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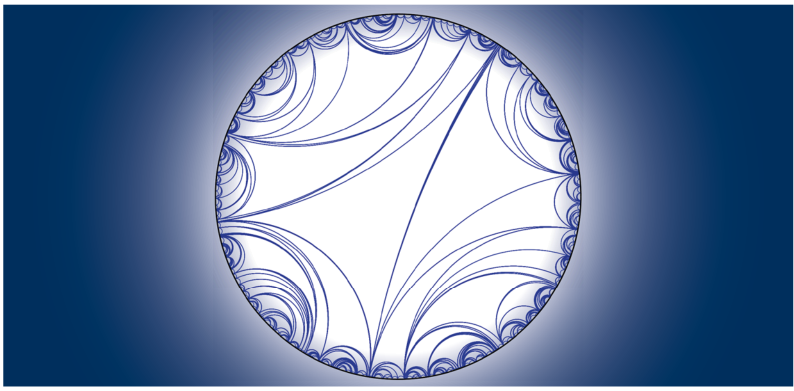

Evgueniya Dyachenko | Nikolai Tarkhanov

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Tel.: +49 (0)331 977 2533 / Fax: 2292
E-Mail: verlag@uni-potsdam.de

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Kontakt:
Institut für Mathematik
Am Neuen Palais 10
14469 Potsdam
Tel.: +49 (0)331 977 1028
WWW: http://www.math.uni-potsdam.de

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SINGULAR PERTURBATIONS OF ELLIPTIC OPERATORS

E. DYACHENKO AND N. TARKHANOV

This paper is dedicated to our teacher L. R. Volevich.

ABSTRACT. We develop a new approach to the analysis of pseudodifferential operators with small parameter $\varepsilon \in (0,1]$ on a compact smooth manifold $\mathcal{X}$. The standard approach assumes action of operators in Sobolev spaces whose norms depend on ε. Instead we consider the cylinder $[0,1] \times \mathcal{X}$ over $\mathcal{X}$ and study pseudodifferential operators on the cylinder which act, by the very nature, on functions depending on ε as well. The action in ε reduces to multiplication by functions of this variable and does not include any differentiation. As but one result we mention asymptotic of solutions to singular perturbation problems for small values of ε.

CONTENTS

1. INTRODUCTION

An excellent introduction into asymptotic phenomena in mathematical physics is the survey [Fri55] which remains to be of current importance.

Most differential equations of physics possess solutions which involve quick transitions, and it is an interesting task to study those features of these equations which make such quick transitions possible. A case in point is Prandtl's ingenious conception of the boundary layer. This is a narrow layer along the surface of a body, traveling in a fluid, across which the flow velocity changes quickly. Prandtl's observation of this quick transition was the starting point for his theory of fluid resistance, see [Pra05].

A large class of discontinuity phenomena in mathematical physics may be interpreted as boundary layer phenomena. There was never any doubt that the

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boundary layer theory gave a proper account of physical reality, but its mathematical aspects remained a puzzle for some time. Only when this theory is fitted into the framework of asymptotic analysis, does its mathematical structure become transparent.

In such a systematic approach one may develop an appropriate quantity in powers of a parameter ε. This expansion is to be set up in such a way that the quantity is continuous for $\varepsilon > 0$ but discontinuous for $\varepsilon = 0$. A series expansion with this character must have peculiar properties. In general these series do not converge. The idea of giving validity to these formal series is classical and it goes back to Poincaré [Poi86].

The boundary layer in linear differential equations has been studied in detail in [VL57]. On using this method it is possible to describe the iteration processes that formally yield an asymptotic representation of the solution for small ε. To prove this asymptotic representation one needs a priori estimates for solutions of boundary value problems in function spaces with weight norms. The well-known method of construction of such estimates (see [Kon66, S. 4] makes it possible to obtain them from uniform estimates of boundary value problems with a small parameter in the higher derivatives.

The asymptotic phenomena of ordinary differential equations have also been studied in connection with nonlinear equations. An interesting problem concerns periodic solutions of a differential equation of the form $\varepsilon u'' = f(u, u')$. The question is what happens with these periodic solutions as $\varepsilon \to 0$, in particular if the limit equation $f(u, u') = 0$ has no periodic solution. Of course there could be no boundary layer effect in the strict sense since there is no boundary. What happens is that the limit function, if it exists, satisfies the equation $f(u, u') = 0$ except at certain points where the derivative u' has a jump discontinuity. Strong results on asymptotic periodic solutions have been obtained by Levinson since 1942, see [Lev50].

2. Loss of initial data

In this section we demonstrate the behaviour of solutions of the initial problem to a first order ordinary differential equation as the parameter ε tends to zero. This question is extremely elementary, but nevertheless leads in a natural way to the boundary layer phenomenon.

To wit,

$$\left\{ \begin{array}{rcll} \varepsilon\, u'(x) + q(x)u(x) & = & f(x) \quad \text{for} & x \in (a, b), \\ u(a) & = & u_0, & \end{array} \right. \tag{2.1}$$

where q and f are continuous functions on the interval $[a, b)$ and ε a small positive parameter. We prescribe an initial value u_0 for the solution of our differential equation at the point a and ask how the solution of this initial value problem behaves as $\varepsilon \to 0$.

Note that for $\varepsilon = 0$ the differential equation reduces to the equation of order zero $qu = f$ in (a, b). One may therefore wonder whether the solution of problem (2.1) approaches the solution

$$\frac{f(x)}{q(x)}$$

of the equation of order zero. Now the solution of the zero order equation is already determined and one cannot expect that the initial condition will be satisfied in the limit. This question and related questions can easily be answered with the aid of explicit formulas.

An elementary calculations shows that

$$u(x) = \exp\Big(-\frac{1}{\varepsilon}\int_a^x q(\vartheta)d\vartheta\Big)u_0 + \frac{1}{\varepsilon}\int_a^x \exp\Big(-\frac{1}{\varepsilon}\int_{x'}^x q(\vartheta)d\vartheta\Big)f(x')dx'$$

for $x \in [a,b)$. The first term on the right-hand side satisfies the homogeneous differential equation $\varepsilon\, u' + qu = 0$ in (a,b) and the initial condition $u(a) = u_0$. If q is positive in (a,b), then this term converges to zero uniformly away from a, as $\varepsilon \to 0$. The second term on the right-hand side is a solution of the inhomogeneous solution $\varepsilon\, u' + qu = f$ in (a,b) and satisfies the homogeneous initial condition $u(a) = 0$.

If the solution of the zero order equation is continuously differentiable in $[a,b)$, then one can transform the formula for the solution u of problem (2.1) to elucidate the character of convergence of u for $\varepsilon \to 0$. Namely, on integrating by parts one obtains

$$\begin{aligned} u(x) &= \frac{f(x)}{q(x)} + \exp\Big(-\frac{1}{\varepsilon}\int_a^x q(\vartheta)d\vartheta\Big)\Big(u_0 - \frac{f(a)}{q(a)}\Big) \\ &- \int_a^x \exp\Big(-\frac{1}{\varepsilon}\int_{x'}^x q(\vartheta)d\vartheta\Big)\Big(\frac{f(x')}{q(x')}\Big)' dx' \end{aligned} \tag{2.2}$$

for all $x \in [a,b)$.

Assume that q is positive in the interval (a,b). Then the second term on the right-hand side of (2.2) converges to zero uniformly in $x \in (a,b)$ bounded from a, when $\varepsilon \to 0$. Moreover, this term vanishes for all $x \in [a,b)$, if the solution of the zero order equation takes on the value u_0 at a. The last term on the right-hand side converges to zero for each $x \in [a,b)$, as $\varepsilon \to 0$, which is due to Lebesgue's dominated convergence theorem. From what has been said it follows that under appropriate conditions the solution of the initial problem converges to the solution of the zero order equation in (a,b) indeed. This solution fails to assume the initial value. The process of losing an initial value takes place through nonuniform convergence. If the parameter ε is small enough, the solution will run near the limit solution except in a small segment at the initial point a where it changes quickly in order, as it were, to retrieve the initial value about to be lost. Thus a "quick transition" is found to occur. It must occur since an initial condition is about to be lost, and this loss in turn is necessary since the order of the differential equation is about to drop, cf. [Fri55].

The leading symbol which controls the asymptotic behaviour of the solution of initial problem (2.1) for $\varepsilon \to 0$ proves to be

$$\sigma^0(x,\xi,\varepsilon) := \imath\varepsilon\xi + a(x)$$

regarded for $(x,\xi) \in T^*[a,b)$ and $\varepsilon \in [0,1)$. We get

$$\begin{aligned} |\sigma^0(x,\xi,\varepsilon)| &= \big(\varepsilon^2|\xi|^2 + |a(x)|^2\big)^{1/2} \\ &\geq c\,\langle\varepsilon\xi\rangle \end{aligned} \tag{2.3}$$

where c is the smaller of the numbers 1 and $\inf|a(x)|$, the infimum being over all $x \in [a,b)$. From (2.3) it follows that $\sigma^0(x,\xi,\varepsilon)$ is different from zero for all $(x,\xi) \in T^*[a,b)$ and $\varepsilon \in [0,1)$, provided that $\inf|a(x)| > 0$. And vice versa, if (2.3) is fulfilled with some constant $c > 0$ independent of $(x, \xi) \in T^*[a, b)$ and $\varepsilon \in [0, 1)$, then $\inf |a(x)| > 0$.

