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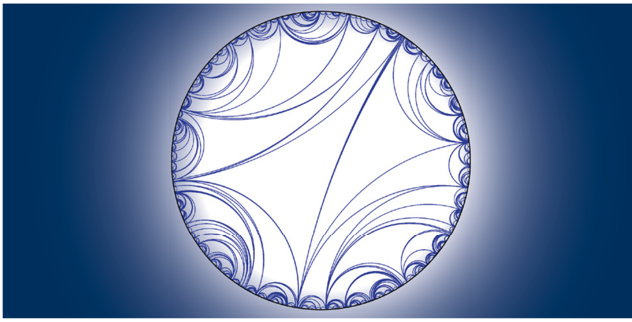

Andrey Pilipenko

An Introduction to Stochastic Differential Equations with Reflection

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To my family

Preface

The main objective of these lecture notes is the study of stochastic equations corresponding to diffusion processes in a domain with a reflection boundary. It is well known that construction of diffusions in the entire Euclidean space is closely related to solutions of stochastic differential equations (SDEs). Consider for simplicity the one-dimensional case and imagine that we have already constructed reflecting diffusion in a half-line $[0,\infty)$ with reflection at 0. Then the corresponding stochastic dynamics should have a stochastic differential of the form

$$\mathrm{d}\xi(t) = a\big(t,\xi(t)\big)\,\mathrm{d}t + b\big(t,\xi(t)\big)\,\mathrm{d}w(t) \tag{1}$$

when it lies inside $(0,\infty)$. The process $\xi(t)$, $t \geq 0$, must be continuous and should somehow reflect when it hits 0. This reflection must be strong enough not to allow ξ to enter the negative half-line. On the other hand, it should be continuous and must disappear when the process enters $(0,\infty)$. Note that the diffusion process has an "infinite" speed, so we cannot just change a speed to the opposite one at the instant of hitting 0.

In the pioneering paper on this topic [64], Skorokhod proposed finding a pair of continuous non-anticipating processes $\xi(t)$ and $l(t)$, $t \geq 0$, such that

$$\xi(t) = \xi_0 + \int_0^t a\big(s,\xi(s)\big)\,\mathrm{d}s + \int_0^t b\big(s,\xi(s)\big)\,\mathrm{d}w(s) + l(t), \quad \text{a.s.}, \tag{2}$$

where $\xi(t) \geq 0$, $t \geq 0$, l is non-decreasing, $l(0) = 0$ and

$$l(t) = \int_0^t \mathrm{1\!I}_{\xi(s)=0}\,\mathrm{d}l(s), \quad t \geq 0.$$

The last requirement means that l may increase only when ξ visits 0.

It turns out that equation (2) for a pair of unknown processes has a unique solution if a and b satisfy linear growth and the Lipschitz conditions in x. Moreover, the process ξ is Markov and the corresponding semigroup matches to some parabolic differential equation with the Neumann boundary condition.

The situation is similar in multidimensional space. The form of the SDE with reflection at the boundary of the multidimensional domain D is the following:

$$\mathrm{d}\xi(t) = a\big(t,\xi(t)\big)\,\mathrm{d}t + \sum_k b_k\big(t,\xi(t)\big)\,\mathrm{d}w_k(t) + v\big(t,\xi(t)\big)\,\mathrm{d}l(t), \quad t \geq 0,$$

where v is a reflecting vector field, $\xi(t) \in \overline{D}$, $t \geq 0$, l is a continuous non-decreasing process such that and $l(t) = \int_0^t \mathbb{1}_{\xi(s)\in\partial D}\,\mathrm{d}l(s)$, $t \geq 0$.

The main topics to discuss here are classical: theorems of existence and uniqueness, continuity of a solution with respect to coefficients and initial data, the Markov property and properties of the corresponding semigroups, relation with PDEs, etc.

A prerequisite needed for reading this book is knowledge of basic facts from stochastic integration and SDE theory.

To simplify the presentation and understanding, results are not given in "the most general form." Rather, the exposition is focused on acquaintance with main ideas and approaches of reflecting SDE theory. Standard technical details from stochastic analysis are sometimes omitted but all theorems are formulated with all assumptions and all hypotheses. If a proof is not included, then the corresponding reference is given. After the main theorems or at the end of sections, I try to give references for further development. However, I do not pretend to offer the complete historical overview, and thus, must apologize if I forget to include a relevant or interesting article.

The material discussed here is the subject of a mini-course on reflecting stochastic differential equations that I have given at the University of Potsdam and the Technical University of Berlin in winter 2013 for PhD students of the Research Training Group 1845 *Stochastic analysis with applications in biology, finance and physics*. I would like to express my sincere gratitude to Dr. Prof. Sylvie Roelly who invited me to give this mini-course and encouraged me to write this manuscript, and to the RTG for the nice opportunity. Many thanks also to Max Schneider for his great help in the English formulation of this text and to Dr. Mathias Rafler for his TEXpertise, which considerably improved the presentation.

Contents

Chapter 1

One-dimensional Skorokhod's problem and reflecting SDEs

1.1 The Skorokhod problem

Let us start with an informal description of a problem we will study. Assume that a particle is located in a positive half-line and that there is a solid wall at the point zero. A particle is driven according to the function f but is "glitching" at times when it intends to go through the wall. Denote by $g(t)$ the position of the particle at the moment $t \geq 0$. If $g(t) > 0, t \in [a,b]$, then increments for g and f should be the same. When $g(t) = 0$, i.e. a particle hits the wall, then all of its "intentions" to go left should be compensated. These compensations should immediately disappear when $g(t) > 0$. The problem is to find g for a given f.

We begin with a precise definition.

Definition 1.1.1 Let $f \in C\big([0,T]\big)$, $f(0) \geq 0$. A pair of continuous functions g and l are called a solution of the Skorokhod problem for f if

1) $g(t) \geq 0, t \in [0,T]$;

2) $g(t) = f(t) + l(t), t \in [0,T]$;

3) $l(0) = 0$, l is non-decreasing;

$$4)\quad \int_0^T \mathbb{I}_{g(s)>0}\, \mathrm{d}\, l(s) = 0. \tag{1.1}$$

Equation (1.1) means that l does not increase when $g(s) > 0$, i.e., l may increase only at those times when $g(s) = 0$. Sometimes this condition is written in the equivalent form

$$4')\quad \int_0^t \mathbb{I}_{g(s)=0}\, \mathrm{d}\, l(s) = l(t),\ t \in [0,T].$$

Theorem 1.1.1 For any $f \in C\big([0,T]\big)$, $f(0) \geq 0$, there is a unique solution to the Skorokhod problem. Moreover,

$$l(t) = -\min_{s\in[0,t]} \big(f(s) \wedge 0\big) = \max_{s\in[0,t]} \big(-f(s) \vee 0\big), \tag{1.2}$$

$$g(t) = f(t) + l(t) = f(t) - \min_{s\in[0,t]} \big(f(s) \wedge 0\big). \tag{1.3}$$

Remark 1.1.1 A set where the function l is increasing may be nowhere dense. This holds, for example, if $f(t)$ is a typical trajectory of a Wiener process.

Proof of Theorem 1.1.1. The proof of existence is straightforward (see Figure 1.1). Let us verify uniqueness.

Proof 1. This proof was proposed by Skorokhod in [64]. Let (g_1, l_1) and (g_2, l_2) be solutions of the Skorokhod problem. Assume that the set $\{t \geq 0\ :\ g_1(t) > g_2(t)\}$ is not empty. Then there exists an interval (a,b) such that $g_1(t) - g_2(t) > 0$, $t \in (a,b)$, and $g_1(a) = g_2(a)$. Observe that $g_1(t) > 0$, $t \in (a,b)$, so $l_1(t)$, $t \in (a,b)$, is constant. Hence $g_1(t) - g_2(t) = l_1(t) - l_2(t)$, $t \in (a,b)$, is a non-increasing positive continuous function such that $g_1(a) - g_2(a) = 0$. This is impossible.

Proof 2. Let (g_1, l_1) and (g_2, l_2) be solutions of the Skorokhod problem. Since $g_1(t) - g_2(t) = l_1(t) - l_2(t)$ is a continuous function of bounded variation, we have

$$\begin{aligned} 0 \leq \big(g_1(t) - g_2(t)\big)^2 &= 2\int_0^t \big(g_1(s) - g_2(s)\big)\, \mathrm{d}\big(l_1(s) - l_2(s)\big) \\ &= 2\int_0^t \mathbb{I}_{g_1(s)=0}\big(g_1(s) - g_2(s)\big)\, \mathrm{d}l_1(s) - 2\int_0^t \mathbb{I}_{g_2(s)=0}\big(g_1(s) - g_2(s)\big)\, \mathrm{d}l_2(s). \end{aligned} \tag{1.4}$$

We used condition (1.1) in the last inequality.

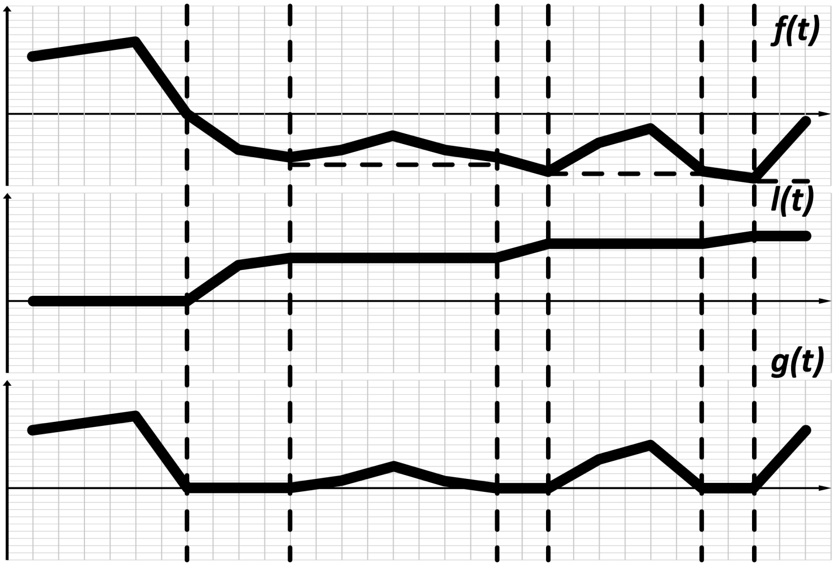

Figure 1.1: The particle driving function f, particle position g and the compensating function l.

The right hand side of (1.4) equals

$$-2\int_0^t \mathbb{I}_{g_1(s)=0} g_2(s)\,\mathrm{d}l_1(s) - 2\int_0^t \mathbb{I}_{g_2(s)=0} g_1(s)\,\mathrm{d}l_2(s) \le 0,$$

because g_1 and g_2 are non-negative and l_1 and l_2 are non-decreasing. So $g_1(t) = g_2(t)$, $t \in [0,T]$, and

$$l_1(t) = f(t) - g_1(t) = f(t) - g_2(t) = l_2(t). \qquad \square$$

By Γ, we denote the map

$$g(\cdot) = \Gamma f(\cdot) := f(\cdot) - \min_{s\in[0,\cdot]} \big(f(s) \wedge 0\big).$$

The map Γ is called the Skorokhod map. It is easy to check the following properties of Γ. We leave the proof for the reader.

Lemma 1.1.1 Skorokhod's map Γ is a continuous function from $C\big([0,T]\big)$ to $C\big([0,T]\big)$, where $C([0,T])$ is equipped with the supremum norm, $\|f\| := \max_{t\in[0,T]} |f(t)|$. Moreover,

1) $\forall\, f_1, f_2 \in C([0,T])$:

$$\|g_1 - g_2\| \le 2\|f_1 - f_2\|,$$
$$\|l_1 - l_2\| \le \|f_1 - f_2\|,$$

where $g_i = \Gamma f_i$, $l_i = f_i - g_i$, $i = 1,2$.

2) $\forall\, \delta > 0 :\ \omega_g(\delta) \le \omega_f(\delta),\ \omega_l(\delta) \le \omega_f(\delta)$,
where $g = \Gamma f,\ \ l = f - g,\ \omega_f(\delta) = \sup_{|s-t|<\delta} |f(s) - f(t)|$ is the modulus of continuity of f.

3) $\forall\, f \in C([0,T])$:

$$\|\Gamma f\| \le 2\|f\|, \quad \|l\| \le \|f\|.$$

Remark 1.1.2 The considerations in this manuscript are restricted to continuous processes; however, it is natural to define the Skorokhod problem for càdlàg functions (see e.g. [13]) or to consider a possibility of a jump-type exit from the boundary (see [55]).

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1.2 Reflecting SDE

Let $\{w(t), t \ge 0\}$ be a Wiener process adapted to a filtration $\{\mathscr{F}_t, t \ge 0\}$, $a = a(t,x)$, $b = b(t,x) : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be measurable functions, $\xi_0 \ge 0$ be $\mathscr{F}_0$-measurable. We will always assume that $\mathscr{F}_t$ is completed by events of null probability and continuous from the right.

The aim of this chapter is to construct a process $\xi(t)$ with values in $[0,\infty)$ that has a stochastic differential of the form

$$\mathrm{d}\xi(t) = a\big(t, \xi(t)\big)\,\mathrm{d}t + b\big(t, \xi(t)\big)\,\mathrm{d}w(t)$$

if $\xi(t) > 0$ and continuously reflects in some sense into the positive half-line when ξ hits 0. Much like the reasoning of the previous chapter, it is natural to give the following definition.

Definition 1.2.1 A pair of continuous $\mathscr{F}_t$-adapted processes $\big(\xi(t), l(t)\big)$, $t \geq 0$, is a solution of the SDE

$$\mathrm{d}\xi(t) = a\big(t, \xi(t)\big)\,\mathrm{d}t + b\big(t, \xi(t)\big)\,\mathrm{d}w(t) + \mathrm{d}l(t), \quad t \geq 0,$$

with reflection at 0 and the initial condition $\xi(0) = \xi_0$, if

1) $\xi(t) \geq 0, t \geq 0$;

2) l is non-decreasing, $l(0) = 0$;

3) $\int_0^t \mathbb{I}_{\xi(s)>0}\,\mathrm{d}l(s) = 0, \quad t \geq 0$;

4) $\xi(t) = \xi_0 + \int_0^t a\big(s, \xi(s)\big)\,\mathrm{d}s + \int_0^t b\big(s, \xi(s)\big)\,\mathrm{d}w(s) + l(t), t \geq 0, \qquad \text{a.s}, \qquad (1.5)$

and all the integrals are well-defined.

Let us discuss a relation between the solution of (1.5) and the Skorokhod problem. Assume that ω is such that (1.5) is satisfied. Denote

$$Y(t) = \xi_0 + \int_0^t a\big(s, \xi(s)\big)\,\mathrm{d}s + \int_0^t b\big(s, \xi(s)\big)\,\mathrm{d}w(s).$$

Then all conditions in Definition 1.2.1 for $Y(t)$ coincide with the conditions in Definition 1.1.1 for $f(t)$. By Theorem 1.1.1,

$$\xi(t) = \big(\Gamma Y\big)(t), \quad t \geq 0, \tag{1.6}$$

and $Y(t)$ is a solution of the following Itô's equation

$$Y(t) = \xi_0 + \int_0^t a\big(s, \Gamma Y(s)\big)\,\mathrm{d}s + \int_0^t b\big(s, \Gamma Y(s)\big)\,\mathrm{d}w(s). \tag{1.7}$$

Remark 1.2.1 For any non-anticipating continuous process $Y(t)$, $t \geq 0$, the process $\big(\Gamma Y\big)(t)$ is also continuous and non-anticipating.

It is easy to see that if $Y(t), t \geq 0$, is a solution of (1.7), then

$$\xi(t) = \Gamma Y(t), \quad l(t) = \xi(t) - Y(t)$$

is a solution of (1.5).

Applying standard results on the solvability of Itô's equation, we get the following existence and uniqueness theorem.

Theorem 1.2.1 Let ξ_0 be a non-negative $\mathscr{F}_0$-adapted random variable. Assume that measurable functions $a = a(t,x)$, $b = b(t,x)$ satisfy the

1) Lipschitz condition in x, uniformly in time:

$$\exists\, L > 0\, \forall\, t \geq 0\, \forall\, x_1, x_2 \in \mathbb{R}_+ : |a(t,x_1) - a(t,x_2)| + |b(t,x_1) - b(t,x_2)| \leq L|x_1 - x_2|;$$

2) linear growth condition in x, uniformly in time:

$$\exists\, C > 0\, \forall\, t \geq 0\, \forall\, x \in \mathbb{R}_+ : |a(t,x)| + |b(t,x)| \leq C\big(1 + |x|\big).$$

Then there exists a unique solution to the reflecting SDE (1.5).

For proof, it is sufficient to notice (see Lemma 1.1.1), that

$$\begin{aligned}
&\forall t > 0\ \forall y_1, y_2 \in C\big([0,t]\big) : \\
&\quad \Big|a\big(t, \big(\Gamma y_1\big)(t)\big) - a\big(t, \big(\Gamma y_2\big)(t)\big)\Big| + \Big|b\big(t, \big(\Gamma y_1\big)(t)\big) - b\big(t, \big(\Gamma y_2\big)(t)\big)\Big| \\
&\qquad \leq L\Big|\big(\Gamma y_1\big)(t) - \big(\Gamma y_2\big)(t)\Big| \leq 2L\|y_1 - y_2\|_{[0,t]};
\end{aligned}$$

where $\|f\|_{[0,t]} := \sup_{s\in[0,t]} |f(t)|$;

$$\begin{aligned}
&\forall\, t > 0\ \forall y \in C\big([0;t]\big) : \\
&\quad \Big|a\big(t, \big(\Gamma y\big)(t)\big)\Big| + \Big|b\big(t, \big(\Gamma y\big)(t)\big)\Big| \leq C\big(1 + |\Gamma y(t)|\big) \leq 2C\big(1 + \|y\|_{[0,t]}\big).
\end{aligned}$$

Let us also give another way to prove uniqueness (cf. proof of Theorem 1.1.1) which will be useful in the multidimensional case. Let (ξ_1, l_1) and (ξ_2, l_2) be solutions of (1.5). Then by Itô's formula,

$$
\begin{aligned}
&\big(\xi_1(t)-\xi_2(t)\big)^2\\
&=\int_0^t \Big\{2\big(\xi_1(z)-\xi_2(z)\big)\big[a\big(z,\xi_1(z)\big)-a\big(z,\xi_2(z)\big)\big]+\big[b\big(z,\xi_1(z)\big)-b\big(z,\xi_2(z)\big)\big]^2\Big\}\,\mathrm{d}z\\
&\qquad +2\int_0^t \big(\xi_1(z)-\xi_2(z)\big)\,\mathrm{d}\big(l_1(z)-l_2(z)\big)\\
&\qquad\qquad +2\int_0^t \big(\xi_1(z)-\xi_2(z)\big)\big[b\big(z,\xi_1(z)\big)-b\big(z,\xi_2(z)\big)\big]\,\mathrm{d}w(z).
\end{aligned}
\tag{1.8}
$$

Similarly to the proof of Theorem 1.1.1, we have

$$
\int_0^t \big(\xi_1(z)-\xi_2(z)\big)\,\mathrm{d}\big(l_1(z)-l_2(z)\big)\le 0,\quad t\ge 0.
$$

It remains to take the expectation in (1.8) and apply Gronwall's lemma. Denote

$$
\tau_n=\inf\big\{t\ge 0:\ |\xi_1(t)|\wedge|\xi_2(t)|\ge n\big\}.
$$

Then

$$
\begin{aligned}
E\big[\xi_1(t\wedge\tau_n)-\xi_2(t\wedge\tau_n)\big]^2 &\le \big(2L+L^2\big)\,E\int_0^{t\wedge\tau_n}\big(\xi_1(z)-\xi_2(z)\big)^2\,\mathrm{d}s\\
&\le \big(2L+L^2\big)\int_0^t E\big[\xi_1(z\wedge\tau_n)-\xi_2(z\wedge\tau_n)\big]^2\,\mathrm{d}z.
\end{aligned}
$$

Gronwall's lemma yields

$$
\forall\, n\ge 0\ \forall\, t>0:\ P\big(\xi_1(t\wedge\tau_n)=\xi_2(t\wedge\tau_n)\big)=1.
$$

Since $\xi_i(t)$ are continuous in t, the last equality implies

$$
P\big(\xi_1(t)=\xi_2(t),t\ge 0\big)=1,
$$

and hence for $k=1,2$:

$$
l_1(t)=\xi_k(t)-\xi_0-\int_0^t a\big(s,\xi_k(s)\big)\,\mathrm{d}s-\int_0^t b\big(s,\xi_k(s)\big)\,\mathrm{d}w(s)=l_2(t),\,t\ge 0,\quad \text{a.}s.\quad\square
$$

Exercise 1.2.1 Assume that a and b satisfy conditions of Theorem 1.2.1. Denote by $\xi_x(t)$

a solution of (1.5) with the initial condition $\xi_x(0) = x$. Prove that

1) $\forall\, p \ge 2\ \exists\, K = K(p,L,C)\ \forall\, t \ge 0\ \forall\, x,y \ge 0\ \ \forall\, z \ge 0$,

$$E \sup_{s\in[0,t]} |\xi_x(s)|^p \le K(1+|x|^p)e^{Kt}, \tag{1.9}$$

$$E \sup_{s\in[0,t]} |\xi_x(s)-x|^p \le K(1+|x|^p)t^{p/2}e^{Kt}; \tag{1.10}$$

$$E \sup_{s\in[0,t]} |\xi_x(s)-\xi_y(s)|^p \le K|x-y|^p e^{Kt}; \tag{1.11}$$

$$E|\xi_x(t+z)-\xi_x(t)|^p \le Kz^{p/2}(1+|x|^p)e^{K(t+z)}. \tag{1.12}$$

Hint: Write Itô's formula, apply Gronwall's lemma, Lemma 1.1.1 and Burkholder's inequality.

