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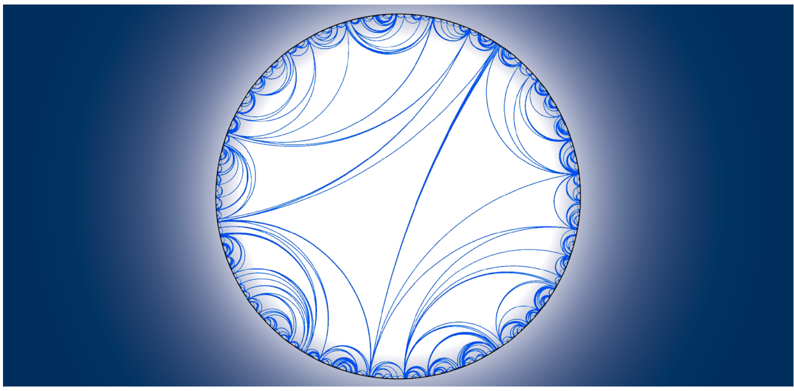

Oskar Sultanov | Leonid Kalyakin | Nikolai Tarkhanov

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Elliptic Perturbations of Dynamical Systems with a Proper Node

O. A. Sultanov [*], L. A. Kalyakin [†], and N. N. Tarkhanov [‡]

April 28, 2014

Contents

Introduction **1**

1 The case of homogeneous differential equation **4**

2 The case of homogeneous boundary condition **7**

3 The case of right-hand side of zero average value **8**

4 Conclusion **9**

5 Appendix **9**

References **12**

Introduction

Consider the first boundary value problem for a second order elliptic equation with small parameter $0 < \varepsilon \ll 1$,

$$\begin{array}{rcll} \ell_\varepsilon u := \varepsilon \Delta u + a_1(x,y)\partial_x u + a_2(x,y)\partial_y u &=& f(x,y), & \text{for} \quad (x,y) \in \Omega, \\ u &=& g(x,y), & \text{for} \quad (x,y) \in \partial\Omega. \end{array} \tag{0.1}$$

Here, $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\}$ is the disk of radius R with centre at the origin, Δ is the Laplace operator in the plane, the coefficients a_1 and a_2 are assumed to be smooth functions in a neighbourhood of the closure of Ω, and f, g are smooth functions in the closure of Ω and at the boundary of Ω, respectively. From a priori estimates of the Schauder type it follows that for every fixed $\varepsilon > 0$ problem (0.1) has a unique solution $u = u(x,y;\varepsilon)$, see for instance [3, Ch. 3]. We are interested in studying the asymptotic behaviour of the solution u as $\varepsilon \to 0$.

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Assume that the dynamical system

$$\begin{array}{rcl} \dot{x} & = & a_1(x,y), \\ \dot{y} & = & a_2(x,y), \end{array} \tag{0.2}$$

$t \geq 0$, has a unique stationary solution $x = 0$, $y = 0$ in Ω. We focus on the case where this solution is asymptotically stable, i.e., the trajectories of the system point towards the domain Ω and tend to the origin.

It should be noted that boundary value problem (0.1) contains a small parameter multiplying the highest order derivatives. It is known [12, 4] that the behaviour of the solution to (0.1) depends on the characteristics of the limit equation $\ell_0 u = f$. The well-known perturbation method for constructing an asymptotic solution of problem (0.1) starts with a "good" solution of the limit problem. If such a solution is available, it can be glued together with a boundary layer constructed by transition to stretched coordinates $x' = \sqrt{\varepsilon}x$ and $y' = \sqrt{\varepsilon}y$. This method falls short of providing an asymptotic solution of problem (0.1), for under the presence of singular point of the vector field $a = (a_1, a_2)$ the limit problem fails to possess any "good" solution. Moreover, the approximation obtained in this way grows exponentially as $\varepsilon \to 0$, which makes the use of boundary layer inefficient. Hence, the study of (0.1) requires more advanced techniques.

Problem (0.1) appears in the study of white noise effect in stability theory of fixed points of dynamical system (0.2). For this purpose one considers the perturbed equations in the form of stochastic differential equations

$$\begin{array}{rcl} dX_t & = & a_1(X_t, Y_t)dt + \sqrt{2\varepsilon}\, dW_t^1, \\ dY_t & = & a_2(X_t, Y_t)dt + \sqrt{2\varepsilon}\, dW_t^2 \end{array} \tag{0.3}$$

under the initial condition $X_0 = x$, $Y_0 = y$. Here, $W_t^1(\omega)$ and $W_t^2(\omega)$ are independent one-dimensional Winer processes defined on a probability space $(\mathcal{X}, \mathcal{A}, P)$, where $\mathcal{X}$ is arbitrary nonempty set, $\mathcal{A}$ a sigma-algebra, and P a probability measure. The solution $X_t(\omega)$, $Y_t(\omega)$ of this system is a stochastic process which depends on the parameter $\varepsilon > 0$. It is well known [11] that the trajectories of (0.3) leave any bounded domain in $\mathbb{R}^2$ with probability one. Hence, there is no stability of the fixed point $x = 0$, $y = 0$ under white noise perturbations. Denote by

