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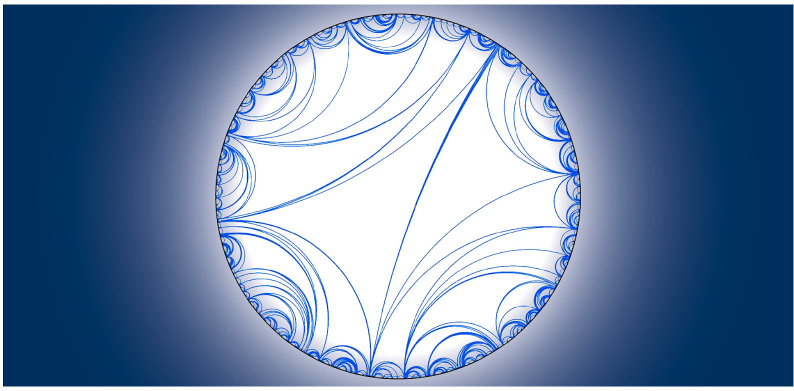

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THE FIRST MIXED PROBLEM FOR THE NONSTATIONARY LAMÉ SYSTEM

O. I. MAKHMUDOV AND N. TARKHANOV

Abstract. We find an adequate interpretation of the Lamé operator within the framework of elliptic complexes and study the first mixed problem for the nonstationary Lamé system.

Contents

1. Introduction

In his work on a systematic dynamical theory of elasticity Gabriel Lamé in mid 1881 derived from Newtonian mechanics his basic equations which are also the conditions for equilibrium. From those he went on to derive what are now known as Lamé equations in elastodynamics

$$\rho u''_{tt} = -\mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u + f, \tag{1.1}$$

where $u : \mathcal{X} \times (0, T) \to \mathbb{R}^3$ is a search-for displacement vector, ρ is the mass density, λ and μ are physical characteristics of the body under consideration called Lamé constants, $\Delta u = -u''_{x_1 x_1} - u''_{x_2 x_2} - u''_{x_3 x_3}$ is the nonnegative Laplace operator in $\mathbb{R}^3$, and f is the density vector of outer forces, see [ES75], [KGBB76], [LL70], [TS82], and elsewhere.

Here, $\mathcal{X}$ stands for a bounded domain in $\mathbb{R}^3$ whose boundary is assumed to be smooth enough. Hence, to specify a particular solution of Lamé equations, we

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consider the first mixed problem for (1.1) in the cylinder $\mathcal{X} \times (O,T)$ by posing the initial conditions

$$\begin{array}{rcll} u(x,0) & = & u_0(x), & \text{for} \quad x \in \mathcal{X}, \\ u'_t(x,0) & = & u_1(x), & \text{for} \quad x \in \mathcal{X}, \end{array} \tag{1.2}$$

on the lower basis of the cylinder and a Dirichlet condition

$$u(x,t) \quad = \quad u_l(x,t), \quad \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T), \tag{1.3}$$

on the lateral surface.

When working in adequate function spaces surviving under restriction to the lateral boundary, one can assume without loss of generality that $u_l \equiv 0$, for if not, one first solves the Dirichlet problem with data on $\partial\mathcal{X} \times [0,T]$ in the class of smooth functions.

To a certain extent the theory of mixed problems for hyperbolic partial differential equations with variable coefficients is a completion of the classical area studying the Cauchy problem and mixed problem for the wave equation. The fundamental idea of J. Leray in the early 1950s is that the energy form corresponding to a hyperbolic operator with simple real characteristics is an elliptic form with parameter, which allows one to obtain estimates in the case of variable coefficients. For a recent account of the theory we refer the reader to Chapter 3 in [GV96]. The energy method for hyperbolic equations takes a considerable part in [GV96]. This method automatically extends to $2b$-parabolic differential equations with variable coefficients. Within the framework of energy method the theories of hyperbolic and parabolic equations can be combined into one theory of operators with dominating principal quasihomogeneous part.

In this paper we apply the theory to the first mixed problem for a generalised Lamé system. While the classical Lamé operator of (1.1) stems from dynamical theory of elasticity, the generalised Lamé system is well motivated by its origin in homological algebra.

2. Generalised Lamé system

The Lamé equations are easily specified within the framework of elliptic complexes on the underlying manifold $\mathcal{X}$. On introducing the de Rham complex of $\mathcal{X}$

$$0 \longrightarrow \Omega^0(\mathcal{X}) \stackrel{d}{\longrightarrow} \Omega^1(\mathcal{X}) \stackrel{d}{\longrightarrow} \Omega^2(\mathcal{X}) \stackrel{d}{\longrightarrow} \Omega^3(\mathcal{X}) \longrightarrow 0$$

we can rewrite system (1.1) in the invariant form

$$\rho u''_{tt} = -\mu \Delta u - (\lambda + \mu)\, dd^*\, u + f \tag{2.1}$$

in the semicylinder $\mathcal{X} \times [0,\infty)$, where $\Delta = d^*d + dd^*$ is the Laplacian of the de Rham complex.

Example 2.1. When restricted to functions, i.e., differential forms of degree $i = 0$, equation (2.1) reads

$$\rho u''_{tt} = -\mu \Delta u + f,$$

which is precisely the wave equation in the cylinder $\mathcal{X} \times (0,T)$.

More generally, let $\mathcal{X}$ be a C^∞ compact manifold with boundary of dimension n. Consider a complex of first order differential operators acting in sections of vector bundles over $\mathcal{X}$,

$$0 \to C^\infty(\mathcal{X}, F^0) \stackrel{A^0}{\to} C^\infty(\mathcal{X}, F^1) \stackrel{A^1}{\to} \ldots \stackrel{A^{N-1}}{\to} C^\infty(\mathcal{X}, F^N) \to 0, \tag{2.2}$$

where $A^i \in \mathrm{Diff}^1(\mathcal{X}; F^i, F^{i-1})$ satisfy $A^{i+1}A^i = 0$ for all $i = 0, 1, \ldots, N-2$. Our basic assumption is that (2.2) is elliptic, i.e., the corresponding complex of principal symbols is exact away from the zero section of the cotangent bundle $T^*\mathcal{X}$, see [Tar95, 1.1.12]. We endow the manifold $\mathcal{X}$ and the vector bundles F^i by Riemannian metrics.

Set

$$F = \bigoplus_{i=0}^{N} F^i$$

and consider two first order differential operators A and A^* in $C^\infty(\mathcal{X}, F)$ given by the $((N+1) \times (N+1))$-matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ A^0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & A^1 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & A^{N-1} & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} 0 & A^{0*} & 0 & \ldots & 0 & 0 \\ 0 & 0 & A^{1*} & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 & A^{N-1*} \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

where $A^i \in \mathrm{Diff}^1(\mathcal{X}; F^{i+1}, F^i)$ stands for the formal adjoint of A^i. It is easily verified that $A \circ A = 0$ and $A^* \circ A^* = 0$ and

$$\Delta := A^*A + AA^* = \begin{pmatrix} \Delta^0 & 0 & \ldots & 0 \\ 0 & \Delta^1 & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & \Delta^N \end{pmatrix}, \tag{2.3}$$

where $\Delta^i = A^{i*}A^i + A^{i-1}A^{i-1*}$ for $i = 0, 1, \ldots, N$ are the so-called Laplacians of complex (2.2). The ellipticity of complex (2.2) just amounts to that of its Laplacians $\Delta^0, \Delta^1, \ldots, \Delta^N$.

Lemma 2.2. *Let r, s be real or complex numbers. Then $rA + sA^* \in \mathrm{Diff}^1(\mathcal{X}; F)$ is elliptic if and only if $rs \neq 0$.*

Proof. Necessity. If at least one of the scalars r and s vanishes then the operator $rA+sA^*$ reduces to a scalar multiple of A or A^*, which operators can not be elliptic because of their nilpotency.

