^1 \frac{d}{dt}\,\mathrm{tr}\,\Omega_t^k\,dt.$$

Show that the form $\mathrm{tr}\,\frac{d}{dt}\,\Omega_t^k$ is exact. To this end, we get

$$\mathrm{tr}\,\frac{d}{dt}\,\Omega_t^k = k\,\mathrm{tr}\,\frac{d}{dt}\,\dot{\Omega}_t\,\Omega_t^{k-1}, \tag{6.3}$$

the dot meaning differentiation in t. Furthermore,

$$\begin{array}{rcl} \dot{\Omega}_t & = & \dot{P}_t\,dP_t\,dP_t\,P_t + P_t\left(d\dot{P}_t\,dP_t + dP_t\,d\dot{P}_t\right)P_t + P_t\,dP_t\,dP_t\,\dot{P}_t \\ & = & \dot{P}_t\,\Omega_t + P_t\,d\left(\dot{P}_t\,dP_t - dP_t\,\dot{P}_t\right)P_t + \Omega_t\,\dot{P}_t. \end{array}$$

When substituting this formula into (6.3), we see that the first and the third summands vanish, for

$$\begin{array}{rcl} P_t\,\dot{P}_t\,\Omega_t & = & 0, \\ \Omega_t\,\dot{P}_t\,P_t & = & 0, \end{array}$$

as follows from $P_t\,\Omega_t = \Omega_t$ and $\Omega_t\,P_t = \Omega_t$ by differentiating the equalities in t, respectively. It follows that

$$\mathrm{tr}\,\frac{d}{dt}\,\Omega_t^k = k\,\mathrm{tr}\,P_t\,d\left(\dot{P}_t\,dP_t - dP_t\,\dot{P}_t\right)P_t\,\Omega_t^{k-1}.$$

It is easily verified that P_t commute with $\dot{P}_t\, dP_t$ and $dP_t\, \dot{P}_t$, whence

$$\begin{array}{rcl} P_t\, d\Big(\dot{P}_t\, dP_t - dP_t\, \dot{P}_t\Big)\, P_t & = & P_t\, d\Big(P_t\Big(\dot{P}_t\, dP_t - dP_t\, \dot{P}_t\Big)\, P_t\Big)\, P_t \\ & = & \partial_t\Big(P_t\Big(\dot{P}_t\, dP_t - dP_t\, \dot{P}_t\Big)\, P_t\Big). \end{array}$$

Finally, since $\partial_t \Omega_t = 0$, we conclude that

$$\begin{array}{rcl} \dfrac{d}{dt}\operatorname{tr}\Omega_t^k & = & k \operatorname{tr}\partial_t\Big(P_t\Big(\dot{P}_t\, dP_t - dP_t\, \dot{P}_t\Big)\, P_t\Big)\, \Omega_t^{k-1} \\ & = & k \operatorname{tr}\partial_t\Big(P_t\Big(\dot{P}_t\, dP_t - dP_t\, \dot{P}_t\Big)\, P_t \Omega_t^{k-1}\Big) \\ & = & k\, d \operatorname{tr} P_t\Big(\dot{P}_t\, dP_t - dP_t\, \dot{P}_t\Big)\, \Omega_t^{k-1}, \end{array}$$

the last equality being due to Lemma 6.5. □

From Lemma 6.6 it follows, in particular, that forms (6.2) define real cohomology classes of $\mathcal{S}$, for each class of equivalent bundles contains an orthogonal projector, for which the curvature matrix Ω is Hermitean.

