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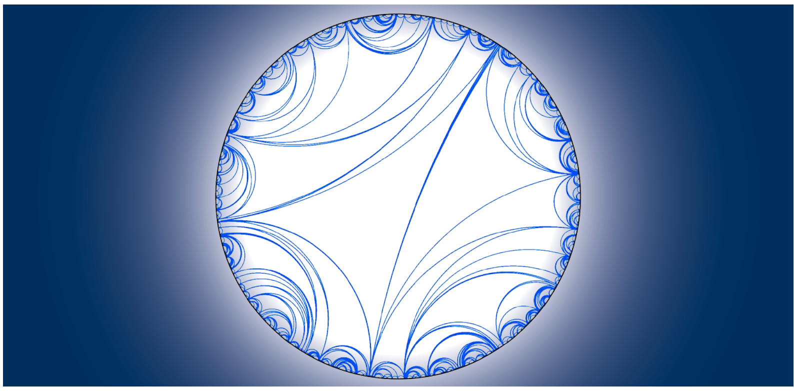

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GENERALISED BELTRAMI EQUATIONS

IBRAHIM LY AND NIKOLAI TARKHANOV

Abstract. We enlarge the class of Beltrami equations by developping a stability theory for the sheaf of solutions of an overdetermined elliptic system of first order homogeneous partial differential equations with constant coefficients in $\mathbb{R}^n$.

Contents

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Introduction

In the theory of quasiconformal mappings of domains in the plane and in real higher-dimensional spaces, the theory of stability of conformal mappings plays an important role (cf. [Lav58], [Bel74], [Res67, Res70, Res78]). The essence of those studies forming the basis of this theory can be summarised as follows. Let f be a quasiconformal mapping and locally close, in some sense, to a conformal mapping. When can we state that f is globally close to a conformal map, in the same sense or in a different one?

For our purpose the stability theory for conformal mappings (cf. [Lav58], [Bel74]), which can be extended to a stability theory for holomorphic mappings, has special significance and brings us to the basic propositions of the theory having the following form.

Theorem 0.1. *There exists a non-negative function δ, defined in $[0,1) \times (0,1)$, such that:*

1) $\delta(\varepsilon,\theta) \to \delta(0,\theta) = 0$ as $\varepsilon \to 0$, and for each $\theta \in (0,1)$;

2) if $f\colon U \to \mathbb{C}$ is a mapping of a domain $U \subset \mathbb{C}$, satisfying a Beltrami equation $(\partial/\partial\bar{z})f = Q\,(\partial/\partial z)f$ with $\sup_U |Q| \le \varepsilon$, then, for each $\theta \in (0,1)$ and disc $B \subset U$, there is a holomorphic mapping $u\colon B \to \mathbb{C}$ such that $|f(\zeta) - u(\zeta)| \le \delta(\varepsilon,\theta)\operatorname{diam} f(B)$ for all $\zeta \in \theta B$.

Here, by θB is meant the smaller concentric disc with the same centre whose radius is θ times that of B.

The function u appearing in this theorem can be chosen to be independent of the parameter θ, but for our purpose this distinction is not important.

Theorem 0.2. *There exists a non-negative function ε, defined in $[0,1) \times (0,1)$, such that:*

1) $\varepsilon(\delta,\theta) \to \varepsilon(0,\theta) = 0$ as $\delta \to 0$, and for each $\theta \in (0,1)$;

2) if $f\colon U \to \mathbb{C}$ is a mapping of a domain U in the complex plane possessing the property that, for each disc $B \subset U$, there is a holomorphic mapping $u\colon B \to \mathbb{C}$ such that $|f(\zeta) - u(\zeta)| \le \delta \operatorname{diam} f(B)$ for all $\zeta \in \theta B$, with some $\delta \in [0,1)$ and $\theta \in (0,1)$ independent of B, then f satisfies a Beltrami equation $(\partial/\partial\bar{z})f = Q\,(\partial/\partial z)f$ with $\sup_U |Q| \le \varepsilon(\delta,\theta)$.

Theorems 0.1 and 0.2 can be proven using general properties of Beltrami systems and quasiconformal mappings (see for instance [Vek62] and [Ahl66]).

These theorems can be considered to be, in a certain sense, reciprocal. They express two facts: 1) holomorphic mappings in the plane are stable, i.e., if a mapping is locally close to a holomorphic mapping, then it is globally close, too; 2) the class of mappings close to being holomorphic coincides with the class of solutions of those Beltrami systems $(\partial/\partial\bar{z})f = Q\,(\partial/\partial z)f$, for which the coefficient Q is "small" enough.

