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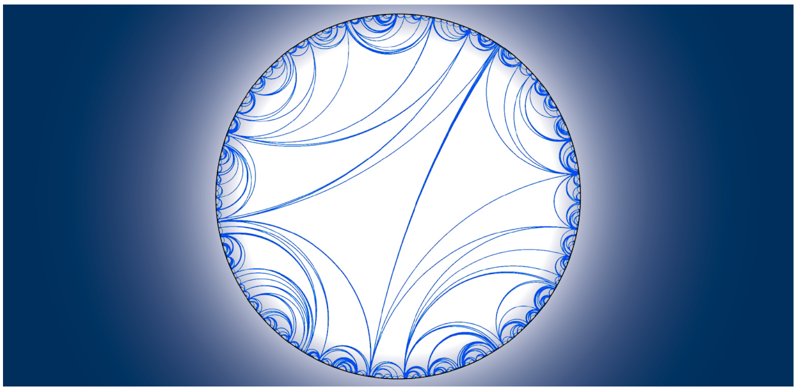

Franco Flandoli | Michael Högele

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A solution selection problem with small stable perturbations

Franco Flandoli, * Michael Högele†

July 14, 2014

Abstract

The zero-noise limit of differential equations with singular coefficients is investigated for the first time in the case when the noise is a general α-stable process. It is proved that extremal solutions are selected and the probability of selection is computed. Detailed analysis of the characteristic function of an exit time form the half-line is performed, with a suitable decomposition in small and large jumps adapted to the singular drift.

Keywords: stochastic differential equations, singular drifts, zero-noise limit, Peano phenomena, non-uniqueness, α-stable process, persistence probabilities, exit problem, selection of solutions.

2010 Mathematical Subject Classification: 60H10; 34A12; 60G52; 60G51; 60F99.

1 Introduction

The zero-noise limit of a stochastic differential equation, with drift vector field b and a Wiener process W, say of the form

$$X_t^\varepsilon = x_0 + \int_0^t b\left(X_s^\varepsilon\right) ds + \varepsilon W_t, \quad t \geqslant 0, \varepsilon > 0, \tag{1.1}$$

is a classical subject of probability, see for instance [10]. When the limit deterministic equation

$$X_t = x_0 + \int_0^t b\left(X_s\right) ds, \quad t \geqslant 0, \tag{1.2}$$

is well posed, usually one has $X_t^\varepsilon \to X_t$ a.s. and typical relevant questions are the speed of convergence and large deviations. On the contrary, when the Cauchy problem (1.2) has more than

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one solution, the first question concerns the *selection*, namely which solutions of (1.2) are selected in the limit and with which probability. This selection problem is still poorly understood and we aim to contribute with the investigation of the case when the noise is an α-stable process.

The case treated until now in the literature is the noise of Wiener type. All known quantitative results are restricted to equations in dimension one. The breakthrough on the subject was due to Bafico and Baldi [1] who solved the selection problem for very general drift b having one point x_0 of singularity. The typical example of b to test the theory is

$$b(x) = \begin{cases} B^+ |x|^{\beta^+} & \text{for} \quad x \geq 0 \\ -B^- |x|^{\beta^-} & \text{for} \quad x < 0. \end{cases} \tag{1.3}$$

where $B^\pm > 0$, $\beta^\pm \in (0,1)$; the deterministic equation (1.2) with $x_0 = 0$ has infinitely many solutions, which are equal to zero on $[0,\infty)$ or on some interval $[0,t_0]$ (possibly $t_0 = 0$) and then, on $[t_0,\infty)$, they are equal either to $C^+ (t-t_0)^{\frac{1}{1-\beta^+}}$ or to $-C^- (t-t_0)^{\frac{1}{1-\beta^-}}$, with $C^\pm$ given in (3.6) of Section 3; a central role will be played by the two *extremal* solution,

$$x^\pm = \pm C^\pm t^{\frac{1}{1-\beta^\pm}}.$$

The paper [1] completely solves the selection problem for this and more general examples, making use of explicit computations on the differential equations satisfied by suitable exit time probabilities; such equations are elliptic PDEs, in general, so they are explicitly solvable only in dimension one (except for particular cases). The final result is that the law P_ε^W, on $C([0,T];\mathbb{R})$, of the unique solution X_t^ε of equation (1.1) with $x_0 = 0$ and b as in (1.3), satisfies

$$P_\varepsilon^W \xrightarrow{w} p^+ \delta_{x^+} + p^- \delta_{x^-}$$

where $p^- = 1 - p^+$

$$p^+ = \begin{cases} 1 & \text{if} \qquad \beta^+ < \beta^- \\ \frac{(B^-)^{-\frac{1}{1+\beta}}}{(B^+)^{-\frac{1}{1+\beta}} + (B^-)^{-\frac{1}{1+\beta}}} & \text{if} \quad \beta^+ = \beta^- =: \beta \\ 0 & \text{if} \qquad \beta^+ > \beta^-. \end{cases} \tag{1.4}$$

This or part of this result was re-proved later on using other approaches, not based on elliptic PDEs but only on tools of stochastic analysis and dynamical arguments, see [5], [24]. These investigations are also motivated by the fact that in dimension greater than one the elliptic PDE approach is not possible.

The aim of this paper is to investigate these questions when the Wiener process W_t is replaced by a general α-stable process L_t. This process satisfies for any $a > 0$ the following self-similarity condition $(L_{at})_{t\geqslant 0} \stackrel{d}{=} (a^{\frac{1}{\alpha}} L_t + \gamma_0 t)_{t\geqslant 0}$, for a drift $\gamma_0 \in \mathbb{R}$ which accounts for the asymmetry of the law of L. The stochastic differential equation, then, takes the form

$$X_t^\varepsilon = x_0 + \int_0^t b\left(X_s^\varepsilon\right) ds + \varepsilon L_t, \qquad t \geqslant 0, \varepsilon > 0. \tag{1.5}$$

Here explicit solution of the elliptic equations for exit time probabilities are not feasible and thus it is again an example where we need to understand the problem with new tools and ideas. This feature is similar to the theory of asymptotic first exit times for equations with regular coefficients and small noise, see [14, 20, 7, 13] for recent progresses in the case of Lévy noise. This requires a careful understanding of the role of small and large jumps, which is conceptually new and interesting; technically the more demanding part is the estimate of the Laplace transform of the exit times. Some ingredients are also inspired by [5].

The final result is the following theorem.

Theorem 1. If $\alpha > 1 - (\beta^+ \wedge \beta^-)$, then

$$P_\varepsilon^L \xrightarrow{w} p^+ \delta_{x^+} + p^- \delta_{x^-}$$

where P_ε^L is the law, on Skorohod space $\mathbb{D}\left([0,T]; \mathbb{R}\right)$, of the unique solution X_t^ε of equation (1.1) with $x_0 = 0$ and $p^+, p^- = 1 - p^+$ are given as follows.

1. For strictly α-stable noise with drift $\gamma_0 = 0$, the probabilities are given by (1.4).

2. For α-stable noise with drift $\gamma_0 \neq 0$, we have the following cases.

   (a) For $\alpha \in (1, 2)$ the probability takes the values of (1.4).

   (b) For $\alpha \leqslant 1$ the probability p^+ is given as

$$p^+ = \begin{cases} 1, & \text{if } \gamma_0 > 0. \\ 0, & \text{if } \gamma_0 < 0. \end{cases}$$

The time interval where this convergence takes place can be chosen to be any bounded interval $[0, T]$, but with a suitable reformulation of the result it may also be an interval which increases like $\left[0, \varepsilon^{-\theta^*}\right]$, for suitable $\theta^* > 0$, see the technical statements below; this is a novelty compared with the literature on the Brownian case. For this purpose we solve an asymptotic first exit problem for the strong solution X^ε of (1.5) from a half-interval. This is a problem in its own right. The proof of this result yields an asymptotic lower bound of X^ε for times beyond the occurrence of the first "large" jump in an appropriate sense as stated in Corollary 5. Before such first large jump, that is on a time scale up to $\varepsilon^{-\theta^*}$ however, the system exhibits the mentioned behavior similar to a Brownian perturbation. Among the other technical novelties, there is the use of the linearized system in order to show that excursions away from the origin are large enough.

In order to understand the role of the drift γ_0 we for $\alpha \leqslant 1$ we would like to mention the following analogous situation. When $\gamma_0 > 0$ (the argument for $\gamma_0 < 0$ is symmetric), consider the following ODE depending on the parameter $\varepsilon \in (0,1)$:

$$x'(t) = b(x(t)) + \varepsilon\gamma_0, \qquad x(0) = 0.$$

The solution, call it $x^\varepsilon(t)$, is unique (in spite of the fact that b is not Lipschitz at $x=0$) and given by $x^\varepsilon(t) = H_\varepsilon^{-1}(t)$ where

$$H_\varepsilon(x) = \int_0^x \frac{dy}{b(y)+\varepsilon\gamma_0}.$$

When $\varepsilon \to 0$, $x^\varepsilon(t)$ converges to $x^+(t)$. We want to remark that this fact is similar to the the result of Theorem 1 for $\alpha \leqslant 1$, but not for $\alpha \in (1,2)$ where the fluctuating part of the noise prevails and the limit is different.

The article is structured as follows. After a brief set of notations, we setup and solve the previously mentioned first exit problem in Section 3. This is carried out for initial values which may approach 0 as a function of ε, however only sufficiently slowly, as $\varepsilon \to 0$. Section 4 zooms into the behavior for very short temporal and spatial scales around the origin and determines the exit probabilities to each side with the help of the self-similarity of the driving Lévy noise. In Section 5 it is shown that an unstable linearized intermediate regime stabilizes the exit direction from the small environment of the origin and rapidly enhances the solution until it reaches the area of initial values for the regime in Section 3. Section 6 concludes the proof of Theorem 1.

2 Preliminaries

For the following notation we refer to Sato [23]. A Lévy process L with values in the real line over a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a stochastic process $L = (L_t)_{t\geqslant 0}$ starting in $0 \in \mathbb{R}$ with independent and identically distributed increments.

The Lévy-Khintchine formula establishes the following representation of the characteristic function of the marginal law of the Lévy process Z. There exists a drift $\gamma \in \mathbb{R}$, $\sigma > 0$ and a σ-finite Borel measure ν on $\mathbb{R}$, the so-called *Lévy measure*, satisfying

$$\nu\{0\} = 0, \qquad \text{and} \qquad \int_{\mathbb{R}} (1 \wedge |u|)^2 \nu(du) < \infty, \tag{2.1}$$

such that for any $t \geqslant 0$ the characteristic function reads

$$\mathbb{E}[e^{i\langle z, L_t\rangle}] = e^{t\psi(z)}, \quad z \in \mathbb{R},$$
$$\psi(z) = i\langle \gamma, z\rangle - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}} (e^{i\langle z,y\rangle} - 1 - i\langle z, y\rangle \mathbf{1}\{|y| \leqslant 1\})\nu(dz). \tag{2.2}$$

The triplet (γ, σ, ν) determines the law of the process L uniquely.

An α-stable process L for $\alpha \in (0, 2)$ is a Lévy process with canonical triplet $(\gamma, 0, \nu)$, where $\gamma \in \mathbb{R}$ and ν is given as

$$\nu(dy) = \frac{c^-}{y^{\alpha+1}}\mathbf{1}\{y < 0\} + \frac{c^+}{y^{\alpha+1}}\mathbf{1}\{y > 0\},$$

where $c^+, c^- \geqslant 0$.

The family of α-stable processes satisfies the following self-similarity property. For an α-stable process L with given Lévy measure ν there is a drift $\gamma_0 \in \mathbb{R}$, such that for any $a > 0$

$$(L_{at})_{t\geqslant 0} \stackrel{d}{=} (a^{\frac{1}{\alpha}} L_t + \gamma_0 t)_{t\geqslant 0}. \tag{2.3}$$

Note that in general γ_0 does not coincide with γ in the Lévy-Chinchine representation. This depends on a chosen cutoff around 0. For details consult [23], Section 14. Instead, for $\alpha > 1$

$$\gamma_0 = \mathbb{E}[L_1] = \int_{\mathbb{R}} y\nu(dy), \tag{2.4}$$

where for $\alpha \in (0, 1)$

$$\gamma_0 = \int_{0<|y|\leqslant 1} y\nu(dy). \tag{2.5}$$

In case of the Cauchy distribution $\alpha = 1$ the drift γ_0 is given as the median of the law of L_1. In fact, in this article we will not exploit the concrete shape of γ_0, but the self-similarity.

An α-stable process L with $\gamma_0 = 0$ is called strictly α-stable. The representations (2.4) and (2.5) show in particular that for any strictly α-stable process, $\alpha \in (0, 2)$ with $\alpha \neq 1$, that $c^+ = c^-$. For $\alpha = 1$ this is a consequence of Theorem 14.7(ii) in [23].

Without loss of generality we shall restrict ourselves to the renormalized case $c^+ + c^- = 1$ in the exposition of this article.

Proposition 2. Let L be an α-stable process $\alpha \in (0, 2)$ over a given filtered probability space. Then equation (1.5) has a unique strong solution, which satisfies the strong Markov property.

The result is not surprising since α-stable distributions are absolutely continuous and the non-uniqueness of the deterministic flow occurs at a single point $\{0\}$, which is a Lebesgue zero set.

3 An exit problem from a half-interval

Theorem 3. For all $\beta \in (0,1)$ and $\alpha \in (0,2)$ there are monotonically increasing, continuous functions $\delta^+_\cdot, \delta^-_\cdot : (0,1) \to (0,1)$ such that the first exit time

$$\tau^{x,\varepsilon} := \inf\{t > 0 \mid X^{\varepsilon,x}_t \in [-\delta^-_\varepsilon, \delta^+_\varepsilon]\}$$

of the solution $X^{x,\varepsilon}$ of (1.5) satisfies for all functions $m_\varepsilon \to \infty$ with $\limsup_{\varepsilon\to 0} m_\varepsilon \varepsilon^\alpha < \infty$, that

$$\sup_{x \notin [-5\delta_\varepsilon, 5\delta_\varepsilon]} \lim_{\varepsilon \to 0} \mathbb{P}(\tau^{x,\varepsilon} \leqslant m_\varepsilon) = 0.$$

Proof. The proof is structured in four parts. After the technical preparation and two essential observations we derive the main recursion. In the last part we conclude.

1) Setting and notation: Let us denote $u(t;x) := X^{x,0}_t$ for convenience. The first observation is the following. Let $\delta > 0$ and $x \in \mathbb{R}$ an initial value with $|x| > \delta$. Then $b\big|_{\mathbb{R}\setminus[-\delta,\delta]}$ satisfies global Lipschitz and growth conditions, such that there exists a unique strong local solution, which lives until to the stopping time

$$\tau^{x,\varepsilon,\delta} := \inf\{t > 0 \mid X^{x,\varepsilon}_t \in [-\delta, \delta]\}.$$

Here the Lipschitz constant depends essentially on δ and explodes as $\delta \searrow 0$. As usually in this situation, we divide the process $L = \eta^\varepsilon + \xi^\varepsilon$ by a ε-dependent threshold $\varepsilon^{-\rho}$, where $\rho \in (0,1)$ is a parameter to be made precise in the sequel. More precisely the compound Poisson process with

$$\eta^\varepsilon_t = \sum_{i=1}^\infty W_i \mathbf{1}\{T_i \leqslant t\}$$

with arrival times $T_i = \sum_{j=1}^i t_i$, where t_i i.i.d. waiting times and i.i.d. "large" jump increments $(W_i)_{i\in\mathbb{N}}$ with the conditional law

$$W_i \sim \frac{1}{\lambda_\varepsilon}\nu(\cdot \cap (\mathbb{R} \setminus [-\varepsilon^{-\rho}, \varepsilon^{-\rho}])) \tag{3.1}$$

$$t_i \sim \mathrm{EXP}(\lambda_\varepsilon) \qquad \text{for } i \in \mathbb{N}, \tag{3.2}$$

where

$$\lambda_\varepsilon = \nu(\mathbb{R} \setminus [-\varepsilon^{-\rho}, \varepsilon^{-\rho}]) = 2\int_{\varepsilon^{-\rho}}^{\infty} \frac{dy}{y^{\alpha+1}} = \frac{2}{\alpha}\varepsilon^{\alpha\rho}, \tag{3.3}$$

and the remaining semi-martingale

$$\xi^\varepsilon = L - \eta^\varepsilon \tag{3.4}$$

with uniformly bounded jumps, which implies the existence of exponential moments. Let us denote by $Y^{x,\varepsilon}$ the solution of

$$Y_t^{x,\varepsilon} = x + \int_0^t b\left(Y_s^{x,\varepsilon}\right) ds + \varepsilon \xi_t^\varepsilon, \tag{3.5}$$

which exists uniquely under the same conditions as does $X^{x,\varepsilon}$. For $\delta > 0$ we fix the notation

$$\begin{aligned} D_\delta &:= \mathbb{R} \setminus [-\delta, \delta] \\ D_\delta^+ &:= (\delta, \infty). \end{aligned}$$

For a function $\delta_\cdot : (0,1) \to (0,1)$ with $\delta_\varepsilon \searrow 0$ to be specified later we fix

$$\begin{aligned} \tau^{x,\varepsilon} &:= \inf\{t > 0 \mid X_{t,x}^\varepsilon \notin D_{\delta_\varepsilon}\} \\ \tau^{x,\varepsilon,-} &:= \inf\{t > 0 \mid X_{t,x}^\varepsilon \notin D_{\delta_\varepsilon}^+\}. \end{aligned}$$

2) Two observations: The following observations reveal the first exit mechanism.

2.1) Up to the first large jump, the deterministic solutions travel sufficiently far: Separation of variables yields the explicit representation for $t \geqslant t'$ and $x \geqslant 0$

$$u(t; t', x) = \Big(B(1-\beta)(t - t') + x^{1-\beta}\Big)^{\frac{1}{1-\beta}}. \tag{3.6}$$

Hence for $z \geqslant x$ and $t' = 0$, we obtain

$$\begin{aligned} P\left(u(T_1; x) \geqslant z\right) &= P\Big(\big(B(1-\beta)T_1 + x^{1-\beta}\big)^{\frac{1}{1-\beta}} \geqslant z\Big) \\ &= P\big(T_1 \geqslant \frac{z^{1-\beta} - x^{1-\beta}}{B(1-\beta)}\big) \\ &= \exp\Big(-(z^{1-\beta} - x^{1-\beta})\frac{\lambda_\varepsilon}{B(1-\beta)}\Big) \\ &= P(Z \geqslant z \mid Z \geqslant x). \end{aligned}$$

This is the tail of the distribution function of a Weibull distributed random variable Z with shape parameter $1 - \beta$ and scaling parameter

$$\Big(\frac{\lambda_\varepsilon}{B(1-\beta)}\Big)^{\frac{1}{1-\beta}} = \frac{\varepsilon^{\frac{\alpha\rho}{1-\beta}}}{B(1-\beta)^{\frac{1}{1-\beta}}}$$

conditioned on the event $\{Z \geqslant x\}$. We define for $\Gamma > 1$ such that $\Gamma < \frac{1}{1-\beta}$ and

$$\gamma_\varepsilon := (\lambda_\varepsilon^{-\frac{1}{\Gamma}} - (3\delta_\varepsilon)^{1-\beta})^{\frac{1}{1-\beta}} \approx_\varepsilon \varepsilon^{-\frac{\alpha}{\Gamma}\frac{\rho}{1-\beta}}.$$

Hence

$$\lim_{\varepsilon\to 0+} \sup_{x\in D_{3\delta_\varepsilon}} \mathbb{P}(u(T_1;x) \geqslant \gamma_\varepsilon) \to 1, \tag{3.7}$$

and

$$\sup_{2\delta_\varepsilon \leqslant x \leqslant \gamma_\varepsilon} \mathbb{P}(u(T_1;x) \leqslant 2\gamma_\varepsilon) \approx_\varepsilon \lambda_\varepsilon^{1-\frac{1}{\Gamma}}.$$

2.2) Control the deviation of the small jump solution from the deterministic solution:
For each $\rho \in (0,1)$ there are functions $\delta_\cdot : (0,1) \to (0,1)$, $r^\cdot : (0,1) \to (0,\infty)$ such that

$$\varepsilon^{\alpha\rho} r^\varepsilon \to \infty \qquad \text{and} \qquad \frac{\delta_\varepsilon}{\varepsilon^{1-\rho} r^\varepsilon} \to \infty.$$

Put in other terms the first result means $r^\varepsilon \gtrsim_\varepsilon \frac{1}{\varepsilon^{\alpha\rho}}$. We define

$$r^\varepsilon := \frac{|\ln(\varepsilon)|^2}{\varepsilon^{\alpha\rho}}. \tag{3.8}$$

For the second expression we have

$$\infty \leftarrow \frac{\delta_\varepsilon}{\varepsilon^{1-\rho} r^\varepsilon} = \frac{\delta_\varepsilon}{\varepsilon^{1-\rho-\alpha\rho}} \frac{1}{\varepsilon^{\alpha\rho} r^\varepsilon}. \tag{3.9}$$