To complete this section we write down local expressions for the curvature matrix and forms (6.2). Let e_i and g_i be frames and coframes of P for some open covering $\{U_i\}$. Then we get $P = e_i g_i$ in U_i whence

$$\Omega = P\, dP\, dP = e_i\left(dg_i\, de_i + (g_i\, de_i)^2\right) g_i.$$

The 2-form $\Omega_i = d(g_i\, de_i) + (g_i\, de_i)^2$ with values in $(r \times r)$-matrices is said to be a local curvature matrix in the given frame and coframe. The forms (6.2) are expressed through the matrix Ω_i in just the same way as through the matrix Ω. To wit,

$$\varphi_{2k} = \operatorname{tr}\Big(-\frac{\Omega_i}{2\pi\imath}\Big)^k = \Big(-\frac{1}{2\pi\imath}\Big)^k \operatorname{tr}\left(d(g_i\, de_i) + (g_i\, de_i)^2\right)^k, \qquad (6.4)$$

as is easy to check.

7. Characteristic classes

Suppose

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

is an analytic function or formal power series in z, and P a vector bundle of rank r with curvature form $\Omega = P\, dP\, dP$. We substitute the matrix $-\Omega/2\pi\imath$ into the series $f(z)$. On defining

$$(-\Omega/2\pi\imath)^0 := P$$

we obtain

$$\begin{array}{rcl} f\Big(-\dfrac{\Omega}{2\pi\imath}\Big) & = & c_0 P + \displaystyle\sum_{k=1}^{\infty} c_k \Big(-\frac{\Omega}{2\pi\imath}\Big)^k \\ & = & P\Big(c_0 E_N + \displaystyle\sum_{k=1}^{\infty} c_k \Big(-\frac{\Omega}{2\pi\imath}\Big)^k\Big). \end{array}$$

Since the entries of the matrix Ω are 2-forms on $\mathcal{S}$, the series is in fact a polynomial of the matrix $-\Omega/2\pi\imath$ whose degree does not exceed the half of the dimension of $\mathcal{S}$. Set

$$\begin{aligned} f_+(P) &= \operatorname{tr} f\Big(-\frac{\Omega}{2\pi\imath}\Big), \\ f_\times(P) &= \operatorname{tr} \Lambda^r f\Big(-\frac{\Omega}{2\pi\imath}\Big), \end{aligned} \tag{7.1}$$

where $\operatorname{tr} \Lambda^r A$ is the sum of all diagonal minor determinants of r-th order of a matrix A, cf. [Fed95, 1.2.2]. The equalities of (7.1) define inhomogeneous differential forms of even degree on $\mathcal{S}$.

Note that $f_\times(P)$ is a polynomial of (6.2). Indeed, $f_\times(P)$ is a symmetric function of the "eigenvalues" of matrix Ω, hence, one can represent $f_\times(P)$ as a polynomial of the elementary symmetric functions of "eigenvalues." These latter are in turn expressed through the forms (6.2) which can be thought of as the sums of the r-th powers of "eigenvalues."

We thus conclude that forms (7.1) are closed and their cohomology classes depend merely on the equivalence class of the bundle P. The cohomology classes of the forms $f_+(P)$ and $f_\times(P)$ are called the additive and multiplicative characteristic classes of the bundle P which correspond to the function $f(z)$, respectively. One can obtain the expressions of forms (7.1) through the local curvature matrices Ω_i. The only difference is that Ω_i^0 is defined to be equal to E_r and in the expression for $f_\times(P)$ the sum of the diagonal minor determinants of r-th order is replaced by the determinant, i.e.

$$f_\times(P) = \det f\Big(-\frac{\Omega_i}{2\pi\imath}\Big).$$

The following properties of characteristic classes are obvious:

$$\begin{aligned} (f_1+f_2)_+(P) &= f_{1+}(P) + f_{2+}(P), & (f_1 f_2)_\times(P) &= f_{1\times}(P) f_{2\times}(P), \\ f_+(P_1 \oplus P_2) &= f_+(P_1) + f_+(P_2); & f_\times(P_1 \oplus P_2) &= f_\times(P_1) f_\times(P_2). \end{aligned}$$

We next show several examples of characteristic classes which we will need in the sequel.