The first order ordinary differential equations satisfying condition (2.3) are called small parameter elliptic. This condition just amounts to saying that the equation is elliptic of order 1 for each fixed $\varepsilon \in (0, 1)$, and it degenerates to a zero order elliptic equation when $\varepsilon \to 0$.

The results discussed in connection with the simple equation of the first order are rather typical and they may frequently serve as a guide in understanding other asymptotic phenomena.

3. A passive approach to operator-valued symbols

Pseudodifferential operators with small parameter are most obviously introduced within the framework of operator-valued symbols. We describe here the so-called "passive" approach to operator-valued symbols which was used in [FST02] for edge and corner theory. In this case it proves to be equivalent to the usual theory based on the edge and corner Sobolev spaces with group action κ_λ. However, it is more convenient to deal with. The passive approach allows one to reduce pseudodifferential operators with operator-valued symbols to the case of integral operators in L^2-spaces, so that the calculus of operator-valued symbols becomes quite similar to that of scalar-valued symbols. To the best of our knowledge it goes back at least as far as [Kar83].

The term "passive" comes from analogy with transformation theory. Recall that a geometrical transformation $y = f(x)$ may be treated either from "active" or "passive" point of view. According to the "active" approach the transformation moves geometrical points $x \mapsto y = f(x)$ while in the "passive" approach the points are fixed and we change only the coordinate system. For example, a linear change $y^i = a^i_j x^j$ (we use the Einstein summation notation) may be thought of as a linear transformation of the space $\mathbb{R}^n$ or as a change of a basis in this space. Of course, both descriptions are equivalent.

We demonstrate this approach by calculus of pseudodifferential operators on a product manifold. Consider $\mathcal{M} = \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X}$, $\mathcal{Y}$ are smooth compact closed manifolds with $\dim \mathcal{X} = n$ and $\dim \mathcal{Y} = m$. Suppose we work with the usual symbol classes $\mathcal{S}^\mu$ on $\mathcal{M}$ and corresponding classes of pseudodifferential operators $\mathcal{L}^\mu$ acting in Sobolev spaces $H^s(\mathcal{M})$. We are aimed at describing these objects using a fibering structure. That is, we would like to introduce appropriate classes of operator-valued symbols on $\mathcal{X}$ with values in pseudodifferential operators on $\mathcal{Y}$ to recover the classes $\mathcal{S}^\mu$ on $\mathcal{M}$. Moreover, we would like to represent $H^s(\mathcal{M})$ as L^2-spaces $L^2(\mathcal{X}, H^s(\mathcal{Y}), \|\cdot\|_\xi)$ to recover the action of pseudodifferential operators from $\mathcal{L}^\mu$ in the spaces $H^s(\mathcal{M})$.

A symbol $a(x, y, \xi, \eta)$ on $\mathcal{M}$ is treated as a symbol on the fiber $\mathcal{Y}$ with estimates depending on the base covariable ξ. To any appearances the estimates might include a group action κ_λ in function spaces on $\mathcal{Y}$. Our present approach is based on a "passive" treatment of the group action κ_λ. The κ_λ does not act on functions, instead we introduce a special family of norms in $H^s(\mathcal{Y})$. In more detail, consider the Sobolev space $H^s(\mathcal{Y})$ with a family of norms $\|\cdot\|_\xi$ depending on a parameter $\xi \in \mathbb{R}^n$,

$$\|u(y)\|^2_\xi = \sum_j \int_{\mathbb{R}^m} |< \xi, \eta >^s \widehat{\psi_j u}(\eta)|^2 \, d\eta. \tag{3.1}$$

Here $\{\mathcal{V}_j\}$ is a coordinate covering of $\mathcal{Y}$, $\{\psi_j\}$ a subordinate partition of unity, $< \upsilon >$ is a smoothed norm function, i.e., $< \upsilon > := f(|\upsilon|)$ where f is a C^∞ function satisfying

$$\begin{array}{rcl} f(|\upsilon|) & \geq & 1, \\ f(|\upsilon|) & \equiv & |\upsilon| \quad \text{for} \quad |\upsilon| \geq 1, \end{array}$$

and $< \xi, \eta >= f(\sqrt{|\xi|^2 + |\eta|^2})$. The norm (3.1) depends, of course, on s but we drop it in the notation.

Next, consider a function $u(x)$ on $\mathcal{X}$ with values in $H^s(\mathcal{Y})$ equipped with the family of norms $\|\cdot\|_\xi$ given by (3.1).

Definition 3.1. *By $L^2(\mathcal{X}, H^s(\mathcal{Y}), \|\cdot\|_\xi)$ is meant the completion of $C^\infty(\mathcal{X}, H^s(\mathcal{Y}))$ with respect to the norm*

$$\|u(x)\|^2 = \sum_i \int_{\mathbb{R}^n} \|\widehat{\varphi_i u}(\xi)\|_\xi^2 \, d\xi. \tag{3.2}$$

Once again $\{\varphi_j\}$ is a partition of unity on $\mathcal{X}$ subordinate to a coordinate covering $\{\mathcal{U}_j\}$ of this manifold. Roughly speaking, (3.2) is an L^2-norm of the scalar-valued function $\|\widehat{\varphi_i u}(\xi)\|_\xi$.

We now are in a position to define the desired symbol classes Σ^m on X with values in pseudodifferential operators on Y.

Definition 3.2. *A smooth function $a(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ whose values are pseudodifferential operators on $\mathcal{Y}$ is said to belong to Σ^μ if, for any $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, the operators $\partial_x^\alpha D_\xi^\beta a(x, \xi) : H^s(\mathcal{Y}) \to H^{s-\mu+\beta}(\mathcal{Y})$ are bounded uniformly in ξ with respect to the norms $\|\cdot\|_\xi$ in both spaces $H^s(\mathcal{Y})$ and $H^{s-\mu+\beta}(\mathcal{Y})$. That is, there are constants $C_{\alpha,\beta}$ independent of ξ, such that*

$$\|\partial_x^\alpha D_\xi^\beta a(x, \xi)\|_\xi \leq C_{\alpha,\beta}. \tag{3.3}$$

Any symbol $a(x, y, \xi, \eta) \in \mathcal{S}^\mu$ defines a symbol $a(x, \xi) \in \Sigma^\mu$ on $\mathcal{X}$ with values in pseudodifferential operators on $\mathcal{Y}$.

We can actually stop at this point. All of what follows is a simple consequence of generalization of these definitions. As mentioned, in a more general context of pseudodifferential operators with operator-valued symbols these techniques was elaborated in [Kar83].

It is easy to see that the norms $\|\cdot\|_\xi$ in $H^s(\mathcal{Y})$ are equivalent for different values $\xi \in \mathbb{R}^n$, but this equivalence is not uniform in ξ. More precisely, on applying Peetre's inequality one sees that the norms vary slowly in ξ.

Lemma 3.3. *There are constants C and q such that*

$$\frac{\|u\|_{\xi_1}}{\|u\|_{\xi_2}} \leq C < \xi_1 - \xi_2 >^q \tag{3.4}$$

for all $\xi_1, \xi_2 \in \mathbb{R}^n$ and smooth functions u on $\mathcal{Y}$. (In fact, we get $C = 2^{|s|}$ and $q = |s|$.)

On the other hand, the norm $\|\cdot\|_\xi$ is independent of the coordinate covering and partition of unity up to uniform equivalence.

Lemma 3.4. *The embedding $\iota : H^{s_2}(\mathcal{Y}) \to H^{s_1}(\mathcal{Y})$ for $s_1 \leq s_2$ admits the following norm estimate*

$$\|\iota\|_\xi \leq C \ < \xi >^{s_1 - s_2} . \tag{3.5}$$

Proof. Since

$$\begin{aligned} \|u_j\|^2_{H^{s_1}(\mathcal{Y}),\xi} &= \int_{\mathbb{R}^m} | < \xi, \eta >^{s_1} \ \widehat{u_j}(\eta)|^2 d\eta \\ &= \int_{\mathbb{R}^m} | < \xi, \eta >^{s_2} \ \widehat{u_j}(\eta)|^2 < \xi, \eta >^{2(s_1 - s_2)} \ d\eta, \end{aligned}$$

estimate (3.5) follows readily from the fact that

$$\begin{aligned} < \xi, \eta >^{s_1 - s_2} &\sim (1 + |\xi|^2 + |\eta|^2)^{(s_1 - s_2)/2} \\ &\leq (1 + |\xi|^2)^{(s_1 - s_2)/2} \\ &\sim < \xi >^{s_1 - s_2}, \end{aligned}$$

for $s_1 - s_2 \leq 0$. □

Theorem 3.5. *Let $a(x,\xi) \in \Sigma^\mu$. If $\mu < 0$, then $a(x,\xi) : H^s(\mathcal{Y}) \to H^s(\mathcal{Y})$ is a bounded operator and its norm satisfies an estimate*

$$\|a(x,\xi)\|_\xi \leq C \ < \xi >^\mu .$$

Proof. By definition, the mapping $a(x,\xi) : H^s(\mathcal{Y}) \to H^{s-\mu}(\mathcal{Y})$ is bounded uniformly in ξ. On applying Lemma 3.4 we conclude moreover that $H^{s-\mu}(\mathcal{Y})$ is embedded into $H^s(\mathcal{Y})$ with estimate $\|\iota\|_\xi \leq C \ < \xi >^\mu$. This gives the desired result. □

This result plays an important role in parameter-dependent theory of pseudodifferential operators.