Therefore a necessary condition for (3.9) to be satisfied is $\delta_\varepsilon \gtrsim_\varepsilon \varepsilon^{1-\rho(1+\alpha)}$. We define

$$\delta_\varepsilon := \varepsilon^{1-\rho(1+\alpha)} |\ln(\varepsilon)|^4. \tag{3.10}$$

For the right-hand side to tend to 0 is equivalent to

$$\rho < \frac{1}{\alpha+1}. \tag{3.11}$$

In particular for all $\alpha \in (0,2)$

$$\alpha\rho < \frac{\alpha}{1+\alpha} < \frac{2}{3} < 1. \tag{3.12}$$

Since ξ^ε has exponential moments we can compensate it

$$\widetilde{\xi}^\varepsilon_t := \xi^\varepsilon_t - t\mathbb{E}[\xi^\varepsilon_1].$$

It is a direct consequence of Lemma 2.1 in [15], which treats the same situation, that for any $c > 0$

$$\mathbb{P}(\sup_{t\in[0,r^\varepsilon]} |\varepsilon\widetilde{\xi}^\varepsilon| > c) \leqslant \exp(-\frac{c}{\varepsilon^{1-\rho} r^\varepsilon}). \tag{3.13}$$

A small direct calculation or Lemma 3.1 in [15] yields that there is constant $h_1 > 0$ such that

$$|\mathbb{E}[\varepsilon\xi^\varepsilon_1]| \leqslant \varepsilon|\gamma_0| + h_1\varepsilon^{1-\rho}.$$

The choice of r^ε in (3.8) and ρ in (3.11) we first obtain that $\varepsilon r^\varepsilon \gamma_0 \to 0$ as $\varepsilon \to 0$ and also

$$h_1 r^\varepsilon \varepsilon^{1-\rho} = \varepsilon^{1-(\alpha+1)\rho} |\ln(\varepsilon)|^2 \leqslant_\varepsilon \varepsilon^{1-(\alpha+1)\rho} |\ln(\varepsilon)|^4 = \delta_\varepsilon. \tag{3.14}$$

Hence for any $\varepsilon > 0$ sufficiently small we have $|r^\varepsilon \mathbb{E}[\varepsilon \xi_1^\varepsilon]| \leqslant \delta_\varepsilon$ and infer

$$\begin{aligned}
\mathbb{P}(\sup_{t\in[0,T_1]} |\varepsilon \xi_t^\varepsilon| > 2c) &= \mathbb{P}(\sup_{t\in[0,T_1]} |\varepsilon \widetilde{\xi}_t^\varepsilon - t\varepsilon\gamma_0| > 2c) \\
&\leqslant \mathbb{P}(\sup_{t\in[0,r^\varepsilon]} |\varepsilon \widetilde{\xi}_t^\varepsilon| > c) + \mathbb{P}(T_1 > r^\varepsilon) \\
&\leqslant \exp(-\frac{c}{\varepsilon^{1-\rho} r^\varepsilon}) + \exp(-\varepsilon^{\alpha\rho} r^\varepsilon).
\end{aligned} \tag{3.15}$$

Denote $V_t^{x,\varepsilon} = Y_t^{x,\varepsilon} - 2c - \varepsilon \xi_t^\varepsilon$. The monotonicity of b on $(0,\infty)$ yields on the events $\{t \in [0,T_1]\}$ and $\{\sup_{t\in[0,T_1]} |\varepsilon \xi_s^\varepsilon| \leqslant 2c\}$ that

$$\begin{aligned}
V_t^{x,\varepsilon} &= x - 2c + \int_0^t b(V_s^{x,\varepsilon} + 2c + \varepsilon \xi_s^\varepsilon) ds \\
&\geqslant x - 2c + \int_0^t b(V_s^{x,\varepsilon}) ds.
\end{aligned}$$

By (3.14) we may set $c = \delta_\varepsilon$ we obtain

$$V_t^{x,\varepsilon} \geqslant x - 2\delta_\varepsilon + \int_0^t b(V_s^{x,\varepsilon}) ds \qquad t \in [0,T_1].$$

Hence an elementary comparison argument implies under these assumptions

$$V_t^{x,\varepsilon} \geqslant u(t; x - 2\delta_\varepsilon), \qquad \text{for all} \quad t \in [0,T_1], \quad x \geqslant 2\delta_\varepsilon.$$

In particular in the preceding setting we take the supremum over all $x \geqslant 4\delta_\varepsilon$ and obtain

$$\begin{aligned}
&\sup_{x\in D^+_{4\delta_\varepsilon}} \mathbb{P}(\sup_{t\in[0,T_1]} (Y_t^{x,\varepsilon} - (u(t;x-2\delta_\varepsilon) - 2\delta_\varepsilon)) < 0) \\
&\leqslant \mathbb{P}(\sup_{t\in[0,T_1]} |\varepsilon \xi^\varepsilon| > 2\delta_\varepsilon) \leqslant \exp(-\frac{\delta_\varepsilon}{\varepsilon^{1-\rho} r^\varepsilon}) + \exp(-\varepsilon^{\alpha\rho} r^\varepsilon) = 2\varepsilon^2.
\end{aligned} \tag{3.16}$$

With the identical reasoning we obtain

$$\begin{aligned}
\sup_{x\geqslant(i-1)\gamma_\varepsilon} \mathbb{P}(\sup_{t\in[0,T_1]} (Y_t^{x,\varepsilon} - (u(t;x-2\delta_\varepsilon) - 2\delta_\varepsilon)) < 0) &\leqslant \mathbb{P}(\sup_{t\in[0,T_1]} |\varepsilon \xi^\varepsilon| > i\gamma_\varepsilon) \\
&\leqslant \exp(-\frac{i\gamma_\varepsilon}{2\varepsilon^{1-\rho} r^\varepsilon}) + \exp(-\frac{i\varepsilon^{\alpha\rho} r^\varepsilon}{2}).
\end{aligned} \tag{3.17}$$

Remark 3.1. In the light of the observations 2.1) and 2.2) it is clear that the exit behavior is mainly determined by the behavior of the large jumps εW_i.

3) Estimate of the Laplace transform of the exit time: We estimate the Laplace transform of the first exit time. Let $\theta > 0$. Then

$$\begin{aligned}
\sup_{x\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\varepsilon^\alpha \tau^{x,\varepsilon,-}}\Big] &= \sum_{k=1}^\infty \sup_{x\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\varepsilon^\alpha \tau^{x,\varepsilon,-}} \mathbf{1}\{\tau^{x,\varepsilon,-} \in (T_{k-1}, T_k]\}\Big] \\
&\leqslant \sum_{k=1}^\infty \sup_{x\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\varepsilon^\alpha T_{k-1}} \mathbf{1}\{\tau^{x,\varepsilon,-} \in (T_{k-1}, T_k]\}\Big] \\
&\leqslant \sum_{k=1}^{n_\varepsilon} \sup_{x\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\varepsilon^\alpha T_{k-1}} \mathbf{1}\{\tau^{x,\varepsilon,-} \in (T_{k-1}, T_k]\}\Big] + \sum_{k=n_\varepsilon}^\infty \mathbb{E}\Big[e^{-\theta\varepsilon^\alpha T_1}\Big]^k \\
&=: \sum_{k=1}^{n_\varepsilon} \mathcal{I}_1(k) + \mathcal{I}_2 =: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

3.1) The infinite remainder: For the second sum we obtain

$$\begin{aligned}
\mathcal{I}_2 &= \sum_{k=n_\varepsilon}^\infty \Big(\frac{1}{1+\frac{\theta\varepsilon^\alpha}{\lambda_\varepsilon}}\Big)^k = \sum_{k=n_\varepsilon}^\infty e^{k\ln\left(1-\frac{\theta\varepsilon^\alpha}{\lambda_\varepsilon}\right)} \lesssim_\varepsilon \sum_{k=n_\varepsilon}^\infty e^{-k\frac{2\theta\varepsilon^\alpha}{\lambda_\varepsilon}} = \frac{e^{-n_\varepsilon\frac{2\theta\varepsilon^\alpha}{\lambda_\varepsilon}}}{1-e^{-\frac{2\theta\varepsilon^\alpha}{\lambda_\varepsilon}}} \\
&\lesssim_\varepsilon \frac{e^{-n_\varepsilon\frac{2\theta\varepsilon^\alpha}{\lambda_\varepsilon}}}{\frac{2\theta\varepsilon^\alpha}{\lambda_\varepsilon}} = e^{-n_\varepsilon\frac{2\theta\varepsilon^\alpha}{\lambda_\varepsilon} - \ln(\frac{2\theta\varepsilon^\alpha}{\lambda_\varepsilon})} =: S_1(\varepsilon).
\end{aligned} \tag{3.18}$$

In order to get $S_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, we need the asymptotics

$$n_\varepsilon \varepsilon^{\alpha(1-\rho)} + \ln(\varepsilon) \to \infty, \tag{3.19}$$

or for simplicity

$$n_\varepsilon \gtrsim_\varepsilon \frac{1}{\varepsilon^{\alpha(1-\rho)}} + |\ln(\varepsilon)|.$$

If we define

$$n_\varepsilon := \frac{|\ln(\varepsilon)|^2}{\varepsilon^{\alpha(1-\rho)}}, \tag{3.20}$$

we obtain

$$S_1(\varepsilon) \approx_\varepsilon \varepsilon^{2+\alpha(1-\rho)} \to 0 \qquad \text{as } \varepsilon \to 0.$$

3.2) Estimate of the main sum: The rest of the proof is devoted to estimate $\sum_{k=0}^{n_\varepsilon} \mathcal{I}_1(k)$. We define the following events for $y \in D^+_{5\delta_\varepsilon}$ and $s, t \geqslant 0$ by

$$\begin{aligned}
A^-_{t,s,y} &:= \{X^{\cdot,\varepsilon}_r \circ \theta_s(y) \in D^+_{\delta_\varepsilon} \text{ for all } r \in [0,t]\}, \\
B^-_{t,s,y} &:= \{X^{\cdot,\varepsilon}_r \circ \theta_s(y) \in D^+_{\delta_\varepsilon} \text{ for all } r \in [0,t) \text{ and } X^{\cdot,\varepsilon}_t \circ \theta_s(y) \notin D^+_{\delta_\varepsilon}\}.
\end{aligned}$$

Recall the waiting times $t_k := T_k - T_{k-1}$ and exploit the decomposition

$$\{\tau^{x,\varepsilon,-} \in (T_{k-1}, T_k]\} = \bigcap_{i=1}^{k-1} A^-_{t_i, T_{i-1}, X_{T_{i-1},x}} \cap \Big(\bigcup_{t\in(0,t_k]} B^-_{t,T_{k-1},x}\Big).$$

3.2.1) Derivation of the recursion for the idealized exit from an unstable point 0**:** We estimate $\mathcal{I}_1(k)$ with the help of the strong Markov property

$$\begin{aligned}
&\mathcal{I}_1(k)\\
&= \sup_{x\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[\mathbb{E}\Big[\prod_{i=1}^{k-1} e^{-\theta\lambda_\varepsilon t_i}\mathbf{1}\Big(A^-_{t_i,T_{i-1},X_{T_{i-1},x}}\Big)\\
&\quad\Big(\mathbf{1}\Big\{u(T_1;x-2\delta_\varepsilon)-2\delta_\varepsilon+\varepsilon W_1 > \lambda_\varepsilon^{-\frac{1}{\Gamma(1-\beta)}}\Big\} + (\mathbf{1}\Big\{u(T_1;x-2\delta_\varepsilon)-2\delta_\varepsilon+\varepsilon W_1 \leqslant \lambda_\varepsilon^{-\frac{1}{\Gamma(1-\beta)}}\Big\}\Big)\\
&\quad\Big(\mathbf{1}\big\{\sup_{t\in[0,T_1]}(Y^{x,\varepsilon,1}_t-(u(t;x-2\delta_\varepsilon)-22\delta_\varepsilon))\geqslant 0\big\}+\mathbf{1}\big\{\sup_{t\in[0,T_1]}(Y^{x,\varepsilon,1}_t-(u(t;x-2\delta_\varepsilon)-2\delta_\varepsilon))<0\big\}\Big)\\
&\qquad\qquad\mathbf{1}\big(\bigcup_{t\in(0,T_k-T_{k-1}]} B^-_{t,T_{k-1},x}\big)\mid\mathcal{F}_{T_1}\Big]\Big]\\
&\leqslant \sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\Big(A^-_{T_1,0,y}\Big)\mathbf{1}\big\{\sup_{t\in[0,T_1]}(Y^{y,\varepsilon,1}_t-(u(t;y-2\delta_\varepsilon)-2\delta_\varepsilon))\geqslant 0\big\}\Big]\\
&\quad\sup_{y\geqslant\gamma_\varepsilon}\mathbb{E}\Big[\prod_{i=1}^{k-1} e^{-\theta\lambda_\varepsilon t_i}\mathbf{1}\Big(A^-_{t_i,T_{i-1},X_{T_{i-1},y}}\Big)\mathbf{1}\Big(\bigcup_{t\in(0,T_k-T_{k-1}]} B^-_{t,T_{k-1},y}\Big)\Big]\\
&\quad+\sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\big\{\sup_{t\in[0,T_1]}(Y^{y,\varepsilon,1}_t-(u(t;y-2\delta_\varepsilon)-2\delta_\varepsilon))<0\big\}\Big]\\
&\quad+\sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\Big\{u(T_1;y-2\delta_\varepsilon)-2\delta_\varepsilon+\varepsilon W_1 \leqslant \lambda_\varepsilon^{-\frac{1}{\Gamma(1-\beta)}}\Big\}\Big]
\end{aligned}$$

where we recall that $\gamma_\varepsilon = (\lambda_\varepsilon^{-\frac{1}{\Gamma}} - (5\delta_\varepsilon)^{1-\beta})^{\frac{1}{1-\beta}}$. Taking a closer look we may identify the preceding inequality as the recursive estimate

$$\begin{aligned}
&\sup_{x\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_{k-1}}\mathbf{1}\{\tau^{x,\varepsilon,-}\in(T_{k-1},T_k]\}\Big]\\
&\leqslant \sup_{y\geqslant\gamma_\varepsilon}\mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_{k-2}}\mathbf{1}\{\tau^{y,\varepsilon,-}\in(T_{k-2},T_{k-1}]\}\Big]\\
&\quad\cdot\sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\Big(A^-_{T_1,0,y}\Big)\mathbf{1}\big\{\sup_{t\in[0,T_1]}(Y^{y,\varepsilon,1}_t-(u(t;y-2\delta_\varepsilon)-2\delta_\varepsilon))\geqslant 0\big\}\Big]\\
&\quad+\sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\big\{\sup_{t\in[0,T_1]}(Y^{y,\varepsilon,1}_t-(u(t;y-2\delta_\varepsilon)-2\delta_\varepsilon))<0\big\}\Big]\\
&\quad+\sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\Big\{u(T_1;y-2\delta_\varepsilon)-2\delta_\varepsilon+\varepsilon W_1 \leqslant \lambda_\varepsilon^{-\frac{1}{\Gamma(1-\beta)}}\Big\}\Big].
\end{aligned}\tag{3.21}$$

The same reasoning yields for all $2 \leqslant i \leqslant k$ the recursive inequality

$$
\begin{aligned}
&\sup_{x \geqslant (i-1)\gamma_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_{k-1}} \mathbf{1}\{\tau^{x,\varepsilon,-} \in (T_{k-1}, T_k]\}\Big] \\
&\leqslant \sup_{y \geqslant i\gamma_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_{k-2}} \mathbf{1}\{\tau^{y,\varepsilon,-} \in (T_{k-2}, T_{k-1}]\}\Big] \\
&\qquad \cdot \sup_{y \geqslant (i-1)\gamma_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1} \mathbf{1}\Big(A^-_{T_1,0,y}\Big) \mathbf{1}\big\{\sup_{t\in[0,T_1]} (Y^{y,\varepsilon,1}_t - (u(t; y-2\delta_\varepsilon) - 2\delta_\varepsilon)) \geqslant 0\big\}\Big] \\
&\qquad + \sup_{y \geqslant (i-1)\gamma_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1} \mathbf{1}\big\{\sup_{t\in[0,T_1]} (Y^{y,\varepsilon,1}_t - (u(t; y-2\delta_\varepsilon) - 2\delta_\varepsilon)) < 0\big\}\Big] \\
&\qquad + \sup_{y \geqslant (i-1)\gamma_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1} \mathbf{1}\Big\{u(T_1; y-2\delta_\varepsilon) - 2\delta_\varepsilon + \varepsilon W_1 \leqslant \lambda_\varepsilon^{-\frac{1}{\Gamma(1-\beta)}}\Big\}\Big].
\end{aligned}
$$

Hence solving the recursion we obtain

$$
\begin{aligned}
\mathcal{I}_1(k) &\leqslant \prod_{j=1}^{k-1} \sup_{y \geqslant (j-1)\gamma_\varepsilon \vee 5\delta_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1} \mathbf{1}\Big(A^-_{T_1,0,y}\Big) \mathbf{1}\big\{\sup_{t\in[0,T_1]} (Y^{y,\varepsilon,1}_t - (u(t; y-2\delta_\varepsilon) - 2\delta_\varepsilon)) \geqslant 0\big\}\Big] \\
&\quad \cdot \sup_{y \geqslant (k-1)\gamma_\varepsilon \vee 5\delta_\varepsilon} \mathbb{P}\Big(\tau^{y,\varepsilon,-} \in (0, T_1]\Big) \\
&\quad + \sum_{i=1}^{k-2} \sup_{y \in (i-1)\gamma_\varepsilon \vee 5\delta_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1} \mathbf{1}\big\{\sup_{t\in[0,T_1]} (Y^{y,\varepsilon,1}_t - (u(t; y-2\delta_\varepsilon) - 2\delta_\varepsilon)) < 0\big\}\Big] \\
&\quad + \sum_{i=1}^{k-2} \sup_{y \geqslant (i-1)\gamma_\varepsilon \vee 5\delta_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1} \mathbf{1}\Big\{u(T_1; y-\delta_\varepsilon) + \varepsilon W_1 \leqslant \lambda_\varepsilon^{-\frac{1}{\Gamma(1-\beta)}}\Big\}\Big]. \qquad (3.22)
\end{aligned}
$$

3.2.2) Estimate of the second sum of the recursion (3.22): By (3.16) and (3.17) there exists $\varepsilon_0 \in (0,1)$ such that for $\varepsilon \in (0, \varepsilon_0]$

$$
\begin{aligned}
&\sum_{k=1}^{n_\varepsilon} \sum_{i=1}^{k-2} \sup_{y \in (i-1)\gamma_\varepsilon \vee 5\delta_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1} \mathbf{1}\big\{\sup_{t\in[0,T_1]} (Y^{y,\varepsilon,1}_t - (u(t; y-2\delta_\varepsilon) - 2\delta_\varepsilon)) < 0\big\}\Big] \\
&\leqslant n_\varepsilon \sum_{i=1}^{\infty} \sup_{y \geqslant (i-1)\gamma_\varepsilon \vee 5\delta_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1} \mathbf{1}\big\{\sup_{t\in[0,T_1]} (Y^{y,\varepsilon,1}_t - (u(t; y-2\delta_\varepsilon) - 2\delta_\varepsilon)) < 0\big\}\Big] \\
&\leqslant 2n_\varepsilon (\exp(-\frac{\delta_\varepsilon}{\varepsilon^{1-\rho} r^\varepsilon}) + \exp(-\varepsilon^{\alpha\rho} r^\varepsilon)) =: S_2(\varepsilon) \searrow 0,
\end{aligned}
$$

with the convention $\sum^{-1} = 0$. We determine the order of S_2

$$
\begin{aligned}
&n_\varepsilon (\exp(-\frac{\delta_\varepsilon}{\varepsilon^{1-\rho} r^\varepsilon}) + \exp(-\varepsilon^{\alpha\rho} r^\varepsilon)) \\
&= |\ln(\varepsilon)|^2 \varepsilon^{-\alpha(1-\rho)} \exp(-\frac{\varepsilon^{1-\rho(1+\alpha)} |\ln(\varepsilon)|^4}{\varepsilon^{1-\rho} \varepsilon^{-\alpha\rho} |\ln(\varepsilon)|^2}) + |\ln(\varepsilon)|^2 \varepsilon^{-\alpha(1-\rho)} \exp(-\varepsilon^{-\alpha\rho} |\ln(\varepsilon)|^2 \varepsilon^{\alpha\rho}) \\
&= 2|\ln(\varepsilon)|^2 \varepsilon^{2-\alpha+\alpha\rho}.
\end{aligned}
$$

3.2.3) Estimate of the third sum in the recursion (3.22): For $i = 0$ and $0 < \varepsilon \leqslant \varepsilon_0$ we perform the core calculation of the article. The idea is the following: $X_t^{x,\varepsilon} \gtrsim_\varepsilon u(t; x - 2\delta) - 2\delta_\varepsilon + \varepsilon W_1 \mathbf{1}\{t = T_1\}$ for all $t \in [0, T_1]$. For small ε and $5\delta_\varepsilon < x \leqslant \gamma_\varepsilon$ the solution $u(T_1, x - 2\delta_\varepsilon) - 2\delta_\varepsilon$ escapes sufficiently far away from x, that is $u(T_1, x - 2\delta_\varepsilon) - 2\delta_\varepsilon \geqslant 2\gamma_\varepsilon$, such that the probability that $u(T_1, x - 2\delta_\varepsilon) - 2\delta_\varepsilon + \varepsilon W_1 < \gamma_\varepsilon$ decays sufficiently fast.