Sufficiency. If both r and s are different from zero then a trivial verification gives

$$\begin{array}{rcl} (s^{-1}A + r^{-1}A^*)(rA + sA^*) & = & AA^* + A^*A, \\ (rA + sA^*)(s^{-1}A + r^{-1}A^*) & = & AA^* + A^*A, \end{array}$$

showing the ellipticity of $rA + sA^*$. □

By generalised Lamé operators related to complex (2.2) are meant the products of two operators of the form $rA + sA^*$, where $rs \neq 0$. These are precisely operators $L \in \mathrm{Diff}^2(\mathcal{X}; F)$ of the form $L = rA^*A + sAA^*$, where $rs \neq 0$. They are elliptic and preserve the grading of complex (2.2) in the sense that if u is a section of F^i, then so is Lu.

Consider the Dirichlet problem for the elliptic operator $\Delta^2 = (A^*A)^2 + (AA^*)^2$ on $\mathcal{X}$ with data

$$\begin{array}{rcl} u & = & 0 \quad \text{at} \quad \partial\mathcal{X}, \\ (A + A^*)u & = & 0 \quad \text{at} \quad \partial\mathcal{X}. \end{array} \tag{2.4}$$

This boundary value problem is elliptic and formally selfadjoint. As usual, it can be treated within the framework of densely defined unbounded operators in the Hilbert space $L^2(\mathcal{X}, F)$, cf. [ST03]. In particular, there is a bounded operator $G : L^2(\mathcal{X}, F) \to H^4(\mathcal{X}, F)$ called the Green operator, such that $u = Gf$ satisfies (2.4) and

$$f = Hf + \Delta^2(Gf) \tag{2.5}$$

for all $f \in L^2(\mathcal{X}, F)$, where H is the orthogonal projection of $L^2(\mathcal{X}, F)$ onto the finite-dimensional subspace of $L^2(\mathcal{X}, F)$ consisting of all $h \in C^\infty(\mathcal{X}, F)$ which satisfy $(A + A^*)h = 0$ in $\mathcal{X}$ and $h = 0$ at $\partial\mathcal{X}$. The Green operator G is actually known to be a pseudodifferential operator of order -4 in Boutet de Monvel's algebra on $\mathcal{X}$.

If $A + A^*$ has the uniqueness property for the global Cauchy problem on $\mathcal{X}$ then $H = 0$.

Lemma 2.3. *Suppose that $L = rA^*A + sAA^*$ is a Lamé operator on $\mathcal{X}'$, where $rs \neq 0$. Then $P = (\Delta^2/L)G$, with $\Delta^2/L = r^{-1}A^*A + s^{-1}AA^*$, is a parametrix of L.*

Proof. By the above, we get

$$\begin{aligned} LP &= L(\Delta^2/L)G \\ &= \Delta^2 G \\ &= I - H \end{aligned}$$

where $H \in \Psi^{-\infty}(\mathcal{X}; F)$. Hence, P is a left parametrix of L. Since L is elliptic, P is also a right parametrix of L in the interior of $\mathcal{X}$. □

Write

$$\begin{aligned} L &= r\Delta + (s - r)AA^* \\ &= -\mu\Delta - (\lambda + \mu)AA^*, \end{aligned}$$

where $r = -\mu$ and $s = -\lambda - 2\mu$. Then for the ellipticity of L it is necessary and sufficient that $\mu \neq 0$ and $\lambda + 2\mu \neq 0$.

3. Wave equation

In the open cylinder $\mathcal{C}_T = \overset{\circ}{\mathcal{X}} \times (0, T)$ for some $T > 0$ we consider the hyperbolic system

$$\rho u''_{tt} = -\mu\Delta u - (\lambda + \mu)AA^*u + f \tag{3.1}$$

for a section u of the bundle $(x, t) \mapsto F^i_x$ over $\mathcal{X} \times [0, T]$, which we write F^i for short, cf. Fig. 1. Assume $\rho = 1$ and $\mu > 0$.

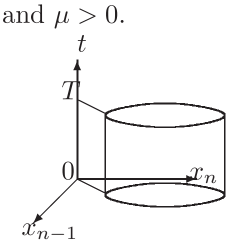

Fig. 1. *A cylinder $\mathcal{C}_T$*

A function $u \in C^2(\mathcal{C}_T, F^i) \cap C^1(\mathcal{X} \times [0,1), F^i)$ satisfying equation (3.1) in $\mathcal{C}_T$, the initial conditions

$$\begin{array}{rcll} u(x,0) & = & u_0(x), & \text{for} \quad x \in \overset{\circ}{\mathcal{X}}, \\ u'_t(x,0) & = & u_1(x), & \text{for} \quad x \in \overset{\circ}{\mathcal{X}}, \end{array} \tag{3.2}$$

on the lower basis of the cylinder and a Dirichlet condition

$$u(x,t) \;=\; u_l(x,t), \quad \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T), \tag{3.3}$$

on the lateral surface is said to be a classical solution of the first mixed problem for the generalised Lamé equation. Since the case of inhomogeneous boundary conditions reduces easily to the case of homogeneous ones, we will assume $u_l \equiv 0$ in the sequel.

Let u be a classical solution of the first mixed problem for the generalised Lamé equation with $f \in L^2(\mathcal{C}_T, F^i)$. Given any $\varepsilon > 0$, we multiply both sides of (3.1) with g^*, where g is an arbitrary smooth function in the closure of $\mathcal{C}_{T-\varepsilon}$ vanishing at the lateral surface and the head of this cylinder, and integrate the resulting equality over $\mathcal{C}_{T-\varepsilon}$. We will write the inner product of the values of f and g at any point $(x,t) \in \mathcal{C}_{T-\varepsilon}$ simply (f,g) when no confusion can arise. Using the Stokes theorem, we get

$$\begin{aligned} \int_{\mathcal{C}_{T-\varepsilon}} (f,g)\, dxdt &= \int_{\mathcal{C}_{T-\varepsilon}} (u''_{tt} + \mu\Delta u + (\lambda{+}\mu) AA^* u, g)\, dxdt \\ = - \int_{\mathcal{X}} (u_1, g) dx &+ \int_{\mathcal{C}_{T-\varepsilon}} (-(u'_t, g'_t) + \mu(Au, Ag) + (\lambda{+}2\mu)(A^*u, A^*g))\, dxdt. \end{aligned}$$

We exploit this identity to introduce the concept of weak solution of the first mixed problem for the generalised Lamé system. We assume that $f \in L^2(\mathcal{C}_T, F^i)$ and $u_1 \in L^2(\mathcal{X}, F^i)$.