Example 7.1. The Chern class $C(P)$ is the multiplicative class corresponding to the function $1+z$, i.e.

$$C(P) = \operatorname{tr} \Lambda^r \Big(P - \frac{\Omega}{2\pi\imath}\Big) = 1 + \sum_{k=1}^{r} \operatorname{tr} \Lambda^k \Big(-\frac{\Omega}{2\pi\imath}\Big).$$

The homogeneous component c_k of this form equal to the sum of the diagonal minor determinants of k-th order of the matrix $-\Omega/2\pi\imath$ defines a cohomology class called the k-th Chern class. It should be noted that each characteristic class of the bundle P can be expressed through the classes c_k, for these latter can be regarded as elementary symmetric functions of the "eigenvalues" of the curvature matrix Ω of P.

Example 7.2. The Chern character $\operatorname{ch} P$ is the additive class corresponding to the function $\exp z$, i.e.

$$\operatorname{ch} P = \operatorname{tr} \exp\Big(-\frac{\Omega}{2\pi\imath}\Big) = r + \sum_{k=1}^{\infty} \frac{1}{k!}\, \varphi_{2k}.$$

Besides of the properties of additive characteristic classes, the Chern character has the property $\operatorname{ch}(P_1 \otimes P_2) = \operatorname{ch} P_1 \operatorname{ch} P_2$, which is an immediate consequence of the fact that the curvature form of the bundle $P_1 \otimes P_2$ can be expressed in the form $\Omega_1 \otimes P_2 + P_1 \otimes \Omega_2$.

Example 7.3. The Todd class $\mathcal{T}(P)$ is the multiplicative class corresponding to the function $z/(1-\exp(-z))$, i.e.

$$\mathcal{T}(P) = \operatorname{tr} \Lambda^r \Big(-\frac{\Omega}{2\pi\imath} \Big) \Big(P - \exp \frac{\Omega}{2\pi\imath} \Big)^{-1}.$$

If the projection matrix P has real entries, the constructions above bear certain peculiarities. Without restriction of generality we can assume that P is symmetric. Then the curvature matrix Ω of the bundle P is skew-symmetric, and so its diagonal minor determinants of odd order vanish. Thus, all the Chern classes c_{2j+1} are zero while the class $(-1)^j c_{2j}$ is said to be the j-th Pontryagin class of the bundle P. Since each characteristic class can be expressed through the Chern classes, we see that the characteristic classes of a real bundle can be expressed through the Pontryagin classes and, therefore, they are defined by differential forms whose degree is a multiple of four.

Let us dwell on the tangent bundles of Riemannian manifolds. The projector P which defines the tangent bundle of a manifold $\mathcal{S}$ is constructed in the following way. Pick an embedding f of $\mathcal{S}$ into a Euclidean space $\mathbb{R}^N$. Given any point $x \in \mathcal{S}$, the differential $f'(x)$ maps the tangent space $T_x\mathcal{S}$ at x onto some subspace of $\mathbb{R}^N$, and so we have defined an orthogonal projector $P(x)$ onto this subspace. The characteristic classes of the bundle P are now defined in the usual way. Notice that these classes are independent on the particular choice of the embedding P. Indeed, if g is an other embedding of $\mathcal{S}$ into $\mathbb{R}^M$, then f and g, if regarded as embeddings of $\mathcal{S}$ into the direct sum $\mathbb{R}^N \oplus \mathbb{R}^M$, are homotopic. Hence it follows that the corresponding projectors are stationary homotopic, from which the invariance of the characteristic classes follows. As usual, the characteristic classes of the tangent bundle $T\mathcal{S}$ obtained in this way are called the characteristic classes of the manifold $\mathcal{S}$

Let $x^1, \ldots, x^{n-1}$ be local coordinates in $\mathcal{S}$, and $R^i_{j,k,l}$ the curvature tensor of the Riemannian metric induced by the embedding of $\mathcal{S}$ into $\mathbb{R}^N$. In the given local chart we define the projector $P(x)$ by choosing the images of the tangent vectors $\partial/\partial x^1, \ldots, \partial/\partial x^{n-1}$ as a frame. From the definition of the curvature tensor it follows that in this frame the entries Ω^i_j of the local curvature matrix Ω take the form

$$\Omega^i_j = \frac{1}{2} R^i_{j,k,l} dx^k dx^l,$$

see [Fed95].

