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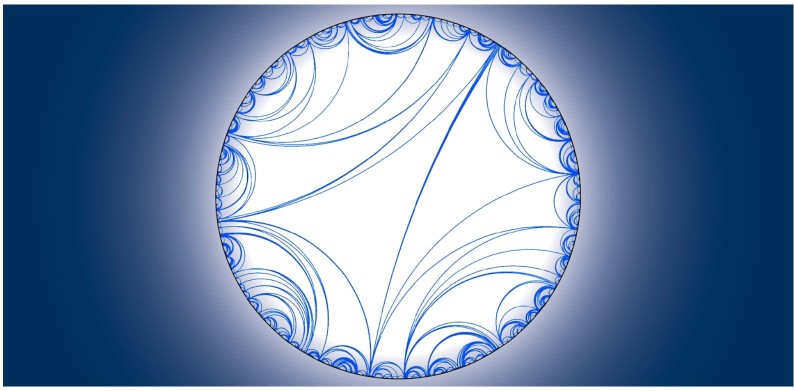

Dmitry Fedchenko | Nikolai Tarkhanov

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A CLASS OF TOEPLITZ OPERATORS IN SEVERAL VARIABLES

D. FEDCHENKO AND N. TARKHANOV

ABSTRACT. We introduce the concept of Toeplitz operator associated with the Laplace-Beltrami operator on a compact Riemannian manifold with boundary. We characterise those Toeplitz operators which are Fredholm, thus initiating the index theory.

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INTRODUCTION

There are a number of ways in which the theory of Toeplitz operators can be generalised to n dimensions, see e.g. [Ven72], [Dou73] and the references given there. The monograph [Upm96] presents much more advanced theory of Toeplitz operators in several complex variables.

The paper [Gui84] describes precisely how Toeplitz operators of "Bergman type" are related to Toeplitz operators of "Szegö type." A remarkable connection between the theory of Toeplitz operators á la [Ven72] and the standard theory of pseudo-differential operators emerged from the work [BG81]. This connection in its broad outlines is elucidated in [Gui84], too.

This work focuses on a new class Toeplitz operators which is more closely related to elliptic theory. The new Toeplitz operators admit very transparent description which motivates strikingly their study. To this end, let A be an $(l \times k)$-matrix of scalar partial differential operators in a neighbourhood of the closed bounded domain $\mathcal{X}$ with smooth boundary in $\mathbb{R}^n$. Assume that A is overdetermined elliptic,

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i.e. the leading symbol of A has rank k away from the zero section of the cotangent bundle of $\mathcal{X}$. Then, given any solution u of the homogeneous equation $Au = 0$ in the interior of $\mathcal{X}$ which has finite order of growth at the boundary, the Cauchy data $t(u)$ of u with respect to A possess weak limit values at the boundary. If A satisfies the so-called uniqueness condition of the local Cauchy problem in a neighbourhood of $\mathcal{X}$, then the solution u is uniquely defined by its Cauchy data at $\partial\mathcal{X}$. Let $t(u) = \{B_j u\}_{j=0,1,\ldots,m-1}$ be a representation of the Cauchy data of u, where m is the order of A. The space of all Cauchy data $B_j u = u_j$ of u at the boundary is effectively described by the condition of orthogonality to solutions of the formal adjoint equation $A^* g = 0$ near $\mathcal{X}$ by means of a Green formula, see [Tar95, § 10.3.4]. In this way we distinguish many Hilbert spaces of vector-valued functions $\oplus u_j$ on $\partial\mathcal{X}$ which represent solutions to $Au = 0$ in the interior of $\mathcal{X}$. In particular, one introduces Hardy spaces H as subspaces of $\oplus L^2(\partial\mathcal{X}, \mathbb{C}^k)$ consisting of the Cauchy data of solutions to $Au = 0$ in the interior of $\mathcal{X}$ with appropriate behaviour at the boundary. Pick such a Hilbert space H. By the above, H is a closed subspace of $\oplus L^2(\partial\mathcal{X}, \mathbb{C}^k)$ and we write Π for the orthogonal projection of $\oplus L^2(\partial\mathcal{X}, \mathbb{C}^k)$ onto H.

On using the Calderón projection one reduces any boundary value problem for solutions of $Au = 0$ in $\mathcal{X}$ to an equation $T\,t(u) = u_0$ in the space H, where T is a continuous selfmap of H. In general, this is a pseudodifferential operator of order zero on the boundary. The simplest of these are direct generalisations of classical Toeplitz operators. To introduce them we assume for simplicity that A is of order one. Then the Cauchy data of a solution u to $Au = 0$ in the interior of $\mathcal{X}$ reduce to the restriction of u to $\partial\mathcal{X}$ in a weak sense, and H is a closed subspace of $L^2(\partial\mathcal{X}, \mathbb{C}^k)$. If A is the Cauchy-Riemann operator then Π just amounts to the Szegö projection.

Given a $(k \times k)$-matrix $E(x)$ of smooth function on $\partial\mathcal{X}$, the operator T_E on H given by $u \mapsto \Pi(Eu)$ is said to be a Toeplitz operator with multiplier E. If E is a scalar multiple of the unit matrix, then the theory of Toeplitz operators T_E is much about the same as the classical theory of Toeplitz operators. Otherwise the theory is much more complicated and even the proof of composition formula for Toeplitz operators is based on studying a subelliptic problem, such as the Neumann problem of D. Spencer. Under some convexity conditions on $\mathcal{X}$ this implies a very useful representation

$$T_E = E - A^* G^1 A E$$

on $\mathcal{X}$.

Let us shortly dwell on the contents of the paper. In Section 1 we discuss the well-known interplay between the theory of Toeplitz operators on the circle and the theory of pseudodifferential operators on the line (Wiener-Hopf operators), cf. [Gui84]. In Section 2 we introduce operators of Laplace type which go back at least as far as the Laplace-Beltrami operator. Section 3 elaborates the concept of Green operator in the Dirichlet problem for a Laplace type equation on a smooth compact manifold with boundary. As a byproduct of this theory we shortly study in Section 4 the Bergman projection related to the Dirichlet problem. In Sections 5 we present the results of F. Berezin [Ber72] and [Ber74] who resurrected the largely forgotten paper [Aro50]. In Section 6 we show an explicit example of reproducing kernel function. Section 7 is of key importance in this work where we bring together analytic and geometric tools to introduce a new class of Toeplitz operators in n dimensions and show the Fredholm property of elliptic Toeplitz operators. Finally, in Section 8 we briefly discuss Toeplitz operators related to the algebra of quaternions $\mathbb{H}$ introduced by W. R. Hamilton (1843). The algebra $\mathbb{H}$ holds a special place in analysis since, according to the Frobenius theorem, it is one of only two finite-dimensional division rings containing the real numbers as a proper subring, the other being the complex numbers.

1. Toeplitz operators in one dimension

Assume that $\mathcal{X}$ is a set with a measure, H a closed subspace of $L^2(\mathcal{X})$, and $\Pi : L^2(\mathcal{X}) \to H$ orthogonal projection.

Let $a : \mathcal{X} \to \mathbb{C}$ be a measurable bounded function. If the rather prosaic-looking operator $M_a : L^2(\mathcal{X}) \to L^2(\mathcal{X})$ given by $u \mapsto au$ is composed with Π, one gets a much more interesting operator

$$T_a : H \to H, \tag{1.1}$$

$u \mapsto \Pi(au)$, which one calls the Toeplitz operator with multiplier a. We will describe in this section two kinds of such operators.

Example 1.1. Take $\mathcal{X}$ to be $\mathbb{S}^1$ equipped with the rotation-invariant measure $d\mu = d\theta/2\pi$, and take for H the Hardy space, i.e. the subspace of $L^2(\mathbb{S}^1)$ consisting of all u satisfying

$$\int_{-\pi}^{\pi} u e^{\imath n\theta} d\mu = 0 \tag{1.2}$$

for all $n = 1, 2, \ldots$. Then the operator (1.1) is the classical Toeplitz operator with multiplier a.

