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Elliptic problems with small parameter

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Frau M.Sc. Evgeniya Dyachenko
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Hauptbetreuer: Prof. Dr. Nikolai Tarkhanov

Gutachter: Prof. Dr. Vladimir Rabinovitch

Gutachter: Prof. Dr. Elmar Schrohe

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Abstract

In this thesis we consider diverse aspects of existence and correctness of asymptotic solutions to elliptic differential and pseudodifferential equations. We begin our studies with the case of a general elliptic boundary value problem in partial derivatives. A small parameter enters the coefficients of the main equation as well as into the boundary conditions. Such equations have already been investigated satisfactory, but there still exist certain theoretical deficiencies. Our aim is to present the general theory of elliptic problems with a small parameter. For this purpose we examine in detail the case of a bounded domain with a smooth boundary. First of all, we construct formal solutions as power series in the small parameter. Then we examine their asymptotic properties. It suffices to carry out sharp two-sided *a priori* estimates for the operators of boundary value problems which are uniform in the small parameter. Such estimates are not obtained in functional spaces used in classical elliptic theory. To circumvent this limitation we exploit norms depending on the small parameter for the functions defined on a bounded domain. Similar norms are widely used in literature, but their properties have not been investigated extensively. Our theoretical investigation shows that the usual elliptic technique can be correctly carried out in these norms. The obtained results also allow one to extend the norms to compact manifolds with boundaries. We complete our investigation by formulating algebraic conditions on the operators and showing their equivalence to the existence of *a priori* estimates.

In the second step, we extend the concept of ellipticity with a small parameter to more general classes of operators. Firstly, we want to compare the difference in asymptotic patterns between the obtained series and expansions for similar differential problems. Therefore we investigate the heat equation in a bounded domain with a small parameter near the time derivative. In this case the characteristics touch the boundary at a finite number of points. It is known that the solutions are not regular in a neighbourhood of such points in advance. We suppose moreover that the boundary at such points can be non-smooth but have cuspidal singularities. We find a formal asymptotic expansion and show that when a set of parameters comes through a threshold value, the expansions fail to be asymptotic.

The last part of the work is devoted to general concept of ellipticity with a small parameter. Several theoretical extensions to pseudodifferential operators have already been suggested in previous studies. As a new contribution we involve the analysis on manifolds with edge singularities which allows us to consider wider classes of perturbed elliptic operators. We examine that introduced classes possess *a priori* estimates of elliptic type. As a further application we demonstrate how developed tools can be used to reduce singularly perturbed problems to regular ones.

Keywords: Elliptic problem, Small parameter, Boundary layer

Contents

Introduction

A small parameter occurs regularly in various branches of mathematics and always plays a key role in Analysis. Although this study is devoted to differential equations, the statement of a problem with small parameters has much in common with a great many diverse fields of study. So, if the statement of a mathematical problem contains a set of parameters, we suppose that one of them, ε, is a non-negative scalar variable and the value $\varepsilon = 0$ can be critical. Roughly speaking, this means that some essential properties of the problem change when ε vanishes. For example, the equation $y^2 + 2\varepsilon y + x^2 = 0$ defines the unique function $y(x) = -\varepsilon + \sqrt{\varepsilon^2 - x^2}$ in the interval $(-\varepsilon, \varepsilon)$ until the parameter ε is nonzero. Otherwise the function $y(x)$ is determined non uniquely, therefore the value $\varepsilon = 0$ is critical.

More specifically, this work is devoted to *linear* partial differential and pseudodifferential equations (*i.e.* PDE and ΨDE, respectively) involving small parameters. We consider their solvability and the asymptotic behaviour of the solutions. Our main attention is given to *elliptic* operators that lose their order at the critical point, since finding solutions to the corresponding equations causes certain difficulties.

The elliptic (as well as parabolic and hyperbolic) partial differential equations for one unknown function has become a mature mathematical subject with a fairly long history. The corresponding classification for PDE of the second order is based on properties of the matrices of the coefficients near to the highest derivatives. The characteristic matrices of elliptic equations have only positive real eigenvalues. Their solutions are typically regular and have *a priori* estimates. The ellipticity for operators of higher orders is defined as the invertibility of the principle symbols (see for definitions on page 24). The most characteristic properties of the second order operators are generalised to higher order (for an introduction to the basic theory of elliptic PDE see [LU68, Mir70, Hoe63]).

For linear systems the notion of ellipticity is a finer matter, there exist several nonequivalent definitions. The class of strongly elliptic systems possesses the properties closest to elliptic equations with one unknown function. It consists of systems, whose characteristic matrices are uniformly positive definite. Most features of elliptic equations with one unknown function allow generalisation to strongly elliptic systems.

The statement of general boundary elliptic problems caused trouble for some time. There are basic examples (see [Sche59]) of solvable problems with non-unique solutions. Therefore, the set of invertible "up to a finite-dimensional space" operators (*i.e.* when an operator A and its adjoint A^* have finite-dimensional kernels, $\dim \operatorname{Ker} A < \infty$, $\dim \operatorname{Ker} A^* < \infty$) could be treated as a proper class. Such operators are called Fredholm. It is shown in [Sche59, Bro59, ADN59] that elliptic equations stated for domains with boundaries have the above-mentioned solvability if the corresponding operators of boundary value problem satisfy certain *a priori* estimates in the Sobolev norms or estimates of Schauder type. These estimates in the general form first appeared in the paper [ADN59].

As a complement to the *a priori* estimates, there also exists a condition for the Fredholm property in the algebraic form. It is known as the *Shapiro-Lopatinskii condition* and was first introduced in [Sha53, Lop53].

In the theory of Fredholm operators the so called *local principle* helps to reduce boundary value problems to systems with constant coefficients. It is based on that the Fredholm property of differential operators with coefficients taken at a fixed point $x \in \overline{\Omega}$ (such property is called sometimes local solvability at x) implies the Fredholm property for equations with coefficients "slightly" different in a neighbourhood of x. As a result, the Fredholm property for operators on a bounded domain Ω may be proved using a finite covering for Ω and elliptic estimates for the equations with "frozen" coefficients. The relation between the local solvability and the Fredholm property has been used for quite a long time, but firstly was shaped in the theorems by Simonenko [Sim65].

Elliptic problems involving small parameters attract great attention amongst mathematicians. It is enough to say that the asymptotic behaviour of the solution to $\varepsilon\Delta\rho = \rho$ was investigated by Riemann [Rie58]. Parameters in differential equations frequently originate from physical quantities. Some physical constants tend to be small or large affecting coefficients in corresponding differential equations. An analytical investigation of such problems is of special value because it is often extremely difficult to solve them numerically.

The theory of differential equations has many places where the presence of a parameter plays a role. For example: spectral analysis, numerical modelling, bifurcation theory, index theorems and so on. This study mainly discusses the matters of convergence and of *regular degeneration*. Let us explain what is exactly meant here.

The presence of a small parameter ε is helpful in obtaining a solution for an equation $A_\varepsilon u = f$ with some operator A_ε. Assuming that ε is variable, the solution is looked upon as power expansion

$$u_\varepsilon(x) = \sum_{i=0}^{\infty} u_i(x)\varepsilon^i. \tag{1}$$

This approach is known as the *asymptotic method*, which proves itself so effectively that if an equation does not explicitly involve small parameters, then often the parameter is introduced artificially. The coefficients u_i are determined as a result of standard procedure of substituting in the equation. The operator in the left hand side is also expanded in the power series $A_\varepsilon = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \dots$. The first approximation u_0 is a solution of the equation $A_0 u_0 = f$. If the resulting series converges it is the true solution for small ε. Otherwise it is desired to be asymptotic at least, *i.e.* to approximate the solution in the sense

$$\left\| u_\varepsilon - \sum_{i=0}^{N} u_i \varepsilon^i \right\| = o(\varepsilon^N),$$

for all natural N. In this context the question of a proper norm naturally arises. The second point is necessary and sufficient conditions on the operator A_ε implying the existence of asymptotic expansions for the solutions. They are required because not every analytic A_ε is analytically invertible with respect to ε.

The situation when the method of small parameter gives true asymptotic expansions for the solutions is called *regular degeneration*. But sometimes, the equation $A_0 u_0 = f$ is not correctly solvable. In such cases, problems are called *singularly perturbed*. In Physics, this usually corresponds to that reaching the critical point $\varepsilon = 0$ by a system implies a dramatic change of its physical properties (if the model remains applicable).

The first ideas of asymptotic series go back to Laplace, Euler, Legendre; but the rigorous theory begins with the works of Poincaré [Poi86] in 1886, where he introduced the notion of asymptotic expansion and valid operations on it. Significant observations were also made by Stokes, Birkhoff, Horn, Wentzel, Kramers, Brillouin. The subject and historical background are explained extensively in [Fri55, Cop65]. The asymptotic methods do not only assist mathematical manipulations, sometimes they help to deduce new equations for Physics. The formal asymptotic solutions bring up other interesting questions. For example, one can consider their limits $\lim_{\varepsilon\to 0} u_\varepsilon$ in comparison with the solutions of the reduced operator $A_0 u = f$. Common sense guides us to seek a convergence under the strongest possible restrictions on the perturbed operator. But even very "good" assumptions on regularity do not definitely answer the question. Simple differential equations already show non-regular behaviour by degeneration. However, the physical evidence motivates one to seek a connection between degenerate and perturbed solutions at least for some "good" class of regular operators. Of course, the convergence may just be postulated by the usage of a special topology, but it is desirable to find spaces similar to those which are commonly used.

As was mentioned above sometimes the formal asymptotic series for differential equations can not be constructed directly. It is hardly observable on compact manifolds without boundary or singularities, but frequently appears in boundary value problems. This phenomenon is known as *boundary layer effect*. Prandtl [Pra05] was the first to decisively outline a deficiency of the classical asymptotic expansion near the boundary in 1904. He solved the problem of the flow around the body in low-viscous fluid and answered the question why viscous forces appear near the body's surface and do not significantly affect the fluid which is far from the body. The study was based on two suggestions. First of all, he divided the domain into two parts: a small strip near the boundary and the inner area. For each of the parts the problem was stated separately. The second finding of Prandtl is concerned with the variable rescaling near the boundary, which allowed him to derive an equation for remaining part of the solution. The function inside the domain is called the *outer expansion* and the extra function near the boundary is called the *inner expansion*.

The resulting equations were simply solvable, but the method was based on physical intuition rather then formal rigor. Although the proposed trick was very fruitful, its mathematical meaning remained unclear for some time. The actual establishment of boundary layer theory dates back to 1930 (see [And05] for the accurate historical background).

The boundary layer technique was later revised and used very intensively. It was successfully applied to a wide class of hydro- and aerodynamic problems, most notably in studies of shock waves, free stream flows, supersonic movement and so on. Since the problems of boundary layer and small parameters are tightly connected to turbulence effects and the theory of stability, it is worth mentioning that there is also a contribution by Reinolds, Rayleigh, Lyapunov, Birkhoff, whose discoveries influenced the topic (for the details see [Schl04]).

The series of mathematical works of the 1950s turned the ideas of Prandtl into a powerful tool. One method of matching inner and outer expansion was initiated by van Dyke, Kaplun, Cole, Lagerstrom and others. In spite of having significant implementations, primarily in hydrodynamics, the matching principle has been criticized for its formalism. The main framework of this technique is performed in the book [Dyke64], for additional applications and bibliography see [Cole96, Nay73, Schl04, Schw02].

A similar method was developed by Vishik and Lyusternik and presented in their paper [VL57]. As in the technique of matched asymptotic expansions, the core of the method also lies in splitting the problem into two parts and stretching variables. But at the same time, Vishik and Lyusternik proposed an explicit algorithm on how to construct boundary corrections. In a great variety of cases it provides an ordinary differential equation which makes further analytical investigations easier. The scheme works eminently well for linear problems, although it is also applicable for nonlinear operators (further development and applications of this method are discussed in the survey of Trenogin [Tre70]).

The Lyusternik-Vishik method is free from the necessity to solve the equation directly or to find a fundamental system of solutions. The boundary effect is explained in [VL57] as a gap between the domain of the perturbed and reduced operator. The discrepancy inside the domain and near the boundary is handled independently with a two-iteration process. The first iteration is the ordinary method of small parameters, the second step offers solving some auxiliary ODE equation for boundary layer functions. Sufficient number of negative roots of its characteristic polynomial implies the vanishing of boundary layer inside the domain. The mathematical clarity of the method allows one to construct asymptotic expansions conveniently.

Both these methods encouraged to a great extent studies of asymptotics. It would be a difficult task to itemize all significant papers, discoveries and computations which were made since 1960s. One can say that advances in hypersonic dynamics, wings prediction, compressible flow and shock wave theory and so on owe much to asymptotic approaches developed during that period.

It is worth noting that the boundary layer effect is not solely connected with the presence of a boundary. When one considers a differential equation on manifolds with singularities, there frequently occur extra correction terms in asymptotics, which have a structure resembling boundary corrections (for instance [Kon67, Il86]). Moreover, some effects that can be interpreted as "inner boundary layer" were found inside the domain [Isa57]. At the present time such questions form distinct branches of studies for singular problems. Some investigations are known as the theory of contrast structures (for the introduction see [BVN70]).

The reduced equation need not be of the same type as the perturbed are, which is typical for problems with small parameters. Partial differential equations have its own specifics. When an equation degenerates to a lower order, the limiting solution cannot satisfy all boundary conditions (that is, the degenerate equations become overdetermined).

All PDE with small parameters could be classified by types of perturbed and degenerate operators. The case when an equation changes its type becomes heavier to deal with, nevertheless other cases turn out to be noteworthy for detailed investigations. The present study is devoted to elliptic problems which remain elliptic when the parameter is equal to zero.

By now, many reasonably stated cases of elliptic degeneration have been considered. The pioneering papers of Lyusternik and Vishik considered operators of the second and higher orders elliptically degenerating into elliptic operators of the lower order. The domain is suggested to posses a piecewise smooth boundary and all coefficients of the involved operators are assumed to be C^1-smooth. It was noted, that the smoothness of coefficients and boundary has an essential impact on the convergence rate of the remainder term. Prior to Lyusternik and Vishik's investigations, second order operators were also touched upon in the works [Ole52, Lev50, Dav56]. For the case of smooth coefficients, the nonparametric Dirichlet, Neumann and Robin boundary conditions were considered. Uniform estimates for the remainder and the rate of convergence were obtained. The smallest of the remainders (in some integral sense) also was studied for problems of elliptic degeneration of 4th order operators.

The paper [EJ66] presents slightly more general results. Its authors apply the maximum principle to obtain the rate of convergence for the solution or perturbed higher order elliptic problems with a Dirichlet condition to the solution of reduced non-elliptic operator. The case $\varepsilon Au + u = f$ with an elliptic operator A and general boundary conditions was studied in [Fri68] and [Bur66], where the rate of convergence is obtained in a Sobolev norm. The results are very close to the ones which were previously obtained with the Lyusternik-Vishik method, but do not entirely rely on this technique.

Among the remarkable achievements closely related to the considered topic, there is the study of the 4th order non-linear elliptic operators by Friedrichs and Stoker [FS41] (this result is also discussed in [Fri55]). In [Dzh65], the asymptotic method was applied to problems with the domain varying by a small parameter. Goldenveizer [Gol66] proposed the theory of thin halls based on the reduced elliptic system of equations. Elliptic degeneration has much in common with parabolic equations with vanishing elliptic parts, parabolic to elliptic degeneration (for a description of such issues see [VL57, Tre70]), but they often require an essentially different technique.

Most of the results mentioned relate to problems on a domain without singularities. Furthermore, the boundary conditions mainly do not explicitly involve small parameters. The ellipticity with small parameter, despite all of mighty achievements, remained patchy as a field of study for some time. Conditions of the regular degeneration were not expressed in terms of uniform *a priori* estimates or another practicable form.

The first definitively systematic approach to parameters in linear equations is contained in theorems published by Huet [Huet61]. It is shown there that regular degeneration follows from the ε-dependent estimates for the principal symbol. In addition, in [Huet61] Huet gave the proof of convergence theorems in a Hilbert space setting. Then he applied the result to differential problems, which is opposite to starting directly with differential equations. The idea proved very elegant in use and it was interpreted later as norms with parameters. Considering the perturbed problems in the most general form dates back to [Fife71], where boundary conditions also include small parameters.

Nevertheless, as a chapter of general elliptic theory, singular perturbations were convincingly discussed in the works of Frank [Fra90, Fra97] and accomplished by Volevich [Vol06]. It should be noted that the paper [Vol06] restricts itself to operators with constant coefficients in the half-space.

It seems that problems with large and small parameters could be studied as one because of the relation $\varepsilon = \frac{1}{\lambda}$, but in fact both theories are developed parallel to each other. Nevertheless, it should be pointed out that a small parameter can be translated into a large parameter and the choice is based on practical purposes. The theory of problems with large parameter was motivated by the study of the resolvent of elliptic operators. In the work [GS95] the algebra with big parameter is used for operators of the Dirac type with non-local boundary conditions. Sometimes big and small parameters are taken into scene simultaneously, the paper [Dem73] deals with such a case.

By now elliptic operators with small parameters are also studied on conic manifolds. For instance, the Vishik-Lyusternik method is adapted for domains with conical points in [Naz81]. However, there has only been little discussion so far about asymptotic expansions for problems on cuspidal domains. Such problems are heavily solved under common assumptions. At the present time, only particular cases have been investigated (*e.g.* [DT13]). Nevertheless, the obtained results suggest speculations on ways to implement diverse concepts of the asymptotic expansion. In particular, the asymptotic expansion obtained in [DT13] fails to converge classically under certain continuous changes of the problem parameters, however this series can be summed up in some generalised sense. So, if one needs to research a wider range of perturbations, then a conceptually different look at a small parameter could help.

This study attempts

(a) to present elliptic problems with small parameters as a part of general PDE theory, to summarise common properties of singular elliptic problems with regular degeneration;

(b) to observe connections between singularly perturbed problems and other types of non-regularity, their distinctions, similarities and interactions;

(c) to treat classic problems for small parameters with new approaches and tools and propose a way to deal with more general classes of parametric dependence.

This essay begins with Preliminaries to the theoretical background, the next three chapters present original results.

Chapter 2. The second chapter deals with an elliptic linear differential equation on a bounded domain supplemented by boundary conditions. The equation and boundary conditions are supposed to depend on a small parameter ε in a given explicit form. The boundary value problem is elliptic for every $\varepsilon \geqslant 0$, but at point $\varepsilon = 0$ the equation has a lower order and some number of boundary conditions could disappear. Therefore solutions can not always be constructed as a formal series (1). To address the issue we introduce boundary corrections. But the validity of the resulting series requires to be verified. In many particular cases there are *a priori* estimates which imply asymptotic property of such expansions. It gives some idea of how to construct a general theory for the problems with parameters, that is to obtain necessary and sufficient conditions on the operator and the boundary conditions to be uniformly invertible.

The earliest such theory was the case of large parameters developed in [Agm62, AV64]. A large parameter is supposed to enter the equation and boundary conditions as a covariable. Such equations remain elliptic when the parameter is formally replaced by the derivation with respect to a new variable. The solvability and *a priori* estimates in the Sobolev spaces with the norms $\|u\|_l := \|u\|_l + \lambda^l \|u\|_0$ were obtained under quite simple conditions. Essentially, these restrictions are that $A(x, \xi, \lambda)$ is elliptic for each λ big enough, and $(A(x, \xi, \lambda), B_j(x', \xi, \lambda))$ has some algebraic property akin to the Shapiro-Lopatinskii condition.

The task to construct a general theory for a small parameter was initiated in the series of works [Fra79, Fra90, Fra97]. These works consider ΨDO with small parameters on the whole space $\mathbb{R}^n$. They include a great number of perturbed operators investigated earlier. He singles out elliptic operators, proves the existence of parametrix and its equivalence to *a priori* estimates of elliptic type in the specific Sobolev spaces normed under

$$\|u\|_{(s_1,s_2,s_3)} = \varepsilon^{-s_1} \left\| (1+|\xi|^2)^{s_2} (1+\varepsilon^2|\xi|^2)^{s_3} F_{x\to\xi} u \right\|_{L^2(\mathbb{R}^n)}. \tag{2}$$

The similar norms with $s_1 = 0$ was earlier used in the paper [Pok68].

The boundary problems on domains with conical points were considered in [Naz81]. However, these papers fall short of providing an explicit Shapiro-Lopatinskii-type condition of ellipticity with a small parameter. Instead, they assume *a priori* estimates for corresponding problems for ordinary differential equations on the half-axis.

Volevich [Vol06] completed the theory by formulating the Shapiro-Lopatinskii conditions with small parameters and proved that it is equivalent to uniform *a priori* estimates in the norms originated from [Pok68]. However, as was already noted, he did it for $\mathbb{R}^n$ and $\mathbb{R}^n_+$. In Chapter 2 we complete his study with the case of a bounded domain.

The investigation begins with construction of a formal solution as series in ε, then the asymptotic properties are studied. We carry out uniform a priori estimates which are sufficient for convergence and justify the obtained expansions. The existence of estimates is equivalent to the *Shapiro-Lopatinskii condition* with small parameter if the operators are elliptic for every ε.

The investigation is based on general elliptic techniques (see [ADN59]). It suggests to study Banach spaces of functions defined in bounded domains and on their boundaries. The local principle enables us to reduce the Fredholm property of operators with "slowly" varying coefficients to that for operators with constant coefficients. Then the operators with coefficients "frozen" at inner and boundary points can be modified to be defined on functions on all of space $\mathbb{R}^n$ or $\mathbb{R}^n_+$, respectively. We prove that these manipulations preserve the Fredholm property and norms allow local techniques.

In Chapter 2 this technique is adapted to the case of problems with parameters. Our study uses norms and estimates derived in [Vol06] for model problems of such types. The resulting estimate justifies the formal asymptotic solution of stated problem.

Chapter 3. Our next goal is to analyse changes in the asymptotic behaviour when a differential problem with a small parameter involves other singularities. Furthermore, we consider interactions between different kinds of singularities. The discussion is restricted to the Dirichlet problem for the heat equation in a bounded domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ containing a small parameter ε multiplying the time derivative.

The points of the boundary at which the tangent is orthogonal to the time axis are characteristic. The boundary $\partial\Omega$ is allowed to have point singularities and, moreover, they can touch characteristics.

It is well known (see *e.g.* [Tay96]) that a boundary value problem behaves pathologically in advance if characteristics meet the boundary. However, if it is possible to cut off the irregular point with a small neighbourhood from the boundary, then the solution, in its turn, splits up in the singular part near the boundary and is regular part far from the boundary. It is similar to looking for a solution to the singularly perturbed problem when it is considered separately near the boundary.

The subject of our interest is the case when the boundary is non-smooth at such points. More specifically, we assume that the boundary $\partial\Omega$ touches the characteristic at a point (x_0, t_0) and in some neighbourhood U the boundary is given by the equation

$$\partial\Omega \cap U = \{(x,t) \in \partial\Omega : t - t_0 = |x - x_0|^p\},$$

where $p > 0$ is called the *contact degree.*

The values $p = 2, 3, \dots$ give us the regular case and we will not pay attention to them. For $p = 1$ the domain $\Omega \cap U$ is a cone and for $p < 1$ it is a cusp. Thereby, the boundary value problem has two types of irregularities at the characteristic point and typically only one of them becomes a topic for a study.

Solutions of partial differential equations are generally not smooth near a point like (x_0, t_0), their asymptotic behaviour depends on the contact degree p.

The case $p > 2$ defines a weak type of singularity for the heat equation, the studies go back to Gevree [Gev13], Petrovskii [Pet34, Pet35] and are concerned with the first boundary value problem. The classical approach is applied in [Gev13] and rests on the potential theory. For general parabolic equations the asymptotics near contact points were derived in [Kon66]. The asymptotic property holds if $p = 2$, this threshold value is also called "regular singularity."

In the case $p < 2$ the singularity is strong, such points are treated in [AT13] for the heat equation with first boundary value problems to get both a regularity theorem and the Fredholm property in weighted Sobolev spaces. Some approaches to strong singularities could also be found in [RST00].

Conic and cuspidal points make questions about properties of solutions more complicated, because local technique is often no longer applicable. In contrast to the case of regular boundaries, the solutions of elliptic equations fail to be infinitely smooth in general, even if right-hand sides and coefficients are infinitely differentiable. Geometrical singularities of the boundary also modify the set of smooth functions. Asymptotics for solutions of homogeneous elliptic equation near a conical point were first derived in [Kon67]. Their terms have the growth $r^\lambda \log^k(r)$ as the distance r to singularity tends to zero.

Usually, conic and cuspidal points are treated with the *blow-up technique.* Through a change of variables the critical neighbourhood is mapped onto a cylinder, what however adds power or more complicated singularities to the coefficients. *A priori* estimates for boundary problems are usually proved in weighted Sobolev spaces (the explicit constructions could be found in [NP94]).

If the norms with small parameters control the loss of smoothness, the weighted norms express that the solution must be smooth far from the singularity, although its derivatives could have power growth in the critical neighbourhood. To study the perturbed elliptic equations on conic domains, in [Naz81] a small parameter is involved also in the weighted Sobolev norm.

In Chapter 3 an explicit asymptotic solution in a neighbourhood of a characteristic point is constructed and it is shown that the convergence depends on the geometrical characteristic p of cusps. Asymptotic expansions have the form of the Puiseux series in fractional powers of t/ε up to an exponential factor. The series diverges even for points far from the boundary when the characteristic number p comes through the threshold value, but the formal series inherits some asymptotic pattern.

Such divergent expansions are not new but a controversial subject in mathematics. The formal divergent series are mighty instrument for approximations (see [Har49]), and it is a rather common situation that the modern tools of mathematics are able to give a physical sense to a formal solution (see for instance [Koz03]).

On the other hand, the ingrained weakness of non-classical asymptotics explains the absence of unique way for its regularisation. However, some variety of non-classical approximations have been developed. Namely, to decompose a function $u \in H$ from some functional space H formally we consider a countable set of subspaces $\{H_n\}_{n=1}^{\infty} \subset H$ named *a scale*. The formal asymptotic expansion of an element is defined via $u \sim \sum_{n=0}^{\infty} u_n$, where each term $u_n \in H_n$. In such a way the difference $u - \sum_{n=0}^{N} u_n$ belongs to the "better" functional space H_{N+1}. An illustration of this concept is the expansion of classical symbols in homogeneous functions in ΨDO algebras. It is clear that the Poincaré and Erdélyi asymptotic expansions are particular cases of these more general formal sums, but the latter are poorly investigated, not useful in numerical computations and used rarely. Further investigations of formal expansions for PDE could specify honourable classes of expansions and construct a proper scale.

Chapter 4. The last chapter aims at generalising the concept of ellipticity with a parameter to pseudodifferential operators, *i.e.* at defining a calculus of ΨDO with a small parameter. This operator algebra must include elliptic differential operators with regular degeneration and be a natural extension of wider classes of pertubed operators.

By now there are a great number of constructed ΨDO algebras with specified properties (see for examples in [CW78, Pla86, Mel90, SS97]). Several algebras with large and small parameters have been also proposed; they differ in how parameters are included in definitions of admissible and homogeneous symbols. These basic definitions prompt the structure of elliptic symbols (some notions of more general ellipticity with parameter are given in [DV98]). The presence of a parameter in symbols often modifies calculus of ΨDO. In particular, the calculation of homogeneous components of a symbol can take into account a parameter. The following two symbol algebras are mostly used.

Algebra with a large parameter generalises the differential operators investigated in [AV64] and is described comprehensively in [Shu87]. The large parameter λ enters into symbols as a covariable. The corresponding operators act boundedly in the Sobolev spaces with norms depending on λ. More general algebra is constructed in the book [Agr90].

Typical symbols of *algebras with small parameter* involve ε as a power factor near the covariables. When ε is zero the symbols form usual algebra of pseudodifferential operators. The corresponding Sobolev norms where pseudodifferential operators map continuously are used in papers by Pokrovskii, Demidov, Nazarov, Frank and others. A symbol a is homogeneous if $a(x, t\xi, t^{-1}\varepsilon) = t^{\nu} a(x, \xi, \varepsilon)$ for positive t. This class includes a vast number of differential operators investigated in classical theory.

Both cases are encompassed by the class of symbols proposed [Fra90]. The corresponding Sobolev norms are defined by (2). Homogeneous symbols are defined as those in algebras with small parameters. As was mentioned above, the treatise of Frank elaborates comprehensively the case of compact manifolds without boundary.