Lemma 3.6. *For each $s \in \mathbb{R}$, it follows that*

$$L^2(\mathcal{X}, H^s(\mathcal{Y}), \|\cdot\|_\xi) \cong H^s(\mathcal{X} \times \mathcal{Y}).$$

As usual, the norm in $H^s(\mathcal{X} \times \mathcal{Y})$ is defined by

$$\|u(x,y)\|^2 = \sum_{i,j} \iint_{\mathbb{R}^n \times \mathbb{R}^m} | < \xi, \eta >^s \ \widehat{\phi_i \psi_j u}(\xi,\eta)|^2 \, d\xi d\eta.$$

For symbols $a(x,\xi) \in \Sigma^m$, we introduce a quantization map $a \mapsto A = Q(a)$ by setting

$$Q(a) = \sum_i \varphi_i(x) \operatorname{Op}(a(x,\xi)) \, \tilde{\varphi}_i(x).$$

Theorem 3.7. *For $a(x,\xi) \in \Sigma^m$, the operator $A = Q(a)$ extends to a bounded mapping*

$$A : L^2(\mathcal{X}, H^s(\mathcal{Y}), \|\cdot\|_\xi) \to L^2(\mathcal{X}, H^{s-\mu}(\mathcal{Y}), \|\cdot\|_\xi).$$

Proof. In Fourier representation $f = Au$ gives

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \hat{a}(\xi - \xi', \xi')\hat{u}(\xi')d\xi'$$

whence

$$\begin{aligned} &\|\hat{f}(\xi)\|_{H^{s-\mu}(\mathcal{Y}),\xi} \\ \leq & \int_{\mathbb{R}^n} \|\hat{a}(\xi - \xi', \xi')\hat{u}(\xi')\|_{H^{s-\mu}(\mathcal{Y}),\xi} d\xi' \\ \leq & \ C \int_{\mathbb{R}^n} < \xi - \xi' >^q \|\hat{a}(\xi - \xi', \xi')\hat{u}(\xi')\|_{H^{s-\mu}(\mathcal{Y}),\xi'} d\xi' \\ \leq & \ C \int_{\mathbb{R}^n} < \xi - \xi' >^q \|\hat{a}(\xi - \xi', \xi')\|_{\mathcal{L}(H^s(\mathcal{Y}),H^{s-\mu}(\mathcal{Y})),\xi'} \|\hat{u}(\xi')\|_{H^s(\mathcal{Y}),\xi'} d\xi' \\ = & \ C \int O(< \xi - \xi' >^{-\infty}) \|\hat{u}(\xi')\|_{H^s(\mathcal{Y}),\xi'} d\xi'. \end{aligned}$$

So, we have reduced the problem to the boundedness of integral operators in L^2 with kernels $O(< \xi - \xi' >^{-\infty})$. This is evident. □

Obviously, the results of this section make sense in much more general context where the spaces $H^s(\mathcal{Y})$ and $H^{s-\mu}(\mathcal{Y})$ on the fibers of $\mathcal{X} \times \mathcal{Y}$ over $\mathcal{X}$ are replaced by abstract Hilbert spaces V and W endowed with slowly varying families of norms parametrised by $\xi \in \mathbb{R}^n$. In this way we obtain a rough class of pseudodifferential operators on $\mathcal{X}$ whose symbols take their values in $\mathcal{L}(V, W)$ with uniformly bounded norms and which map $L^2(\mathcal{X}, V, \|\cdot\|_\xi)$ continuously to $L^2(\mathcal{X}, W, \|\cdot\|_\xi)$. In Section 6 we develop this construction for another well-motivated choice of Hilbert spaces V and W.

4. Operators with small parameter

In this section we apply the "passive" approach on the product manifold $\mathcal{X} \times \mathcal{Y}$, where $\mathcal{X}$ is a smooth compact closed manifold of dimension n and $\mathcal{Y} = \{P\}$ is a one-point manifold.

Our purpose is to describe a calculus of singularly perturbed differential operators on $\mathcal{X}$. They are locally in the form

$$A(x, D, \varepsilon) = \sum_{\substack{|\alpha|-j \leq \mu \\ |\alpha| \leq m}} a_{\alpha,j}(x)\varepsilon^j D^\alpha, \tag{4.1}$$

where $x = (x^1, \ldots, x^n)$ are coordinates in a coordinate patch on $\mathcal{X}$, D is the vector of local derivatives $-\imath\partial_{x^1}, \ldots, -\imath\partial_{x^n}$, $\varepsilon \in (0, 1]$ a small parameter and we use the standard multi-index notation for higher order derivatives in x. Moreover, μ is an integer with $0 \leq \mu < m$. If $\mu = 0$ then (4.1) reduce to the so-called h-pseudodifferential operators which belong to the basic techniques in semiclassical analysis, with $h = \varepsilon$.

Singular perturbations is a maturing mathematical subject with a fairly long history and a strong promise for continued important applications throughout science, see [Pra05], [Bir08], [VL57], [Was66], [MR80], [Fra79b, FW82, FW84], [Fra90], [Hue60], [Naz81], etc. Volevich [Vol06] was the first to present the small parameter theory as a part of general elliptic theory.

Operators of the form (4.1) are given natural domains $H^{r,s}(\mathcal{X})$ to be mapped into $H^{r-m,s-\mu}(\mathcal{X})$, where r, s are arbitrary real numbers. Seemingly these spaces were first introduced in [Dem75]. More precisely, $H^{r,s}(\mathcal{X})$ is the completion of $C^\infty(\mathcal{X})$ with respect to the norm

$$\|u\|_{r,s}^2 = \sum_i \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\langle \varepsilon\xi \rangle^{2(r-s)} \widehat{\varphi_i u}|^2 d\xi,$$

where $\{\varphi_i\}$ is a partition of unity on $\mathcal{X}$ subordinate to a coordinate covering $\{\mathcal{U}_i\}$. By the very definition, $H^{r,s}(\mathcal{X})$ is a Hilbert space whose norm depends on the parameter ε.

Remark 4.1. *The space $H^{r,s}(\mathcal{X})$ is locally identified within abstract edge spaces $H^s(\mathbb{R}^n, V, \varkappa)$ with the group action $\varkappa$ on $V = \mathbb{C}$ given by $\varkappa_\lambda u = \lambda^{s-r} u$ for $\lambda > 0$, see* [ST05].

One easily recovers the spaces $H^{r,s}(\mathcal{X})$ and $H^{r-m,s-\mu}(\mathcal{X})$ as $L^2(\mathcal{X}, V, \|\cdot\|_\xi)$ and $L^2(\mathcal{X}, W, \|\cdot\|_\xi)$, respectively, where $V = \mathbb{C}$ and $W = \mathbb{C}$ are endowed with the families of norms

$$\begin{array}{rcl} \|u\|_\xi &=& |\langle \varepsilon\xi \rangle^{r-s} \langle \xi \rangle^s u|, \\ \|f\|_\xi &=& |\langle \varepsilon\xi \rangle^{(r-m)-(s-\mu)} \langle \xi \rangle^{s-\mu} f| \end{array}$$

parametrised by $\xi \in \mathbb{R}^n$.

Definition 3.2 applies immediately to specify the corresponding spaces $\Sigma^{m,\mu}$ of operator-valued symbols $a(x,\xi,\varepsilon)$ on $T^*\mathbb{R}^n$ depending on the small parameter $\varepsilon \in (0,1]$. We restrict ourselves to those symbols which depend continuously on $\varepsilon \in (0,1]$ up to $\varepsilon = 0$. To wit, let $\mathcal{S}^{m,\mu}$ be the space of all functions $a(x,\xi,\varepsilon)$ of $(x,\xi) \in T^*\mathbb{R}^n$ and $\varepsilon \in (0,1]$, which are C^∞ in (x,ξ) and continuous in ε up to $\varepsilon = 0$, such that

$$|\partial_x^\alpha D_\xi^\beta a(x,\xi,\varepsilon)| \le C_{\alpha,\beta}\, \langle \varepsilon\xi \rangle^{m-\mu} \langle \xi \rangle^{\mu-|\beta|} \tag{4.2}$$

is fulfilled for all multi-indices $\alpha, \beta \in \mathbb{Z}_{\ge 0}^n$, where the constants $C_{\alpha,\beta}$ do not depend on (x,ξ) and ε.

Note that in terms of group action introduced in Remark 4.1 the symbol estimates (4.2) take the form

$$|\tilde{\varkappa}_{\langle \varepsilon\xi \rangle}^{-1} \partial_x^\alpha D_\xi^\beta a(x,\xi,\varepsilon) \varkappa_{\langle \varepsilon\xi \rangle}| \le C_{\alpha,\beta}\, \langle \xi \rangle^{\mu-|\beta|}$$

for all $\alpha, \beta \in \mathbb{Z}_{\ge 0}^n$ and $\xi \in \mathbb{R}^n$ uniformly in $x \in \mathbb{R}^n$ and $\varepsilon \in (0,1]$, cf. [ST05]. Given any fixed $\varepsilon \in (0,1]$, these estimates reveal the order of the operator-valued symbol $a(x,\xi,\varepsilon)$ to be μ. Moreover, they give rise to appropriate homogeneity for symbols $a(x,\xi,\varepsilon)$. To this end, choose $\alpha = 0$, $\beta = 0$ and substitute $\xi \mapsto \lambda\xi$ and $\varepsilon \mapsto \varepsilon/\lambda$ to (4.2), obtaining

$$|a(x,\lambda\xi,\varepsilon/\lambda)| \le C_{0,0}\, \langle \varepsilon\xi \rangle^{m-\mu} \langle \lambda\xi \rangle^\mu$$

for all $\lambda > 1$, which reduces to

$$|\lambda^{-\mu} a(x,\lambda\xi,\varepsilon/\lambda)| \le C_{0,0}\, \langle \varepsilon\xi \rangle^{m-\mu} (\lambda^{-2} + |\xi|^2)^{\mu/2} \tag{4.3}$$

(we use tacitly an equivalent expression $\sqrt{1+|\xi|^2}$ for $\langle \xi \rangle$). If $\lambda \to \infty$ then the right-hand side of (4.3) tends to a constant multiple of $\langle \varepsilon\xi \rangle^{m-\mu} |\xi|^\mu$.