$$\tau_\Omega(\omega) = \inf\{t \geq 0 : (X_t(\omega), Y_t(\omega)) \notin \Omega\}$$

the first exit time from the domain Ω. It is of interest to compute the mean exit time $\mathbb{E}\tau_\Omega$ of stochastic trajectories $(X_t(\omega), Y_t(\omega))$, when the noise intensity is sufficiently small, i.e. $0 < \varepsilon \ll 1$. It is worth pointing out that the solution of elliptic equation (0.1) is associated with certain probabilistic parameters of stochastic trajectories. For instance, if $f \equiv -1$ and $g = 0$, then $u(x, y; \varepsilon) \equiv \tau_\Omega$ (see [10], p. 110). Hence it follows that asymptotic analysis of solution of boundary value problem (0.1) as $\varepsilon \to 0$ is of great importance in the research of dynamical systems (0.2) under white noise perturbation.

As but prime example let us consider boundary value problem (0.1) in dimension one, i.e. $n = 1$,

$$\begin{array}{rcl} \varepsilon u''_{xx} - x u'_x & = & -1, \quad \text{for} \quad x \in (-1, 1), \\ u(-1) & = & 0, \\ u(1) & = & 0, \end{array}$$

where $0 < \varepsilon \ll 1$. The solution is given by

$$u(x;\varepsilon) = -\frac{1}{\varepsilon}\int_{-1}^{x} \exp(s^2/2\varepsilon) \int_0^s \exp(-z^2/2\varepsilon)dzds.$$

Using Laplace's method one finds an asymptotic expansion for the solution as $\varepsilon \to 0$. Namely,

$$u(x;\varepsilon) = e^{1/2\varepsilon}\sqrt{\frac{\varepsilon\pi}{2}}\left(1 + \varepsilon + 3\varepsilon^2 + O(\varepsilon^3)\right),$$

as $\varepsilon \to 0$, the expansion being uniform in x in each interval $|x| \le 1 - \delta$ with $\delta > 0$. It should be noted that the solution has a growing exponential in its asymptotic expansion which does not depend on x in the main term. Apparently, such intriguing effects can also appear in the two-dimensional case to be studied below.

Such problems have been investigated using probabilistic methods. In particular, an exponential estimate for the solution of (0.1) was found in [11] for the case of $f \equiv -1$ and $g = 0$. The work [7] presents a method for obtaining the leading term of asymptotic expansion of the solution u. In [5], one finds a proof of this formula for the case of potential vector fields a, and in [9] a proof for arbitrary vector fields. However, the construction of full asymptotic expansion for solution (0.1) still remains an open problem.

The present paper is devoted to asymptotic analysis of boundary value problem (0.1) in the spatial case, where the characteristics of limit equation bear radial symmetry. In other words, the stationary solution of system (0.2) is actually a proper node. Then equation (0.1) can be written in the usual polar coordinates (r, φ) as

$$\begin{array}{rcll} \varepsilon(\partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\varphi^2)u + b(r)\partial_r u &=& f(r,\varphi), & \text{if } \ r < R, \\ u &=& g(\varphi), & \text{if } \ r = R, \end{array} \tag{0.4}$$

where $b(r) = -r + O(r^2)$ as $r \to 0$. Since the substitution $x = r\cos\varphi$, $y = r\sin\varphi$ has a singular point $r = 0$, there appears an additional boundary condition $|u(0,\varphi;\varepsilon)| < \infty$.

In the particular case $b(r) = -r$ there is an explicit formula for the solution, and so we get full information about its asymptotics. The study of this case allows one to detect a relation between the right-hand side $f(r,\varphi)$ and the appearance of exponential growth in the solution. To wit, an exponential growth in the solution of (0.4) appears only in the case when $f(r,\varphi)$ has nonzero average value. This remark and the construction of asymptotic expansion for solution (0.4) with $b(r) = -r$ constitute our contribution.

The paper contains five sections. By superposition principle, the general solution of problem (0.1) splits into the sum of two functions, the first one satisfying the homogeneous differential equation and inhomogeneous boundary condition and the second functions satisfying the inhomogeneous differential equation and homogeneous boundary condition. In Section 1 we study the boundary value problem in the case where $f(r,\varphi) \equiv 0$. Section 2 is devoted to the case $g(\varphi) \equiv 0$. In Section 3 we treat in detail the case where $b(r) = -r$. After a short conclusion of Section 4 we adduce proofs of main asymptotic formulas in Section 5.