At the same time, Frank's approach is of phenomenological nature, which makes it non-flexible. Being a well developed tool for elliptic linear equations, his algebra is not suitable for nontypical perturbations. The present study pretends to enrich the theory of small parameters with a machinery related to similar problems. In particular, we apply analysis on non-smooth manifolds [SS97, RST00]. For the edge type singularities, for instance, it consists in the following. Given a manifold $\mathcal{M}$ with a conic edge, the spaces of smooth functions $C^{\infty}(\mathcal{M})$ and of linear operators $\mathcal{L}(C^{\infty}(\mathcal{M}))$ can be naturally introduced. Then the manifold is viewed locally close to the edge as a Cartesian product of a cone $\mathcal{C}$ and the edge $\mathcal{E}$. With some auxiliary constructions it is possible to present pseudodifferential operators on the manifold $\mathcal{M}$ as a calculus on $\mathcal{M} \setminus \mathcal{E}$ with symbols defined on the edge $\mathcal{E}$ and taking their values in the space of operators on the cones.

We take into account the idea of operator-valued symbols for algebra with a small parameter. The aim is to obtain symbols on manifolds, taking their values in some function spaces depending on a parameter. To this end, the theoretical scheme established by Karol' [Kar83] is exploited and two approaches to the construction of such symbols are proposed. The first one endows spaces with parameter dependent norms, in the framework of this concept basic algebras are rediscovered. Another approach states that the variable and parameter (x, ε) form a cylindric domain $\mathcal{X} \times [0, \varepsilon_0)$ with an arbitrary manifold $\mathcal{X}$ as a base. On this cylinder ΨDO are introduced in a usual manner and the symbols obtained in this way depend naturally on ε. The last point is to specify symbols within the operator-valued ones defined on $T^*\mathcal{X}$. In the fourth chapter this idea is carried out for the so-called Sobolev spaces with a group action described in [SS97].

The derived symbolic algebras inherit principal properties of regularly perturbed elliptic partial differential equations, involve achievements of preceding studies and are transparently constructed.

Chapter 1

Preliminaries

1.1 Notation

$\mathbb{R}^n$ denotes the real coordinate space of n dimensions, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, *etc.* are points of this space. The length in $\mathbb{R}^n$ is given by $|x| = \sqrt{\sum_{i=1}^n x_i^2}$. We also use the semi-space

$$\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{x_n : x_n \geqslant 0\},$$

and for $x \in \mathbb{R}^n$ we write $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$. Besides the length, we make use of the notion

$$\langle \xi \rangle = \sqrt{1 + |\xi|^2}, \quad \langle \xi; \eta \rangle = \sqrt{1 + |\xi|^2 + |\eta|^2}.$$

The n-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_j \in \{0\} \cup \mathbb{N}^n$, is called a multiindex and its length is equal to $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Given a multiindex α we write

$$D^\alpha = (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

$\mathfrak{D}(f)$ stands for the domain of a function or operator f. The support of a function f is the set

$$\operatorname{supp} f = \overline{\{x \in \mathfrak{D}(f) : f(x) \neq 0\}}.$$

The closure of a set Ω is denoted by $\overline{\Omega}$. Furthermore, if Ω is a domain, its boundary is $\partial\Omega = \overline{\Omega} \setminus \Omega$.

$C^k(\Omega)$ is the class of functions k times continuously differentiable on the set Ω (accordingly, $C^\infty(\Omega)$ denotes the class of infinitely differentiable functions).

When it is clear from the context on which set the functions are considered, we omit Ω in the notation. An excision function $\chi_R \in C^\infty(\mathbb{R}^n)$ is such that

$$0 \leqslant \chi_R(\xi) \leqslant 1 \quad \text{and} \quad \chi_R(\xi) = \begin{cases} 0, |\xi| \leqslant R, \\ 1, |\xi| > 2R. \end{cases}$$

The Fourier transform of function $u(x)$ is defined by

$$\int e^{-ix\xi} u(\xi)\, đ\xi$$

and also denoted as $F_{x\to\xi} u = \tilde{u}$. The inverse Fourier transform is

$$F_{\xi\to x}\tilde{u} = \int e^{ix\xi}\tilde{u}(\xi)\, d\xi.$$

$\mathcal{S}(\mathbb{R}^n_+)$ and $\mathcal{S}(\mathbb{R}^n)$ stand for the Schwartz classes; $\mathcal{S}(\mathbb{R}^n_+)$ is defined as

$$\mathcal{S}(\mathbb{R}^n_+) = \Big\{ f \in C^\infty(\mathbb{R}^n_+) : \sup_{x\in\mathbb{R}^n_+} |x^\alpha \partial_\beta f(x)| \leqslant C_{\alpha,\beta} \text{ with some } C_{\alpha,\beta} > 0 \Big\}$$

and $\mathcal{S}(\mathbb{R}^n)$ has an analogous definition.

The constants in estimates are supposed to be positive by default. All variables in subscripts and superscripts are supposed to be fixed arbitrarily if it does not cause any misunderstanding that their range is discernible from the context. Otherwise the appropriate conditions are stated explicitly.

1.2 Historical background

Although classical analysis serves the application's needs well, PDE theory got its great breakthrough via implementations of functional analysis. ΨDO calculus was developed in the middle of the 1960s through the efforts of Kohn, Nirenberg, Hörmander, Mihlin, Widom, Volevich and others, and originates from the operator methods and Fourier analysis. The so-called symbols of differential operators were the subjects of consideration for a long time, but entered the mainstream only with ΨDO's approach.

The invertible integral transformations make it possible to connect an operator with some function and then manipulate of ordinary functions instead of differential or integral expressions. With that they set a correspondence between algebraic operations on symbols and the manipulations with operators.

For such transformations as the Fourier transform, many symbols of linear integro-differential operators are computable, and the symbols of differential operators are simply polynomials. In the framework of symbols one can treat integral and differential equations in the same way, which highlights the affinity of these two subjects. It is particularly significant that the operator depends on its symbol continuously. It allows us to use homotopical methods and theorems of index preservation. The Fredholm theory of elliptic ΨDO is well-shaped and culminates in the Atiah and Singer index theorem. But the greatest measurable advantage of ΨDO calculus is that it is possible to write down the inverse operator in an explicit form. The operations on the symbol level constitute the micro-local analysis, which is a proper tool for analysis on manifolds. Symbols of elliptic operators are the simplest to use, because they are invertible by definition, and elliptic ΨDO have been examined thoroughly. Many ascertainable facts on compact manifolds without boundary, such as solvability, follow as direct generalisations from the corresponding facts about differential operators. The algebra of boundary problems (Boutet de Monvel's algebra), introduced in [Bou71], accomplished the case of manifolds with boundaries.

There is a close resemblance between many properties of pseudodifferential and differential equations. Naturally, problems and tasks in PDE were elaborated for ΨDE as well, including equations with singularities. Many results were resumed in the terms of symbolic algebras. For instance, Eskin [VE67] provided the factorisation of an arbitrary symbol, analogous to properties of polynomials. As another illustration, let us refer to Pokrovskii [Pok68], who pointed out the class of ΨDO having boundary layers, although the boundary correction is not expressed through exponents, and provided the asymptotic expansions. At the same time, not all theorems for PDE have straightforward extensions to pseudodifferential equations. For instance, the Shapiro-Lopatinskii condition fails to be formulated algebraically. Additionally, ΨDO have certain obstacles to apply, because there a special algebraic structure of symbols should be determined to make analytical calculations fully correct.

Extraneous difficulties occur when a perturbed problem is considered on a non-smooth domain. The most considerable cases of singularities are conic or, more generally, cuspidal points. The irregular points at the boundary make the questions of solvability and correctness more sophisticated, because the geometrical structure near the singularity has much influence on the set of solutions of elliptic equations with zero right-hand sides. For example, the solutions of homogeneous elliptic equations with constant coefficients increase as $r^\lambda \log^m(r)$ near a conical point (where λ is complex and m is a non-negative number). The innovative studies on singular manifolds date back to the early 1960s and could be found in [Esk63, GK70, Gri85]. But the first systematic approach was undertaken by Kondrat'ev in [Kon67], where the Fredholm properties were investigated within the framework of generalized solutions. Further advances were made in [Mel90, KMR97] and others (see also the full bibliography in [KMR97]). The calculus on manifolds with conical points is developed comprehensively in [Mel90, Pla86, Schu91]. Elliptic estimates and Fredholm property for general boundary value problems are presented by Bagirov [BF73], Maz'ya and Plamenevskii [MP73]. The paper [RST00] elaborates the results by Feigin [Fei71] to a general statement of elliptic problems on manifolds with cuspidal edges. The algebra involved in [RST00] has operator-valued symbols.

The algebras of ΨDO's have the advantages to posses symbols. We thus get an algebra of operators $\mathcal{A}$ and algebra of symbols S along with so called quantisation map $a \mapsto Op(a)$ for all $a \in S$. This map is surjective and one chooses a left inverse called the principal symbol mapping (denoted by $A \mapsto \sigma(A)$). The map $\sigma : A \mapsto S$ is multiplicative in the sense that $\sigma(AB) = \sigma(A)\sigma(B)$. Moreover, the symbol $\sigma(A)$ vanishes $\sigma(B)$ if and only if A is compact in the corresponding scale of Sobolev space. Using this property of principal symbol one introduces the notion of ellipticity in operator algebras with symbolic structure. Namely, $A \in \mathcal{A}$ is said to be elliptic if $\sigma(A)$ is invertible we write $\sigma(A)^{-1}$ for the inverse, then a suitable quantisation $Op(\sigma(A)^{-1})$ leads to an operator A satisfying $PA = I$ and $AP = I$ modulo compact operators in the algebra. Such operator P is called parametrix of A and the existence of a parametrix just amounts to the Fredholm property of A. The parametrix construction is especially transparent for polyhomogenuous (or classical) ΨDO.

1.3 Symbols of pseudodifferential operators

This chapter provides background material on pseudodifferential calculus that is used throughout the work. These concepts are presented comprehensively in [Shu87].

Definition 1. *We call a smooth function $a(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ a symbol of the class $S^m(\mathbb{R}^n)$ if $a(x,\xi) = a_1(x,\xi) + a_2(\xi)$ where $\left|\partial_x^\alpha \partial_\xi^\beta a_1(x,\xi)\right| \leqslant C'_{\alpha,\beta}(x)\langle\xi\rangle^{m-|\beta|}$, $\left|\partial_\xi^\beta a_2(\xi)\right| \leqslant C_\beta \langle\xi\rangle^{m-|\beta|}$ for every $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and multiindices α, β. Here $C'_{\alpha,\beta} \in \mathcal{S}(\mathbb{R}^n)$, and $C_\beta > 0$ are constants independent of (x,ξ).*

The pseudodifferential operator corresponding to $a \in S^m$ is given by the formula $\mathrm{Op}(a)u = \int e^{ix\xi} a(x,\xi)\tilde{u}(\xi)d\xi$. The set of operators corresponding to $S^m(\mathbb{R}^n)$ is denoted by Ψ^m and *the complete symbol* of a given operator $A \in \Psi^m$ by $\sigma_A(x,\xi) = a(x,\xi)$. The real number m is called the *order of the pseudodifferential operator* A.

We also use the notations

$$S^{-\infty} = \cap_m S^m, \quad S^\infty = \cup_m S^m, \quad \Psi^{-\infty} = \cap_m \Psi^m, \quad \Psi^\infty = \cup_m \Psi^m.$$

The operators Ψ^∞ map $\mathcal{S}(\mathbb{R}^n)$ continuously to $\mathcal{S}(\mathbb{R}^n)$. The Sobolev space $H^s(\mathbb{R}^n)$ is the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_s^2 = \int \langle\xi\rangle^s |\tilde{u}|^2 d\xi.$$

Theorem 1. *The operators $A \in \Psi^m$ extend to linear bounded maps*

$$A : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n),$$

i.e. *A are defined for each $u \in H^s(\mathbb{R}^n)$ and satisfy the estimate*

$$\|Au\|_s \leqslant C_s \|u\|_{s-m}.$$

1.4 Algebra of pseudodifferential operators

In the sequel we deal with algebras of pseudodifferential operators with elements defined up to smoothing operators.

Definition 2. *An operator $\mathfrak{G} : \mathcal{S}' \to \mathcal{S}'$ is called smoothing if it maps $\mathcal{S}'$ continuously into $H^s(\mathbb{R}^n)$ for each s.*

For such operators the estimate

$$\|\mathfrak{G}u\|_{s+N} \leqslant C\|u\|_s$$

holds for every $u \in H^s(\mathbb{R}^n)$ and $N \in \mathbb{R}$. The operators from $\Psi^{-\infty}$ are smoothing. The converse is false. The class $\Psi^{-\infty}$ defines the equivalence for every $A, B \in \Psi^{\infty}$ naturally so that *A is equal to B modulo smoothing operators*, if $A - B$ is a smoothing operator.

Lemma 1. *If $A = B$ modulo smoothing operators of $\Psi^{-\infty}$ and $A \in \Psi^m$, then $B \in \Psi^m$.*

Now we clarify the idea of asymptotic expansions. Let us consider the sequence of symbols $a_j \in S^{m-j}$, $j = 1, 2, \ldots$, not necessarily convergent. The formal series $\sum_{j=1}^{\infty} a_j$ is called an asymptotic expansion of some symbol a if for each N we have $a - \sum_{j=1}^{N} a_j \in S^{m-N-1}$. The asymptotic expansion of a is denoted as

$$a \sim \sum_{j=1}^{\infty} a_j$$

It is clear, that $a \in S^m$. Any sequence of symbols a_j of decreasing order defines an asymptotic expansion for some symbol a, precisely:

Lemma 2. *Given a sequence $a_j \in S^{m-j}$ there exist a symbol $a \in S^m$ such that*

$$a \sim \sum_{j=1}^{\infty} a_j.$$

The symbol a in this lemma is not uniquely defined, but up to smoothing operators. The spaces Ψ^m are obviously linear, the next lemma implies that they form an algebra.

Lemma 3 (Composition formula)**.** *Let $A \in \Psi^{m_1}$ and $B \in \Psi^{m_2}$ then their composition AB is also a pseudodifferential operator of the class $\Psi^{m_1+m_2}$ with the symbol*

$$\sigma(AB) \sim \sum_{\alpha} \partial_{\xi}^{\alpha}\sigma_A \cdot D_x^{\alpha}\sigma_B/\alpha!.$$

The next two statements allow one to define ΨDO on closed compact manifolds. Firstly, the composition formula implies the *pseudolocal property* of ΨDO. That is, if $\zeta, \theta \in C_0^{\infty}$ and $\theta|_{\operatorname{supp}\zeta} \equiv 1$ then $w \mapsto \zeta \cdot A\big((1-\theta)\cdot w\big)$ is a smoothing operator for every $A \in \Psi^{\infty}$.

Further, let A be a pseudodifferential operator, Ω be some domain in $\mathbb{R}^n$ and $\kappa : \Omega \to \tilde{\Omega}$ be a diffeomorphism. Let also $\zeta, \theta \in C_0^\infty(\Omega)$ be such functions that $\theta(x) = 1$ in a neighbourhood of $\operatorname{supp}\zeta$. The operator $\tilde{A} : C_0^\infty(\tilde{\Omega}) \to C_0^\infty(\tilde{\Omega})$ is correctly determined by the formula

$$\tilde{A}v(y) = \zeta A[\theta u](x), \quad x \in \Omega,$$

where $u = v \circ \kappa$.

Lemma 4 (Change of variables in ΨDO)**.** *The operator $\tilde{A}$ is pseudodifferential of class Ψ^m and its symbol $\sigma_{\tilde{A}} = \tilde{a}(y, \eta)$ is*

$$\tilde{a}(y,\eta) \sim \zeta(x) \sum_{\alpha} \varsigma_\alpha(y,\eta) \partial_\xi^\alpha \sigma_A \Big|_{\substack{\xi = {}^t\kappa'(x)\eta \\ x = \kappa^{-1}(y)}},$$

where ${}^t\kappa'(x)$ denotes the transpose of the Jacobi matrix and $\varsigma_\alpha(y,\eta)$ is given by the formula

$$\varsigma_\alpha(y,\eta) = \frac{1}{\alpha!} D_z^\alpha \exp\left[i\big(\kappa(z) - \kappa(x) - \kappa'(x)(z-x)\big)\right]\Big|_{\substack{z = x \\ x = \kappa^{-1}(y)}}.$$

The next class of symbols has a special meaning and we pay the most attention to it. They are widely known as *classical* and favourable for calculus on manifolds.

Definition 3 (classical symbols)**.** *The set of classical symbols $S^m_{\mathsf{cl}} \subset S^m$ consists of symbols which are asymptotically expanded as*

$$a \sim \sum_{j=-\infty}^{m} \chi_R a_j, \quad a_j \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$$

where $\chi_R(\xi)$ is an excision function and $a_j(x,\xi)$ are homogeneous in variable ξ of order j, i.e. *for every positive real t and $(x,\xi) \in \mathbb{R}^{2n}$, holds $a(x,t\xi) = t^j a(x,\xi)$. The term a_m in asymptotic expansion is called the principal symbol of a.*

The functions a_j in the last definition are known as *homogeneous symbols*, but they are not real symbols, *i.e.* do not belong to S^∞. Also not every ΨDO is classical, but all differential operators belong to $\Psi^\infty_{\mathsf{cl}}$. The principal symbol for each operator $A \in \Psi^\infty_{\mathsf{cl}}$ is defined uniquely and is designated as $\sigma^m(A)$ or a_m. The composition AB of two classical operators $A \in \Psi^{m_1}_{\mathsf{cl}}, B \in \Psi^{m_2}_{\mathsf{cl}}$ is classical as well, and $\sigma^{m_1+m_2}(AB) = \sigma^{m_1}(A)\sigma^{m_2}(B)$. A classical operator A

remains classical under diffeomorphic change of coordinates. Moreover, its principal symbol $\sigma^m(A)$ is invariant with respect to smooth changes of variables. Indeed, for a symbol $a \in S^m_{\mathsf{cl}}$ and a given diffeomorphism $\kappa : \Omega \mapsto \tilde{\Omega}$, $\kappa(x) = y$, Lemma 4 implies that for some small neighbourhood of the point $y \in \tilde{\Omega}$ the principal symbol $\tilde{a}_m(y, \eta)$ of the operator $\kappa \circ \mathrm{Op}(a) \circ \kappa^{-1}$ is computed as

$$\tilde{a}_m(y, \eta) = a_m(x, \xi) \tag{1.1}$$

where on the right-hand side

$$x = \kappa^{-1}(y), \quad \xi = \left[{}^t(\kappa^{-1})'(y)\right]^{-1} \eta.$$

1.5 Ellipticity and regularisers

Definition 4. *A classical pseudodifferential operator $A \in \Psi^m_{\mathsf{cl}}(\mathbb{R}^n)$ is called* elliptic *at a point $x \in \mathbb{R}^n$ if $a_m(x, \xi) \neq 0$ for all $\xi \neq 0$.*

This definition is also equivalent to the estimate

$$|a(x, \xi)| \geqslant C_x |\xi|^m \text{ for sufficiently large } |\xi| \geqslant R. \tag{1.2}$$

An operator $A \in \Psi^m_{\mathsf{cl}}$ is elliptic in a domain $\Omega \subset \mathbb{R}^n$ if it is elliptic at every point $x \in \Omega$. The estimate (1.2) is a characteristic property of elliptic operators.

Definition 5 (Regularisers for an elliptic operator)**.** *Let A be a pseudodifferential operator. Pseudodifferential operators $R_{\mathsf{left}}, R_{\mathsf{right}} \in \Psi^\infty$ are called left and right regularisers for A respectively if*

$$R_{\mathsf{left}} A = I + T_1, \quad A R_{\mathsf{right}} = I + T_2, \tag{1.3}$$

where I is the unit operator and $T_1, T_2 \in \Psi^{-\infty}$. If an operator R satisfies both equalities, then it is called a regulariser for A.

If A has left R_{left} and right R_{right} regularisers, then $R_{\mathsf{left}} = R_{\mathsf{right}}$ modulo smoothing operators and each of them is a regulariser for A.

Theorem 2. *For every elliptic operator $A \in \Psi^m$ there exists an elliptic pseudodifferential operator $R \in \Psi^{-m}$ which is a regulariser for A.*

The following fact connects ellipticity, the existence of regulariser and *a priori* estimates.

Theorem 3 (*A priori* estimate)**.** *For any elliptic $A \in \Psi^m$ there exists a constant $C_s > 0$ such that*

$$\|u\|_s \leqslant C_s(\|Au\|_{s-m} + \|u\|_0)$$

for all $u \in H^s(\mathbb{R}^n)$. Conversely, if the above estimate holds then A is elliptic.

It remains to note that the existence of a regulariser for an operator A implies the estimate above.

1.6 ΨDO on manifolds without boundaries

Hereinafter $\mathcal{X}$ stands for smooth n-dimensional, connected, closed compact manifolds. The cotangent bundle to $\mathcal{X}$ without the zero section is denoted as $T^*\mathcal{X}\backslash 0$. We fix a covering $O = \{O_k\}_{k=1}^K$ of $\mathcal{X}$ and a partition of unity $\{\phi_k\}_{k=1,2,\dots,K}$ subordinate to O. For the points of $\mathcal{X}$ we designate by $x = (x_1, x_2, \dots, x_n)$ their local coordinates given by a chosen chart $\mathcal{O}$.

The scale of Sobolev spaces $H^s(\mathcal{X})$ is defined through the norm

$$\|u\|_{s,\mathcal{X}}^2 = \sum_{i=1}^{K} \|\phi_i u\|_s^2,$$

where $\|\phi_i u\|_s$ is computed in local coordinates. Other partitions of unity determine equivalent norms.

Let us specify pseudodifferential operators among all continuous linear operators acting on functions on $\mathcal{X}$.

Definition 6. *We call an operator A with $\mathfrak{D}(A) \supset C^\infty(\mathcal{X})$ a pseudodifferential operator of class $\Psi^m(\mathcal{X})$ if the following conditions are satisfied.*

(a) Let ζ, θ be arbitrary chosen functions from $C^\infty(\mathcal{X})$ with disjoint supports. Then the composition $\zeta A(\theta\,\cdot\,)$ extends to a smoothing operator on $\mathcal{X}$.

(b) Let some domain in $\mathcal{X}$ be mapped by a chart $\mathcal{O}$ onto $U \subset \mathbb{R}^n$, and let $\Omega \subset \mathcal{O}^{-1}(U)$ be a strictly inner subdomain. Then there is a pseudodifferential operator $A_\Omega \in \Psi^m(\mathbb{R}^n)$ such that if ζ and θ are functions from $C^\infty(\mathcal{X})$ with supports contained in Ω, then in the local coordinates $\zeta A(\theta u)(x) = \zeta(x)(A_\Omega(\theta u))(x)$.

This definition is correct in the following sense.

Suppose that another system of functions $\{\theta_k\}_{k=1}^K$ is subordinated to the covering $O = \{O_k\}$ and $\theta_k \equiv 1$ in a neighbourhood of the support of ϕ_k for each k. Let Ω_k be a strictly inner subdomain of O_k such that $\operatorname{supp}\theta_k \subset \Omega_k$, and let A_{Ω_k} be a corresponding pseudodifferential operator of $\Psi^m(\mathbb{R}^n)$ given by the definition. Then

$$A = \sum_k \phi_k A_{\Omega_k}(\theta_k \cdot\,) + T_1 \quad \text{and} \quad A = \sum_k \theta_k A_{\Omega_k}(\phi_k \cdot\,) + T_2,$$

where T_1 and T_2 are smoothing operators on $\mathcal{X}$. Conversely, if such expansions exist for an operator $A : C^\infty(\mathcal{X}) \to C^\infty(\mathcal{X})$, then A is a pseudodifferential operator.

Operators from Ψ^m map $H^s(\mathcal{X})$ continuously into $H^{s-m}(\mathcal{X})$ for all $s \in \mathbb{R}$, what is a direct consequence of Theorem 1.

We set $\Psi^{-\infty}(\mathcal{X}) = \cap_m \Psi^m(\mathcal{X})$, $\Psi^\infty(\mathcal{X}) = \cup_m \Psi^m(\mathcal{X})$.

The class $\Psi^m_{\mathsf{cl}}(\mathcal{X})$ of classical operators arises naturally when all restrictions $A_\Omega \in \Psi^m(\mathbb{R}^n)$ from Definition 6 are assumed to be classical operators.

The formula (1.1) allows one to consider the principal symbol a_m of an operator $A \in \Psi^m_{\mathsf{cl}}$ as a smooth function defined on the cotangent bundle $T^*\mathcal{X} \setminus 0$. It coincides locally with principal symbols of A_Ω.

Definition 7. *A pseudodifferential operator* $A \in \Psi^m_{\mathsf{cl}}(\mathcal{X})$ *with the principal symbol* $a_m : T^*\mathcal{X} \setminus 0 \to \mathbb{R}$ *is elliptic on* $\mathcal{X}$ *if* $a_m(x,\xi) \neq 0$ *for every* $(x,\xi) \in T^*\mathcal{X} \setminus 0$.

For pseudodifferential operators on manifolds the concept of regulariser is introduced in the same way as for $\mathbb{R}^n$. A pseudodifferential operator R is the regulariser to a pseudodifferential operator $A \in \Psi^m(\mathcal{X})$ if the equalities (1.3) hold with smoothing operators T_1 and T_2 on $\mathcal{X}$. For pseudodifferential operators on closed compact smooth manifolds any smoothing operator is actually a compact operator in any Sobolev space. Therefore the regulariser R is a parametrix of A, i.e., an inverse operator up to compact operators. By the well-known theorem of Atkinson (1953) an operator A possesses a parametrix if and only if it is Fredholm. Therefore, ellipticity is an efficient algebraic characterisation of elliptic operators for compact closed smooth manifolds indeed. The property to be elliptic is equivalent to the existence of a regulariser and the *a priori* estimate. Namely,

Theorem 4. *Let A be a classical elliptic pseudodifferential operator of order m. Then*

(a) there exists an elliptic pseudodifferential operator $R \in \Psi_{\mathsf{cl}}^{-m}(\mathcal{X})$ which is a regulariser for A,

(b) for any s there exists a constant C_s such that

$$\|u\|_{s,\mathcal{X}} \leqslant C_s(\|Au\|_{s-m,\mathcal{X}} + \|u\|_{0,\mathcal{X}}).$$

Conversely, if (a) or (b) holds for a classical operator A then it is elliptic.

1.7 Ellipticity for linear differential boundary value problems

In the work [Bou71] the algebra of boundary value problems is developed for pseudodifferential operators. Since it has a complicated construction we do not present the complete theory of elliptic operators on manifolds with boundaries. The focus of this work is on differential operators in regular situations[1] so we apply the usual definitions.

Let a domain $\Omega \subset \mathbb{R}^n$, differential operators

$$A(x,D)u = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha u,$$
$$B_j(x,D)u = \sum_{|\beta| \leqslant b_j} b_{j,\beta}(x) D^\beta u, \quad j = 1, 2, \ldots, r$$

and functions $f : \Omega \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$ be given.

Recall that the complete and principal symbols of differential operators are polynomials. For the operators A, B the principal symbols look like

$$\sigma_A^m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha,$$
$$\sigma_{B_j}^{b_j}(x',\xi) = \sum_{|\beta|=b_j} b_{j,\beta}(x')\xi^\beta, \quad j = 1, \ldots, r.$$

[1]Here by regularity we mean that boundary surfaces, all coefficients in operators, given functions, etc. possess proper smoothness.

The differential operators corresponding to their principal symbols are called *principal parts.*

The *boundary value problem* consists in finding a function $u(x)$ such that

$$\begin{aligned} A(x,D)u &= f(x), \quad x \in \Omega, \\ B_j(x,D)u\big|_{\partial\Omega} &= g_j(x'), \quad j = 1,2,\dots,r, \quad x' \in \partial\Omega. \end{aligned} \tag{1.4}$$

The last r equalities are called *boundary conditions.* We call the principal parts $\sigma_A^m(x,\xi), \sigma_{B_j}^{b_j}(x',\xi)$ the *inner* and *boundary symbol.* To every boundary value problem (1.4) there corresponds an operator A, B mapping functions $C^\infty(\overline{\Omega})$ to the Cartesian product $H^{s-m}(\Omega) \times \prod_{j=1}^r H^{s-b_j-\frac{1}{2}}(\partial\Omega)$.