Most people are more familiar with it in terms of its matrix representation. If

$$a = \sum_{-\infty}^{\infty} a_n e^{\imath n\theta},$$

then

$$T_a\, e^{\imath n\theta} = \sum_{k=0}^{\infty} a_{k-n} e^{\imath k\theta} \tag{1.3}$$

for $n = 0, 1, \ldots$. In other words, the matrix of T_a, expressed in terms of the basis $\{e^{\imath n\theta}\}_{n=0,1,\ldots}$, is

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Matrices of this form are called Toeplitz matrices. It can be shown that a matrix defines a bounded operator on H if and only if the numbers a_n are the Fourier coefficients of a bounded measurable function a.

It is clear from the matrix representation (1.3) that the composition of two Toeplitz operators is not necessarily a Toeplitz operator. However, we will show that if a^1 and a^2 are smooth functions then $T_{a^1}T_{a^2}$ is very nearly equal to $T_{a^1a^2}$. Recall that by smoothing operators $K : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ are meant operators with smooth Schwartz kernels.

Theorem 1.2. *If a^1 and a^2 are smooth, then $T_{a^1}T_{a^2}$ differs from $T_{a^1a^2}$ by a smoothing operator.*

Proof. Since Π commutes with rotations, there is a distributional function k on $\mathbb{S}^1$, such that

$$\Pi u\,(\theta) = (k * u)(\theta) = \int_{-\pi}^{\pi} k(\theta-\theta')u(\theta')d\mu(\theta').$$

In fact it is clear that

$$k(\theta) = \sum_{n=0}^{\infty} e^{\imath n\theta} = \frac{1}{1-e^{\imath\theta}}.$$

Thus,

$$a\Pi u - \Pi a u = K_a u$$

where

$$\begin{aligned} K_a(\theta,\theta') &= \frac{1}{2\pi}\frac{a(\theta)-a(\theta')}{1-e^{\imath(\theta-\theta')}} \\ &\sim \frac{1}{2\pi\imath}\frac{a(\theta)-a(\theta')}{\theta-\theta'} \end{aligned}$$

for θ' close to θ. This shows that K_a is smooth both in θ and θ'. Now

$$\begin{aligned} T_{a^1}T_{a^2}u &= \Pi(a^1(\Pi(a^2u))) \\ &= \Pi(a^1a^2u) - \Pi(a^1K_{a^2}u) \\ &= \Pi(a^1a^2u) + Ku \end{aligned}$$

with K smoothing. □

By Theorem 1.2, if a is a smooth nonvanishing function on the circle, then $P = T_{a^{-1}}$ is a parametrix for T_a modulo smoothing operators. Hence, $T_a : H \to H$ is a Fredholm operator. Conversely, the condition $a \neq 0$ on $\mathbb{S}^1$ is also necessary for the Fredholm property of T_a.

Actually there is a much more advanced result which is based on an explicit formula for the semicommutator $T_{a^1a^2} - T_{a^1}T_{a^2}$ in terms of the so-called Hankel operators $H_a : H \to H^\perp$, where $H^\perp$ is the orthogonal complement of H in $L^2(\mathbb{S}^1)$. Given a fixed bounded function a on the circle, one defines $T_a u := (I-\Pi)(au)$ for $u \in H$. Then

$$T_{a^1a^2} - T_{a^1}T_{a^2} = (H_{\overline{a^1}})^* H_{a^2},$$

see for instance [Pro88].

One of the main difficulties with dimension > 1 will be finding a substitute for this very simple composition formula.

Example 1.3. Take $\mathcal{X}$ to be the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane and let H be the space of holomorphic functions u on $\mathbb{D}$ satisfying

$$\int_{\mathbb{D}} |u|^2 \,\frac{1}{2\pi\imath} d\bar{z} \wedge dz < \infty.$$

Then the operator (1.1) is also said to be a classical Toeplitz operator with multiplier a.

Note that if the Taylor series of u is

$$\sum_{n=0}^{\infty} u_n z^n,$$

then

$$\int_{\mathbb{D}} |u|^2 \, \frac{1}{2\pi\imath} d\bar{z} \wedge dz = \sum_{n=0}^{\infty} \frac{|u_n|^2}{n+1}.$$

Therefore, H contains as dense subspace those holomorphic functions on $\mathbb{D}$ for which

$$\sum_{n=0}^{\infty} |u_n|^2 < \infty.$$

Denote this space by $H^{1/2}$. This is precisely the space of holomorphic functions on the disk which have well-defined L^2 boundary values. In fact, if $u \in H^{1/2}$, then

$$u \restriction_{\mathbb{S}^1} = \sum_{n=0}^{\infty} u_n \, e^{\imath n\theta}$$

and the right-hand side is in the Hardy space $H^2(\mathbb{S}^1)$. Hence, the operator, that assigns $u \restriction_{\mathbb{S}^1}$ to u, can be thought of as isomorphism of Hilbert spaces

$$H^{1/2}(\mathbb{D}) \stackrel{\cong}{\to} H^2(\mathbb{S}^1). \tag{1.4}$$

Now let a be a smooth function on $\overline{\mathbb{D}}$ and let T_a be the Toeplitz operator (1.1). One can show that T_a maps $H^{1/2}$ into $H^{1/2}$. So, by (1.4), it corresponds to an operator $T_a^\sharp$ on $H^2(\mathbb{S}^1)$. It is natural to ask how operators of this type are related to the Toeplitz operators we considered in Example 1.1. The answer requires a result which is proved in § 9 of [Gui84] in considerably more generality. We write $D : H^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1)$ for the differential operator $(1/\imath)d/d\theta$, i.e. $De^{\imath n\theta} = ne^{\imath n\theta}$ for $n = 0, 1, \ldots$.

Theorem 1.4. *Let a be a smooth function on $\overline{\mathbb{D}}$ and, for $n = 0, 1, \ldots$, let a^n be the restriction to $\mathbb{S}^1$ of $(-\bar{z}\, d/d\bar{z})^n a$. Then*

$$T_a^\sharp \sim \sum_{n=0}^{\infty} (D+1)^{-n} T_{a^n}. \tag{1.5}$$

The symbol "$\sim$" in (1.5) can be interpreted in the following sense. The difference between $T_a^\sharp$ and the first N terms on the right-hand side is "smoothing of order N." In other words, it is a bounded operator from $L^2(\mathbb{S}^1)$ into the space of all $u \in H^2(\mathbb{S}^1)$, such that

$$\sum_{n=0}^{\infty} (|n|^{2N} + 1)\, |u_n|^2 < \infty.$$

It is clear from Theorem 1.4 that we get from Example 1.3 a much larger class of operators on Hardy space than that from Example 1.1.

2. Operators of Laplace type

Let X be a compact Riemannian manifold with boundary of dimension n endowed with a positive-definite metric g. On studying differential operators on manifolds, a key role is played by the operators of Laplace type. These are second-order elliptic operators with leading symbol given by the Riemannian metric of the base space $\mathcal{X}$. They also include the effect of an endomorphism of a vector bundle F over $\mathcal{X}$. An equivalent way of stating this definition is that in any local coordinate system

$$\Delta^F = -\sum_{i,j=1}^{n} g^{i,j}(x) I_F \partial_i \partial_j \tag{2.1}$$

up to a first-order part, where $g^{i,j}(x) = (dx^i, dx^j)_x$ is the metric on the cotangent bundle of $\mathcal{X}$, see [BGV96]. We write simply Δ for the Laplace operator Δ^F if it will cause no confusion.

The operator Δ induces a continuous linear map of $C^\infty(\mathcal{X}, F)$ into itself. To apply Hilbert space methods for the study of the equation $\Delta u = f$ in $\mathcal{X}$, we pass from this map to its closure in $L^2(\mathcal{X}, F)$. Denote by $\mathcal{D}_T$ the set of all sections $u \in L^2(\mathcal{X}, F)$, for which there exists a sequence $\{u_\nu\}$ satisfying 1) $u_\nu \in C^\infty(\mathcal{X}, F)$; 2) $\{u_\nu\}$ converges to u in $L^2(\mathcal{X}, F)$; and 3) $\{\Delta u_\nu\}$ is a Cauchy sequence in $L^2(\mathcal{X}, F)$. The mapping $T : \mathcal{D}_T \to L^2(\mathcal{X}, F)$ defined by $Tu = \lim \Delta u_\nu$, where $\{u_\nu\}$ is a sequence with properties 1)–3), is usually referred to as the maximal operator generated by Δ.