A function $u \in H^1(\mathcal{C}_T, F^i)$ is called a weak solution of the first mixed problem for (3.1) in $\mathcal{C}_T$, if u satisfies

$$\begin{array}{rcll} u(x,0) & = & u_0(x), & \text{for} \quad x \in \overset{\circ}{\mathcal{X}}, \\ u(x,t) & = & 0, & \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T), \end{array}$$

and

$$\int_{\mathcal{C}_T} (-(u'_t, g'_t) + \mu(Au, Ag) + (\lambda{+}2\mu)(A^*u, A^*g))\, dxdt = \int_{\mathcal{X}} (u_1, g) dx + \int_{\mathcal{C}_T} (f,g)\, dxdt \tag{3.4}$$

for all $g \in H^1(\mathcal{C}_T, F^i)$, such that

$$\begin{array}{rcll} g(x,T) & = & 0, & \text{for} \quad x \in \overset{\circ}{\mathcal{X}}, \\ g(x,t) & = & 0, & \text{for} \quad (x,t) \in \partial\mathcal{X} \times (0,T). \end{array} \tag{3.5}$$

Just as classical solution, if u is a weak solution of the first mixed problem for the generalised Lamé system in $\mathcal{C}_T$, then u is a weak solution of the corresponding problem also in the cylinder $\mathcal{C}_{T'}$ with any $T' < T$. Indeed, u belongs to $H^1(\mathcal{C}_{T'}, F^i)$ for all $T' < T$ and it vanishes on the lateral boundary of $\mathcal{C}_{T'}$. Moreover, the identity (3.4) is fulfilled for all $g \in H^1(\mathcal{C}_T, F^i)$ with property (3.5). It is immediately verified that if a function g belongs to $H^1(\mathcal{C}_{T'}, F^i)$, the trace of g at the cross-section $\{t = T'\}$ is zero and $g = 0$ in $\mathcal{C}_T \setminus \mathcal{C}_{T'}$, then $g \in H^1(\mathcal{C}_T)$ and $g(x,T) = 0$ for all x in the interior of $\mathcal{X}$. If moreover $g = 0$ at $\partial\mathcal{X} \times (0,T')$, then g vanishes at the lateral boundary of $\mathcal{C}_T$. Hence it follows that the function u satisfies the integral identity by means of which one defines the weak solution of the corresponding mixed problem in $\mathcal{C}_{T'}$.

Note that we introduced the concept of weak solution of the first mixed problem as natural generalisation of the concept of classical solution (with $f \in L^2(\mathcal{C}_T, F^i)$). We have actually proved that the classical solution of the first mixed problem in $\mathcal{C}_T$ with $f \in L^2(\mathcal{C}_T, F^i)$ is a weak solution of this problem in the smaller cylinder $\mathcal{C}_{T-\varepsilon}$ for any $\varepsilon \in (0,T)$.

Along with classical and weak solutions of the first mixed problem one can introduce the notion of 'almost everywhere' solution. A function u is said to be an 'almost everywhere' solution of the first mixed problem if $u \in H^2(\mathcal{C}_T, F^i)$ satisfies equation (3.1) for almost all $(x,t) \in \mathcal{C}_T$, initial conditions (3.2) for almost all x in the interior of $\mathcal{X}$ and the trace of u on the lateral surface vanishes almost everywhere. From the definition it follows immediately that if the classical solution of the first mixed problem belongs to $H^2(\mathcal{C}_T, F^i)$ then it is also an 'almost everywhere' solution. Moreover, if an 'almost everywhere' solution u of the first mixed problem belongs to the class $C^2(\mathcal{C}_T, F^i) \cap C^1(\mathcal{X} \times [0,1), F^i)$ then u is obviously a classical solution, too.

Every 'almost everywhere' solution of the first mixed problem in $\mathcal{C}_T$ is a weak solution of this problem in $\mathcal{C}_T$. The converse assertion is also true.

Lemma 3.1. *If a weak solution of the first mixed problem belongs to the space $H^2(\mathcal{C}_T, F^i)$ then it is an 'almost everywhere' solution of this problem. If a weak solution of the first mixed problem belongs to $C^2(\mathcal{C}_T, F^i) \cap C^1(\mathcal{X} \times [0,1), F^i)$ then it is a classical solution of this problem.*

Proof. This is a standard fact on functions with generalised derivatives, cf. Lemma 1 in [Mik76, p. 287]. □

We are now in a position to prove a uniqueness theorem for the weak solution of the first mixed problem.

Theorem 3.2. *Suppose $\mu \geq 0$ and $\lambda + 2\mu \geq 0$. Then the first mixed problem for the generalised Lamé system has at most one weak solution.*

Proof. Let $u \in H^1(\mathcal{C}_T, F^i)$ be a weak solution of the first mixed problem with $f = 0$ in $\mathcal{C}_T$ and $u_0 = u_1 = 0$ in the interior of $\mathcal{X}$.

Pick an arbitrary $s \in (0,T)$ and consider the function

$$g(x,t) = \begin{cases} \displaystyle\int_t^s u(x,\theta)d\theta, & \text{if} \quad 0 < t < s, \\ 0, & \text{if} \quad s < t < T, \end{cases}$$

defined in $\mathcal{C}_T$. It is immediately verified that the function g has generalised derivatives

$$g'_{x^j}(x,t) = \begin{cases} \displaystyle\int_t^s u'_{x^j}(x,\theta)d\theta, & \text{if} \quad 0 < t < s, \\ 0, & \text{if} \quad s < t < T, \end{cases}$$

and

$$g'_t(x,t) = \begin{cases} -u(x,t), & \text{if} \quad 0 < t < s, \\ 0, & \text{if} \quad s < t < T, \end{cases}$$

in $\mathcal{C}_T$. Therefore, we get $g \in H^1(\mathcal{C}_T)$. Moreover, g vanishes at the lateral boundary and the head of the cylinder $\mathcal{C}_T$.

Substituting the function g into identity (3.4) yields

$$\int_{\mathcal{C}_s} \Big((u'_t, u) + \mu (Au, \int_t^s Au(\cdot,\theta) d\theta) + (\lambda{+}2\mu)(A^* u, \int_t^s A^* u(\cdot,\theta) d\theta) \Big)\, dxdt = 0$$

for all $s \in (0,T)$. It is obvious that

$$\Re \int_{\mathcal{C}_s} (u'_t, u)\, dxdt = \frac{1}{2} \int_{\mathcal{X}} |u(x,s)|^2 dx.$$

Since

$$\begin{aligned} \int_{\mathcal{C}_s} (Au(x,t), \int_t^s Au(x,\theta)d\theta)\, dxdt &= \int_{\mathcal{X}} \int_0^s (Au(x,t), \int_t^s Au(x,\theta)d\theta)\, dxdt \\ &= \int_{\mathcal{X}} \int_0^s (\int_0^\theta Au(x,t)dt, Au(x,\theta))\, dxd\theta \end{aligned}$$

which transforms to

$$\begin{aligned} \int_{\mathcal{X}} (\int_0^s Au(x,t)dt, \int_0^s Au(x,\theta)d\theta)\, dx - \int_{\mathcal{X}} \int_0^s (\int_\theta^s Au(x,t)dt, Au(x,\theta))\, dxd\theta \\ = \int_{\mathcal{X}} |\int_0^s Au(x,t)dt|^2\, dx - \int_{\mathcal{C}_s} (\int_\theta^s Au(x,t)dt, Au(x,\theta))\, dxd\theta, \end{aligned}$$

we get

$$\Re \int_{\mathcal{C}_s} (Au(x,t), \int_t^s Au(x,\theta)d\theta)\, dxdt = \frac{1}{2} \int_{\mathcal{X}} |\int_0^s Au(x,t)dt|^2\, dx.$$

Similarly we obtain

$$\Re \int_{\mathcal{C}_s} (A^*u(x,t), \int_t^s A^*u(x,\theta)d\theta)\, dxdt = \frac{1}{2} \int_{\mathcal{X}} |\int_0^s A^*u(x,t)dt|^2\, dx$$

whence

$$\int_{\mathcal{X}} |u(x,s)|^2 dx + \mu \int_{\mathcal{X}} |\int_0^s Au(x,t)dt|^2\, dx + (\lambda{+}2\mu) \int_{\mathcal{X}} |\int_0^s A^*u(x,t)dt|^2\, dx = 0 \tag{3.6}$$

for all $s \in (0,T)$.