2) Prove estimates similar to (1.9), (1.11) and (1.12) for the process $l_x(t)$ and

$$E \sup_{s\in[0,t]} l_x^p(s) \le Kt^{p/2}\big(1+|x|^p\big)e^{Kt}. \tag{1.13}$$

3) Prove that there exist modifications of $\xi_x(t)$ and $l_x(t)$ that are continuous in (t,x).
Hint: Apply the Kolmogorov's continuity criterion: If $\varphi(u) = \varphi(u_1,\dots,u_n)$, $|u_i| \le R$, is a random field such that

$$\exists\beta > 0\ \exists C > 0\ \exists\varepsilon > 0\ \forall u,v: \quad E\big|\varphi(u)-\varphi(v)\big|^\beta \le C|u-v|^{n+\varepsilon},$$

then there exists a continuous modification of $\varphi(u_1,\dots,u_n)$.

Take $n = 2$, $u_1 = t$, $u_2 = x$ and apply (1.11) and (1.12) for $\xi_x(t)$.

Let us discuss the Markov property for a solution of reflecting SDE (1.2). The method of proof is the same as for SDEs without reflection.

By $\xi_{x,s}(t), t \in [s,\infty)$, denote a solution of reflecting SDE

$$\mathrm{d}\,\xi_{x,s}(t) = a\big(t,\xi_{x,s}(t)\big)\,\mathrm{d}t + b\big(t,\xi_{x,s}(t)\big)\,\mathrm{d}w(t) + \mathrm{d}\,l_{x,s}(t), \quad t \in [s,\infty), \tag{1.14}$$

with initial condition

$$\xi_{x,s}(s) = x.$$

It follows from Exercise 1.2.1 that $\xi_{x,s}(t)$ has a measurable (and even continuous) modification in (x,t) and that $\xi_{x,s}(t)$ is measurable w.r.t. the σ-algebra $\sigma\big(w(z)-w(s), z\in[s,t]\big)$ completed by events of null probability. Hence $\xi_{x,s}(t)$ is independent of $\xi(s)$ and with probability 1, we have that

$$\xi_{\xi(s),s}(t)=\xi(s)+\int_s^t a\big(z,\xi_{\xi(s),s}(z)\big)\,\mathrm{d}z+\int_s^t b\big(z,\xi_{\xi(s),s}(z)\big)\,\mathrm{d}w(z)+l_{\xi(s),s}(t),\quad t\in[s,\infty).$$

Observe that

$$\xi(t)=\xi(s)+\int_s^t a\big(z,\xi(z)\big)\,\mathrm{d}z+\int_s^t b\big(z,\xi(z)\big)\,\mathrm{d}w(z)+l(t)-l(s).$$

So $\big(\xi(t),l(t)-l(s)\big)$ and $\big(\xi_{\xi(s),s}(t),l_{\xi(s),s}(t)\big)$, $t\in[s,\infty)$, are both solutions of (1.14) with initial condition $\xi(s)$.

It follows from the uniqueness of the solution that

$$\xi(t)=\xi_{\xi(s),s}(t)\quad \text{a.s.}$$

Since $\xi(s)$ is $\mathscr{F}_s$-measurable and $\xi_{x,s}(t)$ is independent of $\mathscr{F}_s$,

$$\begin{aligned}P\big(\xi(t)\in A\big|\mathscr{F}_s\big)&=E\big(\mathrm{I\!I}_{\xi_{\xi(s),s}(t)\in A}\big|\mathscr{F}_s\big)=E\big(\mathrm{I\!I}_{\xi_{x,s}(t)\in A}\big)\big|_{x=\xi(s)}\\&=E\big(\mathrm{I\!I}_{\xi_{\xi(s),s}(t)\in A}\big|\xi(s)\big)=P\big(\xi(t)\in A\big|\xi(s)\big).\end{aligned}$$

Thus, we have proved the following result.

Theorem 1.2.2 Assume that functions a and b satisfy conditions of Theorem 1.2.1. Then $\{\xi(t),t\geq 0\}$ is a Markov process with transition probabilities

$$P\big(\xi(t)\in A\big|\xi(s)=x\big)=P\big(\xi_{x,s}(t)\in A\big).$$

Exercise 1.2.2 Prove that the process $\big\{\big(\xi(t),l(t)\big),t\geq 0\big\}$ is a Markov process. Is the process $\{l(t),t\geq 0\}$ a Markov process?

Exercise 1.2.3 Let $a=0$, $b=1$, i.e., $\xi_x(t)$, $t\geq 0$, is a reflecting Wiener process started at $x\geq 0$. Set $\tau(x)=\inf\{t\geq 0\,:\,\xi_x(t)=0\}$. Prove that $\inf\{t\geq 0\,:\,\xi_x(t)=\xi_y(t)\}=\tau(x\vee y)$ a.s. Describe the behavior of $\xi_x(t)$, $t\geq 0$, as a function of the spatial variable. Consider also the case of non-constant coefficients.

Example 1.2.1 Reflection for diffusions can sometimes be modelised via very intricate stochastic equations (without Skorokhod's construction). For example, let $\xi(t)$, $t \geq 0$, be a Bessel process with parameter d, see [60]. Assume that τ is a stopping time such that $\xi(\tau) > 0$ a.s. Put $\tau_0 = \inf\{t \geq 0 : \ \xi(t) = 0\}$. It is well known that

$$\mathrm{d}\xi(t) = \frac{d-1}{2\sqrt{\xi(t)}}\,\mathrm{d}t + \mathrm{d}w(t), \quad t \in [\tau, \tau_0),$$

where $w(t)$, $t \geq 0$, is some Wiener process.

If $d \in (0,1)$, then $\tau_0 < \infty$ a.s. Since $\xi(t) \geq 0$, $t \geq 0$, it is natural to conjecture that ξ satisfies the following reflecting SDE

$$\mathrm{d}\xi(t) = \frac{d-1}{2\sqrt{\xi(t)}}\,\mathrm{d}t + \mathrm{d}w(t) + \mathrm{d}l(t), \quad t \geq 0.$$

However, this is not true if $d \in (0,1)$. The reason for this is the divergence of the integral $\int_\tau^{\tau_0} 1/\sqrt{\xi(s)}\,\mathrm{d}s = \infty$ a.s. Note that there is some specific SDE for ξ, see [60], Ch. XI, Exercise 1.26. It can be checked that the solution is strong, cf. [5].

1.3 Characterization of reflecting term as a local time

Consider the reflected SDE (1.5). In this section, we show that $l(t)$ is a local time at the point 0 of the process $\xi(t)$. Let us recall basic facts about the local time of a continuous semimartingale (see for ex. [60]).

Let $\{X(t), t \geq 0\}$ be a continuous semimartingale. It can be proved that, almost surely, there exists a limit

$$L^a(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathrm{I\!I}_{[a,a+\varepsilon]}\big(X(s)\big)\,\mathrm{d}\langle X, X\rangle_s$$

for every a and $t \geq 0$. This limit is called the local time of X at a.

1) The Tanaka formula for the local time

$$\begin{aligned} L^a(t) &= 2\left(\big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \int_0^t \mathrm{I\!I}_{X(s)>a}\,\mathrm{d}X(s)\right) \\ &= \big|X(t)-a\big| - \big|X(0)-a\big| - \int_0^t \operatorname{sign}\big(X(s)-a\big)\,\mathrm{d}X(s) \quad \text{a.s.,} \end{aligned} \tag{1.15}$$

where $\operatorname{sign} x = \begin{cases} -1, & x \le 0 \\ 1, & x > 0. \end{cases}$.

2) There exists a modification of $\{L^a(t), a \in \mathbb{R}, t \ge 0\}$ such that the map $(a,t) \mapsto L^a(t)$ is a.s. continuous in t and càdlàg in a (henceforth we consider only this modification).

3) For any measurable non-negative function f, the occupation times formula holds:

$$\int_0^t f\big(X(s)\big)\, \mathrm{d}\langle X,X\rangle_s = \int_{-\infty}^{\infty} f(a) L^a(t)\, \mathrm{d}a \quad \text{a.s.} \tag{1.16}$$

Now let $\xi(t)$ be a solution of (1.5). Then $\xi(t) = \xi^+(t) = (\xi(t)-0)^+$ and by Tanaka's formula:

$$\begin{aligned} L^0(t) &= 2\left(\big(\xi(t)-0\big)^+ - \big(\xi_0 - 0\big)^+ - \int_0^t \mathrm{II}_{\xi(s)>0}\, \mathrm{d}\xi(s) \right) \\ &= 2\left(\xi(t) - \xi_0 - \int_0^t \mathrm{II}_{\xi(s)>0} a\big(s,\xi(s)\big)\, \mathrm{d}s \right. \\ &\qquad \left. - \int_0^t \mathrm{II}_{\xi(s)>0} b\big(s,\xi(s)\big)\, \mathrm{d}w(s) - \int_0^t \mathrm{II}_{\xi(s)>0}\, \mathrm{d}l(s) \right) \\ &= 2\left(\xi(t) - \xi_0 - \int_0^t a\big(s,\xi(s)\big) ds + \int_0^t \mathrm{II}_{\xi(s)=0} a(s,0)\, \mathrm{d}s \right. \\ &\qquad \left. - \int_0^t b\big(s,\xi(s)\big)\, \mathrm{d}w(s) + \int_0^t \mathrm{II}_{\xi(s)=0} b(s,0)\, \mathrm{d}w(s) \right). \end{aligned}$$

Here we used that $\int_0^t \mathrm{II}_{\xi(s)>0}\, \mathrm{d}l(s) = 0$ by definition.

Hence

$$L^0(t) = 2\left(l(t) - \int_0^t \mathrm{II}_{\xi(s)=0}\, a(s,0)\, \mathrm{d}s - \int_0^t \mathrm{II}_{\xi(s)=0}\, b(s,0)\, \mathrm{d}w(s) \right). \tag{1.17}$$

Let us give sufficient conditions ensuring that the process ξ spends zero time at 0. By the occupation times formula (1.16),

$$\int_0^t \mathrm{II}_{\xi(s)=0}\, \mathrm{d}\langle \xi, \xi \rangle_s = \int_{-\infty}^{\infty} \mathrm{II}_{a=0}\, L^a(t)\, \mathrm{d}a = 0 \quad \text{a.s.}$$

So $\int_0^t \mathrm{II}_{\xi(s)=0}\, b^2(s,0)\, \mathrm{d}s = 0$. Assume that $b(s,0) \neq 0$ for λ-a.a. s, where λ is the

Lebesgue measure. Then,

$$P\left(\int_0^t \mathrm{I\!I}_{\xi(s)=0}\ \mathrm{d}s = 0, t \geq 0\right) = 1$$

and both integrals in (1.17) disappear. Hence, we have proved the following theorem.

Theorem 1.3.1 Assume that $b(s,0) \neq 0$ for λ-a.a. $s \geq 0$. Then $l(t) = \frac{1}{2}L^0(t)$, $t \geq 0$, a.s., where $L^0(t)$ is the local time of ξ at 0.

Remark 1.3.1 Under the assumptions of Theorem 1.1.1, $l(t)$ a.s. equals the two-sided local time of ξ at 0 defined by

$$\lim_{\varepsilon\to 0+} (2\varepsilon)^{-1} \int_0^t \mathrm{I\!I}_{[-\varepsilon,\varepsilon]}\big(\xi(s)\big)\ \mathrm{d}\langle \xi, \xi\rangle_s$$

.

Remark 1.3.2 In the pioneering work of Skorokhod [64], the process $l(t)$, $t \geq 0$, was identified with some additive functional of the process $\xi(t)$, $t \geq 0$. It was proved that for almost all points

$$\lim_{\Delta t \downarrow 0} \frac{l(t+\Delta t) - l(t)}{\sqrt{\Delta t}} = \sqrt{\frac{\pi}{8}} b(t,0)\, \mathrm{I\!I}_{\xi(t)=0}$$

and

$$l(t) = \sqrt{\frac{\pi}{8}} \int_{t_0}^t b(s,0)\, \mathrm{I\!I}_{\xi(s)=0} \sqrt{\mathrm{d}s}.$$

The integral in the right hand side was rigorously defined as some limit of integral sums.

Example 1.3.1 (Local time and maximum process) By Tanaka's formula,

$$\mathrm{d}\left|w(t)\right| = \operatorname{sign} w(t)\ \mathrm{d}w(t) + L_w^0(t), \quad t \geq 0.$$

The process $B(t) = \int_0^t \operatorname{sign} w(s)\ \mathrm{d}w(s)$ is a Brownian motion. So

$$\left|w(t)\right| = B(t) + L_w^0(t), \quad t \geq 0.$$

Observe that because $L_w^0(t)$, $t \geq 0$, is a non-decreasing continuous process, it may increase only when w hits zero (see the definition of a local time). So the pair $\big(|w(t)|, L_w^0(t)\big)$ is a solution of Skorokhod's problem for $B(t)$.

By Theorem 1.1.1:

$$\big|w(t)\big| = \big(\Gamma B\big)(t), \quad L_w^0(t) = B(t) - \big(\Gamma B\big)(t) = -\min_{s\in[0,t]} B(s).$$

Since a pair $\big(\Gamma B(\,\cdot\,), \min_{s\in[0,\cdot]} B(s)\big)$ has the same distribution as $\big(\Gamma w(\,\cdot\,), \min_{s\in[0,\cdot]} w(s)\big)$, we have proved the following result.

Theorem 1.3.2 The two processes

$$\big\{\big(|w(t)|, L_w^0(t)\big), t \geq 0\big\} \quad \text{and} \quad \big\{\big(w(t) - \min_{s\in[0,t]} w(s), -\min_{s\in[0,t]} w(s)\big), t \geq 0\big\}$$

have the same distribution.

Exercise 1.3.1 ([23], § 23) Extend functions a and b to negative values of x by

$$a(t,-x) := -a(t,x), \quad b(t,-x) := b(t,x).$$

Assume that $b(t,0) \neq 0$, $t \geq 0$. Let $\xi(t)$, $t \geq 0$, be a solution of the following SDE on $\mathbb{R}$

$$\mathrm{d}\xi(t) = a\big(t,\xi(t)\big)\,\mathrm{d}t + b\big(t,\xi(t)\big)\,\mathrm{d}w(t), \quad t \geq 0.$$

Prove that $|\xi(t)|$, $t \geq 0$, satisfies (1.5) with initial condition $|\xi(0)|$ and a new Wiener process $\tilde{w}(t) = \int_0^t \operatorname{sign}\xi(s)\,\mathrm{d}w(s)$.

• •

1.4 Approximation of reflecting SDEs

Euler's scheme. Let $a, b : \mathbb{R}_+ \to \mathbb{R}$ be Lipschitz functions, $\big(\xi(t), l(t)\big)$ be a solution of reflecting SDE

$$\begin{cases} \mathrm{d}\xi(t) = a\big(\xi(t)\big)\,\mathrm{d}t + b\big(\xi(t)\big)\,\mathrm{d}w(t) + \mathrm{d}l(t), \quad t \geq 0, \\ \xi(0) = \xi_0, \end{cases} \tag{1.18}$$

where ξ_0 is a non-negative $\mathscr{F}_0$-adapted random variable, $E(\xi_0)^2 < \infty$.

Consider the sequence of processes that satisfy the following reflecting equation:

$$\mathrm{d}\,\xi_n(t) = a\big(\xi_n(k/n)\big)\,\mathrm{d}t + b\big(\xi_n(k/n)\big)\,\mathrm{d}w(t) + \mathrm{d}\,l_n(t), \qquad t \in \left[\frac{k}{n}, \frac{k+1}{n}\right],$$

where $\xi_n(t) \geq 0, t \geq 0$; $\xi_n(0) = \xi_0$; $\xi_n(t)$ is continuous in t; $l_n(0) = 0$; l_n is non-decreasing; $\int_0^\infty \mathbb{I}_{\xi_n(s)>0}\,\mathrm{d}\,l_n(s) = 0$. The process $\xi_n(t)$ can be calculated successively

$$\begin{aligned}\xi_n(t) = {} & \xi_n(k/n) + a\big(\xi_n(k/n)\big)\big(t - k/n\big) \\ & + b\big(\xi_n(k/n)\big)\big(w(t) - w(k/n)\big) + l_n(t) - l_n(k/n), \qquad t \in \left[\frac{k}{n}, \frac{k+1}{n}\right],\end{aligned}$$

where $l_n(t)$ is such that $\xi_n(t) \geq 0$ and $l_n(t)$ does not increase when $\xi_n(t) > 0$. This is the Skorokhod problem on $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ for the function

$$\xi_n(k/n) + a\big(\xi_n(k/n)\big)\big(t - k/n\big) + b\big(\xi_n(k/n)\big)\big(w(t) - w(k/n)\big).$$

Observe that $\big(\xi_n(t), l_n(t)\big)$ satisfies the equation

$$\xi_n(t) = \xi_0 + \int_0^t a\big(\xi_n(\varphi_n(s)\big)\,\mathrm{d}s + \int_0^t b\Big(\xi_n\big(\varphi_n(s)\big)\Big)\,\mathrm{d}w(s) + l_n(t), \tag{1.19}$$

where $\varphi_n(s) = k/n$ for $s \in \left[\frac{k}{n}, \frac{k+1}{n}\right)$.