Note that T is well defined. We will think of T as an unbounded operator from $L^2(\mathcal{X}, F)$ to itself, whose domain is $\mathcal{D}_T$. This operator T is densely defined and closed.

From the lemma of du Bois-Reymond and the uniqueness of a weak limit it follows that if $u \in \mathcal{D}_T$ then $Tu = \Delta u$ in the sense of distributions in the interior of $\mathcal{X}$.

We now define T^*, the adjoint of T, as usual for unbounded operators. Namely, let $\mathcal{D}_{T^*}$ be the set of all $g \in L^2(\mathcal{X}, F)$ with the property that there is $v \in L^2(\mathcal{X}, F)$ satisfying $(Tu, g)_{L^2(\mathcal{X},F)} = (u, v)_{L^2(\mathcal{X},F)}$ for $u \in \mathcal{D}_T$. Define $T^* : \mathcal{D}_{T^*} \to L^2(\mathcal{X}, F)$ by $T^* g = v$. The operator T^* is well defined, for $\mathcal{D}_T$ is dense in $L^2(\mathcal{X}, F)$. It is clear that $T^* g$ is in general different from $\Delta^* g$ in the sense of distributions in the interior of $\mathcal{X}$, where Δ^* is the formal adjoint for Δ with respect to the inner product in $L^2(\mathcal{X}, F)$.

Let us introduce operators L^0 and L^1 on $L^2(\mathcal{X}, F)$ with domains $\mathcal{D}_{L^0}$ and $\mathcal{D}_{L^1}$, which better suit the Hilbert structure of $L^2(\mathcal{X}, F)$ than the formal Laplacians $\Delta^*\Delta$ and $\Delta\Delta^*$, respectively. Namely, write $\mathcal{D}_{L^0}$ for the set of all $u \in \mathcal{D}_T$ with the property that $Tu \in \mathcal{D}_{T^*}$. Then the operator $L^0 : \mathcal{D}_{L^0} \to L^2(\mathcal{X}, F)$ is defined by $L^0 u = T^* T u$. Similarly, $\mathcal{D}_{L^1}$ stands for the set of all $g \in \mathcal{D}_{T^*}$ with the property that $T^* g \in \mathcal{D}_T$. Then the operator $L^1 : \mathcal{D}_{L^1} \to L^2(\mathcal{X}, F)$ is defined by the equality $L^1 g = T T^* g$.

We have thus arrived at the dual short complexes of Hilbert spaces and their closed maps

$$\begin{array}{ccccccc} 0 & \to & L^2(\mathcal{X}, F) & \stackrel{T}{\to} & L^2(\mathcal{X}, F) & \to & 0, \\ 0 & \leftarrow & L^2(\mathcal{X}, F) & \stackrel{T^*}{\leftarrow} & L^2(\mathcal{X}, F) & \leftarrow & 0 \end{array}$$

on $\mathcal{X}$. We are led to two dual problems which go back at least as far as the theory of harmonic integrals by Hodge, see for instance [Tar95, Ch. 5]. Assume $i = 0, 1$. Given a section $v \in L^2(\mathcal{X}, F)$, when is there $u \in \mathcal{D}_{L^i}$ such that $L^i u = v$, and how does u depend on v?

The weak orthogonal decomposition is actually the first step in studying the problem. Set

$$\begin{array}{rcl} H^0 & = & \{u \in \mathcal{D}_T \ : \ Tu = 0\}, \\ H^1 & = & \{g \in \mathcal{D}_{T^*} : T^* g = 0\}. \end{array}$$

Since the operators T and T^* are closed, both H^0 and H^1 are closed subspaces of $L^2(\mathcal{X}, F)$. Denote by Π^0 and Π^1 the orthogonal projections of $L^2(\mathcal{X}, F)$ onto H^0 and H^1, respectively.

Lemma 2.1.
1) $u \in H^0$ if and only if $u \in \mathcal{D}_{L^0}$ and $L^0 u = 0$.
2) $g \in H^1$ if and only if $g \in \mathcal{D}_{L^1}$ and $L^1 g = 0$.

Proof. If $u \in H^0$ then obviously $u \in \mathcal{D}_{L^0}$ and $L^0 u = 0$. If $L^0 u = 0$ then $(L^0 u, u)_{L^2(\mathcal{X}, F)} = 0$, and since

$$(L^0 u, u)_{L^2(\mathcal{X}, F)} = \|Tu\|^2_{L^2(\mathcal{X}, F)}$$

we have $u \in H^0$. This proves 1), and the proof of 2) is analogous. □

The operators L^0 and L^1 are selfadjoint, and $(L^0 + I)^{-1}$ and $(L^1 + I)^{-1}$ exist, are bounded and defined everywhere in $L^2(\mathcal{X}, F)$. Combining the selfadjointness of L^0 and L^1 with Lemma 2.1, we obtain immediately the weak orthogonal decompositions

$$\begin{array}{rcl} L^2(\mathcal{X}, F) & = & H^0 \oplus \overline{L^0 \mathcal{D}_{L^0}}, \\ L^2(\mathcal{X}, F) & = & H^1 \oplus \overline{L^1 \mathcal{D}_{L^1}}, \end{array} \tag{2.2}$$

the second summands on the right-hand sides being the closures of subspaces in $L^2(\mathcal{X}, F)$.

The elements of H^0 need not satisfy any additional conditions at the boundary of $\mathcal{X}$. More precisely, H^0 consists of all sections $u \in L^2(\mathcal{X}, F)$ satisfying $\Delta u = 0$ weakly in the interior of $\mathcal{X}$. Hence, H^0 is of infinite dimension unless the boundary of $\mathcal{X}$ is empty. It follows that the first decomposition in (2.2) is not related to any Fredholm operator. On the contrary, the definition of H^1 includes strong boundary conditions for solutions of $T^* g = 0$. Namely, a section $g \in H^2(\mathcal{X}, F)$ belongs to the space H^1 if an only if it satisfies $\Delta^* g = 0$ weakly in the interior of $\mathcal{X}$ and g vanishes up to the first order at the boundary of $\mathcal{X}$. Therefore, the second equality of (2.2) is reminiscent of the weak orthogonal decomposition related to the Dirichlet problem for the Laplace equation $L^1 g = f$ in $\mathcal{X}$. This latter boundary value problems is a well understood classical elliptic boundary value problem, see [ADN59]. The boundary conditions are of Shapiro-Lopatinskii type, and the main a priori estimate reads as follows.

Lemma 2.2. *Let $\mathcal{X}$ be a compact Riemannian manifold and Δ^F a Laplace operator on $\mathcal{X}$ related to a Riemannian vector bundle F. Then there is a constant $c > 0$ such that*

$$\|g\|_{H^2(\mathcal{X}, F)} \leq c \, (\|T^* g\|^2_{L^2(\mathcal{X}, F)} + \|g\|^2_{L^2(\mathcal{X}, F)})^{1/2} \tag{2.3}$$

for all $g \in C^\infty(\mathcal{X}, F)$ vanishing up the the first order at the boundary of $\mathcal{X}$.

Proof. See [ADN59] and elsewhere. □

The norm on the right-hand side of (2.3) is called the Dirichlet norm. Beginning with its classical forms, the norm has been an important technical tool in studying the Dirichlet problem. For $f, g \in \mathcal{D}_{T^*}$, the Dirichlet inner product of f and g is defined by

$$D(f,g) = (T^* f, T^* g)_{L^2(\mathcal{X},F)} + (f,g)_{L^2(\mathcal{X},F)},$$

and the Dirichlet norm is $D(g) = \sqrt{D(g,g)}$. The space $\mathcal{D}_{T^*}$ with the Dirichlet norm is a complete (Hilbert) space. From Lemma 2.2 it follows that $\mathcal{D}_{T^*}$ just amounts to

$$\overset{\circ}{H}{}^2(\mathcal{X},F),$$

the closure in $H^2(\mathcal{X},F)$ of C^∞ sections of F with compact support in the interior of $\mathcal{X}$.