1 The case of homogeneous differential equation

We first observe that the functions g at the boundary of Ω can be though of as functions of $\varphi \in [0, 2\pi]$ satisfying $g(0) = g(2\pi)$. The arc length at the circle $\partial\Omega$ is $ds = R d\varphi$, hence the scalar product in the space $L^2(\partial\Omega)$ looks like that at the unit circle

$$(g, h)_{L^2(\partial\Omega)} = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi)\overline{h(\varphi)} d\varphi.$$

We look for a formal solution to boundary value problem (0.4) in the form of Fourier series

$$u(r, \varphi; \varepsilon) = \sum_{k=-\infty}^{\infty} e^{\imath k\varphi} U_k(r; \varepsilon), \tag{1.1}$$

on the interval $[0, 2\pi]$, $r \in [0, R]$ being a parameter. The coefficients satisfy the boundary value problem

$$\begin{array}{rcll} l_k U_k := \Big(\varepsilon(\partial_r^2 + r^{-1}\partial_r - k^2 r^{-2}) + b(r)\partial_r\Big) U_k(r; \varepsilon) & = & 0, & \text{if} \quad r < R, \\ U_k(R; \varepsilon) & = & G_k, & \\ |U_k(0; \varepsilon)| & < & \infty, & \end{array} \tag{1.2}$$

where

$$G_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-\imath k\varphi} g(\varphi) d\varphi$$

are the Fourier coefficients of the Dirichlet data. For $k \neq 0$ there is an additional boundary condition at the singular point $r = 0$, namely $U_k(0; \varepsilon) = 0$. This condition arises from the claim of continuity of solution.

If $k = 0$, then $U_0(r; \varepsilon) \equiv G_0$, for the solution of (1.2) is unique. For $k \neq 0$, we are able to estimate the solution of (1.2). Indeed, we shall construct a solution in the form

$$U_k(r; \varepsilon) = r^2 G_k / R^2 + V_k(r; \varepsilon).$$

Then V_k ought to satisfy the boundary value problem

$$\begin{array}{rcll} l_k V_k & = & (\varepsilon(k^2 - 4) + 2r^2) \dfrac{G_k}{R^2}, & \text{if} \quad r < R, \\ V_k(R; \varepsilon) & = & 0, & \\ V_k(0; \varepsilon) & = & 0. & \end{array} \tag{1.3}$$

On using the maximum principle for solutions of (1.3) we establish an estimate for $U_k(r; \varepsilon)$, to wit

$$\max_{r \in [0;R]} |U_k(r; \varepsilon)| \leq \Big(6 + \frac{2R^2}{\varepsilon k^2}\Big) |G_k|$$

for all $\varepsilon > 0$. If $g \in C^2$, then series (1.1) and its first and second derivatives converge.

We now construct an asymptotic expansion for the solution of (0.4) in the case $b(r) = -r$. To this end we study the behaviour of the terms of series (1.1) as $\varepsilon \to 0$. Consider the problem

$$\begin{array}{rcll} \partial_r^2 U_k - \Big(\dfrac{r}{\varepsilon} - \dfrac{1}{r}\Big)\partial_r U_k - \dfrac{k^2}{r^2} U_k & = & 0, & \text{if} \quad r < R, \\ U_k(R; \varepsilon) & = & G_k, & \\ |U_k(0; \varepsilon)| & < & \infty. & \end{array} \tag{1.4}$$

Since $U_{-k}(r;\varepsilon) = U_k(r;\varepsilon)$, we restrict our attention to those k which are nonnegative. If $k = 2m$ with a nonnegative integer m, then equation (1.4) has two linearly independent solutions

$$\begin{aligned} \Phi_0(r;\varepsilon) &\equiv 1, \\ \Psi_0(r;\varepsilon) &= \int_r^R \exp\left(\frac{z^2}{2\varepsilon}\right) \frac{dz}{z}, \end{aligned}$$

if $m = 0$, and

$$\begin{aligned} \Phi_{2m}(r;\varepsilon) &= \sum_{j=1}^{m} \frac{a_{2m,2j}\varepsilon^j}{r^{2j}} \Big(\exp\left(\frac{r^2}{2\varepsilon}\right) - \sum_{l=0}^{j} \frac{1}{l!}\left(\frac{r^2}{2\varepsilon}\right)^l \Big), \\ \Psi_{2m}(r;\varepsilon) &= \sum_{j=0}^{m} \frac{b_{2m,2j}\varepsilon^j}{r^{2j}}, \end{aligned}$$

if $m \neq 0$. The coefficients $a_{2m,2j}$ and $b_{2m,2j}$ are uniquely determined from the recurrence relations

$$\begin{aligned} j a_{2m,2j+2} &= 2(j^2 - m^2) a_{2m,2j}, \\ (j+1) b_{2m,2j+2} &= 2(m^2 - j^2) b_{2m,2j} \end{aligned}$$

for $j = 0, 1, \ldots, m-1$, where $a_{2m,2}$ and $b_{2m,0}$ are arbitrary nonzero constants. It is immediately obvious that $\Phi_{2m}(r;\varepsilon) = 0(\xi^{2m})$ as $\xi = r/\sqrt{\varepsilon} \to 0$, because

$$\sum_{j=1}^{m} \frac{a_{2m,2j}}{2^j (j+l)!} \equiv 0$$

for $l = 1, \ldots, m-1$.