Similarly to the proof of Theorem 1.2.1, denote

$$\begin{aligned}Y_n(t) &:= \xi_0 + \int_0^t a\Big(\xi_n\big(\varphi_n(s)\big)\Big)\,\mathrm{d}s + \int_0^t b\Big(\xi_n\big(\varphi_n(s)\big)\Big)\,\mathrm{d}w(s), \\ Y(t) &:= \xi_0 + \int_0^t a\big(\xi(s)\big)\,\mathrm{d}s + \int_0^t b\big(\xi(s)\big)\,\mathrm{d}w(s).\end{aligned}$$

Then

$$\xi_n(t) = \big(\Gamma Y_n\big)(t), \quad \xi(t) = \big(\Gamma Y\big)(t)$$

and

$$\begin{aligned}Y_n(t) &= \xi_0 + \int_0^t a\Big(\big(\Gamma Y_n\big)(\varphi_n(s))\Big)ds + \int_0^t b\Big(\big(\Gamma Y_n\big)(\varphi_n(s))\Big)\,\mathrm{d}w(s), \\ Y(t) &= \xi_0 + \int_0^t a\Big(\big(\Gamma Y\big)(s)\Big)ds + \int_0^t b\Big(\big(\Gamma Y\big)(s)\Big)\,\mathrm{d}w(s).\end{aligned}$$

Exercise 1.4.1 Prove that

$$\sup_n E \sup_{t\in[0,T]} \Big(\big(Y_n(t)\big)^2 + \big(Y(t)\big)^2 \Big) < \infty.$$

Applying the Lipschitz condition for a and b and the Burkholder inequality, we obtain, for $t \in [0,T]$,

$$\begin{aligned}
&E \sup_{s\in[0,t]} (Y(s) - Y_n(s))^2 \\
&\qquad \le CE \int_0^t \Big(\Gamma Y(z) - \Gamma Y_n\big(\varphi_n(z)\big)\Big)^2 \,\mathrm{d}z \\
&\qquad \le 2CE \int_0^t \Big[\Big(\Gamma Y(z) - \Gamma Y\big(\varphi_n(z)\big)\Big)^2 + \Big(\Gamma Y\big(\varphi_n(z)\big) - \Gamma Y_n\big(\varphi_n(z)\big)\Big)\Big]^2 \,\mathrm{d}z \\
&\qquad \le 2C \left(E \int_0^t \Big(\Gamma Y(z) - \Gamma Y\big(\varphi_n(z)\big)\Big)^2 \,\mathrm{d}z + \int_0^t E \sup_{s\in[0,z]} \big(\Gamma Y(s) - \Gamma Y_n(s)\big)^2 \,\mathrm{d}z \right),
\end{aligned}$$

where C is a constant. Applying Gronwall's lemma yields

$$E \sup_{s\in[0,t]} \big(Y(s) - Y_n(s)\big)^2 \le E \int_0^t \Big(\Gamma Y(z) - \Gamma Y\big(\varphi_n(z)\big)\Big)^2 \,\mathrm{d}z\, e^{2Ct}. \tag{1.20}$$

It follows from Exercise 1.4.1 and the Lebesgue dominated convergence theorem that

$$E \sup_{s\in[0,T]} \Big(Y(s) - Y\big(\varphi_n(s)\big)\Big)^2 \to 0, \qquad n \to \infty. \tag{1.21}$$

The application of (1.20), (1.21) and Lemma 1.1.1 yields the convergence

$$\lim_{n\to\infty} E \sup_{s\in[0,T]} \big(Y(s) - Y_n(s)\big)^2 = 0.$$

Exercise 1.4.2 Prove that

$$\forall T > 0\ \forall \varepsilon > 0,\ \exists c > 0 :\ E \sup_{s\in[0,T]} \Big(Y(s) - Y\big(\varphi_n(s)\big)\Big)^2 \le \frac{c}{n^{1-\varepsilon}}. \tag{1.22}$$

Hint: Use that $\int_0^t b\Big(\big(\Gamma Y\big)(s)\Big)\,\mathrm{d}w(s) = B\left(\int_0^t b^2\Big(\big(\Gamma Y\big)(s)\Big)\,\mathrm{d}s\right)$, where B is a Brownian motion. Since a and b are bounded, it suffices to prove (1.22) for B instead of Y.

Estimate (1.22) together with (1.20) to give a rate of convergence for Euler approximations.

Remark 1.4.1 For more on Euler's schemes and estimates for their rates of convergence for reflecting SDEs in multidimensional domains, see, for example, [67, 47, 40, 48, 68].

Penalization method. Let $\xi(t)$, $t \geq 0$, be a solution of (1.18), where a and b are Lipschitz functions. Extend a and b to $(-\infty, 0)$ such that their extensions are again Lipschitz functions. For example, put $a(x) := a(0)$, $b(x) := b(0)$ for $x < 0$. The idea of the penalization method is the following. Let us allow a process to penetrate into the set $(-\infty, 0)$, at which time we add a very large drift term that pushes the process upward. Namely, denote by $\xi_n(t)$ a solution of the SDE

$$\mathrm{d}\,\xi_n(t) = a\big(\xi_n(t)\big)\,\mathrm{d}t + b\big(\xi_n(t)\big)\,\mathrm{d}w(t) + g_n\big(\xi_n(t)\big)\,\mathrm{d}t, \quad t \geq 0,$$

where

$$g_n(x) = \begin{cases} 0 & x \geq 0 \\ -nx & x < 0 \end{cases}. \tag{1.23}$$

Theorem 1.4.1 Assume that $\xi_n(0) = \xi(0) = x \geq 0$. Then

$$\sup_{t\in[0,T]} |\xi_n(t) - \xi(t)| \xrightarrow{P} 0, \qquad n \to \infty, \tag{1.24}$$

$$\sup_{t\in[0,T]} \left| \int_0^t g_n\big(\xi_n(s)\big)\,\mathrm{d}s - l(t) \right| \xrightarrow{P} 0, \qquad n \to \infty. \tag{1.25}$$

Let us sketch the main steps of the proof as a sequence of exercises.

Step 1. Apply Itô's formula to $|\xi_n(t)|^p$, where $p \geq 2$, and prove that

$$\sup_n \sup_{t\in[0,T]} E|\xi_n(t)|^p < \infty, \tag{1.26}$$

$$\sup_n E\left| \int_0^T g_n\big(\xi_n(t)\big)\,\mathrm{d}t \right|^p < \infty. \tag{1.27}$$

Step 2. Use the Burkholder inequality and prove that for any $p \geq 2$

$$\sup_n E \sup_{t\in[0,T]} |\xi_n(t)|^p < \infty. \tag{1.28}$$

Step 3. Use Itô's formula for $p > 2$:

$$
\begin{aligned}
|\xi_n(t)|^p \mathbb{I}_{\xi_n(t)<0} \leq & -np \int_0^t |\xi_n(z)|^p \mathbb{I}_{\xi_n(z)<0}\, \mathrm{d}z \\
& + c \int_0^t \Big(|\xi_n(z)|^{p-2} + |\xi_n(z)|^{p-1} + |\xi_n(z)|^p \Big)\, \mathrm{d}z + M(t),
\end{aligned}
$$

where $M(t) = -\int_0^t p|\xi_n(z)|^{p-1} \mathbb{I}_{\xi_n(z)<0} b\big(\xi_n(z)\big)\, \mathrm{d}w(z)$, $EM(t) = 0$ and c is a constant independent of n. Prove that

$$
c\big(x^{p-1} + x^{p-2}\big) \leq \frac{np}{2} x^p + \frac{c_1}{n^{p/2-1}}, \quad x \geq 0.
$$

Further, make the conclusion that

$$
\begin{gathered}
\sup_{t\in[0,T]} E|\xi_n(t)|^p \mathbb{I}_{\xi_n(t)<0} < \frac{c_2}{n^{p/2-1}}, \\
E\int_0^T n|\xi_n(z)|^p \mathbb{I}_{\xi_n(z)<0}\, \mathrm{d}z < \frac{c_2}{n^{p/2-1}}.
\end{gathered}
$$

Step 4. Apply Burkholder's inequality and prove that

$$
E \sup_{t\in[0,T]} |\xi_n(t)|^p \mathbb{I}_{\xi_n(t)<0} < \frac{c_2}{n^{p/2-1}}. \tag{1.29}
$$

Step 5. By Itô's formula

$$
\begin{aligned}
&\big(\xi_n(t) - \xi_m(t)\big)^2 \\
&\quad \leq \int_0^t 2\big(\xi_n(z) - \xi_m(z)\big)\big(-n\xi_n(z)\mathbb{I}_{\xi_n(z)<0} + m\xi_m(z)\mathbb{I}_{\xi_m(z)<0}\big)\, \mathrm{d}z \\
&\qquad + c_4 \int_0^t \big(\xi_n(z) - \xi_m(z)\big)^2\, \mathrm{d}z + M(t),
\end{aligned} \tag{1.30}
$$

where $M(t)$ is a martingale, $EM(t) = 0$. The first term in the right hand side of (1.30) does not exceed

$$
2(n+m)\int_0^t \xi_n(z)\xi_m(z)\mathbb{I}_{\xi_n(z)<0}\mathbb{I}_{\xi_m(z)<0}\, \mathrm{d}z.
$$

It follows from (1.27) and (1.29) that

$$
\begin{aligned}
E\int_0^T & n\xi_n(z)\xi_m(z)\,\mathbb{I}_{\xi_n(z)<0}\,\mathbb{I}_{\xi_m(z)<0}\,\mathrm{d}z \\
&\le E\int_0^T n|\xi_n(z)|\,\mathbb{I}_{\xi_n(z)<0}\,\mathrm{d}z\cdot \sup_{t\in[0,T]}|\xi_m(t)|\,\mathbb{I}_{\xi_m(t)<0} \\
&\le \left(E\left(\int_0^T n|\xi_n(z)|\,\mathbb{I}_{\xi_n(z)<0}\,\mathrm{d}z\right)^q\right)^{1/q}\cdot E\left(\sup_{t\in[0,T]}|\xi_m(t)|^p\,\mathbb{I}_{\xi_m(t)<0}\right)^{1/p} \\
&\le \frac{c_5}{n^{1/2-1/p}}. \qquad (1.31)
\end{aligned}
$$

Combining Gronwall's lemma, (1.30) and (1.31) yields

$$
\sup_{t\in[0,T]} E\big(\xi_n(t)-\xi_m(t)\big)^2 \le c_6\left(\frac{1}{n^{1/2-1/p}}+\frac{1}{m^{1/2-1/p}}\right).
$$

Applying Burkholder's inequality again, we can get

$$
E\sup_{t\in[0,T]}\big(\xi_n(t)-\xi_m(t)\big)^2 \le c_7\left(\frac{1}{n^{1/2-1/p}}+\frac{1}{m^{1/2-1/p}}\right). \qquad (1.32)
$$

Step 6. It follows from (1.32) that there is a continuous non-anticipating process $\widetilde{\xi}(t)$ such that

$$
E\sup_{t\in[0,T]}\Big(\xi_n(t)-\widetilde{\xi}(t)\Big)^2 \to 0, \qquad n\to\infty. \qquad (1.33)
$$

It remains to verify that $\widetilde{\xi}(t)$ is a solution of (1.18). Indeed, (1.33) yields

$$
\begin{aligned}
E\sup_{t\in[0,T]} & \left(\int_0^t a\big(\xi_n(s)\big)\,\mathrm{d}s-\int_0^t a\big(\widetilde{\xi}(s)\big)\,\mathrm{d}s\right)^2 \\
&+E\sup_{t\in[0,T]}\left(\int_0^t b\big(\xi_n(s)\big)\,\mathrm{d}w(s)-\int_0^t b\big(\widetilde{\xi}(s)\big)\,\mathrm{d}w(s)\right)^2 \to 0, \qquad n\to\infty.
\end{aligned}
$$

This implies that there is a continuous process $\widetilde{l}(t)$, $t\in[0,T]$, such that

$$
E\sup_{t\in[0,T]}\left(-n\int_0^t \xi_n(s)\,\mathbb{I}_{\xi_n(s)<0}\,\mathrm{d}s-\widetilde{l}(t)\right)^2 \to 0, \qquad n\to\infty.
$$

This process is non-negative and nondecreasing as a limit of such processes. It is easy to see that $\int_0^T \mathrm{I\!I}_{\widetilde{\xi}(s)>0}\, \mathrm{d}\widetilde{l}(s) = 0$ a.s.

It follows from (1.29) and (1.33) that $\widetilde{\xi}(t) \geq 0$, $t \in [0,T]$, a.s. Thus, $\left(\widetilde{\xi}(t), \widetilde{l}(t)\right)$ is a solution of (1.18). This completes the proof of Theorem 1.4.1.

Remark 1.4.2 The idea of the proof is taken from [42], where the approximation scheme for reflecting SDE in a multidimensional convex set was considered. Note that we can even find some estimates for a rate of convergence using the reasoning above, see also [48, 68, 39].

A penalization coefficient g_n must not necessarily be of the form (1.23). The statement of Theorem 1.4.1 holds true, for example, for $g_n(x) = g(nx)$, where g is a smooth function such that

$$g(x) = 0, \quad x > 0 \qquad \text{and} \qquad g(x) = \frac{1}{x^2}, \quad x \in (0,1);$$

see [52]. In this case a process $\xi_n(t)$ cannot even reach 0.

Chapter 2

Multidimensional reflecting SDEs

2.1 Warm up calculations. Skorokhod's problem in a half-space

In this section we consider the reflecting problem in a half-space $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times [0,\infty)$. On the one hand, we will see a lot of similarities with the one-dimensional case. On the other hand, we will see possible ways of generalization to more complex domains or non-normal reflections.

As in the previous chapter, let us start with a deterministic Skorokhod's problem.

Definition 2.1.1 Let $f = (f_1,\dots,f_d) \in C\big([0,\infty),\mathbb{R}^d\big)$ be a continuous function such that $f(0) \in \mathbb{R}^d_+$. A pair of continuous functions $g = (g_1,\dots,g_d) \in C\big([0,\infty),\mathbb{R}^d\big)$ and $l \in C\big([0,\infty)\big)$ is called a solution of Skorokhod's reflecting problem in $\mathbb{R}^d_+$ with normal reflection at the boundary $\partial\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \{0\}$ if

1) $g(t) \in \mathbb{R}^d_+$, $t \geq 0$;

2) $g(t) = f(t) + n\,l(t)$, $t \geq 0$, where $n = (0,\dots,0,1)$;

3) l is non-decreasing, $l(0) = 0$,

$$\int_0^\infty \mathbb{I}_{g(s)\notin\partial\mathbb{R}^d_+}\, \mathrm{d}l(s) = 0.$$

It is easily seen from the definition that

$$g_1(t) = f_1(t), \ldots, g_{d-1}(t) = f_{d-1}(t),$$

and

$$g_d(t) = f_d(t) + l(t), \quad \int_0^\infty \mathbb{I}_{g_d(s)>0} \, \mathrm{d}l(s) = 0.$$

Thus, the pair (g_d, l) is the exact solution of the one-dimensional Skorokhod problem for f_d. Therefore

$$l(t) = -\min_{s\in[0,t]} f_d(s) \wedge 0, \quad g_d(t) = f_d(t) - \min_{s\in[0,t]} f_d(s) \wedge 0 = \Gamma f_d(t).$$

Define a multidimensional Skorokhod's map by the same symbol:

$$\Gamma f(t) = \Gamma\big(f_1, \ldots, f_d\big)(t) = \big(f_1(t), \ldots, f_{d-1}(t), \Gamma f_d(t)\big).$$

Obviously, all estimates of Lemma 1.1.1 hold true for this situation.

Let us consider a multidimensional reflecting SDE in $\mathbb{R}^d_+$ with normal reflection at the boundary.

Let $a = a(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$, $b_k = b_k(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$, $k = \overline{1,m}$, be measurable functions, $\{w_k(t), t \ge 0\}$, $k = \overline{1,m}$, be independent Wiener processes adapted to a filtration $\mathscr{F}_t$ and ξ_0 be $\mathscr{F}_0$-measurable. Similarly to Definition 1.2.1, we say that a pair of continuous $\mathscr{F}_t$-adapted processes $\big(\xi(t), l(t)\big)$, $t \ge 0$, is a solution of a reflecting SDE

$$\mathrm{d}\xi(t) = a\big(t, \xi(t)\big)\, \mathrm{d}t + \sum_{k=1}^m b_k\big(t, \xi(t)\big)\, \mathrm{d}w_k(t) + n \;\, \mathrm{d}l(t), \quad t \ge 0,$$

in $\mathbb{R}^d_+$ with normal reflection at the boundary and with initial condition $\xi_0 \in \mathbb{R}^d_+$ if the following conditions are a.s. satisfied:

1) $\xi(t) \in \mathbb{R}^d_+$, $t \ge 0$;

2) $l(0) = 0$, l is non-decreasing;

3) $\int_0^t \mathbb{I}_{\xi(s)\notin\partial\mathbb{R}^d_+} \; \mathrm{d}l(s) = 0$, $t \ge 0$;

4) $$\xi(t)=\xi_0+\int_0^t a\big(s,\xi(s)\big)\,\mathrm{d}s+\sum_{k=1}^m\int_0^t b_k\big(s,\xi(s)\big)\,\mathrm{d}w_k(s)+n\,l(t),\quad t\ge 0, \tag{2.1}$$
and all the integrals are well-defined.

In coordinate form, equation (2.1) is as follows

$$\begin{aligned}
\mathrm{d}\xi_1(t)&=a_1\big(t,\xi(t)\big)\,\mathrm{d}t+\sum_{k=1}^m b_{k,1}\big(t,\xi(t)\big)\,\mathrm{d}w_k(t),\\
&\vdots\\
\mathrm{d}\xi_{d-1}(t)&=a_{d-1}\big(t,\xi(t)\big)\,\mathrm{d}t+\sum_{k=1}^m b_{k,d-1}\big(t,\xi(t)\big)\,\mathrm{d}w_k(t),\\
\mathrm{d}\xi_d(t)&=a_d\big(t,\xi(t)\big)\,\mathrm{d}t+\sum_{k=1}^m b_{k,d}\big(t,\xi(t)\big)\,\mathrm{d}w_k(t)+\mathrm{d}l(t).
\end{aligned}$$

If we denote

$$Y(t)=\xi_0+\int_0^t a\big(s,\xi(s)\big)\,\mathrm{d}s+\sum_{k=1}^m b_k\big(s,\xi(s)\big)\,\mathrm{d}w_k(s),$$

then $\xi(t)=\big(\Gamma Y\big)(t)$ (compare with (1.6), (1.7)) and

$$Y(t)=\xi_0+\int_0^t a\big(s,\big(\Gamma Y\big)(s)\big)\,\mathrm{d}s+\sum_{k=1}^m\int_0^t b_k\big(s,\big(\Gamma Y\big)(s)\big)\,\mathrm{d}w_k(s).$$

Since the multidimensional Skorokhod's map Γ satisfies the Lipschitz condition in a space of continuous functions (see Lemma 1.1.1), we have the following existence and uniqueness theorem. The Markov property can also be proved similarly to Theorem 1.2.2.

Theorem 2.1.1 Assume that functions a and $b_k, k=\overline{1,m}$, satisfy the

1) global Lipschitz condition in x:

$$\exists L\ \forall t\ge 0\ \forall x_1,x_2\in\mathbb{R}_+^d:\ \big|a(t,x_1)-a(t,x_2)\big|+\sum_{k=1}^m\big|b_k(t,x_1)-b_k(t,x_2)\big|\le L|x_1-x_2|;$$

2) linear growth condition in x:

$$\exists C\ \forall t \geq 0\ \forall x \in \mathbb{R}_+^d : \big|a(t,x)\big| + \sum_{k=1}^{m} \big|b_k(t,x)\big| \leq C\big(1+|x|\big).$$

Then there exists a unique solution to (2.1). The process $\xi(t)$, $t \geq 0$, is a Markov process.

Exercise 2.1.1 Solve Exercise 1.2.1 under the conditions of Theorem 2.1.1.

Remark 2.1.1 It is well known, see e.g. [36], that an SDE in the entire Euclidean space $\mathbb{R}^d$ with smooth bounded coefficients generates a flow of diffeomorphisms of $\mathbb{R}^d$. This is not true for reflecting SDEs. Assume for simplicity that coefficients of (2.1) are time-homogeneous, infinite-differentiable with bounded derivatives and that diffusion is not degenerate everywhere. Then for any fixed $t > 0$, a map $\mathbb{R}_+^d \ni x \to \xi_x(t) \in \mathbb{R}_+^d$ is not an injection with probability 1. Moreover, this map does not belong to a class $C^1\left(\mathbb{R}_+^d, \mathbb{R}_+^d\right)$ (see Exercise 1.2.3 for one-dimensional case). Nevertheless $\xi_\cdot(t) \in \bigcap_{p\geq 1} W^1_{p,\mathrm{loc}}(\mathbb{R}_+^d, \mathbb{R}_+^d)$ and its Sobolev derivative satisfies a particular stochastic equation. See results on stochastic reflecting flows in [3, 11, 12, 49, 50, 51, 52, 53, 54].