Lemma 2.3. *Suppose $\mathcal{X}$ is a compact Riemannian manifold and Δ^F a Laplace operator on $\mathcal{X}$ related to a Riemannian vector bundle F. Then the space H^1 is finite dimensional.*

Proof. Observe that if $f, g \in H^1$ then $D(f,g) = (f,g)_{L^2(\mathcal{X},F)}$. Suppose that the dimension of H^1 is infinite. Then there exists an infinite sequence $\{g_\nu\}$ of orthonormal elements in H^1. Since $D(g_\nu) = \|g_\nu\|_{L^2(\mathcal{X},F)} = 1$ the sequence $\{u_\nu\}$ contains a convergent subsequence. But this is at variance with the fact that if $\nu \neq \mu$ then $\|u_\nu - u_\mu\|_{L^2(\mathcal{X},F)} = \sqrt{2}$. □

Lemma 2.4. *There is a constant $c > 0$, such that for all $u \in \mathcal{D}_{T^*}$ orthogonal to H^i we have*

$$\|T^* g\|_{L^2(\mathcal{X},F)} \geq c\, \|g\|_{L^2(\mathcal{X},F)}.$$

Proof. Consider the closed operator $T^* : \mathcal{D}_{T^*} \to L^2(\mathcal{X},F)$. We will prove that the range of T^* is closed. Suppose that $T^*\mathcal{D}_{T^*}$ is not closed. Then there exists a sequence $\{g_\nu\}$ in $\mathcal{D}_{T^*}$, such that $\lim T^* g_\nu = v$ and $v \notin T^*\mathcal{D}_{T^*}$. Set $g'_\nu = g_\nu - \Pi^1 g_\nu$, then g'_ν are orthogonal to H^1 and $\lim T^* g'_\nu = v$. If the norms $\|g'_\nu\|_{L^2(\mathcal{X},F)}$ are bounded then

$$D(g'_\nu) = (\|T^* g'_\nu\|^2_{L^2(\mathcal{X},F)} + \|g'_\nu\|^2_{L^2(\mathcal{X},F)})^{1/2}$$

are bounded, too. By the above, $\{g'_\nu\}$ has a convergent subsequence with a limit g, and since T^* is closed then $T^* g = v$ which contradicts the assumption that $v \notin T^*\mathcal{D}_{T^*}$. Thus by choosing a subsequence, if necessary, we may assume that $\lim \|g'_\nu\|_{L^2(\mathcal{X},F)} = \infty$.

Now set $G_\nu = g'_\nu / \|g'_\nu\|_{L^2(\mathcal{X},F)}$. Then $\lim \|T^* G_\nu\|_{L^2(\mathcal{X},F)} = 0$ and $D(G_\nu)$ are bounded. Therefore $\{G_\nu\}$ has a convergent subsequence $\{G_{\nu_k}\}$, such that

$$\begin{aligned} \lim G_{\nu_k} &= G, \\ \lim T^* G_{\nu_k} &= 0. \end{aligned}$$

Hence $T^* G = 0$ so that $G \in H^1$. Since G_ν is orthogonal to H^1 we have $G = 0$, but $\|G_\nu\|_{L^2(\mathcal{X},F)} = 1$. This contradiction proves that the range $T^*\mathcal{D}_{T^*}$ is closed in $L^2(\mathcal{X},F)$.

Let R be the restriction of T^* to the orthogonal complement of H^1 in $\mathcal{D}_{T^*}$. Then R is one-to-one and has a closed range. By the closed graph theorem, the inverse R^{-1} is bounded. Hence there is $c > 0$ such that $\|Rg\|_{L^2(\mathcal{X},F)} \geq c\, \|g\|_{L^2(\mathcal{X},F)}$. This proves the lemma. □

Theorem 2.5. *Let $\mathcal{X}$ be a compact Riemannian manifold and Δ^F a Laplace operator on $\mathcal{X}$ related to a Riemannian vector bundle F. Then $L^1\mathcal{D}_{L^1}$ is a closed subspace of $L^2(\mathcal{X}, F)$.*

Proof. By Lemma 2.4 there exists $c > 0$ with the property that for all $g \in \mathcal{D}_{L^1}$ which are orthogonal to H^1 we have

$$(L^1 g, g)_{L^2(\mathcal{X},F)} \geq c\, \|g\|^2_{L^2(\mathcal{X},F)},$$

so that $\|L^1 g\|_{L^2(\mathcal{X},F)} \geq c\, \|g\|_{L^2(\mathcal{X},F)}$. Set $f = \lim L^1 g_\nu$. We may assume that g_ν are orthogonal to H^1, and then $\|g_\nu\|_{L^2(\mathcal{X},F)}$ are uniformly bounded. Therefore, $\{g_\nu\}$ has a subsequence whose arithmetic means converge. Denoting this limit by g, we get $f = L^1 g$, which completes the proof. □

On combining Theorem 2.5 with weak orthogonal decomposition (2.2) we get the strong orthogonal decomposition

$$L^2(\mathcal{X}, F) = H^1 \oplus TT^*\mathcal{D}_{L^1}. \tag{2.4}$$

3. Green operator

By Lemma 2.1, equality (2.4) is a direct sum decomposition related to the Fredholm operator $L^1 : \mathcal{D}_{L^1} \to L^2(\mathcal{X}, F)$. In this section we specify the relevant operators.

Let $f \in L^2(\mathcal{X}, F)$, then $f = \Pi^1 f + L^1 g$ where $g \in \mathcal{D}_{L^1}$. The Green operator $G^1 : L^2(\mathcal{X}, F) \to \mathcal{D}_{L^1}$ is defined by

$$G^1 f = g - \Pi^1 g.$$

Note that G^1 is well defined. Indeed, if also $f = \Pi^1 f + L^1 g'$ where $g' \in \mathcal{D}_{L^1}$ then $L^1(g - g') = 0$ whence

$$(g - \Pi^1 g) - (g' - \Pi^1 g') = (g - g') - \Pi^1 (g - g') = 0.$$

We summarise the properties of the Green operator. They are reminiscent of those of the Green operator from Hodge theory on a compact closed manifold, see [Tar95, Ch. 4].

Theorem 3.1. *As defined above, the Green operator $G^1 : L^2(\mathcal{X}, F) \to \mathcal{D}_{L^1}$ possesses the following properties:*

1) G^1 is bounded, selfadjoint, $\Pi^1 G^1 = G^1 \Pi^1 = 0$, and G^1 originates the orthogonal decomposition

$$f = \Pi^1 f + TT^* G^1 f \tag{3.1}$$

for all $f \in L^2(\mathcal{X}, F)$.

2) If $f \in \mathcal{D}_{T^*}$ and $T^* f = 0$ then $T^* G^1 f = 0$.

Proof.

1) The equalities $\Pi^1 G^1 = G^1 \Pi^1 = 0$ and formula (3.1) follow immediately from the definition of G^1.

Further, by the closed graph theorem there exists a constant $c > 0$, such that if $g \in \mathcal{D}_{L^1}$ is orthogonal to H^1 then $\|L^1 g\|_{L^2(\mathcal{X},F)} \geq c\, \|g\|_{L^2(\mathcal{X},F)}$. Applying this to $G^1 f$, we obtain

$$\|G^1 f\|_{L^2(\mathcal{X},F)} \leq \frac{1}{c}\, \|L^1 G^1 f\|_{L^2(\mathcal{X},F)} = \frac{1}{c}\, \|f - \Pi^1 f\|_{L^2(\mathcal{X},F)} \leq \frac{1}{c}\, \|f\|_{L^2(\mathcal{X},F)}.$$

Hence G^1 is bounded.