If $k = 2m+1$ with a nonnegative integer m, a pair of linearly independent solutions to equation (1.4) is given by means of the confluent hypergeometric functions $\tilde{\Phi}(\rho; a, c)$ and $\tilde{\Psi}(\rho; a, c)$, namely

$$\begin{aligned} \Phi_k(r;\varepsilon) &= \Big(\frac{r^2}{2\varepsilon}\Big)^{k/2} \tilde{\Phi}\Big(\frac{r^2}{2\varepsilon}; \frac{k}{2}, k+1\Big), \\ \Psi_k(r;\varepsilon) &= \Big(\frac{r^2}{2\varepsilon}\Big)^{k/2} \tilde{\Psi}\Big(\frac{r^2}{2\varepsilon}; \frac{k}{2}, k+1\Big). \end{aligned}$$

There are integral representations for the special functions $\tilde{\Phi}(\rho; a, c)$ and $\tilde{\Psi}(\rho; a, c)$, to wit

$$\begin{aligned} \tilde{\Phi}(\rho; a, c) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{\rho t} t^{a-1} (1-t)^{c-a-1}\, dt, \\ \tilde{\Psi}(\rho; a, c) &= \frac{1}{\Gamma(a)} \int_0^\infty e^{-\rho t} t^{a-1} (1+t)^{c-a-1}\, dt, \end{aligned}$$

see for instance [1, 8]. Asymptotic analysis of the integrals shows that $\Phi_k(r;\varepsilon)$ is smooth at zero, and it has exponential growth as $\xi \to \infty$. On the other hand, $\Psi_k(r;\varepsilon)$ has a singularity at $r = 0$ and it decreases as $\xi \to \infty$. The behaviour of solutions in a neighborhood of singular points $\xi = 0$ and $\xi = \infty$ can be derived from the formal constructions of [6]. Asymptotic solutions are easily constructed in the form of power series with constant coefficients

$$\Phi_k(r;\varepsilon)e^{-\xi^2/2} = \sum_{j=1}^{\infty} a_{k,2j}\xi^{-2j}, \quad \Psi_k(r;\varepsilon) = \sum_{j=0}^{\infty} b_{k,2j}\xi^{-2j},$$

as $\xi \to \infty$. Substituting these series in (1.4) and equating the coefficients of the same powers of ξ give recurrence relations for determining the coefficients, namely

$$\begin{aligned} a_{k,2j+4} &= \frac{4(j+1)^2 - k^2}{2(j+1)} a_{k,2j+2}, \\ b_{k,2j+2} &= \frac{k^2 - 4j^2}{2(j+1)} b_{k,2j} \end{aligned}$$

for $j \geq 0$. The coefficients $a_{k,2}$, $b_{k,0}$ are arbitrary nonzero constants. In much the same way we construct an asymptotic solution as $\xi \to 0$, to wit

$$\Phi_1(r;\varepsilon) = \xi \sum_{j=0}^{\infty} (\tilde{c}_{1,2j} \ln \xi + c_{1,2j})\xi^{2j}, \qquad \begin{aligned} \Phi_k(r;\varepsilon) &= \xi^k \sum_{j=0}^{\infty} c_{k,2j}\, \xi^{2j}, \\ \Psi_k(r;\varepsilon) &= \xi^{-k} \sum_{j=0}^{\infty} d_{k,2j}\, \xi^{2j}, \end{aligned}$$

where $\tilde{c}_{1,2j}$ and $c_{k,2j}$, $d_{k,2j}$ are constants, $k \geq 1$. Each of the constructed series corresponds to an exact solution, for which that series gives an asymptotic expansion as $\xi \to \infty$ or $\xi \to 0$, cf. [6].

The general solution of (0.4) in the case of $b(r) = -r$ is constructed in the form of linear combination

$$U_k(r;\varepsilon) = C_k(\varepsilon)\Phi_k(r;\varepsilon) + D_k(\varepsilon)\Psi_k(r;\varepsilon),$$

for $k \in \mathbb{Z}$. We choose $D_k(\varepsilon) \equiv 0$ to exclude singularities at zero. Then $C_k(\varepsilon)$ is determined from the boundary condition $U_k(R;\varepsilon) = G_k$, that is,

$$\begin{aligned} C_0(\varepsilon) &\equiv G_0, \\ C_k(\varepsilon) &= G_k/\Phi_k(R;\varepsilon). \end{aligned}$$

Hence it follows that

$$\begin{aligned} U_0(r;\varepsilon) &\equiv G_0, \\ U_k(r;\varepsilon) &= G_k e^{(r^2-R^2)/2\varepsilon}\left(1 + O(\varepsilon)\right), \end{aligned}$$

as $\varepsilon \to 0$, uniformly in $r \in [\varepsilon^{1/2-\delta}, R]$ for any $\delta > 0$.