Definition 8. *A differential operator $A(x,D)u$ of order m is called elliptic in the domain Ω if its principal symbol $\sigma_A^m(x,\xi)$ does not vanish for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.*

In the case of $n = 2$ we put more restrictive conditions on the inner symbol. An operator $A(x,D)$ is properly elliptic if its characteristic polynomial $A(x,\xi' + \tau\xi'')$ considered with respect to τ has precisely $m/2$ roots in the upper half-plane and $m/2$ roots in the lower one for every $x \in \Omega$, $\xi', \xi'' \in \mathbb{R}^n$.

The boundary value problem (1.4) is elliptic if (a) $A(x,D)$ is elliptic (is properly elliptic in the case $n = 2$) and (b) the condition **(I)** holds.

(I) Shapiro-Lopatinskii condition: for any point $x' \in \partial\Omega$ and any non-zero vector ξ tangent to $\partial\Omega$ at x the polynomials $B_j(x', \xi + \tau\nu)$, $j = 1,\dots,r$ in the variable τ are linearly independent modulo the polynomial $\prod_{i=1}^{\frac{m}{2}}(\tau - \tau_i^+(x',\xi))$, where ν is the outward normal to $\partial\Omega$ at x' and τ_i^+ are the roots of the principal symbol $A_0(x', \xi + \tau\nu)$ with positive imaginary parts.

In the simplest case when the elliptic inner symbol $A(x', \xi + \tau\nu)$ has only prime roots and the coefficients in the boundary symbol $B_j(x',\xi)$ form a non-singular square matrix, the ellipticity of boundary value problem means the relation $\frac{m}{2} = r$.

The Shapiro-Lopatinskii condition is also equivalent to that for any fixed $x' \in \partial\Omega$ and $\xi' \in \mathbb{R}^n \setminus \{0\}$ the problem

$$\begin{aligned} A(x',\xi',D_n)v(\xi',x_n) &= 0, \\ B_j(x',\xi',D_n)v(x_n)\big|_{x_n=0} &= \tilde{g}_j, \quad j = 1,2,\dots,r, \end{aligned}$$

has an unique solution in $\mathcal{S}(\mathbb{R}_+)$.

The following fact clarifies the connection between the Fredholm property and *a priori* estimates for elliptic boundary value problems.

Theorem 5. *Suppose that the boundary value problem A, B is elliptic. Then*

(a) for $r = \frac{m}{2}$ and any real $s \geqslant m$ the operator

$$A, B : H^s(\Omega) \mapsto H^{s-m}(\Omega) \times \prod_{i=1}^{\frac{m}{2}} H^{s-b_j-\frac{1}{2}}(\partial\Omega)$$

is a Fredholm operator, i.e. *has finite-dimensional kernel and cokernel.*

(b) for any $s \geqslant m, s \geqslant b_j + \frac{1}{2}$ the inequality

$$\|u\|_s \leqslant C\Big(\|A(x, D)u\|_{s-m} + \sum_{j=1}^{r} \|B_j(x', D)u\|_{s-b_j-\frac{1}{2},\partial\Omega} + \|u\|_0\Big)$$

holds, where the constant C is independent of u.

Chapter 2

Elliptic problems with small parameters for linear differential equations

We consider the family of operators depending on a small parameter $\varepsilon \geqslant 0$ of the following form:

$$\begin{aligned} &\varepsilon^{2m-2\mu} A_{2m}(x,D)u + \varepsilon^{2m-2\mu-1} A_{2m-1}(x,D)u \\ &\qquad + \cdots + \varepsilon A_{2\mu+1}(x,D)u + A_{2\mu}(x,D)u = f(x), \end{aligned} \tag{2.1}$$

supplemented by the boundary conditions

$$\begin{aligned} &\varepsilon^{b_j-\beta_j} B_{j,b_j}(x',D)u + \varepsilon^{b_j-\beta_j-1} B_{j,b_j-1}(x',D)u \\ &\qquad + \cdots + B_{j,\beta_j}(x',D)u = g_j(x'), \quad x' \in \partial\Omega, \quad j = 1,\ldots,m, \end{aligned} \tag{2.2}$$

where $A_{m-i}(x,D)$ and $B_{j,b_j-i}(x',D)$ are differential operators of order $m-i$ and $b_j - i$, respectively, with smooth variable coefficients.

The domain Ω is supposed to be bounded with smooth boundary $\partial\Omega$. This family of operators must be elliptic for each $\varepsilon \geqslant 0$.

We will construct asymptotic expansions and study their convergence. The first task is to introduce main spaces for functions on Ω and at the boundary $\partial\Omega$, we denote them as $H^{r,s}(\Omega)$ and $\mathcal{H}^{\rho,\sigma}(\partial\Omega)$, respectively. These techniques goes back at least as far as Volevich [Vol06] who developed analysis of function spaces for the model problem on $\mathbb{R}^n$ and $\mathbb{R}^n_+$. His results include trace theorem for the spaces $H^{r,s}(\mathbb{R}^n_+)$ and $\mathcal{H}^{\rho,\sigma}(\mathbb{R}^{n-1})$.

Our main goal is to deduce the estimate

$$\|u; H^{r,s}(\Omega)\| \leqslant C \Bigg(\left\|A(x,D,\varepsilon)u; H^{r-2m,s-2\mu}(\Omega)\right\| \\ + \sum_{j=1}^{m} \left\|B_j(xD,\varepsilon)u; \mathcal{H}^{r-b_j-1/2,s-\beta_j-1/2}(\partial\Omega)\right\| + \left\|u; L^2(\Omega)\right\| \Bigg) \tag{2.3}$$

with a constant C independent of u and ε. Such estimates are already proved in [Vol06] for $\Omega = \mathbb{R}^n_+$ and functions $u \in \mathcal{S}(R^n_+)$.

Here is the scheme of the proof for a bounded Ω. **Firstly**, using a finite covering $\{U_i\}$ of $\overline{\Omega}$ by sufficiently small open sets (*e.g.* balls) in $\mathbb{R}^n$, we represent any function $u \in H^{r,s}(\Omega)$ as the sum of functions $u_i \in H^{r,s}(\Omega)$ compactly supported in $U_i \cap \overline{\Omega}$, just setting $u_i = \phi_i u$ for a suitable partition of unity $\{\phi_i\}$ in $\overline{\Omega}$ subordinate to the covering $\{U_i\}$. **Secondly**, for each summand u_i we formulate its own elliptic problem and find *a priori* estimates for its solutions. If U_i does not meet the boundary of Ω, then the support of u_i is a compact subset of Ω and the proof of (2.3) reduces to a global analysis in $\mathbb{R}^n$. For those U_i which intersect the boundary of Ω we choose a change of variables $x = h_i^{-1}(z)$ to rectify the boundary surface within U_i. To wit, $h_i(U_i \cap \Omega) = O_i \cap \mathbb{R}^n_+$, where O_i is an open set in $\mathbb{R}^n$, and so in the coordinates y estimate (2.3) reduces to that in the case $\Omega = \mathbb{R}^n_+$. **Thirdly**, we glue together all *a priori* estimates for u_i, thus obtaining the *a priori* estimate (2.3) for u. In the next sections all components of this scheme are accomplished.

Let us make the following convention. In this chapter A_0 stands for the principal part of the operator A, which is understood here as

$$A_0(x,D,\varepsilon) := \varepsilon^{2m-2\mu} A_{2m,0}(x,D) + \cdots + \varepsilon A_{2\mu+1,0}(x,D) + A_{2\mu,0}(x,D),$$

where $A_{j,0}(x,\xi)$ is the principal homogeneous symbol of the differential operator $A_j(x,D)$ of order j, with $2\mu \leqslant j \leqslant 2m$. The principal symbols $B_{j,0}$ for the operators B_j are defined analogously.

2.1 Formal asymptotic expansions

Our aim here is to construct a formal solution of the equation, *i.e.* a formal series $u_\varepsilon(x) = \sum_{k=1}^{\infty} \phi_k(x,\varepsilon)$, where $\phi_k(x,\varepsilon)$ belong to some asymptotic scale giving a small discrepancy for $\varepsilon \to 0$. To wit for every point $x \in \overline{\Omega}$

$$A(x, D, \varepsilon)\Big(u_\varepsilon(x) - \sum_{k=1}^{N} \phi_k(x, \varepsilon)\Big) = o(\phi_{N+1}(x, \varepsilon)),$$

$$B_j(x', D, \varepsilon)\big(u_\varepsilon - \sum_{k=1}^{N} \phi_k\big)\Big|_{\partial\Omega} = o(\phi_{N+1}(x', \varepsilon)), \quad j = 1, 2, \ldots, m,$$

as $\varepsilon \to 0$. The functional spaces for the operators $A(x, D, \varepsilon)$, $B_j(x', D, \varepsilon)$ are not pointed out deliberately.

Here the functions $\phi_k(x, \varepsilon)$ are chosen as $\phi_k(x)\varepsilon^k$. We look for a solution $u_\varepsilon(x)$ in a bounded domain, a WKB method type expansion might be insufficient and one requires also a boundary layer construction. Therefore, the solution near and far from the boundary $\partial\Omega$ should be looked separately. For this reason, near the boundary new coordinates are introduced. Because of boundness, there is no unified coordinate system, but this obstacle can be avoided if one considers Ω as a manifold. To make all calculations more transparent we leave those technical things behind.

So let us introduce new coordinates $(y_1, y_2, \ldots, y_{n-1}, z)$ in Ω, such that $y \in \partial\Omega$ is a variable on the surface $\partial\Omega$ and z is the distance to $\partial\Omega$. By $A'(y, z, D_y, D_z, \varepsilon)$, $B'(y, D_y, D_z, \varepsilon)$ we denote operators A, B in the new variables.

We are looking for a solution of (A, B) in the form

$$u(x, \varepsilon) = U(x, \varepsilon) + V(y, z/\varepsilon, \varepsilon)$$

where U is the regular part of u and V is the boundary layer. Suppose that the function $V(y, z/\varepsilon, \varepsilon)$ satisfies the following three conditions:

(a) $V(y, z/\varepsilon, \varepsilon)$ is a sufficiently smooth solution of the homogeneous equation $AV = 0$;

(b) $V(y, z/\varepsilon, \varepsilon)$ depends on the "fast" variable $t = z/\varepsilon$;

(c) $V(y, z/\varepsilon, \varepsilon)$ differs from zero only in a small strip near the boundary $\partial\Omega$.

The outer expansion and the inner expansion are looked for as the formal asymptotic series

$$U(x, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k u_k(x), \quad V(y, z/\varepsilon, \varepsilon) = \varepsilon^{l_\nu} \sum_{k=0}^{\infty} \varepsilon^k v_k(y, z/\varepsilon). \qquad (2.4)$$

The role of the factor ε^{l_ν} will be clarified later. To obtain the outer expansion $U(x,\varepsilon)$ the standard procedure of small parameter method is applied. We substitute the series into the equation $A(x,D,\varepsilon)u = f$ and collect the terms with the same power of ε. It gives us the system

$$\begin{aligned} A_{2\mu}u_0 &= f, \\ A_{2\mu}u_k &= -\sum_{i=1}^{k} A_{2\mu+i}u_{k-i} \end{aligned} \tag{2.5}$$

for unknowns u_k. To determine the coefficients $v_k(y,z/\varepsilon)$ of the inner expansion we apply the operator $A'(y,z,D_y,D_z,\varepsilon)$ to $V(y,z/\varepsilon,\varepsilon)$. Condition (a) implies

$$\sum_{k=0}^{\infty} \varepsilon^k A'(y,z,D_y,D_z,\varepsilon)(v_k(y,z/\varepsilon)) = 0.$$

The operator A' splits up as

$$A'(y,z,D_y,D_z,\varepsilon) = \sum_{k=\mu}^{m-\mu} \Big(A'_{2k}(y,z,D_y,D_z,\varepsilon)\Big) + \check{A}(y,z,D_y,D_z,\varepsilon),$$

where $A'_{2k}(y,z,D_y,D_z,\varepsilon)$ is homogeneous of degree $2k$ and ε enters A'_{2k} with degree $2k-2\mu$. Let us rewrite the operator $A'(y,z,D_y,D_z,\varepsilon)$ in the variables $(y, z/\varepsilon = t)$. For the homogeneous part A'_{2k} of degree $2k$ we have

$$A'_{2k}(y,z,D_y,D_z,\varepsilon) = \varepsilon^{-2\mu} A'_{2k}(y,\varepsilon t,\varepsilon D_y,D_t).$$

On expanding $A'_{2k}(y,\varepsilon t,\varepsilon D_y,D_t)$ as the Taylor series about the point $(y,0,0,D_t)$ we obtain

$$A'_{2k}(y,z,D_y,D_z,\varepsilon) = \varepsilon^{-2\mu}\Big(A''_k(y,0,0,D_t) + \sum_{l=1}^{\infty} \varepsilon^l A_{k,l}(y,t,D_y,D_t)\Big),$$

where the operators $A_{k,l}$ have smooth coefficients.

The rest part $\check{A}$ is expanded at (y,t) as

$$\check{A}(y,t,D_y,D_t,\varepsilon) = \varepsilon^{-2\mu} \sum_{s\geqslant 1} \check{A}_s(y,t,D_y,D_t)\varepsilon^s.$$

Therefore,

$$A'(y,z,D_y,D_z,\varepsilon) = \varepsilon^{-2\mu}\left(A''(y,0,0,D_t) + \sum_{l=1}^{\infty} \varepsilon^l A_l(y,t,D_y,D_t)\right),$$

A_l depends on $A_{k,l}$ and $\check{A}_k$ linearly.

As a result, we obtain the following equations for v_k

$$\begin{aligned} &A''(y,0,0,D_t)v_0(y,t) = 0; \\ &A''(y,0,0,D_t)v_k(y,t) = -\sum_{l=1}^{k} A_l(y,t,D_y,D_t)v_{k-l}, \quad k \geqslant 1. \end{aligned} \tag{2.6}$$

This is a recurrent system of $(2m)$th order. Now we substitute the partial sums

$$U_n = \sum_{k=0}^{n} \varepsilon^k u_k(x) \quad \text{and} \quad V_n = \varepsilon^{l_\nu} \sum_{k=0}^{n} \varepsilon^k v_k(y,t)$$

into the original equation and boundary conditions and find the discrepancy. For $A(x,D,\varepsilon)$, it looks like

$$\begin{aligned} &A(x,D,\varepsilon)(u(x,\varepsilon) - U_n - V_n) \\ &\quad = f - A_{2\mu}(x,D)u_0 - \big(A(x,D,\varepsilon)U_n - A_{2\mu}(x,D)u_0 + A'(y,z,D_y,D_z,\varepsilon)V_n\big) \\ &\quad = O(\varepsilon^{n+1}). \end{aligned}$$

Note, that terms u_k, v_k are not constrained yet, they are chosen arbitrary from the space of fundamental solutions of equations (2.5) and (2.6).

Now let us examine the discrepancy at the boundary. The obtained expansions must satisfy the boundary conditions,

$$B'_j(y,D_y,D_z,\varepsilon)(U(y,z,\varepsilon) + V(y,z/\varepsilon,\varepsilon)) = g_j(y), \quad j = 1,\dots,m.$$

The term $U(y,z)$ gives for each j

$$B'_j(y,D_y,D_z,\varepsilon)U(y,z,\varepsilon) = B'_{j,\beta_j}(y,D_y,D_z)u_0(y,z) + o(\varepsilon).$$

Since the equations (2.5) has the order 2μ, we have to exclude $m-\mu$ equations for u_0. The inner expansion is responsible for the rest. Rewriting boundary conditions $\{B_j\}$ into variables (y,t) and expanding it in the Taylor series at $(y,0,D_t)$ results in

$$B'_j(y,D_y,D_z,\varepsilon) = \varepsilon^{-\beta_j}\left(\sum_{k=\beta_j}^{b_j-\beta_j} B'_{j,k}(y,0,D_t) + \sum_{s\geqslant 1} \check{B}_s(y,D_y,D_t)\varepsilon^s\right)$$

For convenience, let us suppose that the boundary conditions are ordered so that $\beta_1 \leqslant \beta_2 \leqslant \beta_3 \leqslant \cdots \leqslant \beta_{m-1} \leqslant \beta_m$. Computing a contribution of $V(y,t,\varepsilon)$

to the boundary value gives us

$$
\begin{aligned}
&B'_j(y, D_y, D_t, \varepsilon)V(y, t, \varepsilon) \\
&\quad = \varepsilon^{l_\nu - \beta_j}\Bigg(\sum_{k=\beta_j}^{b_j} B'_{j,k}(y, 0, D_t)v_0 + \sum_{k=\beta_j}^{b_j} B'_{j,k}(y, 0, D_t)\sum_{s\geqslant 1}\varepsilon^s v_s(y, t) \\
&\qquad + \sum_{s\geqslant 1}\varepsilon^s \check{B}_s(y, D_y, D_t)\sum_{k=0}^{\infty}\varepsilon^k v_k(y, t)\Bigg) \\
&\quad = \varepsilon^{l_\nu - \beta_j}\Bigg(\sum_{k=\beta_j}^{b_j} B'_{j,k}(y, 0, D_t)v_0 + \sum_{s\geqslant 1}\varepsilon^s \check{B}_s(y, D_y, D_t)v_0(y, t) \\
&\qquad + \sum_{s\geqslant 1}\varepsilon^s \check{B}'_s(y, D_y, D_t)v_s(y, t)\Bigg),
\end{aligned}
\tag{2.7}
$$

where the operators $\check{B}'_s(y, D_y, D_t)$ are obtained by collecting terms like ε^s. Now we supply the system (2.5) with μ first boundary conditions

$$
(B'_{j,\beta_j}(y, D_y, D_z))u_0\big|_{\partial\Omega} = g_j(y), \quad j = 1, \dots, \mu. \tag{2.8}
$$

Further, we wish $u_0 + v_0$ to give discrepancy $o(\varepsilon)$ in boundary conditions. It may be done in several ways, however, we choose the following. First we set $l_\nu = \beta_{\mu+1}$. If $\beta_\mu < \beta_{\mu+1}$ then (2.7) implies

$$
B'_j(y, D_y, D_t, \varepsilon)V(y, t, \varepsilon) = o(\varepsilon) \quad \text{for} \quad j \leqslant \mu,
$$

so the choice of v_k, $k = 0, 1, \dots$, do not spoil the desired rate of convergence in the first μ boundary conditions. Otherwise there exists such μ_ν that $\beta_{\mu-\mu_\nu-1} < \beta_{\mu-\mu_\nu} = \beta_{\mu-\mu_\nu+1} = \beta_{\mu-\mu_\nu+2} = \dots = \beta_\mu = \beta_{\mu+1}$.

Taking the aforesaid into consideration, we state the following boundary conditions for v_0

$$
\begin{aligned}
&(B'_j(y, 0, D_t, \varepsilon))v_0\big|_{\partial\Omega} = g_j(y), && \beta_j = l_\nu \text{ and } j \geqslant \mu + 1; \\
&(B'_j(y, 0, D_t, \varepsilon))v_0\big|_{\partial\Omega} = 0, && \beta_j > l_\nu; \\
&(B'_j(y, 0, D_t, \varepsilon))v_0\big|_{\partial\Omega} = g_j(y) - (B'_j(y, D_y, D_t, \varepsilon))u_0\big|_{\partial\Omega}, && \beta_{\mu-\mu_\nu} \leqslant j \leqslant \beta_\mu.
\end{aligned}
\tag{2.9}
$$

Under these conditions, the expansion defined by the boundary value problem (2.5), (2.8) and by the Cauchy problem (2.6), (2.9) has the discrepancy $o(\varepsilon)$.

The last task is to extinguish the negative exponents $\varepsilon^{l_\nu - \beta_j}$; it means to put appropriate conditions on the next terms of expansions. Namely,

$$
\check{B}'_s(y, D_y, D_t)v_s\big|_{\partial\Omega} = -\check{B}_s(y, D_y, D_t)v_0\big|_{\partial\Omega}, \quad s = 1, 2, \dots, \beta_j - l_\nu.
$$

It shows that reducing discrepancy in boundary conditions needs taking into account more terms of asymptotic expansions.

2.2 A priori estimates

This section is aimed to prove the convergence of formal series (2.4). In suggestions that formal asymptotic expansions are defined correctly, we start the investigation with precise restrictions on the problem (2.1)–(2.2). First of all the diferential operators (2.1) and (2.2) are supposed to have smooth coefficients and the boundary Ω is also smooth. The operator (A, B) is postulated to be an *elliptic problem with small parameter*, *i.e.* to satisfy two restrictions pointed below.

(II) Small parameter ellipticity condition: in the case when dimension of $\Omega \subset \mathbb{R}^n$ is $n > 2$, the operator $A_0(x_0, D, \varepsilon)$ is called to be *small parameter elliptic* if at every point $x_0 \in \overline{\Omega}$ its principal polynomial $A_0(x_0, \xi, \varepsilon)$ admits the estimate

$$|A_0(x_0, \xi, \varepsilon)| \geqslant c_{x_0} |\xi|^{2\mu} (1 + \varepsilon|\xi|)^{2m-2\mu}$$

from below.

In the case $n = 2$ the polynomial $A_0(x, \xi', \xi_n, \varepsilon)$ considered with respect to the variable ξ_n is assumed to possess exactly m roots in the upper complex half-plane and m roots in the lower half-plane, for every $x \in \Omega$, $\varepsilon > 0$, $\xi' \in \mathbb{R}^{n-1}$.

(III) Shapiro-Lopatinskii condition with small parameter: boundary value problem $(A(x, D, \varepsilon), B(x', D, \varepsilon))$ satisfies the usual Shapiro-Lopatinskii condition **(I)** for each fixed $x' \in \partial\Omega$ and $\varepsilon \in [0, \varepsilon_0)$. This condition means that the polynomials $B_j(x', \xi, \varepsilon)$ are linearly independent modulo $A(x', \xi, \varepsilon)$ for each point $x' \in \partial\Omega$ and $\varepsilon \geqslant 0$.

2.3 The main spaces

On the Sobolev spaces $H^r(\mathbb{R}^n)$ and $H^\rho(\mathbb{R}^{n-1})$, where $r, \rho > 0$, we consider the norms

$$\|u\|_{r,s} = \left\| (1 + |\xi|^2)^{s/2} (1 + \varepsilon^2|\xi|^2)^{(r-s)/2} \tilde{u} \right\|_{L^2},$$
$$\|u\|_{\rho,\sigma,\mathbb{R}^{n-1}} = \|u\|_{L^2(\mathbb{R}^{n-1})} + \left\| |\eta|^\sigma (1 + \varepsilon^2|\eta|^2)^{(\rho-\sigma)/2} \tilde{u} \right\|_{L^2(\mathbb{R}^{n-1})}.$$

Then $H^{r,s}(\mathbb{R}^n)$ consists of all functions $u \in H^r(\mathbb{R}^n)$ which have the norm $\|u; H^{r,s}(\mathbb{R}^n)\| := \|u\|_{r,s}$ finite, and $H^{r,s}(\mathbb{R}^n_+)$ is the factor space $H^{r,s}(\mathbb{R}^n)/H^{r,s}_-(\mathbb{R}^n)$ where $H^{r,s}_-(\mathbb{R}^n)$ is the subspace of $H^{r,s}(\mathbb{R}^n)$ consisting of all functions with the support in $\{x \in \mathbb{R}^n : x_n \leqslant 0\}$. As usual, the factor space is endowed with the canonical norm

$$\big\|[u]; H^{r,s}(\mathbb{R}^n_+)\big\| = \inf_{u \in [u]} \|u\|_{r,s}.$$

When it does not cause misunderstanding we denote this norm simply by $\|u\|_{r,s}$. Analogously, given some domain Ω let us introduce the spaces of functions defined in it. To wit,

$$H^{r,s}(\Omega) := H^{r,s}(\mathbb{R}^n)/H^{r,s}_{\mathbb{R}^n\setminus\Omega}(\mathbb{R}^n)$$

where functions of $H^{r,s}_{\mathbb{R}^n\setminus\Omega}(\mathbb{R}^n)$ are supported outside the domain Ω. This space is also given the canonical norm $\|u; H^{r,s}(\Omega)\|$, which we denote sometimes by $\|u\|_{r,s}$ for short.

Lemma 5. *Let f be a smooth function in $\mathbb{R}^n$, such that $f(x) = 1$ for $x \in \Omega$. Then $\|u; H^{r,s}(\Omega)\| = \|fu; H^{r,s}(\Omega)\|$.*

Lemma 6. *If $u \in H^{r,s}(\mathbb{R}^n)$ and $\operatorname{supp} u \subset \overline{\Omega}$, then $\|u; H^{r,s}(\Omega)\| = \|u; H^{r,s}(\mathbb{R}^n)\|$.*

For positive integer numbers s and $r \geqslant s$ the space $H^{r,s}(\Omega)$ proves to be the completion of $C^\infty(\overline{\Omega})$ with respect to the norm $\|u; H^{r,s}(\Omega)\|_{r,s}$. The elliptic technique used in this chapter includes the “rectification” of the boundary. Therefore, the invariance of $\|\cdot\|_{r,s}$ with respect to a change of variables is one of the key points. For every fixed $\varepsilon \geqslant 0$, the norms $\|\cdot\|_{r,s}$ are the ordinary Sobolev norms and the main question is what kind of coordinate transformations save the form of the dependence of $\|\cdot\|_{r,s}$ on ε. The following statement displays how ε enters into the norms $\|\cdot\|_{r,s}$.

Lemma 7. *For natural r and s satisfying $r \geqslant s$, the squared norm $\|u; H^{r,s}(\Omega)\|^2$ has a representation of the form*

$$\sum_{\substack{i=0 \\ i \text{ is even}}}^{r} a_{r,s,i}(\varepsilon)\|\Delta^{i/2}u\|^2_{L^2(\Omega)} + \sum_{\substack{i=1 \\ i \text{ is odd}}}^{r} a_{r,s,i}(\varepsilon)\|\nabla^i u\|^2_{L^2(\Omega)},$$

where $a_{r,s,i}(\varepsilon)$ are polynomials of degree $2i$ and $a_{r,s,0}(\varepsilon) \neq 0$.

Proof. Applying the binomial formula we get

$$\left(1+|\xi|^2\right)^s = \sum_{i=0}^{s} C_s^i |\xi|^{2i} \qquad \text{and} \qquad \left(1+\varepsilon^2|\xi|^2\right)^{r-s} = \sum_{i=0}^{r-s} C_{r-s}^i \varepsilon^{2i} |\xi|^{2i}.$$

Hence, on multiplying the left-hand sides of these equalities we obtain

$$\left(1+|\xi|^2\right)^s \left(1+\varepsilon^2|\xi|^2\right)^{r-s} = \sum_{i=0}^{r} a_{r,s,i}(\varepsilon)|\xi|^{2i}$$

where

$$a_{r,s,i}(\varepsilon) = \sum_{j=0}^{i} C_s^{i-j} C_{r-s}^{j} \varepsilon^{2j}. \tag{2.10}$$

Here we assume $C_r^k = 0$ when $k > r$. If $\varepsilon = 0$ or $r = s$, then $a_{r,s,i} = C_s^i$. Therefore, $a_{r,s,i}(\varepsilon) \neq 0$ for all ε and $0 \leqslant i \leqslant r$. As a consequence, we get

$$\|u\|_{r,s}^2 = \sum_{i=0}^{r} a_{r,s,i}(\varepsilon) \left\| |\xi|^i \tilde{u} \right\|_{L^2}^2 .$$

Furthermore,

$$\||\xi|^i \tilde{u}\|_{L^2}^2 = \begin{cases} \left\|\Delta^{i/2} u\right\|_{L^2}^2, & \text{if } i \text{ is even,} \\ \|\nabla^i u\|_{L^2}^2, & \text{if } i \text{ is odd,} \end{cases}$$

which establishes the lemma. □

Now everything is prepared for proving the invariance of the norm $\|\cdot\|_{r,s}$ with respect to local changes of variables $x = T(y)$.