Finally, the selfadjointness of G^1 follows immediately from that of the operator L^1, for

$$(G^1 f, g)_{L^2(\mathcal{X},F)} = (G^1 f, L^1 G^1 g)_{L^2(\mathcal{X},F)} = (L^1 G^1 f, G^1 g)_{L^2(\mathcal{X},F)} = (f, G^1 g)_{L^2(\mathcal{X},F)}.$$

2) Suppose $f \in \mathcal{D}_{T^*}$. Then from (3.1) we get $TT^* G^1 f \in \mathcal{D}_{T^*}$. If $T^* f = 0$ then $T^* TT^* G^1 f = 0$. Hence it easily follows that $T^* G^1 f = 0$. □

By the very definition of the Green operator, if $g \in \mathcal{D}_{L^1}$ then $G^1 L^1 g = g - \Pi^1 g$ whence

$$g = \Pi^1 g + G^1 \, TT^* g \tag{3.2}$$

in $\mathcal{X}$, which is a counterpart to (3.1). Since G^1 is selfadjoint, equalities (3.1) and (3.2) are actually equivalent.

Remark 3.2. From the spectral invariance of Boutet de Monvel's algebra [BdM71] it follows that the Green operator G^1 is a pseudodifferential operator of order -4 in this algebra on $\mathcal{X}$.

4. Bergman projection

For compact manifolds with boundary $\mathcal{X}$ the subspace H^0 is usually infinite dimensional. However, the following result still holds.

Theorem 4.1. *Let $\mathcal{X}$ be a compact Riemannian manifold and Δ^F a Laplace operator on $\mathcal{X}$ related to a Riemannian vector bundle F. Then $L^0 \mathcal{D}_{L^0}$ is a closed subspace of $L^2(\mathcal{X}, F)$.*

Proof. It is sufficient to prove that there exists a constant $c > 0$ with the property that $\|L^0 u\|_{L^2(\mathcal{X},F)} \geq c \, \|u\|_{L^2(\mathcal{X},F)}$ for all $u \in \mathcal{D}_{L^0}$ which are orthogonal to H^0.

First, if $u \in \mathcal{D}_{L^0}$ then $Tu \in \mathcal{D}_{T^*}$ and Tu is orthogonal to H^1. Thus, by Lemma 2.4, we obtain

$$\begin{aligned} \|T^* Tu\|_{L^2(\mathcal{X},F)} &= \|L^0 u\|_{L^2(\mathcal{X},F)} \\ &\geq c \, \|Tu\|_{L^2(\mathcal{X},F)}. \end{aligned}$$

Further, since u is orthogonal to H^0, then by the weak orthogonal decomposition (2.2) $u \in \overline{L^0 \mathcal{D}_{L^0}}$. Hence, for each $\varepsilon > 0$ there exists $v \in \mathcal{D}_{L^0}$ with the property that $\|u - L^0 v\|_{L^2(\mathcal{X},F)} < \varepsilon$. Thus,

$$\begin{aligned} \|u\|^2_{L^2(\mathcal{X},F)} &\leq |(L^0 v, u)_{L^2(\mathcal{X},F)}| + \varepsilon \, \|u\|_{L^2(\mathcal{X},F)} \\ &\leq \|Tv\|_{L^2(\mathcal{X},F)} \|Tu\|_{L^2(\mathcal{X},F)} + \varepsilon \, \|u\|_{L^2(\mathcal{X},F)} \\ &\leq \frac{1}{c^2} \, \|L^0 v\|_{L^2(\mathcal{X},F)} \|L^0 u\|_{L^2(\mathcal{X},F)} + \varepsilon \, \|u\|_{L^2(\mathcal{X},F)} \\ &\leq \frac{1}{c^2} \, \|u\|_{L^2(\mathcal{X},F)} \|L^0 u\|_{L^2(\mathcal{X},F)} + \varepsilon \, (\frac{1}{c^2} \, \|L^0 u\|_{L^2(\mathcal{X},F)} + \|u\|_{L^2(\mathcal{X},F)}). \end{aligned}$$

Since ε can be made arbitrarily small by choosing $L^0 v$ close enough to u, we obtain $\|L^0 u\|_{L^2(\mathcal{X},F)} \geq c^2 \, \|u\|_{L^2(\mathcal{X},F)}$, which concludes the proof. □

Let $u \in L^2(\mathcal{X}, F)$, then $u = \Pi^0 u + L^0 v$ where $v \in \mathcal{D}_{L^0}$. The Green operator $G^0 : L^2(\mathcal{X}, F) \to \mathcal{D}_{L^0}$ is defined by

$$G^0 u = v - \Pi^0 v.$$

Note that G^0 is well defined. Indeed, if also $u = \Pi^0 u + L^0 v'$ where $v' \in \mathcal{D}_{L^0}$ then $L^0(v - v') = 0$ whence

$$(v - \Pi^0 v) - (v' - \Pi^0 v') = (v - v') - \Pi^0 (v - v') = 0.$$

Theorem 4.2. *As defined above, the Green operator $G^0 : L^2(\mathcal{X}, F) \to \mathcal{D}_{L^0}$ possesses the following properties:*

1) G^0 is bounded, selfadjoint, $\Pi^0 G^0 = G^0 \Pi^0 = 0$, and G^0 originates the orthogonal decomposition

$$u = \Pi^0 u + T^* T G^0 u \tag{4.1}$$

for all $u \in L^2(\mathcal{X}, F)$.

2) If $u \in \mathcal{D}_T$ then $TG^0 u = G^1 T u$.

Proof. The proof is is analogous to that of Theorem 3.1, the only difference being in the stronger assertion 2). Let $u \in \mathcal{D}_T$. Then we have $Tu = TT^*TG^0 u$ on the one hand, which is due to formula (4.1). On the other hand, $Tu \in L^2(\mathcal{X}, F)$, and so $Tu = TT^* G^1 T u$, which is due to formula (3.1), for Tu is orthogonal to H^1. Comparing these two equalities we conclude that $L^0(TG^0 u - G^1 T u) = 0$. Since $TG^0 u - G^1 T u$ is orthogonal to the null space of L^0, it follows that $TG^0 u = G^1 T u$, as desired. □

The next result follows readily from from Theorems 4.2. It yields a useful formula for the projection Π^0.

Corollary 4.3. *Suppose $\mathcal{X}$ is a compact Riemannian manifold and Δ^F a Laplace operator on $\mathcal{X}$ related to a Riemannian vector bundle F. Then*

$$u = \Pi^0 u + T^* G^1 T u \tag{4.2}$$

for any section $u \in \mathcal{D}_T$.

Since the operator Δ is elliptic, it follows that Δ is hypoelliptic in the interior of $\mathcal{X}$ whence H^0 just amounts to the closed subspace of $L^2(\mathcal{X}, F)$ which consists of smooth solutions of $\Delta u = 0$ in the interior of $\mathcal{X}$. Thus, the operator Π^0 is a generalisation of the Bergman projector from complex analysis. By Corollary 4.3 we obtain

$$\Pi^0 = I - T^* G^1 T.$$

A priori estimates for solutions of elliptic systems imply that for each point x in the interior of X the "evaluation functional" $\delta_x(u) = u(x)$ is bounded on H^0. Therefore, H^0 is a Hilbert space with reproducing kernel, the concept being introduced in [Aro50].

Pick a complete orthonormal system $\{e_\nu\}_{\nu=1,2,\ldots}$ in H^0. If $u \in H^0$ then this section decomposes into the Fourier series $u = \sum c_\nu e_\nu$ over this basis which converges in the $L^2(\mathcal{X}, F)$-norm, and hence uniformly along with all derivatives on compact subsets of the interior of $\mathcal{X}$. In the interior of $\mathcal{X} \times \mathcal{X}$ we now consider the kernel function

$$K(x, y) = K_{\Pi^0}(x, y) = \sum_{\nu=0}^{\infty} *_F \, e_\nu(x) \otimes e_\nu(y), \tag{4.3}$$

where $*_F$ is the Hodge star operator associated with Riemannian bundle F.

Theorem 4.4. *Series (4.3) converges uniformly along with all derivatives on compact subsets of the interior of* $\mathcal{X} \times \mathcal{X}$*, so that* K_{Π^0} *is a* C^∞ *section of* $F^* \boxtimes F$ *over the interior of* $\mathcal{X} \times \mathcal{X}$*. If* $x \in \mathcal{X}$ *is fixed, then this series actually converges in the norm of* $L^2(\mathcal{X}, F_x^* \otimes F)$*.*

Proof. This is a very particular case of Lemma 11.2.13 in [Tar95]. □

The following formula can be therefore thought of as analogue of the classical Bergman formula for holomorphic functions of one complex variable, see the survey [Ber50].