3.2.3.1) Estimate of the backbone decomposition of the first exit event: Due to the independence of T_1 and W_1 we may calculate for $\gamma_\varepsilon^*(x) = \frac{(2\gamma_\varepsilon + 2\delta_\varepsilon)^{1-\beta} - (x - \delta_\varepsilon)^{1-\beta}}{B(1-\beta)}$

$$
\begin{aligned}
&\sup_{5\delta_\varepsilon < x \leqslant \gamma_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1} \mathbf{1}\Big(u(T_1; x - 2\delta_\varepsilon) - 2\delta_\varepsilon + \varepsilon W_1 \leqslant \gamma_\varepsilon\Big)\Big] \\
&\leqslant \sup_{5\delta_\varepsilon < x \leqslant \gamma_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1} \mathbf{1}\big(u(T_1; x - 2\delta_\varepsilon) - 2\delta_\varepsilon + \varepsilon W_1 \leqslant \gamma_\varepsilon\big) \mathbf{1}\big(u(T_1; x - 2\delta_\varepsilon) > 2\gamma_\varepsilon + \delta_\varepsilon\big)\Big] \\
&\quad + \sup_{5\delta_\varepsilon < x \leqslant \gamma_\varepsilon} \mathbb{P}(u(T_1; x - 2\delta_\varepsilon) \leqslant 2\gamma_\varepsilon + 2\delta_\varepsilon) \\
&= \sup_{5\delta_\varepsilon < x \leqslant \gamma_\varepsilon} \int_{\gamma_\varepsilon^*(x)}^{\infty} \mathbb{P}(u(t; x - 2\delta_\varepsilon) - 2\delta_\varepsilon + \varepsilon W_1 \leqslant \gamma_\varepsilon) \lambda_\varepsilon e^{-\lambda_\varepsilon t} dt + \sup_{5\delta_\varepsilon < x \leqslant \gamma_\varepsilon} \mathbb{P}(u(T_1; x - 2\delta_\varepsilon) \leqslant 2\gamma_\varepsilon + 2\delta_\varepsilon)
\end{aligned}
\tag{3.23}
$$

The second term is known from (3.7) and tends to 0, hence it remains to calculate the first one.

$$
\begin{aligned}
&\sup_{5\delta_\varepsilon < x \leqslant \gamma_\varepsilon} \int_{\gamma_\varepsilon^*(x)}^{\infty} \mathbb{P}(u(t; x - 2\delta_\varepsilon) - 2\delta_\varepsilon + \varepsilon W_1 \leqslant \gamma_\varepsilon) \lambda_\varepsilon e^{-\lambda_\varepsilon t} dt \\
&= \sup_{5\delta_\varepsilon < x \leqslant \gamma_\varepsilon} \int_{\gamma_\varepsilon^*(x)}^{\infty} \nu((-\infty, \frac{1}{\varepsilon}(\gamma_\varepsilon - (u(t; x - 2\delta_\varepsilon) - 2\delta_\varepsilon)]) e^{-\lambda_\varepsilon t} dt \\
&= \sup_{5\delta_\varepsilon < x \leqslant \gamma_\varepsilon} \int_{\gamma_\varepsilon^*(x)}^{\infty} \nu((-\infty, \frac{1}{\varepsilon}(\gamma_\varepsilon + 2\delta_\varepsilon - (B(1-\beta)t + (x - 2\delta_\varepsilon)^{1-\beta})^{\frac{1}{1-\beta}})]) e^{-\lambda_\varepsilon t} dt \\
&= \sup_{5\delta_\varepsilon < x \leqslant \gamma_\varepsilon} \frac{\alpha}{4} \frac{\varepsilon^\alpha}{\lambda_\varepsilon} \int_{\gamma_\varepsilon^*}^{\infty} \frac{1}{((B(1-\beta)t + (x - 2\delta_\varepsilon)^{1-\beta})^{\frac{1}{1-\beta}} - (\gamma_\varepsilon + \delta_\varepsilon))^\alpha} \lambda_\varepsilon e^{-\lambda_\varepsilon t} dt \\
&\leqslant \frac{\alpha}{4} \frac{\varepsilon^\alpha}{\lambda_\varepsilon} \frac{1}{\gamma_\varepsilon^\alpha}.
\end{aligned}
\tag{3.24}
$$

The term

$$
\frac{\varepsilon^\alpha}{\lambda_\varepsilon} \frac{1}{\gamma_\varepsilon^\alpha} \approx_\varepsilon \varepsilon^{\alpha(1-\rho) + \frac{\alpha^2 \rho}{\Gamma(1-\beta)}},
$$

converges to 0 as $\varepsilon \to 0$. This gives an estimate for the last term in (3.21). The last term in (3.22) deals with initial values $(i-1)\gamma_\varepsilon < x \leqslant i\gamma_\varepsilon$. We obtain for

$$
\gamma_\varepsilon^*(i, x) := \frac{((i+1)\gamma_\varepsilon + \delta_\varepsilon)^{1-\beta} - x^{1-\beta}}{B(1-\beta)}
$$

with the analogous calculations the following estimate

$$
\begin{aligned}
&\sup_{(i-1)\gamma_\varepsilon<x\leqslant i\gamma_\varepsilon}\mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\Big(u(T_1;x-2\delta_\varepsilon)-2\delta_\varepsilon+\varepsilon W_1\leqslant\gamma_\varepsilon\Big)\Big]\\
&=\sup_{(i-1)\gamma_\varepsilon<x\leqslant i\gamma_\varepsilon}\frac{\alpha}{4}\frac{\varepsilon^\alpha}{\lambda_\varepsilon}\int\limits_{\gamma_\varepsilon^*(i,x)}^{\infty}\frac{1}{((B(1-\beta)t+(x-2\delta_\varepsilon)^{1-\beta})^{\frac{1}{1-\beta}}-(\gamma_\varepsilon-2\delta_\varepsilon))^\alpha}\lambda_\varepsilon e^{-\lambda_\varepsilon t}dt\\
&\quad+\sup_{(i-1)\gamma_\varepsilon<x\leqslant i\gamma_\varepsilon}\mathbb{P}(u(T_1;x-2\delta_\varepsilon)\leqslant(i+1)\gamma_\varepsilon+2\delta_\varepsilon)\\
&\leqslant\frac{\alpha}{4}\frac{\varepsilon^\alpha}{\lambda_\varepsilon}\frac{1}{\gamma_\varepsilon^\alpha i^\alpha}+\sup_{(i-1)\gamma_\varepsilon<x\leqslant i\gamma_\varepsilon}\mathbb{P}(u(T_1;x-2\delta_\varepsilon)\leqslant(i+1)\gamma_\varepsilon+2\delta_\varepsilon).
\end{aligned}\tag{3.25}
$$

Combining the estimates (3.23), (3.24) and (3.25) we obtain for any $C>1$

$$
\begin{aligned}
&\sum_{i=1}^{k-2}\sup_{y\geqslant(i-1)\gamma_\varepsilon\vee 5\delta_\varepsilon}\mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\Big\{u(T_1;y-2\delta_\varepsilon)-2\delta_\varepsilon+\varepsilon W_1\leqslant\lambda_\varepsilon^{-\frac{1}{2(1-\beta)}}\Big\}\Big]\\
&=\sum_{i=1}^{k-2}\sup_{j\geqslant i}\sup_{(j-1)\gamma_\varepsilon\vee 5\delta_\varepsilon<y\leqslant j\gamma_\varepsilon}\mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\Big\{u(T_1;y-2\delta_\varepsilon)-2\delta_\varepsilon+\varepsilon W_1\leqslant\lambda_\varepsilon^{-\frac{1}{2(1-\beta)}}\Big\}\Big]\\
&\lesssim_\varepsilon\sum_{i=1}^{k-2}\sup_{j\geqslant i}\Big(\frac{\alpha}{4}\frac{\varepsilon^\alpha}{\lambda_\varepsilon}\frac{1}{\gamma_\varepsilon^\alpha j^\alpha}+\sup_{(j-1)\gamma_\varepsilon<x\leqslant j\gamma_\varepsilon}\mathbb{P}(u(T_1;x-2\delta_\varepsilon)\leqslant(j+1)\gamma_\varepsilon+2\delta_\varepsilon)\Big)\\
&\lesssim_\varepsilon\sum_{i=1}^{k-2}\Big(\frac{\alpha}{4}\frac{\varepsilon^\alpha}{\lambda_\varepsilon}\frac{1}{\gamma_\varepsilon^\alpha i^\alpha}+C(1-\exp(-[(i+1)^{1-\beta}-i^{1-\beta}]\frac{\gamma_\varepsilon^{1-\beta}\lambda_\varepsilon}{B(1-\beta)}))\Big)\\
&\leqslant\sum_{i=1}^{k-2}\Big(\frac{\alpha}{4}\frac{\varepsilon^\alpha}{\lambda_\varepsilon}\frac{1}{\gamma_\varepsilon^\alpha i^\alpha}+C[(i+1)^{1-\beta}-i^{1-\beta}]\frac{\gamma_\varepsilon^{1-\beta}\lambda_\varepsilon}{B(1-\beta)}\Big)\\
&\leqslant\sum_{i=1}^{k-2}\Big(\frac{\alpha}{4}\frac{\varepsilon^\alpha}{\lambda_\varepsilon}\frac{1}{\gamma_\varepsilon^\alpha i^\alpha}+\frac{C}{B}\frac{\gamma_\varepsilon^{1-\beta}\lambda_\varepsilon}{i^\beta}\Big)\\
&=\frac{\alpha}{4}\frac{\varepsilon^\alpha}{\lambda_\varepsilon}\frac{1}{\gamma_\varepsilon^\alpha}\sum_{i=1}^{k-2}\frac{1}{i^\alpha}+\frac{C}{B}\gamma_\varepsilon^{1-\beta}\lambda_\varepsilon\sum_{i=1}^{k-2}\frac{1}{i^\beta}\\
&\leqslant C\frac{\alpha}{4}\frac{\varepsilon^\alpha}{\lambda_\varepsilon}\frac{1}{\gamma_\varepsilon^\alpha}k^{1-\alpha}+\frac{C}{B}\gamma_\varepsilon^{1-\beta}\lambda_\varepsilon k^{1-\beta}.
\end{aligned}\tag{3.26}
$$

Hence we may sum up

$$
\begin{aligned}
&\sup_{5\delta_\varepsilon<x\leqslant\gamma_\varepsilon}\mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\Big(u(T_1;x-2\delta_\varepsilon)-2\delta_\varepsilon+\varepsilon W_1\leqslant\gamma_\varepsilon\Big)\Big]\\
&+\sum_{k=2}^{n_\varepsilon}\sum_{i=1}^{k-2}\sup_{y\geqslant(i-1)\gamma_\varepsilon\vee 5\delta_\varepsilon}\mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\Big\{u(T_1;y-2\delta_\varepsilon)-2\delta_\varepsilon+\varepsilon W_1\leqslant\lambda_\varepsilon^{-\frac{1}{\Gamma(1-\beta)}}\Big\}\Big]\\
&\lesssim_\varepsilon\frac{\varepsilon^\alpha}{\lambda_\varepsilon}\frac{1}{\gamma_\varepsilon^\alpha}+\lambda_\varepsilon^{1-\frac{1}{\Gamma}}+\frac{C\alpha}{4}\frac{\varepsilon^\alpha}{\lambda_\varepsilon}\frac{1}{\gamma_\varepsilon^\alpha}(n_\varepsilon)^{2-\alpha}+\frac{C}{1-\beta}\gamma_\varepsilon^{1-\beta}\lambda_\varepsilon(n_\varepsilon)^{2-\beta}=:S_3(\varepsilon)
\end{aligned}\tag{3.27}
$$

3.2.3.2) Conditions on parameters in order to establish the convergence $S_3(\varepsilon)\to 0$:

- We check the order of the second to last expression on the right-hand side

$$\varepsilon^{\alpha(1-\rho(1-\frac{\alpha}{\Gamma(1-\beta)}))} n_\varepsilon^{2-\alpha} \approx_\varepsilon \varepsilon^{\alpha(1-\rho(1-\frac{\alpha}{\Gamma(1-\beta)}))-\alpha(2-\alpha)(1-\rho)} |\ln(\varepsilon)|^{2(2-\alpha)}$$
$$= \varepsilon^{\alpha[(1-\rho(1-\frac{\alpha}{\Gamma(1-\beta)}))-(2-\alpha)(1-\rho)]} |\ln(\varepsilon)|^{2(2-\alpha)}.$$

The essential sign of the exponent hence is given as the sign of

$$(1-\rho) + \frac{\rho\alpha}{\Gamma(1-\beta)} - (2-\alpha)(1-\rho) = (\alpha-1)(1-\rho) + \frac{\rho\alpha}{\Gamma(1-\beta)}. \tag{3.28}$$

  - For $1 \leqslant \alpha < 2$ the sign is positive, since all terms are nonnegative and the last term is positive.
  - For $0 < \alpha < 1$ we calculate that the positivity of (3.28)

$$0 < -(1-\alpha)(1-\rho) + \frac{\rho\alpha}{2(1-\beta)} = -(1-\alpha) + \rho[\frac{\alpha}{2(1-\beta)} + (1-\alpha)]$$

    is equivalent to

$$\rho_0(\alpha,\beta) := \frac{\Gamma(1-\alpha)(1-\beta)}{\Gamma(1-\alpha)(1-\beta)+\alpha} < \rho$$

    where the right-hand side is strictly less than 1. Hence in this case the sign is positive if we choose $\rho_0 < \rho < 1$.

- For the second expression on the right-hand side we obtain

$$\gamma_\varepsilon^{1-\beta}\lambda_\varepsilon(n_\varepsilon)^{2-\beta} \approx_\varepsilon \varepsilon^{-(1-\beta)\frac{\alpha\rho}{\Gamma(1-\beta)}}\varepsilon^{\alpha\rho}\varepsilon^{-\alpha(1-\rho)(2-\beta)}|\ln(\varepsilon)|^{2-\beta} = \varepsilon^{\alpha\rho(1-\frac{1}{\Gamma})-\alpha(1-\rho)(1-\beta)}|\ln(\varepsilon)|^{2-\beta}.$$

The positivity of the exponent depends on the sign of

$$0 < (1-\frac{1}{\Gamma})\rho - (1-\rho)(1-\beta) = \rho((1-\frac{1}{\Gamma}) + (1-\beta)) - (1-\beta),$$

which is equivalent to

$$\rho > \frac{(1-\frac{1}{\Gamma})(1-\beta)}{(1-\frac{1}{\Gamma})(1-\beta)+1} =: \rho_1(\beta).$$

Since $\rho_1(\beta) < 1$ for all $\rho_1 < \rho < 1$ the second exponent is also positive.

3.2.3.3) Verify the compatibility of the choice of convergent parameters: We check that the parameters β and α are compatible with $\rho < \frac{1}{1+\alpha}$ in (3.29), which ensures that $\delta_\varepsilon \to 0$, as

$\varepsilon \to 0$. The first convergence in (3.27) yields

$$
\begin{aligned}
&\rho_0 = \frac{\Gamma(1-\alpha)(1-\beta)}{\Gamma(1-\alpha)(1-\beta)+\alpha} < \frac{1}{1+\alpha} \\
\Leftrightarrow \quad &\Gamma(1+\alpha)(1-\alpha)(1-\beta) < \Gamma(1-\alpha)(1-\beta)+\alpha \\
\Leftrightarrow \quad &\Gamma(1-\beta) - \Gamma(1-\beta)\alpha^2 < \Gamma(1-\beta) - \Gamma\alpha(1-\beta)+\alpha \\
\Leftrightarrow \quad &-\Gamma(1-\beta)\alpha^2 < -\Gamma\alpha(1-\beta)+\alpha \\
\Leftrightarrow \quad &0 < \Gamma(1-\beta)\alpha^2 + \Gamma\alpha\beta - (\Gamma-1)\alpha = \alpha(\Gamma(1-\beta)\alpha - (\Gamma-1-\Gamma\beta)) \\
\Leftrightarrow \quad &0 < \Gamma(1-\beta)\alpha - (\Gamma-1-\Gamma\beta) \\
\Leftrightarrow \quad &0 < \Gamma\alpha - \frac{\Gamma-1-\Gamma\beta}{1-\beta} \\
\Leftrightarrow \quad &\frac{\Gamma-1-\Gamma\beta}{\Gamma(1-\beta)} < \alpha,
\end{aligned}
$$

where the left hand side < 0, since $\Gamma < \frac{1}{1-\beta}$ and it does not impose a restriction on α. The second condition gives

$$
\rho_1 = \frac{(1-\frac{1}{\Gamma})(1-\beta)}{(1-\frac{1}{\Gamma})(1-\beta)+1} < \frac{1}{1+\alpha} \quad \Leftrightarrow \quad \frac{(1-\frac{1}{\Gamma})(1-\beta)+1}{(1-\frac{1}{\Gamma})(1-\beta)} > 1+\alpha \quad \Leftrightarrow \quad \frac{1}{(1-\frac{1}{\Gamma})(1-\beta)} > \alpha,
$$

In order to get rid of restrictions on α we calculate

$$
2 \leqslant \frac{1}{(1-\frac{1}{\Gamma})(1-\beta)} \quad \Leftrightarrow \quad (1-\frac{1}{\Gamma}) \leqslant \frac{1}{2(1-\beta)} \quad \Leftrightarrow \quad \Gamma \leqslant \frac{2(1-\beta)}{2(1-\beta)-1}.
$$

Choosing $\Gamma := \frac{1}{2}\Big(1+\frac{1}{2}\Big(\frac{1}{1-\beta}+\frac{2(1-\beta)}{2(1-\beta)-1}\Big)\Big)$ we can always choose

$$
\rho := \frac{1}{2}\Big(\rho_1(\beta) + \frac{1}{1+\alpha}\Big), \tag{3.29}
$$

satisfying all conditions required before.