Since $\mu \geq 0$ and $\mu + 2\lambda \geq 0$, we conclude from (3.6) that

$$\int_{\mathcal{X}} |u(x,s)|^2 dx = 0$$

for all $s \in (0,T)$, and so $u = 0$ in $\mathcal{C}_T$, as desired. □

As mentioned, a classical solution of the first mixed problem is also a weak solution of this problem in $\mathcal{C}_{T-\varepsilon}$ for each $\varepsilon \in (0,T)$. Hence, Theorem 3.2 implies the uniqueness of classical solution as well. Furthermore, since almost everywhere solutions are weak solutions, we also deduce that, if $\mu \geq 0$ and $\mu + 2\lambda \geq 0$, then the first mixed problem for the generalised Lamé system has at most one almost everywhere solution.

4. Existence of a weak solution

We now turn to showing the existence of solutions of the first mixed problem for the generalised Lamé system. To this end we use the Fourier method which consists in looking the solution of the mixed problem in the form of series over eigenfunctions of the corresponding elliptic boundary value problem.

Let v be a weak eigenfunction of the first boundary value problem for the generalised Lamé system

$$\begin{array}{rcll} -\mu\Delta v - (\lambda+\mu)AA^* v & = & \varkappa v & \text{in } \overset{\circ}{\mathcal{X}}, \\ v & = & 0 & \text{at } \partial\mathcal{X}, \end{array} \tag{4.1}$$

where $\varkappa$ is a corresponding eigenvalue. This just amounts to saying that

$$\int_{\mathcal{X}} (-\mu(Av, Ag)_x - (\lambda+2\mu)(A^*v, A^*g)_x)\, dx - \varkappa \int_{\mathcal{X}} (v, g)_x dx = 0 \tag{4.2}$$

for all $g \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$.

Consider the orthonormal system $(v_k)_{k=1,2,\dots}$ in $L^2(\mathcal{X}, F^i)$ consisting of all weak eigenfunction of problem (4.1). Let $(\varkappa_k)_{k=1,2,\dots}$ be the sequence of corresponding eigenvalues. As usual we think of this sequence as nonincreasing sequence with $\varkappa_1 < 0$ and each eigenvalue repeats himself in accord with his multiplicity. The system $(v_k)_{k=1,2,\dots}$ is known to be an orthonormal basis in $L^2(\mathcal{X}, F^i)$ and $\vartheta_k \to -\infty$ when $k \to \infty$. Moreover, the first eigenvalue $\varkappa_1$ is strongly negative, if $\mu > 0$ and $\lambda + 2\mu > 0$.

Suppose that the initial data u_0 and u_1 in (3.2) belong to $L^2(\mathcal{X}, F^i)$, and f belongs to $L^2(\mathcal{C}_T, F^i)$. By the Fubini theorem, $f(\cdot, t) \in L^2(\mathcal{X}, F^i)$ holds for almost all $t \in (0, T)$. We represent the functions u_0 and u_1 and the function $f(\cdot, t)$ for almost all $t \in (0, T)$ as Fourier series over the system $(v_k)_{k=1,2,\dots}$ of eigenfunction of problem (4.1). To wit,

$$u_0(x) = \sum_{k=1}^{\infty} u_{0,k} v_k(x), \quad u_1(x) = \sum_{k=1}^{\infty} u_{1,k} v_k(x),$$

where $u_{0,k} = (u_0, v_k)_{L^2(\mathcal{X}, F^i)}$ and $u_{1,k} = (u_1, v_k)_{L^2(\mathcal{X}, F^i)}$ for $k = 1, 2, \dots$. By the Parseval equality, we get

$$\begin{array}{rcl} \displaystyle\sum_{k=1}^{\infty} |u_{0,k}|^2 & = & \|u_0\|^2_{L^2(\mathcal{X}, F^i)}, \\ \displaystyle\sum_{k=1}^{\infty} |u_{1,k}|^2 & = & \|u_1\|^2_{L^2(\mathcal{X}, F^i)}. \end{array} \tag{4.3}$$

Similarly we get

$$f(x, t) = \sum_{k=1}^{\infty} f_k(t) v_k(x),$$

where $f_k(t) = \displaystyle\int_{\mathcal{X}} (f(\cdot, t), v_k)_x dx$ for $k = 1, 2, \dots$. Since

$$|f_k(t)|^2 \le \int_{\mathcal{X}} |f(\cdot, t)|^2 dx \int_{\mathcal{X}} |v_k|^2 dx = \int_{\mathcal{X}} |f(\cdot, t)|^2 dx,$$

it follows that $f_k \in L^2(0,T)$ for all $k = 1, 2, \ldots$. Moreover,

$$\sum_{k=1}^{\infty} |f_k(t)|^2 = \int_{\mathcal{X}} |f(\cdot,t)|^2 dx$$

holds for almost all $t \in (0,T)$, which is due to the Parseval equality. This yields readily

$$\sum_{k=1}^{\infty} \int_0^T |f_k(t)|^2 dt = \int_{\mathcal{C}_T} |f(x,t)|^2 dx dt. \tag{4.4}$$

Take first the k th harmonics $u_{0,k}v_k$ and $u_{1,k}v_k$ as initial data in (3.2), and the function $f_k(t)v_k(x)$ as function in the right-hand of (3.1), where $k = 1, 2, \ldots$. Consider the function

$$u_k(x,t) = w_k(t)v_k(x), \tag{4.5}$$

where

$$w_k(t) = u_{0,k} \cos \sqrt{-\varkappa_k} t + u_{1,k} \frac{\sin \sqrt{-\varkappa_k} t}{\sqrt{-\varkappa_k}} + \int_0^t f_k(t') \frac{\sin \sqrt{-\varkappa_k}(t-t')}{\sqrt{-\varkappa_k}} dt'.$$

Note that this formula still makes sense if $\varkappa_k = 0$, for the limit of the right-hand side exists as $\varkappa_k \to 0$. The function w_k belongs obviously to $H^2(0,T)$, satisfies the initial conditions $w_k(0) = u_{0,k}$ and $w_k'(0) = u_{1,k}$ and is a solution of the ordinary differential equation

$$w_k'' - \varkappa_k w_k = f_k \tag{4.6}$$

for almost all $t \in (0,T)$.

Our next objective is to show that if v_k is an eigenfunction of problem (4.1) corresponding to the eigenvalue $\varkappa_k$ then $u_k(x,t)$ is a weak solution of the first mixed problem for the equation

$$u_{tt}''(x,t) = -\mu \Delta u(x,t) - (\lambda+\mu) AA^* u(x,t) + f_k(t)v_k(x)$$

in $\mathcal{C}_T$ with initial data

$$\begin{array}{rcll} u(x,0) & = & u_{0,k}v_k(x), & \text{for} \quad x \in \mathring{\mathcal{X}}, \\ u_t'(x,0) & = & u_{1,k}v_k(x), & \text{for} \quad x \in \mathring{\mathcal{X}}. \end{array}$$

Indeed, the function u_k given by (4.5) belongs to $H^1(\mathcal{C}_T, F^i)$, satisfies the initial conditions and vanishes at the lateral boundary of the cylinder. It remains to show that

$$\int_{\mathcal{C}_T} \left(-((u_k)_t', g_t') + \mu(Au_k, Ag) + (\lambda+2\mu)(A^*u_k, A^*g)\right) dx dt$$
$$= \int_{\mathcal{X}} u_{1,k}(v_k, g) dx + \int_{\mathcal{C}_T} f_k(t)(v_k, g)\, dx dt$$

for all $g \in H^1(\mathcal{C}_T, F^i)$ satisfying (3.5). It is sufficient to establish the above identity only for functions $g \in C^1(\overline{\mathcal{C}}_T, F^i)$ satisfying (3.5).