Lemma 4.2. *Suppose the limit*

$$\sigma^{\mu}(a)(x,\xi,\varepsilon) = \lim_{\lambda\to\infty} \lambda^{-\mu} a(x,\lambda\xi,\varepsilon/\lambda),$$

exists for some $x \in \mathbb{R}^n$ *and all* $\xi \in \mathbb{R}^n$, $\varepsilon > 0$. *Then* $\sigma^{\mu}(a)(x,\xi,\varepsilon)$ *is homogeneous of degree* μ *in* (ξ, ε^{-1}).

It is worth pointing out that $\sigma^{\mu}(a)(x,\xi,\varepsilon)$ is actually defined on the whole semi-axis $\varepsilon > 0$.

Proof. Let $s > 0$. Then

$$\sigma^{\mu}(a)(x,s\xi,\varepsilon/s) = \lim_{\lambda\to\infty} \lambda^{-\mu} a(x,\lambda s\xi,\varepsilon/\lambda s),$$

and so on setting $\lambda' = \lambda s$ we get

$$\begin{aligned} \sigma^{\mu}(a)(x,s\xi,\varepsilon/s) &= \lim_{\lambda'\to\infty} s^{\mu}\lambda'^{-\mu} a(x,\lambda'\xi,\varepsilon/\lambda') \\ &= s^{\mu}\sigma^{\mu}(a)(x,\xi,\varepsilon), \end{aligned}$$

as desired. □

In particular, Lemma 4.2 applies to the full symbol of the differential operator $A(x,D,\varepsilon)$ given by (4.1).

Example 4.3. By the very origin the full symbol $a(x,\xi,\varepsilon)$ of (4.1) belongs to the class $\mathcal{S}^{m,\mu}$ and

$$\begin{aligned} \sigma^{\mu}(a)(x,\xi,\varepsilon) &= \lim_{\lambda\to\infty} \lambda^{-\mu}\Big(\sum_{\substack{|\alpha|-j\le\mu \\ |\alpha|\le m}} a_{\alpha,j}(x)\xi^{\alpha}\varepsilon^{j}\,\lambda^{|\alpha|-j}\Big) \\ &= \sum_{\substack{|\alpha|-j=\mu \\ |\alpha|\le m}} a_{\alpha,j}(x)\xi^{\alpha}\varepsilon^{j} \end{aligned}$$

is well defined.

In fact, the full symbol of any differential operator $A(x,D,\varepsilon)$ of the form (4.1) expands as finite sum of homogeneous symbols of decreasing degree with step 1. More generally, one specifies the subspaces $\mathcal{S}^{m,\mu}_{\mathrm{phg}}$ in $\mathcal{S}^{m,\mu}$ consisting of all polyhomogeneous symbols, i.e., those admitting asymptotic expansions in homogeneous symbols. To introduce polyhomogeneous symbols more precisely, we need a purely technical result.

Lemma 4.4. *Let* a *be a* C^{∞} *function of* $(x,\xi) \in T^*\mathbb{R}^n \setminus \{0\}$ *and* $\varepsilon > 0$ *satisfying* $|\partial_x^{\alpha} D_{\xi}^{\beta} a(x,\xi,1)| \le C_{\alpha,\beta}\,\langle\xi\rangle^{m-|\beta|}$ *for* $|\xi| \ge 1$ *and* $\alpha,\beta \in \mathbb{Z}^n_{\ge 0}$. *If* a *is homogeneous of degree* μ *in* (ξ,ε^{-1}), *then* $\chi a \in \mathcal{S}^{m,\mu}$ *for any excision function* $\chi = \chi(\xi)$ *for the origin in* $\mathbb{R}^n$.

Proof. Since each derivative $\partial_x^{\alpha} D_{\xi}^{\beta} a$ is homogeneous of degree $\mu - |\beta|$ in (ξ,ε^{-1}), it suffices to prove estimate (4.2) only for $\alpha = \beta = 0$. We have to show that there is a constant $C > 0$, such that

$$|\chi(\xi)a(x,\xi,\varepsilon)| \le C\,\langle\varepsilon\xi\rangle^{m-\mu}\langle\xi\rangle^{\mu}$$

for all $(x,\xi) \in T^*\mathbb{R}^n$ and $\varepsilon \in (0,1]$. Such an estimate is obvious if ξ varies in a compact subset of $\mathbb{R}^n$, for χ vanishes in a neighbourhood of $\xi = 0$. Hence, there is no restriction of generality in assuming that $|\xi| \geq R$, where $R > 1$ is large enough, so that $\chi(\xi) \equiv 1$ for $|\xi| \geq R$.

We distinguish two cases, namely $\varepsilon \leq \langle \xi \rangle^{-1}$ and $\varepsilon > \langle \xi \rangle^{-1}$. In the first case we immediately get

$$\begin{array}{rcl} |a(x,\xi,\varepsilon)| & = & \langle \xi \rangle^{\mu} |a(x,\xi/\langle \xi \rangle, \varepsilon \langle \xi \rangle)| \\ & \leq & C\, \langle \xi \rangle^{\mu}, \end{array}$$

where C is the supremum of $|a(x,\xi',\varepsilon')|$ over all x, $1/\sqrt{2} \leq |\xi'| \leq 1$ and $\varepsilon' \in [0,1]$. Moreover, $\langle \varepsilon\xi \rangle^{m-\mu}$ is bounded from below by a positive constant independent of ξ and ε, for $\varepsilon|\xi| \leq 1$. This yields $|a(x,\xi,\varepsilon)| \leq C'\, \langle \varepsilon\xi \rangle^{m-\mu} \langle \xi \rangle^{\mu}$ with some new constant C', as desired.

Assume that $\varepsilon > \langle \xi \rangle^{-1}$. Then $\varepsilon^{-1} < \langle \xi \rangle$ whence

$$\begin{array}{rcl} |a(x,\xi,\varepsilon)| & = & |\varepsilon^{-\mu}\, a(x,\varepsilon\xi,1)| \\ & \leq & C\, \varepsilon^{-\mu}\, \langle \varepsilon\xi \rangle^{m} \\ & = & C\, \langle \varepsilon\xi \rangle^{m-\mu}\, (\varepsilon^{-1} \langle \varepsilon\xi \rangle)^{\mu} \end{array}$$

with C a constant independent of x, ξ and ε. If $\mu > 0$ then the factor $(\varepsilon^{-1}\langle \varepsilon\xi \rangle)^{\mu}$ is estimated by

$$(\varepsilon^{-2} + |\xi|^2)^{\mu/2} \leq 2^{\mu/2}\, \langle \xi \rangle^{\mu}.$$

If $\mu < 0$ then this estimate is obvious, even without the factor $2^{\mu/2}$. This establishes the desired estimate. □

The family $\mathcal{S}^{m-j,\mu-j}$ with $j = 0, 1, \ldots$ is used as usual to define asymptotic sums of homogeneous symbols. A symbol $a \in \mathcal{S}^{m,\mu}$ is said to be polyhomogeneous if there is a sequence $\{a_{\mu-j}\}_{j=0,1,\ldots}$ of smooth function of $(x,\xi) \in T^*\mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$ satisfying $|\partial_x^{\alpha} D_{\xi}^{\beta} a_{\mu-j}(x,\xi,1)| \leq C_{\alpha,\beta}\, \langle \xi \rangle^{m-j-|\beta|}$ for $|\xi| \geq 1$ and $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, such that every $a_{\mu-j}$ is homogeneous of degree $\mu - j$ in (ξ, ε^{-1}) and a expands as asymptotic sum

$$a(x,\xi,\varepsilon) \sim \chi(\xi) \sum_{j=0}^{\infty} a_{\mu-j}(x,\xi,\varepsilon) \qquad (4.4)$$

in the sense that $a - \chi \sum_{j=0}^{N} a_{\mu-j} \in \mathcal{S}^{m-N-1,\mu-N-1}$ for all $N = 0, 1, \ldots$.

The appropriate concept in abstract algebra to describe expansions like (4.4) is that of filtration. To wit,

$$\mathcal{S}^{m,\mu}_{\mathrm{phg}} \sim \bigoplus_{j=0}^{\infty} \left(\mathcal{S}^{m-j,\mu-j}_{\mathrm{phg}} \ominus \mathcal{S}^{m-j-1,\mu-j-1}_{\mathrm{phg}} \right).$$

Each symbol $a \in \mathcal{S}^{m,\mu}_{\mathrm{phg}}$ possesses a well-defined principal homogeneous symbol of degree μ, namely $\sigma^{\mu}(a) := a_{\mu}$. To construct an algebra of pseudodifferential operators on $\mathcal{X}$ with symbolic structure one need not consider full asymptotic expansions like (4.4). It suffices to ensure that the limit $\sigma^{\mu}(a)$ exists and the difference $a - \chi\sigma^{\mu}(a)$ belongs to $\mathcal{S}^{m-1,\mu-1}$. For more details we refer the reader to Section 3.3 in [Fra90].