It remains to estimate the sum of the Fourier series as $\varepsilon \to 0$ in order to get asymptotics of the solution to boundary value problem (0.4). Note that the maximum principle applies to boundary value problem (1.4) to give an estimate for $U_k(r;\varepsilon)$, where $k \neq 0$. More precisely, we obtain

$$|U_k(r;\varepsilon)| \leq e^{(r^2-R^2)/2\varepsilon}|G_k|\Big(1 + \frac{3r^2 + \varepsilon|k^2-4|}{|r^2 - \varepsilon k^2|}\Big)$$

for all $\varepsilon > 0$ and $r \in [0; R]$. Hence, the solution of (0.4) in the case of $b(r) = -r$ can be expressed as convergent Fourier series

$$u(r,\varphi;\varepsilon) = G_0 + e^{(r^2-R^2)/2\varepsilon} \sum_{k \neq 0} e^{\imath k\varphi}\tilde{U}_k(r;\varepsilon). \tag{1.5}$$

Theorem 1.1 *Let $f \equiv 0$, $g \in C^2(\partial\Omega)$, and $b(r) = -r$. Then, for each $\varepsilon > 0$, the solution $u \in C^2(\overline{\Omega})$ of boundary value problem (0.4) has the form (1.5) and $u(r,\varphi;\varepsilon) = G_0 + o(1)$, as $\varepsilon \to 0$, uniformly in $r \in [\varepsilon^{1/2-\delta}; R-\delta]$ and $\varphi \in (0, 2\pi]$, for any $\delta > 0$.*

2 The case of homogeneous boundary condition

As in Section 1, the solution is constructed in the form (1.1). Then its coefficients fulfill the boundary value problem

$$\begin{array}{rcl} l_k U_k(r,\varepsilon) & = & F_k(r), \quad \text{for} \quad r < R, \\ U_k(R;\varepsilon) & = & 0, \\ |U_k(0;\varepsilon)| & < & \infty, \end{array} \tag{2.1}$$

where $F_k(r)$ are the Fourier coefficients of the function $f(r,\varphi)$ on the interval $\varphi \in [0, 2\pi]$.

For $k = 0$, problem (2.1) can be solved explicitly, which yields

$$U_0(r;\varepsilon) = \frac{1}{\varepsilon}\int_R^r s^{-1} e^{\theta(s)/\varepsilon} \int_0^s z e^{-\theta(z)/\varepsilon} F_0(z)\,dz\,ds$$

with $\theta(r) = -\int_0^r b(z)dz$.

If $k \neq 0$, then there is an a priori estimate for solutions. Note that the assumption on the continuity of solution $u(r,\varphi;\varepsilon)$ in the disk Ω implies readily $U_k(0;\varepsilon) = 0$ for all $\varepsilon > 0$ and $k \neq 0$. On using the maximum principle we immediately obtain

$$\max_{r\in[0;R]} |U_k(r;\varepsilon)| \le \frac{R^2}{k^2\varepsilon} \max_{r\in[0;R]} |F_k(r)| \tag{2.2}$$

for all $\varepsilon > 0$. It follows that the Fourier series (1.1) converges together with the first and second derivatives.

Our next concern will be the asymptotics of constructed solution $u(r,\varphi;\varepsilon)$ as $\varepsilon \to 0$. Let $f \in C^2(\overline{\Omega})$. Then, for any $k > 0$, the Fourier coefficient $F_k(r)$ possesses asymptotics

$$F_k(r) = r^{|k|}\left(F_{k,0} + rF_{k,1} + O(r^2)\right),$$

as $r \to 0$.

Note that if $b(r) = -r$, $\theta(r) = r^2/2$ and $F_0(r)$ does not vanish identically, then $U_0(r;\varepsilon)$ has exponential growth as $\varepsilon \to 0$. To wit,

$$U_0(r;\varepsilon) = -c(\varepsilon)\, e^{R^2/2\varepsilon} \sum_{j=1}^{\infty} \frac{\omega_{2j}\,\varepsilon^j}{R^{2j}},$$

as $\varepsilon \to 0$, uniformly in $r \in [\varepsilon^{1/2-\delta}, R-\delta]$, for any $\delta > 0$, where

$$\begin{array}{rcl} c(\varepsilon) & = & F_{0,0} + \varepsilon^{1/2}\sqrt{2\pi}F_{0,1} + O(\varepsilon), \\ \omega_{2j} & = & (2j-2)!!. \end{array}$$

The remaining coefficients $U_k(r;\varepsilon)$ with $k \neq 0$ fulfill inequalities (2.2). Furthermore, there is an available estimate for the remainder of the Fourier series

$$U(r,\varphi;\varepsilon) - U_0(r;\varepsilon) = O(\varepsilon^{-1}) \tag{2.3}$$

as $\varepsilon \to 0$, uniformly in $(r,\varphi) \in \Omega$.