2.2 Skorokhod's problem in a domain. Definition and preliminaries

As we have seen in the previous section, if one is able to prove nice properties of the Skorokhod map, then the theorem on existence and uniqueness for reflecting SDEs can be easily proved. We now give an abstract definition of the Skorokhod problem in any domain. We even assume the possibility of a multivalued reflecting vector field. This may be useful if we consider domains with a non-smooth boundary; for example, if we need to define a direction of a reflection when a process visits a vertex of a corner.

Let $D \subset \mathbb{R}^d$ be an open set with a boundary ∂D. By $\overline{D}$, denote the closure of D. Assume that, for any $x \in \partial D$, a non-empty set of reflecting directions K_x is given. We will assume that $|v| \neq 0$ for any $v \in K_x$. Let $f \in C\big([0,\infty), \mathbb{R}^d\big)$, $f(0) \in \overline{D}$.

Definition 2.2.1 A pair of continuous functions $(g,l) : [0,\infty) \to \mathbb{R}^d \times [0,\infty)$ is a solution of the Skorokhod problem for (D,K,f) if, for any $t \geq 0$,

1) $g(t) = f(t) + \int_0^t \nu\big(g(s)\big)\, \mathrm{d}l(s),$ (2.2)

2) $g(t) \in \overline{D}$, $\nu\big(g(t)\big) \in K_{g(t)}$ if $g(t) \in \partial D$,

3) l is a non-decreasing function, $l(0) = 0$, and

$$\int_0^t \Big|\nu\big(g(s)\big)\Big|\, \mathrm{d}l(s) < \infty, \quad \int_0^t 1_{g(s)\notin\partial G}\, \mathrm{d}l(s) = 0, \quad t \geq 0.$$

If the Skorokhod problem has a unique solution, we denote g by Γf and call Γ the Skorokhod map.

Remark 2.2.1 Sometimes it is convenient to assume that $|\nu| = 1$ for $\nu \in K_x$, $x \in \partial D$. If we denote $\varphi(t) := \int_0^t \nu\big(g(s)\big)\, \mathrm{d}l(s)$, then $\varphi(t) = \int_0^t \nu\big(g(s)\big)\, \mathrm{d}|\varphi|(s)$, where $|\varphi|(t)$ is the total variation of φ on $[0,t]$.

Remark 2.2.2 Let $K'_x = \{c(x)\nu(x) :\ \nu(x) \in K\}$, where c is a positive continuous function. Then (g,l) is a solution of the Skorokhod problem for (D,K,f) iff (g,l') is a solution of the Skorokhod problem for (D,K',f), where $l'(t) = \int_0^t c^{-1}\big(g(s)\big)\, \mathrm{d}l(s)$.

Remark 2.2.3 If the set K_x contains only one element $\nu(x)$, then we interpret (2.2) as a reflecting problem with reflection along a vector field ν. In particular, the case $D = \mathbb{R}^d_+$ with normal reflection was considered in the previous section. We will also sometimes say that g is a solution of (2.2) without mentioning l and conditions 2) and 3) of Definition 2.2.1.

Now let $\{\mathscr{F}_t, t \geq 0\}$ be a filtration satisfying usual assumptions; $\{w_k(t), t \geq 0\}$, $k = \overline{1,m}$, be independent Wiener processes adapted to $\{\mathscr{F}_t, t \geq 0\}$; $a = a(t,x)$, $b_k = b_k(t,x)$, $k = \overline{1,m}$, be measurable functions; $\xi_0 \in \overline{D}$ be $\mathscr{F}_0$-measurable random variable.

Definition 2.2.2 A pair $\big(\xi(t), l(t)\big)$ of continuous $\mathscr{F}_t$-adapted processes is a solution of a reflecting SDE

$$\mathrm{d}\xi(t) = a\big(t,\xi(t)\big)\, \mathrm{d}t + \sum_{k=1}^m b_k\big(t,\xi(t)\big)\, \mathrm{d}w_k(t) + \nu\big(\xi(t)\big)\, \mathrm{d}l(t), \quad t \geq 0, \tag{2.3}$$

with initial condition $\xi(0) = \xi_0$ if

$$\xi(t) \in \overline{D}, \quad t \geq 0;$$

l is non-decreasing, $l(0) = 0$,

$$\int_0^t \Big| v\big(\xi(s)\big) \Big| \, \mathrm{d}l(s) < \infty, \quad \int_0^t \mathrm{I\!I}_{\xi(s) \notin \partial D} \, \mathrm{d}l(s) = 0, \quad t \geq 0,$$

where $v\big(\xi(s)\big) \in K_{\xi(s)}$ if $\xi(s) \in \partial D$; for $t \geq 0$,

$$\xi(t) = \xi_0 + \int_0^t a\big(s, \xi(s)\big) \, \mathrm{d}s + \sum_{k=1}^m \int_0^t b_k\big(s, \xi(s)\big) \, \mathrm{d}w_k(s) + \int_0^t v\big(\xi(s)\big) \, \mathrm{d}l(s), \quad (2.4)$$

and all the integrals in (2.4) are well defined.

Remark 2.2.4 Sometimes we say that "$\xi(t)$ is a solution of (2.3)." In this case we always have the process $l(t)$ in mind as well.

Remark 2.2.5 Together with (2.2) and (2.4), equations with a time-dependent vector field $v = v(t, x)$ can be considered (the corresponding definition is similar). This situation is investigated in §3.1, 3.2; however, almost all results of this chapter (except §2.5) have natural generalisations to the case of time-dependent reflection.

We have seen in the previous section that there is a strong relationship between properties of Skorokhod's problem and reflecting SDEs. Namely, assume that Skorokhod's problem (2.2) has a unique solution and the Skorokhod map Γ is a continuous mapping in $C\big([0,\infty), \mathbb{R}^d\big)$. This implies that the map $\Gamma : C\big([0,\infty), \mathbb{R}^d\big) \to C\big([0,\infty), \mathbb{R}^d\big)$ is non-anticipative. Observe that the process $\xi(t)$ from (2.3) is a solution of the Skorokhod problem for (D, K, η) with

$$\eta(t) = \xi_0 + \int_0^t a\big(s, \xi(s)\big) \, \mathrm{d}s + \sum_{k=1}^m \int_0^t b_k\big(s, \xi(s)\big) \, \mathrm{d}w_k(s), \quad t \geq 0. \quad (2.5)$$

That is, $\eta(t)$, $t \geq 0$, is a solution of Itô's equation

$$\eta(t) = \xi_0 + \int_0^t a\big(s, \Gamma\eta(s)\big) \, \mathrm{d}s + \sum_{k=1}^m \int_0^t b_k\big(s, \Gamma\eta(s)\big) \, \mathrm{d}w_k(s). \quad (2.6)$$

Vice versa, if η is a solution of (2.6), then $\xi = \Gamma\eta$ is a solution of (2.4).

Remark 2.2.6 Since Γ is a non-anticipative map in $C\big([0,\infty), \mathbb{R}^d\big)$, for any continuous $\mathscr{F}_t$-adapted process $\eta(t)$, $t \geq 0$, the process $\big(\Gamma\eta\big)(t)$, $t \geq 0$, is also continuous and $\mathscr{F}_t$-adapted.

Exercise 2.2.1 Prove Theorem 2.1.1 for a general domain D and reflection directions K_x if Γ is a Lipschitz map in a space of continuous functions.

Remark 2.2.7 Notice that the local Lipschitz condition is usually sufficient for uniqueness of a solution to Itô's SDE. If we could additionally ensure that a solution does not blow up, then we are able to prove the global existence too. This means that an investigation of properties of deterministic Skorokhod's maps such as continuity, the Lipschitz condition or the local Lipschitz condition, etc., is very important for the study of reflecting SDEs. We already know that the Skorokhod map in a half-plane with normal reflection is Lipschitz continuous. In the next section, we use a change of variables and localization to reduce some reflecting SDEs to the known case.

Example 2.2.1 (Skorokhod's problem with oblique reflection in a half-space) Let $D=\mathbb{R}^{d-1}\times(0,\infty)$, $v(x)$, $x\in\partial\mathbb{R}^d_+$, be a vector field with values in $\mathbb{R}^d$, f be a continuous function with values in $\mathbb{R}^d$, $f(0)\in\mathbb{R}^d_+$.

Consider the Skorokhod problem

$$g(t)=f(t)+\int_0^t v\big(g(s)\big)\,\mathrm{d}l(s), \tag{2.7}$$

where l is a non-decreasing continuous function, $l(0)=0$,

$$l(t)=\int_0^t \mathrm{I\!I}_{g(s)\in\partial\mathbb{R}^d_+}\,\mathrm{d}l(s).$$

Assume that the scalar product $(v(x),n)$ is equal to 1 where $n=(0,\dots,0,1)$, i.e.,

$$v(x)=\big(v_1(x),\dots,v_{d-1}(x),1\big). \tag{2.8}$$

Let us write (2.7) in the coordinate form

$$\begin{aligned}
g_1(t)&=f_1(t)+\int_0^t v_1\big(g(s)\big)\,\mathrm{d}l(s), && t\ge 0,\\
&\vdots\\
g_{d-1}(t)&=f_{d-1}(t)+\int_0^t v_{d-1}\big(g(s)\big)\,\mathrm{d}l(s), && t\ge 0,\\
g_d(t)&=f_d(t)+l(t), && t\ge 0.
\end{aligned} \tag{2.9}$$

The last equality in (2.9) means that g_d is a solution of the one-dimensional Skorokhod problem. Therefore

$$l(t) = -\min_{s\in[0,t]} \big(f_d(s)\wedge 0\big)$$

and

$$g_d(t) = f_d(t) - \min_{s\in[0,t]} \big(f_d(s)\wedge 0\big).$$

The first $(d-1)$ equations in (2.9) are integral equations. If v satisfies the Lipschitz condition, then there is a unique solution to (2.9).

Exercise 2.2.2 Assume that ν satisfies the Lipschitz condition. Let $\{f^{(n)}\}$ converge to f uniformly on $[0,T]$. Prove the uniform convergence

$$l^{(n)} \rightrightarrows l \quad \text{and} \quad g^{(n)} \rightrightarrows g \quad \text{as } n\to\infty,$$

on $[0,T]$.

Exercise 2.2.3 Assume that v is continuous and $\big(v(x),n\big) < 0$ for all x from some open set of $\partial\mathbb{R}^d_+$. Construct an example when a solution of the Skorokhod problem does not exist.

Exercise 2.2.4 Let $v(x) = v = \text{const}$, $x\in\mathbb{R}^d_+$, $(v,n) > 0$. Find a solution of the Skorokhod problem and prove that Skorokhod's map is Lipschitz continuous.

Assume that v is a Lipschitz function and that $\big(v(x),n\big) = 1$, $x\in\partial\mathbb{R}^d_+$. Let us make the traditional estimates in (2.9) and try to prove that Skorokhod's map is Lipschitz continuous.

Let $f^{(1)}, f^{(2)} \in C\big([0,T],\mathbb{R}^d\big)$, $f^{(1)}(0), f^{(2)}(0)\in\mathbb{R}^d_+$. We know from §1.1 that

$$\|l^{(1)} - l^{(2)}\| \le 2\|f^{(1)} - f^{(2)}\|,$$

where $\|f\| = \max_{t\in[0,T]} |f(t)|$. We have

$$\begin{aligned}
\big|g^{(1)}(t) - g^{(2)}(t)\big| \\
&\le 2\big|f^{(1)}(t) - f^{(2)}(t)\big| + \left|\int_0^t v\big(g^{(1)}(s)\big)\,\mathrm{d}l^{(1)}(s) - \int_0^t v\big(g^{(2)}(s)\big)\,\mathrm{d}l^{(2)}(s)\right| \\
&\le \big|f^{(1)}(t) - f^{(2)}(t)\big| + c\int_0^t \big|g^{(1)}(s) - g^{(2)}(s)\big|\,\mathrm{d}l^{(1)}(s)
\end{aligned}$$

$$+\left|\int_0^t v\big(g^{(2)}(s)\big)\,\mathrm{d}\left(l^{(1)}(s)-l^{(2)}(s)\right)\right|, \tag{2.10}$$

where c is the Lipschitz constant of v. If we remove the third term in the right hand side of (2.10) and apply Gronwall's lemma to (2.10), then the bound for $|g^{(1)}(t)-g^{(2)}(t)|$ will be at least $\exp\left\{cl^{(1)}(t)\right\}\left|f^{(1)}(t)-f^{(2)}(t)\right|$. Recall that $l^{(1)}(t)=-\min_{s\in[0,t]}\left(f_d^{(1)}(s)\wedge 0\right)$ depends on $f^{(1)}$.

Therefore, the Lipschitz constant might be non-global and might depend on $\|f\|$. But in a reflecting SDE, the role of f has an unknown process $\eta(t)$, that depends on ξ, see (2.5). This adds some difficulties to the study of reflecting SDEs in a general domain.

Note that we forgot to consider the third term in (2.10). It is questionable that a function of the form

$$\int_0^t v\big(g(s)\big)\,\mathrm{d}\left(l^{(1)}(s)-l^{(2)}(s)\right)$$

can be bounded by $\text{const}\cdot\|f^{(1)}-f^{(2)}\|$ if we do not have some specific assumptions on g. This example shows that in very simple situations, the traditional approach might be inapplicable. Surely, some other way to show the Lipschitz property may exist.

2.3 Weak solution of reflecting SDEs. Existence and convergence

Definition 2.3.1 We say that (2.3) has a weak solution if there exists a probability space with filtration $(\Omega,\mathscr{F},P,\mathscr{F}_t)$, $\mathscr{F}_t$-Wiener processes $\{w_k(t),t\geq 0\}$, $k=\overline{1,m}$ and processes $\{\xi(t),t\geq 0\}$, $\{\varphi(t),t\geq 0\}$ that satisfy all conditions given in §2.2.

Theorem 2.3.1 Assume that the Skorokhod map of a reflecting problem for (D,K,f) is uniquely defined and continuous in f. In other words, if $f_n\in C\big([0,\infty),\mathbb{R}^d\big)$, $f_n(0)\in\overline{D}$, $n\geq 0$ are such that

$$\forall\, T>0:\ \max_{t\in[0,T]}\left|f_n(t)-f_0(t)\right|\to 0,\quad n\to\infty,$$

then

$$\forall\, T>0:\ \max_{t\in[0,T]}\left|\big(\Gamma f_n\big)(t)-\big(\Gamma f_0\big)(t)\right|\to 0,\quad n\to\infty.$$

Suppose that $a = a(t,x)$, $b_k = b_k(t,x)$ are bounded functions which are continuous in (t,x). Then there exists a weak solution to equation (2.3).

Proof. Let $\varphi_n(t) = \frac{k}{n}$, $t \in \left[\frac{k}{n}, \frac{k+1}{n}\right)$. Consider a sequence of reflecting SDEs

$$\mathrm{d}\,\xi_n(t) = a\Big(t, \xi_n\big(\varphi_n(t)\big)\Big)\,\mathrm{d}t + \sum_{k=1}^{m} b_k\Big(t, \xi_n\big(\varphi_n(t)\big)\Big)\,\mathrm{d}w_k(t) + \nu\big(\xi_n(t)\big)\,\mathrm{d}l_n(t), \quad t \ge 0, \tag{2.11}$$

with the initial condition $\xi_n(0) = \xi_0$. Observe that (2.11) can be solved successively on each interval $\left[\frac{k}{n}, \frac{k+1}{n}\right)$ similarly to §1.4. The existence and uniqueness of a solution to the Skorokhod problem ensures the existence and uniqueness of a solution to (2.11).

Lemma 2.3.1 Let $\big\{\alpha_n(t), t \in [0,T]\big\}$, $n \ge 1$ be a sequence of measurable $\mathscr{F}_t$-adapted processes. Assume that

$$\sup_n \sup_t \sup_\omega \big|\alpha_n(t,\omega)\big| < \infty.$$

Then distributions of sequences $\big\{\int_0^t \alpha_n(s)\,\mathrm{d}s, t \in [0,T]\big\}_{n\ge1}$ and $\big\{\int_0^t \alpha_n(s)\,\mathrm{d}w_k(s), t \in [0,T]\big\}_{n\ge1}$ are weakly relatively compact in $C\big([0,T],\mathbb{R}^d\big)$.

A proof can be found in, e.g. [30].

Denote

$$\eta_n(t) = \xi_0 + \int_0^t a\Big(s, \xi_n\big(\varphi_n(s)\big)\Big)\,\mathrm{d}s + \sum_{k=1}^{m} \int_0^t b_k\Big(s, \xi_n\big(\varphi_n(s)\big)\Big)\,\mathrm{d}w_k(s).$$

Then,

$$\xi_n(t) = \Gamma\eta_n(t).$$

It follows from Lemma 2.3.1 that if the distributions of

$$\int_0^t a\Big(s, \xi_n\big(\varphi_n(s)\big)\Big)\,\mathrm{d}s, \quad \int_0^t b_k\Big(s, \xi_n\big(\varphi_n(s)\big)\Big)\,\mathrm{d}w_k(s), \qquad n \ge 1,$$

are weakly relatively compact in $C\big([0,T],\mathbb{R}^d\big)$, then so are the distributions of $\{\eta_n, n \ge 1\}$. Since Γ is continuous, distributions of $\{\xi_n, n \ge 1\}$ are also weakly relatively compact. Select a subsequence $\{n_k, k \ge 1\}$ such that all mentioned sequences are weakly convergent. Without loss of generality, we will assume that these sequences are weakly convergent themselves.

We need the following Skorokhod's representation theorem; see [65] or [31] for more information.

Theorem 2.3.2 Let $\{\zeta_n, n \geq 1\}$ be a sequence of random elements with values in a complete separable metric space. Assume that the distributions of $\{\zeta_n, n \geq 1\}$ converge weakly. Then, there is a probability space and a sequence of random elements $\{\widetilde{\zeta}_n, n \geq 1\}$ defined on it such that

1) $\widetilde{\zeta}_n \stackrel{d}{=} \zeta_n$, $n \geq 1$,

2) the sequence $\{\widetilde{\zeta}_n, n \geq 1\}$ converges almost surely.

Let us apply Skorokhod's representation theorem to the sequence

$$\left\{ \left(\eta_n(\,\cdot\,), \xi_n(\,\cdot\,), \int_0^{\cdot} a\Big(s, \xi_n\big(\varphi_n(s)\big)\Big)\, \mathrm{d}s, \int_0^{\cdot} b_k\Big(s, \xi_n\big(\varphi_n(s)\big)\Big)\, \mathrm{d}w_k(s), w_k(\,\cdot\,), \right.\right.$$
$$\left.\left. k = \overline{1,m} \right) \right\}_{n \geq 1},$$

where the metric space is the space of functions which are continuous on $[0,T]$ with values in $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{dm} \times \mathbb{R}^m$.