Lemma 8. *Let $r, s \in \mathbb{Z}_{\geqslant 0}$ satisfy $r \geqslant s$. The norm $\|u; H^{r,s}(\Omega)\|$ is invariant with respect to any local changes of variables in Ω of the form $x = T(y)$, such that*

(a) $T : U \to U'$ is a C^r-diffeomorphism of domains U and U' in $\mathbb{R}^n$, both U and U' intersecting Ω;

(b) $T(U \cap \overline{\Omega}) = U' \cap \overline{\Omega}$;

(c) $T(U \cap \partial\Omega) = U' \cap \partial\Omega$.

Our task is to prove that there is a constant $C > 0$ independent of ε, with the property that

$$\|T^* u; H^{r,s}(\Omega)\| \leqslant C \, \|u; H^{r,s}(\Omega)\| \tag{2.11}$$

for all smooth functions u in the closure of Ω supported in some compact set $K \subset U' \cap \overline{\Omega}$. Here, by $T^* u(y) := u(T(y))$ is meant the pullback of u by the diffeomorphism T. If u is supported in K, then T^*u is supported in $T^{-1}(K)$, which is a compact subset of $U' \cap \overline{\Omega}$ by the properties of T. Since this applies to the inverse $T^{-1} : U' \to U$, it follows from (2.11) that the space $H^{r,s}(\Omega)$ survives under the local C^r -diffeomerphisms of $\overline{\Omega}$.

Proof. For the proof we make use of another norm in $H^{r,s}(\Omega)$ which is obviously equivalent to $\|u; H^{r,s}(\Omega)\|$ and more convenient here. To wit,

$$\|u; H^{r,s}(\Omega)\| \cong \sum_{|\alpha| \leqslant r} a_{r,s,|\alpha|}(\varepsilon) \|\partial^\alpha u; L^2(\Omega)\| \tag{2.12}$$

or

$$\|u; H^{r,s}(\Omega)\| \cong \sum_{i=0}^{r} a_{r,s,i}(\varepsilon)\, \|u; H^i(\Omega)\|,$$

as is easy to verify, where $a_{r,s,i}(\varepsilon)$ are the polynomials of Lemma 7. Fix a compact set K in $U' \cap \overline{\Omega}$. As mentioned, if u is a smooth function in $\overline{\Omega}$ with the support in K, then T^*u is a smooth function in $\overline{\Omega}$ supported in $T^{-1}(K) \subset U \cap \overline{\Omega}$. Obviously,

$$\begin{aligned}\|T^*u; H^{r,s}(\Omega)\| &= \|u \circ T; H^{r,s}(U \cap \Omega)\| \\ &= \sum_{|\alpha| \leqslant r} a_{r,s,|\alpha|}(\varepsilon) \|\partial^\alpha (u \circ T); L^2(U \cap \Omega)\|.\end{aligned}$$

By the chain rule,

$$\partial_y^\alpha \big(u(T(y))\big) = \sum_{0 \neq \beta \leqslant \alpha} c_{\alpha,\beta}(y) \left(\partial_x^\beta u\right) (T(y))$$

for any multiindex α with $|\alpha| \leqslant r$. Here, the coefficients $c_{\alpha,\beta}(y)$ are polynomials of degree $|\beta|$ of partial derivatives of $T(y)$ up to order $|\alpha| - |\beta| + 1 \leqslant r$. Since $T : U \to U'$ is a diffeomorphism of class C^r, all the $c_{\alpha,\beta}(y)$ are bounded on the compact set $T^{-1}(K)$ and the Jacobian $\det T'(y)$ does not vanish on $T^{-1}(K)$. This implies

$$\begin{aligned}\|T^*u; H^{r,s}(\Omega)\| &\leqslant c \sum_{|\alpha| \leqslant r} a_{r,s,|\alpha|}(\varepsilon) \sum_{\beta \leqslant \alpha} \|(\partial_x^\beta u) \circ T; L^2(T^{-1}(K))\| \\ &\leqslant c \sum_{|\alpha| \leqslant r} a_{r,s,|\alpha|}(\varepsilon) \sum_{\beta \leqslant \alpha} \|\partial_x^\beta u; L^2(K)\|,\end{aligned}$$

where $c = c(T, r, K)$ is a constant independent of u and different in diverse applications. Interchanging the sums in α and β yields

$$\|T^* u; H^{r,s}(\Omega)\| \leqslant c \sum_{|\beta| \leqslant r} \Big(\sum_{\substack{|\alpha| \leqslant r \\ \alpha \geqslant \beta}} a_{r,s,|\alpha|}(\varepsilon) \Big) \sum_{0 \neq \beta \leqslant \alpha} \|\partial^\beta u; L^2(\Omega)\|$$

for all smooth functions u in $\overline{\Omega}$ with the support in K.

Therefore, if there is a constant $C > 0$ such that

$$\sum_{\substack{|\alpha| \leqslant r \\ \alpha \geqslant \beta}} a_{r,s,|\alpha|}(\varepsilon) \leqslant C\, a_{r,s,|\beta|}(\varepsilon)$$

for each multiindex β of norm $|\beta| \leqslant r$, then the lemma follows. Since

$$\sum_{\substack{|\alpha| \leqslant r \\ \alpha \geqslant \beta}} a_{r,s,|\alpha|}(\varepsilon) \leqslant c \sum_{i=|\beta|}^{r} a_{r,s,i}(\varepsilon)$$

with c a constant dependent only on r and n, we are left with the task to show that there is a constant $C > 0$ independent of ε, such that

$$\sum_{i=i_0}^{r} a_{r,s,i}(\varepsilon) \leqslant C\, a_{r,s,i_0}(\varepsilon)$$

for all $i_0 = 0, 1, \ldots, r$. This latter estimate is in turn fulfilled if we show that

$$a_{r,s,i}(\varepsilon) \leqslant C\, a_{r,s,i-1}(\varepsilon) \tag{2.13}$$

for all $i = 1, \ldots, r$, where C is a constant independent of $\varepsilon \in [0, 1]$. By the formula (2.10),

$$a_{r,s,i}(\varepsilon) = \sum_{j=0}^{i-s-1} C_s^{i-j} C_{r-s}^{j} \varepsilon^{2j},$$

hence, the estimate (2.13) is fulfilled for sufficiently small $\varepsilon > 0$ with any constant C greater than C_s^i / C_s^{i-1}. Since (2.13) is valid for all ε in any interval $[\varepsilon_0, 1]$ with $\varepsilon_0 > 0$, the proof is complete. □

Further we use the trace theorem proved in [Vol06].

Theorem 6. *For $r > l + 1/2$ and $s \geqslant 0$, $s \neq l + 1/2$, we have*

$$\|D_n^l u(\cdot, 0); \mathcal{H}^{r-l-1/2, s-l-1/2}(\mathbb{R}^{n-1})\| \leqslant c\, \|u; H^{r,s}(\mathbb{R}^n_+)\|$$

with c a constant independent of ε.

The spaces $\mathcal{H}^{\rho,\sigma}(\partial\Omega)$ are defined by locally rectifying the boundary surface. Since the boundary is compact, there is a finite covering $\{U_i\}_{i=1}^N$ of $\partial\Omega$ consisting of sufficiently small open subsets U_i of $\mathbb{R}^n$. Let $\{\phi_i\}$ be a partition of unity in a neighbourhood of $\partial\Omega$ subordinate to this covering. If U_i is small enough, there is a smooth diffeomorphism h_i of U_i onto an open set O_i in $\mathbb{R}^n$, such that $h_i(U_i \cap \Omega) = O_i \cap \mathbb{R}^n_+$ and $h_i(U_i \cap \partial\Omega) = O_i \cap \mathbb{R}^{n-1}$, where $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n : x_n = 0\}$. The transition mappings $T_{i,j} = h_i^{-1} \circ h_j$ prove to be local diffeomorphism of Ω, as explained in Lemma 8. For any smooth function u on the boundary the norm $\left\| \left(h_i^{-1}\right)^* (\phi_i u); \mathcal{H}^{\rho,\sigma}\left(\mathbb{R}^{n-1}\right) \right\|$ is obviously well defined and we set

$$\|u; \mathcal{H}^{\rho,\sigma}(\partial\Omega)\| := \sum_{i=1}^{N} \left\| \left(h_i^{-1}\right)^* (\phi_i u); \mathcal{H}^{\rho,\sigma}(\mathbb{R}^{n-1}) \right\|, \tag{2.14}$$

where $\left(h_i^{-1}\right)^* (\phi_i u) = (\phi_i u) \circ h_i^{-1}$. As usual, the space $\mathcal{H}^{\rho,\sigma}(\partial\Omega)$ is introduced to be the completion of $C^\infty(\partial\Omega)$ with respect to the norm (2.14).

When combined Lemma 8 with the trace theorem 6 for $H^{r,s}(\mathbb{R}^n_+)$ and $\mathcal{H}^{\rho,\sigma}(\mathbb{R}^{n-1})$, a familiar trick readily shows that the Banach spaces $\mathcal{H}^{\rho,\sigma}(\partial\Omega)$ are actually independent of the particular choice of the covering of $\partial\Omega$ by coordinate patches $\{U_i\}$ in $\mathbb{R}^n$, the special coordinate system $h_i : U_i \to \mathbb{R}^n$ in U_i and the partition of unity $\{\phi_i\}$ in a neighbourhood of $\partial\Omega$ subordinate to the covering $\{U_i\}$. Any other choice of these data leads to an equivalent norm (2.14) in $C^\infty(\partial\Omega)$.

2.4 Auxiliary results

Estimates for model problems on $\mathbb{R}^n$ and $\mathbb{R}^n_+$. The estimate 2.3 was derived by Volevich [Vol06] for the half-space $\mathbb{R}^n_+$. More precisely his results are expressed in

Theorem 7. *Let the boundary value problem*

$$\begin{aligned} A(D,\varepsilon)u &= \varepsilon^{2m-2\mu} A_{2m}(D)u + \varepsilon^{2m-2\mu-1} A_{2m-1}(D)u + \cdots \\ &\quad + \varepsilon A_{2\mu+1}(D)u + A_{2\mu}(D)u = f(x), \quad x \in \mathbb{R}^n_+; \\ B_j(D,\varepsilon)u\big|_{x_n=0} &= \varepsilon^{b_j-\beta_j} B_{j,b_j}(D)u + \varepsilon^{b_j-\beta_j-1} B_{j,b_j-1}(D)u + \cdots \\ &\quad + B_{j,\beta_j}(D)u\big|_{x_n=0} = g_j(x'), \quad x' \in \mathbb{R}^{n-1}, \quad j = 1,\dots,m, \end{aligned}$$

where $A(D,\varepsilon), B_j(D,\varepsilon)$ have constant coefficients. The following statements are equivalent

(a) *$A(D,\varepsilon)$ is a properly elliptic operator, $A(D,\varepsilon), B_j(D,\varepsilon)$ satisfies Shapiro-Lopatinskii condition with small parameter;*

(b) *For each solution $u \in H^{r,s}(\mathbb{R}^n_+)$ there is the estimate*

$$\begin{aligned}&\|u; H^{r,s}(\mathbb{R}^n_+)\| \\ &\quad \leqslant C\Big(\big\|A(D,\varepsilon)u; H^{r-2m,s-2\mu}(\mathbb{R}^n_+)\big\| \\ &\qquad + \sum_{j=1}^{m} \big\|B_j(D,\varepsilon)u\big|_{x_n=0}; \mathcal{H}^{r-b_j-1/2,s-\beta_j-1/2}(\mathbb{R}^{n-1})\big\| + \big\|u; L^2(\mathbb{R}^n_+)\big\|\Big),\end{aligned}$$

where C does not depend on ε.

In the case of $\mathbb{R}^n$ the similar fact is

Theorem 8. *The solutions $u \in \mathcal{S}(\mathbb{R}^n_+)$ of the elliptic with small parameter equation $A(D,\varepsilon)u = f$, $f \in H^{r-2m,s-2\mu}(\mathbb{R}^n)$, satisfy the estimate*

$$\|u; H^{r,s}(\mathbb{R}^n)\| \leqslant C\, \big\|A(D,\varepsilon)u; H^{r-2m,s-2\mu}(\mathbb{R}^n)\big\|$$

where C does not depend on u and ε. Conversely, if the a priori *estimate holds then operator $A(D,\varepsilon)$ is elliptic.*

Invariance of norms. According to the usual local techniques of elliptic theory, the theory of elliptic boundary value problems with small parameter include only three additional estimates uniform in the parameter. To wit,

(a) the invariance of the norm with respect to local changes of variables on the compact manifold $\overline{\Omega}$;

(b) estimates of the form $\varepsilon^k \|\partial^\alpha u\|_{r,s} \leqslant c\, \|u\|_{r',s'}$ with c independent of ε;

(c) inequalities like $\|u\|_{r,s} \leqslant \delta\, \|u\|_{r',s'} + C(\delta)\, \|u\|_{L^2}$ with $r' \geqslant r$, $s' \geqslant s$ and $\delta > 0$ a fixed arbitrary small parameter.

As usual, we write α, β and γ for multiindices. By $\beta \leqslant \alpha$ is meant that $\beta_i \leqslant \alpha_i$ for all $i = 1, \ldots, n$. We first recall several basic inequalities concerning Sobolev spaces. Directly from the multinomial theorem we obtain

$$|\xi^\alpha| \leqslant \frac{1}{C_n^\alpha}\, |\xi|^{|\alpha|/2}, \tag{2.15}$$

where C_n^α is the multinomial coefficient. This inequality, if combined with the Plancherel theorem, yields

$$\|\partial^\alpha u\|_{L^2} \leqslant \frac{1}{C_n^\alpha} \|\Delta^{|\alpha|/2} u\|_{L^2}$$

for all $u \in H^{|\alpha|} := H^{|\alpha|}(\mathbb{R}^n)$, where $\Delta^{|\alpha|/2}$ is a fractional power of the Laplace operator in $\mathbb{R}^n$.

Besides, we use the following consequence of the embedding theorem for the Sobolev spaces (see *e.g.* [Bur98]).

Theorem 9. *Suppose u is a square integrable function with compact support in $\mathbb{R}^n$ and $\alpha \in \mathbb{Z}^n_{\geqslant 0}$ is fixed. If, in addition, the weak derivatives $\partial^\beta u$ are square integrable for all $\beta \leqslant \alpha$, then*

$$\|\partial^\beta u\|_{L^2} \leqslant C \|\partial^\alpha u\|_{L^2},$$

where $C = \sup\{|x|^2 : x \in \operatorname{supp} u\}$.

We also need some basic inequalities for the norms $\| \cdot \|_{r,s}$.

Lemma 9. *Let $u \in H^{r,s}(\mathbb{R}^n)$ be a function with compact support, $k \geqslant 1$ an integer and α a multiindex. Then:*

(a) We have $\varepsilon \|u\|_{r,s} \leqslant c \|u\|_{r+1,s}$, where c depends on the support of u but not on u and ε.

(b) If $k > |\alpha|$, then $\varepsilon^k \|\partial^\alpha u\|_{r,s} \leqslant c \|u\|_{r+k,s}$, with the constant c being independent of u and ε.

(c) If $k \leqslant |\alpha|$, then $\varepsilon^k \|\partial^\alpha u\|_{r,s} \leqslant c \|u\|_{r+|\alpha|,s+|\alpha|-k}$, where c is independent of u and ε.

Proof. Using the expression for the norm in $H^{r,s}(\mathbb{R}^n)$ we get

$$\begin{aligned} \varepsilon \left\|\Delta^{1/2} u\right\|_{r,s} &= \varepsilon \left\| |\xi| \left(1+|\xi|^2\right)^{s/2} \left(1+\varepsilon^2|\xi|^2\right)^{(r-s)/2} \tilde{u} \right\|_{L^2} \\ &\leqslant \|u\|_{r+1,s}. \end{aligned}$$

As $\|u\|_{r,s} \leqslant c \|\Delta^{1/2} u\|_{r,s}$, the part (a) is true.

The part (b) is proved in much the same way if one applies $k - |\alpha|$ times what has already been proved in the part (a).

To prove the part (c) we split the majorising factor as $\varepsilon^k|\xi|^{|\alpha|} = (\varepsilon|\xi|)^k\,|\xi|^{|\alpha|-k}$. The first factor contributes with order k to the terms with ε while the second one does $|\alpha| - k$ to the others. □

The part (b) actually holds for all function in $H^{r+k,s}$ even if u fails to be of compact support.

Lemma 10. *Let δ be an arbitrary small positive number. Then there is a constant $C(\delta)$, such that*

$$\|u\|_{r-1,s-1} \leqslant \delta\,\|u\|_{r,s} + C(\delta)\,\|u\|_{L^2}$$

for all $u \in H^{r,s}(\mathbb{R}^n)$.

Proof. Set $\langle\xi\rangle = \sqrt{1+|\xi|^2}$ for $\xi \in \mathbb{R}^n$. Given any $R > 0$, we obtain

$$\begin{aligned}\|u\|^2_{r-1,s-1} &= \int_{|\xi|>R} \frac{\langle\xi\rangle^{2s}}{\langle\xi\rangle^2}\langle\varepsilon\xi\rangle^{2(r-s)}|\tilde{u}|^2 d\xi + \int_{|\xi|\leqslant R}\langle\xi\rangle^{2(s-1)}\langle\varepsilon\xi\rangle^{2(r-s)}|\tilde{u}|^2 d\xi \\ &\leqslant \frac{1}{1+R^2}\,\|u\|^2_{r,s} + \left(1+R^2\right)^{s-1}\left(1+\varepsilon^2R^2\right)^{r-s}\,\|u\|^2_{L^2},\end{aligned}$$

Choosing $R > 0$ in such a way that $\delta^2 \leqslant (1+R^2)^{-1}$, we establish the estimate. □

2.5 Local estimates in the interior

This following fact stated the *a priori* estimates for the functions with slowly varying coefficients.

Theorem 10. *For every $x_0 \in \Omega$ there exists a neighbourhood U_{x_0} in Ω and a constant C independent of ε, such that*

$$\|u\|_{r,s} \leqslant C\left(\|A(x,D,\varepsilon)u\|_{r-2m,s-2\mu} + \|u\|_{L^2}\right) \tag{2.16}$$

for all functions $u \in H^{r,s}(\Omega)$ with compact support in U_{x_0}, where $r \geqslant 2m$, $s \geqslant 2\mu$ are integer.

Proof. If $u \in H^{r,s}(\Omega)$ is compactly supported in Ω, it can be thought of as an element of $H^{r,s}(\mathbb{R}^n)$ as well. The norm of u in $H^{s,r}(\Omega)$ just amounts to the norm of u in $H^{s,r}(\mathbb{R}^n)$. Hence, Theorem 7 applies if $A(x,D,\varepsilon)$ has constant coefficients, as is the case *e.g.* for $A_0(x_0,D,\varepsilon)$, the principal part of $A(x,D,\varepsilon)$

with coefficients frozen at x_0. According to Theorem 8, there is a constant $C > 0$ independent of ε, such that

$$\|u\|_{r,s} \leqslant C \, \|A_0(x_0, D, \varepsilon)u\|_{r-2m,s-2\mu} \tag{2.17}$$

for all functions $u \in H^{r,s}(\Omega)$ of compact support in Ω.

We are thus left with the task to majorise the right-hand side of (2.17) by that of (2.16) uniformly in $\varepsilon \in [0, 1]$ on functions with compact support in U_{x_0}. To this end, we write

$$A_0(x_0, D, \varepsilon) = A(x, D, \varepsilon) - \big(A(x, D, \varepsilon) - A_0(x, D, \varepsilon)\big) - \big(A_0(x, D, \varepsilon) - A_0(x_0, D, \varepsilon)\big)$$

whence

$$\begin{aligned} \|A_0(x_0, D, \varepsilon)u\|_{r-2m,s-2\mu} \leqslant & \; \|A(x, D, \varepsilon)u\|_{r-2m,s-2\mu} \\ & + \|(A(x, D, \varepsilon) - A_0(x, D, \varepsilon))u\|_{r-2m,s-2\mu} \\ & + \|(A_0(x, D, \varepsilon) - A_0(x_0, D, \varepsilon))u\|_{r-2m,s-2\mu}. \end{aligned} \tag{2.18}$$

Our next concern will be to estimate the last two summands on the right-hand side of (2.18). We begin with the first of these two. By the very structure of the operator $A(x, D, \varepsilon)$, the difference $A(x, D, \varepsilon) - A_0(x, D, \varepsilon)$ is the sum of terms of the form

$$\varepsilon^{2m-2\mu-k} a_{k,\beta}(x) \partial^\beta u,$$

where $k = 0, 1, \ldots, 2m - 2\mu$, $|\beta| \leqslant 2m - k - 1$ and $a_{k,\beta}$ are smooth functions in the closure of Ω (cf. (2.24)). Hence, the reasoning used in the proof of Theorem 12 shows that the second summand on the right-hand side of (2.18) is dominated uniformly in $\varepsilon \in [0, 1]$ by the norm $\|u\|_{r-1,s-1}$. On applying Lemma 10 we conclude that

$$\|(A(x, D, \varepsilon) - A_0(x, D, \varepsilon))u\|_{r-2m,s-2\mu} \leqslant \delta \, \|u\|_{r,s} + C(\delta) \, \|u\|_{L^2}, \tag{2.19}$$

where $\delta > 0$ is an arbitrarily small parameter and $C(\delta)$ depends only on δ but not on u and ε.

It remains to estimate the last summand on the right-hand side of (2.18). Let us write

$$A_0(x, D, \varepsilon) = \sum_{2\mu \leqslant |\beta| \leqslant 2m} \varepsilon^{|\beta| - 2\mu} A_{0,\beta}(x) \partial^\beta,$$

where $A_{0,\beta}$ are smooth functions on the closure of Ω. Then

$$\begin{aligned}&\|(A_0(x,D,\varepsilon)-A_0(x_0,D,\varepsilon))u\|_{r-2m,s-2\mu}\\ &\quad\leqslant \sum_{2\mu\leqslant|\beta|\leqslant 2m} \varepsilon^{|\beta|-2\mu}\|(A_{0,\beta}(x)-A_{0,\beta}(x_0))\partial^\beta u\|_{r-2m,s-2\mu}.\end{aligned}$$

To evaluate the summands we invoke the equivalent expression for the norm in $H^{r-2m,s-2\mu}(\Omega)$ given by (2.12). The typical term is

$$a_{r-2m,s-2\mu,|\alpha|}(\varepsilon)\,\varepsilon^{|\beta|-2\mu}\,\|\partial^\alpha\left((A_{0,\beta}(x)-A_{0,\beta}(x_0))\partial^\beta u\right)\|_{L^2(\Omega)}$$

with $|\alpha|\leqslant r-2m$ and $a_{r-2m,s-2\mu,|\alpha|}(\varepsilon)$ are the polynomials defined in of (2.10). By the Leibniz formula,

$$\partial^\alpha\left((A_{0,\beta}(x)-A_{0,\beta}(x_0))\partial^\beta u\right)=(A_{0,\beta}(x)-A_{0,\beta}(x_0))\partial^{\alpha+\beta}u+[\partial^\alpha,A_{0,\beta}]\,\partial^\beta u,$$

where the commutator $[\partial^\alpha,A_{0,\beta}]$ is a differential operator of order $|\alpha|-1$ with coefficients smooth in $\overline{\Omega}$. Observe that $|\alpha|+|\beta|\leqslant r$. Arguing as above we derive easily an estimate like (2.19) for the sum

$$\sum_{|\alpha|\leqslant r-2m} a_{r-2m,s-2\mu,|\alpha|}(\varepsilon)\,\varepsilon^{|\beta|-2\mu}\,\|[\partial^\alpha,A_{0,\beta}]u\|_{L^2(\Omega)}$$

whenever $u\in H^{r,s}(\Omega)$ is of compact support in Ω.

It is the term

$$a_{r-2m,s-2\mu,|\alpha|}(\varepsilon)\,\varepsilon^{|\beta|-2\mu}\,\|(A_{0,\beta}(x)-A_{0,\beta}(x_0))\partial^{\alpha+\beta}u\|_{L^2(\Omega)}$$

that admits a desired estimate only in the case if the support of u is small enough. (Recall that u is required to have compact support in U_{x_0}.) Since the coefficients $A_{0,\beta}(x)$ are Lipschitz continuous in $\overline{\Omega}$, for any arbitrarily small $\delta'>0$ there is a positive $\varrho=\varrho(\delta')$, such that

$$\left\|(A_{0,\beta}(x)-A_{0,\beta}(x_0))\partial^{\alpha+\beta}u\right\|_{L^2(\Omega)}\leqslant\delta'\,\left\|\partial^{\alpha+\beta}u\right\|_{L^2(\Omega)}$$

for all functions $u\in H^{r,s}(\Omega)$ with compact support in $B(x_0,\varrho)$, the ball of radius ϱ with the centre at x_0.

Summarising we conclude that for each $\delta>0$ there is a constant $C=C(\delta)$ independent of ε, such that

$$\|(A_0(x,D,\varepsilon)-A_0(x_0,D,\varepsilon))u\|_{r-2m,s-2\mu}\leqslant\delta\,\|u\|_{r,s}+C(\delta)\,\|u\|_{L^2}\qquad(2.20)$$

for all functions $u \in H^{r,s}(\Omega)$ with compact support in $B(x_0, \varrho)$, provided that $\varrho = \varrho(\delta)$ is sufficiently small. Needless to say that $C(\delta)$ need not coincide with the similar constant of the inequality (2.19). However, we may assume this without loss of generality.

On gathering estimates (2.18) and (2.19), (2.20) and substituting them into (2.16) we arrive at

$$(1 - 2C\delta)\, \|u\|_{r,s} \leqslant C \Big(\|A(x,D,\varepsilon)u\|_{r-2m,s-2\mu} + 2C(\delta)\, \|u\|_{L^2} \Big)$$

for all $u \in H^{r,s}(\Omega)$ with the compact support in $B(x_0, \varrho)$. Of course, this latter inequality does not yield any estimate for $\|u\|_{r,s}$ unless $1 - 2C\delta > 0$. Thus, choosing $\delta < 1/2C$ we get

$$\|u\|_{r,s} \leqslant \frac{2CC(\delta)}{1-2C\delta} \big(\|A(x,D,\varepsilon)u\|_{r-2m,s-2\mu} + \|u\|_{L^2} \big),$$

if $C(\delta) \geqslant 1/2$. □

2.6 The case of boundary points

Localisation at a boundary point $x_0 \in \partial\Omega$ requires not only small parameter ellipticity of the operator $A(x, D, \varepsilon)$ but also the Shapiro-Lopatinskii condition with small parameter.

Theorem 11. *For every point $x_0 \in \partial\Omega$ there is a neighbourhood U_{x_0} in $\mathbb{R}^n$, such that*

$$\begin{aligned} &\|u\|_{r,s} \\ &\leqslant C \Big(\|A(x,D,\varepsilon)u\|_{r-2m,s-2\mu} + \sum_{j=1}^{m} \big\|B_j(x',D,\varepsilon)u\big|_{\partial\Omega}\big\|_{r-b_j-1/2,s-\beta_j-1/2} + \|u\|_{L^2(\Omega)} \Big) \end{aligned} \tag{2.21}$$

for all functions $u \in H^{r,s}(\Omega)$ with compact support in $U_{x_0} \cap \Omega$, where C is a constant independent of both u and $\varepsilon \in [0,1]$.

Proof. Choose a neighbourhood U of x_0 in $\mathbb{R}^n$ and a diffeomorphism $z = h(x)$ of U onto an a neighbourhood O of the origin $0 = h(x_0)$ in $\mathbb{R}^n$ with the property that $h(U \cap \Omega) = O \cap \mathbb{R}^n_+$ and $h(U \cap \partial\Omega) = \{z \in O : z_n = 0\}$. If $u \in H^{r,s}(\Omega)$ is a function with compact support in $U \cap \overline{\Omega}$, then the pullback $\tilde{u} = (h^{-1})^* u$ belongs to $H^{r,s}(\mathbb{R}^n_+)$ and has compact support in $O \cap \overline{\mathbb{R}^n_+}$, which is due to Lemma 8.