The class of polyhomogeneous symbols $\mathcal{S}^{m,\mu}_{\mathrm{phg}}$ with $\mu < 0$ gains in interest if we realise that

$$a(x,\xi,\varepsilon) \sim \varepsilon^{-\mu}\chi(\xi)\sum_{j=0}^{\infty}\varepsilon^{j}\, a_{\mu-j}(x,\varepsilon\xi,1)$$

where $a_{\mu-j}(x,\varepsilon\xi,1)$ are homogeneous functions of degree 0 in (ξ,ε^{-1}). Thus, any symbol $a \in \mathcal{S}^{m,\mu}_{\mathrm{phg}}$ with $\mu < 0$ factors through the power $\varepsilon^{-\mu}$ which vanishes up to order $-\mu$ at $\varepsilon = 0$.

We may now quantise symbols $a \in \mathcal{S}^{m,\mu}$ as pseudodifferential operators on $\mathcal{X}$ in just the same way as in Section 3. The space of operators $A = Q(a)$ with symbols $a \in \mathcal{S}^{m,\mu}$ is denoted by $\Psi^{m,\mu}(\mathcal{X})$.

Theorem 4.5. *Let $A \in \Psi^{m,\mu}(\mathcal{X})$. For any $r, s \in \mathbb{R}$, the operator A extends to a bounded mapping*

$$A : H^{r,s}(\mathcal{X}) \to H^{r-m,s-\mu}(\mathcal{X})$$

whose norm is independent of $\varepsilon \in [0,1]$.

Proof. This is a consequence of Theorem 3.7. □

Let $\Psi^{m,\mu}_{\mathrm{phg}}(\mathcal{X})$ stand for the subspace of $\Psi^{m,\mu}(\mathcal{X})$ consisting of those operators which have polyhomogeneous symbols. For $A \in \Psi^{m,\mu}_{\mathrm{phg}}(\mathcal{X})$, the principal homogeneous symbol of degree μ is defined by $\sigma^{\mu}(A) = \sigma^{\mu}(a)$, where $A = Q(a)$. If $\sigma^{\mu}(A) = 0$ then A belongs actually to $\Psi^{m-1,\mu-1}_{\mathrm{phg}}(\mathcal{X})$. Hence, the mapping $A : H^{r,s}(\mathcal{X}) \to H^{r-m,s-\mu}(\mathcal{X})$ is compact, for it factors through the compact embedding

$$H^{r-m+1,s-\mu+1}(\mathcal{X}) \hookrightarrow H^{r-m,s-\mu}(\mathcal{X}).$$

Theorem 4.6. *If $A \in \Psi^{m,\mu}_{\mathrm{phg}}(\mathcal{X})$ and $B \in \Psi^{n,\nu}_{\mathrm{phg}}(\mathcal{X})$, then $BA \in \Psi^{m+n,\mu+\nu}_{\mathrm{phg}}(\mathcal{X})$ and $\sigma^{\mu+\nu}(BA) = \sigma^{\nu}(B)\sigma^{\mu}(A)$.*

Proof. See for instance Proposition 3.3.3 in [Fra90]. □

As usual, an operator $A \in \Psi^{m,\mu}_{\mathrm{phg}}(\mathcal{X})$ is called elliptic if its symbol $\sigma^{\mu}(A)(x,\xi,\varepsilon)$ is invertible for all $(x,\xi) \in T^{*}\mathcal{X} \setminus \{0\}$ and $\varepsilon \in [0,1]$.

Theorem 4.7. *An operator $A \in \Psi^{m,\mu}_{\mathrm{phg}}(\mathcal{X})$ is elliptic if and only if it possesses a parametrix $P \in \Psi^{-m,-\mu}_{\mathrm{phg}}(\mathcal{X})$, i.e. $PA = I$ and $AP = I$ modulo operators in $\Psi^{-\infty,-\infty}(\mathcal{X})$.*

Proof. The necessity of ellipticity follows immediately from Theorem 4.6, for the equalities $PA = I$ and $AP = I$ modulo $\Psi^{-\infty,-\infty}(\mathcal{X})$ imply that $\sigma^{-\mu}(P)$ is the inverse of $\sigma^{\mu}(A)$.

Conversely, look for a parametrix $P = Q(p)$ for $A = Q(a)$, where $p \in \mathcal{S}^{-m,-\mu}_{\mathrm{phg}}$ has asymptotic expansion $p \sim p_{-\mu} + p_{-\mu-1} + \ldots$. The ellipticity of A just amounts to saying that

$$\sigma^{\mu}(A)(x,\xi,\varepsilon) \geq c\,\langle\varepsilon\xi\rangle^{m-\mu}|\xi|^{\mu}$$

for all $(x,\xi) \in T^{*}\mathcal{X} \setminus \{0\}$ and $\varepsilon \in [0,1]$, where the constant $c > 0$ does not depend on x, ξ and ε. Hence, $p_{-\mu} := (\sigma^{\mu}(A))^{-1}$ gives rise to a "soft" parametrix $P^{(0)} = Q(\chi p_{-\mu})$ for A. More precisely, $P^{(0)} \in \Psi^{-m,-\mu}_{\mathrm{phg}}(\mathcal{X})$ satisfies $P^{(0)}A = I$ and

$AP^{(0)} = I$ modulo $\Psi^{-1,-1}(\mathcal{X})$. Now, the standard techniques of pseudodifferential calculus applies to improve the discrepancies $P^{(0)}A - I$ and $AP^{(0)} - I$, see for instance [ST05]. □

To sum up the homogeneous components $p_{-\mu-j}$ with $j = 0, 1, \ldots$, one uses a trick of L. Hörmander for asymptotic summation of symbols, see Theorem 3.6.3 in [Fra90].

Corollary 4.8. *Assume that $A \in \Psi^{m,\mu}_{\mathrm{phg}}(\mathcal{X})$ is an elliptic operator on $\mathcal{X}$. Then, for any $r, s \in \mathbb{R}$ and any large $R > 0$, there is a constant $C > 0$ independent of ε, such that*

$$\|u\|_{r,s} \leq C\ (\|Au\|_{r-m,s-\mu} + \|u\|_{-R,-R})$$

whenever $u \in H^{r,s}(\mathcal{X})$.

Proof. Let $P \in \Psi^{-m,-\mu}_{\mathrm{phg}}(\mathcal{X})$ be a parametrix of A given by Theorem 4.7. Then we obtain

$$\begin{array}{rcl} \|u\|_{r,s} & = & \|P(Au) + (I - PA)u\|_{r,s} \\ & \leq & \|P(Au)\|_{r,s} + \|(I - PA)u\|_{r,s} \end{array}$$

for all $u \in H^{r,s}(\mathcal{X})$. To complete the proof it is now sufficient to use the mapping properties of pseudodifferential operators P and $I - PA$ formulated in Theorem 4.5. □

5. Ellipticity with large parameter

Setting $\lambda = 1/\varepsilon$ we get a "large" parameter. The theory of problems with large parameter was motivated by the study of the resolvent of elliptic operators. Both theories are parallel to each other. Substituting $\varepsilon = 1/\lambda$ to (4.1) and multiplying A by $\lambda^{m-\mu}$ yields

$$\tilde{A}(x, D, \lambda) = \sum_{\substack{|\alpha|+j \leq m \\ j \leq m-\mu}} \tilde{a}_{\alpha,j}(x) \lambda^j D^\alpha$$

in local coordinates in $\mathcal{X}$. For this operator the ellipticity with large parameter leads to the inequality

$$|\sum_{\substack{|\alpha|+j = m \\ j \leq m-\mu}} \tilde{a}_{\alpha,j}(x) \lambda^j \xi^\alpha| \geq c\, \langle \lambda, \xi \rangle^{m-\mu} |\xi|^\mu,$$

which is a generalization of the Agmon-Agranovich-Vishik condition of ellipticity with parameter corresponding to $\mu = 0$, see [AV64], [Vol06] and the references given there.

6. Another approach to parameter-dependent theory

In this section we develop another approach to pseudodifferential operators with small parameter which stems from analysis on manifolds with singularities. In this ares the role of small parameter is played by the distance to singularities and it has been usually chosen as a local coordinate. Thus, the small parameter is included into functions under study as independent variable and the action of operators include also that in the small parameter. Geometrically this approach corresponds to analysis on the cylinder $\mathcal{C} = \mathcal{X} \times [0, 1]$ over a compact closed manifold $\mathcal{X}$ of dimension n, see Fig. 1. Subject to the problem its base $\varepsilon = 0$ can be thought

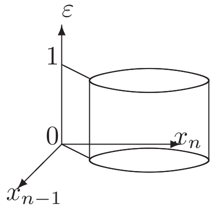

FIG. 1. *A cylinder* $\mathcal{C} = \mathcal{X} \times [0, 1)$ *over* $\mathcal{X}$

of as singular point blown up by a singular transformation of coordinates. In this case one restricts the study to functions which are constant on the base, taking on the values 0 or ∞. In our problem the base is regarded as part of the boundary $\mathcal{X} \times \{0\}$ of the cylinder $\mathcal{C}$, and so we distinguish the values of functions on the base. The top $\mathcal{X} \times \{1\}$ is actually excluded from consideration by a particular choice of function spaces on the segment $\mathcal{Y} = [0, 1]$, for we are interested in local analysis at $\varepsilon = 0$.