We obtain a sequence of copies

$$X_n = \Big(\widetilde{\eta}_n, \widetilde{\xi}_n, \widetilde{A}_n, \widetilde{B}_{n,k}, \widetilde{w}_{n,k}, k = \overline{1,m}\Big), \quad n \geq 1$$

that converges almost surely as $n \to \infty$. Let $\widetilde{\mathscr{F}_t^n}$ be a filtration generated by $\{X_n(s), s \in [0,t]\}$, completed by sets of null probability. It is easy to see that $\big\{\widetilde{w}_{n,k}(t), t \in [0,T]\big\}$, $k = \overline{1,m}$, are independent $\widetilde{\mathscr{F}_t^n}$-Wiener processes (they may depend on n). It can be seen that $\widetilde{\xi}_n = \Gamma \widetilde{\eta}_n$,

$$\widetilde{A}_n(t) = \int_0^t a\Big(s, \widetilde{\xi}_n\big(\varphi_n(s)\big)\Big)\, \mathrm{d}s \quad \text{a.s.,} \tag{2.12}$$

$$\widetilde{B}_{n,k}(t) = \int_0^t b_k\Big(s, \widetilde{\xi}_n\big(\varphi_n(s)\big)\Big)\, \mathrm{d}\widetilde{w}_{n,k}(s) \quad \text{a.s.} \tag{2.13}$$

By $X = \Big(\widetilde{\eta}, \widetilde{\xi}, \widetilde{A}, \widetilde{B}_k, \widetilde{w}_k, k = \overline{1,m}\Big)$ denote the limit of $\{X_n\}$. Let us pass to the limit in the equation

$$\widetilde{\eta}_n(t) = \widetilde{\xi}_n(0) + \int_0^t a\Big(s, \widetilde{\xi}_n\big(\varphi_n(s)\big)\Big)\,\mathrm{d}s + \sum_{k=1}^m \int_0^t b_k\Big(s, \widetilde{\xi}_n\big(\varphi_n(s)\big)\Big)\,\mathrm{d}\widetilde{w}_{n,k}(s).$$

We have uniform convergence, i.e.

$$\widetilde{\xi}_n(\,\cdot\,) \rightrightarrows \widetilde{\xi}(\,\cdot\,), \qquad n \to \infty \quad \text{a.s.}$$

By continuity of a in x, boundedness of a and Lebesgue's dominated convergence theorem, we have

$$\int_0^{\cdot} a\Big(s, \widetilde{\xi}_n\big(\varphi_n(s)\big)\Big)\,\mathrm{d}s \rightrightarrows \int_0^{\cdot} a\big(s, \widetilde{\xi}(s)\big)\,\mathrm{d}s, \quad n \to \infty \quad \text{a.s.} \tag{2.14}$$

Since Γ is continuous, convergence (in the space of continuous functions)

$$\widetilde{\eta}_n \to \widetilde{\eta}, \qquad n \to \infty \quad \text{a.s.}$$

yields

$$\Gamma\widetilde{\eta}_n \to \Gamma\widetilde{\eta}, \qquad n \to \infty \quad \text{a.s.}$$

Note that $\Gamma\widetilde{\eta}_n = \widetilde{\xi}_n \circ \varphi_n \to \widetilde{\xi}$, $n \to \infty$ a.s. So $\Gamma\widetilde{\eta} = \widetilde{\xi}$ a.s. To conclude the proof, it suffices to verify that $\widetilde{\eta}(t)$ satisfies the SDE

$$\widetilde{\eta}(t) = \widetilde{\xi}(0) + \int_0^t a\big(s, \Gamma\widetilde{\eta}(s)\big)\,\mathrm{d}s + \sum_{k=1}^m \int_0^t b_k\big(s, \Gamma\widetilde{\eta}(s)\big)\,\mathrm{d}\widetilde{w}_k(s).$$

It remains to check the convergence of stochastic integrals

$$\int_0^t b_k\Big(s, \widetilde{\xi}_n\big(\varphi_n(s)\big)\Big)\,\mathrm{d}\widetilde{w}_{n,k}(s) \to \int_0^t b_k\big(s, \widetilde{\xi}(s)\big)\,\mathrm{d}\widetilde{w}_k(s), \qquad n \to \infty. \tag{2.15}$$

The application of the following result completes the proof of Theorem 2.3.1. □

Theorem 2.3.3 ([65]) Let $\{\bar{w}_n(t), t \in [0,T]\}$, $n \geq 0$, be a sequence of $\mathscr{G}_t^n$-Wiener processes, where $\mathscr{G}_t^n$ are some filtrations. Assume that $\mathscr{G}_t^n$-adapted measurable processes

$g_n(t), t \in [0,T], n \geq 0$, are such that

$$\int_0^T E g_n^2(s)\, \mathrm{d}s < \infty, \quad n \geq 0,$$

and

$$\int_0^T E\big(g_n(s) - g_0(s)\big)^2\, \mathrm{d}s < \infty.$$

Then

$$E\left[\int_0^T g_n(s)\, \mathrm{d}\bar{w}_n(s) - \int_0^T g_0(s)\, \mathrm{d}\bar{w}_0(s)\right]^2 \to 0, \quad n \to \infty.$$

Remark 2.3.1 We assume boundedness of a and b_k in Theorem 2.3.1 only to have simple conditions ensuring weak compactness of integrals and the possibility to pass to the limit in (2.14) and (2.15). If we have some growth conditions for a and b_k and some a priori moments estimates of solutions of (2.11), then the proof can be done similarly.

Remark 2.3.2 The corresponding idea of the proof of a weak solution existence for an SDE without reflection belongs to Skorokhod, see [64, 66]. For the proof for a reflecting SDE in a domain, see [45, 61]. Another effective method to prove the existence or convergence of weak solutions is based on an investigation of a submartingale problem proposed by Stroock and Varadhan [71], see §3.2 further.

The following theorem gives us a continuous dependence of the solution of a reflecting SDE on its equation's coefficients.

Theorem 2.3.4 Let $\xi_n(t),\ t \in [0,T], n \geq 0$, be weak (or strong) solutions to the SDE

$$\begin{cases} \mathrm{d}\,\xi_n(t) = a_n\big(t, \xi_n(t)\big)\, \mathrm{d}t + \sum_{k=1}^{m} b_{n,k}\big(s, \xi_n(t)\big)\, \mathrm{d}w_k(t) + v\big(\xi_n(t)\big)\, \mathrm{d}l_n(t), \quad t \geq 0 \\ \xi_n(0) = \varepsilon_n, \end{cases} \tag{2.16}$$

where ε_n are $\mathscr{F}_0$-measurable and the reflection vector field v is the same for all $n \geq 0$. Assume that

1) functions $\{a_n, b_{n,k}\}$ are bounded by the same constant:

$$\sup_{t\in[0,T]} \sup_x \sup_n \left(\big|a_n(t,x)\big| + \sum_{k=1}^{m} \big|b_{n,k}(t,x)\big| \right) \leq c < \infty;$$

2) coefficients converge locally uniformly in x, i.e.

$$\forall N\ \forall t \in [0,T]: \lim_{n\to\infty} \sup_{|x|\le N} \left(\left|a_n(t,x) - a_0(t,x)\right| + \sum_{k=1}^{m} \left|b_{n,k}(t,x) - b_{0,k}(t,x)\right| \right) = 0; \tag{2.17}$$

3) for any $t \in [0,T]$, functions $a_0(t,\cdot)$ and $b_{0,k}(t,\cdot)$ are continuous in x;

4) Skorokhod's map for reflection vector field v is continuous;

5) the initial conditions converge: $\varepsilon_n \to \varepsilon_0$, $n \to \infty$ weakly (or in probability);

6) equation (2.16) has a unique weak (or strong) solution for $n = 0$.

Then we have convergence $\xi_n \to \xi_0$, $n \to \infty$ in distribution in $C\big([0,T],\mathbb{R}^d\big)$ (or uniformly in probability if the solutions were strong).

Proof. Similarly to the proof of the previous theorem, a sequence of processes

$$X_n = \left(\eta_n(\cdot), \xi_n(\cdot), \int_0^{\cdot} a_n\big(s,\xi_n(s)\big)ds, \int_0^{\cdot} b_{n,k}\big(s,\xi_n(s)\big)\,\mathrm{d}w_k(s), w_k(\cdot), k = \overline{1,m} \right), \quad n \ge 1$$

is weakly relatively compact in a space of continuous functions on $[0,T]$.

To prove the convergence, it suffices to verify that, for any subsequence $\{\xi_{n_k}\}$, there is a sub-subsequence $\{\xi_{n_{k_l}}\}$ that converges to ξ_0. So, without loss of generality we may assume that a sequence $\{X_n, n \ge 1\}$ is weakly convergent. We will show that the limit is X_0.

By Skorokhod's representation theorem, construct a sequence of copies $\widetilde{X}_n$ that converges almost surely (in the space of continuous functions):

$$\left(\widetilde{\eta}_n, \widetilde{\xi}_n, \int_0^{\cdot} a_n\big(s,\widetilde{\xi}_n(s)\big)\,\mathrm{d}s, \int_0^{\cdot} b_{n,k}\big(s,\widetilde{\xi}_n(s)\big)\,\mathrm{d}\widetilde{w}_{n,k}(s), \widetilde{w}_{n,k}(\cdot), k = \overline{1,m} \right)$$
$$\to \left(\widetilde{\eta}_0, \widetilde{\xi}_0, A_0, B_k, \widetilde{w}_k, k = \overline{1,m} \right), \qquad n \to \infty \quad \text{a.s.}$$

It follows from the assumptions of the theorem that

$$\int_0^t a_n\big(s,\widetilde{\xi}_n(s)\big)\,\mathrm{d}s \to \int_0^t a_0\big(s,\widetilde{\xi}_0(s)\big)\,\mathrm{d}s, \qquad n \to \infty \quad \text{a.s.} \tag{2.18}$$

and by Theorem 2.3.3, we have that

$$\int_0^t b_{n,k}\big(s,\widetilde{\xi}_n(s)\big)\,\mathrm{d}\widetilde{w}_{n,k}(s) \xrightarrow{L_2} \int_0^t b_{0,k}\big(s,\widetilde{\xi}_0(s)\big)\,\mathrm{d}\widetilde{w}_k(s), \qquad n\to\infty. \tag{2.19}$$

By continuity of Γ,

$$\widetilde{\xi}_n = \Gamma\widetilde{\eta}_n \to \Gamma\widetilde{\eta}_0, \qquad n\to\infty \quad \text{a.s.}$$

So $\Gamma\widetilde{\eta}_0 = \widetilde{\xi}_0$ and $\widetilde{\xi}_0$ is a solution of the limit reflecting SDE, and as the solution must be unique, $\widetilde{\xi}_0 \stackrel{d}{=} \xi_0$. Hence the convergence

$$\widetilde{\xi}_n \to \widetilde{\xi}_0, \qquad n\to\infty \quad \text{a.s.}$$

in $C\big([0,T],\mathbb{R}^d\big)$ also yields the weak convergence

$$\xi_n \Rightarrow \xi_0, \qquad n\to\infty.$$

If all solutions $\{\xi_n, n\ge 0\}$ are strong, consider a sequence

$$X_n = \big(\eta_n,\eta_0,\xi_n,\xi_0,w_k,k=\overline{1,m}\big)$$

and a corresponding sequence of copies

$$\widetilde{X}_n = \big(\widetilde{\eta}_n,\widetilde{\eta}_{n,0},\widetilde{\xi}_n,\widetilde{\xi}_{n,0},\widetilde{w}_{n,k},k=\overline{1,m}\big)$$

that converges a.s. to

$$\widetilde{X}_0 = \big(\widetilde{\eta}_0,\widetilde{\eta}_{0,0},\widetilde{\xi}_0,\widetilde{\xi}_{0,0},\widetilde{w}_k,k=\overline{1,m}\big).$$

As before

$$\widetilde{\xi}_n \to \widetilde{\xi}_0, \qquad n\to\infty \quad \text{a.s.}$$

and

$$\widetilde{\xi}_{n,0} \to \widetilde{\xi}_{0,0}, \qquad n\to\infty \quad \text{a.s.},$$

where $\widetilde{\xi}_0$ and $\widetilde{\xi}_{0,0}$ are solutions of the same reflecting SDE with the same Wiener processes $\widetilde{w}_k$.

By uniqueness of the strong solution, we have the equality $\widetilde{\xi}_0 = \widetilde{\xi}_{0,0}$ a.s. Therefore the convergence

$$\widetilde{\xi}_n - \widetilde{\xi}_{n,0} \to 0, \qquad n \to \infty \quad \text{a.s.},$$

implies the convergence in probability

$$\xi_n - \xi_0 \xrightarrow{P} 0, \qquad n \to \infty$$

because $\widetilde{\xi}_n - \widetilde{\xi}_{n,0} \stackrel{d}{=} \xi_n - \xi_0$. □

Remark 2.3.3 The assumption of locally uniform convergence of coefficients in (2.17) can be relaxed. Indeed, we used (2.17) only in the proof of (2.18) and (2.19). To verify them, it is sufficient to prove convergence in probability for $a_n\big(s, \widetilde{\xi}_n(s)\big) \to a_0\big(s, \widetilde{\xi}_0(s)\big)$ and $b_{n,k}\big(s, \widetilde{\xi}_n(s)\big) \to b_{0,k}\big(s, \widetilde{\xi}_0(s)\big)$ as $n \to \infty$. If we have some uniform a priori estimates of transition densities for $\xi_n(s)$, $n \geq 0$, then pointwise convergence of the coefficients is sufficient. This follows from the next result.

Lemma 2.3.2 Let X and Y be complete separable metric spaces and $(\Omega, \mathscr{F}, P)$ be a probability space. Let $\eta_n : \Omega \to X$, $h_n : X \to Y$, $n \geq 0$, be measurable mappings such that

1) $\eta_n \to \eta_0$, $n \to \infty$, in probability;
2) $h_n \to h_0$, $n \to \infty$, in measure ν, where ν is a probability measure on X;
3) for all $n \geq 1$, the distribution P_{η_n} of η_n is absolutely continuous w.r.t. the measure ν;
4) the sequence of densities $\left\{\frac{\mathrm{d}P_{\eta_n}}{\mathrm{d}\nu} : n \geq 1\right\}$ is uniformly integrable w.r.t. the measure ν.

Then $h_n(\eta_n) \to h_0(\eta_0)$, $n \to \infty$, in probability.

The proof can be found, for example, in [8], Corollary 9.9.11 or [16], Lemma 2.

• • • • • • • • • • •

2.4 Localization

Let $D \subset \mathbb{R}^d$ be an open set with a smooth boundary ∂D and $\xi(t)$, $t \geq 0$, be a solution of the reflecting SDE in $\overline{D}$:

$$\begin{cases} \mathrm{d}\xi(t) = a\big(t, \xi(t)\big)\,\mathrm{d}t + \sum_{k=1}^{m} b_k\big(t, \xi(t)\big)\,\mathrm{d}w_k(t) + \nu\big(\xi(t)\big)\,\mathrm{d}l(t), \quad t \geq 0, \\ \xi(0) = \xi_0, \end{cases} \tag{2.20}$$

where v is a single-valued vector field. For simplicity, we assume that a and b_k are locally bounded.

Assume that there exists a twice continuously differentiable function $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ such that φ is an injection in some neighborhood of $\overline{D}$. This implies that the inverse map is also C^2 in some neighborhood of $\varphi(\overline{D})$.

Denote $D' = \varphi(D)$, $\xi'(t) = \varphi\big(\xi(t)\big)$. Then by Itô's formula,

$$\begin{aligned} \mathrm{d}\,\xi'(t) &= L_t\varphi\big(t,\xi(t)\big)\,\mathrm{d}t + \sum_{k=1}^{m} \nabla\varphi\big(\xi(t)\big) b_k\big(t,\xi(t)\big)\,\mathrm{d}w_k(t) + \nabla\varphi\big(\xi(t)\big) v\big(\xi(t)\big)\,\mathrm{d}l(t) \\ &= a'\big(t,\xi'(t)\big)\,\mathrm{d}t + \sum_{k=1}^{m} b'_k\big(t,\xi'(t)\big)\,\mathrm{d}w_k(t) + v'\big(\xi'(t)\big)\,\mathrm{d}l(t), \end{aligned} \tag{2.21}$$

where

$$L_t f(x) = \sum_{i=1}^{d} a_i(t,x)\frac{\partial f(x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{m} b_{ki}(t,x) b_{kj}(t,x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j},$$

$$\begin{cases} a'(t,y) = L_t\varphi\big(t,\varphi^{-1}(y)\big), \\ b'_k(t,y) = \nabla\varphi\big(\varphi^{-1}(y)\big) b_k\big(t,\varphi^{-1}(y)\big), \\ v'(y) = \nabla\varphi\big(\varphi^{-1}(y)\big) v\big(\varphi^{-1}(y)\big). \end{cases} \tag{2.22}$$

Observe that $\partial D' = \partial\big(\varphi(D)\big) = \varphi(\partial D)$, so

$$\int_0^t \mathbb{I}_{\xi'(s)\in D'}\,\mathrm{d}l(s) = 0, \quad t \ge 0.$$

We may consider (2.21) as a reflecting SDE in $\overline{D'}$ with a reflection vector field $v'(y)$, $y \in \partial D'$. Vice versa, if $\xi'(t)$, $t \ge 0$, is a solution of (2.21) with a', b'_k and v' from (2.22), then $\xi(t)$, $t \ge 0$, is a solution of (2.20). Hence, if we are able to find a change of variables φ such that (2.21) has a unique solution, then (2.20) has a unique solution. For example, we know from Theorem 2.1.1 that if $D' = \mathbb{R}^d_+$, $v' = n$, and a' and b'_k are of linear growth and satisfy the Lipschitz condition, then there exists a unique solution to (2.21). We reduce to this case (at least locally) for a reflecting equation with Lipschitz a and b_k if v is sufficiently smooth and ∂D is a smooth manifold.

It is certainly possible that the global map φ does not exist. Suppose that $D \subset \mathbb{R}^d$ is a bounded open set, $D = \cup_{k=1}^n D_k$, where D_k are open and either $\overline{D}_k \subset D$ or there exists a

C^3-diffeomorphism $\varphi_k : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\varphi_k(\overline{D}_k) = \{x \in \mathbb{R}^d_+ : |x| \le 1\}, \tag{2.23}$$

$$\varphi_k(\partial D_k \cap \partial D) = \{x \in \partial\mathbb{R}^d_+ : |x| \le 1\}, \tag{2.24}$$

$$\nabla\varphi_k(x)\nu(x) = n, \quad x \in \partial D_k \cap \partial D. \tag{2.25}$$

Assume that a and $b_k, k = \overline{1,m}$, are bounded and Lipschitz in x. Solution of (2.20) can be constructed as follows, see [2]. Let $\xi(0) \in D_{i_0}$. If $\overline{D}_{i_0} \subset D$, then the last term in (2.20) has a value of 0 before the instant τ_0 when ξ exits D_{i_0}. So (2.20) is a usual SDE without reflection and ξ is well defined up to τ_0.

If $\overline{D}_{i_0} \not\subset D$, then we make a change of variables

$$\xi_0'(t) := \varphi_{i_0}\big(\xi(t)\big).$$

The process ξ_0' satisfies a reflecting SDE with Lipschitz coefficients in $B = \{x \in \mathbb{R}^d_+ : |x| < 1\}$, where the reflecting vector field at $\{x \in \partial\mathbb{R}^d_+ : |x| < 1\}$ is $n = (0,\ldots,0,1)$. So ξ_0' and ξ are uniquely defined up to the stopping time

$$\tau_0 = \inf\{s \ge 0 : \ \xi_0'(s) \notin B\} = \inf\{s \ge 0 : \ \xi(s) \in D \setminus D_{i_0}\}.$$

If $\tau_0 = \infty$, then ξ is already constructed. If $\tau_0 < \infty$ and $\xi(\tau_0) \in D_{i_1}$, then, similarly to the previous reasoning, we can uniquely extend ξ up to the moment τ_1,

$$\tau_1 = \inf\{s \ge \tau_0 : \xi(s) \in D \setminus D_{i_1}\},$$

and so on. Therefore ξ is uniquely defined up to $\tau_\infty = \lim_{n\to\infty} \tau_n$.

Exercise 2.4.1 Prove, under the assumptions made on D, a and b_k that

$$P(\tau_\infty = +\infty) = 1.$$

If D is unbounded and can be represented as an enumerable union of D_k with the properties described above, and coefficients a and b_k, $k = \overline{1,m}$, are locally bounded and locally Lipschitz in x, then, using the same reasoning, the process ξ can be uniquely constructed up to τ_∞.