Thus, the differential forms defining the characteristic classes of the manifold $\mathcal{S}$ are expressed through the curvature tensor. In particular, the Todd class of the manifold $\mathcal{S}$ is given by the formula

$$\mathcal{T}(T\mathcal{S}) = \det \Big(-\frac{\Omega}{2\pi\imath} \Big) \Big(E_{n-1} - \exp \frac{\Omega}{2\pi\imath} \Big)^{-1}, \tag{7.2}$$

where Ω is the matrix with entries Ω^i_j. Since the tangent bundle is real, the homogeneous components of the Todd class are differential forms whose degree is a multiple of four, i.e.

$$\mathcal{T}(T\mathcal{S}) = 1 + \sum_{j=1}^{\left[\frac{n-1}{4}\right]} \mathcal{T}_{4j},$$

where $\mathcal{T}_{4j}$ are forms of degree $4j$.

8. Fedosov index formula

Denote by $B^*\mathcal{S}$ the subbundle of the cotangent bundle $T^*\mathcal{S}$ whose fibre over $x \in \mathcal{S}$ is the unit ball in $T_x^*\mathcal{S}$. The boundary of $B^*\mathcal{S}$ in $T^*\mathcal{S}$ is the cosphere bundle $S^*\mathcal{S}$.

Assume that σ is a smooth function on $S^*\mathcal{S}$ with values in invertible matrices of type $N \times N$. In the Atiyah-Singer theorem the so-called difference construction is of great importance. When applied to our case, it assigns to the matrix σ a vector bundle P of rank N over $T^*\mathcal{S}$ with transition function σ and compact support. The compactness of the support of P means that P is constant away from a compact subset of $T^*\mathcal{S}$. We now construct the bundle P explicitly and evaluate its character. Strictly speaking, in the construction of Atiyah-Singer P is a virtual bundle, i.e., an element of the group $K(T^*\mathcal{S})$, see [Ati65]. To not introduce the group $K(T^*\mathcal{S})$, we deviate from the standard definition of the difference construction, see [Pal65], [Ati65], etc.

Given a point $x \in \mathcal{S}$, we identify any vector $\xi \in T_x^*\mathcal{S}$ with the pair (ω, r), where $\omega = \xi/|\xi|$ and $r = |\xi|$. The pairs (x, ξ) with zero vector $\xi \in T_x^*\mathcal{S}$ define the zero section of the bundle $T^*\mathcal{S}$. Note that at these points the coordinate ω is determined not uniquely while $r = 0$. We also endow each fibre $T_x^*\mathcal{S}$ with a point at infinity which we denote by (ω, ∞). Once again the coordinate ω is not uniquely determined for such points. We write $\hat{T}^*\mathcal{S}$ for the bundle $T^*\mathcal{S}$ whose fibres are completed by points at infinity. The pairs (x, ξ) with $\xi = (\omega, \infty)$ form the so-called infinite section of $\hat{T}^*\mathcal{S}$.

Consider the covering of $\hat{T}^*\mathcal{S}$ which is formed by two open sets U_0 and U_∞. The set U_0 consists of all points of $\hat{T}^*\mathcal{S}$ except for the points at infinity, i.e. U_0 coincides with $T^*\mathcal{S}$. The set U_∞ consists of all points of $\hat{T}^*\mathcal{S}$ except for the zero section. With the help of frames and coframes over $\hat{T}^*\mathcal{S}$ we now define a vector bundle of rank N over $\hat{T}^*\mathcal{S}$, such that the transition matrix for these frames coincides with σ on $S^*\mathcal{S}$.