Basically there are two possibilities to develop a calculus of pseudodifferential operators on the cylinder $\mathcal{C}$. Either one thinks of them as pseudodifferential operators on $\mathcal{X}$ with symbols taking on their values in an operator algebra on $[0, 1]$. Or one treats them as pseudodifferential operators on the segment $[0, 1]$ whose symbols are pseudodifferential operators on $\mathcal{X}$. Singularly perturbed problems require the first approach with symbols taking on their values in multiplication operators in $\mathcal{L}(V, W)$, where

$$\begin{array}{rcl} V & = & L^2([0, 1], \varepsilon^{-2\gamma}), \\ W & = & L^2([0, 1], \varepsilon^{-2\gamma}) \end{array}$$

with $\gamma \in \mathbb{R}$.

Any continuous function $a \in C[0, 1]$ induces the multiplication operator $u \mapsto au$ on $L^2([0, 1], \varepsilon^{-2\gamma})$ that is obviously bounded. Moreover, the norm of this operator is equal to the supremum norm of a in $C[0, 1]$. Hence, $C[0, 1]$ can be specified as a closed subspace of $\mathcal{L}(V, W)$.

Pick real numbers μ and s. We endow the spaces V and W with the families of norms

$$\begin{array}{rcl} \|u\|_\xi & = & \|\langle\xi\rangle^s \varkappa_{\langle\xi\rangle}^{-1} u\|_{L^2([0,1],\varepsilon^{-2\gamma})}, \\ \|f\|_\xi & = & \|\langle\xi\rangle^{s-\mu} \tilde{\varkappa}_{\langle\xi\rangle}^{-1} f\|_{L^2([0,1],\varepsilon^{-2\gamma})} \end{array}$$

parametrised by $\xi \in \mathbb{R}^n$, where

$$\begin{array}{rcl} (\varkappa_\lambda u)(\varepsilon) & = & \lambda^{-\gamma+1/2}\, u(\lambda\varepsilon), \\ (\tilde{\varkappa}_\lambda f)(\varepsilon) & = & \lambda^{-\gamma+1/2}\, f(\lambda\varepsilon) \end{array}$$

for $\lambda \le 1$.

The space $L^2(\mathcal{X}, V, \|\cdot\|_\xi)$ is defined to be the completion of $C^\infty(\mathcal{X}, V)$ with respect to the norm

$$\|u\|_{s,\gamma}^2 = \sum_i \int_{\mathbb{R}^n} \|\widehat{\varphi_i u}\|_\xi^2 d\xi,$$

where $\{\varphi_i\}$ is a C^∞ partition of unity on $\mathcal{X}$ subordinate to a finite coordinate covering $\{\mathcal{U}_i\}$.

Remark 6.1. *The space* $L^2(\mathcal{X}, V, \|\cdot\|_\xi)$ *is locally identified within abstract edge spaces* $H^s(\mathbb{R}^n, V, \varkappa)$ *with the group action* $\varkappa$ *on* $V = L^2([0, 1], \varepsilon^{-2\gamma})$ *defined above, see* [ST05].

In a similar way one introduces the space $L^2(\mathcal{X}, W, \|\cdot\|_\xi)$ whose norm is denoted by $\|\cdot\|_{s-\mu,\gamma}$. Set

$$\begin{array}{rcl} H^{s,\gamma}(\mathcal{C}) & = & L^2(\mathcal{X}, V, \|\cdot\|_\xi), \\ H^{s-\mu,\gamma}(\mathcal{C}) & = & L^2(\mathcal{X}, W, \|\cdot\|_\xi), \end{array}$$

which will cause no confusion since the right-hand sides coincide for $\mu = 0$, as is easy to check. We are thus led to a scale of function spaces on the cylinder $\mathcal{C}$ which are Hilbert.

Our next objective is to describe those pseudodifferential operators on $\mathcal{C}$ which map $H^{s,\gamma}(\mathcal{C})$ continuously into $H^{s-\mu,\gamma}(\mathcal{C})$. To this end we specify the definition of symbol spaces, see (3.3). If $a(x, \xi, \varepsilon)$ is a function of $(x, \xi) \in T^*\mathbb{R}^n$ and $\varepsilon \in [0, 1]$, which is smooth in (x, ξ) and continuous in ε, then a straightforward calculation shows that

$$\|\partial_x^\alpha D_\xi^\beta a(x, \xi, \varepsilon)\|_{\mathcal{L}(V,W),\xi} = \langle\xi\rangle^{-\mu} \sup_{\varepsilon\in[0,1]} |(\partial_x^\alpha D_\xi^\beta a)(x, \xi, \varepsilon/\langle\xi\rangle)|$$

holds on all of $T^*\mathbb{R}^n$. We now denote by $\mathcal{S}^\mu$ the space of all functions $a(x, \xi, \varepsilon)$ of $(x, \xi) \in T^*\mathbb{R}^n$ and $\varepsilon \in [0, 1]$, which are smooth in (x, ξ) and continuous in ε and satisfy

$$|(\partial_x^\alpha D_\xi^\beta a)(x, \xi, \varepsilon/\langle\xi\rangle)| \le C_{\alpha,\beta}\, \langle\xi\rangle^{\mu-|\beta|} \tag{6.1}$$

for all multi-indices $\alpha, \beta \in \mathbb{Z}^n_{\ge 0}$, where $C_{\alpha,\beta}$ are constants independent of (x, ξ) and ε.

In terms of group action introduced in Remark 6.1 the symbol estimates (6.1) take the form

$$\|\tilde{\varkappa}^{-1}_{\langle\xi\rangle} \partial_x^\alpha D_\xi^\beta a(x, \xi, \varepsilon) \varkappa_{\langle\xi\rangle}\|_{\mathcal{L}(L^2([0,1],\varepsilon^{-2\gamma}))} \le C_{\alpha,\beta}\, \langle\xi\rangle^{\mu-|\beta|}$$

for all $(x, \xi) \in T^*\mathbb{R}^n$ and $\alpha, \beta \in \mathbb{Z}^n_{\ge 0}$, cf. [ST05]. In particular, the order of the symbol a is μ. Moreover, using group actions in fibers V and W gives a direct way to the notion of homogeneity in the calculus of operator-valued symbols on $T^*\mathbb{R}^n$. Namely, a function $a(x, \xi, \varepsilon)$, defined for $(x, \xi) \in T^*\mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$, is said to be homogeneous of degree μ if the equality $a(x, \lambda\xi, \varepsilon) = \lambda^\mu \tilde{\varkappa}_\lambda a(x, \xi, \varepsilon) \varkappa_\lambda^{-1}$ is fulfilled for all $\lambda > 0$. It is easily seen that a is homogeneous of degree μ with respect to the group actions $\varkappa$ and $\tilde{\varkappa}$ if and only if $a(x, \lambda\xi, \varepsilon/\lambda) = \lambda^\mu a(x, \xi, \varepsilon)$ for all $\lambda > 0$, i.e. a is homogeneous of degree μ in (ξ, ε^{-1}). Thus, we recover the homogeneity of symbols invented in Section 4.

Lemma 6.2. *Assume that the limit*

$$\sigma^\mu(a)(x, \xi, \varepsilon) = \lim_{\lambda\to\infty} \lambda^{-\mu} \tilde{\varkappa}_\lambda^{-1} a(x, \lambda\xi, \varepsilon) \varkappa_\lambda$$

exists for some $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$ and $\varepsilon > 0$. Then $\sigma^\mu(a)(x, \xi, \varepsilon)$ is homogeneous of degree μ.

Proof. Let $s > 0$ and let $u = u(\varepsilon)$ be an arbitrary function of V. By the definition of group action, we get

$$\begin{array}{rcl} \sigma^\mu(a)(x, s\xi, \varepsilon)u & = & \lim\limits_{\lambda\to\infty} \lambda^{-\mu} \tilde{\varkappa}_\lambda^{-1} a(x, \lambda s\xi, \varepsilon) \varkappa_\lambda u \\ & = & s^\mu \tilde{\varkappa}_s \Big(\lim\limits_{\lambda'\to\infty} (\lambda')^{-\mu} \tilde{\varkappa}_{\lambda'}^{-1} a(x, \lambda'\xi, \varepsilon) \varkappa_{\lambda'} \Big) \varkappa_s^{-1} u, \end{array}$$

the second equality being a consequence of substitution $\lambda' = \lambda s$. Since the expression in the parentheses just amounts to $\sigma^\mu(a)(x, \xi, \varepsilon)$, the lemma follows. □

The function $\sigma^\mu(a)$ defined away from the zero section of the cotangent bundle $T^*\mathcal{C}$ is called the principal homogeneous symbol of degree μ of a. We also use this designation for the operator $A = Q(a)$ on the cylinder which is a suitable quantization of a.

Example 6.3. As defined above, the principal homogeneous symbol of differential operator (4.1) is

$$\begin{aligned} \sigma^\mu(A)(x,\xi,\varepsilon) &= \lim_{\lambda\to\infty} \lambda^{-\mu}\Big(\sum_{\substack{|\alpha|-j\le\mu \\ |\alpha|\le m}} a_{\alpha,j}(x)(\lambda\xi)^\alpha \left(\tilde{\varkappa}_\lambda^{-1}\varepsilon^j\varkappa_\lambda\right)\Big) \\ &= \sum_{\substack{|\alpha|-j=\mu \\ |\alpha|\le m}} a_{\alpha,j}(x)\xi^\alpha\varepsilon^j\varkappa_\lambda, \end{aligned}$$

cf. Example 4.3.

Now one introduces the subspaces $\mathcal{S}^\mu_{\mathrm{phg}}$ in $\mathcal{S}^\mu$ consisting of all polyhomogeneous symbols, i.e., those admitting asymptotic expansions in homogeneous symbols. To do this, we need an auxiliary result.