Example 2.4.1 Let $D = \mathbb{R}^d_+$. Assume that a sufficiently smooth vector field $v(x)$, $x \in \partial\mathbb{R}^d_+$, satisfies (cf. Example 2.2.1)

$$\big(v(x), n\big) = 1, \quad x \in \partial\mathbb{R}^d_+. \tag{2.26}$$

We are going to find a diffeomorphism $\varphi : \mathbb{R}^d_+ \to \mathbb{R}^d_+$ such that

$$\nabla\varphi(x)v(x) = n, \quad x \in \partial\mathbb{R}^d_+.$$

One of the ways to do this is the following. Extend v to $\mathbb{R}^d$ such that this extension is also smooth and $v_d(x) = 1$. Consider the first order PDE

$$\nabla\varphi(x)v(x) = n, \quad x \in \mathbb{R}^d, \tag{2.27}$$

where $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is such that

$$\varphi(x_1,\dots,x_{d-1},0) = (x_1,\dots,x_{d-1},0),$$

i.e.

$$\varphi(x) = x, \quad x \in \partial\mathbb{R}^d_+.$$

Denote

$$\bar{x} = (x_1,\dots,x_{d-1}), \quad t = x_d, \quad \varphi = (\varphi_1,\dots,\varphi_d)$$
$$\bar{v}(\bar{x}) = \big(v_1(x_1,\dots,x_{d-1},0),\dots,v_{d-1}(x_1,\dots,x_{d-1},0)\big).$$

Recall that $v_d(x) = 1$. Then equation (2.27) is equivalent to d systems of first order PDEs:

$$\begin{cases} \dfrac{\partial\varphi_k(\bar{x},t)}{\partial t} + \displaystyle\sum_{i=1}^{d-1} v_i(\bar{x},t)\dfrac{\partial\varphi_k(\bar{x},t)}{\partial x_i} = 0, \quad k = \overline{1,d-1}, \\ \bar{\varphi}_k(\bar{x},0) = x_k, \quad k = \overline{1,d-1}, \end{cases} \tag{2.28}$$

$$\begin{cases} \dfrac{\partial\varphi_d(\bar{x},t)}{\partial t} + \displaystyle\sum_{i=1}^{d-1} v_i(\bar{x},t)\dfrac{\partial\varphi_d(\bar{x},t)}{\partial x_i} = 1, \\ \varphi_d(\bar{x},0) = 0. \end{cases} \tag{2.29}$$

By $\bar{u}(\bar{x},t) = \big(\bar{u}_1(\bar{x},t),\dots,\bar{u}_{d-1}(\bar{x},t)\big)$, denote a solution of the following ordinary differential equation in $\mathbb{R}^{d-1}$:

$$\frac{\partial \bar{u}(\bar{x},t)}{\partial t} + \bar{v}\big(\bar{u}(\bar{x},t)\big) = 0$$

with initial condition

$$\bar{u}(\bar{x},0) = \bar{x}.$$

It is well known (see e.g. [17]) that, for any $g \in C^1(\mathbb{R}^{d-1})$, the function $\psi(\bar{x},t) = g\big(\bar{u}(\bar{x},t)\big)$ is a unique solution of the equation

$$\frac{\partial \psi(\bar{x},t)}{\partial t} + \sum_{i=1}^{d-1} v_i(\bar{x}) \frac{\partial \psi(\bar{x},t)}{\partial x_i} = 0$$

with initial condition

$$\psi(\bar{x},0) = g(\bar{x}).$$

So

$$\varphi_k(\bar{x},t) = \bar{u}_k(\bar{x},t), \quad k = \overline{1,d-1}.$$

If $k = d$, then a solution of (2.29) is

$$\varphi_d(\bar{x},t) = t.$$

It is easy to see that if v is r times continuously differentiable and has a bounded derivative, then the constructed mapping φ is C^r-diffeomorphism of $\mathbb{R}^d$ and

$$\varphi\big(\mathbb{R}^d_+\big) = \mathbb{R}^d_+, \quad \varphi\big(\partial\mathbb{R}^d_+\big) = \partial\mathbb{R}^d_+,$$
$$\nabla\varphi(x)v(x) = n, \quad x \in \partial\mathbb{R}^d_+.$$

Hence, sufficient conditions ensuring existence and uniqueness of a solution to the initial reflecting SDE in $\mathbb{R}^d_+$ are

1) a and b_k, $k = \overline{1,m}$, are globally Lipschitz in x;

2) a and b_k, $k = \overline{1,m}$, satisfy a linear growth condition in x;

3) $v \in C^3_b\big(\partial\mathbb{R}^d_+\big)$.

Exercise 2.4.2 Let $G \subset \mathbb{R}^d$ be an open set. Let there be a $x^* \in \partial D$ such that ∂D in a neighborhood of x^* has a form $\Big\{\big(\bar{x}, g(\bar{x})\big), \bar{x} = (x_1, \ldots, x_{d-1}) \in U\Big\}$, where $g : U \subset \mathbb{R}^{d-1} \to \mathbb{R}$ is a smooth function. Construct a neighborhood D_{x^*} and a map $\varphi = \varphi_{x^*}(x)$, $x \in D_{x^*}$, that satisfies properties (2.23), (2.24) and (2.25).

Hint: Let $y = y(x,t)$, $x \in \mathbb{R}^d$, $t \geq 0$, be a solution of the ordinary differential equation

$$\begin{cases} \dfrac{\partial y}{\partial t} = v(y), & t \geq 0, \\ y(x,0) = x, & x \in \mathbb{R}^d. \end{cases}$$

Define a mapping

$$x = (\bar{x}, x_d) \to \psi(x) := y\Big(\big(\bar{x}, g(\bar{x})\big), x_d\Big).$$

Verify that, if $g \in C^1$ and $v \in C^1$, then there exists an $\varepsilon > 0$ such that ψ is a C^1-diffeomorphism of the ball $B\big((\bar{x}^*, 0), \varepsilon\big)$ and

1) $\psi\Big(B\big((\bar{x}^*, 0), \varepsilon\big) \cap \{x_d > 0\}\Big) \subset D$, $\psi\Big(B\big((\bar{x}^*, 0), \varepsilon\big) \cap \{x_d = 0\}\Big) \subset \partial D$,
2) the inverse map $\varphi := \psi^{-1}$ is such that $\nabla\varphi(x)v(x) = n$ for all $x \in \partial D$ from some neighborhood of x^*.

Summing up all of the reasoning in this section, we may prove the following result.

Theorem 2.4.1 Let D be a bounded set with C^3 boundary. Assume that a and b_k, $k = \overline{1,m}$, are bounded functions on $\overline{D}$ satisfying the Lipschitz condition in x; let v be a C^3-function on ∂D. Then there exists a unique solution to (2.20). The process $\xi(t)$, $t \geq 0$, is a Markov process.

Exercise 2.4.3 Formulate an analogue of Theorem 2.4.1 for unbounded D. Find some sufficient conditions ensuring that a solution does not blow up almost surely.

Exercise 2.4.4 Construct a localization procedure for a time-dependent reflecting vector field $v = v(t,x)$.

2.5 Properties of multidimensional Skorokhod's problem

In this section, we give results on existence, uniqueness and continuity for a solution of the deterministic Skorokhod problem. The most general results have been obtained for the normal reflection, i.e., if $K_x = N_x$, where the set N_x of inward normal unit vectors at $x \in \partial D$ is defined by

$$N_x = \bigcup_{r>0} N_{x,r},$$
$$N_{x,r} = \big\{ n \in \mathbb{R}^d : |n| = 1, B(x - rn, r) \cap D = \varnothing \big\}.$$

If ∂D is a smooth manifold, then N_x naturally consists of the only vector which is the unit inward normal to the manifold. In the general case, it is possible that N_x is empty or infinite.

Definition 2.5.1 A set D satisfies the uniform exterior sphere condition if

$$\exists r_0 > 0\ \forall x \in \partial D : \ N_x = N_{x,r_0} \neq \varnothing. \tag{2.30}$$

Exercise 2.5.1 Prove that a convex set satisfies the uniform exterior sphere condition with any $r_0 > 0$.

Exercise 2.5.2 Prove that $n \in N_{x,r}$ if and only if

$$\forall y \in \overline{D} : \ (y - x, n) + \frac{1}{2r} |y - x|^2 \geq 0. \tag{2.31}$$

Hint: $B(x - rn, r) \cap D = \varnothing$ if and only if $\forall y \in \overline{D} : \ |y - (x - rn)| \geq r$.

It can be proved (see [41]) that the uniform exterior sphere condition ensures uniqueness for the deterministic normal reflection Skorokhod problem. Let us show that this condition implies the pathwise uniqueness for a solution of a reflecting SDE if its coefficients satisfy the Lipschitz condition.

Theorem 2.5.1 Assume that a domain $D \subset \mathbb{R}^d$ satisfies the uniform sphere condition property; let $a, b_k : [0,\infty) \times \overline{D} \to \mathbb{R}^d$, $k = \overline{1,m}$, satisfy the Lipschitz condition in x. Let $\xi_i(t)$, $l_i(t)$, $t \geq 0$, $i = 1,2$, be solutions of

$$\mathrm{d}\xi(t) = a\big(t, \xi(t)\big)\,\mathrm{d}t + \sum_{k=1}^{m} b_k\big(t, \xi(t)\big)\,\mathrm{d}w_k(t) + n\big(\xi(t)\big)\,\mathrm{d}l(t), \quad t \geq 0, \tag{2.32}$$

where $n(x) \in N_x$, $\xi_1(0) = \xi_2(0)$. Then $P\big(\xi_1(t) = \xi_2(t),\ l_1(t) = l_2(t),\ t \geq 0\big) = 1$.

Proof. Let us apply Itô's formula and use the Lipschitz condition:

$$\begin{aligned}
&\big|\xi_1(t)-\xi_2(t)\big|^2\exp\big\{-C\big(l_1(t)+l_2(t)+t\big)\big\}\\
&\quad\le\int_0^t (L-C)\big|\xi_1(s)-\xi_2(s)\big|^2\exp\big\{-C\big(l_1(s)+l_2(s)+s\big)\big\}\,\mathrm{d}s\\
&\qquad+\int_0^t\Big(2\big(\xi_1(s)-\xi_2(s),n(\xi_1(s))\big)-C\big|\xi_1(s)-\xi_2(s)\big|^2\Big)\,\mathrm{d}l_1(s)\\
&\quad+\int_0^t\Big(2\big(\xi_2(s)-\xi_1(s),n(\xi_2(s))\big)-C\big|\xi_1(s)-\xi_2(s)\big|^2\Big)\,\mathrm{d}l_2(s)+M(t),
\end{aligned}\tag{2.33}$$

where L depends on the Lipschitz constant and $M(t)$, $t\ge 0$, is a continuous local martingale, $M(0)=0$. It follows from (2.31) that the second and third terms in the right hand side of (2.33) are non-positive if $C>\frac{1}{r_0}$. So for $C>\frac{1}{r_0}\vee L$, the process

$$\eta(t)=\big|\xi_1(t)-\xi_2(t)\big|^2\exp\big\{-C\big(l_1(t)+l_2(t)+t\big)\big\}$$

is a non-negative continuous supermartingale with $\eta(0)=0$. Hence $\eta(t)=0$, $t\ge 0$, a.s. and $\xi_1(t)=\xi_2(t)$, $t\ge 0$, a.s. □

Exercise 2.5.3 Prove that $l_1(t)=l_2(t)$, $t\ge 0$, a.s. (Note that even if $\xi_1(t)=\xi_2(t)$, theoretically different representatives for $n\big(\xi_1(t)\big)$ and $n\big(\xi_2(t)\big)$ can be taken in (2.32).)

Theorem 2.5.2 ([61]) Assume that a domain D satisfies

a) the uniform exterior sphere condition,

b) the uniform cone condition:

$$\exists\delta>0\ \exists\alpha\in[0,1)\ \forall x\in\partial D\ \exists l_x, |l_x|=1\ \forall y\in B(x,\delta)\cap\partial D:\quad C(y,l_x,\alpha)\cap B(x,\delta)\subset\overline{D},$$

where $C(y,l_x,\alpha)=\big\{z\in\mathbb{R}^d:(z-y,l_x)\ge\alpha|z-y|\big\}$.

Then for any $f\in C\big(\mathbb{R}^d,\mathbb{R}^d\big)$ with $f(0)\in\overline{D}$, there exists a unique solution to the Skorokhod normal reflection problem

$$g(t)=f(t)+\int_0^t n\big(g(s)\big)\,\mathrm{d}l(s),\quad t\ge 0.\tag{2.34}$$

Moreover, g depends continuously on (t, f).

A proof for convex D was considered in [72]. Existence of a solution for (2.34) for arbitrary D satisfying assumptions of the theorem can be verified using approximations by step functions; see details in [61]. Namely, assume at first that f is a step function,

$$f(t) = f(t_k), \quad t \in [t_k, t_{k+1}),$$

where $0 = t_0 < t_1 < \ldots$, $\lim\limits_{n\to\infty} t_n = \infty$, $\sup_{k\ge 0} \left| f(t_{k+1}) - f(t_k) \right| < r_0$. Set

$$g(t) = \begin{cases} f(0), & 0 \le t < t_1 \\ \overline{g(t_{k-1}) + f(t_k) - f(t_{k-1})}, & t_k \le t < t_{k+1},\ k \ge 1, \end{cases}$$

where $\bar{x}$ is such that $|\bar{x} - x| = \inf\left\{|y - x| : y \in \overline{D}\right\}$ (if $x \in \overline{D}$, then $x = \bar{x}$). Condition (2.30) ensures that $\bar{x}$ exists and is unique if $\operatorname{dist}(x, D) < r_0$. Set

$$\varphi(t) = \begin{cases} 0, & 0 \le t < t_1 \\ \varphi(t_{k-1}) + \overline{g(t_{k-1}) + f(t_k) - f(t_{k-1})} & \\ \qquad\qquad - g(t_{k-1}) - f(t_k) + f(t_{k+1}), & t_k \le t < t_{k-1},\ k \ge 1. \end{cases}$$

Then

$$g(t) = f(t) + \int_{[0,t]} n\big(g(s)\big)\, \mathrm{d}l(s),$$

where $l(t) = \operatorname{Var} \varphi\big|_0^t$. If f is continuous, then define

$$f_n(t) = f(t_{n,k}), \quad t \in [t_{n,k}, t_{n,k+1}),$$

where $0 = t_{n,0} < t_{n,1} < \ldots$, $\lim\limits_{n\to\infty} \sup\limits_k (t_{n,k+1} - t_{n,k}) = 0$. Certainly one needs some a priori estimates for solutions of the Skorokhod problem to check that the limit of $\{g_n, n \ge 1\}$ constructed for $\{f_n, n \ge 1\}$ exists and satisfies (2.34).

Another way to prove the existence of a solution of the Skorokhod problem is the penalization method. Assume that ∂D is smooth and that there exists a function $p \in C^2(\mathbb{R}^d)$ such that $p(x) = 0$, $x \in \overline{D}$; $p(x) > 0$, $x \notin \overline{D}$; $p(x) = \big(\operatorname{dist}(x, \overline{D})\big)^2$ for x in a neighborhood of $\overline{D}$. Then a solution of (2.34) can be approximated by solutions of the following equations

$$g_\varepsilon(t) = f(t) + \varepsilon^{-1} \int_0^t \nabla p\big(g_\varepsilon(s)\big)\, \mathrm{d}s.$$

See details in [41], where it is also proved that the Skorokhod map satisfies the Hölder property of order $1/2$.

The question of the existence, uniqueness and properties of a solution to the oblique Skorokhod problem is a difficult one, see [28, 18, 15, 20], where the Skorokhod problem was considered in non-smooth domains, in particular in orthants or polyhedras. Some conditions ensuring that the Skorokhod map is a Lipschitz function on the input function f were obtained in the above sources, which also considered non-normal reflection. This had specific structure sometimes; however, such reflection vector fields appear naturally in some limit models of queuing theory. See also [19, 6] for SDEs with oblique reflection in non-smooth domains. Note the paper [57], where some generalization of the Skorokhod problem was considered. There, the condition that a reflection term has bounded variation is relaxed.

Theorem 2.5.3 ([61]) Assume that D satisfies conditions of Theorem 2.5.2 and that a and b_k, $k = \overline{1,m}$, are bounded continuous functions. Then there exists a weak solution to (2.32).

To prove the existence of a weak solution, we may use Theorem 2.5.2 together with the Euler approximations and compactness arguments, similarly to §1.4, §2.3.

By the Yamada–Watanabe theorem, pathwise uniqueness and existence of the weak solution imply the existence of a unique strong solution. This theorem was proved for SDEs without reflection, but the proof remains true also for reflecting SDEs. So Theorems 2.5.1 and 2.5.3 yield the following result.

Theorem 2.5.4 Assume that D satisfies the conditions of Theorem 2.5.2. If a and b_k, $k = \overline{1,m}$, satisfy the Lipschitz condition, then there exists a unique strong solution to (2.32).

Remark 2.5.1 If all the coefficients of a reflecting SDE are smooth and ∂D is a smooth manifold, then the existence and uniqueness of a solution can be obtained by localization, see §2.4.

Chapter 3

Other approaches to reflecting SDEs

3.1 Reflecting SDEs and PDEs

Consider the reflecting SDE

$$\mathrm{d}\xi(t) = a\big(\xi(t)\big)\,\mathrm{d}t + \sum_{k=1}^{m} b_k\big(\xi(t)\big)\,\mathrm{d}w_k(t) + v\big(\xi(t)\big)\,\mathrm{d}l(t). \tag{3.1}$$

Recall that under some regularity of the coefficients, the process $\xi(t)$, $t \geq 0$, is a Markov process, see §1.2, §2.4. We can therefore construct a semigroup corresponding to this Markov process, use Kolmogorov's equations and so on. One of the aims of this section is to associate Partial Differential Equations (PDEs) with expectations of some functionals of reflecting SDEs.

Introduce a second order differential operator

$$L = \sum_{i=1}^{d} a_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} \sigma_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \tag{3.2}$$

where

$$\sigma_{ij}(x) = \sum_{k=1}^{m} b_{ki}(x) b_{kj}(x). \tag{3.3}$$

Further, in this section we will always assume that all reflecting SDEs have weak solutions.

Exercise 3.1.1 Let $f \in C_b^2(\overline{D})$ be such that $\frac{\partial f(x)}{\partial v} = 0, x \in \partial D$. Assume that a, b_k, $k = \overline{1,m}$, v are bounded and continuous functions. Set $u(t,x) = E_x f\big(\xi(t)\big)$. Apply Itô's formula and prove that

$$\forall x \in \overline{D},\ \frac{\partial u(t,x)}{\partial t}\bigg|_{t=0} = Lf(x). \tag{3.4}$$

Exercise 3.1.1 gives us an idea that the generator of a semigroup $T_t f(x) = E_x f\big(\xi(t)\big)$, $t \geq 0$, equals L, where

$$\mathscr{D}(L) = C_b^2(\overline{D}) \cap \left\{ f \ :\ \frac{\partial f(x)}{\partial v} = 0,\ x \in \partial D \right\}. \tag{3.5}$$

Certainly, a lot of problems should be discussed if we are going to apply the Hille-Yosida theorem or to determine a semigroup (see the analytical approach to reflecting SDEs in [9, 63, 73, 74, 56]). For example,

- ◇ We have to determine a Banach space on which T_t acts. The natural choices could be, e.g., a space of bounded continuous functions, a space of Hölder continuous functions or a space of Sobolev differentiable functions.
- ◇ The limit $\frac{\partial u(t,x)}{\partial t}\Big|_{t=0} = \lim\limits_{t \to 0+} \frac{u(t,x) - u(0,x)}{t}$ in (3.4) is point-wise. To determine the generator, it should be a limit in a proper Banach space.
- ◇ It is usually difficult to describe the whole domain of the generator. It would be nice to show that $\mathscr{D}(L)$ defined in (3.5) is rich enough to determine a semigroup.

We are not going to proceed with the steps above and show the one-to-one correspondence between a reflecting SDE and a semigroup with the generator L. However, we do not actually need this to prove many results including, for example, the Feynman-Kac formula.