Let $f_1(r)$ and $f_2(r)$ be C^∞ functions with real values, such that $f_1^2 + f_2^2 = 1$ and $f_1(r) = 0$ for $r < 1/2$ and $f_1(r) = 1$ for $r > 1$. Define matrices e_0 and g_0 on U_0, setting

$$e_0(x, (\omega, r)) = \begin{pmatrix} f_2(r) E_N \\ f_1(r)\sigma(x, \omega) \end{pmatrix}, \quad g_0(x, (\omega, r)) = \begin{pmatrix} f_2(r) E_N & f_1(r)\sigma^{-1}(x, \omega) \end{pmatrix},$$

where E_N is the identity matrix of type $N \times N$. Analogously we introduce matrices e_∞ and g_∞ on U_∞, to wit

$$e_\infty(x, (\omega, r)) = \begin{pmatrix} f_2(r)\sigma^{-1}(x, \omega) \\ f_1(r) E_N \end{pmatrix}, \quad g_\infty(x, (\omega, r)) = \begin{pmatrix} f_2(r)\sigma(x, \omega) & f_1(r) E_N \end{pmatrix}.$$

It is easily seen that

$$\begin{aligned} g_0 e_0 &= E_N, \\ g_\infty e_\infty &= E_N. \end{aligned}$$

In the intersection $U_0 \cap U_\infty$, i.e., for $0 < r < \infty$, we get

$$\begin{array}{rcl} e_0 & = & e_\infty \sigma, \\ g_0 & = & \sigma^{-1} g_\infty, \end{array}$$

hence it follows that e_0, e_∞ and g_0, g_∞ are local frames and coframes of some vector bundle over $\hat{T}^* \mathcal{S}$. The corresponding projector looks like

$$P(x,(\omega,r)) = e_0 g_0 = e_\infty g_\infty = \begin{pmatrix} (f_2(r))^2 E_N & f_1(r) f_2(r) \sigma^{-1}(x,\omega) \\ f_1(r) f_2(r) \sigma(x,\omega) & (f_1(r))^2 E_N \end{pmatrix}.$$

We now evaluate the Chern character of the bundle P. By formula (6.4), we obtain

$$\varphi_{2k} = \mathrm{tr} \Big(-\frac{\Omega}{2\pi\imath} \Big)^k = \Big(-\frac{1}{2\pi\imath} \Big)^k \mathrm{tr} \left(d(g_0\, de_0) + (g_0\, de_0)^2 \right)^k.$$

Furthermore, $g_0\, de_0 = f_1^2\, \sigma^{-1} d\sigma$, and so

$$d(g_0\, de_0) + (g_0\, de_0)^2 = -f_1^2 f_2^2\, (\sigma^{-1} d\sigma)^2 + df_1^2\, \sigma^{-1} d\sigma \qquad (8.1)$$

where we have used the fact that $d\sigma^{-1} = -\sigma^{-1} d\sigma \sigma^{-1}$. On substituting (8.1) into the formula for φ_{2k} and evaluating the k-th power we observe that all summands that contain at least two factors df_1^2 vanish, for $df_1^2 \wedge df_1^2 = 0$. Hence it follows immediately that

$$\varphi_{2k} = \Big(\frac{1}{2\pi\imath} \Big)^k \left((f_1 f_2)^{2k} \mathrm{tr}\, (\sigma^{-1} d\sigma)^{2k} - k\, (f_1 f_2)^{2k-2} df_1^2\, \mathrm{tr}\, (\sigma^{-1} d\sigma)^{2k-1} \right).$$

If $k \geq 1$, then the differential form $\mathrm{tr}\, (\sigma^{-1} d\sigma)^{2k}$ vanishes, for

$$\mathrm{tr}\, (\sigma^{-1} d\sigma)^{2k} = \mathrm{tr}\, (\sigma^{-1} d\sigma)^{2k-1} (\sigma^{-1} d\sigma) = -\mathrm{tr}\, (\sigma^{-1} d\sigma)(\sigma^{-1} d\sigma)^{2k-1} = 0.$$