3.2.4) Estimate of the first sum of the recursion (3.22): It remains to estimate the expression

$$
\begin{aligned}
&\sum_{k=1}^{n_\varepsilon} \prod_{j=1}^{k-1} \sup_{y \geqslant (j-1)\gamma_\varepsilon \vee 5\delta_\varepsilon} \mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1} \mathbf{1}\Big(A^-_{T_1,0,y}\Big)\mathbf{1}\big\{\sup_{t\in[0,T_1]} (Y^{y,\varepsilon,1}_t - (u(t;y-2\delta_\varepsilon)-2\delta_\varepsilon)) \geqslant 0\big\}\Big] \\
&\qquad \cdot \sup_{y \geqslant (k-1)\gamma_\varepsilon \vee 5\delta_\varepsilon} \mathbb{P}\Big(\tau^{y,\varepsilon,-} \in (0,T_1]\Big).
\end{aligned}
$$

3.2.4.1) We estimate the factors one by one: For $j \geqslant 2$

$$
\begin{aligned}
&\sup_{y \geqslant (j-1)\gamma_\varepsilon \vee 5\delta_\varepsilon} \mathbb{E}\Big[\mathbf{1}\Big(A^-_{T_1,0,y}\Big)\mathbf{1}\big\{\inf_{t\in[0,T_1]} (Y^{y,\varepsilon,1}_t - (u(t;y-2\delta_\varepsilon)-2\delta_\varepsilon)) \geqslant 0\big\}\Big] \\
&\quad \lesssim_\varepsilon 1 - (1-C)\mathbb{P}(\varepsilon W_1 < -(j-1)\gamma_\varepsilon) \\
&\quad = 1 - \frac{(1-C)}{2}\Big(\frac{\varepsilon}{(j-1)\gamma_\varepsilon}\Big)^{\alpha\rho} \qquad (3.30)
\end{aligned}
$$

and for $j = 1$

$$\begin{aligned}
&\sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[\mathbf{1}\Big(A^-_{T_1,0,y}\Big)\mathbf{1}\big\{\inf_{t\in[0,T_1]}(Y^{y,\varepsilon,1}_t - (u(t;y-2\delta_\varepsilon) - 2\delta_\varepsilon)) \geqslant 0\big\}\Big] \\
&\leqslant \sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[\mathbf{1}\Big(A^-_{T_1,0,y}\Big)\mathbf{1}\big\{\inf_{t\in[0,T_1]}(Y^{y,\varepsilon,1}_t - (u(t;y-2\delta_\varepsilon) - 2\delta_\varepsilon)) \geqslant 0\big\}\mathbf{1}\{u(T_1,y-2\delta_\varepsilon) \geqslant 2\gamma_\varepsilon\}\Big] \\
&\quad + \sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{P}(u(t;y) \leqslant 2\gamma_\varepsilon + 2\delta_\varepsilon) \\
&\quad \lesssim_\varepsilon 1 - (1-C)\mathbb{P}(\varepsilon W_1 < -\gamma_\varepsilon) + C\lambda_\varepsilon^{1-\frac{1}{\Gamma}} \\
&\quad \lesssim_\varepsilon 1 - \frac{(1-C)}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho} + C\varepsilon^{\alpha\rho(1+\frac{1}{\Gamma})}. \qquad (3.31)
\end{aligned}$$

We estimate for $k \geqslant 2$ with the help of (3.16)

$$\begin{aligned}
&\sup_{y\geqslant(k-1)\gamma_\varepsilon} \mathbb{P}\Big(\tau^{x,\varepsilon,-} \in (0,T_1]\Big) \\
&\leqslant \mathbb{P}\Big(W_1 < -(k-1)\frac{\gamma_\varepsilon}{\varepsilon}\Big) + \sup_{y\geqslant(k-1)\gamma_\varepsilon} \mathbb{P}\Big(\sup_{t\in[0,T_1]}(Y^{y,\varepsilon,1}_t - (u(t;y) - 2\delta_\varepsilon)) > 0\Big) \\
&\leqslant \frac{1}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho}\frac{1}{(k-1)^{\alpha\rho}} + \sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{P}\Big(\sup_{t\in[0,T_1]}(Y^{y,\varepsilon,1}_t - (u(t;y) - 2\delta_\varepsilon)) > 0\Big) \qquad (3.32) \\
&\leqslant \frac{1}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho}\frac{1}{(k-1)^{\alpha\rho}} + 2\varepsilon^2 \qquad (3.33)
\end{aligned}$$

whereas for $k = 1$

$$\begin{aligned}
&\sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{P}\Big(\tau^{x,\varepsilon,-} \in (0,T_1]\Big) \\
&\leqslant \frac{1}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho} + \sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{P}\Big(\sup_{t\in[0,T_1]}(Y^{y,\varepsilon,1}_t - (u(t;y) - 2\delta_\varepsilon)) > 0\Big) + \sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{P}(u(T_1,y) \leqslant \gamma_\varepsilon + 2\delta_\varepsilon) \\
&\leqslant \frac{1}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho} + \sup_{y\in D^+_{5\delta_\varepsilon}} \mathbb{P}\Big(\sup_{t\in[0,T_1]}(Y^{y,\varepsilon,1}_t - (u(t;y) - 2\delta_\varepsilon)) > 0\Big) + \frac{2}{B(1-\beta)}\lambda_\varepsilon^{1-\frac{1}{\Gamma}}, \qquad (3.34)
\end{aligned}$$

where the last term is known from (3.7).

3.2.4.2) Estimate of the entire sum: Collecting the previous (3.30), (3.31), (3.33), (3.34) and for the small noise estimate (3.17) together with (3.26) we continue

$$\begin{aligned}
&\sum_{k=1}^{n_\varepsilon}\prod_{j=1}^{k-1}\sup_{y\geqslant(j-1)\gamma_\varepsilon\vee 5\delta_\varepsilon}\mathbb{E}\Big[e^{-\theta\lambda_\varepsilon T_1}\mathbf{1}\Big(A^-_{T_1,0,y}\Big)\mathbf{1}\{\sup_{t\in[0,T_1]}(Y_t^{y,\varepsilon,1}-(u(t;y-2\delta_\varepsilon)-2\delta_\varepsilon))\geqslant 0\}\Big]\\
&\qquad\cdot\sup_{y\geqslant(k-1)\gamma_\varepsilon\vee 5\delta_\varepsilon}\mathbb{P}\Big(\tau^{y,\varepsilon,-}\in(0,T_1]\Big)\\
&\leqslant\frac{1}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho}+\sup_{y\in D^+_{5\delta_\varepsilon}}\mathbb{P}\Big(\sup_{t\in[0,T_1]}(Y_t^{y,\varepsilon,1}-(u(t;y)-2\delta_\varepsilon))>0\Big)+\frac{2}{B(1-\beta)}\lambda_\varepsilon^{1-\frac{1}{\Gamma}}\\
&\quad+\frac{1}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho}\sum_{k=1}^{n_\varepsilon}\Big(1-\frac{(1-C)}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho}\Big)^{k-1}\frac{1}{k^{\alpha\rho}}\\
&\quad+C\varepsilon^2\sum_{k=2}^{n_\varepsilon}\Big(1-\frac{(1-C)}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho}\Big)^{k-2}\frac{1}{k^{\alpha\rho}}.
\end{aligned}$$

We identify

$$\frac{1}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho}\sum_{k=1}^{n_\varepsilon}\Big(1-\frac{(1-C)}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho}\Big)^{k-1}\frac{1}{k^{\alpha\rho}}\lesssim_\varepsilon\varepsilon^\kappa\,\mathrm{Li}_{\alpha\rho}\Big(1-\frac{(1-C)}{2}\varepsilon^\kappa\Big),$$

where

$$\kappa=\alpha\rho(1+\frac{\alpha\rho}{\Gamma(1-\beta)})$$

and $\mathrm{Li}_a(x)=\sum_{k=1}^\infty\frac{x^k}{k^a}$ is the polylogarithm function with parameter $a\in\mathbb{R}$ and $x\in(0,1)$, a well-known analytic extension of the logarithm. Recall that $\alpha\rho<\frac{\alpha}{1+\alpha}<1$ due to (3.12). By the following representation [16], Section 25.12, for $a\neq\mathbb{N}$ and $0<x<1$, given by

$$\mathrm{Li}_a(x)=\Gamma(1-a)(\ln(\frac{1}{x}))^{a-1}+\sum_{n=0}^\infty\zeta(a-n)\frac{(\ln(x))^n}{n!},\tag{3.35}$$

we obtain that for $a\in(0,1)$

$$\lim_{x\nearrow 1}\mathrm{Li}_a(x)/(1-x)^{a-1}=\Gamma(1-a).$$

Hence there is $C>0$ such that for $\varepsilon\in(0,\varepsilon_0)$ sufficiently small

$$\begin{aligned}
&\frac{1}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho}\sum_{k=1}^{n_\varepsilon}\Big(1-\frac{(1-C)}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho}\Big)^{k-1}\frac{1}{k^{\alpha\rho}}\\
&\leqslant\varepsilon^\kappa\,\mathrm{Li}_{\alpha\rho}\Big(1-\frac{(1-C)}{2}\varepsilon^\kappa\Big)\\
&\leqslant C\varepsilon^\kappa\varepsilon^{-\kappa(1-\alpha\rho)}=\varepsilon^{\kappa\alpha\rho}=S_4(\varepsilon)\searrow 0.
\end{aligned}$$

The same polylogarithmic asymptotics is carried out for

$$\begin{aligned}
&C\varepsilon^2 \sum_{k=2}^{n_\varepsilon} \Big(1 - \frac{(1-C)}{2}\Big(\frac{\varepsilon}{\gamma_\varepsilon}\Big)^{\alpha\rho}\Big)^{k-2} \frac{1}{k^{\alpha\rho}} \\
&\leqslant C\varepsilon^2 \operatorname{Li}_{\alpha\rho}\Big(1 - \frac{(1-C)}{2}\varepsilon^{\kappa}\Big) \\
&\leqslant C\varepsilon^2 \varepsilon^{-\kappa(1-\alpha\rho)} = \varepsilon^{2+\kappa\alpha\rho-\kappa} = S_5(\varepsilon) \searrow 0,
\end{aligned}$$

since due to $\Gamma(1-\beta) < 1$

$$2 - (\frac{\alpha\rho}{\Gamma(1-\beta)} + 1)(\alpha\rho - 1) \geqslant 2 - (\alpha\rho - 1)(\alpha\rho + 1) = 2 - (\alpha\rho^2 - 1) = 3 - (\alpha\rho)^2 > 0.$$

4) Estimate of the exit probabilities: For all $m > 0$

$$\begin{aligned}
\sup_{y \in D^+_{5\delta_\varepsilon}} \mathbb{P}(\tau^{y,\varepsilon} \leqslant m) &\leqslant \sup_{y \in D^+_{5\delta_\varepsilon}} \mathbb{P}(\tau^{y,\varepsilon,-} \leqslant m) \\
&= \sup_{y \in D^+_{5\delta_\varepsilon}} \mathbb{P}(e^{-\theta\varepsilon^\alpha \tau^{y,\varepsilon,-}} \geqslant e^{\theta\varepsilon^\alpha m}) \\
&\leqslant \sup_{y \in D^+_{5\delta_\varepsilon}} \mathbb{E}\Big[e^{-\theta\varepsilon^\alpha \tau^{y,\varepsilon,-}}\Big] e^{\theta\varepsilon^\alpha m} \\
&\leqslant \underbrace{C(S_1(\varepsilon) + S_2(\varepsilon) + S_3(\varepsilon) + S_4(\varepsilon) + S_5(\varepsilon))}_{=:S(\varepsilon)} e^{\theta\varepsilon^\alpha m}.
\end{aligned}$$

Replacing m by m_ε with $\limsup_{\varepsilon\to 0} m_\varepsilon \varepsilon^\alpha < \infty$ we obtain

$$\sup_{y \in D^+_{5\delta_\varepsilon}} \mathbb{P}(\tau^{y,\varepsilon} \leqslant m_\varepsilon) \lesssim_\varepsilon S(\varepsilon) \to 0.$$

S_6 can be chosen to be a monotonic function. This finishes the proof.

□

Remark 4. In Theorem 3 we have not specified the rates of convergence. For all

$$\rho_2(\alpha, \beta) < \rho < \frac{1}{1+\alpha}. \tag{3.36}$$

we have that the upper bound of the desired exit probability is of order

$$S(\varepsilon) \approx_\varepsilon S_1(\varepsilon) \vee S_2(\varepsilon) \vee S_3(\varepsilon) \vee S_4(\varepsilon) \vee S_5(\varepsilon),$$

where we collect

$$\begin{aligned}
S_1(\varepsilon) &\approx_\varepsilon \varepsilon^{2+\alpha(1-\rho)} \\
S_2(\varepsilon) &\approx_\varepsilon \varepsilon^{2-\alpha+\alpha\rho} \\
S_3(\varepsilon) &\approx_\varepsilon \varepsilon^{\alpha(1-\rho)+\frac{\alpha^2\rho}{\Gamma(1-\beta)}} + \varepsilon^{\alpha\rho(1-\frac{1}{\Gamma})} + \varepsilon^{\alpha[(1-\rho(1-\frac{\alpha}{\Gamma(1-\beta)}))-(2-\alpha)(1-\rho)]} + \varepsilon^{\alpha\rho(1-\frac{1}{\Gamma})-\alpha(1-\rho)(1-\beta)}.
\end{aligned}$$

For convenience we write $\kappa = \alpha\rho(1 + \frac{\alpha\rho}{2(1-\beta)})$ and further collect

$$S_4(\varepsilon) \approx_\varepsilon \varepsilon^{\kappa\alpha\rho}$$
$$S_5(\varepsilon) \approx_\varepsilon \varepsilon^{3-\alpha\rho}.$$

Since S_1, S_2, S_4, S_5 are of order greater or equal than $(\alpha\rho)^2$, the lowest order is $S_3(\varepsilon)$.

$$S(\varepsilon) \approx_\varepsilon S_3(\varepsilon).$$

Taking a close look at (3.27) the first term of S_3 is of larger order than the third term. In the same way, the second term of S_3 is obviously of larger order than the fourth term and can be neglected asymptotically. Hence we obtain the polynomial order

$$S(\varepsilon) \approx_\varepsilon \varepsilon^{\alpha[(1-\rho(1-\frac{\alpha}{\Gamma(1-\beta)}))-(2-\alpha)(1-\rho)]} + \varepsilon^{\alpha\rho(1-\frac{1}{\Gamma})-\alpha(1-\rho)(1-\beta)}. \tag{3.37}$$

The second term we are interested is δ_ε, which determines the proximity of the initial values to 0, is of polynomial order

$$\delta_\varepsilon \approx_\varepsilon \varepsilon^{1-\rho(1+\alpha)}. \tag{3.38}$$

Corollary 5. Let the assumptions of the last theorem be satisfied and ρ being chosen according to (3.36) and $\limsup_{\varepsilon\to 0} m_\varepsilon \varepsilon^\alpha < \infty$. Construct recursively

$$\begin{aligned}
U_t^{x,\varepsilon,1} &:= \Big(u(t; x - \delta_\varepsilon) - \delta_\varepsilon + W_1 \mathbf{1}\{t = T_1\}\Big) \wedge \gamma_\varepsilon, && t \in [0, T_1]\\
U_t^{x,\varepsilon,n+1} &:= \Big(u(t - T_n; U_{T_n}^{x,\varepsilon,n} - \delta_\varepsilon) - \delta_\varepsilon + W_{n+1}\mathbf{1}\{t = T_{n+1} - T_n\}\Big) \wedge \gamma_\varepsilon, && t \in (0, T_{n+1} - T_n]\\
Z_t^{x,\varepsilon} &:= \sum_{n=1}^{\infty} U_t^{x,\varepsilon,n} \mathbf{1}\{t \in (T_n, T_{n+1}]\}, && t \geqslant 0.
\end{aligned}$$

where the arrival times T_n of the large jump increments W_n are defined in (3.1), (3.2) and (3.3). Then

$$\liminf_{\varepsilon\to 0} \inf_{x\geqslant 5\delta_\varepsilon} \mathbb{P}(\sup_{t\in[0,m_\varepsilon]} X_t^{x,\varepsilon} - Z_t^{x,\varepsilon} \geqslant 0) = 1$$

This is a mere reformulation of the proof of Theorem 3. The process we compare $X^{\varepsilon,x}$ to the deterministic solution $u(\cdot; x)$, starting in x with large heavy-tailed jump increments $(T_n^\varepsilon, W_n^\varepsilon \wedge \gamma_\varepsilon)$, where the increments W_n^ε are cut-off from below by a value γ_ε. The choice of γ_ε has to satisfy two things: First, the deterministic trajectory has to overcome it during the waiting time $T_{n+1}^\varepsilon - T_n^\varepsilon$ with a probability tending to 1. Second, for larger and larger initial values $i\gamma_\varepsilon < x \leqslant (i+1)\gamma_\varepsilon$, the probability that $u(t, x) + \varepsilon W_i \leqslant \gamma_\varepsilon$ has to decrease for growing i and decreasing ε with a sufficiently large.

Corollary 6. Let the assumptions of Theorem 3 be satisfied and δ_ε being chosen according to (3.10). Then for any $m_\cdot : (0,1) \to (0,\infty)$ satisfying $\lim_{\varepsilon\to 0} m_\varepsilon \varepsilon^{\alpha\rho} = 0$. and $c > 0$ we have

$$\inf_{x \geqslant 5\delta_\varepsilon} \mathbb{P}(\sup_{t\in[0,m_\varepsilon]} X_t^{x,\varepsilon} - x_t^+ \geqslant -c) = 1.$$

Proof. First we obtain by a comparison argument that for all $x \geqslant 2\delta_\varepsilon$

$$u(t;x) \geqslant x_t^+ \qquad t \geqslant 0.$$

Secondly we observe that $U_t^{x,\varepsilon 1} = u(t;x)$ for $t < T_1$ and $\mathbb{P}(T_1 \geqslant m_\varepsilon) = e^{-m_\varepsilon \lambda_\varepsilon} \approx_\varepsilon e^{-m_\varepsilon \varepsilon^{\alpha\rho}} \to 1$. Hence combining these findings with inequality (3.16) we obtain

$$\lim_{\varepsilon\to 0} \inf_{x\geqslant 5\delta_\varepsilon} \mathbb{P}(\sup_{t\in[0,m_\varepsilon]} X_t^{x,\varepsilon} - x_t^+ > -\delta_\varepsilon) = 0.$$

$\square$

Lemma 7. Let the assumptions of the Theorem 3 be satisfied and δ_ε being chosen according to (3.10). Then for any $m_\cdot : (0,1) \to (0,\infty)$ satisfying $\lim_{\varepsilon\to 0} m_\varepsilon \varepsilon^{\alpha\rho} = 0$. and $c > 0$ we have

$$\inf_{x\geqslant 5\delta_\varepsilon} \mathbb{P}(\sup_{t\in[0,m_\varepsilon]} X_t^{x,\varepsilon} - x_t^+ < \delta_\varepsilon^{\frac{\beta^2}{2}}) = 1.$$

Proof. First choose ρ we choose according to (3.36) and $x \geqslant 5\delta_\varepsilon$. Recall for $t \in [0, T_1]$ the notation

$$X_t^{\varepsilon,x} = Y_t^{\varepsilon,x} + \varepsilon W_1 \mathbf{1}\{t = T_1\}$$

and

$$V_t^{x,\varepsilon} = Y_t^{x,\varepsilon} - \varepsilon \xi_t^\varepsilon.$$

The subadditivity of $b(y) = B|y|^\beta$ on $(0,\infty)$ yields on the events $\{t < T_1\}$ and $\{\sup_{t\in[0,T_1]} |\varepsilon \xi_s^\varepsilon| \leqslant \delta_\varepsilon\}$ that

$$\begin{aligned} V_t^{\varepsilon,x} &\leqslant x + \int_0^t b(V_s^{x,\varepsilon}) ds + B\delta_\varepsilon^\beta t \\ &\leqslant x + B\delta_\varepsilon^\beta \widetilde{m}_\varepsilon + \int_0^t b(V_s^{x,\varepsilon}) ds, \end{aligned}$$

where $\widetilde{m}_\varepsilon := \delta_\varepsilon^{-\frac{1}{2}\beta} \wedge r_\varepsilon$ with $\delta_\varepsilon = \varepsilon^{1-\rho(1+\alpha)} |\ln(\varepsilon)|^4$ in (3.10) and $r_\varepsilon = \varepsilon^{-\alpha\rho} |\ln(\varepsilon)|^2$ defined in (3.8).

Then Bihari's inequality [19], Theorem 8.3, implies for $x = 5\delta_\varepsilon$

$$
\begin{aligned}
&\sup_{t\in[0,\widetilde{m}_\varepsilon]} V_t^{\varepsilon,x} - x_t^+ \\
&\leqslant \sup_{t\in[0,\widetilde{m}_\varepsilon]} \Big[\Big((1-\beta)Bt + (5\delta_\varepsilon + B\delta_\varepsilon^\beta \widetilde{m}_\varepsilon)^{1-\beta}\Big)^{\frac{1}{1-\beta}} - ((1-\beta)Bt)^{\frac{1}{1-\beta}}\Big] \\
&\leqslant \Big[\Big((1-\beta)B\widetilde{m}_\varepsilon + (5\delta_\varepsilon + B\delta_\varepsilon^\beta \widetilde{m}_\varepsilon))^{1-\beta}\Big)^{\frac{1}{1-\beta}} - ((1-\beta)B\widetilde{m}_\varepsilon)^{\frac{1}{1-\beta}}\Big] \\
&\leqslant 2^{\frac{1}{1-\beta}-1}(5\delta_\varepsilon + B\delta_\varepsilon^\beta \widetilde{m}_\varepsilon)^{1-\beta} \to 0.
\end{aligned}
$$

Note that the bound of the right-hand side is of order

$$
(5\delta_\varepsilon + B\delta_\varepsilon^\beta \widetilde{m}_\varepsilon)^{1-\beta} \lesssim_\varepsilon \delta_\varepsilon^{\frac{\beta(1-\beta)}{2}}.
$$

Hence for any $c > 0$ there is $\varepsilon_0 \in (0,1)$ such that for $\varepsilon \in (0, \varepsilon_0]$ we have

$$
\inf_{x\geqslant 5\delta_\varepsilon} \mathbb{P}(X_t^{\varepsilon,x} - x_t^+ > \delta_\varepsilon^{\frac{\beta^2}{2}}) \leqslant 1 - \mathbb{P}(T_1 > r_\varepsilon) - \mathbb{P}(\sup_{t\in[0,r_\varepsilon]} |\varepsilon\xi_t^\varepsilon| > \delta_\varepsilon) \to 1,
$$

as $\varepsilon \to 0$. □

Combining Corollary 6 and Lemma 7 we obtain the main result of this section.