We denote by $\langle f(r,\varphi)\rangle$ the average value of the function $f(r,\varphi)$ on the interval $[0,2\pi]$, i.e.

$$\langle f(r,\varphi)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(r,\varphi)d\varphi = F_0(r).$$

Theorem 2.1 *Suppose that $b \in C^1[0,R]$ and $f \in C^1(\overline{\Omega})$, $g \equiv 0$. Then, for any $\varepsilon > 0$, the solution $u \in C^2(\overline{\Omega})$ of boundary value problem* (0.4) *has the form* (1.1). *If moreover $\langle f(r,\varphi)\rangle$ is different from zero, then the solution has asymptotics*

$$u(r,\varphi;\varepsilon) = e^{\theta(R)/\varepsilon}\frac{\varepsilon^2 F_{0,0}}{Rb(R)}\left(1 + O(\varepsilon^{1/2})\right),$$

as $\varepsilon \to 0$, uniformly in $r \in [\varepsilon^{1/2-\delta}, R-\delta]$ and $\varphi \in (0,2\pi]$, for any $\delta > 0$.

If $\langle f(r,\varphi)\rangle = 0$, then $U_0(r;\varepsilon) \equiv 0$ and it is necessary to analyse the behaviour of $U_k(r;\varepsilon)$ as $\varepsilon \to 0$ to construct asymptotics of the solution. A rough estimate $O(\varepsilon^{-1})$ follows from the maximum principle. However, this estimate can be specified in the particular case $b(r) = -r$ by evaluating the asymptotics of the Fourier coefficients $U_k(r;\varepsilon)$, as $\varepsilon \to 0$. To do this, one ought to investigate certain Laplace-type integrals, see [2].

3 The case of right-hand side of zero average value

Using variation of constants, one easily obtains a particular solution $V_k(r;\varepsilon)$ to inhomogeneous differential equation (2.1). That is

$$V_0(r;\varepsilon) = \frac{1}{\varepsilon}\int_0^r s^{-1}e^{s^2/2\varepsilon}\int_0^s ze^{-z^2/2\varepsilon}F_0(z)dz\,ds = 0,$$

for $F_0(r) = 0$. If $k \neq 0$ then

$$\begin{aligned} &V_k(r;\varepsilon) \\ &= \Phi_k(r;\varepsilon)\int_R^r ze^{-z^2/2\varepsilon}F_k(z)\Psi_k(z;\varepsilon)dz - \Psi_k(r;\varepsilon)\int_0^r ze^{-z^2/2\varepsilon}F_k(z)\Phi_k(z;\varepsilon)dz. \end{aligned}$$

The solution of (2.1) with $b(r) = -r$ is constructed in the form

$$U_k(r;\varepsilon) = V_k(r;\varepsilon) + C_k(\varepsilon)\Phi_k(r;\varepsilon) + D_k(\varepsilon)\Psi_k(r;\varepsilon), \tag{3.1}$$

for $k \in \mathbb{Z}$. Once again we set $D_k(\varepsilon) \equiv 0$ to exclude singularities at zero. Then $C_0(\varepsilon) \equiv 0$ and

$$C_k(\varepsilon) = \frac{\Psi_k(R;\varepsilon)}{\Phi_k(R;\varepsilon)}\int_0^R ze^{-z^2/2\varepsilon}F_k(z)\Phi_k(z;\varepsilon)\,dz$$

for $k \neq 0$. Formula (3.1) allows one to derive asymptotics of the Fourier coefficients $U_k(r;\varepsilon)$, as $\varepsilon \to 0$ (see Section 5). More precisely, we get

$$U_k(r;\varepsilon) = \ln\varepsilon \sum_{j=2}^{\infty} \tau_{k,j}(r) r^{-j} \varepsilon^{j/2} + \sum_{j=2}^{\infty} \sigma_{k,j}(r) r^{-j} \varepsilon^{j/2}, \qquad (3.2)$$

as $\varepsilon \to 0$, uniformly in $r \in [\varepsilon^{1/2-\delta}; R-\delta]$, for any $\delta > 0$. The coefficients $\sigma_{k,j}(r)$ and $\tau_{k,j}(r)$ are bounded functions of $r \in [0, R]$.