Theorem 4.5. *If* $u \in H^0$ *then*

$$u(x) = (u, K(x, \cdot))_{L^2(\mathcal{X},F)}$$

for all $x \in \overset{\circ}{\mathcal{X}}$*.*

Proof. Let $u = \sum c_\mu e_\mu$. Then by the previous theorem we get for fixed x in the interior of $\mathcal{X}$

$$\begin{aligned}(u, K(x, \cdot))_{L^2(\mathcal{X},F)} &= \sum_{\mu,\nu} c_\mu \,(e_\mu, e_\nu)_{L^2(\mathcal{X},F)}\, e_\nu(x)\\ &= \sum_{\nu} c_\nu \,(e_\nu, e_\nu)_{L^2(\mathcal{X},F)}\, e_\nu(x)\\ &= u(x),\end{aligned}$$

and the proof is complete. □

In order to discover the properties of Π^0 one might study the Green operator G^1, see [BS76].

5. Berezin's theory

The results we describe in this section are due to F. Berezin and are extracted from his papers [Ber72] and [Ber74]. These papers seem to be more familiar to physicists than to mathematicians.

Let $\mathcal{X}$ be a set equipped with a measure μ, H a Hilbert subspace of $L^2(\mathcal{X}, \mu)$ and $\Pi : L^2(\mathcal{X}, \mu) \to H$ orthogonal projection. Recall that, for a a bounded measurable function on $\mathcal{X}$, the Toeplitz operator with multiplier a is the operator $T_a = \Pi M_a$ on H.

Suppose that there is a measurable function $K(x, y)$ on $\mathcal{X} \times \mathcal{X}$, such that

$$(\Pi u)(x) = \int_{\mathcal{X}} K(x, y) u(y) d\mu(y) \tag{5.1}$$

for all $u \in L^2(\mathcal{X}, \mu)$, and suppose that, for fixed x, the function $e_x(y) = K(x, y)^*$ is itself in $L^2(\mathcal{X}, \mu)$. In this case we say, that $K(x, y)$ is a reproducing kernel. Such kernels were first studied systematically by N. Aronszajn in the mid-forties, see [Aro50]. However, his work seems to have been largely forgotten about until it was resurrected by F. Berezin.

Notice that by $e_x(y) = K(x, y)^*$ the equation (5.1) can be written in the form

$$u(x) = (u, e_x) \tag{5.2}$$

for all $u \in H$. In other words, Π is definable by a reproducing kernel if and only if the evaluation map $u \mapsto u(x)$ is a continuous functional on H for all $x \in \mathcal{X}$. Since

Π is selfadjoint, we get $K(x,y) = K(y,x)^*$. Therefore, if $u \in H$, we can rewrite (5.1) in the form

$$\begin{aligned} u(x) &= \int_{\mathcal{X}} u(y)K(y,x)^* d\mu(y) \\ &= \int_{\mathcal{X}} (u, e_y)\; e_y(x) d\mu(y) \end{aligned}$$

or symbolically

$$u = \int_{\mathcal{X}} (u, e_y)\; e_y d\mu(y). \tag{5.3}$$

Moreover, taking the inner product of this equality with a function $v \in H$, we get immediately

$$(u, v) = \int_{\mathcal{X}} (u, e_y)\; \overline{(v, e_y)}\, d\mu(y). \tag{5.4}$$

This formula is reminiscent of the Plancherel formula. Indeed, if $\{e_n\}_{n=1,2,\ldots}$ is an orthonormal basis of H, then just as above

$$u = \sum_{n=1}^{\infty} (u, e_n)\, e_n$$

and

$$(u, v) = \sum_{n=1}^{\infty} (u, e_n)\, \overline{(v, e_n)}.$$

Therefore, even though the functions e_y are neither orthonormal nor linearly independent, one can regard them as a kind of "generalised orthonormal basis" of H.

Suppose, in particular, that $A : H \to H$ is a bounded linear operator. The "matrix entries" of this operator are (Ae_x, e_y) and, if A and B are two operators, it is easy to check that the analogue of the usual rule for matrix multiplication, namely

$$(AB\, e_x, e_y) = \int_{\mathcal{X}} (Be_x, e_z)\, (A\, e_z, e_y)\, d\mu(z), \tag{5.5}$$

is valid. (Just apply (5.4) to $u = Be_x$ and $v = A^* e_y$.)

Following F. Berezin we will define the symbol of an operator A to be the quantity

$$\sigma(A)(y) = \frac{(Ae_y, e_y)}{(e_y, e_y)}. \tag{5.6}$$

Except for the normalising factor (e_y, e_y), the symbol $\sigma(A)$ can be thought of as the "diagonal entries" of the matrix representing A in the basis $\{e_y\}$. In many interesting situation A is determined by its symbol and (5.5) can be interpreted as "composition formula" for symbols.

If A is a Toeplitz operator with multiplier a, then

$$(Ae_y, e_y) = (\Pi(ae_y), e_y) = (ae_y, \Pi e_y) = (ae_y, e_y)$$

since $e_y \in H$. Thus, the symbol of A can be expressed in terms of a by the integral formula

$$\begin{aligned}\sigma(A)(y) &= \frac{1}{(e_y, e_y)} \int_{\mathcal{X}} a(x) e_y(x) (e_y(x))^* d\mu(x) \\ &= \frac{1}{(e_y, e_y)} \int_{\mathcal{X}} |K(x,y)|^2 a(x) d\mu(x).\end{aligned}$$

The most interesting aspect of Berezin's theory has to do with the application of this machinery to concrete examples. One very interesting example is that of a strongly pseudoconvex domain $\mathcal{X}$ in a higher-dimensional complex space. If one takes for H the space of holomorphic functions in $\mathcal{X}$ of class $L^2(\mathcal{X})$, the orthogonal projection of $L^2(\mathcal{X}, \mu)$ on H is the Bergman operator, and is known to have the reproducing property. Moreover, its kernel $K(x,y)$ is holomorphic in x and anti-holomorphic in y. So, if $A : H \to H$ is a bounded operator, its symbol $\sigma(A)(x)$ can, by (5.6), be extended uniquely to a function defined in a neighbourhood of the diagonal in $\mathcal{X} \times \mathcal{X}$ in such a way that it is holomorphic in the first variable and anti-holomorphic in the second variable. In fact, such an extension is given explicitly by

$$\frac{(Ae_x, e_y)}{(e_x, e_y)}.$$

It is clear, therefore, that the symbol determines (Ae_x, e_y) on a neighbourhood of the diagonal and, by analyticity, on all of $\mathcal{X} \times \mathcal{X}$. Thus, by (5.3), the symbol of A determines A itself. The composition formula (5.5) can be made very explicit if $\mathcal{X}$ is the unit ball in $\mathbb{C}^n$ or more generally a "homogeneous complex domain" in the sense of Cartan.

6. Fock projection

The example we will mainly be interested in in this section is a special case of Berezin's theory. Let $\mathcal{X} = \mathbb{C}^n$ be equipped with the Gaussian measure

$$d\mu = \frac{1}{(2\pi\imath)^n} e^{-|z|^2} d\bar{z} \wedge dz. \tag{6.1}$$

The space of holomorphic functions on $\mathbb{C}^n$ which are square summable with respect to (6.1) is called Fock space. To accord with conventional usage we will denote it by F rather than H. Let us determine the reproducing kernel associated with orthogonal projection of $L^2(\mathbb{C}^n, \mu)$ onto F. By $e_x(y) = K(x,y)^*$, this amounts to determining the functions e_λ for all values $\lambda \in \mathbb{C}^n$. We will do this by first observing that

$$\begin{aligned}\int_{\mathbb{C}^n} z_k u \bar{v} e^{-|z|^2} d\bar{z} \wedge dz &= \int_{\mathbb{C}^n} u \bar{v} \Big(-\partial_{\bar{z}_k} e^{-|z|^2} \Big) d\bar{z} \wedge dz \\ &= \int_{\mathbb{C}^n} u \overline{\Big(\partial_{z_k} v \Big)} e^{-|z|^2} d\bar{z} \wedge dz\end{aligned}$$

for all $u, v \in F$. (Since u is holomorphic, $\partial_{\bar{z}_k} u = 0$.) This shows that ∂_{z_k} is the transpose of the operator of multiplication by z_k. We will use this to deduce