Corollary 8. Let the assumptions of the Theorem 3 be satisfied and δ_ε chosen as in (3.10). Then there exists $\theta^* > 0$ such that

$$
\lim_{\varepsilon\to 0+} \sup_{x\geqslant 5\delta_\varepsilon} \mathbb{P}(\sup_{t\in[0,\varepsilon^{-\theta^*}]} |X_t^{\varepsilon,x} - x_t^+| > \delta_\varepsilon^{\frac{\beta^2}{2}}) = 0.
$$

4 The solution leaves a small environment of the origin in a short time

Let us denote by $(X_t)_{t\geqslant 0}$ the strong solution $(X_t^{\varepsilon,0})_{t\geqslant 0}$ of system (1.5) with initial value $x = 0$. In addition we stipulate for $r_1, r_2 > 0$

$$\tau^\varepsilon(r_1, r_2) := \inf\{t > 0 : X_t \leqslant -r_1 \text{ or } X_t \geqslant r_2\}. \tag{4.1}$$

and abbreviate for convenience $\tau_{r_1,r_2} = \tau^\varepsilon(r_1, r_2)$.

4.1 Typical noise induced exit from a neighborhood of the origin

Proposition 9. There are monotonically increasing functions $\Theta_\cdot^+, \Theta_\cdot^-, t_\cdot : (0,1) \to (0,1)$ with $\lim_{\varepsilon\to 0+}\Theta_\varepsilon^+ = \lim_{\varepsilon\to 0+}\Theta_\varepsilon^- = \lim_{\varepsilon\to 0+} t_\varepsilon = 0$, such that for any function $\hat{t}_\cdot : (0,1) \to (0,\infty)$ satisfying $\lim_{\varepsilon\to 0}\hat{t}_\varepsilon / t_\varepsilon = +\infty$ we have

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\tau_{\Theta_\varepsilon^- - \varepsilon t_\varepsilon\gamma_0, \Theta_\varepsilon^+ + \varepsilon t_\varepsilon \gamma_0} > \hat{t}_\varepsilon\right) = 0.$$

We omit the iteration argument by Markov property. The key result is the followin.

Lemma 10. Under the previous assumptions and

$$\alpha > 1 - (\beta^+ \wedge \beta^-)$$

we have the following statement. There are monotonically increasing functions $\Theta_\cdot^+, \Theta_\cdot^-, t_\cdot : (0,1) \to (0,1)$ with $\lim_{\varepsilon\to 0+}\Theta_\varepsilon^+ = \lim_{\varepsilon\to 0+}\Theta_\varepsilon^- = \lim_{\varepsilon\to 0+} t_\varepsilon = 0$, such that we have

$$\lim_{\varepsilon\to 0+}\mathbb{P}\left(\tau_{\Theta_\varepsilon^- - \varepsilon t_\varepsilon\gamma_0, \Theta_\varepsilon^+ + \varepsilon t_\varepsilon \gamma_0} > t_\varepsilon\right) < 1.$$

Proof. Assume there are $\Theta_\varepsilon^+, \Theta_\varepsilon^-, t_\varepsilon$ as in the statement of the lemma and let us abbreviate for convenience $\sigma = \tau_{\Theta_\varepsilon^- - \varepsilon t_\varepsilon\gamma_0, \Theta_\varepsilon^+ + \varepsilon t_\varepsilon \gamma_0}$. The definition of the event $\{\sigma > t_\varepsilon\}$ implies

$$-\Theta_\varepsilon^- \leqslant X_t - \varepsilon t \gamma_0 \leqslant \Theta_\varepsilon^+ \qquad \forall t \in [0, t_\varepsilon].$$

Therefore, we infer from the event $\{\sigma > t_\varepsilon\}$ for $t \in [0, t_\varepsilon]$ that

$$\begin{aligned}
\varepsilon L_t + \varepsilon t \gamma_0 &= X_t - \int_0^t b(X_s)ds \\
&\leqslant X_t + B^- \int_0^t (X_s)^{\beta^-} ds \\
&\leqslant \Theta_\varepsilon^+ + B^- t_\varepsilon (\Theta_\varepsilon^-)^{\beta^-}.
\end{aligned}$$

Analogously we obtain

$$\varepsilon L_t + \varepsilon t\gamma_0 \geqslant -\Theta_\varepsilon^- - B^+ t_\varepsilon (\Theta_\varepsilon^+)^{\beta^+}.$$

If we now impose that the non-linear term is asymptotically smaller, that is for instance $\Theta_\varepsilon^\beta t_\varepsilon^{1-\vartheta}$, $\vartheta \in (0,1)$, than the boundary Θ_ε

$$\begin{aligned} B^+ t_\varepsilon (\Theta_\varepsilon^+)^{\beta^+} &= \Theta_\varepsilon^- t_\varepsilon^{1-\vartheta} \\ B^- t_\varepsilon (\Theta_\varepsilon^-)^{\beta^-} &= \Theta_\varepsilon^+ t_\varepsilon^{1-\vartheta} \end{aligned} \tag{4.2}$$

it follows

$$-(1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^- \leqslant \varepsilon L_t + \varepsilon t\gamma_0 \leqslant (1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^+, \qquad t \in [0, t_\varepsilon],$$

and in particular $-(1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^- \leqslant \varepsilon L_{t_\varepsilon} + \varepsilon t_\varepsilon \gamma_0 \leqslant (1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^+$. As a first case we may assume that $\Theta_\varepsilon^+/\Theta_\varepsilon^- \to 0$ as $\varepsilon \to 0$.

Case A: If we stipulate for $\vartheta \in (0,1)$

$$\Theta_\varepsilon^\circ = \frac{\varepsilon t_\varepsilon^{\frac{1}{\alpha}}}{1+t_\varepsilon^{1-\vartheta}} \tag{4.3}$$

this yields

$$\begin{aligned} &\mathbb{P}\Big(-(1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^- \leqslant \varepsilon L_{t_\varepsilon} + \varepsilon t_\varepsilon\gamma_0 \leqslant (1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^+ \Big) \\ &= \mathbb{P}\Big(-(1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^- \leqslant \varepsilon t_\varepsilon^{\frac{1}{\alpha}} L_1 \leqslant (1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^+ \Big) \\ &= \mathbb{P}\Big(-(1+t_\varepsilon^{1-\vartheta})\frac{\Theta_\varepsilon^-}{\Theta_\varepsilon^+}\frac{\Theta_\varepsilon^+}{\varepsilon t_\varepsilon^{\frac{1}{\alpha}}} \leqslant L_1 \leqslant (1+t_\varepsilon^{1-\vartheta})\frac{\Theta_\varepsilon^+}{\varepsilon t_\varepsilon^{1/\alpha}} \Big) \\ &= \mathbb{P}\Big(-\frac{\Theta_\varepsilon^-}{\Theta_\varepsilon^+} \leqslant L_1 \leqslant 1 \Big) \xrightarrow{\varepsilon\to 0} \mathbb{P}\Big(-\infty < L_1 \leqslant 1 \Big) > 0. \end{aligned}$$

Case B: If we stipulate for $\vartheta \in (0,1)$

$$\Theta_\varepsilon^* = \frac{\varepsilon t_\varepsilon^{\frac{1}{\alpha}}}{1+t_\varepsilon^{1-\vartheta}} \tag{4.4}$$

we obtain

$$\begin{aligned} &\mathbb{P}\Big(-(1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^- \leqslant \varepsilon L_{t_\varepsilon} + \varepsilon t_\varepsilon\gamma_0 \leqslant (1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^+ \Big) \\ &= \mathbb{P}\Big(-(1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^- \leqslant \varepsilon t_\varepsilon^{\frac{1}{\alpha}} L_1 \leqslant (1+t_\varepsilon^{1-\vartheta})\Theta_\varepsilon^+ \Big) \\ &= \mathbb{P}\Big(-(1+t_\varepsilon^{1-\vartheta})\frac{\Theta_\varepsilon^-}{\varepsilon t_\varepsilon^{\frac{1}{\alpha}}} \leqslant L_1 \leqslant (1+t_\varepsilon^{1-\vartheta})\frac{\Theta_\varepsilon^+}{\Theta_\varepsilon^-}\frac{\Theta_\varepsilon^-}{\varepsilon t_\varepsilon^{1/\alpha}} \Big) \\ &= \mathbb{P}\Big(-1 \leqslant L_1 \leqslant \frac{\Theta_\varepsilon^+}{\Theta_\varepsilon^-} \Big) \xrightarrow{\varepsilon\to 0} \mathbb{P}\Big(-1 \leqslant L_1 \leqslant 0 \Big) > 0. \end{aligned}$$

The case when $\Theta_\varepsilon^+/\Theta_\varepsilon^- \to c \in (0,\infty)$ is treated analogously. The proof concludes with the following remark which shows that for any exponent $\alpha \in (0,2)$, any powers $\beta^+,\beta^- \in (0,1)$ satisfying $\alpha \geqslant 1-(\beta^+\wedge\beta^-)$ and $\varepsilon \in (0,1)$ the system (4.2) together either with (4.3) or (4.4) as a unique solution $(\Theta_\varepsilon^+,\Theta_\varepsilon^-,t_\varepsilon)$. $\square$

Remark 11. We solve the equations for t_ε, Θ_ε^+ and Θ_ε^- in the previous lemma for any $\vartheta \in (0,1)$. We start with the system (4.2) which implies by reinsertion

$$\begin{aligned}\Theta_\varepsilon^- &= t_\varepsilon^\vartheta B^+(\Theta_\varepsilon^+)^{\beta^+}\\ &= t_\varepsilon^\vartheta B^+(t_\varepsilon^\vartheta B^-(\Theta_\varepsilon^-)^{\beta^-})^{\beta^+}\\ &= B^+(B^-)^{\beta^+} t_\varepsilon^{\vartheta(1+\beta^+)}(\Theta_\varepsilon^-)^{\beta^+\beta^-},\end{aligned}$$

and

$$\begin{aligned}&(\Theta_\varepsilon^-)^{1-\beta^+\beta^-} = B^+(B^-)^{\beta^+} t_\varepsilon^{\vartheta(1+\beta^+)}\\ \Leftrightarrow\qquad &\Theta_\varepsilon^- = \Big(B^+(B^-)^{\beta^+} t_\varepsilon^{\vartheta(1+\beta^+)}\Big)^{\frac{1}{1-\beta^+\beta^-}} = (B^+)^{\frac{1}{1-\beta^+\beta^-}}\ (B^-)^{\frac{\beta^+}{1-\beta^+\beta^-}}\ t_\varepsilon^{\frac{\vartheta(1+\beta^+)}{1-\beta^+\beta^-}}\end{aligned}$$

and by symmetry

$$\begin{aligned}&(\Theta_\varepsilon^+)^{1-\beta^+\beta^-} = B^-(B^+)^{\beta^-} t_\varepsilon^{\vartheta(1+\beta^-)}\\ \Leftrightarrow\qquad &\Theta_\varepsilon^+ = \Big(B^-(B^+)^{\beta^-} t_\varepsilon^{\vartheta(1+\beta^-)}\Big)^{\frac{1}{1-\beta^+\beta^-}} = (B^-)^{\frac{1}{1-\beta^+\beta^-}}\ (B^+)^{\frac{\beta^-}{1-\beta^+\beta^-}}\ t_\varepsilon^{\frac{\vartheta(1+\beta^-)}{1-\beta^+\beta^-}}.\end{aligned}$$

Denote by $\beta^\circ := \beta^+\wedge\beta^-$ and $\beta^* := \beta^+\vee\beta^-$. The last two formulas yield

$$\begin{aligned}\Theta_\varepsilon^* &:= \Theta_\varepsilon^+\vee\Theta_\varepsilon^- = (B^\circ)^{\frac{1}{1-\beta^\circ\beta^*}}\ (B^*)^{\frac{\beta^\circ}{1-\beta^\circ\beta^*}}\ t_\varepsilon^{\frac{\vartheta(1+\beta^\circ)}{1-\beta^\circ\beta^*}}\\ \Theta_\varepsilon^\circ &:= \Theta_\varepsilon^+\wedge\Theta_\varepsilon^- = (B^*)^{\frac{1}{1-\beta^\circ\beta^*}}\ (B^\circ)^{\frac{\beta^*}{1-\beta^\circ\beta^*}}\ t_\varepsilon^{\frac{\vartheta(1+\beta^*)}{1-\beta^\circ\beta^*}}\end{aligned}$$

As a consequence, we obtain for $\beta^\circ < \beta^*$

$$\lim_{\varepsilon\to 0+} \Theta_\varepsilon^\circ/\Theta_\varepsilon^* = 0. \tag{4.5}$$

and for $\beta = \beta^* = \beta^\circ$

$$\frac{\Theta_\varepsilon^\circ}{\Theta_\varepsilon^*} = \frac{(B^*)^{\frac{1}{1-\beta^2}}\ (B^\circ)^{\frac{\beta}{1-\beta^2}}}{(B^\circ)^{\frac{1}{1-\beta^2}}\ (B^*)^{\frac{\beta}{1-\beta^2}}} = \Big(\frac{B^\circ}{B^*}\Big)^{-\frac{1}{1+\beta}}. \tag{4.6}$$

Case A: Minimum complement: We first complement the system (4.2) by equation (4.3). Inserting in

$$\Leftrightarrow\qquad \varepsilon = (B^*)^{\frac{1}{1-\beta^*\beta^\circ}}\ (B^\circ)^{\frac{\beta^*}{1-\beta^*\beta^\circ}}\ t_\varepsilon^{\frac{\vartheta(1+\beta^*)}{1-\beta^*\beta^\circ}-\frac{1}{\alpha}}.$$

We examine the exponent

$$\frac{\vartheta(1+\beta^*)}{1-\beta^*\beta^\circ} - \frac{1}{\alpha} = \frac{\vartheta\alpha(1+\beta^*) - 1 + \beta^*\beta^\circ}{\alpha(1-\beta^*\beta^\circ)} = \frac{\vartheta\alpha - 1 + \vartheta\alpha\beta^* + \beta^*\beta^\circ}{\alpha(1-\beta^*\beta^\circ)}$$
$$= \frac{\vartheta\alpha - 1 + \beta^*(\vartheta\alpha + \beta^\circ)}{\alpha(1-\beta^*\beta^\circ)} \geqslant \frac{\vartheta\alpha - 1 + \beta^*}{\alpha(1-\beta^*\beta^\circ)} > 0,$$

since $\vartheta\alpha + \beta^\circ > 1$ and therefore $\vartheta\alpha + \beta^* > 1$ we have

$$\varepsilon = (B^*)^{\frac{1}{1-\beta^*\beta^\circ}} \, (B^\circ)^{\frac{\beta^*}{1-\beta^*\beta^\circ}} \, t_\varepsilon^{\frac{\vartheta\alpha+\beta^*(\vartheta\alpha+\beta^\circ)-1}{\alpha(1-\beta^*\beta^\circ)}}$$
$$\Leftrightarrow \qquad t_\varepsilon = \frac{\varepsilon^{\frac{\alpha(1-\beta^*\beta^\circ)}{\vartheta\alpha+\beta^*(\vartheta\alpha+\beta^\circ)-1}}}{(B^\circ)^{\frac{\alpha\beta^*}{\alpha+\beta^*(\alpha+\beta^\circ)-1}} \, (B^*)^{\frac{\alpha}{\alpha+\beta^*(\alpha+\beta^\circ)-1}}} = \bar{C}\varepsilon^{\frac{\alpha(1-\beta^*\beta^\circ)}{\vartheta\alpha+\beta^*(\vartheta\alpha+\beta^\circ)-1}}. \tag{4.7}$$

We obtain

$$\Theta_\varepsilon^+ = (B^-)^{\frac{1}{1-\beta^\circ\beta^*}} \, (B^+)^{\frac{\beta^-}{1-\beta^\circ\beta^*}} \, t_\varepsilon^{\frac{\vartheta(1+\beta^-)}{1-\beta^\circ\beta^*}} = \frac{(B^-)^{\frac{1}{1-\beta^\circ\beta^*}} \, (B^+)^{\frac{\beta^-}{1-\beta^\circ\beta^*}}}{(B^\circ)^{\frac{\alpha\beta^*}{\alpha+\beta^*(\alpha+\beta^\circ)-1}} \, (B^*)^{\frac{\alpha}{\alpha+\beta^*(\alpha+\beta^\circ)-1}}} \, \varepsilon^{\frac{\vartheta\alpha(1+\beta^-)}{\vartheta\alpha+\beta^*(\vartheta\alpha+\beta^\circ)-1}}$$
$$=: C^+\varepsilon^{\frac{\vartheta\alpha(1+\beta^-)}{\vartheta\alpha+\beta^*(\vartheta\alpha+\beta^\circ)-1}}$$

and

$$\Theta_\varepsilon^- = \frac{(B^+)^{\frac{1}{1-\beta^\circ\beta^*}} \, (B^-)^{\frac{\beta^+}{1-\beta^\circ\beta^*}}}{(B^\circ)^{\frac{\alpha\beta^*}{\alpha+\beta^*(\alpha+\beta^\circ)-1}} \, (B^*)^{\frac{\alpha}{\alpha+\beta^*(\alpha+\beta^\circ)-1}}} \, \varepsilon^{\frac{\vartheta\alpha(1+\beta^+)}{\vartheta\alpha+\beta^*(\vartheta\alpha+\beta^\circ)-1}} =: C^-\varepsilon^{\frac{\vartheta\alpha(1+\beta^+)}{\vartheta\alpha+\beta^*(\vartheta\alpha+\beta^\circ)-1}}.$$

Case B: Maximum We set $\vartheta = 1$ and complement (4.2) by (4.4) we may insert again

$$\varepsilon = (B^\circ)^{\frac{1}{1-\beta^\circ\beta^*}} \, (B^*)^{\frac{\beta^\circ}{1-\beta^\circ\beta^*}} \, t_\varepsilon^{\frac{1+\beta^\circ}{1-\beta^\circ\beta^*}-\frac{1}{\alpha}}$$

We examine the exponent

$$\frac{1+\beta^\circ}{1-\beta^*\beta^\circ} - \frac{1}{\alpha} = \frac{\alpha(1+\beta^\circ) - 1 + \beta^*\beta^\circ}{\alpha(1-\beta^*\beta^\circ)} = \frac{\alpha - 1 + \alpha\beta^\circ + \beta^*\beta^\circ}{\alpha(1-\beta^*\beta^\circ)}$$
$$= \frac{\alpha - 1 + \beta^\circ(\alpha + \beta^*)}{\alpha(1-\beta^*\beta^\circ)} \geqslant \frac{\alpha - 1 + \beta^\circ}{\alpha(1-\beta^*\beta^\circ)} > 0,$$

since $\alpha + \beta^\circ > 1$. Analogously we have

$$\varepsilon = (B^*)^{\frac{1}{1-\beta^*\beta^\circ}} \, (B^\circ)^{\frac{\beta^*}{1-\beta^*\beta^\circ}} \, t_\varepsilon^{\frac{\alpha+\beta^\circ(\alpha+\beta^*)-1}{\alpha(1-\beta^*\beta^\circ)}}$$
$$\Leftrightarrow \qquad t_\varepsilon = \frac{\varepsilon^{\frac{\alpha(1-\beta^*\beta^\circ)}{\alpha+\beta^\circ(\alpha+\beta^*)-1}}}{(B^\circ)^{\frac{\alpha\beta^*}{\alpha+\beta^\circ(\alpha+\beta^*)-1}} \, (B^*)^{\frac{\alpha}{\alpha+\beta^\circ(\alpha+\beta^*)-1}}} = \bar{C}\varepsilon^{\frac{\alpha(1-\beta^*\beta^\circ)}{\alpha+\beta^\circ(\alpha+\beta^*)-1}}. \tag{4.8}$$

We obtain

$$\Theta_\varepsilon^+ = (B^-)^{\frac{1}{1-\beta^\circ\beta^*}} \, (B^+)^{\frac{\beta^-}{1-\beta^\circ\beta^*}} \, t_\varepsilon^{\frac{1+\beta^-}{1-\beta^\circ\beta^*}} = \frac{(B^-)^{\frac{1}{1-\beta^\circ\beta^*}} \, (B^+)^{\frac{\beta^-}{1-\beta^\circ\beta^*}}}{(B^\circ)^{\frac{\alpha\beta^*}{\alpha+\beta^\circ(\alpha+\beta^*)-1}} \, (B^*)^{\frac{\alpha}{\alpha+\beta^\circ(\alpha+\beta^*)-1}}} \, \varepsilon^{\frac{\alpha(1+\beta^-)}{\alpha+\beta^\circ(\alpha+\beta^*)-1}}$$
$$=: C^+\varepsilon^{\frac{\alpha(1+\beta^-)}{\alpha+\beta^\circ(\alpha+\beta^*)-1}}$$

and

$$\Theta_\varepsilon^- = \frac{(B^+)^{\frac{1}{1-\beta^\circ\beta^*}} \, (B^-)^{\frac{\beta^+}{1-\beta^\circ\beta^*}}}{(B^\circ)^{\frac{\alpha\beta^*}{\alpha+\beta^\circ(\alpha+\beta^*)-1}} \, (B^*)^{\frac{\alpha}{\alpha+\beta^\circ(\alpha+\beta^*)-1}}} \, \varepsilon^{\frac{\alpha(1+\beta^+)}{\alpha+\beta^\circ(\alpha+\beta^*)-1}} =: C^- \varepsilon^{\frac{\alpha(1+\beta^+)}{\alpha+\beta^\circ(\alpha+\beta^*)-1}}.$$

These calculations establish the existence and uniqueness of $(\Theta_\varepsilon^+, \Theta_\varepsilon^-, t_\varepsilon)$ as claimed in Lemma 10 for any $\varepsilon > 0$.