By (4.5) and integration by parts,

$$\begin{array}{rcl} \displaystyle\int_{\mathcal{C}_T} ((u_k)_t', g_t')\, dx dt & = & \displaystyle\int_{\mathcal{X}} \Big(v_k, \int_0^T w_k'(t) g_t' dt\Big)_x dx \\ & = & \displaystyle\int_{\mathcal{X}} \Big(v_k, -u_{1,k} g(x,0) - \int_0^T w_k''(t) g dt\Big)_x dx \end{array}$$

which reduces, by (4.6), to

$$-\int_{\mathcal{X}} u_{1,k}(v_k, g(x,0))_x\, dx - \varkappa_k \int_{\mathcal{C}_T} (u_k, g)\, dxdt - \int_{\mathcal{C}_T} f_k(t)(v_k, g)\, dxdt.$$

Hence, the desired identity follows from (4.2), for

$$\begin{aligned}
&\int_{\mathcal{C}_T} (\mu(Au_k, Ag) + (\lambda{+}2\mu)(A^*u_k, A^*g))\, dxdt \\
= &\int_0^T w_k(t)\Big(\int_{\mathcal{X}} (\mu(Av_k, Ag)_x + (\lambda{+}2\mu)(A^*v_k, A^*g)_x)\, dx\Big) dt \\
= &-\int_0^T w_k(t)\Big(\varkappa_k \int_{\mathcal{X}} (v_k, g)_x dx\Big) dt,
\end{aligned}$$

as desired.

If one takes the partial sums

$$\sum_{k=1}^{N} u_{0,k} v_k(x), \ \ \sum_{k=1}^{N} u_{1,k} v_k(x)$$

of the Fourier series for the functions u_0 and u_1, respectively, as initial data and the partial sum

$$\sum_{k=1}^{N} f_k(t) v_k(x)$$

of the Fourier series for f as the right-hand side of the equation, then the weak solution of the first mixed problem is

$$s_N(x,t) = \sum_{k=1}^{N} u_k(x,t) = \sum_{k=1}^{N} w_k(t) v_k(x).$$

In particular, the function s_N satisfies the identity

$$\begin{aligned}
&\int_{\mathcal{C}_T} (-((s_N)'_t, g'_t) + \mu(As_N, Ag) + (\lambda{+}2\mu)(A^*s_N, A^*g))\, dxdt \\
= &\int_{\mathcal{X}} \Big(\sum_{k=1}^{N} u_{1,k} v_k, g\Big)\, dx + \int_{\mathcal{C}_T} \Big(\sum_{k=1}^{N} f_k(t) v_k, g\Big)\, dxdt
\end{aligned} \tag{4.7}$$

for all $g \in H^1(\mathcal{C}_T, F^i)$ satisfying (3.5).

Thus it is to be expected that under certain assumptions on u_0, u_1 and f the solution of the first mixed problem for the generalised Lamé system can be represented as series

$$u(x,t) = \sum_{k=0}^{\infty} w_k(t) v_k(x), \tag{4.8}$$

where $(v_k)_{k=1,2,\ldots}$ are weak eigenfunctions of problem (4.1).

Theorem 4.1. *Let* $u_0 \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$, $u_1 \in L^2(\mathcal{X}, F^i)$ *and* $f \in L^2(\mathcal{C}_T, F^i)$. *Then the first mixed problem possesses a weak solution given by series (4.8) which converges in* $H^1(\mathcal{C}_T, F^i)$. *Moreover,*

$$\|u\|_{H^1(\mathcal{C}_T,F^i)} \le C \left(\|f\|_{L^2(\mathcal{C}_T,F^i)} + \|u_0\|_{H^1(\mathcal{X},F^i)} + \|u_1\|_{L^2(\mathcal{X},F^i)} \right) \tag{4.9}$$

with C a constant independent of u_0, u_1 and f.

Proof. From the formula for w_k it follows that

$$|w_k(t)| \leq |u_{0,k}| + \frac{1}{\sqrt{|\lambda_k|}}\, |u_{1,k}| + \frac{1}{\sqrt{|\lambda_k|}} \int_0^T |f_k(t')|\, dt'$$

for all $t \in [0,T]$ and $k = 1, 2, \ldots$. Hence,

$$\begin{aligned} |w_k(t)|^2 &\leq 3\, |u_{0,k}|^2 + \frac{3}{|\lambda_k|}\, |u_{1,k}|^2 + \frac{3}{|\lambda_k|} \Big(\int_0^T |f_k(t')|\, dt' \Big)^2 \\ &\leq c(T) \Big(|u_{0,k}|^2 + |\lambda_k|^{-1}\, |u_{1,k}|^2 + |\lambda_k|^{-1} \int_0^T |f_k(t')|^2\, dt' \Big). \end{aligned} \tag{4.10}$$

Furthermore, since

$$|w_k'(t)| \leq \sqrt{|\lambda_k|}\, |u_{0,k}| + |u_{1,k}| + \int_0^T |f_k(t')|\, dt'$$

for all $t \in [0,T]$, we get

$$|w_k'(t)|^2 \leq c(T) \Big(|\lambda_k|\, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2\, dt' \Big). \tag{4.11}$$

Since the function u_0 belongs to $\mathring{H}^1(\mathcal{X}, F^i)$, its Fourier series over the orthonormal system $(v_k)_{k=1,2,\ldots}$ converges to u_0 actually in the $H^1(\mathcal{X}, F^i)$-norm, see Theorem 3 in [Mik76, p. 181] and elsewhere. Moreover, there is a constant $c > 0$ with the property that

$$\sum_{k=1}^{\infty} |\lambda_k|\, |u_{0,k}|^2 \leq c\, \|u_0\|^2_{H^1(\mathcal{X},F^i)} \tag{4.12}$$

for all $u_0 \in \mathring{H}^1(\mathcal{X}, F^i)$.

Consider the partial sum $s_N(x,t)$ of Fourier series (4.8). Since both w_k and w_k' are continuous on $[0,T]$, for each fixed $t \in [0,T]$, the function s_N and its derivative in t belong to

$$\mathring{H}^1(\mathcal{X}, F^i).$$

To study the values of $t \mapsto s_N(\cdot, t)$ in $\mathring{H}^1(\mathcal{X}, F^i)$, it is convenient to endow this space with the so-called Dirichlet scalar product

$$D(v,g) = \int_{\mathcal{X}} \Big(\mu (Av, Ag)_x + (\lambda{+}2\mu)(A^*v, A^*g)_x \Big) dx$$

and the Dirichlet norm $D(v) := \sqrt{D(v,v)}$. The system

$$\Big(\frac{v_k}{\sqrt{-\varkappa_k}} \Big)_{k=1,2,\ldots}$$

is obviously orthonormal with respect to the Dirichlet scalar product. By (4.10), if $1 \leq M < N$, then

$$\begin{aligned}
D(s_N(\cdot,t) - s_M(\cdot,t))^2 &= D\Big(\sum_{k=M+1}^{N} w_k(t) v_k \Big)^2 \\
&= \sum_{k=M+1}^{N} |w_k(t)|^2 \, |\lambda_k| \\
&\leq c(T) \sum_{k=M+1}^{N} \Big(|\lambda_k| \, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2 \, dt' \Big)
\end{aligned}$$

for all $t \in [0,T]$. Similarly, using (4.11), we get

$$\begin{aligned}
& \| (s_N)'_t(\cdot,t) - (s_M)'_t(\cdot,t) \|^2_{L^2(\mathcal{X},F^i)} \\
&= \| \sum_{k=M+1}^{N} w'_k(t) v_k \|^2_{L^2(\mathcal{X},F^i)} \\
&= \sum_{k=M+1}^{N} |w'_k(t)|^2 \\
&\leq c(T) \sum_{k=M+1}^{N} \Big(|\lambda_k| \, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2 \, dt' \Big)
\end{aligned}$$

for $t \in [0,T]$. Here, $c(T)$ stands for a constant which depends on T but not on M and N, and which can be different in diverse applications.