On setting

$$\begin{aligned} A^\sharp &:= (h^{-1})^* A h^*, \\ B_j^\sharp &:= (h^{-1})^* B_j h^* \end{aligned}$$

for $j = 1, \dots, m$, we obtain the pullbacks of the operators A and B_j under the diffeomorphism $h : U \cap \overline{\Omega} \to O \cap \overline{\mathbb{R}^n_+}$. It is easily seen that $A^\sharp$ and $B_j^\sharp$ are differential operators with small parameter $\varepsilon \in [0,1]$ on $O \cap \overline{\mathbb{R}^n_+}$ in the sense explained above. We write $\tilde{A} := A^\sharp$ and $\tilde{B}_j := B_j^\sharp$ for short. Since the spaces $H^{r,s}(\Omega)$ and $\mathcal{H}^{\rho,\sigma}(\partial\Omega)$ are invariant under local diffeomorphisms of Ω, it follows that estimate (2.21) is equivalent to

$$\begin{aligned} & \|\tilde{u}\|_{r,s} \\ \leqslant\ & C \left(\|\tilde{A}(z,D,\varepsilon)\tilde{u}\|_{r-2m,s-2\mu} + \sum_{j=1}^m \|\tilde{B}_j(z',D,\varepsilon)\tilde{u}\|_{r-b_j-1/2,s-\beta_j-1/2} + \|\tilde{u}\|_{L^2(\Omega)} \right) \end{aligned} \tag{2.22}$$

for all functions $\tilde{u} \in H^{r,s}(\mathbb{R}^n_+)$ with compact support in $O \cap \overline{\mathbb{R}^n_+}$, where C is a constant independent of $\tilde{u}$ and ε.

From the transformation formula for principal symbols of differential operators it follows that the problem

$$\begin{cases} \tilde{A}_0(0,D,\varepsilon)\tilde{u} = \tilde{f} & \text{for} \quad z_n > 0, \\ \tilde{B}_{j,0}(0,D,\varepsilon)\tilde{u} = \tilde{u}_j & \text{for} \quad z_n = 0, \end{cases}$$

where $j = 1, \dots, m$, satisfies both the ellipticity condition and the Shapiro-Lopatinskii condition with small parameter in the half-space. We now apply the main result of [Vol06] which says that there is a constant $C > 0$ independent of ε, such that the inequality

$$\begin{aligned} & \|\tilde{u}\|_{r,s} \\ \leqslant\ & C \left(\|\tilde{A}_0(0,D,\varepsilon)\tilde{u}\|_{r-2m,s-2\mu} + \sum_{j=1}^m \|\tilde{B}_{j,0}(0,D,\varepsilon)\tilde{u}\|_{r-b_j-1/2,s-\beta_j-1/2} + \|\tilde{u}\|_{L^2(\mathbb{R}^n_+)} \right) \end{aligned}$$

holds true for all functions $\tilde{u} \in H^{r,s}(\mathbb{R}^n_+)$ with compact support in the closed half-space.

Estimate (2.22) follows from the latter estimate in much the same way as estimate (2.16) does from (2.17), see the proof of Theorem 10. The only difference consists in evaluating the boundary terms. However, estimates on the boundary are reduced readily to those in the half-space if one exploits the embedding theorem, see Theorem 6. Namely,

$$\begin{aligned}&\|(\tilde{B}_j(z', D, \varepsilon) - \tilde{B}_{j,0}(0, D, \varepsilon))\tilde{u};\ \mathcal{H}^{r-b_j-1/2, s-\beta_j-1/2}(\mathbb{R}^{n-1})\| \\ &\quad \leqslant c\, \|(\tilde{B}_j(z', D, \varepsilon) - \tilde{B}_{j,0}(0, D, \varepsilon))\tilde{u};\ H^{r-b_j, s-\beta_j}(\mathbb{R}^n_+)\|\end{aligned}$$

with c a constant independent of $\tilde{u}$ and ε. □

2.7 The a priori elliptic estimate in a bounded domain

Now we are in a position to present the main result of the present chapter related to the Fredholm property and *a priori* estimate for boundary value problems of the form (2.1)–(2.2). Recall that the problems must satisfy conditions mentioned in the section 2.2. Let $r \geqslant 2m$ and $s \geqslant 2\mu$.

Theorem 12. *The boundary value problem* (2.1)-(2.2) *is elliptic with small parameter if and only if there is the estimate*

$$\begin{aligned}&\|u\|_{r,s} \\ &\leqslant C\left(\|A(x, D, \varepsilon)u\|_{r-2m, s-2\mu} + \sum_{j=1}^{m} \|B_j(x', D, \varepsilon)u\|_{r-b_j-\frac{1}{2}, s-\beta_j-\frac{1}{2}} + \|u\|_{L^2(\Omega)}\right)\end{aligned} \tag{2.23}$$

with C a constant independent of u and ε.

Proof. Our proof exploits the scheme pointed out in the begin of this chapter. For each point $x_0 \in \Omega$ we choose a neighbourhood U_{x_0} in Ω in which the estimate of Theorem 10 holds. And for each point $x_0 \in \partial\Omega$ we choose a neighbourhood U_{x_0} in $\mathbb{R}^n$, such that the estimate of Theorem 11 is valid. Shrinking U_{x_0}, if necessary, one can assume that the surface $U_{x_0} \cap \partial\Omega$ can be rectified by some diffeomorphism $h_i : U_{x_0} \to \mathbb{R}^n$, as explained above. The family $\{U_{x_0}\}_{x_0 \in \overline{\Omega}}$ is an open covering of $\overline{\Omega}$, hence it contains a finite family $\{U_i\}$ which covers $\overline{\Omega}$. Fix a C^∞ partition of unity $\{\phi_i\}$ in a neighbourhood of $\overline{\Omega}$ subordinate to the covering $\{U_i\}$.

Given any $u \in H^{r,s}(\Omega)$, we get

$$u = \sum_i u_i$$

in Ω, where $u_i := \phi_i u$ belongs to $H^{s,r}(\Omega)$ and $\operatorname{supp} u_i \subset U_i \cap \overline{\Omega}$. By assumption, for any function u_i estimate (2.23) holds with a constant C depending on i. As the family $\{U_i\}$ is finite, there is no restriction of generality in assuming that C does not depend on i. Hence,

$$\|u\|_{r,s} \leqslant C$$
$$\times \sum_i \Big(\|A(x,D,\varepsilon)u_i\|_{r-2m,s-2\mu} + \sum_{j=1}^{m} \|B_j(x',D,\varepsilon)u_i\|_{r-b_j-\frac{1}{2},s-\beta_j-\frac{1}{2}} + \|u_i\|_{L^2(\Omega)} \Big).$$

By the Leibniz formula,

$$A(x,D,\varepsilon)u_i = \phi_i\, A(x,D,\varepsilon)u + [A,\phi_i]\, u,$$
$$B_j(x',D,\varepsilon)u_i = \phi_i\, B_j(x,D,\varepsilon)u + [B_j,\phi_i]\, u,$$

where $[A,\phi_i]u = A(\phi_i u) - \phi_i A u$ is the commutator of A and the operator of multiplication with ϕ_i, and similarly for $[B_j,\phi_i]$. The commutators are known to be differential operators of order less than that of A and B_j, respectively. From the structure of the operator $A(x,D,\varepsilon)$ we see that the summands of $[A,\phi_i]u$ are of the form

$$\varepsilon^{2m-2\mu-k} a_{k,\beta}(x)\partial^\beta u, \tag{2.24}$$

where $k = 0,1,\ldots,2m-2\mu$, $|\beta| \leqslant 2m-k-1$ and $a_{k,\beta}$ are smooth functions in the closure of Ω independent of u.

To estimate the norm of (2.24) in $H^{r-2m,s-2\mu}$, we apply Lemma 9 and consider separately the cases

$$2m-2\mu-k > |\beta|,$$
$$2m-2\mu-k \leqslant |\beta|.$$

If *e.g.* $|\beta| \geqslant 2m-2\mu-k$, then

$$\varepsilon^{2m-2\mu-k} \big\|a_{k,\beta}\partial^\beta u\big\|_{r-2m,s-2\mu} \leqslant c\, \varepsilon^{2m-2\mu-k} \big\|a_{k,\beta}\partial^\beta u\big\|_{r-2m+|\beta|,s-2m+|\beta|+k},$$

where $|\beta|-2m+k \leqslant -1$. It follows that

$$\varepsilon^{2m-2\mu-k} \big\|a_{k,\beta}\partial^\beta u\big\|_{r-2m,s-2\mu} \leqslant c\, \big\|u\big\|_{r-1,s-1}$$

with c a constant independent of u and ε. Such terms are handled by Lemma 10. Analogously we estimate the summands (2.24) with $2m-2\mu-k > |\beta|$ and the commutators $[B_j,\phi_i]$, which establishes (2.23). □

2.8 Conclusions

The Poisson (or stationary heat/diffusion) equation has solutions which are the stable states in diffusion of matter or energy. Such processes arise in Physics (*e.g.* of solid matter) very often, with no exception for the singularly perturbed case. This fundamental value of the heat\diffusion equation motivated the study of the second chapter. Let us put forward one example.

> **Example** (The Poisson equation)**.** In a solid homogeneous medium the stable distribution of temperature $T(x)$, where $x = (x_1, x_2, x_3)$ is the spatial coordinate, is determined by the Poisson equation
>
> $$-\varkappa \sum_{k=1}^{3} \frac{\partial^2 T}{\partial x_k^2}(x) = -\varkappa \Delta T(x) = \omega(x).$$
>
> Here $\omega(x)$ describes sources of heating, and the constant $\varkappa$ denotes the thermal conductivity of the material. The case of low-conductive media ($\varkappa \ll 1$) and active heat sources ($\omega(x) \gg 1$ in a neighbourhood of some point x_0) corresponds to the equation with small coefficient. That is, the conductivity $\varkappa$ can be accepted as a small parameter.

Solving more general systems under some special conditions frequently give rise to elliptic equations of higher order. For example, if we assume that $\omega(x)$ in the example above also satisfies the Poisson equation, we obtain the Sophie Germain equation, which arises in the theory of small vibrations of thin plates as well as in the study of the stream functions in 2D flows of viscous incompressible fluid. Or more generally, if we take $\omega(x, T, D^\alpha T) = \omega_0(x) + \varepsilon^{|\alpha|-2} A(x, D)T$, then the case of elliptic operator $A(x, D)$ corresponds to equations dependent on small ε, a particular case of what was investigated above.

In Section 2.1 we have constructed a formal asymptotic solution of the general elliptic boundary vaule problem (2.1)-(2.2) and the estimate (2.3) proves they are true asymptotic expansions. The main theoretical result of this chapter is that the stated Shapiro-Lopatinskii condition **(III)** and ellipticity with small parameter **(II)** are equivalent to the *a priori* estimate (2.3). The estimate is more complicated for checking in certain situations.

One of the central points is the study of norms $\|u; H^{r,s}(\Omega)\|$ and $\|u; \mathcal{H}^{\rho,\sigma}(\partial\Omega)\|$ in which the estimates are obtained. The most important result is their invariance with respect to the smooth change of variables given by Lemma 8, which allows one to define these norms on manifolds.

The results of this chapter are obtained under assumption that all coefficients in (2.1)–(2.2) and boundary $\partial\Omega$ are smooth. A similar investigation was given in [Naz81] for domains with conic points (the distinctions are pointed on page 11) and asymptotic expansions are also obtained in the form of a regular part and boundary layer. Observe that the conditions **(II)** and **(III)** determine the behaviour of the solutions for a small ε, so principally different types of asymptotics might be discovered when a change of the equation type occurs. The next chapter investigates the case when a parabolic equation degenerates into elliptic.

Chapter 3

Asymptotics of solutions to the heat equation on a bounded domain

We consider the first boundary value problem for the heat equation in a bounded domain $\Omega \subset \mathbb{R} \times \mathbb{R}^n$

$$\begin{aligned} \varepsilon \frac{\partial u}{\partial t} - \Delta u = f \quad &\text{in} \quad \Omega, \\ u = u_0 \quad &\text{at} \quad \partial\Omega. \end{aligned} \tag{3.1}$$

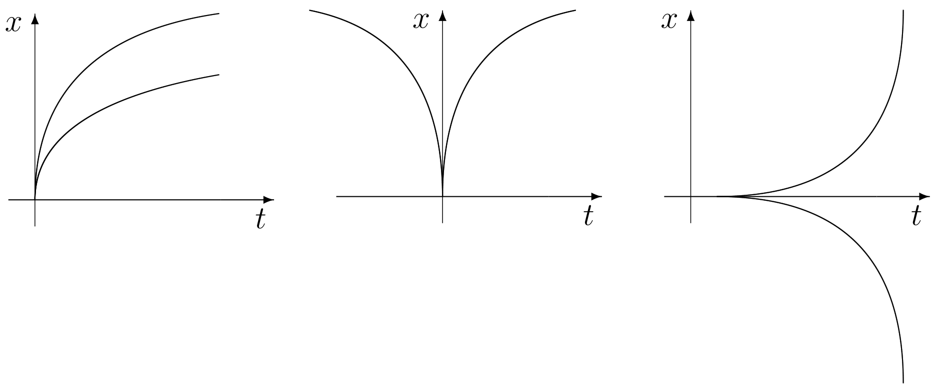
The boundary of $\partial\Omega$ is assumed to be C^∞ curve except for a finite number of points, $\varepsilon \in [0, \varepsilon_0)$ is a small parameter, f and u_0 are given functions. To begin with, we specify more precisely the boundary conditions at peculiar points. The boundary surface is given by zeros of some smooth function $\Phi(t, x)$ of two variables $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, such that $\nabla\Phi(t, x) \neq 0$ for all $(t, x) \in \partial\Omega$.

By the local principle (see *e.g.* [Rab69]), the Fredholm property of problem (3.1) in suitable function spaces is equivalent to the local invertibility of this problem at each point of the closure of Ω. We restrict our attention to the points where the boundary touches the characteristics, for the heat equation they are just horizontal hyperplanes. The set of all such points is designated as Σ.

The reasoning coming from the theory of singularities for algebraic curves allows us to consider the boundary $\partial\Omega$ near characteristic points as a solution of the algebraic equation

$$\Phi(t,x) = \sum_{i_1+i_2+\cdots+i_{n+1}\leqslant p} a_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} t^{i_{n+1}} = 0, \tag{3.2}$$

where $a_{i_1,i_2,\ldots,i_n}$ are fixed arbitrary real numbers, $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. This equation reveals different patterns for the boundary curve near the characteristic. The examples below illustrate some of them:



(a) $t^7 - xt^3 - x^2 - x^2t^4 + x^3 = 0$ (b) $t^3 - x^2 = 0$ (c) $x^3 - t^2 = 0$

Figure 3.1: Some patterns for boundary curve near the characteristic $t = 0$.

On the page 12 we mentioned several types of singular points which have already been considered in the literature. Our studies deal with points $(x,t) \in \Sigma$, such that

(a) the equation (3.2) is solvable with respect to t in some small neighbourhood $U_{(x,t)}$;

(b) the boundary $\partial\Omega$ is described locally in $U_{(x,t)}$ by equation

$$t = \sum_{i_1+i_2+\cdots+i_n=p} a_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}, \tag{3.3}$$

where the right-hand side preserve the sign close to $x = 0$.

On the figure 3.1b we show one such curve. The number p is called the *touch degree*. Our aim is to construct formal asymptotic series for the solution of (3.1). The case $p < 2$ is more interesting since it is poorly investigated. The solution is supposed to satisfy the heat equation in every inner point of the domain Ω and be equal u_0 at $\partial\Omega$ except points of Σ.

Equation (3.1) is first considered for the case $\Omega \subset \mathbb{R}^2$ and then the obtained asymptotics are generalised to higher dimensions.

3.1 One dimensional heat equation

To this point we formulate the first boundary value problem for the heat equation as follows

$$\begin{aligned} \varepsilon u'_t - u''_{x,x} &= f \quad \text{in} \quad \Omega, \\ u &= u_0 \quad \text{at} \quad \partial\Omega \setminus \Sigma, \end{aligned} \tag{3.4}$$

where $\varepsilon \in (0, \varepsilon_0)$ is a small parameter.

Suppose the domain Ω is described in a neighbourhood of a point $(x_0, t_0) \in \Sigma$ by the inequality

$$t - t_0 > |x - x_0|^p, \tag{3.5}$$

where p is a positive real number. There is no loss of generality in assuming that (x_0, t_0) is the origin and $|x - x_0| \leqslant 1$.

We introduce new coordinates (ω, r) with the aid of

$$\begin{aligned} x &= t^{1/p}\, \omega, \\ t &= \varepsilon r, \end{aligned} \tag{3.6}$$

where $|\omega| < 1$ and $r \in (0, 1/\varepsilon)$. It is clear that the new coordinates are singular at $r = 0$, for the entire segment $[-1, 1]$ on the ω-axis is blown down into the origin by (3.6). The rectangle $(-1, 1) \times (0, 1/\varepsilon)$ transforms under the change of coordinates (3.6) into the part of the domain Ω nearby (x_0, t_0) lying below the line $t = 1$. Note that for $\varepsilon \to 0$ the rectangle $(-1, 1) \times (0, 1/\varepsilon)$ stretches to the whole half-strip $(-1, 1) \times (0, \infty)$.

In the domain of coordinates (ω, r) problem (3.4) reduces to an ordinary differential equation with respect to the variable r with operator-valued coefficients. More precisely, under transformation (3.6) the derivatives in t and x change by the formulas

$$\begin{aligned} \varepsilon \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial r} - \frac{1}{r} \frac{\omega}{p} \frac{\partial u}{\partial \omega}, \\ \frac{\partial u}{\partial x} &= \frac{1}{(\varepsilon r)^{1/p}} \frac{\partial u}{\partial \omega}, \end{aligned}$$

and so (3.4) transforms into

$$\begin{aligned} r^Q\, U'_r - \frac{1}{\varepsilon^Q} U''_{\omega,\omega} - r^{Q-1} \frac{\omega}{p} U'_\omega &= r^Q F \quad \text{in} \quad (-1, 1) \times (0, 1/\varepsilon), \\ U &= U_0 \quad \text{at} \quad \{\pm 1\} \times (0, 1/\varepsilon), \end{aligned} \tag{3.7}$$

where $U(\omega, r)$ and $F(\omega, r)$ are pullbacks of $u(x,t)$ and $f(x,t)$ under transformation (3.6), respectively, and

$$Q = \frac{2}{p}.$$

We are interested in the local solvability of problem (3.7) near the edge $r = 0$ in the rectangle $(-1,1) \times (0, 1/\varepsilon)$. Note that the ordinary differential equation degenerates at $r = 0$, since the coefficient $r^{2/p}$ of the higher order derivative in r vanishes at $r = 0$. For the parameter values $\varepsilon > 0$, the exponent Q is of crucial importance for specifying the ordinary differential equation. If $p = 2$ then it is a Fuchs-type equation, these are also called regular singular equations. The Fuchs-type equations fit well into an algebra of pseudodifferential operators based on the Mellin transform. If $p > 2$, then the singularity of the equation at $r = 0$ is weak and so regular theory of finite smoothness applies. In the case $p < 2$ the degeneracy at $r = 0$ is strong and the equation can not be treated except by the theory of slowly varying coefficients [RST00].

3.2 Formal asymptotic solution

To determine appropriate function spaces in which a solution of problem (3.7) is sought, one constructs formal asymptotic solutions of the corresponding homogeneous problem. That is

$$\begin{aligned} r^Q\, U'_r - \frac{1}{\varepsilon^Q}\, U''_{\omega,\omega} - r^{Q-1}\frac{\omega}{p}\, U'_\omega = 0 \quad \text{in} \quad (-1,1)\times(0,\infty), \\ U(\pm 1, r) = 0 \quad \text{on} \quad (0,\infty). \end{aligned} \tag{3.8}$$

We first consider the case $p \neq 2$. We look for a formal solution to (3.8) of the form

$$U(\omega, r) = e^{S(r)}\, V(\omega, r), \tag{3.9}$$

where S is a differentiable function of $r > 0$ and V expands as a formal Puiseux series with nontrivial principal part

$$V(\omega, r) = \frac{1}{r^{\mathfrak{e}N}} \sum_{j=0}^{\infty} V_{j-N}(\omega)\, r^{\mathfrak{e}j},$$

the complex exponent N and real exponent $\mathfrak{e}$ have to be determined. Perhaps the factor $r^{-\mathfrak{e}N}$ might be included into the definition of $\exp S$ as $\exp(-\mathfrak{e}N \ln r)$, however, we prefer to highlight the key role of Puiseux series. Substituting

(3.9) into (3.8) yields

$$\begin{aligned} r^Q\ (S'V + V_r') - \frac{1}{\varepsilon^Q} V''_{\omega,\omega} - r^{Q-1}\frac{\omega}{p} V'_\omega &= 0 \quad \text{in} \quad (-1,1)\times(0,\infty), \\ V(\pm 1, r) &= 0 \quad \text{on} \quad (0,\infty). \end{aligned}$$

In order to reduce this boundary value problem to an eigenvalue problem we require the function S to satisfy the eikonal equation $r^Q S' = \lambda$ with a complex constant λ. This implies

$$S(r) = \lambda \frac{r^{1-Q}}{1-Q}$$

up to an inessential constant to be included into a factor of $\exp S$. In this manner the problem reduces to

$$\begin{aligned} r^Q V_r' - \frac{1}{\varepsilon^Q} V''_{\omega,\omega} - r^{Q-1}\frac{\omega}{p} V'_\omega &= -\lambda V \quad \text{in} \quad (-1,1)\times(0,\infty), \\ V(\pm 1, r) &= 0 \qquad \text{on} \quad (0,\infty). \end{aligned} \tag{3.10}$$

If $\mathfrak{e} = \dfrac{Q-1}{k}$ for some natural number k, then

$$\begin{aligned} r^Q V_r' &= \sum_{j=k}^{\infty} \mathfrak{e}(j-N-k)\, V_{j-N-k} r^{\mathfrak{e}(j-N)}, \\ V''_{\omega,\omega} &= \sum_{j=0}^{\infty} V''_{j-N} r^{\mathfrak{e}(j-N)}, \\ r^{Q-1} V'_\omega &= \sum_{j=k}^{\infty} V'_{j-N-k} r^{\mathfrak{e}(j-N)}, \end{aligned}$$

as is easy to check. On substituting these equalities into (3.10) and equating the coefficients of the same powers of r we get two collections of Sturm-Liouville problems

$$\begin{aligned} -\frac{1}{\varepsilon^Q} V''_{j-N} + \lambda V_{j-N} &= 0 \quad \text{in} \quad (-1,1), \\ V_{j-N} &= 0 \quad \text{at} \quad \mp 1, \end{aligned} \tag{3.11}$$

for $j = 0, 1, \ldots, k-1$, and

$$\begin{aligned} -\frac{1}{\varepsilon^Q} V''_{j-N} + \lambda V_{j-N} &= \frac{\omega}{p} V'_{j-N-k} - \mathfrak{e}(j-N-k)\, V_{j-N-k} \quad \text{in} \quad (-1,1), \\ V_{j-N} &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{at} \quad \mp 1, \end{aligned} \tag{3.12}$$

for $j = mk, mk+1, \ldots, mk + (k-1)$, where m takes on all natural values.

Given any $j = 0, 1, \ldots, k-1$, the Sturm-Liouville problem (3.11) has obviously simple eigenvalues

$$\lambda_n = -\frac{1}{\varepsilon^Q}\Big(\frac{\pi}{2}n\Big)^2$$

for $n = 1, 2, \ldots$, a nonzero eigenfunction corresponding to λ_n being $\sin\frac{\pi}{2}n(\omega+1)$. It follows that

$$V_{j-N}(\omega) = c_{j-N}\,\sin\frac{\pi}{2}n(\omega+1),$$

for $j = 0, 1, \ldots, k-1$, where c_{j-N} are constant. Without restriction of generality we can assume that the first coefficient V_{-N} in the Puiseux expansion of V is different from zero. Hence, $V_{j-N} = c_{j-N}V_{-N}$ for $j = 1, \ldots, k-1$. For simplicity of notation, we drop the index n.

On having determined the functions $V_{-N}, \ldots, V_{k-1-N}$, we turn our attention to problems (3.12) with $j = k, \ldots, 2k-1$. Set

$$f_{j-N} = \frac{\omega}{p}\,V'_{j-N-k} - \mathfrak{e}(j-N-k)\,V_{j-N-k},$$

then for the inhomogeneous problem (3.12) to possess a nonzero solution V_{j-N} it is necessary and sufficient that the right-hand side f_{j-N} be orthogonal to all solutions of the corresponding homogeneous problem, to wit V_{-N}. The orthogonality refers to the scalar product in $L^2(-1,1)$. Let us evaluate the scalar product (f_{j-N}, V_{-N}). We get

$$(f_{j-N}, V_{-N}) = c_{j-N-k}\Big(\frac{1}{p}\,(\omega V'_{-N}, V_{-N}) - \mathfrak{e}(j-N-k)\,(V_{-N}, V_{-N})\Big)$$

and

$$\begin{aligned}(\omega V'_{-N}, V_{-N}) &= \omega\,|V_{-N}|^2\,\Big|_{-1}^{1} - (V_{-N}, V_{-N}) - (V_{-N}, \omega V'_{-N})\\ &= -(V_{-N}, V_{-N}) - (\omega V'_{-N}, V_{-N}),\end{aligned}$$

the latter equality being due to the fact that V_{-N} is real-valued and vanishes at ± 1. Hence,

$$(\omega V'_{-N}, V_{-N}) = -\frac{1}{2}\,(V_{-N}, V_{-N})$$

and

$$(f_{j-N}, V_{-N}) = -c_{j-N-k}\,\Big(\frac{1}{2p} + \mathfrak{e}(j-N-k)\Big)(V_{-N}, V_{-N}) \qquad (3.13)$$

for $j = k, \ldots, 2k-1$.

Since $V_{-N} \neq 0$, the condition $(f_{j-N}, V_{-N}) = 0$ fulfills for $j = k$ if and only if

$$\mathfrak{e}N = \frac{1}{2p}. \tag{3.14}$$

Under this condition, problem (3.12) with $j = k$ is solvable and its general solution has the form

$$V_{k-N} = V_{k-N,0} + c_{k-N} V_{-N},$$

where $V_{k-N,0}$ is a particular solution of (3.12) and c_{k-N} an arbitrary constant. Moreover, for $(f_{j-N}, V_{-N}) = 0$ to fulfill for $j = k+1, \ldots, 2k-1$ it is necessary and sufficient that $c_{1-N} = \cdots = c_{k-1-N} = 0$, *i.e.*, all of $V_{1-N}, \ldots, V_{k-1-N}$ vanish. This in turn implies that $f_{k+1-N} = \cdots = f_{2k-1-N} = 0$, whence $V_{j-N} = c_{j-N} V_{-N}$ for all $j = k+1, \ldots, 2k-1$, where c_{j-N} are arbitrary constants. We choose the constants $c_{k-N}, \ldots, c_{2k-1}$ in such a way that the solvability conditions of the next k problems are fulfilled.

More precisely, we consider the problem (3.12) for $j = 2k$, the right-hand side being

$$\begin{aligned} f_{2k-N} &= \Big(\frac{\omega}{p} V'_{k-N,0} - \mathfrak{e}(k-N)\, V_{k-N,0}\Big) + c_{k-N} \Big(\frac{\omega}{p} V'_{-N} - \mathfrak{e}(k-N)\, V_{-N}\Big) \\ &= \Big(\frac{\omega}{p} V'_{k-N,0} - \mathfrak{e}(k-N)\, V_{k-N,0}\Big) + c_{k-N} \Big(f_{k-N} - \mathfrak{e}k\, V_{-N}\Big). \end{aligned}$$

Combining (3.13) and (3.14) we conclude that

$$\begin{aligned} (f_{k-N} - \mathfrak{e}k\, V_{-N}, V_{-N}) &= -\mathfrak{e}k\, (V_{-N}, V_{-N}) \\ &= (1-Q)\, (V_{-N}, V_{-N}) \end{aligned}$$

is different from zero. Hence, the constant c_{k-N} can be uniquely defined in such a way that $(f_{2k-N}, V_{-N}) = 0$. Moreover, the functions $f_{2k+1-N}, \ldots, f_{3k-1-N}$ are orthogonal to V_{-N} if and only if $c_{k+1-N} = \cdots = c_{2k-1-N} = 0$. It follows that V_{j-N} vanishes for each $j = k+1, \ldots, 2k-1$.