Lemma 6.4. *Let a be a C^∞ function of $(x,\xi) \in T^*\mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$ with $a \equiv 0$ for $|x| \gg 1$. If a is homogeneous of degree μ, then $\chi a \in \mathcal{S}^\mu$ for any excision function $\chi = \chi(\xi)$ for the origin in $\mathbb{R}^n$.*

Proof. Since each derivative $\partial_x^\alpha D_\xi^\beta a$ is homogeneous of degree $\mu - |\beta|$, it suffices to prove estimate (6.1) only for $\alpha = \beta = 0$. We have to show that there is a constant $C > 0$, such that

$$\|\tilde{\varkappa}_{\langle\xi\rangle}^{-1}\left(\chi(\xi)a(x,\xi,\varepsilon)\right)\varkappa_{\langle\xi\rangle}\|_{\mathcal{L}(L^2([0,1],\varepsilon^{-2\gamma}))} \le C\,\langle\xi\rangle^\mu$$

for all $(x,\xi) \in T^*\mathbb{R}^n$ and $\varepsilon \in [0,1]$. Such an estimate is obvious if ξ varies in a compact subset of $\mathbb{R}^n$, for χ near $\xi = 0$. Hence, we may assume without loss of generality that $|\xi| \ge R$, where $R > 1$ is sufficiently large, so that $\chi(\xi) \equiv 1$ for $|\xi| \ge R$. Then

$$\begin{aligned} \|\tilde{\varkappa}_{\langle\xi\rangle}^{-1}\left(\chi(\xi)a(x,\xi,\varepsilon)\right)\varkappa_{\langle\xi\rangle}\|_{\mathcal{L}(L^2([0,1],\varepsilon^{-2\gamma}))} &= \|a(x,\xi,\varepsilon/\langle\xi\rangle)\|_{\mathcal{L}(L^2([0,1],\varepsilon^{-2\gamma}))} \\ &\le C\,\langle\xi\rangle^\mu, \end{aligned}$$

where

$$C = \sup_{(x,\xi)\in T^*\mathbb{R}^n} \|a(x,\xi/\langle\xi\rangle,\varepsilon)\|_{\mathcal{L}(L^2([0,1],\varepsilon^{-2\gamma}))}.$$

From conditions imposed on a it follows that the supremum is finite, which completes the proof. □

In contrast to Lemma 4.4 no additional conditions are imposed here on a except for homogeneity. This might testify to the fact that the symbol classes $\mathcal{S}^\mu$ give the best fit to the study of operators (4.1).

The family $\mathcal{S}^{\mu-j}$ with $j = 0, 1, \ldots$ is used in the usual way to define asymptotic sums of homogeneous symbols. A symbol $a \in \mathcal{S}^\mu$ is called polyhomogeneous if there is a sequence $\{a_{\mu-j}\}_{j=0,1,\ldots}$ of smooth function of $(x,\xi) \in T^*\mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$, such that every $a_{\mu-j}$ is homogeneous of degree $\mu - j$ in (ξ, ε^{-1}) and a expands as asymptotic sum

$$a(x,\xi,\varepsilon) \sim \chi(\xi) \sum_{j=0}^{\infty} a_{\mu-j}(x,\xi,\varepsilon) \tag{6.2}$$

in the sense that $a - \chi \sum_{j=0}^{N} a_{\mu-j} \in \mathcal{S}^{\mu-N-1}$ for all $N = 0, 1, \ldots$.

Each symbol $a \in \mathcal{S}^m_{\mathrm{phg}}$ admits a well-defined principal homogeneous symbol of degree μ, namely $\sigma^\mu(a) := a_\mu$. We quantise symbols $a \in \mathcal{S}^\mu$ as pseudodifferential operators on $\mathcal{X}$ similarly to Section 3. Write $\Psi^\mu(\mathcal{C})$ for the space of all operators $A = Q(a)$ with $a \in \mathcal{S}^\mu$.

Theorem 6.5. *Let $A \in \Psi^\mu(\mathcal{C})$. For any $s, \gamma \in \mathbb{R}$, the operator A extends to a bounded mapping*

$$A : H^{s,\gamma}(\mathcal{C}) \to H^{s-\mu,\gamma}(\mathcal{C}).$$

Proof. This is a consequence of Theorem 3.7. □

Let $\Psi^\mu_{\mathrm{phg}}(\mathcal{C})$ stand for the subspace of $\Psi^\mu(\mathcal{C})$ consisting of all operators with polyhomogeneous symbols. For $A = Q(a)$ of $\Psi^\mu_{\mathrm{phg}}(\mathcal{C})$, the principal homogeneous symbol of degree μ is defined by $\sigma^\mu(A) = \sigma^\mu(a)$. If $\sigma^\mu(A) = 0$ then A belongs to $\Psi^{\mu-1}_{\mathrm{phg}}(\mathcal{C})$. When combined with Theorem 6.6 stated below, this result allows one to describe those operators A on the cylinder which are invertible modulo operators of order $-\infty$.

Theorem 6.6. *If $A \in \Psi^\mu_{\mathrm{phg}}(\mathcal{C})$ and $B \in \Psi^\nu_{\mathrm{phg}}(\mathcal{C})$, then $BA \in \Psi^{\mu+\nu}_{\mathrm{phg}}(\mathcal{C})$ and $\sigma^{\mu+\nu}(BA) = \sigma^\nu(B)\sigma^\mu(A)$.*

Proof. This is a standard fact of calculus of pseudodifferential operators with operator-valued symbols. □

As usual, an operator $A \in \Psi^\mu_{\mathrm{phg}}(\mathcal{C})$ is called elliptic if $\sigma^\mu(A)(x,\xi,\varepsilon)$ is invertible for all (x,ξ,ε) away from the zero section of the cotangent bundle $T^*\mathcal{C}$ of the cylinder.

Theorem 6.7. *An operator $A \in \Psi^\mu_{\mathrm{phg}}(\mathcal{C})$ is elliptic if and only if there is an operator $P \in \Psi^{-\mu}_{\mathrm{phg}}(\mathcal{C})$, such that both $PA = I$ and $AP = I$ are fulfilled modulo operators of $\Psi^{-\infty}(\mathcal{C})$.*

Proof. The necessity of ellipticity follows immediately from Theorem 6.6, for the equalities $PA = I$ and $AP = I$ modulo $\Psi^{-\infty}(\mathcal{C})$ imply that $\sigma^{-\mu}(P)$ is the inverse of $\sigma^\mu(A)$.

Conversely, look for an inverse $P = Q(p)$ for $A = Q(a)$ modulo $\Psi^{-\infty}(\mathcal{C})$, where $p \in \mathcal{S}^{-\mu}_{\mathrm{phg}}$ has asymptotic expansion $p \sim p_{-\mu} + p_{-\mu-1} + \ldots$. The ellipticity of A just amounts to saying that

$$\sigma^\mu(A)(x,\xi,\varepsilon) \geq c\,|\xi|^\mu$$

for all $(x,\xi) \in T^*\mathcal{X} \setminus \{0\}$ and $\varepsilon \in [0,1]$, where the constant $c > 0$ does not depend on x, ξ and ε. Hence, $p_{-\mu} := (\sigma^\mu(A))^{-1}$ gives rise to a "soft" inverse $P^{(0)} = Q(\chi p_{-\mu})$

for A. More precisely, $P^{(0)} \in \Psi^{-\mu}_{\mathrm{phg}}(\mathcal{C})$ satisfies $P^{(0)}A = I$ and $AP^{(0)} = I$ modulo operators of $\Psi^{-1}(\mathcal{C})$. Now, the standard techniques of pseudodifferential calculus applies to improve the discrepancies $P^{(0)}A - I$ and $AP^{(0)} - I$, see for instance [ST05]. □

We avoid the designation "parametrix" for P since the operators of $\Psi^{-\infty}(\mathcal{C})$ need not be compact in $H^{s,\gamma}(\mathcal{C})$.

Corollary 6.8. *Assume that $A \in \Psi^{\mu}_{\mathrm{phg}}(\mathcal{C})$ is an elliptic operator on $\mathcal{C}$. Then, for any $s, \gamma \in \mathbb{R}$ and any large $R > 0$, there is a constant $C > 0$ independent of ε, such that*

$$\|u\|_{s,\gamma} \leq C\ (\|Au\|_{s-\mu,\gamma} + \|u\|_{-R,\gamma})$$

whenever $u \in H^{s,\gamma}(\mathcal{C})$.