Definition 12. Let $\alpha \in (0,2)$ and $\beta^+, \beta^- \in (0,1)$ given satisfying $\alpha > 1 - (\beta^+ \wedge \beta^-)$.

1. For strictly α-stable noise L, that is $\gamma_0 = 0$, we define the family $(\Theta_{\varepsilon,\vartheta}^+, \Theta_{\varepsilon,\vartheta}^-, t_{\varepsilon,\vartheta})_{\varepsilon,\vartheta\in(0,1)}$ as in Case A.

2. For stable, but not strictly α-stable noise L, that is $\gamma_0 \neq 0$, and

   (a) $\alpha > 1$ we define we define $(\Theta_{\varepsilon,\vartheta}^+, \Theta_{\varepsilon,\vartheta}^-, t_{\varepsilon,\vartheta})_{\varepsilon,\vartheta\in(0,1)}$ as in Case A,
   whereas for additional

   (b) $1 - (\beta^+ \wedge \beta^-) < \alpha < 1$ we define $(\Theta_{\varepsilon,\vartheta}^+, \Theta_{\varepsilon,\vartheta}^-, t_{\varepsilon,\vartheta})_{\varepsilon,\vartheta\in(0,1)}$ as in Case B.

For notational convenience we will immediately drop once and for all the dependence on ϑ, whenever possible.

4.2 The exit locations from a neighborhood of the origin

Assume a parameter ϑ fixed and denote by $\chi := \chi_\varepsilon := \tau_{\Theta_\varepsilon^+, \Theta_\varepsilon^-}$ as defined in (4.1) and $(\Theta_\varepsilon^+, \Theta_\varepsilon^-, t_\varepsilon)_{\varepsilon\in(0,1)}$ defined by Definition 12 and Lemma 10. In this subsection we determine the asymptotic probabilities

$$\mathbb{P}(X_\chi^\varepsilon \geqslant \Theta_\varepsilon^+) \quad \text{and} \quad \mathbb{P}(X_\chi^\varepsilon \leqslant -\Theta_\varepsilon^-)$$

in the limit of small ε.

4.2.1 Strictly α-stable perturbations, $\gamma_0 = 0$

Proposition 13. Consider the case of symmetric roots $\beta = \beta^+ = \beta^-$ and

$$\alpha > 1 - \beta$$

and the parametrized family of functions $(\Theta^+_{\varepsilon,\vartheta}, \Theta^-_{\varepsilon,\vartheta}, t_{\varepsilon,\vartheta})_{\varepsilon,\vartheta\in(0,1]}$ determined in Definition 12. Then there is ϑ^* such that $(\Theta^+_{\varepsilon,\vartheta^*}, \Theta^-_{\varepsilon,\vartheta^*}, t_{\varepsilon,\vartheta^*})_{\varepsilon\in(0,1)}$

$$\lim_{\varepsilon\to 0} \mathbb{P}(X_\chi^\varepsilon \geqslant \Theta_\varepsilon^+) = \begin{cases} 1 & \text{if } \beta^+ < \beta^- \\ \left(1 + \left(\frac{B^+}{B^-}\right)^{-\frac{1}{1+\beta}}\right)^{-1} & \text{if } \beta = \beta^+ = \beta^- \\ 0 & \text{if } \beta^+ < \beta^-. \end{cases}$$

The proof is concluded after the following two lemmas at the end of this subsection.

We decompose X^ε into the sum of V^ε and εL, where $V_t^\varepsilon := X_t^\varepsilon - \varepsilon L_t$. It satisfies

$$V_t^\varepsilon = \int_0^t b(V_s^\varepsilon + \varepsilon L_s) ds, \qquad t \geqslant 0.$$

Lemma 14. Consider the case of symmetric roots $\beta = \beta^+ = \beta^-$ and

$$\alpha > 1 - \beta$$

and the parametrized family of functions $(\Theta^+_{\varepsilon,\vartheta}, \Theta^-_{\varepsilon,\vartheta}, t_{\varepsilon,\vartheta})_{\varepsilon,\vartheta\in(0,1]}$ determined in Definition 12. Then there is ϑ^* such that $(\Theta^+_{\varepsilon,\vartheta^*}, \Theta^-_{\varepsilon,\vartheta^*}, t_{\varepsilon,\vartheta^*})_{\varepsilon\in(0,1)}$

$$\lim_{\varepsilon\to 0+} |\mathbb{P}(X_\chi^\varepsilon \geqslant \Theta_\varepsilon^+) - \mathbb{P}(\varepsilon L_\chi \geqslant \Theta_\varepsilon^+)| = 0$$

$$\lim_{\varepsilon\to 0+} |\mathbb{P}(X_\chi^\varepsilon \leqslant -\Theta_\varepsilon^-) - \mathbb{P}(\varepsilon L_\chi \leqslant -\Theta_\varepsilon^-)| = 0.$$

Proof. By symmetry of the argument it is enough to treat the first statement. For convenience we drop all superscripts and leave ϑ unspecified for the moment. By decomposition $Y_t = V_t + \varepsilon L_t$ we obtain for any $g > 0$

$$\mathbb{P}(X_\chi^\varepsilon \geqslant \Theta_\varepsilon^+, \chi \leqslant \hat{t}_\varepsilon) \leqslant \mathbb{P}(\varepsilon L_\chi \geqslant \Theta_\varepsilon^+(1 - \varepsilon^\gamma)) + \mathbb{P}(\sup_{t\in[0,\hat{t}_\varepsilon]} |V_s^\varepsilon| > \Theta_\varepsilon^\circ \varepsilon^g) + \mathbb{P}(\chi > \hat{t}_\varepsilon).$$

Proposition 9 send the last term tends to 0 as $\varepsilon \to 0$. We define $\hat{t}_\varepsilon = t_\varepsilon |\ln(\varepsilon)|$. It is therefore enough to show that there exists $\vartheta \in (0,1)$ such that

$$\lim_{\varepsilon \to 0+} \mathbb{P}(\sup_{t \in [0, \hat{t}_\varepsilon]} |V_t^\varepsilon| > \Theta_\varepsilon^+ \wedge \Theta_\varepsilon^-) \to 0.$$

It is enough to show that for t_ε determined in in Proposition 9 that there is $\gamma > 0$

$$\mathbb{P}(\sup_{t \in [0, t_\varepsilon]} |V_t^\varepsilon| > \Theta_\varepsilon^\circ \varepsilon^g) \to 0, \qquad \text{as } \varepsilon \to 0.$$

Note that V^ε is continuous and $V_0^\varepsilon = 0$. Recall from (4.8) $t_\varepsilon \approx \varepsilon^{\frac{\vartheta\alpha(1-\beta^2)}{\vartheta\alpha+\beta-1+\beta(\vartheta\alpha+\beta-1)}}$. The first order approximation of $V_t^\varepsilon \doteq \varepsilon L_t$ for $t \in [0, \hat{t}_\varepsilon]$ and the self-similarity

$$\sup_{t \in [0, \hat{t}_\varepsilon]} |\varepsilon L|^\beta \stackrel{d}{=} \varepsilon^\beta \hat{\varepsilon}^{\frac{\beta}{\alpha}} \sup_{t \in [0,1]} |L_t|^\beta \stackrel{d}{=} \varepsilon^\beta \hat{t}_\varepsilon^{\frac{\beta}{\alpha}} |L_1|^\beta$$

yields

$$\begin{aligned} &\mathbb{P}(\sup_{t \in [0, \hat{t}_\varepsilon]} |\varepsilon L_t| > \Theta_\varepsilon^\circ \varepsilon^g) \\ &\leqslant \mathbb{P}(2(B^+ \vee B^-) \varepsilon^{\beta^\circ} \hat{t}_\varepsilon^{1+\frac{\beta^\circ}{\alpha}} |L_1|^\beta > \Theta_\varepsilon^\circ \varepsilon^g). \end{aligned}$$

That is, we check if there is a parameter $\vartheta \in (0,1)$ which allows for the choice of $g = g(\vartheta) > 0$ such that

$$\varepsilon^\beta \hat{t}_\varepsilon^{\frac{\alpha+\beta}{\alpha}} \lesssim_\varepsilon t_\varepsilon^{\frac{\vartheta}{1-\beta}}.$$

In other words, since the logarithm is dominated by any polynomial order, we check if

$$\varepsilon^\beta t_\varepsilon^{\frac{\alpha+\beta}{\alpha} - \frac{\vartheta}{1-\beta}} \to 0.$$

Please note that t_ε also depends on ϑ. We check the positivity of the exponent

$$2g := \beta + \frac{\vartheta\alpha(1-\beta^2)}{\vartheta\alpha+\beta-1+\beta(\vartheta\alpha+\beta-1)} \Big(\frac{\alpha+\beta}{\alpha} - \frac{\vartheta(1+\beta)}{1-\beta^2} \Big).$$

We calculate

$$\begin{aligned} &\beta + \frac{\vartheta(\alpha+\beta)(1-\beta^2) - \vartheta^2\alpha(1+\beta)}{\vartheta\alpha+\beta-1+\beta(\vartheta\alpha+\beta-1)} \\ &= \frac{\vartheta\alpha\beta + \beta^2 - \beta + \vartheta\alpha\beta^2 + \beta^3 - \beta^2 + \vartheta\alpha + \vartheta\beta - \vartheta\alpha\beta^2 - \vartheta\beta^3 - \vartheta^2\alpha - \vartheta^2\alpha\beta}{\vartheta\alpha+\beta-1+\beta(\vartheta\alpha+\beta-1)} \end{aligned}$$

Under the positivity assumption $\vartheta\alpha + \beta - 1 > 0$ for the denominator we check the sign of the enumerator

$$\begin{aligned}
&\vartheta\alpha\beta + \beta^2 - \beta + \vartheta\alpha\beta^2 + \beta^3 - \beta^2 + \vartheta\alpha + \vartheta\beta - \vartheta\alpha\beta^2 - \vartheta\beta^3 - \vartheta^2\alpha - \vartheta^2\alpha\beta \\
&= \vartheta\alpha\beta - \beta + \beta^3 + \vartheta\alpha + \vartheta\beta - \vartheta\beta^3 - \vartheta^2\alpha - \vartheta^2\alpha\beta \\
&= (1-\vartheta)\beta^3 - (1-\vartheta)\beta + \vartheta(1-\vartheta)\alpha\beta + \vartheta(1-\vartheta)\alpha \\
&= -\alpha(1+\beta)\vartheta^2 + (\beta - \beta^3 + \alpha\beta + \alpha)\vartheta + \beta^3 - \beta \\
&= -\alpha(1+\beta)(\vartheta - 1)(\vartheta - \frac{\beta(1-\beta^2)}{\alpha(1+\beta)}).
\end{aligned}$$

The assumption $\frac{\beta(1-\beta)}{\alpha} \leqslant 1$ is satisfied for $\alpha > \beta(1-\beta)$, which is true since $\alpha > 1-\beta$. Hence for any $\frac{\beta(1-\beta)}{\alpha} < \vartheta^* < 1$ we have $g > 0$. □

Lemma 15. Assume $\beta^+ > \beta^-$ the parametrized family of functions $(\Theta^+_{\varepsilon,1}, \Theta^-_{\varepsilon,1}, t_{\varepsilon,1})_{\varepsilon\in(0,1]}$ determined in Definition 12. Then there exists $g > 0$ such that for $\hat{t}_\varepsilon := t_\varepsilon|\ln(\varepsilon)|, \varepsilon \in (0,1)$ we have

$$\mathbb{P}(\sup_{t\in[0,\hat{t}_\varepsilon]} (V^\varepsilon_t)_+ > \Theta^+_\varepsilon \varepsilon^g) \to 0.$$

Proof. $\beta^* = \beta^+$. The self-similarity

$$\sup_{t\in[0,\hat{t}_\varepsilon]} (\varepsilon L_t)^{\beta^*}_+ \stackrel{d}{=} \varepsilon^{\beta^*} \hat{t}_\varepsilon^{\frac{\beta^*}{\alpha}} (L_1)^{\beta^*}_+$$

yields

$$\mathbb{P}(\sup_{t\in[0,\hat{t}_\varepsilon]} (\varepsilon L_t)^{\beta^*}_+ > \Theta^*_\varepsilon \varepsilon^g) \leqslant \mathbb{P}(\sup_{t\in[0,\hat{t}_\varepsilon]} \varepsilon^{\beta^*} \hat{t}_\varepsilon^{\frac{\beta^*}{\alpha}} (L_1)^{\beta^*}_+ > \Theta^*_\varepsilon \varepsilon^g)$$

We check whether

$$\varepsilon^{\beta^*} t_\varepsilon^{\frac{\alpha+\beta^*}{\alpha} - \frac{1+\beta^\circ}{1-\beta^*\beta^\circ}} \to 0, \text{ as } \varepsilon \to 0.$$

Check the exponent

$$\begin{aligned}
&\beta^* + \frac{\alpha(1-\beta^\circ\beta^*)}{\alpha + \beta^* - 1 + \beta^*(\alpha + \beta^\circ - 1)}\Big(\frac{\alpha+\beta^*}{\alpha} - \frac{1+\beta^\circ}{1-\beta^*\beta^\circ}\Big) \\
&= \frac{\beta^*(\alpha + \beta^* - 1 + \beta^*(\alpha+\beta^\circ-1)) + (\alpha+\beta^*)(1-\beta^*\beta^\circ) - \alpha(1+\beta^\circ)}{\alpha+\beta^*-1+\beta^*(\alpha+\beta^\circ-1)}
\end{aligned} \tag{4.9}$$

By assumption the denominator is positive. The enumerator behaves as

$$\begin{aligned}
&\beta^*(\alpha + \beta^* - 1 + \beta^*(\alpha+\beta^\circ-1)) + (\alpha+\beta^*)(1-\beta^*\beta^\circ) - \alpha(1+\beta^\circ) \\
&= \alpha\beta^* + (\beta^*)^2 - \beta^* + \alpha(\beta^*)^2 + \beta^\circ(\beta^*)^2 - (\beta^*)^2 + \alpha + \beta^* - \alpha\beta^\circ\beta^* - \beta^\circ(\beta^*)^2 - \alpha - \alpha\beta^\circ \\
&= \alpha\beta^* + \alpha(\beta^*)^2 - \alpha\beta^\circ\beta^* - \alpha\beta^\circ \\
&= \alpha(\beta^* - \beta^\circ) + \alpha\beta^*(\beta^* - \beta^\circ) > 0.
\end{aligned}$$

We set $2g$ equal to the expression in (4.9). $\square$

In the sequel we determine $\lim_{\varepsilon\to 0+}\mathbb{P}(L_\chi \geqslant \Theta_\varepsilon^+)$. The exit problem of $\varepsilon\xi^\kappa$ from $(-\Theta_\varepsilon^-, \Theta_\varepsilon^+)$ will be treated in the spirit of the Brownian case as for instance in the book of Revuz and Yor [22]. For this purpose denote by $\kappa \in \mathbb{R}$ and ξ^κ the Lévy process driven by $\nu\big|_{(-\varepsilon^{-\kappa},\varepsilon^{-\kappa})}$. Note that for

$$\tau_\kappa := \inf\{t > 0 \mid |\Delta_t L| > \varepsilon^{-\kappa}\},$$

we have $\xi_t^\kappa = L_t$ on the event $A_t := \{t \leqslant \tau_\kappa\}$ for any $t \geqslant 0$. Furthermore we have $X_t^\varepsilon = Y_t^\varepsilon$ on A_t, where Y^ε is the original system driven by ξ^κ instead of L, that is

$$Y_t^\varepsilon = \int_0^t b(Y_s^\varepsilon)ds + \varepsilon\xi_t^\kappa.$$

We fix the constant

$$\kappa = -\frac{4}{\alpha + \beta^\circ - 1} \tag{4.10}$$

and consider the Lévy martingale $(\xi_t^\kappa)_{t\geqslant 0}$. This choice allows to verify that jumps beyond the threshold ε^κ occurr after t_ε, with a probability mass which tends to 1. More precisely, since

$$\int_{\varepsilon^{-\kappa}}^\infty \frac{dy}{y^{\alpha+1}} = \frac{-1}{\alpha} y^{-\alpha}\Big|_{\varepsilon^{-\kappa}}^\infty = \frac{1}{\alpha}\varepsilon^{\kappa\alpha}$$

we have

$$\mathbb{P}(\tau_\kappa(\varepsilon) > t_\varepsilon) = \exp(-\frac{1}{\alpha}\varepsilon^{-\kappa\alpha} t_\varepsilon) \to 1, \qquad \text{as } \varepsilon \to 0.$$

As a second crucial feature we obtain

$$\varepsilon^{1-\kappa}/\Theta_\varepsilon^\circ = \varepsilon^{\frac{3+\alpha+\beta^*}{\alpha+\beta^*-1} - \frac{\alpha+\alpha\beta^*}{\alpha+\beta^*-1+\beta^*(\alpha+\beta^\circ-1)}} \to 0, \qquad \text{as } \varepsilon \to 0+.$$

We define the for $r^+, r^- > 0$ and $\varepsilon > 0$ the hitting times of $\mathbb{R}\setminus(-\Theta_\varepsilon^-, \Theta_\varepsilon^+)$

$$\begin{aligned} \sigma_{r^+}^+ &:= \inf\{t > 0 \mid \varepsilon\xi_t^\kappa \geqslant r\}, \\ \sigma_{r^-}^- &:= \inf\{t > 0 \mid \varepsilon\xi_t^\kappa \leqslant -r\}, \\ \sigma_{r^+,r^-} &:= \sigma_{r^+}^+ \wedge \sigma_{r^-}^-. \end{aligned} \tag{4.11}$$

Lemma 16. Under these assumptions we obtain

$$\lim_{\varepsilon\to 0}\mathbb{P}(\sigma_{\Theta_\varepsilon^+}^+ < \sigma_{\Theta_\varepsilon^-}^-) = \lim_{\varepsilon\to 0}\frac{\Theta_\varepsilon^-}{\Theta_\varepsilon^+ + \Theta_\varepsilon^-}.$$

Proof. For $r_1, r_2 > 0$ and $n \in \mathbb{N}$ given we fix

$$\begin{aligned}
\bar\sigma_\varepsilon^+ &:= \sigma_{r^+}^+ \wedge n,\\
\bar\sigma_\varepsilon^- &:= \sigma_{r^-}^- \wedge n,\\
\bar\sigma_\varepsilon &:= \sigma_{r^+,r^-}.
\end{aligned}$$

This yields the estimates

$$\begin{aligned}
&\varepsilon\xi^\varepsilon_{\bar\sigma^+} \leqslant r^+ + \varepsilon^{1-\kappa} \quad \text{and} \quad \varepsilon\xi^\varepsilon_{\bar\sigma^+ -} > r^+ \qquad \text{a.s. on the event } \{\bar\sigma^+ \leqslant n\}\\
&\varepsilon\xi^\varepsilon_{\bar\sigma^-} \geqslant -(r^- + \varepsilon^{1-\kappa}) \quad \text{and} \quad \varepsilon\xi^\varepsilon_{\bar\sigma^- -} < -r^- \qquad \text{a.s. on the event } \{\bar\sigma^- \leqslant n\}.
\end{aligned}$$