Furthermore, the differential form $\mathrm{tr}\, (\sigma^{-1} d\sigma)^{2k-1}$ is closed. Indeed, using the identity $d\sigma^{-1} = -\sigma^{-1} d\sigma \sigma^{-1}$ we get

$$d\, \mathrm{tr}\, (\sigma^{-1} d\sigma)^{2k-1} = -(2k-1)\, \mathrm{tr}\, (\sigma^{-1} d\sigma)^{2k} = 0,$$

as desired. Therefore, we can write φ_{2k} in the form

$$\varphi_{2k} = -\frac{k}{(2\pi\imath)^k}\, d \left(g_k\, \mathrm{tr}\, (\sigma^{-1} d\sigma)^{2k-1} \right)$$

where

$$g_k(r) = \int_0^r (f_1^2 f_2^2)^{k-1}\, df_1^2.$$

For $r > 1$ we get

$$g_k(r) = g_k(1) = \int_0^1 (f_1^2 f_2^2)^{k-1}\, df_1^2 = \int_0^1 (s(1-s))^{k-1}\, ds = B(k,k) = \frac{((k-1)!)^2}{(2k-1)!}.$$

Setting $h_k(r) = \dfrac{g_k(r)}{g_k(1)}$ yields finally

$$\frac{1}{k!}\, \varphi_{2k} = -\frac{1}{(2\pi\imath)^k}\, \frac{(k-1)!}{(2k-1)!} d \left(h_k\, \mathrm{tr}\, (\sigma^{-1} d\sigma)^{2k-1} \right). \qquad (8.2)$$

Form (8.2) has compact support, since $h_k(r) = 1$ for all $r > 1$, and so $\varphi_{2k} = 0$. In place of the function $h_k(r)$ we can take any function $h(r)$ which vanishes in a neighbourhood of $r = 0$ and is equal to 1 for $r > 1$. This will not affect the cohomology class of φ_{2k}.

For each $k = 1, 2, \ldots$, we introduce the closed differential form

$$\omega_{2k-1} = \frac{1}{(2\pi\imath)^k} \frac{(k-1)!}{(2k-1)!} \operatorname{tr} (\sigma^{-1} d\sigma)^{2k-1}$$

on the cosphere bundle $S^*\mathcal{S}$. By (8.2), the formula for the Chern character takes the form

$$\operatorname{ch} P = N - d\Big(h \sum_{k=1}^{n-1} \omega_{2k-1}\Big). \tag{8.3}$$

Since the projector P is constant in a neighbourhood of the zero and infinite sections of $\hat{T}^*\mathcal{S}$, one can assume that the bundle P is defined over a suspension over $S^*\mathcal{S}$. We have nowhere used the fact that $S^*\mathcal{S}$ is a bundle whose fibres are spheres. Hence, everything of what has been said is still valid if one replaces $S^*\mathcal{S}$ by an arbitrary manifold and $\hat{T}^*\mathcal{S}$ by its suspension. In particular, let σ be a mapping of the sphere $\mathbb{S}^{2n-3}$ to the group of invertible matrices. It defines a bundle P over the sphere $\mathbb{S}^{2n-2}$ which is a suspension over $\mathbb{S}^{2n-3}$. From the Bott periodicity theorem it follows that the only stationary invariant of the mapping P is the Chern character of P on $\mathbb{S}^{2n-2}$, see for instance [Pal65], [Fed95], etc. On integrating $d\,(h\omega_{2n-3})$ over $\mathbb{S}^{2n-2}$ and using the Stokes formula we deduce that this invariant just amounts to

$$\int_{\mathbb{S}^{2n-3}} \frac{1}{(2\pi\imath)^{n-1}} \frac{(n-2)!}{(2n-3)!} \operatorname{tr} (\sigma^{-1} d\sigma)^{2n-3}$$

up to the sign.

We are now in a position to rewrite the Atiyah-Singer formula in terms of differential forms à la Fedosov [Fed70].