Lemma 6.1. *The function e_λ satisfies the equation*

$$\partial_{z_k} e_\lambda = \bar{\lambda}_k e_\lambda. \tag{6.2}$$

Proof. If $u \in F$, then by (5.2)

$$(z_k u, e_\lambda) = \lambda_k u(\lambda) = \lambda_k (u, e_\lambda) = \left(u, \bar{\lambda}_k e_\lambda\right).$$

On the other hand,

$$(z_k u, e_\lambda) = (u, \partial_{z_k} e_\lambda).$$

Hence, the inner product with u of the right-hand and left-hand sides of (6.2) is the same for all u. □

The most general solution of (6.2) is of the form $c_\lambda e^{\bar{\lambda} z}$, hence e_λ has to be of this form. To determine the constant c_λ, note that

$$(e_\lambda, e_\lambda) = e_\lambda(\lambda) = c_\lambda\, e^{|\lambda|^2}. \tag{6.3}$$

On the other hand,

$$\begin{aligned}\left(e^{\bar{\lambda} z}, e^{\bar{\lambda} z}\right) &= \frac{1}{(2\pi\imath)^n} \int_{\mathbb{C}^n} e^{\bar{\lambda} z + \lambda \bar{z} - z\bar{z}}\, d\bar{z} \wedge dz \\ &= \frac{e^{|\lambda|^2}}{(2\pi\imath)^n} \int_{\mathbb{C}^n} e^{-|\lambda - z|^2}\, d\bar{z} \wedge dz \\ &= \frac{e^{|\lambda|^2}}{(2\pi\imath)^n} \int_{\mathbb{C}^n} e^{-|z|^2}\, d\bar{z} \wedge dz \\ &= e^{|\lambda|^2}.\end{aligned}$$

Thus, $\left(e^{\bar{\lambda} z}, e^{\bar{\lambda} z}\right) = e^{|\lambda|^2}$. Combining this with (6.3) we deduce that $|c_\lambda|^2 = c_\lambda$, i.e. $c_\lambda = 1$. We have thus proved that

$$e_\lambda(z) = e^{\bar{\lambda} z},$$

and from $e_x(y) = K(x, y)^*$ we obtain

Theorem 6.2. *The reproducing kernel associated with the "Fock projection" is* $K(z, w) = e^{z\bar{w}}$.

Let a be a bounded measurable function on $\mathbb{C}^n$ and let $T_a : F \to F$ be the Toeplitz operator with multiplier a. Our next task is to compute the symbol $\sigma(A)(y)$ of T_a. As defined above,

$$\sigma(A)(\lambda) = \frac{1}{(e_\lambda, e_\lambda)} \int_{\mathbb{C}^n} |K(\lambda, z)|^2 a(z) d\mu(z)$$

and by Theorem 6.2 the right-hand side is

$$\frac{e^{-|\lambda|^2}}{(2\pi\imath)^n} \int_{\mathbb{C}^n} e^{\bar{\lambda} z + \lambda \bar{z} - |z|^2} a(z) d\bar{z} \wedge dz$$

or

$$\frac{1}{(2\pi\imath)^n} \int_{\mathbb{C}^n} e^{-|\lambda - z|^2} a(z)\, d\bar{z} \wedge dz.$$

Notice however that this is just the standard formula for the solution at time $t = 1$ of the heat equation on $\mathbb{C}^n = \mathbb{R}^{2n}$ with initial data a. To wit,

$$\sigma(T_a)(\lambda) = (e^{-\Delta} a)(\lambda) \tag{6.4}$$

where $\Delta = \sum_{k=1}^{n} \partial_{\bar{z}_k} \partial_{z_k}$.

7. Toeplitz operators in n dimensions

Suppose $\mathcal{X}$ is a compact Riemannian manifold and Δ^F a Laplace operator on $\mathcal{X}$ related to a Riemannian vector bundle F. From now on we write $\Pi = \Pi^0$ for the orthogonal projection of $L^2(\mathcal{X}, F)$ onto the closed subspace $H = H^0$ which consists of all sections $u \in L^2(\mathcal{X}, F)$ satisfying $\Delta u = 0$ weakly in the interior of $\mathcal{X}$. By the above, H is a Hilbert space with reproducing kernel and the elements of H are actually smooth in the interior of $\mathcal{X}$. This space is of infinite dimension unless $\mathcal{X}$ is compact and closed.

Pick an endomorphism E of the vector bundle F. By definition, this is a family of linear selfmaps $E(x)$ of the fibres F_x of F parametrised by the points $x \in \mathcal{X}$ of the base and depending smoothly on x. If the bundle F is trivial over an open set $U \subset \mathcal{X}$, then E can be represented by a $(k \times k)$-matrix of smooth functions on U, where k is the rank of F.

Definition 7.1. By a Toeplitz operator in H with symbol E is meant the map $T_E : H \to H$ given by $T_E u = \Pi(Eu)$ for $u \in H$.

Obviously, if E is the identity endomorphism, then $T_E u = u$ for all $u \in H$, i.e. T_E is the identity operator on H. To study the operator algebra generated by Toeplitz operators, the main technical tool is formula (4.2).

Lemma 7.2. *Assume that E^1 and E^2 are bundle endomorphisms of F. Then $T_{E^2E^1} = T_{E^2}T_{E^1}$ modulo compact operators on H.*

Proof. By formula (4.2),

$$\begin{aligned} T_{E^1} &= E^1 - T^*G^1TE^1, \\ T_{E^2} &= E^2 - T^*G^1TE^2 \end{aligned}$$

whence

$$\begin{aligned} T_{E^2}(T_{E^1}u) &= \left(E^2 - T^*G^1TE^2\right)\left(E^1u - T^*G^1TE^1u\right) \\ &= E^2E^1u - T^*G^1TE^2E^1u - E^2T^*G^1TE^1u + T^*G^1TE^2T^*G^1TE^1u \\ &= T_{E^2E^1}u - T_{E^2}\left(T^*G^1TE^1u\right) \end{aligned}$$

for all $u \in H$.

If $u \in H$ then $Tu = 0$ and so $TE^1u = [T, E^1]u$, where $[T, E^1] := TE^1 - E^1T$ is the commutator of T and E^1. Since the leading symbol of T is given by a diagonal matrix, it follows that the commutator $[T, E^1]$ is a first order differential operator on sections of F. Hence

$$T_{E^2}\, T^*G^1TE^1 = T_{E^2}\, T^*G^1[T, E^1]$$

and so this mapping factors through the compact embedding $H^1(\mathcal{X}, F) \to L^2(\mathcal{X}, F)$, which is due to the Rellich theorem.

Summarising we conclude that $T_{E^2E^1} - T_{E^2}T_{E^1}$ is a compact operator on H, as desired. □

We are now in a position to characterise those Toeplitz operators on H which have Fredholm property. Recall that a bounded operator in Banach spaces is Fredholm is and only if it possess a parametrix.

Theorem 7.3. *If E is a bundle isomorphism of F, then the Toeplitz operator T_E in H is Fredholm.*

Proof. Suppose E is a bundle isomorphism of F and E^{-1} is the inverse endomorphism of F. By Lemma 7.2, we get

$$\begin{array}{rcl} T_{E^{-1}} T_E & = & I, \\ T_E T_{E^{-1}} & = & I \end{array}$$

modulo compact operators on H. Hence, the Toeplitz operator $T_{E^{-1}}$ is a parametrix of T_E. □

The condition $\det E(x) \neq 0$ for all $x \in \mathcal{X}$ is clearly not necessary for the Fredholm property of T_E. In particular, if the boundary of $\mathcal{X}$ is empty, then H is finite dimensional and so each linear operator in H is Fredholm and has index zero.

The problem of obtaining a general formula for the index of a Toeplitz operator T_E with Fredholm property is still open. However, if $\mathcal{X}$ is a bounded domain with smooth boundary in $\mathbb{R}^n$, it is to be expected that the techniques of Fedosov [Fed74] applies to yield an analytic formula for the index.