Continuing in this fashion we construct a sequence of functions $V_{j-N}(\omega, \varepsilon)$, for $j = 0, 1, \ldots$, satisfying equations (3.11) and (3.12). The functions $V_{j-N}(\omega, \varepsilon)$ are defined uniquely up to a common constant factor c_{-N}. They depend smoothly on the parameter ε^p. Moreover, V_{j-N} vanishes identically unless

$j = mk$ with $m = 0, 1, \ldots$. Therefore,

$$
\begin{aligned}
V(\omega, r, \varepsilon) &= \frac{1}{r^{\mathfrak{e}N}} \sum_{m=0}^{\infty} V_{mk-N}(\omega, \varepsilon)\, r^{\mathfrak{e}mk} \\
&= \frac{1}{r^{Q/4}} \sum_{m=0}^{\infty} \tilde{V}_m(\omega, \varepsilon)\, r^{(Q-1)m}
\end{aligned}
$$

is a unique (up to a constant factor) formal asymptotic solution of problem (3.10) corresponding to $\lambda = \lambda_n$.

Theorem 13. *Let $p \neq 2$. Then an arbitrary formal asymptotic solution of homogeneous problem (3.8) has the form*

$$
U(\omega, r, \varepsilon) = \frac{c}{r^{Q/4}} \exp\left(\lambda \frac{r^{1-Q}}{1-Q}\right) \sum_{m=0}^{\infty} \frac{\tilde{V}_m(\omega, \varepsilon)}{r^{(1-Q)m}},
$$

where λ is one of eigenvalues $\lambda_n = -\dfrac{1}{\varepsilon^Q}\left(\dfrac{\pi}{2} n\right)^2$.

Proof. The theorem follows readily from (3.9). □

In the original coordinates (x, t) close to the point (x_0, t_0) in Ω the formal asymptotic solution looks like

$$
u(x, t, \varepsilon) = c \left(\frac{\varepsilon}{t}\right)^{Q/4} \exp\left(\frac{\lambda}{1-Q}\left(\frac{t}{\varepsilon}\right)^{1-Q}\right) \sum_{m=0}^{\infty} \tilde{V}_m\left(\frac{x}{t^{1/p}}, \varepsilon\right)\left(\frac{\varepsilon}{t}\right)^{(1-Q)m} \tag{3.15}
$$

for $\varepsilon > 0$. If $1 - Q > 0$, *i.e.* $p > 2$, expansion (3.15) behaves in much the same way as boundary layer expansion in singular perturbation problems, since the eigenvalues are all negative. The threshold value $p = 2$ is a turning contact order under which the boundary layer degenerates.

3.3 The exceptional case $p = 2$

In this section we consider the case $p = 2$ in detail. For $p = 2$, problem (3.8) takes the form

$$
\begin{aligned}
r\, U'_r - \frac{1}{\varepsilon} U''_{\omega,\omega} - \frac{\omega}{2} U'_\omega &= 0 \quad \text{in} \quad (-1, 1) \times (0, \infty), \\
U(\pm 1, r) &= 0 \quad \text{on} \quad (0, \infty).
\end{aligned} \tag{3.16}
$$

The problem is specified as Fuchs-type equation on the half-axis with coefficients in boundary value problems on the interval $[-1,1]$. Such equations have been well understood, see [NP94] and elsewhere.

If one searches for a formal solution to (3.16) of the form $U(\omega,r)=e^{S(r)}V(\omega,r)$, then the eikonal equation $rS'=\lambda$ gives $S(r)=\lambda\ln r$, and so $e^{S(r)}=r^{\lambda}$, where λ is a complex number. It makes therefore no sense to looking for $V(\omega,r)$ being a formal Puiseux series in fractional powers of r. The choice $\mathfrak{e}=(Q-1)/k$ no longer works, and so a good substitute for a fractional power of r is the function $1/\ln r$. Thus,

$$V(\omega,r)=\sum_{j=0}^{\infty}V_{j-N}(\omega)\Big(\frac{1}{\ln r}\Big)^{j-N}$$

has to be a formal asymptotic solution of

$$\begin{aligned} r\,V'_r-\frac{1}{\varepsilon}\,V''_{\omega,\omega}-\frac{\omega}{2}\,V'_\omega &= -\lambda V \quad \text{in} \quad (-1,1)\times(0,\infty),\\ V(\pm1,r) &= 0 \qquad\quad \text{on} \quad (0,\infty), \end{aligned}$$

N being a nonnegative integer. Substituting the series for $V(\omega,r)$ into these equations and equating the coefficients of the same powers of $\ln r$ yields two collections of Sturm-Liouville problems

$$\begin{aligned} -\frac{1}{\varepsilon}\,V''_{-N}-\frac{\omega}{2}\,V'_{-N}+\lambda\,V_{-N} &= 0 \quad \text{in} \quad (-1,1),\\ V_{-N} &= 0 \quad \text{at} \quad \mp1, \end{aligned} \tag{3.17}$$

for $j=0$, and

$$\begin{aligned} -\frac{1}{\varepsilon}\,V''_{j-N}-\frac{\omega}{2}\,V'_{j-N}+\lambda\,V_{j-N} &= (j-N-1)V_{j-N-1} \quad \text{in} \quad (-1,1),\\ V_{j-N} &= 0 \qquad\qquad\qquad\quad\ \text{at} \quad \mp1, \end{aligned} \tag{3.18}$$

for $j\geqslant 1$.

Problem (3.17) has a nonzero solution V_{-N} if and only if λ is an eigenvalue of the operator

$$v\mapsto\frac{1}{\varepsilon}\,v''+\frac{\omega}{2}\,v$$

whose domain consists of all functions $v\in H^2(-1,1)$ vanishing at ∓1. Then, equalities (3.18) for $j=1,\dots,N$ mean that $V_{-N+1},\dots,V_0$ are actually root functions of the operator corresponding to the eigenvalue λ. In other words, $V_{-N},\dots,V_0$ is a Jordan chain of length $N+1$ corresponding to the eigenvalue λ.

Note that for $j = N+1$ the right-hand side of (3.18) vanishes, and so $V_1, V_2, \ldots$ is also a Jordan chain corresponding to the eigenvalue λ. This suggests that the series breaks beginning at $j = N+1$. Moreover, a familiar argument shows that problem (3.17) has eigenvalues

$$\lambda_n = -\frac{1}{\varepsilon}\Big(\frac{\pi}{2}n\Big)^2 + o\Big(\frac{1}{\varepsilon}\Big)$$

for $n = 1, 2, \ldots$, which are simple if ε is small enough. Hence it follows that $N = 0$ and

$$V_0(\omega, \varepsilon) = c_0 \sin\frac{\pi}{2}n(\omega+1) + o(1)$$

for $\varepsilon \to 0$.

Theorem 14. *Suppose $p = 2$. Then an arbitrary formal asymptotic solution of homogeneous problem (3.8) has the form $U(\omega, r, \varepsilon) = r^\lambda V_0(\omega, \varepsilon)$, where λ is one of the eigenvalues λ_n.*

Proof. The theorem follows immediately from the above discussion. □

In the original coordinates (x, t) near the point (x_0, t_0) in Ω the formal asymptotic solution proves to be

$$u(x, t, \varepsilon) = c\,\Big(\frac{\varepsilon}{t}\Big)^{-\lambda} V_0\Big(\frac{x}{t^{1/2}}, \varepsilon\Big)$$

for $\varepsilon > 0$. This expansion behaves similarly to boundary layer expansion in singular perturbation problems, since the eigenvalues are negative provided that ε is sufficiently small.

3.4 Degenerate problem

If $\varepsilon = 0$ then the homogeneous problem corresponding to local problem (3.7) degenerates to

$$\begin{aligned} U''_{\omega,\omega} &= 0 \quad \text{in} \quad (-1,1)\times(0,\infty), \\ U &= 0 \quad \text{at} \quad \{\pm 1\}\times(0,\infty). \end{aligned} \tag{3.19}$$

Substituting the general solution $U(\omega, r) = U_1(r)\omega + U_0(r)$ of the differential equation into the boundary conditions implies readily $U \equiv 0$ in the half-strip, *i.e.* (3.19) has only zero solution.

Corollary 1. *If $p \geq 2$ then the formal asymptotic solution of (3.8) converges to zero uniformly in $t > 0$ bounded away from zero, as $\varepsilon \to 0$. Moreover, for $p > 2$ it vanishes exponentially.*

Proof. This follows immediately from Theorems 13 and 14. □

On the contrary, if $p < 2$ then the formal asymptotic solution of problem (3.8) hardly converges, as $\varepsilon \to 0$.

3.5 Generalisation to higher dimensions

The explicit formulas obtained above generalise easily to the evolution equation related to the bth power of the Laplace operator in $\mathbb{R}^n$, where b is a natural number. Consider the first boundary value problem for the operator $\varepsilon\partial_t + (-iD)^b$ in a bounded domain $\Omega \subset \mathbb{R}^{n+1}$. Note that the choice of sign $(-1)^b$ is explained exceptionally by our wish to deal with parabolic (not backward parabolic) equation. By $\varepsilon > 0$ is meant a small parameter.

The boundary of Ω is assumed to be C^∞ except for a finite number of characteristic points. These are those points of $\partial\Omega$ at which the boundary touches with a hyperplane in $\mathbb{R}^{n+1}$ orthogonal to the t-axis. As above, we restrict our attention to analysis of the Dirichlet problem near a characteristic point given by condition 3.3. The first boundary value problem for the evolution equation in Ω is formulated as follows: Let Σ be the set of all characteristic points of the boundary of Ω. Given any functions f in $\Omega \to \mathbb{R}$ $u_0, u_1, \ldots, u_{b-1}$ on $\partial\Omega \setminus \Sigma$, find a function u on $\overline{\Omega} \setminus \Sigma$ satisfying

$$\begin{aligned} \varepsilon u_t' + (-iD)^b u &= f \quad \text{in} \quad \Omega, \\ \partial_\nu^j u &= u_j \quad \text{at} \quad \partial\Omega \setminus \Sigma, \end{aligned} \tag{3.20}$$

for $j = 0, 1, \ldots, b-1$, where ∂_ν is the derivative along the outward unit normal vector of the boundary. We focus upon a characteristic point O of the boundary which is assumed to be the origin in $\mathbb{R}^{n+1}$.

Suppose the domain Ω is described in a neighbourhood of the origin by the inequality

$$t > \mathfrak{f}(x), \tag{3.21}$$

where $\mathfrak{f}$ is a smooth function of $x \in \mathbb{R}^n \setminus 0$ homogeneous of degree $p > 0$. We blow up the domain Ω at O by introducing new coordinates $(\omega, r) \in D \times (0, 1/\varepsilon)$

with the aid of

$$\begin{aligned} x &= t^{1/p}\,\omega, \\ t &= \varepsilon r, \end{aligned} \tag{3.22}$$

where D is the domain in $\mathbb{R}^n$ consisting of those $\omega \in \mathbb{R}^n$ which satisfy $f(\omega) < 1$. Under this change of variables the domain Ω nearby O transforms into the half-cylinder $D \times (0, \infty)$, the cross-section $D \times \{0\}$ blowing down into the origin by (3.22). Note that for $\varepsilon \to 0$ the cylinder $D \times (0, 1/\varepsilon)$ stretches into the whole half-cylinder $D \times (0, \infty)$.

In the domain of coordinates (ω, r) problem (3.20) reduces to an ordinary differential equation with respect to the variable r with operator-valued coefficients. It is easy to see that under transformation (3.22) the derivatives in t and x change by the formulas

$$\begin{aligned} \varepsilon\, u'_t &= u'_r - \frac{1}{p}\frac{1}{r}\,(\omega, u'_\omega), \\ u'_{x_k} &= \frac{1}{(\varepsilon r)^{1/p}}\, u'_{\omega_k} \end{aligned}$$

for $k = 1, \ldots, n$, where $(\omega, u'_\omega) = \displaystyle\sum_{k=1}^{n} \omega_k \frac{\partial u}{\partial \omega_k}$ stands for the Euler derivative. Thus, (3.20) transforms into

$$\begin{aligned} r^Q\, U'_r + \frac{1}{\varepsilon^Q}\,(-iD_\omega)^b U - \frac{1}{p} r^{Q-1}\,(\omega, U'_\omega) &= r^Q F \quad \text{in} \quad D \times (0, 1/\varepsilon), \\ \partial_\nu^j\, U &= U_j \qquad \text{at} \quad \partial D \times (0, 1/\varepsilon) \end{aligned} \tag{3.23}$$

for $j = 0, 1, \ldots, b-1$, where $U(\omega, r)$ and $F(\omega, r)$ are pullbacks of $u(x, t)$ and $f(x, t)$ under transformation (3.22), respectively, and

$$Q = \frac{2b}{p}.$$

We are interested in the local solvability of problem (3.23) near the base $r = 0$ in the cylinder $D \times (0, 1/\varepsilon)$. Note that the ordinary differential equation degenerates at $r = 0$, since the coefficient r^Q of the higher order derivative in r vanishes at $r = 0$. The theory of [RST00] still applies to characterise those problems (3.23) which are locally invertible.

To describe function spaces which give the best fit for solutions of problem (3.23), one constructs formal asymptotic solutions of the corresponding homogeneous problem. That is

$$\begin{aligned} r^Q\,U'_r + \frac{1}{\varepsilon^Q}\,(-iD_\omega)^b U - \frac{1}{p} r^{Q-1}\,(\omega, U'_\omega) = 0 \quad &\text{in} \quad D\times(0,\infty), \\ \partial^\alpha_\omega U = 0 \quad &\text{on} \quad \partial D\times(0,\infty) \end{aligned} \tag{3.24}$$

for all $|\alpha| \leqslant b-1$.

We assume that $p \neq 2b$. Similar arguments apply to the case $p = 2b$, the only difference being in the choice of the Ansatz, see Section 3.3. We look for a formal solution to (3.24) of the form $U(\omega,r) = e^{S(r)}\,V(\omega,r)$, where S is a differentiable function of $r>0$ and V expands as a formal Puiseux series with nontrivial principal part

$$V(\omega,r) = \frac{1}{r^{\mathfrak{e}N}}\sum_{j=0}^{\infty} V_{j-N}(\omega)\,r^{\mathfrak{e}j},$$

where N is a complex number and $\mathfrak{e}$ a real exponent to be determined.

On substituting $U(\omega,r)$ into (3.8) we extract the eikonal equation $r^Q S' = \lambda$ for the function $S(r)$, where λ is a (possibly complex) constant to be defined. For $Q \neq 1$ this implies

$$S(r) = \lambda\,\frac{r^{1-Q}}{1-Q}$$

up to an inessential constant factor. In this way the problem reduces to

$$\begin{aligned} r^Q\,V'_r + \frac{1}{\varepsilon^Q}\,(-iD_\omega)^b V - \frac{1}{p} r^{Q-1}\,(\omega, V'_\omega) = -\lambda V \quad &\text{in} \quad D\times(0,\infty), \\ \partial^\alpha_\omega V = 0 \qquad &\text{on} \quad \partial D\times(0,\infty) \end{aligned} \tag{3.25}$$

for all $|\alpha| \leqslant b-1$.

Analysis similar to that in Section 3.2 shows that a right choice of $\mathfrak{e}$ is $\mathfrak{e} = (Q-1)/k$ for some natural number k. On substituting the formal series for $V(\omega,r)$ into (3.25) and equating the coefficients of the same powers of r we get two collections of problems

$$\begin{aligned} \frac{1}{\varepsilon^Q}\,(-iD)^b V_{j-N} + \lambda\,V_{j-N} = 0 \quad &\text{in} \quad D, \\ \partial^\alpha\,V_{j-N} = 0 \quad &\text{at} \quad \partial D \end{aligned} \tag{3.26}$$

for all $|\alpha| \leqslant b-1$, where $j = 0, 1, \ldots, k-1$, and

$$\begin{aligned} \frac{1}{\varepsilon^Q}(-iD)^b V_{j-N} + \lambda V_{j-N} &= \frac{1}{p}(\omega, V'_{j-N-k}) - \mathfrak{e}(j-N-k)\, V_{j-N-k} && \text{in} \quad D, \\ \partial^\alpha V_{j-N} &= 0 && \text{at} \quad \partial D \end{aligned} \tag{3.27}$$

for all $|\alpha| \leqslant b-1$, where $j = k, k+1, \ldots, 2k-1$, and so on.

Given any $j = 0, 1, \ldots, k-1$, problem (3.26) is essentially an eigenvalue problem for the strongly nonnegative operator $(-iD)^b$ in $L^2(D)$ whose domain consists of all functions of $H^{2b}(D)$ vanishing up to order $b-1$ at ∂D. The eigenvalues of the latter operator are known to be all positive and form a nondecreasing sequence $\lambda'_1, \lambda'_2, \ldots$ which converges to ∞. Hence, (3.26) admits nonzero solutions only for

$$\lambda_n = -\frac{1}{\varepsilon^Q}\lambda'_n$$

where $n = 1, 2, \ldots$.

In general, the eigenvalues $\{\lambda'_n\}$ fail to be simple. The generic simplicity of the eigenvalues of the Dirichlet problem for self-adjoint elliptic operators with respect to variations of the boundary have been investigated by several authors, see [PP08] and the references given there. We focus on an eigenvalue λ'_n of multiplicity 1, in which case the formal asymptotic solution is especially simple. By the above, this condition is not particularly restrictive.

If $\lambda = \lambda_n$, there is a nonzero solution $e_n(\omega)$ of this problem which is determined uniquely up to a constant factor. This yields

$$V_{j-N}(\omega) = c_{j-N}\, e_n(\omega),$$

for $j = 0, 1, \ldots, k-1$, where c_{j-N} are constant. Without restriction of generality we can assume that the first coefficient V_{-N} in the Puiseux expansion of V is different from zero. Hence, $V_{j-N} = c_{j-N} V_{-N}$ for $j = 1, \ldots, k-1$. For simplicity of notation, we drop the index n.

On taking the functions $V_{-N}, \ldots, V_{k-1-N}$ for granted, we now turn to problems (3.12) with $j = k, \ldots, 2k-1$. Set

$$f_{j-N} = \frac{1}{p}(\omega, V'_{j-N-k}) - \mathfrak{e}(j-N-k)\, V_{j-N-k},$$

then for the inhomogeneous problem (3.27) to admit a nonzero solution V_{j-N} it is necessary and sufficient that the right-hand side f_{j-N} be orthogonal to all solutions of the corresponding homogeneous problem, to wit V_{-N}. The orthogonality refers to the scalar product in $L^2(D)$. Let us evaluate the scalar product (f_{j-N}, V_{-N}). We get

$$(f_{j-N}, V_{-N}) = c_{j-N-k}\Big(\frac{1}{p}\,((\omega, V'_{-N}), V_{-N}) - \mathfrak{e}(j-N-k)\,(V_{-N}, V_{-N})\Big)$$

and, by Stokes' formula,

$$\begin{aligned}((\omega, V'_{-N}), V_{-N}) &= \int_{\partial D} |V_{-N}|^2 (\omega, \nu)\, ds - \sum_{k=1}^{n} \int_D V_{-N} \frac{\partial}{\partial \omega_k}(\omega_k \overline{V_{-N}})\, d\omega \\ &= -n\|V_{-N}\|^2 - ((\omega, V'_{-N}), V_{-N}),\end{aligned}$$

the latter equality being due to the fact that V_{-N} is real-valued and vanishes at ∂D. Hence,

$$((\omega, V'_{-N}), V_{-N}) = -\frac{n}{2}\,\|V_{-N}\|^2$$

and

$$(f_{j-N}, V_{-N}) = -c_{j-N-k}\left(\frac{n}{2p} + \mathfrak{e}(j-N-k)\right)\|V_{-N}\|^2 \qquad (3.28)$$

for $j = k, \ldots, 2k-1$.

Since $V_{-N} \neq 0$, the condition $(f_{j-N}, V_{-N}) = 0$ fulfills for $j = k$ if and only if

$$\mathfrak{e}N = \frac{n}{2p}. \qquad (3.29)$$

Under this condition, problem (3.27) with $j = k$ is solvable and its general solution has the form

$$V_{k-N} = V_{k-N,0} + c_{k-N} V_{-N},$$

where $V_{k-N,0}$ is a particular solution of (3.27) and c_{k-N} an arbitrary constant. Moreover, for $(f_{j-N}, V_{-N}) = 0$ to fulfill for $j = k+1, \ldots, 2k-1$ it is necessary and sufficient that $c_{1-N} = \cdots = c_{k-1-N} = 0$, *i.e.* all of $V_{1-N}, \ldots, V_{k-1-N}$ vanish. This in turn implies that $f_{k+1-N} = \cdots = f_{2k-1-N} = 0$, whence $V_{j-N} = c_{j-N} V_{-N}$ for all $j = k+1, \ldots, 2k-1$, where c_{j-N} are arbitrary constants. We choose the constants $c_{k-N}, \ldots, c_{2k-1}$ in such a way that the solvability conditions of the next k problems are fulfilled.

More precisely, we consider the problem (3.27) for $j = 2k$, the right-hand side being

$$f_{2k-N} = \Big(\frac{1}{p}(\omega, V'_{k-N,0}) - \mathfrak{e}(k-N)V_{k-N,0}\Big) + c_{k-N}\Big(\frac{1}{p}(\omega, V'_{-N}) - \mathfrak{e}(k-N)V_{-N}\Big)$$
$$= \Big(\frac{1}{p}(\omega, V'_{k-N,0}) - \mathfrak{e}(k-N)V_{k-N,0}\Big) + c_{k-N}\Big(f_{k-N} - \mathfrak{e}kV_{-N}\Big).$$

Combining (3.28) and (3.29) we conclude that

$$(f_{k-N} - \mathfrak{e}k\,V_{-N}, V_{-N}) = -\mathfrak{e}k\,(V_{-N}, V_{-N})$$
$$= (1-Q)\,(V_{-N}, V_{-N})$$

is different from zero. Hence, the constant c_{k-N} can be uniquely defined in such a way that $(f_{2k-N}, V_{-N}) = 0$. Moreover, the functions $f_{2k+1-N}, \dots, f_{3k-1-N}$ are orthogonal to V_{-N} if and only if $c_{k+1-N} = \dots = c_{2k-1-N} = 0$. It follows that V_{j-N} vanishes for each $j = k+1, \dots, 2k-1$.

Continuing in this manner we construct a sequence of functions $V_{j-N}(\omega, \varepsilon)$, for $j = 0, 1, \dots$, satisfying equations (3.26) and (3.27). The functions $V_{j-N}(\omega, \varepsilon)$ are defined uniquely up to a common constant factor c_{-N}. They depend smoothly on the parameter ε^p. Moreover, V_{j-N} vanishes identically unless $j = mk$ with $m = 0, 1, \dots$. Therefore,

$$V(\omega, r, \varepsilon) = \frac{1}{r^{\mathfrak{e}N}} \sum_{m=0}^{\infty} V_{mk-N}(\omega, \varepsilon)\, r^{\mathfrak{e}mk}$$
$$= \frac{1}{r^{n/2p}} \sum_{m=0}^{\infty} \tilde{V}_m(\omega, \varepsilon)\, r^{(Q-1)m}$$

is a unique (up to a constant factor) formal asymptotic solution of problem (3.25) corresponding to $\lambda = \lambda_n$. Summarising, we arrive at the following generalisation of Theorem 13.

Theorem 15. *Let $p \neq 2b$. Then an arbitrary formal asymptotic solution of homogeneous problem (3.24) has the form*

$$U(\omega, r, \varepsilon) = \frac{c}{r^{n/2p}} \exp\Big(\lambda\, \frac{r^{1-Q}}{1-Q}\Big) \sum_{m=0}^{\infty} \frac{\tilde{V}_m(\omega, \varepsilon)}{r^{(1-Q)m}},$$

where λ is one of eigenvalues $\lambda_n = -\dfrac{1}{\varepsilon^Q}\lambda'_n$.

Thus, the construction of formal asymptotic solution U of general problem (3.20) follows by the same method as in Section 3.2.

In the original coordinates (x,t) close to the point O in Ω the formal asymptotic solution looks like

$$u(x,t,\varepsilon) = c\left(\frac{\varepsilon}{t}\right)^{n/2p} \exp\left(-\frac{\lambda'}{\varepsilon}\frac{t^{1-Q}}{1-Q}\right)\sum_{m=0}^{\infty}\tilde{V}_m\left(\frac{x}{t^{1/p}},\varepsilon\right)\left(\frac{\varepsilon}{t}\right)^{(1-Q)m} \tag{3.30}$$

for $\varepsilon > 0$. If $1 - Q > 0$, *i.e.* $p > 2b$, expansion (3.30) behaves in much the same way as boundary layer expansion in singular perturbation problems, since the eigenvalues are all negative. The threshold value $p = 2b$ is a turning contact order under which the boundary layer degenerates.

The computations of this section obviously extend both to eigenvalues λ_n of higher multiplicity and arbitrary self-adjoint elliptic operators $A(x,D)$ in place of $(-iD)^b$. When solving nonhomogeneous equations (3.27), one chooses the only solution which is orthogonal to all solutions of the corresponding homogeneous problem (3.26). This special solution actually determines what is known as the Green operator. However, formula (3.30) becomes less transparent. And so we omit the details.

3.6 Conclusions

The asymptotic property of derived formal expansions follows from [AT13]. For $p < 2b$, expansion (3.30) fails to be asymptotic in small $\varepsilon > 0$, even if (x,t) is bounded away from the boundary of Ω. An asymptotic character of this series can only be revealed on using parameter dependent norms. Indeed, if $\varepsilon \to 0$, then the summands on the right-hand side of (3.30) increase unless the quotient t/ε does not exceed 1. Hence, ε is allowed to tend to zero only under the condition that $t/\varepsilon < 1$. Then expansion (3.30) still reveals certain asymptotic character. Within the framework of analysis on manifolds with singularities one exploits the weighted norms

$$\left(\int_D \exp\left(2\gamma\frac{1}{t^Q}\frac{t}{\varepsilon}\right)\left(\frac{t}{\varepsilon}\right)^{-2\mu}|u(x,t,\varepsilon)|^2 dxdt\right)^{1/2}$$

on functions defined near the singular point, where γ and μ are real numbers.

Chapter 4

Algebra of ΨDO with a small parameter

This chapter generalises the concept of ellipticity with a small parameter to pseudodifferential operators on a smooth closed compact manifold $\mathcal{X}$. It means in turn to specify the symbols of ΨDO depending on small parameters and algebraic operations on them. This can be done in different ways. For example, one can treats the symbols $a(x, \xi, \varepsilon)$ as functions of one more variable which is supposed small. Another approach is to consider symbols defined on the cotangent bundle $T^*\mathcal{X}$ but taking their values in some functional spaces depending on ε. We use the last idea with the methods going back to [Kar83] and propose two examples of ΨDO calculus with operator-valued symbols. However small parameters enter the algebras differently. In the first algebra ε plays "passive" role, in another construction the small variable is suggested to be "active".

The term "passive" comes from analogy with transformation theory. Recall that a geometrical transformation $y = f(x)$ may be treated either from an "active" or "passive" point of view. According to the "active" approach the transformation moves geometrical points $x \mapsto y = f(x)$ while in the "passive" approach the points are fixed and we only change the coordinate system. For example, a linear change $y^i = a^i_j x^j$ (we use the Einstein summation notation) may be thought of as a linear transformation of the space $\mathbb{R}^n$ or as a change of a basis in this space. Of course, both descriptions are equivalent.

In the sequel we apply both concepts to ΨDO calculus for closed compact manifolds.

4.1 A passive approach to operator-valued symbols

We describe here the “passive” approach to operator-valued symbols which allows one to reduce pseudodifferential operators with operator-valued symbols to the case of integral operators in abstract L^2-spaces, so that the calculus of operator-valued symbols becomes quite similar to that of scalar-valued symbols.

We demonstrate this approach by calculus of pseudodifferential operators on a product manifold. Consider $\mathcal{M} = \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X}$, $\mathcal{Y}$ are smooth compact closed manifolds with $\dim \mathcal{X} = n$ and $\dim \mathcal{Y} = m$. Suppose we work with the usual symbol classes $\mathcal{S}^\mu$ on $\mathcal{M}$ and corresponding classes of pseudodifferential operators $\mathcal{L}^\mu$ acting in Sobolev spaces $H^s(\mathcal{M})$. We are aimed at describing these objects using a fibering structure. That is, we would like to introduce appropriate classes of operator-valued symbols on $\mathcal{X}$ with values in pseudodifferential operators on $\mathcal{Y}$ to recover the classes $\mathcal{S}^\mu$ on $\mathcal{M}$. Moreover, we would like to represent $H^s(\mathcal{M})$ as L^2-spaces $L^2(\mathcal{X}, H^s(\mathcal{Y}), \|\cdot\|_\xi)$ to recover the action of pseudodifferential operators from $\mathcal{L}^\mu$ in the spaces $H^s(\mathcal{M})$.