The manifold $T^*\mathcal{S}$ possesses a canonical orientation, namely, if $x^1, \ldots, x^{n-1}$ are local coordinates in $\mathcal{S}$ and $\xi_1, \ldots, \xi_{n-1}$ are the corresponding coordinates in the fibres $T^*_x\mathcal{S}$, then the orientation of $T^*\mathcal{S}$ is given by the coordinate sequence $\xi_1, x^1, \ldots, \xi_{n-1}, x^{n-1}$.

Let E_Ψ be an elliptic pseudodifferential operator in $L^2(\mathcal{S}, \mathbb{C}^k)$. Its principal homogeneous symbol σ restricted to $S^*\mathcal{S}$ is an invertible matrix-valued function which determines a vector bundle P over $T^*\mathcal{S}$ with compact support. We introduce an inhomogeneous differential form $\operatorname{ch}\sigma$ with compact support on $T^*\mathcal{S}$ by the formula $\operatorname{ch}\sigma := \operatorname{ch} P - N$, where N is the rank of the bundle P. In (7.2) we defined the Todd class $\mathcal{T}(T\mathcal{S})$ of $\mathcal{S}$. Here we think of $\mathcal{T}(T\mathcal{S})$ as differential form on $T^*\mathcal{S}$ which does not depend on ξ.

The Atiyah-Singer formula states that the index of E_Ψ just amounts to the value of the homogeneous component of degree $2n-2$ of the product $\operatorname{ch}\sigma\,\mathcal{T}(T\mathcal{S})$ on the main cycle $T^*\mathcal{S}$, i.e.,

$$\operatorname{ind} E_\Psi = \int_{T^*\mathcal{S}} (\operatorname{ch}\sigma\,\mathcal{T}(T\mathcal{S}))_{2n-2}\,.$$

From (8.3) it follows that

$$\operatorname{ch}\sigma = -d\Big(h \sum_{k=1}^{n-1} \omega_{2k-1}\Big),$$

which yields

$$(\mathrm{ch}\,\sigma\,\mathcal{T}\,(T\mathcal{S}))_{2n-2} = -d\left(h\,\omega_{2n-3} + h\sum_{j=1}^{\left[\frac{n-1}{4}\right]}\omega_{2n-3-4j}\,\mathcal{T}_{4j}\right).$$

Notice that the differential form $(\mathrm{ch}\,\sigma\,\mathcal{T}\,(T\mathcal{S}))_{2n-2}$ vanishes for $r > 1$. Hence, in the equality for $\mathrm{ind}\,E_\Psi$ we can replace the integral over $T^*\mathcal{S}$ by the integral over the coball bundle $B^*\mathcal{S}$, i.e. over $r < 1$. This gives

$$\mathrm{ind}\,E_\Psi = -\int_{B^*\mathcal{S}} d\left(h\,\omega_{2n-3} + h\sum_{j=1}^{\left[\frac{n-1}{4}\right]}\omega_{2n-3-4j}\,\mathcal{T}_{4j}\right).$$

On applying the Stokes formula to the latter integral we arrive at the explicit formula

$$\mathrm{ind}\,E_\Psi = -\int_{S^*\mathcal{S}} \left(\omega_{2n-3} + \sum_{j=1}^{\left[\frac{n-1}{4}\right]}\omega_{2n-3-4j}\,\mathcal{T}_{4j}\right), \tag{8.4}$$

for $h(1) = 1$.

Remark 8.1. In the case where all $\mathcal{T}_k$ with $k > 0$ are exact, formula (8.4) takes the simplest form mentioned in Section 5. Namely,

$$\mathrm{ind}\,E_\Psi = -\int_{S^*\mathcal{S}} \omega_{2n-3}.$$

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(Dmitry Fedchenko) Institute of Mathematics, Siberian Federal University, Svobodny Prospekt 79, 660041 Krasnoyarsk, Russia
E-mail address: dfedchenk@gmail.com

(Nikolai Tarkhanov) Institute of Mathematics, University of Potsdam, Am Neuen Palais 10, 14469 Potsdam, Germany
E-mail address: tarkhanov@math.uni-potsdam.de