Let $\xi(t)$, $t \geq 0$, be a weak solution of the non-homogeneous reflecting SDE

$$\mathrm{d}\xi(t) = a\big(t,\xi(t)\big)\,\mathrm{d}t + \sum_{k=1}^{m} b_k\big(t,\xi(t)\big)\,\mathrm{d}w_k(t) + v\big(t,\xi(t)\big)\,\mathrm{d}l(t). \tag{3.6}$$

Denote

$$L_t = \sum_i a_i(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j}\sigma_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j},$$
$$\sigma_{ij}(t,x) = \sum_k b_{ki}(t,x)b_{kj}(t,x).$$

Theorem 3.1.1 (Kolmogorov's equation for a semigroup) Let $u(t,x)$ be a solution of the PDE

$$\frac{\partial u(t,x)}{\partial t} + L_t u(t,x) = 0, \quad x \in \overline{D},\ t < T, \tag{3.7}$$

with the boundary condition

$$\frac{\partial u(t,x)}{\partial v} = 0, \quad x \in \partial D,\ t < T, \tag{3.8}$$

and the terminal condition

$$u(T,x) = f(x), \quad x \in \overline{D}. \tag{3.9}$$

Assume that functions f and b_k are bounded and

$$u \in C_b^{1,2}\big([0,T]\times\overline{D}\big). \tag{3.10}$$

Then

$$u(t,x) = E_{t,x} f\big(\xi(T)\big) := E\Big(f\big(\xi(T)\big)\Big|\xi(t) = x\Big). \tag{3.11}$$

Proof. By Itô's formula

$$\begin{aligned} f\big(\xi(T)\big) &= u\big(T,\xi(T)\big) \\ &= u\big(t,\xi(t)\big) + \int_t^T \left(\frac{\partial u(s,\xi(s))}{\partial s} + L_s u\big(s,\xi(s)\big)\right) \mathrm{d}s \\ &\quad + \int_t^T \big(\nabla u(s,\xi(s)), v(\xi(s))\big)\, \mathrm{d}l(s) \\ &\quad + \sum_{k=1}^m \int_t^T \big(\nabla u(s,\xi(s)), b_k(s,\xi(s))\big)\, \mathrm{d}w_k(s), \quad t \le T. \end{aligned} \tag{3.12}$$

The second and the third terms in the right hand side of (3.12) are equal to zero. Let us calculate the conditional expectation in (3.12), given $\xi(t) = x$. Since ∇u and b_k are bounded, we have

$$E\left(\int_t^T \big(\nabla u(s,\xi(s)), b_k(s,\xi(s))\big)\,\mathrm{d}w_k(s)\,\Big|\,\xi(t)=x\right)=0. \tag{3.13}$$

Hence representation (3.11) is proved. □

Remark 3.1.1 The assumptions for Theorem 3.1.1 are very strong. They can be relaxed in various ways. For example, we supposed boundedness of ∇u and b_k simply to ensure (3.13). If we can guarantee (3.13), then to obtain (3.11), it is sufficient to assume, for example, that $u \in C^{1,2}\big([0,T)\times\overline{D}\big)\cap C_b\big([0,T]\times\overline{D}\big)$. Moreover, Itô's formula can sometimes even be applied for $u \in W_p^{1,2}$. If this is the case, nothing changes in the proof.

Remark 3.1.2 We do not prove the existence of the solution for the PDE in Theorem 3.1.1. The corresponding theory is well developed for domains D with a smooth boundary if L_t satisfies strong ellipticity conditions. For existence and uniqueness of a solution, it is sufficient to assume that all coefficients of the equation are bounded and Hölder continuous, see for example [21, 38, 22]. In this case, the solution can be written as

$$u(t,x)=\int_D G(t,x,T,y)f(y)\,\mathrm{d}y,$$

where G is the Green function. Moreover, $\int_D G(t,x,T,y)\,\mathrm{d}y=1$ and $G(t,x,T,y)\geq 0$ and satisfies the Chapman-Kolmogorov equation. It can be proved under the same (and even weaker) assumptions that there exists a unique weak solution of (3.6) and this solution is a Markov process, see §3.2. So the Green function G is the transition density of ξ.

Assume now that the coefficients are homogeneous in time. Let us use the following change of variables:

$$g(t,x):=u(T-t,x).$$

It is easy to see that $g(t,x)$ satisfies the initial value parabolic PDE

$$\frac{\partial g(t,x)}{\partial t}=Lg(t,x),\quad t>0,\ x\in\overline{D},$$
$$g(0,x)=f(x),\quad x\in\overline{D},$$

with boundary condition

$$\frac{\partial g(t,x)}{\partial v} = 0, \quad x \in \partial D,\ t > 0.$$

So

$$g(t,x) = E\Big(f\big(\xi(T)\big)\Big|\xi(T-t) = x\Big) = E\Big(f\big(\xi(t)\big)\Big|\xi(0) = x\Big) =: E_x f\big(\xi(t)\big).$$

If $p(t,x,y)$ is the transition density of ξ, then we have the representation

$$g(t,x) = \int_D f(y)p(t,x,y)\,\mathrm{d}y.$$

Theorem 3.1.2 (Feynman-Kac formula) Assume that a bounded continuous function $u(t,x)$ satisfies PDE

$$\frac{\partial u(t,x)}{\partial t} + L_t u(t,x) = r(t,x)u(t,x), \quad t < T,\ x \in \overline{D},$$
$$u(T,x) = f(x), \quad x \in \overline{D},$$

and the boundary condition

$$\frac{\partial u(t,x)}{\partial v} = 0, \quad x \in \partial D,\ t < T,$$

where r is either a bounded or non-negative function. Suppose that for any $t \le T$, $x \in \overline{D}$,

$$E_{t,x}\left(\sum_{k=1}^{m}\int_t^T \big(\nabla u(s,\xi(s)), b_k(s,\xi(s))\big)^2\,\mathrm{d}s\right) < \infty.$$

Then

$$u(t,x) = E_{t,x}\left(f\big(\xi(T)\big)\exp\left\{-\int_t^T r\big(s,\xi(s)\big)\,\mathrm{d}s\right\}\right). \tag{3.14}$$

Exercise 3.1.2 Apply Itô's formula to $u\big(s,\xi(s)\big)\exp\big\{-\int_t^s r\big(z,\xi(z)\big)\,\mathrm{d}z\big\}$, $s \in [t,T]$, and prove (3.14).

Exercise 3.1.3 Assume that the operator $L_t = L$ is homogeneous in time and $u(t,x)$ satisfies the equation

$$\frac{\partial u(t,x)}{\partial t} = Lu(t,x) - r(t,x)u(t,x), \quad t > 0,\ x \in \overline{D},$$
$$u(0,x) = f(x), \quad x \in \overline{D},$$
$$\frac{\partial u(t,x)}{\partial v} = 0, \quad x \in \partial D,\ t > 0.$$

Prove that

$$u(t,x) = E_x f\big(\xi(t)\big) \exp\left\{ -\int_0^t r\big(t-s, \xi(s)\big)\, \mathrm{d}s \right\}.$$

Let $K, D \subset \mathbb{R}^d$ be open sets, $\overline{K} \subset D$ and $\xi(t)$, $t \geq 0$, be a weak solution of the homogeneous in time equation (3.1). By τ_K, denote the hitting time of $\overline{K}$

$$\tau_K = \inf\big\{t \geq 0 : \xi(t) \in \overline{K}\big\}.$$

Theorem 3.1.3 Suppose that all coefficients of (3.1) are bounded, continuous functions and that D is bounded. Assume that there exists $u \in C^2\big(\overline{D} \setminus K\big)$ satisfying the elliptic PDE

$$\begin{cases} Lu(x) = -1, & x \in \overline{D} \setminus K, \\ \dfrac{\partial u(x)}{\partial v} = 0, & x \in \partial D, \\ u(x) = 0, & x \in \partial K. \end{cases}$$

Then $\tau_K < \infty$ a.s. and

$$u(x) = E_x \tau_K.$$

Proof. By Itô's formula,

$$u\big(\xi(\tau_K \wedge t)\big) = u\big(\xi(0)\big) - \int_0^{\tau_K \wedge t} \mathrm{d}s + \sum_{k=1}^m \int_0^{\tau_K \wedge t} \big(\nabla u(\xi(s)), b_k(\xi(s))\big)\, \mathrm{d}w_k(s).$$

Take the conditional expectation given $\xi(0) = x$ to obtain

$$E_x u\big(\xi(\tau_K \wedge t)\big) = u(x) - E_x(\tau_K \wedge t). \tag{3.15}$$

By the monotone convergence theorem,

$$E_x\tau_K = \lim_{t\to\infty} E_x(\tau_K \wedge t).$$

Since u is bounded (the set $\overline{D}\setminus K$ is a compact), the limit in the right hand side of (3.15) is finite. So $\tau_K < \infty$ a.s. and

$$u(x) - E_x\tau_K = \lim_{t\to\infty} E_x u\big(\xi(\tau_K \wedge t)\big) = E_x u\big(\xi(\tau_K)\big) = 0.$$

Theorem 3.1.3 is proved. □

Exercise 3.1.4 Let $D = \{x \in \mathbb{R}^d : |x| < R\}$, $K = \{x \in \mathbb{R}^d : |x| < r\}$, where $0 < r < R$. Assume that ξ is a solution to the SDE

$$\mathrm{d}\xi(t) = \mathrm{d}w(t) - \frac{\xi(t)}{|\xi(t)|}\,\mathbb{I}_{\xi(t)\in\partial D}\,\mathrm{d}l(t),$$

where $w(t), t \geq 0$, is a Wiener process in $\mathbb{R}^d$, i.e., $\xi(t)$ is a reflecting Brownian motion in a ball D with a normal reflection at the boundary. Find $E_x\tau_K$.

Hint: Search for a function $u(x)$ that takes the form $\alpha|x|^2 + \beta|x|^{2-d} + \gamma$ if $d \geq 3$ and $u(x) = \alpha|x|^2 + \beta\ln|x| + \gamma$ if $d = 2$.

Exercise 3.1.5 Let ξ be a solution of the one-dimensional reflecting SDE with a reflecting barrier at x_1 so that

$$\mathrm{d}\xi(t) = a\big(\xi(t)\big)\,\mathrm{d}t + b\big(\xi(t)\big)\,\mathrm{d}w(t) + \mathbb{I}_{\xi(t)=x_1}\,\mathrm{d}l(t),$$

where a and b are continuous functions, $b(x) > 0$, $x \geq x_1$. Find $E_x\tau_{x_2}$, where $x_1 \leq x \leq x_2$ and

$$\tau_{x_2} = \inf\{t \geq 0 : \xi(t) = x_2\}.$$

(We assume in all exercises that the reflecting SDEs have solutions.)

Exercise 3.1.6 Assume that there exists a function $u \in C^2\big(\overline{D}\setminus K\big)$ such that

$$Lu(x) \leq -1, \quad x \in \overline{D}\setminus K,$$
$$\frac{\partial u(x)}{\partial v} \leq 0, \qquad x \in \partial D.$$

Prove that the solution of (3.1) is such that $E_x \tau_K \le 2\sup_y \big|u(y)\big|$.

Exercise 3.1.7 Assume that coefficients of (3.1) are bounded and continuous functions, D is a bounded, connected and open set with sufficiently smooth boundary, $K \subset D$ is a non-empty open set, $\big(v(x), n(x)\big) > 0$, $x \in \partial D$ and L satisfies the strong ellipticity condition

$$\exists C > 0 \; \forall x \in \overline{D} \setminus K \; \forall \lambda \in \mathbb{R}^d : \sum_{i,j} \sigma_{ij}(x)\lambda_i\lambda_j \ge c|\lambda|^2.$$

Prove that $\sup\limits_{x\in D} E_x \tau_K < \infty$.

Theorem 3.1.4 Let assumptions of Theorem 3.1.3 be satisfied. Suppose that $u \in C^2\big(\overline{D} \setminus K\big)$ is such that

$$Lu(x) = 0, \qquad x \in \overline{D} \setminus K, \tag{3.16}$$

$$u(x) = 0, \qquad x \in \partial K, \tag{3.17}$$

$$\frac{\partial u(x)}{\partial v} = -f(x), \quad x \in \partial D, \tag{3.18}$$

where f is a continuous function. Then

$$u(x) = E_x \int_0^{\tau_K} f\big(\xi(t)\big) \, \mathrm{d}l(t).$$

Exercise 3.1.8 Prove Theorem 3.1.4.

Exercise 3.1.9 Find $E_x l(\tau_K)$ for the reflecting SDE from Exercise 3.1.4.

Exercise 3.1.10 Find $E_x l(\tau_{x_2})$ for the reflecting SDE from Exercise 3.1.5.

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3.2 Submartingale problem

Stroock and Varadhan proposed a very effective method of construction and investigation of diffusion processes connected with solving some martingale problems, see [69, 70]. It is convenient to associate the study of diffusions with boundaries with some submartingale problem.

Let $\xi(t)$, $t \geq t_0$, be a solution of a reflecting SDE in $\overline{G} \subset \mathbb{R}^d$

$$\mathrm{d}\xi(t) = a\big(t,\xi(t)\big)\,\mathrm{d}t + \sum_{k=1}^{m} b_k\big(t,\xi(t)\big)\,\mathrm{d}w_k(t) + \gamma\big(t,\xi(t)\big)\,\mathrm{d}l(t), \quad t \geq t_0. \tag{3.19}$$

Assume for simplicity that a, b_k and γ are bounded. Introduce the operators

$$L_t = \sum_{i=1}^{d} a_i(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} \sigma_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}, \quad t \geq t_0,\ x \in \overline{G},$$

$$J_t = \sum_{i=1}^{d} \gamma_i(t,x)\frac{\partial}{\partial x_i}, \quad t \geq t_0,\ x \in \partial G,$$

where $\sigma_{ij}(t,x) = \sum_{k=1}^{m} b_{ki}(t,x)b_{kj}(t,x)$.

It follows from Itô's formula that, for any $f \in C_0^{1,2}\big([t_0,\infty) \times \mathbb{R}^d\big)$ such that

$$J_t f(t,x) \geq 0, \quad t \geq t_0,\ x \in \partial G, \tag{3.20}$$

the process

$$f\big(t,\xi(t)\big) - \int_{t_0}^{t} \left(\frac{\partial f\big(s,\xi(s)\big)}{\partial s} + L_s f\big(s,\xi(s)\big) \right) \mathrm{d}s, \quad t \geq t_0, \tag{3.21}$$

is a submartingale.

It turns out that the converse statement is also true. That is, if (3.21) holds for all $f \in C_0^{1,2}\big([t_0,\infty) \times \mathbb{R}^d\big)$ satisfying (3.20), then ξ is a (weak) solution to (3.19).

Let us introduce the general setup and assumptions. We follow Stroock and Varadhan [71]. Suppose that $G \subset \mathbb{R}^d$ is a non-empty set such that for some $\varphi \in C_b^2\big(\mathbb{R}^d\big)$

$$G = \{x \in \mathbb{R}^d : \varphi(x) > 0\}, \quad \partial G = \{x \in \mathbb{R}^d : \varphi(x) = 0\},$$

and $\big|\nabla\varphi(x)\big| \geq 1$, $x \in \partial G$.

Let $\sigma = \big(\sigma_{ij}(t,x)\big)_{i,j=1}^{d}$ be a bounded, measurable function with values in a set of symmetric non-negative definite $d \times d$ matrices; $a : [t_0,\infty) \times G \to \mathbb{R}^d$ be bounded and measurable; $\gamma : [t_0,\infty) \times \partial G \to \mathbb{R}^d$ be a bounded, continuous function such that

$$\inf_{t\geq t_0,\ x\in\partial G} \big(\gamma(t,x), \nabla\varphi(x)\big) > 0;$$

$\rho : [t_0,\infty) \times \partial G \to [0,\infty)$ be bounded and continuous.

Definition 3.2.1 A probability measure P on $C\big([t_0,\infty),\mathbb{R}^d\big)$ equipped with the Borel σ-algebra solves the submartingale problem on G for coefficients a, σ, γ and ρ if

1) the coordinate process $\xi(t)$, $t \geq t_0$, is such that

$$P\big(\xi(t) \in \overline{G}\big) = 1, \quad t \geq t_0; \tag{3.22}$$

2) for any $f \in C_0^{1,2}\big([t_0,\infty) \times \mathbb{R}^d\big)$ such that

$$\rho \frac{\partial f}{\partial t} + J_t f \geq 0 \quad \text{on } [t_0,\infty) \times \partial G,$$

the process

$$f\big(t,\xi(t)\big) - \int_{t_0}^{t} 1\!\!\mathrm{I}_G\big(\xi(s)\big) \left(\frac{\partial f\big(s,\xi(s)\big)}{\partial s} + L_s f\big(s,\xi(s)\big) \right) \mathrm{d}s, \quad t \geq t_0, \tag{3.23}$$

is a P-submartingale. If $P\big(\xi(t_0) = x_0\big) = 1$, then we denote the measure P by P_{t_0,x_0}.

Remark 3.2.1 Functions a and σ from the definition are defined only for $x \in G$. In contrast to (3.21), there is an extra indicator function $1\!\!\mathrm{I}_G\big(\xi(s)\big)$ in the integral in (3.23). However, if

$$P\left(\int_{t_0}^{\infty} 1\!\!\mathrm{I}_{\partial G}\big(\xi(s)\big)\, \mathrm{d}s = 0 \right) = 1 \tag{3.24}$$

and $\rho \equiv 0$, then the processes in (3.21) and (3.23) are equal a.s.

Theorem 3.2.1 A measure P solves the martingale problem for a, σ, ρ and γ if and only if there exists a continuous, non-decreasing and non-anticipating function $l : [t_0,\infty) \times C\big([t_0,\infty),\mathbb{R}^d\big) \to [0,\infty)$ such that

1) the process

$$M(t) = \xi(t) - \int_{t_0}^{t} 1\!\!\mathrm{I}_G\big(\xi(s)\big) a\big(s,\xi(s)\big)\, \mathrm{d}s - \int_{t_0}^{t} 1\!\!\mathrm{I}_{\partial G}\big(\xi(s)\big) \gamma\big(s,\xi(s)\big)\, \mathrm{d}l(s), \quad t \geq t_0,$$

is a continuous P-martingale with square characteristics

$$\langle M_i, M_j\rangle(t) = \int_0^t \mathbb{I}_G\big(\xi(s)\big)\sigma_{ij}\big(s,\xi(s)\big)\,\mathrm{d}s, \quad t \ge t_0,\ P\text{-a.s.};$$

2) $$\int_{t_0}^t \mathbb{I}_{\partial G}\big(\xi(s)\big)\,\mathrm{d}s = \int_{t_0}^t \rho\big(s,\xi(s)\big)\,\mathrm{d}l(s), \quad t \ge t_0,\ P\text{-a.s.}; \tag{3.25}$$

The proof follows from [71], Theorem 2.5, where only a positive definite σ is considered, see also [27].

Remark 3.2.2 Instead of Definition 3.2.1, the equivalent martingale problem is also sometimes considered. It is required that there is a non-anticipative, non-decreasing, continuous process $l(t)$, $t \ge t_0$, such that $l(0) = 0$, $l(t) = \int_0^t \mathbb{I}_{\xi(s)\in\partial D}\,\mathrm{d}l(s)$, (3.25) and (3.22) hold and, for any $f \in C_0^{1,2}\big([t_0,\infty)\times\mathbb{R}^d\big)$,

$$\begin{aligned} f\big(t,\xi(t)\big) - \int_{t_0}^t \left(\frac{\partial f\big(s,\xi(s)\big)}{\partial s} + L_s f\big(s,\xi(s)\big)\right)\mathrm{d}s \\ - \int_{t_0}^t \left[\left(\frac{\partial f\big(s,\xi(s)\big)}{\partial s} + J_s f\big(s,\xi(s)\big)\right)\right. \\ \left. - \rho\big(s,\xi(s)\big)\left(\frac{\partial f\big(s,\xi(s)\big)}{\partial s} + L_s f\big(s,\xi(s)\big)\right)\right]\mathrm{d}l(s), \quad t \ge t_0, \end{aligned} \tag{3.26}$$

is a P-martingale. Moreover, it was shown in [27] that in (3.26), it suffices to consider only time-homogeneous functions f, i.e., for all $f \in C_b^2\big(\overline{D}\big)$,

$$\begin{aligned} f\big(\xi(t)\big) - \int_{t_0}^t L_s f\big(s,\xi(s)\big)\,\mathrm{d}s \\ - \int_{t_0}^t \Big(J_s f\big(s,\xi(s)\big) - \rho\big(s,\xi(s)\big)L_s f\big(\xi(s)\big)\Big)\,\mathrm{d}l(s), \quad t \ge t_0, \end{aligned} \tag{3.27}$$

is a P-martingale.