Theorem 3.1 *Let $b(r) = -r$. Suppose $f \in C^1(\overline{\Omega})$ has zero average value on the interval $[0, 2\pi]$ and $g \equiv 0$. Then the Fourier coefficients of the solution $u \in C^2(\overline{\Omega})$ to boundary value problem* (0.4) *have asymptotics* (3.2) *uniformly in $r \in [\varepsilon^{1/2-\delta}, R-\delta]$ for all $\delta > 0$.*

4 Conclusion

We construct an explicit formal solution of Dirichlet problem (0.1) and establish its asymptotic character, as $\varepsilon \to 0$. If $\langle f(r,\varphi)\rangle \neq 0$, then the solution grows exponentially, as ε tends to zero. If $\langle f(r,\varphi)\rangle = 0$, then the solution has power-logarithmic asymptotics.

5 Appendix

Here we compute asymptotic estimates for the solution $U_k(r;\varepsilon)$ of boundary value problem (2.1), as $\varepsilon \to 0$. All asymptotic series written here are uniform with respect to the parameter $r \in [\varepsilon^{1/2-\delta}, R-\delta]$, where $\delta > 0$ is an arbitrary small number.

We rewrite $U_k(r;\varepsilon)$ as

$$U_k(r;\varepsilon) = \Phi_k(r;\varepsilon)\left(J_k^2(r;\varepsilon) + C_k(\varepsilon)\right) + \Psi_k(r;\varepsilon)\left(J_k^1(0;\varepsilon) - J_k^1(r;\varepsilon)\right),$$

where

$$\begin{aligned} J_k^1(r;\varepsilon) &:= \int_R^r z e^{-z^2/2\varepsilon} F_k(z) \Phi_k(z;\varepsilon)\,dz, \\ J_k^2(r;\varepsilon) &:= \int_R^r z e^{-z^2/2\varepsilon} F_k(z) \Psi_k(z;\varepsilon)\,dz. \end{aligned}$$

The functions $J_k^1(r;\varepsilon)$ and $J_k^2(r;\varepsilon)$ are Laplace-type integrals bearing asymptotic estimates

$$\begin{aligned} J_k^1(r;\varepsilon) &= \sum_{l=1}^{\infty} \alpha_{k,l}(r) \xi^{-2l}, \\ J_k^2(r;\varepsilon) &= e^{-\xi^2/2} \sum_{l=1}^{\infty} \beta_{k,l}(r) \xi^{-2l}, \end{aligned}$$

as $\xi \to \infty$. The coefficients $\beta_{k,l}(r)$ are linear combinations of $F_k(r)$ and its derivatives, in particular,

$$\begin{aligned} \beta_{k,1}(r) &= -r^2 F_k(r), \\ \beta_{k,2}(r) &= -b_{k,2} r^2 F_k(r) - r^3 F_k'(r), \end{aligned}$$

etc., while

$$\alpha_{k,l}(r) = a_{k,2l} r^{2l} \int_R^r z^{1-2l} F_k(z) dz.$$

The functions $\alpha_{k,l}(r)$ and $\beta_{k,l}(r)$ are bounded. The construction of asymptotic expansion of $J_k^1(0;\varepsilon)$ is slightly more complicated. Let $k = 2m$ with a nonnegative integer m. Then, using an explicit representation for the integrand, we find that

$$J_{2m}^1(0;\varepsilon) = \sum_{j=1}^{m} \tilde{A}_{2m,2j} \varepsilon^j + \tilde{J}_{2m}^1(\varepsilon) \varepsilon^{m+1},$$

where

$$\begin{aligned}
\tilde{A}_{2m,2j} &= -a_{2m,2j} \int_0^R z^{1-2j} F_{2m}(z)\, dz, \\
\tilde{J}_{2m}^1(\varepsilon) &= \sum_{j=1}^{m} a_{2m,2j} \sum_{l=0}^{j} \int_0^{R/\sqrt{\varepsilon}} \frac{z^{1+2(|m|-j+l)}}{2^l l!} \tilde{F}_{2m}(\sqrt{\varepsilon} z) e^{-z^2/2}\, dz \;=\; O(1),
\end{aligned}$$

as $\varepsilon \to 0$. Here, $\tilde{F}_{2m}(r) = r^{-2|m|} F_{2m}(r) = O(1)$, as $r \to 0$.

If $k = 2m+1$, then

$$F_k(z) = z^{|k|} \Big(\sum_{l=0}^{n} F_{k,l} z^l + \tilde{F}_{k,n+1}(z) \Big),$$

where $\tilde{F}_{k,n+1}(z) = O(z^{n+1})$ as $z \to 0$. Furthermore, we get

$$\Phi_k(z;\varepsilon) = e^{\xi^2/2} \Big(\sum_{j=1}^{N} a_{k,j} \xi^{-2j} + \tilde{\Phi}_{k,N+1}(\xi) \Big),$$

where $\tilde{\Phi}_{k,N+1}(\xi) = O(\xi^{-2N-2})$, as $\xi \to \infty$. It follows that