A symbol $a(x, y, \xi, \eta)$ on $\mathcal{M}$ is treated as a symbol on the fiber $\mathcal{Y}$ with estimates depending on the base covariable ξ. Our approach consists in equipping spaces $H^s(\mathcal{Y})$ with a special family of norms. In more detail, consider the Sobolev space $H^s(\mathcal{Y})$ with the norms $\|\cdot\|_\xi$ depending on a parameter $\xi \in \mathbb{R}^n$.

$$\|u(y)\|_\xi^2 = \sum_j \int_{\mathbb{R}^m} | < \xi, \eta >^s \widehat{\psi_j u}(\eta)|^2 \, d\eta. \tag{4.1}$$

Here $\{\psi_j\}$ is a partition of unity subordinating some coordinate covering O_j. The norm (4.1) depends, of course, on s but we drop it in the notation.

Next, consider a function $u(x)$ on $\mathcal{X}$ with values in $H^s(\mathcal{Y})$ equipped with the family of norms $\|\cdot\|_\xi$ given by (4.1).

Definition 9. *By $L^2(\mathcal{X}, H^s(\mathcal{Y}), \|\cdot\|_\xi)$ is meant the completion of $C^\infty(\mathcal{X}, H^s(\mathcal{Y}))$ with respect to the norm*

$$\|u(x)\|^2 = \sum_i \int_{\mathbb{R}^n} \|\widehat{\phi_i u}(\xi)\|_\xi^2 \, d\xi. \tag{4.2}$$

Once again $\{\phi_j\}$ is a partition of unity on $\mathcal{X}$ subordinate to a coordinate covering $\{O_j\}$ of this manifold. Roughly speaking, (4.2) is an L^2-norm of the scalar-valued function $\|\widehat{\phi_i u}(\xi)\|_\xi$.

We now are in a position to define the desired symbol classes Σ^m on $\mathcal{X}$ with values in pseudodifferential operators on $\mathcal{Y}$.

Definition 10. *A function $a(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ whose values are pseudodifferential operators on $\mathcal{Y}$ is said to belong to Σ^μ if, for any $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, the operators $\partial_x^\alpha D_\xi^\beta a(x,\xi) : H^s(\mathcal{Y}) \to H^{s-\mu+\beta}(\mathcal{Y})$ are bounded uniformly in ξ with respect to the norms $\|\cdot\|_\xi$ in both spaces $H^s(\mathcal{Y})$ and $H^{s-\mu+\beta}(\mathcal{Y})$. That is, there are constants $C_{\alpha,\beta}$ independent of ξ, such that*

$$\|\partial_x^\alpha D_\xi^\beta a(x,\xi)\|_\xi \leqslant C_{\alpha,\beta}. \tag{4.3}$$

Any symbol $a(x,y,\xi,\eta) \in \mathcal{S}^\mu$ defines a symbol $a(x,\xi) \in \Sigma^\mu$ on $\mathcal{X}$ with values in pseudodifferential operators on $\mathcal{Y}$.

We can actually stop at this point. All of what follows is a simple consequence of generalization of these definitions. As mentioned, in a more general context of pseudodifferential operators with operator-valued symbols these techniques was elaborated in [Kar83].

It is easy to see that the norms $\|\cdot\|_\xi$ in $H^s(\mathcal{Y})$ are equivalent for different values $\xi \in \mathbb{R}^n$, but this equivalence is not uniform in ξ. More precisely, on applying Peetre's inequality one sees that the norms vary slowly in ξ.

Lemma 11. *There are constants C and q such that*

$$\frac{\|u\|_{\xi_1}}{\|u\|_{\xi_2}} \leqslant C \langle \xi_1 - \xi_2 \rangle^q \tag{4.4}$$

for all $\xi_1, \xi_2 \in \mathbb{R}^k$ and smooth functions u on $\mathcal{Y}$. (In fact, we get $C = 2^{|s|}$ and $q = |s|$.)

On the other hand, the norm $\|\cdot\|_\xi$ is independent of the coordinate covering and partition of unity up to uniform equivalence.

Lemma 12. *The embedding $\iota : H^{s_2}(\mathcal{Y}) \to H^{s_1}(\mathcal{Y})$ for $s_1 \leqslant s_2$ admits the following norm estimate*

$$\|\iota\|_\xi \leqslant C \langle \xi \rangle^{s_1 - s_2}. \tag{4.5}$$

Proof. Since

$$\begin{aligned}\|u_j\|^2_{H^{s_1}(Y),\xi} &= \int_{\mathbb{R}^m} |\langle \xi, \eta \rangle^{s_1} \widehat{u_j}(\eta)|^2 d\eta \\ &= \int_{\mathbb{R}^m} |\langle \xi, \eta \rangle^{s_2} \widehat{u_j}(\eta)|^2 \langle \xi, \eta \rangle^{2(s_1 - s_2)} d\eta,\end{aligned}$$

estimate (4.5) follows readily from the fact that

$$\begin{aligned}\langle \xi, \eta \rangle^{s_1 - s_2} &\sim (1 + |\xi|^2 + |\eta|^2)^{(s_1 - s_2)/2} \\ &\leqslant (1 + |\xi|^2)^{(s_1 - s_2)/2} \\ &\sim \langle \xi \rangle^{s_1 - s_2},\end{aligned}$$

for $s_1 - s_2 \leqslant 0$. □

Theorem 16. *Let $a(x, \xi) \in \Sigma^\mu$. If $\mu < 0$, then $a(x, \xi) : H^s(\mathcal{Y}) \to H^s(\mathcal{Y})$ is a bounded operator and its norm satisfies an estimate*

$$\|a(x, \xi)\|_\xi \leqslant C \langle \xi \rangle^\mu.$$

Proof. By definition, the mapping $a(x, \xi) : H^s(\mathcal{Y}) \to H^s(\mathcal{Y})$ is bounded uniformly in ξ. On applying Lemma 12 we conclude moreover that $H^{s-\mu}(\mathcal{Y})$ is embedded into $H^s(\mathcal{Y})$ with estimate $\|\iota\|_\xi \leqslant C \langle \xi \rangle^\mu$. This gives the desired result. □

This result plays an important role in parameter-dependent theory of pseudodifferential operators.

Lemma 13. *For each $s \in \mathbb{R}$, it follows that*

$$L^2(\mathcal{X}, H^s(\mathcal{Y}), \|\cdot\|_\xi) \cong H^s(\mathcal{X} \times \mathcal{Y}).$$

As usual, the norm in $H^s(\mathcal{X} \times \mathcal{Y})$ is defined by

$$\|u(x, y)\|^2 = \sum_{i,j} \iint_{\mathbb{R}^n \times \mathbb{R}^m} |\langle \xi, \eta \rangle^s \widehat{\phi_i \psi_j u}(\xi, \eta)|^2 \, d\xi d\eta.$$

For symbols $a(x, \xi) \in \Sigma^m$, we introduce a quantization map $a \mapsto A = Q(a)$ by setting

$$Q(a) = \sum_i \phi_i(x) \operatorname{Op}(a(x, \xi)) \, \theta_i(x).$$

where $\theta_i(x) = 1$ for $x \in \operatorname{supp} \phi_i$.

Theorem 17. *For $a(x, \xi) \in \Sigma^m$, the operator $A = Q(a)$ extends to a bounded mapping*

$$A : L^2(\mathcal{X}, H^s(\mathcal{Y}), \|\cdot\|_\xi) \to L^2(\mathcal{X}, H^{s-\mu}(\mathcal{Y}), \|\cdot\|_\xi).$$

Proof. In Fourier representation $f = Au$ gives

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \hat{a}(\xi - \xi', \xi')\hat{u}(\xi')d\xi'$$

whence

$$\begin{aligned}
&\|\hat{f}(\xi)\|_{H^{s-\mu}(\mathcal{Y}),\xi} \\
&\quad \leqslant \int_{\mathbb{R}^n} \|\hat{a}(\xi - \xi', \xi')\hat{u}(\xi')\|_{H^{s-\mu}(\mathcal{Y}),\xi} d\xi' \\
&\quad \leqslant C \int_{\mathbb{R}^n} \langle \xi - \xi' \rangle^q \|\hat{a}(\xi - \xi', \xi')\hat{u}(\xi')\|_{H^{s-\mu}(\mathcal{Y}),\xi'} d\xi' \\
&\quad \leqslant C \int_{\mathbb{R}^n} \langle \xi - \xi' \rangle^q \|\hat{a}(\xi - \xi', \xi')\|_{\mathcal{L}(H^s(\mathcal{Y}),H^{s-\mu}(\mathcal{Y})),\xi'} \|\hat{u}(\xi')\|_{H^s(\mathcal{Y}),\xi'} d\xi' \\
&\quad = C \int O(\langle \xi - \xi' \rangle^{-\infty}) \|\hat{u}(\xi')\|_{H^s(\mathcal{Y}),\xi'} d\xi'.
\end{aligned}$$

So, we have reduced the problem to the boundedness of integral operators in L^2 with kernels $O(\langle \xi - \xi' \rangle^{-\infty})$. This is evident. □

Obviously, the results of this section make sense in much more general context where the spaces $H^s(\mathcal{Y})$ and $H^{s-\mu}(\mathcal{Y})$ on the fibers of $\mathcal{X} \times \mathcal{Y}$ over $\mathcal{X}$ are replaced by abstract Hilbert spaces V and W endowed with slowly varying families of norms parameterized by $\xi \in \mathbb{R}^n$. In this way we obtain a rough class of pseudodifferential operators on $\mathcal{X}$ whose symbols take their values in $\mathcal{L}(V, W)$ with uniformly bounded norms and which map $L^2(\mathcal{X}, V, \|\cdot\|_\xi)$ continuously to $L^2(\mathcal{X}, W, \|\cdot\|_\xi)$. In Section 4.4 we develop this construction for another well-motivated choice of Hilbert spaces V and W.

4.2 Operators with small parameter

In this section we apply the "passive" approach on the product manifold $\mathcal{X} \times \mathcal{Y}$, where $\mathcal{X}$ is a smooth compact closed manifold of dimension n and $\mathcal{Y} = \{P\}$ is a one-point manifold.

Our purpose is to describe a calculus involving differential operators of type (2.1) on $\mathcal{X}$. They are already investigated in norms $H^{r,s}$ which easily extend(see 1.6) to the $H^{r,s}(\mathcal{X})$ with $r, s \in \mathbb{R}$.

One easily recovers the spaces $H^{r,s}(\mathcal{X})$ and $H^{r-m,s-\mu}(\mathcal{X})$ as $L^2(\mathcal{X}, V, \|\cdot\|_\xi)$ and $L^2(\mathcal{X}, W, \|\cdot\|_\xi)$, respectively, where $V = \mathbb{C}$ and $W = \mathbb{C}$ are endowed with the families of norms

$$\|u\|_\xi = |\langle \varepsilon\xi\rangle^{r-s}\langle\xi\rangle^s u|,$$
$$\|f\|_\xi = |\langle \varepsilon\xi\rangle^{(r-m)-(s-\mu)}\langle\xi\rangle^{s-\mu} f|$$

parameterized by $\xi \in \mathbb{R}^n$.

Definition 10 applies immediately to specify the corresponding spaces $\Sigma^{m,\mu}$ of operator-valued symbols $a(x,\xi,\varepsilon)$ on $T^*\mathbb{R}^n$ depending on the small parameter $\varepsilon \in (0,1]$. We restrict ourselves to those symbols which depend continuously on $\varepsilon \in (0,1]$ up to $\varepsilon = 0$. To wit, let $\mathcal{S}^{m,\mu}$ be the space of all functions $a(x,\xi,\varepsilon)$ of $(x,\xi) \in T^*\mathbb{R}^n$ and $\varepsilon \in (0,1]$, which are C^∞ in (x,ξ) and continuous in ε up to $\varepsilon = 0$, such that

$$|\partial_x^\alpha D_\xi^\beta a(x,\xi,\varepsilon)| \leqslant C_{\alpha,\beta}\, \langle\varepsilon\xi\rangle^{m-\mu}\langle\xi\rangle^{\mu-|\beta|} \tag{4.6}$$

is fulfilled for all multi-indices $\alpha,\beta \in \mathbb{Z}^n_{\geq 0}$, where the constants $C_{\alpha,\beta}$ do not depend on (x,ξ) and ε.

Let us define appropriate homogeneity for symbols $a(x,\xi,\varepsilon)$. Our choice is motivated by corresponding property of operators (2.1)

Definition 11. *Suppose that for $a \in S^{m,\mu}$ the uniform with respect to $\xi \in \mathbb{R}^n$ limit*

$$\sigma^\mu(a)(x,\xi,\varepsilon) = \lim_{\lambda\to\infty} \lambda^{-\mu} a(x,\lambda\xi,\varepsilon/\lambda),$$

exists for some $x \in \mathbb{R}^n$ and $\varepsilon > 0$. Then $\sigma^\mu(a)(x,\xi,\varepsilon)$ is homogeneous of degree μ in (ξ,ε^{-1}).

It is worth pointing out that $\sigma^\mu(a)(x,\xi,\varepsilon)$ is actually defined on the whole semiaxis $\varepsilon > 0$. Indeed, let $s > 0$. Then

$$\sigma^\mu(a)(x,s\xi,\varepsilon/s) = \lim_{\lambda\to\infty} \lambda^{-\mu} a(x,\lambda s\xi,\varepsilon/\lambda s),$$

and so on setting $\lambda' = \lambda s$ we get

$$\begin{aligned}\sigma^\mu(a)(x,s\xi,\varepsilon/s) &= \lim_{\lambda'\to\infty} s^\mu \lambda'^{-\mu} a(x,\lambda'\xi,\varepsilon/\lambda') \\ &= s^\mu \sigma^\mu(a)(x,\xi,\varepsilon),\end{aligned}$$

as desired.

Example. By the very origin the complete symbol $a(x,\xi,\varepsilon)$ of (2.1) belongs to the class $\mathcal{S}^{m,\mu}$ and

$$\begin{aligned}\sigma^\mu(a)(x,\xi,\varepsilon) &= \lim_{\lambda\to\infty} \lambda^{-\mu}\Big(\sum_{\substack{|\alpha|-j\leqslant\mu\\ |\alpha|\leqslant m}} a_{\alpha,j}(x)\xi^\alpha\varepsilon^j\,\lambda^{|\alpha|-j}\Big)\\ &= \sum_{\substack{|\alpha|-j=\mu\\ |\alpha|\leqslant m}} a_{\alpha,j}(x)\xi^\alpha\varepsilon^j\end{aligned}$$

is well defined.

In fact, the complete symbol of any differential operator $A(x,D,\varepsilon)$ of the form (2.1) expands as finite sum of homogeneous symbols of decreasing degree with step 1. More generally, one specifies the subspaces $\mathcal{S}^{m,\mu}_{\mathsf{cl}}$ in $\mathcal{S}^{m,\mu}$ consisting of all classical symbols, *i.e.* those admitting asymptotic expansions in homogeneous symbols. To introduce classical symbols more precisely, we need a purely technical result.

Lemma 14. *Let a be a C^∞ function of $(x,\xi)\in T^*\mathbb{R}^n\setminus\{0\}$ and $\varepsilon>0$ satisfying $|\partial_x^\alpha D_\xi^\beta a(x,\xi,1)|\leqslant C_{\alpha,\beta}\,\langle\xi\rangle^{m-|\beta|}$ for $|\xi|\geq 1$ and $\alpha,\beta\in\mathbb{Z}^n_{\geq 0}$. If a is homogeneous of degree μ in (ξ,ε^{-1}), then $\chi a\in\mathcal{S}^{m,\mu}$ for any excision function $\chi=\chi(\xi)$ for the origin in $\mathbb{R}^n$.*

Proof. Since each derivative $\partial_x^\alpha D_\xi^\beta a$ is homogeneous of degree $\mu-|\beta|$ in (ξ,ε^{-1}), it suffices to prove estimate (4.6) only for $\alpha=\beta=0$. We have to show that there is a constant $C>0$, such that

$$|\chi(\xi)a(x,\xi,\varepsilon)|\leqslant C\,\langle\varepsilon\xi\rangle^{m-\mu}\langle\xi\rangle^\mu$$

for all $(x,\xi)\in T^*\mathbb{R}^n$ and $\varepsilon\in(0,1]$. Such an estimate is obvious if ξ varies in a compact subset of $\mathbb{R}^n$, for χ vanishes in a neighbourhood of $\xi=0$. Hence, there is no restriction of generality in assuming that $|\xi|\geq R$, where $R>1$ is large enough, so that $\chi(\xi)\equiv 1$ for $|\xi|\geq R$.

We distinguish two cases, namely $\varepsilon\leqslant\langle\xi\rangle^{-1}$ and $\varepsilon>\langle\xi\rangle^{-1}$. In the first case we immediately get

$$\begin{aligned}|a(x,\xi,\varepsilon)| &= \langle\xi\rangle^\mu|a(x,\xi/\langle\xi\rangle,\varepsilon\langle\xi\rangle)|\\ &\leqslant C\,\langle\xi\rangle^\mu,\end{aligned}$$

where C is the supremum of $|a(x,\xi',\varepsilon')|$ over all x, $1/\sqrt{2}\leqslant|\xi'|\leqslant 1$ and $\varepsilon'\in[0,1]$. Moreover, $\langle\varepsilon\xi\rangle^{m-\mu}$ is bounded from below by a positive constant independent of ξ and ε, for $\varepsilon|\xi| \leqslant 1$. This yields $|a(x,\xi,\varepsilon)| \leqslant C' \langle \varepsilon\xi\rangle^{m-\mu}\langle\xi\rangle^{\mu}$ with some new constant C', as desired.

Assume that $\varepsilon > \langle\xi\rangle^{-1}$. Then $\varepsilon^{-1} < \langle\xi\rangle$ whence

$$\begin{aligned} |a(x,\xi,\varepsilon)| &= |\varepsilon^{-\mu}\, a(x,\varepsilon\xi,1)| \\ &\leqslant C\,\varepsilon^{-\mu}\,\langle\varepsilon\xi\rangle^{m} \\ &= C\,\langle\varepsilon\xi\rangle^{m-\mu}\,(\varepsilon^{-1}\langle\varepsilon\xi\rangle)^{\mu} \end{aligned}$$

with C a constant independent of x, ξ and ε. If $\mu > 0$ then the factor $(\varepsilon^{-1}\langle\varepsilon\xi\rangle)^{\mu}$ is estimated by

$$(\varepsilon^{-2} + |\xi|^2)^{\mu/2} \leqslant 2^{\mu/2}\,\langle\xi\rangle^{\mu}.$$

If $\mu < 0$ then this estimate is obvious, even without the factor $2^{\mu/2}$. This establishes the desired estimate. □

The family $\mathcal{S}^{m-j,\mu-j}$ with $j = 0, 1, \ldots$ is used as usual to define asymptotic sums of homogeneous symbols. A symbol $a \in \mathcal{S}^{m,\mu}$ is said to be classical if there is a sequence $\{a_{\mu-j}\}_{j=0,1,\ldots}$ of smooth function of $(x,\xi) \in T^*\mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$ satisfying $|\partial_x^{\alpha} D_{\xi}^{\beta} a_{\mu-j}(x,\xi,1)| \leqslant C_{\alpha,\beta}\,\langle\xi\rangle^{m-j-|\beta|}$ for $|\xi| \geq 1$ and $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, such that every $a_{\mu-j}$ is homogeneous of degree $\mu - j$ in (ξ, ε^{-1}) and a expands as asymptotic sum

$$a(x,\xi,\varepsilon) \sim \chi(\xi) \sum_{j=0}^{\infty} a_{\mu-j}(x,\xi,\varepsilon) \tag{4.7}$$

in the sense that $a - \chi \sum_{j=0}^{N} a_{\mu-j} \in \mathcal{S}^{m-N-1,\mu-N-1}$ for all $N = 0, 1, \ldots$.

The appropriate concept in abstract algebra to describe expansions like (4.7) is that of filtration. To wit,

$$\mathcal{S}^{m,\mu}_{\mathsf{cl}} \sim \bigoplus_{j=0}^{\infty} \left(\mathcal{S}^{m-j,\mu-j}_{\mathsf{cl}} \ominus \mathcal{S}^{m-j-1,\mu-j-1}_{\mathsf{cl}}\right).$$

Each symbol $a \in \mathcal{S}^{m,\mu}_{\mathsf{cl}}$ possesses a well-defined principal homogeneous symbol of degree μ, namely $\sigma^{\mu}(a) := a_{\mu}$. To construct an algebra of pseudodifferential operators on $\mathcal{X}$ with symbolic structure one need not consider full asymptotic expansions like (4.7). It suffices to ensure that the limit $\sigma^{\mu}(a)$ exists and the difference $a - \chi\sigma^{\mu}(a)$ belongs to $\mathcal{S}^{m-1,\mu-1}$. For more details we refer to Section 3.3 in [Fra90].

We may now quantise symbols $a \in \mathcal{S}^{m,\mu}$ as pseudodifferential operators on $\mathcal{X}$ in just the same way as in Section 4.1. The space of operators $A = Q(a)$ with symbols $a \in \mathcal{S}^{m,\mu}$ is denoted by $\Psi^{m,\mu}(\mathcal{X})$.

Theorem 18. *Let $A \in \Psi^{m,\mu}(\mathcal{X})$. For any $r, s \in \mathbb{R}$, the operator A extends to a bounded mapping*

$$A : H^{r,s}(\mathcal{X}) \to H^{r-m,s-\mu}(\mathcal{X})$$

whose norm is independent of $\varepsilon \in [0, 1]$.

Proof. This is a consequence of Theorem 17. □

Let $\Psi^{m,\mu}_{\mathsf{cl}}(\mathcal{X})$ stand for the subspace of $\Psi^{m,\mu}(\mathcal{X})$ consisting of those operators which have classical symbols. For $A \in \Psi^{m,\mu}_{\mathsf{cl}}(\mathcal{X})$, the principal homogeneous symbol of degree μ is defined by $\sigma^\mu(A) = \sigma^\mu(a)$, where $A = Q(a)$. If $\sigma^\mu(A) = 0$ then A belongs actually to $\Psi^{m-1,\mu-1}_{\mathsf{cl}}(\mathcal{X})$. Hence, the mapping $A : H^{r,s}(\mathcal{X}) \to H^{r-m,s-\mu}(\mathcal{X})$ is compact, for it factors through the compact embedding

$$H^{r-m+1,s-\mu+1}(\mathcal{X}) \hookrightarrow H^{r-m,s-\mu}(\mathcal{X}).$$

Theorem 19. *If $A \in \Psi^{m,\mu}_{\mathsf{cl}}(\mathcal{X})$ and $B \in \Psi^{n,\nu}_{\mathsf{cl}}(\mathcal{X})$, then $BA \in \Psi^{m+n,\mu+\nu}_{\mathsf{cl}}(\mathcal{X})$ and $\sigma^{\mu+\nu}(BA) = \sigma^\nu(B)\sigma^\mu(A)$.*

Proof. See for instance Proposition 3.3.3 in [Fra90]. □

As usual, an operator $A \in \Psi^{m,\mu}_{\mathsf{cl}}(\mathcal{X})$ is called elliptic if its symbol $\sigma^\mu(A)(x, \xi, \varepsilon)$ is invertible for all $(x, \xi) \in T^*\mathcal{X} \setminus \{0\}$ and $\varepsilon \in [0, 1]$.

Theorem 20. *An operator $A \in \Psi^{m,\mu}_{\mathsf{cl}}(\mathcal{X})$ is elliptic if and only if it possesses a parametrix $P \in \Psi^{-m,-\mu}_{\mathsf{cl}}(\mathcal{X})$,* i.e. *$PA = I$ and $AP = I$ modulo operators in $\Psi^{-\infty,-\infty}(\mathcal{X})$.*

Proof. The necessity of ellipticity follows immediately from Theorem 19, for the equalities $PA = I$ and $AP = I$ modulo $\Psi^{-\infty,-\infty}(\mathcal{X})$ imply that $\sigma^{-\mu}(P)$ is the inverse of $\sigma^\mu(A)$.

Conversely, look for a parametrix $P = Q(p)$ for $A = Q(a)$, where $p \in \mathcal{S}^{-m,-\mu}_{\mathsf{cl}}$ has asymptotic expansion $p \sim p_{-\mu} + p_{-\mu-1} + \cdots$. The ellipticity of A just amounts to saying that

$$\sigma^\mu(A)(x, \xi, \varepsilon) \geq c\, \langle \varepsilon\xi \rangle^{m-\mu} |\xi|^\mu$$

for all $(x,\xi) \in T^*\mathcal{X} \setminus \{0\}$ and $\varepsilon \in [0,1]$, where the constant $c > 0$ does not depend on x, ξ and ε. Hence, $p_{-\mu} := (\sigma^\mu(A))^{-1}$ gives rise to a "soft" parametrix $P^{(0)} = Q(\chi p_{-\mu})$ for A. More precisely, $P^{(0)} \in \Psi_{\mathrm{cl}}^{-m,-\mu}(\mathcal{X})$ satisfies $P^{(0)}A = I$ and $AP^{(0)} = I$ modulo $\Psi^{-1,-1}(\mathcal{X})$. Now, the standard techniques of pseudodifferential calculus applies to improve the discrepancies $P^{(0)}A - I$ and $AP^{(0)} - I$, see for instance [ST05]. □

To sum up the homogeneous components $p_{-\mu-j}$ with $j = 0,1,\ldots$, one uses a trick of L. Hörmander for asymptotic summation of symbols, see Theorem 3.6.3 in [Fra90].

Corollary 2. *Assume that $A \in \Psi_{\mathrm{cl}}^{m,\mu}(\mathcal{X})$ is an elliptic operator on $\mathcal{X}$. Then, for any $r,s \in \mathbb{R}$ and any large $R > 0$, there is a constant $C > 0$ independent of ε, such that*

$$\|u\|_{r,s} \leqslant C \left(\|Au\|_{r-m,s-\mu} + \|u\|_{-R,-R}\right)$$

whenever $u \in H^{r,s}(\mathcal{X})$.

Proof. Let $P \in \Psi_{\mathrm{cl}}^{-m,-\mu}(\mathcal{X})$ be a parametrix of A given by Theorem 20. Then we obtain

$$\begin{aligned} \|u\|_{r,s} &= \|P(Au) + (I - PA)u\|_{r,s} \\ &\leqslant \|P(Au)\|_{r,s} + \|(I - PA)u\|_{r,s} \end{aligned}$$

for all $u \in H^{r,s}(\mathcal{X})$. To complete the proof it is now sufficient to use the mapping properties of pseudodifferential operators P and $I - PA$ formulated in Theorem 18. □

4.3 Ellipticity with large parameter

Setting $\lambda = 1/\varepsilon$ we get a "large" parameter. Substituting $\varepsilon = 1/\lambda$ to (2.1) and multiplying A by $\lambda^{m-\mu}$ yields

$$\tilde{A}(x,D,\lambda) = \sum_{\substack{|\alpha|+j\leqslant m \\ j\leqslant m-\mu}} \tilde{a}_{\alpha,j}(x)\lambda^j D^\alpha$$

in local coordinates in $\mathcal{X}$. For this operator the ellipticity with large parameter leads to the inequality

$$\Big| \sum_{\substack{|\alpha|+j=m \\ j\leqslant m-\mu}} \tilde{a}_{\alpha,j}(x)\lambda^j\xi^\alpha \Big| \geq c\,\langle\lambda,\xi\rangle^{m-\mu}|\xi|^\mu,$$

which is a generalization of the Agmon-Agranovich-Vishik condition of ellipticity with parameter corresponding to $\mu = 0$, see [AV64], [Vol06] and the references given there.