Proof. Let $P \in \Psi^{-\mu}_{\mathrm{phg}}(\mathcal{C})$ be the inverse of A up to operators of $\Psi^{-\infty}(\mathcal{C})$ given by Theorem 4.7. Then we obtain

$$\begin{aligned} \|u\|_{s,\gamma} &= \|P(Au) + (I - PA)u\|_{s,\gamma} \\ &\leq \|P(Au)\|_{s,\gamma} + \|(I - PA)u\|_{s,\gamma} \end{aligned}$$

for all $u \in H^{s,\gamma}(\mathcal{C})$. To complete the proof it is now sufficient to use the mapping properties of pseudodifferential operators P and $I - PA$ formulated in Theorem 4.5. □

We finish this section by evaluating the local norm in $H^{s,\gamma}(\mathcal{C})$ to compare this scale with the scale $H^{r,s}(\mathcal{X})$ used in Section 4. This norm is equivalent to that in $L^2(\mathbb{R}^n, V, \|\cdot\|_\xi)$, which is

$$\begin{aligned} \|u\|^2_{s,\gamma} &= \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \|\varkappa^{-1}_{\langle\xi\rangle} \hat{u}(\xi)\|^2_{L^2([0,1],\varepsilon^{-2\gamma})} d\xi \\ &= \int_{\mathbb{R}^n} \langle\xi\rangle^{2s+2\gamma-1} \int_0^1 \varepsilon^{-2\gamma} |\hat{u}(\xi, \varepsilon/\langle\xi\rangle)|^2 d\varepsilon d\xi. \end{aligned}$$

Substituting $\varepsilon' = \varepsilon/\langle\xi\rangle$ yields

$$\begin{aligned} \|u\|^2_{s,\gamma} &= \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \Big(\int_0^{1/\langle\xi\rangle} (\varepsilon')^{-2\gamma} |\hat{u}(\xi, \varepsilon')|^2 d\varepsilon' \Big) d\xi \\ &= \int_0^1 (\varepsilon')^{-2(\gamma+\Delta\gamma)} \Big(\int_{\langle\xi\rangle \leq 1/\varepsilon'} (\varepsilon' \langle\xi\rangle)^{2\Delta\gamma} \langle\xi\rangle^{2(s-\Delta\gamma)} |\hat{u}(\xi, \varepsilon')|^2 d\xi \Big) d\varepsilon', \end{aligned}$$

which is close to $\displaystyle\int_0^1 \varepsilon^{-2(\gamma+\Delta\gamma)} \|u\|^2_{H^{s,s-\Delta\gamma}(\mathcal{X})} d\varepsilon$ with any $\Delta\gamma \in \mathbb{R}$.

Remark 6.9. *The modern theory of pseudodifferential operators on manifolds with singularities allows one to study the problem for compact manifolds $\mathcal{X}$ with boundary as well.*

7. Regularization of singularly perturbed problems

The idea of constructive reduction of elliptic singular perturbations to regular perturbations goes back at least as far as [FW82]. For the complete bibliography see [Fra90, p. 531].

The calculus of pseudodifferential operators with small parameter developed in Section 4 allows one to reduce the question of the invertibility of elliptic operators $A \in \Psi^{m,\mu}_{\mathrm{phg}}(\mathcal{X})$ acting from $H^{r,s}(\mathcal{X})$ into $H^{r-m,s-\mu}(\mathcal{X})$ to that of the invertibility of their limit operators at $\varepsilon = 0$ acting in usual Sobolev spaces $H^s(\mathcal{X}) \to H^{s-\mu}(\mathcal{X})$. To shorten notation, we write $A(\varepsilon)$ instead of $A(x, D, \varepsilon)$, and so $A(0) \in \Psi^{\mu}_{\mathrm{phg}}(\mathcal{X})$ is the reduced operator.

Given any $f \in H^{r-m,s-\mu}(\mathcal{X})$, consider the inhomogeneous equation $A(\varepsilon)u = f$ on $\mathcal{X}$ for an unknown function $u \in H^{r,s}(\mathcal{X})$. We first assume that $u \in H^{r,s}(\mathcal{X})$ satisfies $A(\varepsilon)u = f$ in $\mathcal{X}$.

Since the symbol $\sigma^{\mu}(A(\varepsilon))(x, \xi, \varepsilon)$ is invertible for all $(x, \xi) \in T^*\mathcal{X} \setminus \{0\}$ and $\varepsilon \in [0, 1]$, it follows that $A(0)$ is an elliptic operator of order μ. Hence, the Hodge theory applies to $A(0)$. According to this theory, there is an operator $G \in \Psi^{-\mu}_{\mathrm{phg}}(\mathcal{X})$ satisfying

$$\begin{array}{rcl} u & = & H^0 u + G A(0) u, \\ f & = & H^1 f + A(0) G f \end{array} \tag{7.1}$$

for all distributions u and f on $\mathcal{X}$, where H^0 and H^1 are $L^2(\mathcal{X})$-orthogonal projections onto the null-spaces of $A(0)$ and $A(0)^*$, respectively. (Observe that the null-spaces of $A(0)$ and $A(0)^*$ are actually finite dimensional and consist of C^∞ functions.)

Applying $G \in \Psi^{-\mu,-\mu}_{\mathrm{phg}}(\mathcal{X})$ to both sides of the equality $A(0)u + (A(\varepsilon) - A(0))u = f$ on $\mathcal{X}$ we obtain

$$u - H^0 u = Gf - G\,(A(\varepsilon) - A(0))\,u \tag{7.2}$$

for each $u \in H^{r,s}(\mathcal{X})$. (We have used the first equality of (7.1)). This is a far-reaching generalization of formula (2.2), for the function $Gf \in H^{r-(m-\mu),s}(\mathcal{X})$ is a solution of the unperturbed equation $A(0)Gf = f$, which is due to the second equality of (7.1). Let $\mu \le m$. Since the "coefficients" of $A(\varepsilon)$ are continuous up to $\varepsilon = 0$, it follows that $(A(\varepsilon) - A(0))\,u$ converges to zero in $H^{r-m,s-\mu}(\mathcal{X})$ as $\varepsilon \to 0$. By continuity, $G\,(A(\varepsilon) - A(0))\,u$ converges to zero in $H^{r-(m-\mu),s}(\mathcal{X})$, and so $u - H^0 u \in H^{r,s}(\mathcal{X})$ converges to Gf in $H^{r-(m-\mu),s}(\mathcal{X})$ as $\varepsilon \to 0$. If $\mu > m$ then in the same manner we can see that $u - H^0 u \in H^{r,s}(\mathcal{X})$ converges to Gf in $H^{r,s}(\mathcal{X})$ as $\varepsilon \to 0$.

The solution u of $A(\varepsilon)u = f$ need not converge to the solution Gf of the reduced equation, for both solutions are not unique. Formula (7.2) describes the limit of the component $u - H^0 u$ of u which is orthogonal to the space of solutions of the homogeneous equation $A(0)u = 0$. This results gains in interest if the equation $A(0)u = 0$ has only zero solution, i.e. $H^0 = 0$. The task is now to show that from the unique solvability of the reduced equation it follows that $A(\varepsilon)u = f$ is uniquely solvable if ε is small enough.

Theorem 7.1. *Suppose that $A(\varepsilon) \in \Psi^{m,\mu}(\mathcal{X})$ is elliptic. If the reduced operator $A(0) : H^s(\mathcal{X}) \to H^{s-\mu}(\mathcal{X})$ is an isomorphism uniformly with respect to $\varepsilon \in [0, 1]$, then $A(\varepsilon) : H^{r,s}(\mathcal{X}) \to H^{r,s-\mu}(\mathcal{X})$ is an isomorphism, too, for all $\varepsilon \in [0, \varepsilon_0]$ with sufficiently small ε_0.*

Proof. We only clarify the operator theoretic aspects of the proof. For symbol constructions we refer the reader to Corollary 3.14.10 in [Fra90] and the comments after its proof given there.

To this end, write $I = GA(0) + (I - GA(0))$ whence

$$\begin{aligned} A(\varepsilon) &= A(0) + (A(\varepsilon) - A(0))\, GA(0) + (A(\varepsilon) - A(0))(I - GA(0)) \\ &= (I + (A(\varepsilon) - A(0))G)\, A(0) + (A(\varepsilon) - A(0))(I - GA(0)) \end{aligned}$$

for all $\varepsilon \in [0, 1]$. As mentioned, the difference $A(\varepsilon) - A(0)$ is small if $\varepsilon \le 1$ is small enough. Hence, the operator

$$\begin{aligned} Q(\varepsilon) &= I + (A(\varepsilon) - A(0))G \\ &= H^0 + A(\varepsilon)G \end{aligned}$$

is invertible in the scale $H^{r,s}(\mathcal{X})$, provided that $\varepsilon \in [0, \varepsilon_0]$ where $\varepsilon_0 \le 1$ is sufficiently small.

If the operator $A(0) \in \Psi^\mu(\mathcal{X})$ is invertible in the scale of usual Sobolev spaces on $\mathcal{X}$, then the product $Q(\varepsilon)A(0)$ is invertible for all $\varepsilon \in [0, \varepsilon_0]$. Hence, by decreasing ε_0 if necessary, we conclude readily that $A(\varepsilon)$ is invertible for all $\varepsilon \in [0, \varepsilon_0]$, as desired. □

The proof above gives more, namely

$$\begin{aligned} A(\varepsilon) &= Q(\varepsilon)A(0) + S^0(\varepsilon), \\ &= A(0)Q(\varepsilon) + S^1(\varepsilon), \end{aligned} \qquad (7.3)$$

where $S^0(\varepsilon)$ and $S^1(\varepsilon)$ have at most the same order as $A(\varepsilon)$ and are infinitesimally small if $\varepsilon \to 0$.

From (7.3) it follows immediately that for the operator $A(\varepsilon)$ to be invertible for small ε it is necessary and sufficient that $A(0)$ would be invertible. Any representation of a singularly perturbed operator $A(\varepsilon)$ in the form (7.3) is called a regularization of $A(\varepsilon)$.

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(Evgueniya Dyachenko) Universität Potsdam, Institut für Mathematik, Am Neuen Palais 10, 14469 Potsdam, Germany

E-mail address: `dyachenk@uni-potsdam.de`

(Nikolai Tarkhanov) Universität Potsdam, Institut für Mathematik, Am Neuen Palais 10, 14469 Potsdam, Germany

E-mail address: `tarkhanov@math.uni-potsdam.de`