$$J_{2m+1}^1(0;\varepsilon) = I_k^1(\varepsilon) + I_k^2(\varepsilon) + I_k^3(\varepsilon) + I_k^4(\varepsilon),$$

where

$$\begin{aligned}
I_k^1(\varepsilon) &:= -\sum_{l=0}^{n} F_{k,l} \int_0^R z^{|k|+1+l} \sum_{j=1}^{N_l} \frac{a_{k,2j} \varepsilon^j}{z^{2j}} dz, \\
I_k^2(\varepsilon) &:= -\sum_{l=0}^{n} F_{k,l} \int_0^R z^{|k|+1+l} \tilde{\Phi}_{k,N_l+1}\Big(\frac{z}{\sqrt{\varepsilon}}\Big) dz, \\
I_k^3(\varepsilon) &:= -\sum_{l=1}^{N} a_{k,2l} \varepsilon^l \int_0^R z^{|k|+1-2j} \tilde{F}_{k,n+1}(z) dz, \\
I_k^4(\varepsilon) &:= -\int_0^R z^{|k|+1} \tilde{F}_{k,n+1}(z) \tilde{\Phi}_{k,N+1}\Big(\frac{z}{\sqrt{\varepsilon}}\Big) dz,
\end{aligned}$$

$N > 1$ and $n > 0$.

For each $l \geq 0$ we choose $N_l = [(l+k+1)/2]$ in $I_k^1(\varepsilon)$. Then

$$I_k^1(\varepsilon) = \sum_{j=1}^{N_n} \tilde{\lambda}_{k,2j} \varepsilon^j,$$

where $\tilde{\lambda}_{k,2j}$ are constants. We consider the terms of $I_k^2(\varepsilon)$ with $l = 2p$ and $l = 2p+1$, to wit

$$\begin{aligned} I_{k,2p}^2 &= \int_0^R z^{2|m|+2p+2} \tilde{\Phi}_{k,|m|+p+2}\Big(\frac{z}{\sqrt{\varepsilon}}\Big) dz \\ &= \varepsilon^{|m|+p+3/2} \Big(\int_0^\infty - \int_{R/\sqrt{\varepsilon}}^\infty \Big) \xi^{2|m|+2p+2} \tilde{\Phi}_{k,|m|+p+2}(\xi) d\xi \\ &= \nu_{k,2p} \varepsilon^{|m|+p+3/2} + \sum_{j=|m|+p+2}^\infty \lambda_{k,j} \varepsilon^j \end{aligned}$$

and

$$\begin{aligned} & I_{k,2p+1}^2 \\ &= \int_0^R z^{2|m|+2p+3} \tilde{\Phi}_{k,|m|+p+2}\Big(\frac{z}{\sqrt{\varepsilon}}\Big) dz \\ &= \varepsilon^{|m|+p+2} \Big(\int_0^1 + \int_1^{R/\sqrt{\varepsilon}} \Big) \xi^{2|m|+2p+3} \Big(a_{k,2|m|+2p+4} \xi^{-2|m|-2p-4} + \tilde{\Phi}_{k,|m|+p+3}(\xi) \Big) d\xi \\ &= (\nu_{k,2p+1} + \mu_{k,2p+1} \ln \varepsilon) \, \varepsilon^{|m|+p+2} + \sum_{j=|m|+p+3}^\infty \lambda_{k,j} \varepsilon^j. \end{aligned}$$

The coefficients $\lambda_{k,j}$, $\nu_{k,j}$, $\mu_{k,2p+1}$ can be computed explicitly and they do not depend on ε. This gives an asymptotic estimate for the sum

$$I_k^1(\varepsilon) + I_k^2(\varepsilon) = \ln \varepsilon \sum_{j=2}^\infty \tilde{\mu}_{k,j} \varepsilon^{j/2} + \sum_{j=2}^\infty \tilde{\lambda}_{k,j} \varepsilon^{j/2},$$

as $\varepsilon \to 0$. In $I_k^3(\varepsilon)$ and $I_k^4(\varepsilon)$ we choose $N = N_{n-1}$ for $n > 1$. Then

$$I_k^3(\varepsilon) + I_k^4(\varepsilon) = \sum_{j=1}^{N_{n-1}} \tilde{\lambda}_{k,2j} \varepsilon^j + O(\varepsilon^{N_{n-1}+1}),$$

as $\varepsilon \to 0$, $n > 1$. Thus, we arrive at an asymptotic estimate for $J_{2m+1}^1(0;\varepsilon)$

$$J_{2m+1}^1(0;\varepsilon) = \ln \varepsilon \sum_{j=2}^\infty A_{2m+1,j} \varepsilon^{j/2} + \sum_{j=2}^\infty B_{2m+1,j} \varepsilon^{j/2},$$

with some constants $A_{2m+1,j}$ and $B_{2m+1,j}$ independent of ε. From this the desired asymptotic expansion for $U_k(r;\varepsilon)$ follows.

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