On integrating these two inequalities in $t \in [0,T]$ and summing up them we obtain immediately

$$\| s_N - s_M \|^2_{H^1(\mathcal{C}_T,F^i)} \leq c(T) \sum_{k=M+1}^{N} \Big(|\lambda_k| \, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2 \, dt' \Big) \quad (4.13)$$

for all $1 \leq M < N$. Combining (4.13) with (4.3), (4.4) and (4.12) we conclude that $(s_N)_{N=1,2,\ldots}$ is a Cauchy sequence in $H^1(\mathcal{C}_T, F^i)$. Therefore, series (4.8) converges in this space and to a function $u(x,t)$ in $H^1(\mathcal{C}_T, F^i)$. Obviously, u satisfies the initial conditions (3.2) and vanishes at the lateral boundary of $\mathcal{C}_T$. Letting $N \to \infty$ in (4.7) we deduce that u is a weak solution of the first mixed problem for the generalised Lamé system.

In much the same way we derive inequalities

$$\begin{aligned}
D(s_N(\cdot,t))^2 &= D\Big(\sum_{k=1}^{N} w_k(t) v_k \Big)^2 \\
&= \sum_{k=1}^{N} |w_k(t)|^2 \, |\lambda_k| \\
&\leq c(T) \sum_{k=1}^{N} \Big(|\lambda_k| \, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2 \, dt' \Big)
\end{aligned}$$

and

$$
\begin{aligned}
\|(s_N)'_t(\cdot,t)\|^2_{L^2(\mathcal{X},F^i)} &= \|\sum_{k=1}^{N} w'_k(t)v_k\|^2_{L^2(\mathcal{X},F^i)} \\
&= \sum_{k=1}^{N} |w'_k(t)|^2 \\
&\leq c(T)\sum_{k=1}^{N}\Big(|\lambda_k|\,|u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2\,dt'\Big)
\end{aligned}
$$

for all $t \in [0,T]$ and $N \geq 1$. Integrating these inequalities in $t \in [0,T]$, summing up them and using (4.3), (4.4) and (4.12) we establish estimate (4.9), thus completing the proof. □

5. Galerkin method

There are also other proofs of the existence of weak solutions to mixed problems which do not exploit eigenfunctions. In this section we present the so-called Galerkin method which allows one to also construct an approximate solution of the mixed problem. In contrast to the Fourier method, the Galerkin method applies also in the case where the coefficients of A depend not only on the space variables but also on the time t.

As before, we assume $u_0 \in \overset{\circ}{H}{}^1(\mathcal{X},F^i)$, $u_1 \in L^2(\mathcal{X},F^i)$ and $f \in L^2(\mathcal{C}_T)$. Pick an arbitrary system $(v_k)_{k=1,2,\ldots}$ in $C^2(\mathcal{X},F^i)$ which satisfies $v_k = 0$ at $\partial\mathcal{X}$ and is complete in

$$\overset{\circ}{H}{}^1(\mathcal{X},F^i).$$

Given any integer $N \geq 1$, we solve problem (3.1), (3.2) and (3.3) with $u_l = 0$ in the finite-dimensional subspace V_N of $L^2(\mathcal{X},F^i)$ spanned by functions $v_1,\ldots,v_N$. More precisely, we look for a function u_N in $H^2(\mathcal{C}_T,F^i)$, such that $u_N(\cdot,t)$ belongs to the subspace V_N for any fixed $t \in [0,T]$, u_N satisfies conditions (3.2) with initial data

$$
\begin{aligned}
u_{0,N}(x) &= \sum_{k=1}^{N} u_{0,k}v_k(x), \\
u_{1,N}(x) &= \sum_{k=1}^{N} u_{1,k}v_k(x)
\end{aligned}
$$

being orthogonal projections of u_0 and u_1 onto V_N, respectively, and the orthogonal projections of $(u_N)''_{tt} + \mu\Delta u_N + (\lambda+\mu)AA^*u_N$ and f onto V_N coincide for almost all $t \in [0,T]$. (Note that the orthogonality refers here to the inner product of $L^2(\mathcal{X},F^i)$.)

We thus search for functions $w_1(t),\ldots,w_N(t)$ in $H^2(0,T)$ satisfying $w_k(0) = u_{0,k}$ and $w'_k(0) = u_{1,k}$ for all $k = 1,\ldots,N$, and such that

$$u_N(x,t) = \sum_{k=1}^{N} w_k(t)v_k(x)$$

fulfills

$$\int_{\mathcal{X}} ((u_N)''_{tt} + \mu\Delta u_N + (\lambda+\mu)AA^*u_N, v_k)_x\,dx = \int_{\mathcal{X}} (f,v_k)_x\,dx \qquad (5.1)$$

for almost all $t \in [0,T]$ (for which $f(\cdot,t) \in L^2(\mathcal{X},F^i)$), where $k = 1,\ldots,N$. The Galerkin method consists in approximating the solution u of mixed problem (3.1), (3.2) and (3.3) with $u_l = 0$ by solutions u_N of the projected problems. To substantiate this method one ought to show that each projected problem has a unique solution u_N and the sequence $(u_N)_{N=1,2,\ldots}$ converges in some sense (weakly in $H^1(\mathcal{C}_T,F^i)$) to u.

For simplicity, we restrict ourselves to the case of homogeneous initial conditions $u_0 = 0$ and $u_1 = 0$. Then the coefficients $u_{0,k}$ and $u_{1,k}$ vanish and we are lead to the system

$$\begin{array}{rcl} w_k(0) & = & 0, \\ w_k'(0) & = & 0 \end{array} \tag{5.2}$$

for all $k = 1,\ldots,N$.

Equations (5.1) constitute a system of second order linear ordinary differential equations with constant coefficients for unknown functions $w_1(t),\ldots,w_N(t)$. To wit,

$$\sum_{j=1}^{N} \bigl(w_j''(t)\,(v_j,v_k)_{L^2(\mathcal{X},F^i)} + w_j(t)\,D(v_j,v_k)\bigr) = f_k(t) \tag{5.3}$$

for $k = 1,\ldots,N$, where

$$f_k(t) = \int_{\mathcal{X}} (f(\cdot,t),v_k)_x dx$$

belongs to $L^2(\mathcal{X},F^i)$.

Our task is to prove that system (5.3) has a unique solution $w_1,\ldots,w_N$ with components in $H^1(0,T)$ satisfying initial conditions (5.2). Since the system $v_1,\ldots,v_N$ is linearly independent for all integer $N \geq 1$, the (Gram-Schmidt) determinant of the $(N\times N)$-matrix with entries $(v_j,v_k)_{L^2(\mathcal{X},F^i)}$ is different from zero. Hence, system (5.3) can be resolved with respect to the higher order derivatives. It follows that problem (5.3), (5.2) reduces to the initial problem of canonical form on $[0,T]$, namely

$$\begin{array}{rcll} W'(t) & = & AW(t) + F(t), & \text{if} \quad t \in (0,T), \\ W(0) & = & 0, & \end{array} \tag{5.4}$$

where $W = (w',w)^T$ and

$$A = -\begin{pmatrix} 0 & \bigl((v_j,v_k)_{L^2(\mathcal{X},F^i)}\bigr)^{-1} (D(v_j,v_k)) \\ E_N & 0 \end{pmatrix}.$$

The components of the $2N$-column $F(t)$ belong to $L^2(0,T)$. We look for a solution W of problem (5.4) in $H^1((0,T),\mathbb{C}^{2N})$. As usual, we replace this problem by the equivalent system of integral equations

$$W(t) = \int_0^t AW(t')dt' + \int_0^t F(t')dt', \tag{5.5}$$

the free term on the right-hand side belonging to $H^1((0,T),\mathbb{C}^{2N})$ and so being continuous on $[0,T]$. If $W \in H^1((0,T),\mathbb{C}^{2N})$ is a solution of (5.4), then it is continuous on $[0,T]$ and satisfies equation (5.5). Conversely, if $W : [0,T] \to \mathbb{C}^{2N}$ is a continuous solution of equation (5.5), then it is actually of class $H^1((0,T),\mathbb{C}^{2N})$ and satisfies (5.4). And the existence and uniqueness of a continuous solution to equation (5.4) is a direct consequence of the Banach fixed point theorem. We have thus proved that system (5.3) has a unique solution $w_1, \ldots, w_N$ in $H^1(0,T)$ satisfying (5.2).