One can also introduce more general Toeplitz operators on H which are of the form $T_\Psi u := \Pi(\Psi u)$, where $\Psi \in \Psi^m_{\mathrm{phg}}(\mathcal{X}; F)$ is a polyhomogeneous pseudodifferential operator of order $m \leq 0$ in sections of the vector bundle F on $\mathcal{X}$. However, this topic exceeds the scope of this paper and we refer the reader to [BdM79] and [BG81].

8. Concluding remarks

In much the same way we may study a Toeplitz operator related to a Dirac operator A in a neighbourhood of a closed bounded domain $\mathcal{X}$ with smooth boundary in $\mathbb{R}^n$. By this is meant any $(l \times k)$-matrix of first order scalar partial differential operators near $\mathcal{X}$ satisfying $A^* A = -\Delta E_k$ up to a first-order part, where Δ is the standard (nonpositive) Laplace operator in $\mathbb{R}^n$ and E_k the unit $(k \times k)$-matrix. On writing

$$A = \sum_{j=1}^{n} A_j(x) \partial_j$$

up to a zero-order part, we obtain the equations $A_i^* A_j + A_j^* A_i = -2\delta_{i,j} E_k$ for $1 \leq i \leq j \leq n$, where $\delta_{i,j}$ is the Kronecker delta.

The matrices $A_1, \ldots, A_n$ are thus generators of a noncommutative associative algebra which is called the Clifford algebra. Clifford algebras have important applications in a variety of fields including geometry, theoretical physics and digital image processing.

In the sequel we restrict our attention to the special Clifford algebra corresponding to the case $n = 4$. This latter is called the algebra of quaternions and denoted by $\mathbb{H}$. To wit,

$$A = \begin{pmatrix} \partial_1 & -\partial_2 & -\partial_3 & -\partial_4 \\ \partial_2 & \partial_1 & -\partial_4 & \partial_3 \\ \partial_3 & \partial_4 & \partial_1 & -\partial_2 \\ \partial_4 & -\partial_3 & \partial_2 & \partial_1 \end{pmatrix}.$$

The operator A is obviously elliptic and its fundamental solution of convolution type is given by $\Phi(x) = -A^* e(x)$, where

$$e(x) = \frac{-1}{(2\pi)^2} \frac{1}{|x|^2}$$

is the standard fundamental solution of convolution type of the Laplace operator Δ in $\mathbb{R}^4$.

Theorem 8.1. *Let $u_0 \in L^2(\partial\mathcal{X}, \mathbb{R}^4)$. In order that there be a solution u to $Au = 0$ in the interior of $\mathcal{X}$, which has finite order of growth at $\partial\mathcal{X}$ and coincides with u_0 on $\partial\mathcal{X}$, it is necessary and sufficient that*

$$\int_{\partial\mathcal{X}} (A(\nu)u_0, g)_y \, ds = 0 \tag{8.1}$$

for all solutions of the formal adjoint equation $A^* g = 0$ near $\mathcal{X}$, where ds is the surface measure on $\partial\mathcal{X}$ and $\nu(y)$ the unit outward normal vector of $\partial\mathcal{X}$ at a point y.

Proof. See Theorem 10.3.14 in [Tar95]. □

We denote by H the (closed) subspace of $L^2(\partial\mathcal{X}, \mathbb{R}^4)$, consisting of all functions u satisfying the orthogonality conditions (8.1). The elements of H can be actually specified as solutions to $Au = 0$ of Hardy class H^2 in the interior of $\mathcal{X}$, see [Tar95, 11.2.2]. The orthogonal projection Π of $L^2(\partial\mathcal{X}, \mathbb{R}^4)$ onto H is therefore an analogue of Szegö projection.

To specify the class of multipliers we describe all (4×4)-matrices $E(x)$ commuting with $A_1 = E_4, A_2, \ldots, A_4$.

Lemma 8.2. *A (4×4)-matrix E of real numbers commutes with A if and only if E is of the form*

$$E = \begin{pmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & \delta & -\gamma \\ \gamma & -\delta & \alpha & \beta \\ \delta & \gamma & -\beta & \alpha \end{pmatrix}, \tag{8.2}$$

where α, β, γ and δ are arbitrary real numbers.

Proof. This is verified by straightforward calculation. □

From Lemma 8.2 it follows readily that matrices E of the form (8.2) constitute an unital algebra. Since $\det E = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2$, a matrix E is invertible if and only if $E \neq 0$.

Let $E(x)$ be a (4×4)-matrix of the form (8.2) whose entries are bounded functions on the boundary. By a Toeplitz operator T_E with multiplier E is meant the operator $u \mapsto \Pi(Eu)$ in H.

To study the algebra generated by Toeplitz operators we introduce a singular Cauchy type integral

$$\mathcal{C}u = -\text{p.v.} \int_{\partial\mathcal{X}} (A(\nu)u, \Phi(x - \cdot))_y ds$$

for $x \in \partial\mathcal{X}$, where $u \in L^2(\partial\mathcal{X}, \mathbb{R}^4)$. The principal value of the integral on the right hand side exists for almost all $x \in \partial\mathcal{X}$ and it induces a bounded linear operator in $L^2(\partial\mathcal{X}, \mathbb{R}^4)$.

Lemma 8.3. *The operators $(1/2)I \pm \mathcal{C}$ are orthogonal projections on the space $L^2(\partial\mathcal{X}, \mathbb{R}^4)$.*

Proof. This follows from the equality $\mathcal{C}^2 = 1/4\, I$ by a trivial verification, cf. for instance [Tar06]. □

Using Lemma 8.3 we establish a very useful formula for the projection Π, namely, $\Pi = (1/2)I + \mathcal{C}$. Thus,

$$\begin{array}{rcl} T_{E^1} & = & (1/2)E^1 + \mathcal{C}E^1, \\ T_{E^2} & = & (1/2)E^2 + \mathcal{C}E^2, \end{array}$$

and so

$$T_{E^2}T_{E^1} = T_{E^2E^1} - [\mathcal{C}, E^2]\left((1/2)E^1 - \mathcal{C}E^1\right). \tag{8.3}$$

Lemma 8.4. *Assume that E^1 and E^2 are (4×4)-matrices of smooth functions on $\partial\mathcal{X}$. Then $T_{E^2E^1} = T_{E^2}T_{E^1}$ modulo compact operators on H.*

Proof. If a (4×4)-matrix E commutes with A, it is of the form (8.2), and so the adjoint matrix E^* has the same form. Hence it follows that E^* commutes with A, and so E commutes with the adjoint A^*, too. Now an elementary analysis shows readily that

$$\begin{array}{rcl} \mathcal{C}(Eu)(x) & = & -\text{p.v.} \displaystyle\int_{\partial\mathcal{X}} E(y)\,\Phi(x-y)A(\nu(y))u(y)ds \\ & = & E(\mathcal{C}u)(x) + \displaystyle\int_{\partial\mathcal{X}} (A(\nu(y))(E(x)-E(y))u, \Phi(x-y))_y ds \end{array}$$

holds almost everywhere on the boundary for all $u \in L^2(\partial\mathcal{X}, \mathbb{R}^4)$. In particular, if $u \in H$, then

$$\mathcal{C}(Eu)(x) = (1/2)\,(Eu)(x) + \int_{\partial\mathcal{X}} (A(\nu(y))(E(x)-E(y))u, \Phi(x-y))_y ds$$

for almost all $x \in \partial\mathcal{X}$. From these two equalities we conclude that the remainder $[\mathcal{C}, E^2]\left((1/2)E^1 - \mathcal{C}E^1\right)$ in (8.3) is a pseudodifferential operator of order -2 on the surface $\partial\mathcal{X}$. □

Lemma 8.4 allows one to develop the Fredholm theory of Toeplitz operators with multipliers of the form (8.2) in just the same way as in Section 7.

One deduces from the proof of Lemma 8.4 that $T_{E^2E^1} = T_{E^2}T_{E^1}$ holds actually up to trace operators, if $n \leq 2$. Hence, the results of [HH75] apply to evaluate the index of Fredholm Toeplitz operators with multipliers of the form (8.2) in the case $n \geq 2$.

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