Remark 3.2.3 If $\sigma_{ij}(t,x) = \sum_{k=1}^m b_{ki}(t,x)b_{kj}(t,x)$, then there are independent Wiener processes $\{w_k(t), t \ge t_0\}$, $k = \overline{1,m}$ which may be defined on some extension of the probability space such that

$$d\xi(t) = \mathbb{I}_G\big(\xi(t)\big)\Big(a\big(t,\xi(t)\big)\,dt + \sum_{k=1}^{m} b_k\big(t,\xi(t)\big)\,dw_k(t)\Big) + \mathbb{I}_{\partial G}\big(\xi(t)\big)\gamma\big(t,\xi(t)\big)\,dl(t), \quad t \geq t_0. \quad (3.28)$$

As in Remark 3.2.1, if $\rho \equiv 0$ and (3.24) holds, then $\xi(t)$ also satisfies (3.19).

Exercise 3.2.1 Prove that $l(t)$, $t \geq t_0$, is uniquely determined from (3.28), up to a null set, if $l(t_0) = 0$ and $\int_{t_0}^{t} \mathbb{I}_{\xi(s)\in G}\,dl(s) = 0$, $t \geq t_0$.

Remark 3.2.4 Let ξ be a solution of (3.19) and

$$\sum_{k=1}^{m} \big(b_k(t,x), \nabla\varphi(x)\big)^2 > 0, \quad t \geq t_0,\ x \in \partial G.$$

Then, similarly to the one-dimensional case (see §1.3), it can be shown that (3.24) holds true. See also [62] where the process $l(t)$ is associated with an analogue of a local time on ∂G.

Remark 3.2.5 Function ρ corresponds to the case when the boundary ∂G is “elastic.” A solution for positive ρ can be obtained from the case $\rho \equiv 0$ by some transformation of time. This transformation slows down time in the proper way when $\xi(t)$ with $\rho \equiv 0$ visits ∂G, see [29]. It also should be noted that if $\rho > 0$, then the solution of (3.28) may not be a strong solution even if all coefficients are C^∞ [14].

Theorem 3.2.2 (Existence of a solution; [71], Theorem 3.1) Suppose that, in addition to all assumptions on a, σ, ρ, γ, φ and G, the function σ is continuous and $\sigma(t,x)$ is positive definite for all t and x. Then there is a solution P to the submartingale problem, starting from $x_0 \in \overline{G}$ at time t_0.

Theorem 3.2.3 (Uniqueness of a solution; [71], Theorems 5.5 and 5.7) Let assumptions of Theorem 3.2.2 be satisfied, γ be locally Lipschitz in (t,x), and either $\rho \equiv 0$ or ρ is bounded and locally Lipschitz. Then a solution to the submartingale problem is unique for any starting point. Moreover, the corresponding solution P_{t_0,x_0} depends measurably on (t_0,x_0).

Remark 3.2.6 If we have a unique solution of a submartingale problem, then this solution is a strong Markov process. The proof is similar to the case of the martingale problem, e.g. [70].

Example 3.2.1 Let $G = [0,\infty)$, $\gamma \equiv 1$, $\sigma \equiv 1$, $a \equiv 0$, $\rho \equiv 0$, ξ_1 be a Wiener process reflected at 0 and ξ_2 be a Wiener process stopped at 0. Then ξ_1 is a solution to the submartingale problem (and satisfies the martingale problem discussed in Remark 3.2.2) but ξ_2 does not satisfy the submartingale problem. However

$$f\big(\xi_2(t)\big) - \frac{1}{2}\int_0^t \mathbb{I}_{\xi_2(s)>0} f''\big(\xi_2(s)\big)\,\mathrm{d}s, \quad t \geq 0,$$

is a submartingale for any $f \in C_0^2\big(\mathbb{R}\big), f'(0) \geq 0$. This example shows that, for homogeneous in time coefficients, we cannot restrict the definition of the submartingale problem to $f \in C_0^2\big(\mathbb{R}^d\big)$ such that $\rho(x)f(x) + Jf(x) \geq 0$, $x \in \partial G$.

Remark 3.2.7 All coefficients in the previous theorems were assumed to be bounded only to enable convenient formulations and corresponding proofs in [71]. A localization for the submartingale problem can be done. For example, it was proved in [71], Theorem 5.6, that if coefficients a, σ, ρ, γ and a', σ', ρ', γ' from Theorem 3.2.3 coincide in a set $[t_0, t_0 + \varepsilon] \times S$, where S is a neighborhood of x_0, then solutions of the corresponding submartingale problems coincide until the exit time from this set.

Remark 3.2.8 The boundary conditions for the submartingale problem discussed in Definition 3.2.1 and Theorems 3.2.1, 3.2.2 and 3.2.3 are not the general ones, see [80, 81]. For example, one may have some diffusion term during a time when the process visits a boundary or the jump exit from ∂D, among other conditions. It is also interesting to consider reflecting SDEs with Lévy noise or the corresponding submartingale problems. However these problems exceed the scope of this manuscript, see e.g. [79, 46, 1, 29, 43, 44, 4, 76, 56, 32, 33, 34, 35].

Remark 3.2.9 Let D be a cone with a smooth surface and O be its vertex. Suppose that coefficients of a reflecting SDE are smooth. Then we can construct a strong solution to this SDE until it visits O. If we do not assume that the reflection coefficient satisfies the uniform exterior sphere and the uniform cone conditions, then we have no instruments to construct reflecting diffusion in D after the instant of hitting O. It turns out that instead of requirements on a reflection at the vertex, we may make the natural assumption that the time spent at the vertex equals zero a.s., see the corresponding submartingale problem and construction of reflected Brownian motion in a cone in [77, 78, 37]. Note that constructed processes are not always semimartingales (compare with Example 1.2.1).

3.3 Reflecting SDEs and the Queueing Theory

The aim of this section is to show a possible relationship between reflecting SDEs and some models of queueing theory, which is a subject of very intensive investigations. We consider a limit theorem for one simple model of queueing theory that leads us to a reflection SDE and discuss possible generalizations.

Consider a queueing system $M/M/1/m$, i.e., assume that

a) there is one device that processes requests;

b) the maximal number of requests in a system equals m (buffer size); if a new request arrives and the buffer is overloaded, then this request is discarded;

c) inter-arrival times are exponential i.i.d. with intensity α, service times are exponential i.i.d. with intensity μ.

Denote by $X(t) = X^{\alpha,\mu}(t)$ the number of requests in the system at an instant $t \geq 0$. It is well known that $X(t)$ is a birth and death Markov process with a state space $E = \{0, 1, \ldots, m\}$, birth intensity α and death intensity μ.

The process $X(t)$ can be represented as

$$X^{\alpha,\mu}(t) = X^{\alpha,\mu}(0) + N_\alpha(t) - N_\mu(t) + L_0^{\alpha,\mu}(t) - L_m^{\alpha,\mu}(t), \tag{3.29}$$

where $N_\alpha(t)$ and $N_\mu(t)$, $t \geq 0$, are independent Poisson processes with intensities α and μ respectively; $L_0^{\alpha,\mu}(t)$ and $L_m^{\alpha,\mu}(t)$ are jump processes with possible jumps equal to 1; $L_0^{\alpha,\mu}(t)$ jumps only when $N_\mu(t) - N_\mu(t-) = 1$ and $X^{\alpha,\mu}(t) = 0$; $L_m^{\alpha,\mu}(t)$ jumps only when $N_\alpha(t) - N_\alpha(t-) = 1$ and $X^{\alpha,\mu}(t) = m$.

The process $N_\alpha(t)$ can be interpreted as the number of requests which have arrived before t; $L_m^{\alpha,\mu}(t)$ is the number of discarded requests before t; $N_\mu(t)$ is the cumulative service capacity over $[0,t]$; $L_0^{\alpha,\mu}(t)$ is the cumulative lost service capacity over $[0,t]$; $\big(N_\mu(t) - L_0^{\alpha,\mu}(t)\big)$ is the number of processed requests before t.

It is well known that

$$\frac{N_\alpha(t) - \alpha t}{\sqrt{\alpha}} \Rightarrow w(t), \quad \alpha \to \infty, \tag{3.30}$$

in a space $D\big([0,\infty)\big)$, where $w(t)$, $t \geq 0$, is a Wiener process.

Assume now that $\mu = \mu(\alpha)$ and $m = m(\alpha)$ are such that

$$\mu(\alpha) = \alpha + c\sqrt{\alpha} + o(\sqrt{\alpha}), \quad \alpha \to \infty, \tag{3.31}$$

$$m(\alpha) = d\sqrt{\alpha} + o(\sqrt{\alpha}), \qquad \alpha \to \infty, \tag{3.32}$$

where c and d are constants. Then

$$\frac{N_\alpha(t) - N_\mu(t)}{\sqrt{\alpha}} \Rightarrow \sqrt{2}w(t) - ct, \quad \alpha \to \infty,$$

in $D\big([0,\infty)\big)$.

Equation (3.29) can be considered as the Skorokhod problem (for step functions) with reflecting barriers at 0 and m. It can be proved that if (3.31) and (3.32) are satisfied and $X^{\alpha,\mu}(0) \Rightarrow x$, $\alpha \to \infty$, then

$$\left(\frac{X^{\alpha,\mu}(\cdot)}{\sqrt{\alpha}}, \frac{L_0^{\alpha,\mu}(\cdot)}{\sqrt{\alpha}}, \frac{L_m^{\alpha,\mu}(\cdot)}{\sqrt{\alpha}}\right) \Rightarrow \big(\xi(\cdot), l_0(\cdot), l_d(\cdot)\big), \tag{3.33}$$

in $D\big([0,\infty)\big)$, where (ξ, l_0, l_d) is a solution of the Skorokhod problem in $[0,d]$;

$$\xi(t) = x + \sqrt{2}w(t) - ct + l_0(t) - l_d(t), \quad t \ge 0;\ \xi(t) \in [0,d]; \tag{3.34}$$

$l_0(0) = l_d(0) = 0$; and l_0 and l_d are continuous, non-decreasing processes such that

$$\int_0^t \mathrm{I\!I}_{\xi(s)>0}\, \mathrm{d}l_0(s) = \int_0^t \mathrm{I\!I}_{\xi(s)<d}\, \mathrm{d}l_d(s) = 0, \quad t \ge 0.$$

So, the process $l_d(t)$ is an analogue of the number of lost requests during time t. We will find a limit $l_d(t)/t$ as $t \to +\infty$. This limit characterizes the average number of lost packets per unit of time. But before we start to calculate the limit, let us make a remark on possible generalizations.

Remark 3.3.1 An analogue of (3.30) and (3.33) is valid not only for a Poisson process of arrivals or processing. The Donsker invariance principle holds under various assumptions, see for example [7]. Diffusion approximations can be applied for group arrivals or even non-independent group arrivals (but certainly with some mixing condition), arrival and service rates may depend on the queue length, multi-server queueing systems can be considered, etc. Another generalization of the model described above is to assume that requests may have different priorities. Then the study of such queueing systems is associated with some Skorokhod's problem on an orthant or simplex. For further references, see e.g. [10, 25, 26, 58, 59, 82, 83].

Remark 3.3.2 If there is a long range dependence between requests, then the fractional Brownian process may arise under some scaling of the arrival process [75]. The limit behavior of $l(t)$ as $t \to \infty$ for a reflected fractional Brownian process cannot be studied by the methods described here and the corresponding problem is very difficult to investigate.

Consider the following generalization of (3.34)

$$\xi(t) = x + at + bw(t) + l_0(t) - l_d(t), \quad t \geq 0, \tag{3.35}$$

where $b \neq 0$ and $a \neq 0$.

Introduce the stopping times

$$\begin{aligned} \sigma_0 &= \inf\big\{t \geq 0 : \xi(t) = 0\big\}, \\ \tau_k &= \inf\big\{t \geq \sigma_k : \xi(t) = d\big\}, \\ \sigma_{k+1} &= \inf\big\{t \geq \tau_k : \xi(t) = 0\big\}, \quad k \geq 0. \end{aligned}$$

Observe that $\big\{\sigma_{k+1} - \sigma_k, k \geq 0\big\}$ are i.i.d. and $\big\{l_d(\sigma_{k+1}) - l_d(\sigma_k), k \geq 0\big\}$ are i.i.d. It can be shown (see §3.1) that $E\sigma_k < \infty$ and $El_d(\sigma_k) < \infty$.

So, by the law of large numbers,

$$\begin{aligned} \lim_{n\to\infty} \frac{l_d(\sigma_n)}{\sigma_n} &= \lim_{n\to\infty} \frac{l_d(\sigma_0) + \sum\limits_{k=0}^{n-1} \big(l_d(\sigma_{k+1}) - l_d(\sigma_k)\big)}{\sigma_0 + \sum\limits_{k=0}^{n-1} \big(\sigma_{k+1} - \sigma_k\big)} \\ &= \frac{E\big(l_d(\sigma_1) - l_d(\sigma_0)\big)}{E\big(\sigma_1 - \sigma_0\big)} = \frac{E_0 l_d(\sigma_1)}{E_0 \sigma_1} \quad \text{a.s.} \end{aligned}$$

Since $E(\sigma_1 - \sigma_0) < \infty$ and $E\big(l_d(\sigma_1) - l_d(\sigma_0)\big) < \infty$, it can be also deduced that

$$\lim_{t\to+\infty} \frac{l_d(t)}{t} = \lim_{n\to\infty} \frac{l_d(\sigma_n)}{\sigma_n} = \frac{E_0 l_d(\sigma_1)}{E_0 \sigma_1} \quad \text{a.s.} \tag{3.36}$$

The process l_d does not increase before ξ reaches d, so $E_0 l_d(\sigma_1) = E_d l_d(\sigma_0)$. The last expectation equals $u(d)$, where the functions $u(x)$ satisfies the equation (see Theorem 3.1.4)

$$Lu(x) = 0, \quad x \in [0,d], \tag{3.37}$$

with boundary conditions

$$u(0) = 0, \tag{3.38}$$

$$u'(d) = 1, \tag{3.39}$$

where

$$Lu(x) = au'(x) + \frac{b^2}{2}u''(x).$$

The general solution of (3.37) is $K_1 + K_2 \exp\{-2ax/b^2\}$. Boundary conditions (3.38) and (3.39) take the form

$$\begin{cases} K_1 + K_2 = 0 \\ -\dfrac{2K_2 a}{b^2}\exp\{-2ad/b^2\} = 1. \end{cases}$$

So

$$K_2 = -\frac{b^2}{2a}\exp\{2ad/b^2\},$$

$$E_0 l_d(\sigma_1) = \frac{b^2}{2a}\big(\exp\{2ad/b^2\} - 1\big). \tag{3.40}$$

To find $E_0\sigma_1$, observe that

$$E_0\sigma_1 = E_0\tau_1 + E_d\sigma_0.$$

By Theorem 3.1.3, we have that $E_0\tau_1 = u(0)$, where

$$\begin{cases} Lu(x) = -1, \quad x \in [0,d], \\ u(d) = 0, \\ u'(0) = 0. \end{cases}$$

Hence

$$u(x) = -\frac{x}{a} + K_1 + K_2 \exp\{-2ax/b^2\}.$$

It follows from the boundary conditions that

$$\begin{cases} -\dfrac{d}{a} + K_1 + K_2 \exp\{-2ad/b^2\} = 0, \\ -\dfrac{1}{a} - \dfrac{2aK_2}{b^2} = 0 \end{cases}$$

and

$$E_0 \tau_1 = u(0) = \frac{d}{a} + \frac{b^2}{2a^2}\big(\exp\{-2ad/b^2\} - 1\big). \tag{3.41}$$

Exercise 3.3.1 Check directly that the right hand side of (3.41) is positive.

Similarly to (3.41), we obtain

$$E_d \sigma_0 = -\frac{d}{a} + \frac{b^2}{2a^2}\big(\exp\{-2ad/b^2\} - 1\big). \tag{3.42}$$

Formulas (3.36), (3.40), (3.41) and (3.42) yield

$$\begin{aligned} \lim_{t\to+\infty} \frac{l_d(t)}{t} &= \frac{\frac{b^2}{2a}\big(\exp\{-2ad/b^2\} - 1\big)}{\frac{b^2}{2a^2}\big(\exp\{-2ad/b^2\} + \exp\{2ad/b^2\} - 2\big)} \\ &= \frac{a}{1 - \exp\{-2ad/b^2\}} \quad \text{a.s.} \end{aligned} \tag{3.43}$$

In particular, if $a = -c$ and $b = \sqrt{2}$ as in (3.34), then

$$\lim_{t\to\infty} \frac{l_d(t)}{t} = \frac{c}{\exp(cd) - 1}. \tag{3.44}$$

Exercise 3.3.2 Find $E_0 l_d(\sigma_1)$, $E_0 \tau_1$ and the limit $\lim_{t\to\infty} \dfrac{l_d(t)}{t}$ if $a = 0$, $b \neq 0$.

As we have noted, equation (3.35) may appear as a limit of queueing systems under very general assumptions. Certainly, it is almost impossible to obtain a nice formula for the average number of lost requests of general queueing systems. However, for a model $M/M/1/m$, the limit $\lim_{t\to\infty} L_m^{\alpha,\mu}(t)/t$ can be computed easily. Let us find this limit and compare with (3.43).

The process $X^{\alpha,\mu}(t)$, $t \geq 0$, is a birth-and-death Markov process. Thus, its stationary distribution $\pi = (\pi_0, \dots, \pi_m)$ satisfies the relation

$$\pi_k = (\alpha/\mu)^k \pi_0.$$

So

$$\pi_k = \frac{\left(\frac{\alpha}{\mu}\right)^k \left(1 - \frac{\alpha}{\mu}\right)}{1 - \left(\frac{\alpha}{\mu}\right)^{m+1}}.$$

Therefore

$$\lim_{t\to\infty} \frac{L_m^{\alpha,\mu}(t)}{t} = \alpha \pi_m = \frac{\left(\frac{\alpha}{\mu}\right)^m \left(1 - \frac{\alpha}{\mu}\right)}{1 - \left(\frac{\alpha}{\mu}\right)^{m+1}}.$$

If α, μ and m satisfy (3.31) and (3.32), then the last expression is equivalent to $\frac{c\sqrt{\alpha}}{\exp(cd)-1}$. This completely agrees with (3.33) and (3.44).

Exercise 3.3.3 Find $\lim_{t\to\infty} \frac{l_d(t)}{t}$ for the reflecting SDE

$$\mathrm{d}\,\xi(t) = a\big(\xi(t)\big)\,\mathrm{d}t + b\big(\xi(t)\big)\,\mathrm{d}w(t) + \mathrm{d}\,l_0(t) - \mathrm{d}\,l_d(t),$$

where a and b are Lipschitz continuous and $b(x) > 0$, $x \in [0, d]$, see [24].

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Index

These lecture notes are intended as a short introduction to diffusion processes on a domain with a reflecting boundary for graduate students, researchers in stochastic analysis and interested readers.

Specific results on stochastic differential equations with reflecting boundaries such as existence and uniqueness, the Markov property, relation to partial differential equations and submartingale problems are given. An extensive list of references to current literature is included.

This book has its origins in a mini-course the author gave at the University of Potsdam and at the Technical University of Berlin in winter 2013.

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