Multiply equality (5.1) by $w_k'(t)$, integrate over $t \in (0,t')$, where t' is an arbitrary number of $[0,T]$, and sum up for $k = 1, \ldots, N$. Then we get

$$\int_{\mathcal{C}_{t'}} ((u_N)''_{tt} + \mu \Delta u_N + (\lambda{+}\mu) A A^* u_N, (u_N)'_t)_x \, dxdt = \int_{\mathcal{C}_{t'}} (f, (u_N)'_t)_x \, dxdt. \quad (5.6)$$

Using the Stokes formula one transforms the real part of the left-hand side of this equality to

$$\frac{1}{2} \int_{\mathcal{X}} \left(|(u_N)'_t\,(x,t')|^2 + \mu\, |A u_N\,(x,t')|^2 + (\lambda{+}2\mu)\, |A^* u_N\,(x,t')|^2 \right) dx$$

for all $t' \in [0,T]$. On the subspace $H^1_b(\mathcal{C}_T, F^i)$ of $H^1(\mathcal{C}_T, F^i)$ consisting of those functions which vanish on the laternal boundary of $\mathcal{C}_T$ and its base, the norm can be equivalently given by

$$\|u\|^2_{H^1_b(\mathcal{C}_T,F^i)} = \int_{\mathcal{C}_T} |u'_t|^2 dxdt + \int_0^T D(u(\cdot,t))^2 dt,$$

where $D(v)$ is the Dirichlet norm of $v \in \mathring{H}^1(\mathcal{X}, F^i)$. Hence,

$$\Re \int_0^T dt' \int_{\mathcal{C}_{t'}} ((u_N)''_{tt} + \mu \Delta u_N + (\lambda{+}\mu) A A^* u_N, (u_N)'_t)_x \, dxdt = \frac{1}{2}\, \|u_N\|^2_{H^1_b(\mathcal{C}_T,F^i)}$$

and equality (5.6) yields

$$\begin{aligned} \|u_N\|^2_{H^1_b(\mathcal{C}_T,F^i)} &= 2\,\Re \int_0^T dt' \int_{\mathcal{C}_{t'}} (f, (u_N)'_t)_x \, dxdt \\ &= 2\,\Re \int_{\mathcal{C}_T} (T-t)\, (f, (u_N)'_t)_x \, dxdt \\ &\leq 2T\, \|f\|_{L^2(\mathcal{C}_T,F^i)} \|u_N\|_{H^1_b(\mathcal{C}_T,F^i)} \end{aligned}$$

whence

$$\|u_N\|_{H^1_b(\mathcal{C}_T,F^i)} \leq 2T\, \|f\|_{L^2(\mathcal{C}_T,F^i)}.$$

We have thus proved that the set of functions u_N, where $N = 1, 2, \ldots$, is bounded in the Hilbert space $H^1_b(\mathcal{C}_T, F^i)$. Therefore, this set is weakly compact in $H^1_b(\mathcal{C}_T, F^i)$, i.e., it has a subsequence which converges weakly in $H^1_b(\mathcal{C}_T, F^i)$ to a function $u \in H^1_b(\mathcal{C}_T, F^i)$. By abuse of notation, we continue to write u_N for this subsequence.

We claim that u is the desired weak solution of the first mixed problem for the generalised Lamé system. To show this it is sufficient to verify that the integral identity

$$\int_{\mathcal{C}_T} (-(u'_t, g'_t) + \mu (Au, Ag) + (\lambda{+}2\mu)(A^*u, A^*g))\, dxdt = \int_{\mathcal{C}_T} (f, g)\, dxdt \quad (5.7)$$

holds for all $g \in H^1(\mathcal{C}_T, F^i)$ which vanish at the laternal boundary of $\mathcal{C}_T$ and the cylinder head, cf. (3.4) with $u_1 = 0$. Let us introduce the temporary notation $H^1_c(\mathcal{C}_T, F^i)$ for the (obviously, closed) subspace of $H^1(\mathcal{C}_T, F^i)$ consisting of all such g. It is actually sufficient to establish (5.7) for all g in a complete subset Σ of $H^1_c(\mathcal{C}_T, F^i)$.

As Σ we take the set of all functions of the form $z(t)v_k(x)$, where $k \geq 1$ is an integer and $z(t)$ a smooth function on $[0,T]$ satisfying $z(T)=0$. We first show that equality (5.7) is true for each function $g(x,t)=z(t)v_k(x)$ and then that the linear combinations of such functions are dense in $H^1_c(\mathcal{C}_T,F^i)$. To this end, we multiply equality (5.1) by $z(t)$, integrate it over $t \in (0,T)$ and apply the Stokes formula, obtaining

$$\int_{\mathcal{C}_T} (-((u_N)'_t, g'_t) + \mu(Au_N, Ag) + (\lambda{+}2\mu)(A^* u_N, A^* g))_x \, dxdt = \int_{\mathcal{C}_T} (f,g)_x \, dxdt$$

for all $N \geq k$, where $g = zv_k$. This implies readily (5.7), for $u_N \to u$ weakly in $H^1(\mathcal{C}_T,F^i)$.

Our next goal is to show that the linear hull of Σ is dense in $H^1_c(\mathcal{C}_T,F^i)$. To do this it is sufficient to prove that each function $g \in C^2(\overline{\mathcal{C}}_T,F^i)$ vanishing at the laternal boundary of the cylinder and its head (the set of such functions is dense in $H^1_c(\mathcal{C}_T,F^i)$) can be approximated in the $H^1(\mathcal{C}_T,F^i)$-norm by linear combinations of functions in Σ. This last assertion is actually well known within the framework of theory of Sobolev spaces. For a proof, we refer the reader to [Mik76, p. 302] and elsewhere.

Remark 5.1. Since the weak solution of the first mixed problem exists and is unique, not only a subsequence but also the sequence $(u_N)_{N=1,2,\ldots}$ itself converges weakly in $H^1(\mathcal{C}_T,F^i)$ to u.

6. Regularity of weak solutions

Assume that the boundary $\partial\mathcal{X}$ of $\mathcal{X}$ is of class C^s for some integer $s \geq 1$. Then the eigenfunctions $(v_k)_{k=1,2,\ldots}$ of problem (4.1) belong to $H^s(\mathcal{X},F^i)$ and satisfy the boundary conditions

$$L^i v_k \;=\; 0 \quad \text{at} \quad \partial\mathcal{X} \tag{6.1}$$

for $i = 0,1,\ldots,\left[\dfrac{s-1}{2}\right]$.