4.4 Another approach to parameter-dependent theory

In this section we develop another approach to pseudodifferential operators with small parameter which stems from analysis on manifolds with singularities. In this ares the role of small parameter is played by the distance to singularities and it has been usually chosen as a local coordinate. Thus, the small parameter is included into functions under study as independent variable and the action of operators include also that in the small parameter. Geometrically this approach corresponds to analysis on the cylinder $\mathcal{C} = \mathcal{X} \times [0, 1]$ over a compact closed manifold $\mathcal{X}$ of dimension n, see Fig. 4.1. Subject to the problem its base $\varepsilon = 0$

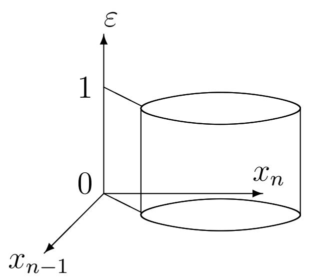


Figure 4.1: *A cylinder* $\mathbb{C} = \mathcal{X} \times [0, 1)$ *over* $\mathcal{X}$

can be thought of as singular point blown up by a singular transformation of coordinates. In this case one restricts the study to functions which are constant on the base, taking on the values 0 or ∞. In our problem the base is regarded as part of the boundary $\mathcal{X} \times \{0\}$ of the cylinder $\mathbb{C}$, and so we distinguish the values of functions on the base. The top $\mathcal{X} \times \{1\}$ is actually excluded from consideration by a particular choice of function spaces on the segment $\mathcal{Y} = [0, 1]$, for we are interested in local analysis at $\varepsilon = 0$.

Basically there are two possibilities to develop a calculus of pseudodifferential operators on the cylinder $\mathbb{C}$. Either one thinks of them as pseudodifferential operators on $\mathcal{X}$ with symbols taking on their values in an operator algebra on $[0, 1]$. Or one treats them as pseudodifferential operators on the segment $[0, 1]$ whose symbols are pseudodifferential operators on $\mathcal{X}$. Singularly perturbed problems require the first approach with symbols taking on their values in multiplication operators in $\mathcal{L}(V, W)$, where

$$V = L^2([0,1], \varepsilon^{-2\gamma}),$$
$$W = L^2([0,1], \varepsilon^{-2\gamma})$$

with $\gamma \in \mathbb{R}$.

Any continuous function $a \in C[0,1]$ induces the multiplication operator $u \mapsto au$ on $L^2([0,1], \varepsilon^{-2\gamma})$ that is obviously bounded. Moreover, the norm of this operator is equal to the supremum norm of a in $C[0,1]$. Hence, $C[0,1]$ can be specified as a closed subspace of $\mathcal{L}(V, W)$.

Pick real numbers μ and s. We endow the spaces V and W with the families of norms

$$\|u\|_\xi = \|\langle\xi\rangle^s \varkappa_{\langle\xi\rangle}^{-1} u\|_{L^2([0,1],\varepsilon^{-2\gamma})},$$
$$\|f\|_\xi = \|\langle\xi\rangle^{s-\mu} \tilde{\varkappa}_{\langle\xi\rangle}^{-1} f\|_{L^2([0,1],\varepsilon^{-2\gamma})}$$

parameterized by $\xi \in \mathbb{R}^n$, where

$$(\varkappa_\lambda u)(\varepsilon) = \lambda^{-\gamma+1/2}\, u(\lambda\varepsilon),$$
$$(\tilde{\varkappa}_\lambda f)(\varepsilon) = \lambda^{-\gamma+1/2}\, f(\lambda\varepsilon)$$

for $\lambda \leqslant 1$.

The space $L^2(\mathcal{X}, V, \|\cdot\|_\xi)$ is defined to be the completion of $C^\infty(\mathcal{X}, V)$ with respect to the norm

$$\|u\|_{s,\gamma}^2 = \sum_i \int_{\mathbb{R}^n} \|\widehat{\phi_i u}\|_\xi^2 d\xi,$$

where $\{\phi_i\}$ is a C^∞ partition of unity on $\mathcal{X}$ subordinate to a finite coordinate covering $\{\mathcal{U}_i\}$.

Remark. *The space $L^2(\mathcal{X}, V, \|\cdot\|_\xi)$ is locally identified within abstract edge spaces $H^s(\mathbb{R}^n, V, \varkappa)$ with the group action $\varkappa$ on $V = L^2([0,1], \varepsilon^{-2\gamma})$ defined above, see [ST05].*

In a similar way one introduces the space $L^2(\mathcal{X}, W, \|\cdot\|_\xi)$ whose norm is denoted by $\|\cdot\|_{s-\mu,\gamma}$. Set

$$H^{s,\gamma}(\mathbb{C}) = L^2(\mathcal{X}, V, \|\cdot\|_\xi),$$
$$H^{s-\mu,\gamma}(\mathbb{C}) = L^2(\mathcal{X}, W, \|\cdot\|_\xi),$$

which will cause no confusion since the right-hand sides coincide for $\mu = 0$, as is easy to check. We are thus led to a scale of function spaces on the cylinder $\mathbb{C}$ which are Hilbert.

Our next objective is to describe those pseudodifferential operators on $\mathbb{C}$ which map $H^{s,\gamma}(\mathbb{C})$ continuously into $H^{s-\mu,\gamma}(\mathbb{C})$. To this end we specify the definition of symbol spaces, see (4.3). If $a(x,\xi,\varepsilon)$ is a function of $(x,\xi) \in T^*\mathbb{R}^n$ and $\varepsilon \in [0,1]$, which is smooth in (x,ξ) and continuous in ε, then a straightforward calculation shows that

$$\|\partial_x^\alpha D_\xi^\beta a(x,\xi,\varepsilon)\|_{\mathcal{L}(V,W),\xi} = \langle\xi\rangle^{-\mu} \sup_{\varepsilon\in[0,1]} |(\partial_x^\alpha D_\xi^\beta a)(x,\xi,\varepsilon/\langle\xi\rangle)|$$

holds on all of $T^*\mathbb{R}^n$. We now denote by $\mathcal{S}^\mu$ the space of all functions $a(x,\xi,\varepsilon)$ of $(x,\xi) \in T^*\mathbb{R}^n$ and $\varepsilon \in [0,1]$, which are smooth in (x,ξ) and continuous in ε and satisfy

$$|(\partial_x^\alpha D_\xi^\beta a)(x,\xi,\varepsilon/\langle\xi\rangle)| \leqslant C_{\alpha,\beta}\,\langle\xi\rangle^{\mu-|\beta|} \tag{4.8}$$

for all multi-indices $\alpha,\beta \in \mathbb{Z}^n_{\geq 0}$, where $C_{\alpha,\beta}$ are constants independent of (x,ξ) and ε.

In terms of group action introduced in Remark 4.4 the symbol estimates (4.8) take the form

$$\|\tilde{\varkappa}^{-1}_{\langle\xi\rangle}\partial_x^\alpha D_\xi^\beta a(x,\xi,\varepsilon)\varkappa_{\langle\xi\rangle}\|_{\mathcal{L}(L^2([0,1],\varepsilon^{-2\gamma}))} \leqslant C_{\alpha,\beta}\,\langle\xi\rangle^{\mu-|\beta|}$$

for all $(x,\xi) \in T^*\mathbb{R}^n$ and $\alpha,\beta \in \mathbb{Z}^n_{\geq 0}$, cf. [ST05]. In particular, the order of the symbol a is μ. Moreover, using group actions in fibers V and W gives a direct way to the notion of homogeneity in the calculus of operator-valued symbols on $T^*\mathbb{R}^n$. Namely, a function $a(x,\xi,\varepsilon)$, defined for $(x,\xi) \in T^*\mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$, is said to be homogeneous of degree μ if the equality $a(x,\lambda\xi,\varepsilon) = \lambda^\mu \tilde{\varkappa}_\lambda a(x,\xi,\varepsilon)\varkappa_\lambda^{-1}$ is fulfilled for all $\lambda > 0$. It is easily seen that a is homogeneous of degree μ with respect to the group actions $\varkappa$ and $\tilde{\varkappa}$ if and only if $a(x,\lambda\xi,\varepsilon/\lambda) = \lambda^\mu a(x,\xi,\varepsilon)$ for all $\lambda > 0$, *i.e.* a is homogeneous of degree μ in (ξ,ε^{-1}). Thus, we recover the homogeneity of symbols invented in Section 4.2.

Lemma 15. *Assume that the limit*

$$\sigma^\mu(a)(x,\xi,\varepsilon) = \lim_{\lambda\to\infty} \lambda^{-\mu}\tilde{\varkappa}_\lambda^{-1} a(x,\lambda\xi,\varepsilon)\varkappa_\lambda$$

exists for some $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$ and $\varepsilon > 0$. Then $\sigma^\mu(a)(x,\xi,\varepsilon)$ is homogeneous of degree μ.

Proof. Let $s > 0$ and let $u = u(\varepsilon)$ be an arbitrary function of V. By the definition of group action, we get

$$\begin{aligned}\sigma^\mu(a)(x, s\xi, \varepsilon)u &= \lim_{\lambda\to\infty} \lambda^{-\mu}\tilde{\varkappa}_\lambda^{-1} a(x, \lambda s\xi, \varepsilon)\varkappa_\lambda u \\ &= s^\mu \tilde{\varkappa}_s \Big(\lim_{\lambda'\to\infty} (\lambda')^{-\mu}\tilde{\varkappa}_{\lambda'}^{-1} a(x, \lambda'\xi, \varepsilon)\varkappa_{\lambda'} \Big) \varkappa_s^{-1} u,\end{aligned}$$

the second equality being a consequence of substitution $\lambda' = \lambda s$. Since the expression in the parentheses just amounts to $\sigma^\mu(a)(x, \xi, \varepsilon)$, the lemma follows. □

The function $\sigma^\mu(a)$ defined away from the zero section of the cotangent bundle $T^*\mathbb{C}$ is called the principal homogeneous symbol of degree μ of a. We also use this designation for the operator $A = Q(a)$ on the cylinder which is a suitable quantization of a.

Example. As defined above, the principal homogeneous symbol of differential operator (2.1) is

$$\begin{aligned}\sigma^\mu(A)(x, \xi, \varepsilon) &= \lim_{\lambda\to\infty} \lambda^{-\mu} \Big(\sum_{\substack{|\alpha|-j\leqslant\mu \\ |\alpha|\leqslant m}} a_{\alpha,j}(x)(\lambda\xi)^\alpha \left(\tilde{\varkappa}_\lambda^{-1}\varepsilon^j \varkappa_\lambda\right) \Big) \\ &= \sum_{\substack{|\alpha|-j=\mu \\ |\alpha|\leqslant m}} a_{\alpha,j}(x)\xi^\alpha \varepsilon^j \varkappa_\lambda.\end{aligned}$$

Now one introduces the subspaces $\mathcal{S}^\mu_{\mathsf{cl}}$ in $\mathcal{S}^\mu$ consisting of all classical symbols, *i.e.* those admitting asymptotic expansions in homogeneous symbols. To do this, we need an auxiliary result.

Lemma 16. *Let a be a C^∞ function of $(x, \xi) \in T^*\mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$ with $a \equiv 0$ for $|x| \gg 1$. If a is homogeneous of degree μ, then $\chi a \in \mathcal{S}^\mu$ for any excision function $\chi = \chi(\xi)$ for the origin in $\mathbb{R}^n$.*

Proof. Since each derivative $\partial_x^\alpha D_\xi^\beta a$ is homogeneous of degree $\mu - |\beta|$, it suffices to prove estimate (4.8) only for $\alpha = \beta = 0$. We have to show that there is a constant $C > 0$, such that

$$\|\tilde{\varkappa}_{\langle\xi\rangle}^{-1} \left(\chi(\xi)a(x, \xi, \varepsilon)\right) \varkappa_{\langle\xi\rangle}\|_{\mathcal{L}(L^2([0,1],\varepsilon^{-2\gamma}))} \leqslant C\, \langle\xi\rangle^\mu$$

for all $(x, \xi) \in T^*\mathbb{R}^n$ and $\varepsilon \in [0, 1]$. Such an estimate is obvious if ξ varies in a compact subset of $\mathbb{R}^n$, for χ near $\xi = 0$. Hence, we may assume without of generality that $|\xi| \geq R$, where $R > 1$ is sufficiently large, so that $\chi(\xi) \equiv 1$ for $|\xi| \geq R$. Then

$$\begin{aligned}\|\tilde{\varkappa}_{\langle\xi\rangle}^{-1} \left(\chi(\xi)a(x,\xi,\varepsilon)\right) \varkappa_{\langle\xi\rangle}\|_{\mathcal{L}(L^2([0,1],\varepsilon^{-2\gamma}))} &= \|a(x,\xi,\varepsilon/\langle\xi\rangle)\|_{\mathcal{L}(L^2([0,1],\varepsilon^{-2\gamma}))} \\ &\leqslant C\,\langle\xi\rangle^{\mu},\end{aligned}$$

where

$$C = \sup_{(x,\xi)\in T^*\mathbb{R}^n} \|a(x,\xi/\langle\xi\rangle,\varepsilon)\|_{\mathcal{L}(L^2([0,1],\varepsilon^{-2\gamma}))}.$$

From conditions imposed on a it follows that the supremum is finite, which completes the proof. □

In contrast to Lemma 14 no additional conditions are imposed here on a except for homogeneity. This might testify to the fact that the symbol classes $\mathcal{S}^{\mu}$ give the best fit to the study of operators (2.1).

The family $\mathcal{S}^{\mu-j}$ with $j = 0, 1, \dots$ is used in the usual way to define asymptotic sums of homogeneous symbols. A symbol $a \in \mathcal{S}^{\mu}$ is called classical if there is a sequence $\{a_{\mu-j}\}_{j=0,1,\dots}$ of smooth function of $(x,\xi) \in T^*\mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$, such that every $a_{\mu-j}$ is homogeneous of degree $\mu - j$ in (ξ, ε^{-1}) and a expands as asymptotic sum

$$a(x,\xi,\varepsilon) \sim \chi(\xi) \sum_{j=0}^{\infty} a_{\mu-j}(x,\xi,\varepsilon) \tag{4.9}$$

in the sense that $a - \chi \sum_{j=0}^{N} a_{\mu-j} \in \mathcal{S}^{\mu-N-1}$ for all $N = 0, 1, \dots$.

Each symbol $a \in \mathcal{S}^{\mu}_{\mathsf{cl}}$ admits a well-defined principal homogeneous symbol of degree μ, namely $\sigma^{\mu}(a) := a_{\mu}$. We quantise symbols $a \in \mathcal{S}^{\mu}$ as pseudodifferential operators on $\mathcal{X}$ similarly to Section 1.6. Write $\Psi^{\mu}(\mathbb{C})$ for the space of all operators $A = Q(a)$ with $a \in \mathcal{S}^{\mu}$.

Theorem 21. *Let $A \in \Psi^{\mu}(\mathbb{C})$. For any $s, \gamma \in \mathbb{R}$, the operator A extends to a bounded mapping*

$$A : H^{s,\gamma}(\mathbb{C}) \to H^{s-\mu,\gamma}(\mathbb{C}).$$

Proof. This is a consequence of Theorem 17. □

Let $\Psi^{\mu}_{\mathsf{cl}}(\mathbb{C})$ stand for the subspace of $\Psi^{\mu}(\mathbb{C})$ consisting of all operators with classical symbols. For $A = Q(a)$ of $\Psi^{\mu}_{\mathsf{cl}}(\mathbb{C})$, the principal homogeneous symbol of degree μ is defined by $\sigma^\mu(A) = \sigma^\mu(a)$. If $\sigma^\mu(A) = 0$ then A belongs to $\Psi^{\mu-1}_{\mathsf{cl}}(\mathbb{C})$. When combined with Theorem 22 stated below, this result allows one to describe those operators A on the cylinder which are invertible modulo operators of order $-\infty$.

Theorem 22. *If $A \in \Psi^\mu_{\mathsf{cl}}(\mathbb{C})$ and $B \in \Psi^\nu_{\mathsf{cl}}(\mathbb{C})$, then $BA \in \Psi^{\mu+\nu}_{\mathsf{cl}}(\mathbb{C})$ and $\sigma^{\mu+\nu}(BA) = \sigma^\nu(B)\sigma^\mu(A)$.*

Proof. This is a standard fact of calculus of pseudodifferential operators with operator-valued symbols. □

As usual, an operator $A \in \Psi^\mu_{\mathsf{cl}}(\mathbb{C})$ is called elliptic if $\sigma^\mu(A)(x,\xi,\varepsilon)$ is invertible for all (x,ξ,ε) away from the zero section of the cotangent bundle $T^*\mathbb{C}$ of the cylinder.

Theorem 23. *An operator $A \in \Psi^\mu_{\mathsf{cl}}(\mathbb{C})$ is elliptic if and only if there is an operator $P \in \Psi^{-\mu}_{\mathsf{cl}}(\mathbb{C})$, such that both $PA = I$ and $AP = I$ are fulfilled modulo operators of $\Psi^{-\infty}(\mathbb{C})$.*

Proof. The necessity of ellipticity follows immediately from Theorem 22, for the equalities $PA = I$ and $AP = I$ modulo $\Psi^{-\infty}(\mathbb{C})$ imply that $\sigma^{-\mu}(P)$ is the inverse of $\sigma^\mu(A)$.

Conversely, look for an inverse $P = Q(p)$ for $A = Q(a)$ modulo $\Psi^{-\infty}(\mathbb{C})$, where $p \in \mathcal{S}^{-\mu}_{\mathsf{cl}}$ has asymptotic expansion $p \sim p_{-\mu} + p_{-\mu-1} + \cdots$. The ellipticity of A just amounts to saying that

$$\sigma^\mu(A)(x,\xi,\varepsilon) \geq c\,|\xi|^\mu$$

for all $(x,\xi) \in T^*\mathcal{X} \setminus \{0\}$ and $\varepsilon \in [0,1]$, where the constant $c > 0$ does not depend on x, ξ and ε. Hence, $p_{-\mu} := (\sigma^\mu(A))^{-1}$ gives rise to a "soft" inverse $P^{(0)} = Q(\chi p_{-\mu})$ for A. More precisely, $P^{(0)} \in \Psi^{-\mu}_{\mathsf{cl}}(\mathbb{C})$ satisfies $P^{(0)}A = I$ and $AP^{(0)} = I$ modulo operators of $\Psi^{-1}(\mathbb{C})$. Now, the standard techniques of pseudodifferential calculus applies to improve the discrepancies $P^{(0)}A - I$ and $AP^{(0)} - I$, see for instance [ST05]. □

Corollary 3. *Assume that $A \in \Psi^\mu_{\mathsf{cl}}(\mathbb{C})$ is an elliptic operator on $\mathbb{C}$. Then, for any $s, \gamma \in \mathbb{R}$ and any large $R > 0$, there is a constant $C > 0$ independent of ε, such that*

$$\|u\|_{s,\gamma} \leqslant C\ (\|Au\|_{s-\mu,\gamma} + \|u\|_{-R,\gamma})$$

whenever $u \in H^{s,\gamma}(\mathbb{C})$.

Proof. Let $P \in \Psi_{\mathrm{cl}}^{-\mu}(\mathbb{C})$ be the inverse of A up to operators of $\Psi^{-\infty}(\mathbb{C})$ given by Theorem 20. Then we obtain

$$\begin{aligned}\|u\|_{s,\gamma} &= \|P(Au) + (I - PA)u\|_{s,\gamma} \\ &\leqslant \|P(Au)\|_{s,\gamma} + \|(I - PA)u\|_{s,\gamma}\end{aligned}$$

for all $u \in H^{s,\gamma}(\mathbb{C})$. To complete the proof it is now sufficient to use the mapping properties of pseudodifferential operators P and $I - PA$ formulated in Theorem 18. □

We finish this section by evaluating the local norm in $H^{s,\gamma}(\mathbb{C})$ to compare this scale with the scale $H^{r,s}(\mathcal{X})$ used in Section 4.2. This norm is equivalent to that in $L^2(\mathbb{R}^n, V, \|\cdot\|_\xi)$, which is

$$\begin{aligned}\|u\|_{s,\gamma}^2 &= \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \|\varkappa_{\langle\xi\rangle}^{-1} \hat{u}(\xi)\|^2_{L^2([0,1],\varepsilon^{-2\gamma})} d\xi \\ &= \int_{\mathbb{R}^n} \langle\xi\rangle^{2s+2\gamma-1} \int_0^1 \varepsilon^{-2\gamma} |\hat{u}(\xi, \varepsilon/\langle\xi\rangle)|^2 d\varepsilon d\xi.\end{aligned}$$

Substituting $\varepsilon' = \varepsilon/\langle\xi\rangle$ yields

$$\begin{aligned}\|u\|_{s,\gamma}^2 &= \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \Big(\int_0^{1/\langle\xi\rangle} (\varepsilon')^{-2\gamma} |\hat{u}(\xi, \varepsilon')|^2 d\varepsilon' \Big) d\xi \\ &= \int_0^1 (\varepsilon')^{-2(\gamma+\Delta\gamma)} \Big(\int_{\langle\xi\rangle \leqslant 1/\varepsilon'} (\varepsilon'\langle\xi\rangle)^{2\Delta\gamma} \langle\xi\rangle^{2(s-\Delta\gamma)} |\hat{u}(\xi, \varepsilon')|^2 d\xi \Big) d\varepsilon',\end{aligned}$$

which is close to $\displaystyle\int_0^1 \varepsilon^{-2(\gamma+\Delta\gamma)} \|u\|^2_{H^{s,s-\Delta\gamma}(\mathcal{X})} d\varepsilon$ with any $\Delta\gamma \in \mathbb{R}$.

4.5 Regularisation of singularly perturbed problems

The idea of constructive reduction of elliptic singular perturbations to regular perturbations goes back at least as far as [FW82]. For the complete bibliography see [Fra90, p. 531].

The calculus of pseudodifferential operators with small parameter developed in Section 4.2 allows one to reduce the question of the invertibility of elliptic operators $A \in \Psi_{\mathrm{cl}}^{m,\mu}(\mathcal{X})$ acting from $H^{r,s}(\mathcal{X})$ into $H^{r-m,s-\mu}(\mathcal{X})$ to that of the invertibility of their limit operators at $\varepsilon = 0$ acting in usual Sobolev spaces

$H^s(\mathcal{X}) \to H^{s-\mu}(\mathcal{X})$. To shorten notation, we write $A(\varepsilon)$ instead of $A(x, D, \varepsilon)$, and so $A(0) \in \Psi^{\mu}_{\mathsf{cl}}(\mathcal{X})$ is the reduced operator.

Given any $f \in H^{r-m,s-\mu}(\mathcal{X})$, consider the inhomogeneous equation $A(\varepsilon)u = f$ on $\mathcal{X}$ for an unknown function $u \in H^{r,s}(\mathcal{X})$. We first assume that $u \in H^{r,s}(\mathcal{X})$ satisfies $A(\varepsilon)u = f$ in $\mathcal{X}$.

Since the symbol $\sigma^{\mu}(A(\varepsilon))(x, \xi, \varepsilon)$ is invertible for all $(x, \xi) \in T^*\mathcal{X} \setminus 0$ and $\varepsilon \in [0, \varepsilon_0)$, it follows that $A(0)$ is an elliptic operator of order μ.

According the Hodge theory, there is an operator $G \in \Psi^{-\mu}_{\mathsf{cl}}(\mathcal{X})$ satisfying

$$\begin{aligned} u &= H^0 u + GA(0)u, \\ f &= H^1 f + A(0)Gf \end{aligned} \tag{4.10}$$

for all distributions u and f on $\mathcal{X}$, where H^0 and H^1 are $L^2(\mathcal{X})$-orthogonal projections onto the null-spaces of $A(0)$ and $A(0)^*$, respectively. (Observe that the null-spaces of $A(0)$ and $A(0)^*$ are actually finite dimensional and consist of C^∞ functions.)

Acting by the operator $G \in \Psi^{-\mu,-\mu}_{\mathsf{cl}}(\mathcal{X})$ to both sides of the equality $A(0)u + (A(\varepsilon) - A(0))u = f$ on $\mathcal{X}$ we obtain

$$u - H^0 u = Gf - G\left(A(\varepsilon) - A(0)\right)u \tag{4.11}$$

for each $u \in H^{r,s}(\mathcal{X})$. (We have used the first equality of (4.10)). To this point the existence of composition $G\left(A(\varepsilon) - A(0)\right)$ is crucial, essentially it is an analogue to condition II. If $\left(A(\varepsilon) - A(0)\right)u$ converges to zero in $H^{r-m,s-\mu}(\mathcal{X})$ as $\varepsilon \to 0$ and $m \geq \mu$ then by continuity, $G\left(A(\varepsilon) - A(0)\right)u$ converges to zero in $H^{r-(m-\mu),s}(\mathcal{X})$, and so $u - H^0 u \in H^{r,s}(\mathcal{X})$ converges to Gf in $H^{r-(m-\mu),s}(\mathcal{X})$ as $\varepsilon \to 0$. The case $m < \mu$ is analysed in the same manner, but $u - H^0 u$ converges to Gf in the spaces $H^{r,s}(\mathcal{X})$ while $\|A(\varepsilon)u - A(0)u\|_{r,s} \to 0$ for every $u \in H^{r,s}$.

The solution u of $A(\varepsilon)u = f$ need not converge to the solution Gf of the reduced equation, for both solutions are not unique. Formula (4.11) describes the limit of the component $u - H^0 u$ of u which is orthogonal to the space of solutions of the homogeneous equation $A(0)u = 0$. This result gains in interest if the equation $A(0)u = 0$ has only zero solution, *i.e.* $H^0 = 0$. The task is now to show that from the unique solvability of the reduced equation it follows that $A(\varepsilon)u = f$ is uniquely solvable if ε is small enough.

Theorem 24. *Suppose that $A(\varepsilon) \in \Psi^{m,\mu}(\mathcal{X})$ is elliptic. If the reduced operator $A(0) : H^s(\mathcal{X}) \to H^{s-\mu}(\mathcal{X})$ is an isomorphism uniformly with respect to*

$\varepsilon \in [0, \varepsilon_0)$, then $A(\varepsilon) : H^{r,s}(\mathcal{X}) \to H^{r,s-\mu}(\mathcal{X})$ is an isomorphism, too, for all $\varepsilon \in [0, \varepsilon_0)$ with sufficiently small ε_0.

Proof. We only clarify the operator theoretic aspects of the proof. For symbol constructions we refer the reader to Corollary 3.14.10 in [Fra90] and the comments after its proof given there.

To this end, write $I = GA(0) + (I - GA(0))$ whence

$$\begin{aligned} A(\varepsilon) &= A(0) + (A(\varepsilon) - A(0))\, GA(0) + (A(\varepsilon) - A(0))(I - GA(0)) \\ &= (I + (A(\varepsilon) - A(0))G)\, A(0) + (A(\varepsilon) - A(0))(I - GA(0)) \end{aligned}$$

for all $\varepsilon \in [0, \varepsilon_0)$. As mentioned, the difference $A(\varepsilon) - A(0)$ is small if $\varepsilon \leqslant 1$ is small enough. Hence, the operator

$$\begin{aligned} Q(\varepsilon) &= I + (A(\varepsilon) - A(0))G \\ &= H^1 + A(\varepsilon)G \end{aligned}$$

is invertible in the scale $H^{r,s}(\mathcal{X})$, provided that $\varepsilon \in [0, \varepsilon_0)$ where $\varepsilon_0 \leqslant 1$ is sufficiently small.

If the operator $A(0) \in \Psi^{\mu}(\mathcal{X})$ is invertible in the scale of usual Sobolev spaces on $\mathcal{X}$, then the product $Q(\varepsilon)A(0)$ is invertible for all $\varepsilon \in [0, \varepsilon_0)$. Hence, by decreasing ε_0 if necessary, we conclude readily that $A(\varepsilon)$ is invertible for all $\varepsilon \in [0, \varepsilon_0)$, as desired. □

The proof above gives more, namely

$$\begin{aligned} A(\varepsilon) &= Q(\varepsilon)A(0) + S^0(\varepsilon), \\ &= A(0)Q(\varepsilon) + S^1(\varepsilon), \end{aligned} \tag{4.12}$$

where $S^0(\varepsilon)$ and $S^1(\varepsilon)$ have at most the same order as $A(\varepsilon)$ and are infinitesimally small if $\varepsilon \to 0$. The obtained relations are regularisation for operator $A(\varepsilon)$, they imply importants properties related to asymptotics behaviour of solutions. Firstly, it follows immediately that for the operator $A(\varepsilon)$ to be invertible for small ε it is necessary and sufficient that $A(0)$ would be invertible. Secondly, for any g_ϵ such that $\lim_{\varepsilon \to 0} A(\varepsilon) g_\varepsilon = 0$ follows $\lim_{\varepsilon \to 0} g_\varepsilon = 0$. It means, in particular, that formal expansions obtained with method of small parameter are asymptotic indeed.

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