Applying the optional stopping theorem we obtain

$$\begin{aligned}
0 &= \mathbb{E}[\varepsilon\xi^\varepsilon_{\sigma_{r_1,r_2}}]\\
&= \mathbb{E}[\varepsilon\xi^\varepsilon_{\sigma_{r^+,r^-}}(\mathbf{1}\{\sigma^+_{r^+} < \sigma^-_{r^-}\} \cap \{\sigma_\varepsilon \leqslant n\} + \mathbf{1}\{\sigma^+_{r^+} \geqslant \sigma^-_{r^-}\} \cap \{\sigma_\varepsilon \leqslant n\})]\\
&= \mathbb{E}[\varepsilon\xi^\varepsilon_{\sigma^+_{r^+}}\mathbf{1}\{\sigma^+_{r^+} < \sigma^-_{r^-}\} \cap \{\sigma_\varepsilon \leqslant n\} + \varepsilon\xi^\varepsilon_{\sigma^-_{r^-}}\mathbf{1}\{\sigma^+_{r^+} \geqslant \sigma^-_{r^-}\} \cap \{\sigma_\varepsilon \leqslant n\}]
\end{aligned}$$

and estimate

$$\begin{aligned}
0 &= \mathbb{E}[\varepsilon\xi^\varepsilon_{\sigma^+_{r^+}}\mathbf{1}\{\sigma^+_{r^+} < \sigma^-_{r^-}\} \cap \{\sigma_\varepsilon \leqslant n\} + \varepsilon\xi^\varepsilon_{\sigma^-_{r^-}}\mathbf{1}\{\sigma^-_{r^-} \leqslant \sigma^+_{r^+}\} \cap \{\sigma_\varepsilon \leqslant n\}]\\
&\leqslant (r^+ + \varepsilon^{1-\kappa})\mathbb{P}(\sigma^+_{r^+} < \sigma^-_{r^-}) - r^-\mathbb{P}(\{\sigma^-_{r^-} \leqslant \sigma^+_{r^+}\} \cap \{\sigma_\varepsilon \leqslant n\})
\end{aligned}$$

and analogously

$$\begin{aligned}
0 &= \mathbb{E}[\varepsilon\xi^\varepsilon_{\sigma^+_{r^+}}\mathbf{1}\{\sigma^+_{r^+} < \sigma^-_{r^-}\} \cap \{\sigma_\varepsilon \leqslant n\} + \varepsilon\xi^\varepsilon_{\sigma^-_{r^-}}\mathbf{1}\{\sigma^-_{r^-} \leqslant \sigma^+_{r^+}\} \cap \{\sigma_\varepsilon \leqslant n\}]\\
&\geqslant r^+\mathbb{P}(\{\sigma^+_{r^+} < \sigma^-_{r^-}\} \cap \{\sigma_\varepsilon \leqslant n\}) - (r^- + \varepsilon^{1-\kappa})\mathbb{P}(\sigma^-_{r^-} \leqslant \sigma^+_{r^+}).
\end{aligned}$$

Letting n tend to ∞ we obtain

$$\begin{aligned}
0 &\leqslant (r^+ + \varepsilon^{1-\kappa})\mathbb{P}(\sigma^+_{r^+} < \sigma^-_{r^-}) - r^-\mathbb{P}(\sigma^-_{r^-} \leqslant \sigma^+_{r^+})\\
0 &\geqslant r^+\mathbb{P}(\sigma^+_{r^+} < \sigma^-_{r^-}) - (r^- + \varepsilon^{1-\kappa})\mathbb{P}(\sigma^-_{r^-} \leqslant \sigma^+_{r^+}).
\end{aligned}$$

The choice of κ entails that r^+ replaced by Θ^+_ε leads to

$$\varepsilon^{1-\kappa} \lesssim_\varepsilon \Theta^+_\varepsilon = C^+\varepsilon^{\frac{\vartheta\alpha(1+\beta^-)}{\vartheta\alpha+\beta^*(\vartheta\alpha+\beta^\circ)-1}},$$

and analogously for r^- being replaced by Θ^-_ε. Hence

$$\begin{aligned}
0 &\leqslant (\Theta^+_\varepsilon + \varepsilon^{1-\kappa})\mathbb{P}(\sigma^+_{\Theta^+_\varepsilon} < \sigma^-_{\Theta^-_\varepsilon}) - \Theta^-_\varepsilon(1 - \mathbb{P}(\sigma^+_{\Theta^+_\varepsilon} < \sigma^-_{\Theta^-_\varepsilon}))\\
0 &\geqslant \Theta^+_\varepsilon\mathbb{P}(\sigma^+_{\Theta^+_\varepsilon} < \sigma^-_{\Theta^-_\varepsilon}) - (\Theta^-_\varepsilon + \varepsilon^{1-\kappa})(1 - \mathbb{P}(\sigma^+_{\Theta^+_\varepsilon} < \sigma^-_{\Theta^-_\varepsilon}))
\end{aligned}$$

eventually leading to

$$\frac{\Theta_\varepsilon^-}{\Theta_\varepsilon^+ + \Theta_\varepsilon^- + \varepsilon^{1-\kappa}} \lesssim_\varepsilon \mathbb{P}(\sigma_{\Theta_\varepsilon^+}^+ < \sigma_{\Theta_\varepsilon^-}^-) \lesssim_\varepsilon \frac{\Theta_\varepsilon^- + \varepsilon^{1-\kappa}}{\Theta_\varepsilon^+ + \Theta_\varepsilon^- + \varepsilon^{1-\kappa}}.$$

□

Proof. of Proposition 13: We start with the case $\beta = \beta^+ = \beta^-$. The result is a direct combination of Lemma 19, and Lemma 16. We calculate the limit

$$\lim_{\varepsilon\to 0} \mathbb{P}(X_\chi^\varepsilon \geqslant \Theta_\varepsilon^+) = \lim_{\varepsilon\to 0} \frac{\Theta_\varepsilon^-}{\Theta_\varepsilon^- + \Theta_\varepsilon^+} = \frac{(B^+)^{\frac{1}{1-\beta^2}} (B^-)^{\frac{\beta}{1-\beta^2}}}{(B^+)^{\frac{1}{1-\beta^2}} (B^-)^{\frac{\beta}{1-\beta^2}} + (B^+)^{\frac{\beta}{1-\beta^2}} (B^-)^{\frac{1}{1-\beta^2}}} = \Big(1 + \Big(\frac{B^-}{B^+}\Big)^{\frac{1}{1+\beta}}\Big)^{-1}.$$

For $\beta^+ > \beta^-$ and $\hat{t}_\varepsilon |\ln(\varepsilon)|$ Lemma 20 guarantees the existence of a constant $g > 0$ such that

$$\begin{aligned}
\mathbb{P}(X_\chi^\varepsilon \geqslant \Theta_\varepsilon^+) &\leqslant \mathbb{P}(X_\chi^\varepsilon \geqslant \Theta_\varepsilon^+, \chi \leqslant \hat{t}_\varepsilon) + \mathbb{P}(\chi > \hat{t}_\varepsilon) \\
&\leqslant \mathbb{P}(\sup_{t\in[0,\hat{t}_\varepsilon]} (V_t^\varepsilon)_+^{\beta^+} + \varepsilon L_\chi \geqslant \Theta_\varepsilon^+) + \mathbb{P}(\chi > \hat{t}_\varepsilon) \\
&\leqslant \mathbb{P}(\sup_{t\in[0,\hat{t}_\varepsilon]} (V_t^\varepsilon)_+^{\beta^+} \geqslant \Theta_\varepsilon^+ \varepsilon^g) + \mathbb{P}(\varepsilon L_\chi \geqslant \Theta_\varepsilon^+(1 - \varepsilon^g)) \to 0,
\end{aligned}$$

as $\varepsilon \to 0+$. Eventually the relation $\lim_{\varepsilon\to 0+} \mathbb{P}(X_\chi^\varepsilon \leqslant -\Theta_\varepsilon^-) = 1 - \lim_{\varepsilon\to 0+} \mathbb{P}(X_\chi^\varepsilon \geqslant \Theta_\varepsilon^+)$ finishes the proof. □

4.2.2 The general α-stable case $\gamma_0 \neq 0$

We decompose X^ε given as the strong solution of (1.5) into the sum of V^ε and εL, where

$$V_t^\varepsilon := X_t^\varepsilon - \varepsilon L_t - \varepsilon t \gamma_0.$$

It satisfies $\mathbb{P}$-a.s.

$$V_t^\varepsilon = \int_0^t b(V_s^\varepsilon + \varepsilon L_s + \varepsilon t\gamma_0) ds, \quad t \geqslant 0.$$

The main result of this subsection is determines the $\lim_{\varepsilon\to 0+} \mathbb{P}(L_\chi \geqslant \Theta_\varepsilon^+)$.

Proposition 17. For $\gamma_0 \neq 0$, $\alpha \in (0,2)$, $\beta^+, \beta^- \in (0,1)$ satisfying $\alpha \neq 1$ and $\alpha > 1 - \beta^+ \wedge \beta^\circ$ we consider the parametrized family of functions $(\Theta_{\varepsilon,\vartheta}^+, \Theta_{\varepsilon,\vartheta}^-, t_{\varepsilon,\vartheta})_{\varepsilon,\vartheta\in(0,1]}$ determined in Definition 12. Then there is ϑ^* such that $(\Theta_{\varepsilon,\vartheta^*}^+, \Theta_{\varepsilon,\vartheta^*}^-, t_{\varepsilon,\vartheta^*})_{\varepsilon\in(0,1)}$ such that

$$\lim_{\varepsilon\to 0} \mathbb{P}(X_\chi^\varepsilon \geqslant \Theta_\varepsilon^+) = \begin{cases} \begin{cases} 1, & \text{if } \beta^+ < \beta^- \\ \Big(1 + \Big(\frac{B^+}{B^-}\Big)^{-\frac{1}{1+\beta}}\Big)^{-1}, & \beta = \beta^+ = \beta^- \\ 0, & \text{if } \beta^+ > \beta^- \end{cases} & \text{and } \alpha \in (1,2). \\ \begin{cases} 1, & \text{if } \gamma_0 > 0 \\ 0, & \text{if } \gamma_0 < 0 \end{cases} & \text{and } \alpha \in (0,1]. \end{cases}$$

The proof will be completed at the end of this subsection after a sequence of lemmas. The appearance of the drift γ_0 changes the picture dramatically if $\alpha \leqslant 1$.

We treat the case $\alpha \leqslant 1$:

Lemma 18. Under the assumptions of Proposition 17 and $1-\beta^\circ < \alpha < 1$ and $\vartheta = 1$ we have the following

$$\lim_{\varepsilon\to 0+} \mathbb{P}(X^\varepsilon_\chi \geqslant \Theta^+_\varepsilon) = \begin{cases} 1, & \text{if } \gamma_0 > 0, \\ 0 & \text{if } \gamma_0 < 0. \end{cases}$$

Proof. Recall that for $\hat{t}_\varepsilon = t_\varepsilon |\ln(\varepsilon)|$

$$\Theta^\pm_\varepsilon \leqslant \Theta^*_\varepsilon = \varepsilon t_\varepsilon^{\frac{1}{\alpha}} \lesssim_\varepsilon \varepsilon t_\varepsilon. \tag{4.12}$$

Without loss of generality we assume $\gamma_0 > 0$. Then for $\Theta^+_\varepsilon = \Theta^*_\varepsilon$ we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} \mathbb{P}(X^\varepsilon_\chi \geqslant \Theta^+_\varepsilon) &= \lim_{\varepsilon\to 0} \mathbb{P}(X^\varepsilon_\chi \geqslant \Theta^+_\varepsilon \text{ and } \chi \leqslant \hat{t}_\varepsilon) \\
&\geqslant \lim_{\varepsilon\to 0} \mathbb{P}(X^\varepsilon_\chi - \varepsilon\chi\gamma_0 \geqslant \Theta^+_\varepsilon - \varepsilon t_\varepsilon \gamma_0 \text{ and } \chi \leqslant \hat{t}_\varepsilon) \\
&= \lim_{\varepsilon\to 0} \mathbb{P}(X^\varepsilon_\chi - \varepsilon\chi\gamma_0 \geqslant \varepsilon t_\varepsilon - \varepsilon \hat{t}_\varepsilon \gamma_0 \text{ and } \chi \leqslant \hat{t}_\varepsilon) \\
&= \lim_{\varepsilon\to 0} \mathbb{P}(X^\varepsilon_\chi - \varepsilon\chi\gamma_0 \geqslant -\Theta^-_\varepsilon \text{ and } \chi \leqslant t_\varepsilon) = 1,
\end{aligned}$$

since $\varepsilon \hat{t}_\varepsilon \gamma_0 / \Theta^-_\varepsilon \to \infty$ as $\varepsilon \to 0+$ by Remark 11. The last equality of the preceding display is due to Lemma 10. Due to relation (4.12) the case $\Theta^+_\varepsilon = \Theta^\circ_\varepsilon$ yields the same result. □

We treat the case $\alpha > 1$: In the following lemma we will not exclude $\alpha = 1$. Since $\alpha \geqslant 1$ we main present a proof only based on the self-similarity of L.

Keeping in mind that $X_t = V_t + \varepsilon t\gamma_0 + \varepsilon L_t$. First note that for $\alpha > 1$ the upper bound for the drift satisfies

$$|\gamma_0| \varepsilon \hat{t}_\varepsilon \lesssim_\varepsilon \varepsilon t_\varepsilon^{\frac{1}{\alpha}} = \Theta^\circ_\varepsilon$$

by Definition 12, Case A, if $\hat{t}_\varepsilon = t_\varepsilon |\ln(\varepsilon)|, \varepsilon \in (0,1)$. By the virtually the same proofs as in the strictly stable case we obtain the symmetric situation of the general stable case.

Lemma 19. Consider the case of symmetric roots $\beta = \beta^+ = \beta^-$ and the parametrized family of functions $(\Theta^+_{\varepsilon,\vartheta}, \Theta^-_{\varepsilon,\vartheta}, t_{\varepsilon,\vartheta})_{\varepsilon,\vartheta\in(0,1]}$ determined in Definition 12, Case A. Then there is ϑ^* such that $(\Theta^+_{\varepsilon,\vartheta^*}, \Theta^-_{\varepsilon,\vartheta^*}, t_{\varepsilon,\vartheta^*})_{\varepsilon\in(0,1)}$ with

$$\lim_{\varepsilon\to 0+} |\mathbb{P}(X^\varepsilon_\chi \geqslant \Theta^+_\varepsilon + \varepsilon t_\varepsilon \gamma_0) - \mathbb{P}(\varepsilon L_\chi \geqslant \Theta^+_\varepsilon)| = 0$$

$$\lim_{\varepsilon\to 0+} |\mathbb{P}(X^\varepsilon_\chi \leqslant -\Theta^-_\varepsilon - \varepsilon t_\varepsilon \gamma_0) - \mathbb{P}(\varepsilon L_\chi \leqslant -\Theta^-_\varepsilon)| = 0.$$

In the same way we obtain.

Lemma 20. Assume $\beta^+ > \beta^-$ for the parametrized family of functions $(\Theta^+_{\varepsilon,1}, \Theta^-_{\varepsilon,1}, t_{\varepsilon,1})_{\varepsilon\in(0,1]}$ determined in Definition 12. Then there exists $g > 0$ such that for $\hat{t}_\varepsilon := t_\varepsilon|\ln(\varepsilon)|, \varepsilon \in (0,1)$ we have

$$\mathbb{P}(\sup_{t\in[0,\hat{t}_\varepsilon]} (V^\varepsilon_t)_+ > \Theta^+_\varepsilon \varepsilon^g) \to 0.$$

In the sequel we determine the remaining question. We identify for $\alpha > 1$ the limit $\lim_{\varepsilon\to 0} \mathbb{P}(L_\chi \geqslant \Theta^+)$.

Lemma 21. Under the assumptions of Lemma 16 we obtain

$$\lim_{\varepsilon\to 0} \mathbb{P}(\sigma^+_{\Theta^+_\varepsilon} < \sigma^-_{\Theta^-_\varepsilon}) = \lim_{\varepsilon\to 0} \frac{\Theta^-_\varepsilon}{\Theta^+_\varepsilon + \Theta^-_\varepsilon}, \text{ as } \varepsilon \to 0.$$

A word about the proof. Again it is virtually identical to the proof of Lemma 16, only replacing the $\varepsilon\xi^\kappa_t$ by $\varepsilon\widetilde{\xi^\kappa}$, where

$$\widetilde{\xi}^\kappa_t := \xi^\kappa_t - \mathbb{E}[\xi^\kappa_t], \quad t \geqslant 0, \varepsilon > 0,$$

is the protagonist martingale. The drift is of order ε

$$|\mathbb{E}[\varepsilon\xi^\kappa_t]| \lesssim_\varepsilon \varepsilon \lesssim_\varepsilon \varepsilon^{1-\kappa} \lesssim_\varepsilon \Theta^\circ_\varepsilon,$$

and can be treated as an additional perturbation of higher order, eventually leading to the bounds

$$\frac{\Theta^-_\varepsilon + \varepsilon t_\varepsilon \gamma_0 + \mathbb{E}[\varepsilon\xi^\varepsilon]}{\Theta^+_\varepsilon + \Theta^-_\varepsilon + 2\varepsilon t_\varepsilon\gamma_0 + 2\mathbb{E}[\varepsilon\xi^\varepsilon] + \varepsilon^{1-\kappa}} \lesssim_\varepsilon \mathbb{P}(\sigma^+_{\Theta^+_\varepsilon} < \sigma^-_{\Theta^-_\varepsilon}) \lesssim_\varepsilon \frac{\Theta^-_\varepsilon + \varepsilon t_\varepsilon\gamma_0 + \mathbb{E}[\varepsilon\xi^\varepsilon] + \varepsilon^{1-\kappa}}{\Theta^+_\varepsilon + \Theta^-_\varepsilon + 2t_\varepsilon\gamma_0 + 2\mathbb{E}[\varepsilon\xi^\varepsilon] + \varepsilon^{1-\kappa}},$$

which leave the asymptotic behavior intact.

5 The linearized dynamics enhances the regime close to the origin

We already know by Section 3 that for initial values $x \geqslant -5\delta_\varepsilon$ the law $\mathbb{P} \circ X^{\varepsilon,x} \to \delta_{x^+}$ uniformly on larger and larger time scales. Section 4.2 establishes for the parametrized family of functions $(\Theta^+_{\varepsilon,\vartheta}, \Theta^-_{\varepsilon,\vartheta}, t_{\varepsilon,\vartheta})_{\varepsilon,\vartheta\in(0,1]}$ for appropriate $\vartheta \in (0,1]$ that for initial values $x \in (-\Theta^-_\varepsilon, \Theta^+_\varepsilon)$ the solution $X^{x,\varepsilon}$ exits the interval $(-\Theta^-_\varepsilon, \Theta^+_\varepsilon)$ in time $\widetilde{t}_\varepsilon$ almost surely as long as $\lim_{\varepsilon\to 0} \widetilde{t}_\varepsilon / t_\varepsilon \to 0$. In order to fill the gap between

$$\Theta^\pm_\varepsilon = \varepsilon^{\frac{\alpha(1+\beta^\pm)}{\alpha+\beta^\circ-1+\beta^*(\alpha+\beta^\circ-1)}} \lesssim_\varepsilon 3\varepsilon^{1-\rho(1+\alpha)} = 3\delta_\varepsilon,$$

we consider the linearized dynamics. Due to monotonicity we may restrict ourselves to the case $\vartheta = 1$. The main result tells us that with a probability tending to 1, the solution exits on the outer boundary of $[-6\delta_\varepsilon, -\Theta^-_\varepsilon] \cup [\Theta^+_\varepsilon, 6\delta_\varepsilon]$. We treat each subinterval individually with out loss of generality $[\Theta^+_\varepsilon, 6\delta_\varepsilon]$. Again by an elementary comparison principle it is enough to consider the case when the drift $\gamma_0 < 0$ acts against the repulsive force of the root. The case of $\gamma_0 \geqslant 0$ follow then automatically.
For $\varepsilon > 0$ and $x \in [\Theta^+_\varepsilon, 5\delta_\varepsilon]$ denote

$$\upsilon^{x,\varepsilon} := \inf\{t > 0 \mid X^{\varepsilon,x}_t \geqslant 6\delta_\varepsilon\}.$$

Proposition 22. For $\beta = \beta^\circ = \beta^+ \geqslant \beta^-$ and $\gamma_0 < 0$ we consider the parametrized family of functions $(\Theta^+_{\varepsilon,\vartheta}, \Theta^-_{\varepsilon,\vartheta}, t_{\varepsilon,\vartheta})_{\varepsilon,\vartheta\in(0,1]}$ determined in Definition 12. Then for any $\vartheta \in (0,1]$ there is an increasing, continuous function $s_\cdot : (0,1) \to (0,1)$ with $s_\varepsilon \to 0$ as $\varepsilon \to 0$, such that

$$\lim_{\varepsilon\to 0} \sup_{x\geqslant\Theta^+_\varepsilon} \mathbb{P}(\upsilon^{x,\varepsilon} > s_\varepsilon) = 0.$$

Proof. **1.** We introduce the time s_ε with $s_\varepsilon \to 0$, as $\varepsilon \to 0$, which will be determined below. For an appropriate choice of a parameter $\pi \in \mathbb{R}$ we denote the time

$$\widetilde{T}_\pi = \widetilde{T}_\pi(\varepsilon) := \inf\{t > 0 \mid |\Delta_t L| > \varepsilon^{-\pi}\}.$$