Let $H^s_{\mathcal{D}}(\mathcal{X},F^i)$ stand for the subspace of $H^s(\mathcal{X},F^i)$ consisting of all functions v satisfying (6.1). We put additional restrictions on the data of the problem to attain to a classical solution. More precisely, we require that $u_0 \in H^s_{\mathcal{D}}(\mathcal{X},F^i)$, $u_1 \in H^{s-1}_{\mathcal{D}}(\mathcal{X},F^i)$ and f belongs to the subspace of $H^{s-1}(\mathcal{C}_T,F^i)$ consisting of all functions satisfying

$$L^i f \;=\; 0 \quad \text{at} \quad \partial\mathcal{X} \times (0,T) \tag{6.2}$$

for $i = 0,1,\ldots,\left[\dfrac{s}{2}\right]-1$.

For $s=1$, the latter equations are empty and we arrive at $f \in L^2(\mathcal{X},F^i)$, as above.

Theorem 6.1. *Under the above hypotheses, series (4.8) converges to the weak solution $u(x,t)$ in $H^s(\mathcal{X},F^i)$ uniformly in $t \in [0,T]$. Given any $j = 1,\ldots,s$, the series obtained from (4.8) by the j-fold termwise differentiation in t converges in $H^{s-j}(\mathcal{X},F^i)$ uniformly in $t \in [0,T]$. Moreover, there is a constant $c > 0$ independent of t, such that*

$$\sum_{j=0}^{s} \| \sum_{k=1}^{\infty} w_k^{(j)}(t) v_k \|^2_{H^{s-j}(\mathcal{X},F^i)} \leq c \left(\|u_0\|^2_{H^s(\mathcal{X},F^i)} + \|u_1\|^2_{H^{s-1}(\mathcal{X},F^i)} + \|f\|^2_{H^{s-1}(\mathcal{C}_T,F^i)} \right) \tag{6.3}$$

for all $t \in [0,T]$.

Proof. The proof of this theorem runs similarly to the proof of Theorem 3 of [Mik76, p. 305], if one exploits the techniques developed earlier in Sections 3 and 4. □

By (6.3), if $1 \leq M < N$, then

$$\sup_{t\in[0,T]} \Big\| \sum_{k=M+1}^{N} w_k^{(j)}(t) v_k \Big\|^2_{H^{s-j}(\mathcal{X},F^i)} \to 0$$

as $M \to \infty$. Hence, the partial sums of series (4.8) converge in $H^s(\mathcal{C}_T, F^i)$ and from (6.3) it follows that

$$\|u\|_{H^s(\mathcal{C}_T,F^i)} \leq c' \left(\|u_0\|_{H^s(\mathcal{X},F^i)} + \|u_1\|_{H^{s-1}(\mathcal{X},F^i)} + \|f\|_{H^{s-1}(\mathcal{C}_T,F^i)} \right). \qquad (6.4)$$

Corollary 6.2. *Under the above hypotheses, the weak solution of the first mixed problem for the generalised Lamé system belongs to* $H^s(\mathcal{C}_T, F^i)$. *Moreover, series (4.8) converges to the weak solution in the* $H^s(\mathcal{C}_T, F^i)$ *-norm and inequality (6.4) holds true.*

From Corollary 6.2 with $s = 2$ it follows that the weak solution of the first mixed problem belongs to $H^2(\mathcal{C}_T, F^i)$, and so it is a solution almost everywhere. If moreover $s > n/2 + 2$, then the weak solution u belongs to the space $C^2(\overline{\mathcal{C}}_T, F^i)$, which is due to the Sobolev embedding theorem, and so u is a classical solution of the problem.

Note that along with the smoothness of u_0, u_1 and f Theorem 6.1 assumes that u_0 satisfies (6.1), u_1 satisfies (6.1) with s replaced by $s-1$, and f satisfies (6.2). The conditions are actually necessary. To show this, suppose $s \geq 2$. Since $u_0(x) = u(x,0)$ is represented by series (4.8) which converges in $H^s(\mathcal{X}, F^i)$, and $u_1(x) = u'_t(x,0)$ is represented by series (4.8) which is differentiated termwise in t and converges in $H^{s-1}(\mathcal{X}, F^i)$, we conclude readily that u_0 satisfies (6.1) and u_1 satisfies (6.1) with s replaced by $s-1$. Furthermore, since series (4.8) converges to u in $H^s(\mathcal{C}_T, F^i)$, the series obtained from (4.8) by termwise applying the operators L and the second derivative in t converge in $H^{s-2}(\mathcal{C}_T, F^i)$ to Lu and u''_{tt}, respectively. Hence, if $s \geq 3$, then $f = u''_{tt} - Lu$ satisfies equalities (6.2) with s replaced by $s-1$. In case s is even, the last condition of (6.2) is superfluous indeed, see Corollary 2 in [Mik76, p. 311].

However, if one wants to prove the smoothness of the weak solution of the first mixed problem rather than the convergence of the Fourier series in the corresponding spaces, then conditions (6.1) and (6.2) can be essentially relaxed, see Theorem 3' in [Mik76, p. 323].

7. Reduction to Schrödinger equation

There is a Lie algebraic connection between the wave equation and the Schrödinger equation. This allows us to construct solutions of hyperbolic equations from solutions of Schrödinger equation.

By the above, the unbounded operator $-L$ in $L^2(\mathcal{X}, F^i)$, whose domain is the set of all sections $v \in H^2(\mathcal{X}, F^i)$ vanishing at $\partial\mathcal{X}$, is closed, selfadjoint and positive, i.e. we have $-L \geq cI$ where c is a positive constant. Denote by $\sqrt{-L}$ the square root of $-L$ and impose upon the domain $\mathcal{D}_{\sqrt{-L}}$ of this operator a Hilbert space structure by identifying it with the range of $\sqrt{-L}$, i.e. the norm in $\mathcal{D}_{\sqrt{-L}}$ just amounts to

$$D(v) = \|\sqrt{-L}v\|_{L^2(\mathcal{X},F^i)}.$$

We now split the solution of the first mixed problem (3.1) and (3.2), (3.3) (with $u_l = 0$) into two parts. To wit, we are looking for two differentiable functions $F, U : [0,T] \to L^2(\mathcal{X}, F^i)$ with values in $\mathcal{D}_{\sqrt{-1}}$ (i.e. curves in $L^2(\mathcal{X}, F^i)$), which satisfy

$$\begin{array}{rcl} F'_t &=& -\imath\sqrt{-L}\,F + f, \quad \text{for} \quad t \in (0,T), \\ F(0) &=& u_1 - \imath\sqrt{-L}\,u_0, \end{array} \tag{7.1}$$

and

$$\begin{array}{rcl} U'_t &=& \imath\sqrt{-L}\,U + F, \quad \text{for} \quad t \in (0,T), \\ U(0) &=& u_0. \end{array} \tag{7.2}$$

If $U : [0,T] \to L^2(\mathcal{X}, F^i)$ is twice differentiable in $t \in (0,1)$, then combining (7.1) and (7.2) yields

$$\begin{array}{rcl} U''_{tt} &=& \imath\sqrt{-L}\,U'_t + F'_t \\ &=& \imath\sqrt{-L}\left(\imath\sqrt{-L}\,U + F\right) - \imath\sqrt{-L}\,F + f \\ &=& LU + f \end{array}$$

in $(0,T)$ and

$$\begin{array}{rcl} U(0) &=& u_0, \\ U'(0) &=& \imath\sqrt{-L}\,u_0 + F(0) \quad = \quad u_1. \end{array}$$

It follows that $u = U$ is a solution of the first mixed problem for the generalised Lamé system in $\mathcal{C}_T$.

It is worth pointing out that $\pm\imath\sqrt{-L}$ are skew-symmetric operators in $L^2(\mathcal{X}, F^i)$. For direct constructions along more classical lines we refer the reader to [Fri54], [Fri58], [FL65], [Agr69].

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