For convenience we write shorthand Θ_ε for Θ^+_ε and β, B for β^+, B^+. Then on the events $\{\widetilde{T}_\pi > s_\varepsilon\}$ and $\{\sup_{t\in[0,s_\varepsilon]} |\varepsilon L_t - \varepsilon t\gamma_0| \leqslant \frac{B^+}{2}\Theta_\varepsilon^{\beta^+} s_\varepsilon\}$ we have for $t \in [0, s_\varepsilon]$

$$\begin{aligned} X^{\varepsilon,x}_t &= x + \int_0^t b(X^{\varepsilon,x}_s) ds + \varepsilon L_t \\ &\geqslant \Theta_\varepsilon + B \int_0^t \Big[\Theta^\beta_\varepsilon + (X^{\varepsilon,x}_s - \Theta_\varepsilon) \frac{(6\delta_\varepsilon)^\beta - \Theta^\beta_\varepsilon}{6\delta_\varepsilon - \Theta_\varepsilon}\Big] ds + \varepsilon(L_t - t\gamma_0) + \varepsilon t\gamma_0. \end{aligned}$$

Hence for $W_t := W_t^{\varepsilon,x} := X_t^{\varepsilon,x} - \Theta_\varepsilon - \varepsilon t\gamma_0$ and $\tau_0 := \inf\{t > 0 \mid W_t < 0\}$

$$
\begin{aligned}
W_t &\geqslant B\int_0^t \Big[\Theta_\varepsilon^\beta + (W_s + \varepsilon s\gamma_0)\frac{(6\delta_\varepsilon)^\beta - \Theta_\varepsilon^\beta}{6\delta_\varepsilon - \Theta_\varepsilon}\Big]ds + \varepsilon(L_t - t\gamma_0)\\
&\geqslant B\Theta_\varepsilon^\beta t - \sup_{s\in[0,t]}|\varepsilon(L_s - \gamma_0 s)| + B\int_0^t W_s\Big[\frac{(6\delta_\varepsilon)^\beta - \Theta_\varepsilon^\beta}{6\delta_\varepsilon - \Theta_\varepsilon}\Big]ds + \frac{\gamma_0 B}{2}\Big[\frac{(6\delta_\varepsilon)^\beta - \Theta_\varepsilon^\beta}{6\delta_\varepsilon - \Theta_\varepsilon}\Big]\varepsilon t^2\\
&\geqslant \frac{B}{2}\Theta_\varepsilon^\beta t + B\int_0^t W_s\Big[\frac{(6\delta_\varepsilon)^\beta - \Theta_\varepsilon^\beta}{6\delta_\varepsilon - \Theta_\varepsilon}\Big]ds + \frac{\gamma_0 B}{2}\Big[\frac{(6\delta_\varepsilon)^\beta - \Theta_\varepsilon^\beta}{6\delta_\varepsilon - \Theta_\varepsilon}\Big]\varepsilon t^2\\
&\gtrsim_\varepsilon \frac{B}{2}\Theta_\varepsilon^\beta t + \frac{\gamma_0 B}{12}\frac{\varepsilon t^2}{\delta_\varepsilon^{1-\beta}} + \frac{B}{6}\frac{1}{\delta_\varepsilon^{1-\beta}}\int_0^t W_s ds.
\end{aligned}
$$

Note that W is a (random) continuous function with $W_0 > 0$ such that $W_0 > 0$ in a small neighborhood of 0, that is $\tau_0 > 0$. A classical non-autonomous Gronwall inequality from below yields for $t < \tau_0$ that

$$
\begin{aligned}
W_t \geqslant\ & \frac{B}{2}\Theta_\varepsilon^\beta t + \frac{\gamma_0 B}{6}\frac{\varepsilon t^2}{\delta_\varepsilon^{1-\beta}}\\
&+ \frac{B}{2}\Theta_\varepsilon^\beta \exp\Big(\frac{B}{6}\frac{t}{\delta_\varepsilon^{1-\beta}}\Big)\int_0^t s\exp\Big(-\frac{B}{6}\frac{s}{\delta_\varepsilon^{1-\beta}}\Big)ds\\
&+ \frac{\gamma_0 B}{12}\frac{\varepsilon}{\delta_\varepsilon^{1-\beta}}\exp\Big(\frac{B}{6}\frac{t}{\delta_\varepsilon^{1-\beta}}\Big)\int_0^t s^2\exp\Big(-\frac{B}{6}\frac{s}{\delta_\varepsilon^{1-\beta}}\Big)ds
\end{aligned}
$$

and by direct calculation

$$
\begin{aligned}
W_t \geqslant\ & \frac{B}{2}\Theta_\varepsilon^\beta t - \frac{|\gamma_0|B}{6}\frac{\varepsilon t^2}{\delta_\varepsilon^{1-\beta}}\\
&+ \frac{18}{B}\Theta_\varepsilon^\beta\delta_\varepsilon^{2(1-\beta)}\exp\Big(\frac{B}{6}\frac{t}{\delta_\varepsilon^{1-\beta}}\Big)\Big(1 - (1 + \frac{B}{6}\frac{t}{\delta_\varepsilon^{1-\beta}})\exp\big(-\frac{B}{6}\frac{t}{\delta_\varepsilon^{1-\beta}}\big)\Big)\\
&- |\gamma_0|\frac{18}{B^2}\varepsilon\delta_\varepsilon^{3(1-\beta)}\exp\Big(\frac{B}{6}\frac{t}{\delta_\varepsilon^{1-\beta}}\Big)\Big(2 - (2 + \frac{B}{6}\frac{t}{\delta_\varepsilon^{1-\beta}})\frac{B}{6}\frac{t}{\delta_\varepsilon^{1-\beta}}\exp\big(-\frac{B}{6}\frac{t}{\delta_\varepsilon^{1-\beta}}\big)\Big).
\end{aligned}
$$

We set $s_\varepsilon = \frac{6}{B}\delta_\varepsilon^{\frac{1-\beta}{2}}$. This choice yields for any $C > 0$ a constant $\varepsilon_0 \in (0,1)$ such that $0 < \varepsilon \leqslant \varepsilon_0$

$$
\max\{(1 + \frac{B}{6}\frac{s_\varepsilon}{\delta_\varepsilon^{1-\beta}})\exp\big(-\frac{B}{6}\frac{s_\varepsilon}{\delta_\varepsilon^{1-\beta}}\big), (2 + \frac{B}{6}\frac{s_\varepsilon}{\delta_\varepsilon^{1-\beta}})\frac{B}{6}\frac{s_\varepsilon}{\delta_\varepsilon^{1-\beta}}\exp\big(-\frac{B}{6}\frac{s_\varepsilon}{\delta_\varepsilon^{1-\beta}}\big)\} \leqslant C.
$$

Therefore for $\varepsilon \in (0, \varepsilon_0]$

$$
\begin{aligned}
X_{s_\varepsilon}^{\varepsilon,x} \geqslant\ & \Theta_\varepsilon - \varepsilon s_\varepsilon|\gamma_0| + \frac{B}{2}\Theta_\varepsilon^\beta s_\varepsilon - \frac{|\gamma_0|B}{6}\frac{\varepsilon s_\varepsilon^2}{\delta_\varepsilon^{1-\beta}}\\
&+ \Big[\Big(1 - C\Big)\frac{18}{B}\Theta_\varepsilon^\beta\delta_\varepsilon^{2(1-\beta)} - \Big(2 - C\Big)\frac{18|\gamma_0|}{B^2}\varepsilon\delta_\varepsilon^{3(1-\beta)}\Big]\exp\Big(\frac{B}{6}\frac{s_\varepsilon}{\delta_\varepsilon^{1-\beta}}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
X^{\varepsilon,x}_{s_\varepsilon} &\geqslant \Theta_\varepsilon - \varepsilon s_\varepsilon|\gamma_0| + \frac{B}{2}\Theta_\varepsilon^\beta s_\varepsilon - \frac{|\gamma_0|B}{6}\frac{\varepsilon s_\varepsilon^2}{\delta_\varepsilon^{1-\beta}} \\
&\qquad + \delta_\varepsilon^{2(1-\beta)}\Theta_\varepsilon^\beta\Big[\frac{18}{B}\Big(1-C\Big) - \frac{18|\gamma_0|}{B^2}\Big(2-C\Big)\frac{\varepsilon\delta_\varepsilon}{\Theta_\varepsilon^\beta}\Big]\exp\Big(\frac{B}{6}\frac{s_\varepsilon}{\delta_\varepsilon^{1-\beta}}\Big) \\
&\gtrsim_\varepsilon \exp\Big(\delta_\varepsilon^{-\frac{1-\beta}{3}}\Big) \gtrsim_\varepsilon 6\delta_\varepsilon.
\end{aligned}
$$

This implies for $\pi < 0$ sufficiently small that

$$
\begin{aligned}
\mathbb{P}(\upsilon^{\varepsilon,x} > s_\varepsilon) &\leqslant \mathbb{P}(\sup_{t\in[0,s_\varepsilon]}|\varepsilon\widetilde{\xi}^\varepsilon(t)| > \frac{B}{2}\Theta_\varepsilon^\beta s_\varepsilon) + \mathbb{P}(\widetilde{T}_\pi > s_\varepsilon) \\
&\leqslant \exp(-\frac{B}{2}\frac{\Theta_\varepsilon^\beta}{\varepsilon^{1-\pi}}) + \exp(-\varepsilon^{\alpha\pi}s_\varepsilon) \\
&\lesssim_\varepsilon \exp\big(-\frac{B}{2}\varepsilon^{\frac{2\beta}{1+\beta}(1+\frac{2}{\alpha+\beta-1}((1-\frac{\alpha}{2})+\ln(1-\frac{\alpha}{2})))-1-(-\pi))}\big) - \varepsilon^{-\alpha(-\pi)}\delta_\varepsilon^{\frac{1-\beta}{2}}\big).
\end{aligned}
$$

A particular choice of π is given by

$$
\pi = -\Big|\frac{2\beta}{1+\beta}(1+\frac{2}{\alpha+\beta-1}((1-\frac{\alpha}{2})+\ln(1-\frac{\alpha}{2})))\Big|,
$$

which is finite since $1-\beta < \alpha < 2$. $\square$

6 The solution selection problem: Proof of the main theorem

By Corollary 8 the time scale of convergence is bounded by $\delta_\varepsilon^{-\frac{\beta^2}{2}} \wedge r_\varepsilon$. By definition (3.10) and (3.8) there is $\theta^* > 0$ such that $\varepsilon^{-\theta^*}/(\delta_\varepsilon^{-\frac{\beta^2}{2}} \wedge r_\varepsilon) \to 0$ as $\varepsilon \to 0$. Recall $(\Theta^+_{\varepsilon,\vartheta}, \Theta^-_{\varepsilon,\vartheta}, t_{\varepsilon,\vartheta})_{\varepsilon,\vartheta\in(0,1]}$ defined by Definition 12 and Lemma 10 and the respective hitting times as defined by (4.1)

$$
\begin{aligned}
\tau_{\Theta^+_\varepsilon,\Theta^-_\varepsilon}(\varepsilon,x) &= \inf\{t>0 \mid X^{\varepsilon,x}_t < -\Theta^-_\varepsilon \text{ or } X^{\varepsilon,x}_t \geqslant \Theta^+_\varepsilon\}\\
\sigma_{\delta^+_\varepsilon,\delta^-_\varepsilon}(\varepsilon,x) &= \inf\{t>0 \mid X^{x,\varepsilon}_t < -6\delta^-_\varepsilon \text{ or } X^{x,\varepsilon}_t > 6\delta^+_\varepsilon\},
\end{aligned}
$$

where we dropped the dependence on ϑ. Fix a time scale $\hat t_\varepsilon = t_\varepsilon|\ln(\varepsilon)|$ chosen according to Proposition 9 with respect to t_ε and s_ε determined by Proposition 22.

Since all other dependencies are clear we shall write shorthand $\tau = \tau_{\Theta^+_\varepsilon,\Theta^-_\varepsilon}(\varepsilon,0)$ and $\sigma^x = \sigma_{\delta^+_\varepsilon,\delta^-_\varepsilon}(\varepsilon,x)$ and $X^x = X^{\varepsilon,x}$. We use the strong Markov property of X^x to control the exit from the neighborhood $(-\Theta^-_\varepsilon, \Theta^+_\varepsilon)$ of the origin.

$$
\begin{aligned}
&\mathbb{E}[f((X^0_t)_{t\in[0,\varepsilon^{-\theta^*}]})]\\
&= \mathbb{E}[\mathbb{E}[f((X^0_t)_{t\in[0,\varepsilon^{-\theta^*}]})\mathbf{1}\{\tau \leqslant \varepsilon^{-\theta^*}\}\big(\mathbf{1}\{X^0_\tau \geqslant \Theta^+_\varepsilon\} + \mathbf{1}\{X^0_\tau \leqslant -\Theta^-_\varepsilon\}\big) \mid \mathcal{F}_\tau]]\\
&\qquad + \mathbb{P}(\tau > \varepsilon^{-\theta^*})\\
&\leqslant \mathbb{P}(X^0_\tau \geqslant \Theta^+_\varepsilon) \sup_{x\geqslant\Theta^+_\varepsilon} \mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}-\tau]})\mathbf{1}\{\tau \leqslant \varepsilon^{-\theta^*}\}]\\
&\qquad + \mathbb{P}(X^0_\tau \leqslant -\Theta^-_\varepsilon) \sup_{x\leqslant-\Theta^-_\varepsilon} \mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}-\tau]})\mathbf{1}\{\tau \leqslant \varepsilon^{-\theta^*}\}]\\
&\qquad + \mathbb{P}(\tau > \varepsilon^{-\theta^*}).
\end{aligned}
$$

Proposition 13 and Proposition 17 choose for the strictly α-stable case, $\gamma_0 = 0$ and the general α-stable case, $\gamma_0 \neq 0$ an appropriate ϑ, that the probability $\mathbb{P}(X^0_\tau \geqslant \Theta^+_\varepsilon)$ tends to p^+ as $\varepsilon \to 0$ as given the statement of Theorem 1. The last term tends to 0 due to Proposition 9. We first consider the positive branch.

$$
\begin{aligned}
&\sup_{x\geqslant\Theta^+_\varepsilon} \mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}-\tau]})\mathbf{1}\{\tau \leqslant \varepsilon^{-\theta^*}\}]\\
&\leqslant \sup_{x\geqslant\Theta^+_\varepsilon} \mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}]})]\\
&\leqslant \max\{\sup_{\Theta^+_\varepsilon\leqslant x<6\delta^+_\varepsilon} \mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}]})], \sup_{x\geqslant6\delta^+_\varepsilon} \mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}]})]\}
\end{aligned}
$$

We treat the first term

$$\begin{aligned}
&\sup_{\Theta_\varepsilon^+\leqslant x<6\delta_\varepsilon^+} \mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}]})]\\
&\leqslant \sup_{\Theta_\varepsilon^+\leqslant x<6\delta_\varepsilon^+} \mathbb{E}[f((X^{\varepsilon,x})_{t\in[0,\varepsilon^{-\theta^*}]})\mathbf{1}\{\sigma^x\leqslant \varepsilon^{-\theta^*}\}\mathbf{1}\{X^x_{\sigma^x}\geqslant 6\delta_\varepsilon\}]\\
&\quad+\sup_{\Theta_\varepsilon^+\leqslant x<6\delta_\varepsilon^+}\mathbb{P}(X^x_{\sigma^x}<\Theta_\varepsilon^+)+\sup_{\Theta_\varepsilon^+\leqslant x<6\delta_\varepsilon^+}\mathbb{P}(\sigma^x>\varepsilon^{-\theta^*}),
\end{aligned}$$

where the last two terms tend to 0 as $\varepsilon\to 0$. We continue once again with the help of the strong Markov property

$$\begin{aligned}
&\sup_{\Theta_\varepsilon^+\leqslant x<6\delta_\varepsilon^+} \mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}]})\mathbf{1}\{\sigma^x\leqslant \varepsilon^{-\theta^*}\}\mathbf{1}\{X^x_{\sigma^\varepsilon}\geqslant 6\delta_\varepsilon^+\}]\\
&\leqslant \sup_{x\geqslant 6\delta_\varepsilon^+}\mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}-\sigma^x]})]\\
&\leqslant \sup_{x\geqslant 6\delta_\varepsilon^+}\mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}]})].
\end{aligned}$$

First let f be uniformly continuous with respect to $\mathbb{D}([0,\infty);\mathbb{R})$ equipped with the uniform norm. We denote by Ξ the uniform module of continuity of f. By Corollary 8 for $\theta^*=\kappa$ defined there we have

$$\begin{aligned}
&\sup_{x\geqslant 6\delta_\varepsilon^+}\mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}]})\{\sup_{t\in[0,\varepsilon^{-\theta}]}|X^x_t-x^+_t|\leqslant(\delta_\varepsilon^+)^{\frac{(\beta^+)^2}{2}}\}]+\sup_{x\geqslant 6\delta_\varepsilon^+}\mathbb{P}(\sup_{t\in[0,\varepsilon^{-\theta}]}|X^x_t-x^+_t|>(\delta_\varepsilon^+)^{\frac{(\beta^+)^2}{2}})\\
&\leqslant f((x^+_t)_{t\in[0,\varepsilon^{-\theta^*}]})+\Xi(\delta_\varepsilon^{\frac{(\beta^+)^2}{2}})+\sup_{x\geqslant 6\delta_\varepsilon^+}\mathbb{P}(\sup_{t\in[0,\varepsilon^{-\theta}]}|X^x_t-x^+_t|>(\delta_\varepsilon^+)^{\frac{(\beta^+)^2}{2}}).
\end{aligned}$$

Corollary 8 yields that the last term converges to 0. For the case of general case of f not uniformly continuous, we define the cutoff function $f_m(x):=f(x)\mathbf{1}\{-m\leqslant x\leqslant m\}$, which is uniformly continuous and finally send m to infinity, which is justified by the Beppo-Levi theorem.

We prove the lower bound. Let f be uniformly continuous.

$$\begin{aligned}
&\mathbb{E}[f((X^0_t)_{t\in[0,\varepsilon^{-\theta^*}]})]\\
&\mathbb{E}[f((X^0_t)_{t\in[0,\varepsilon^{-\theta^*}]})(\mathbf{1}\{\tau\leqslant\hat t_\varepsilon\}+\mathbf{1}\{\tau>\hat t_\varepsilon\})]\\
&=\mathbb{E}[\mathbb{E}[f((X^{\varepsilon,0}_t)_{t\in[0,\varepsilon^{-\theta^*}]})\mathbf{1}\{\tau\leqslant\hat t_\varepsilon\}\big(\mathbf{1}\{X^0_\tau\geqslant\Theta_\varepsilon^+\}+\mathbf{1}\{X^0_\tau\leqslant-\Theta_\varepsilon^-\}\big)\mid\mathcal{F}_\tau]]\\
&\geqslant \mathbb{P}(X^0_\tau\geqslant\Theta_\varepsilon^+)\sup_{x\geqslant\Theta_\varepsilon^+}\mathbb{E}[f((X^{\varepsilon,x})_{t\in[0,\varepsilon^{-\theta^*}-\tau]})\mathbf{1}\{\tau\leqslant\hat t_\varepsilon\}]\\
&\qquad+\mathbb{P}(X^0_\tau\geqslant-\Theta_\varepsilon^-)\sup_{x\leqslant-\Theta_\varepsilon^-}\mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta^*}-\tau]})\mathbf{1}\{\tau\leqslant\hat t_\varepsilon\}]\\
&\geqslant p^+\sup_{x\geqslant\Theta_\varepsilon^+}\mathbb{E}[f((X^{\varepsilon,x})_{t\in[0,\varepsilon^{-\theta^*}-\hat t_\varepsilon]})]+p^-\sup_{x\leqslant-\Theta_\varepsilon^-}\mathbb{E}[f((X^{\varepsilon,x})_{t\in[0,\varepsilon^{-\theta^*}-\hat t_\varepsilon]})].
\end{aligned}$$

We continue with the positive branch

$$
\begin{aligned}
&\sup_{x\geqslant\Theta_\varepsilon^+}\mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta*}-\hat{t}_\varepsilon]})]\\
&\geqslant \sup_{x\geqslant\Theta_\varepsilon^+}\mathbb{E}[f((X^{\varepsilon,x})_{t\in[0,\varepsilon^{-\theta*}-\hat{t}_\varepsilon]})\mathbf{1}\{X^x_{\sigma^x}\geqslant 6\delta_\varepsilon^+\}\mathbf{1}\{\sigma^x\leqslant s_\varepsilon\}]\\
&\geqslant \sup_{x\geqslant 6\delta_\varepsilon^+}\mathbb{E}[f((X^x)_{t\in[0,\varepsilon^{-\theta*}-\hat{t}_\varepsilon-s_\varepsilon]})\mathbf{1}\{\sup_{t\in[0,\varepsilon^{-\theta*}]}|X^x_t-x^+_t|\leqslant\delta_\varepsilon^{\frac{(\beta^+)^2}{2}}\}]\\
&\geqslant f((x^+_t)_{t\in[0,\varepsilon^{-\theta*}-\hat{t}_\varepsilon-s_\varepsilon]})-\Xi((\delta_\varepsilon^+)^{\frac{(\beta^+)^2}{2}}).
\end{aligned}
$$

The negative branch is treated analogously. For a function f not uniformly continuous we use the same truncation argument as before. This proves the desired result.

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