Similar facts provide the basis of stability theory for conformal mappings in real higher-dimensional spaces. The role of mappings which are close to holomorphic ones is played in this case by the mappings whose quasiconformality coefficients are close to 1 (cf. [Kop82]). In the '80s, Kopylov [Kop82] presented a series of papers which together form a stability theory for holomorphic mappings in higher-dimensional complex spaces, analogous to the theory of planar and real higher-dimensional cases discussed above.

It was a very natural idea of Kopylov to extend the stability theory to those mappings which are given by solutions of a system of partial differential equations in $\mathbb{R}^n$. Since the mapping classes he studied should be invariant under translations and homotheties, the differential equations were required to be of constant coefficients and homogeneous. As but one step in this direction he suggested his MS student Bezrukova to investigate a class of solutions of the well-known Moisil-Theodoresco system in $\mathbb{R}^3$. Her results published in [Bez83] contain in particular a canonical construction of Beltrami equations related to the Moisil-Theodoresco system. The paper [Bez83] was explained in the context of solutions to general overdetermined elliptic systems in [DK85] and [Tar85]. In [DK85], there is no canonical construction of Beltrami equations but a substitution by a system of rough structure which could hardly be specified within Beltrami equations. The paper [Tar85] already gives such a canonical construction, however, it applies only to elliptic (i.e., quadratic) systems. Complete proof of the results of [Tar85] were first published in the book [Tar95] where also a canonical construction of Beltrami equations was conjectured for overdetermined elliptic systems, see Section 9.3.4 *ibid.* The present paper is aimed at proving this conjecture. Namely, we develop the stability theory for a class of solutions of an overdetermined elliptic system of first order homogeneous partial differential equations with constant coefficients in $\mathbb{R}^n$. As a by-product, we get a canonical construction of Beltrami equations related to such a system (cf. also [Bez83], [Tar85], [DK85]).

1. The stability concept

1.1. **Basic classes of mappings.** Let $\mathcal{S}$ be a class of mappings from open sets in the space $\mathbb{R}^n$ into the space $\mathbb{R}^k$. We assume that each open set U in $\mathbb{R}^n$ is the domain of definition of at least one mapping $u \in \mathcal{S}$.

For an open set $U \subset \mathbb{R}^n$, we denote by $\mathcal{S}(U)$ the set of mappings from U to $\mathbb{R}^k$, belonging to $\mathcal{S}$. In the sequel, we shall endow the class $\mathcal{S}$ with some of the following properties:

$\mathcal{P}_1$) The class $\mathcal{S}$ consists of locally bounded mappings, i.e., for each open set $U \subset \mathbb{R}^n$, the mappings in $\mathcal{S}(U)$ are bounded on compact subsets of U.

$\mathcal{P}_2$) The class $\mathcal{S}$ is invariant under translations and homotheties of the spaces $\mathbb{R}^n$ and $\mathbb{R}^k$, i.e., if $T_1 x = \delta_1 x + x^0$, $T_2 y = \delta_2 y + y^0$, where δ_1, δ_2 are positive constants and $x^0 \in \mathbb{R}^n$, $y^0 \in \mathbb{R}^k$ are fixed vectors, then, for each open set $U \subset \mathbb{R}^n$, the composition $T_2 \circ u \circ T_1$ is in $\mathcal{S}(T_1^{-1}(U))$ whenever $u \in \mathcal{S}(U)$.

$\mathcal{P}_3$) Given an arbitrary open set $U \subset \mathbb{R}^n$, any uniformly bounded family in $\mathcal{S}(U)$ is equicontinuous on compact subsets of U.

$\mathcal{P}_4$) The class $\mathcal{S}$ is closed under uniform convergence on compact subsets of the domains of definition.

$\mathcal{P}_5$) If $u \in \mathcal{S}(U)$, where $U \subset \mathbb{R}^n$ is open, then the restriction of u to each open subset Ω of U belongs to $\mathcal{S}(\Omega)$.

$\mathcal{P}_6$) If $u : U \to \mathbb{R}^k$ is locally of class $\mathcal{S}$, i.e., for each point $x \in U$, there is a neighborhood $U(x) \subset U$ of x such that $u|_{U(x)} \in \mathcal{S}$, then $u \in \mathcal{S}$.

Note that an equivalent formulation of $\mathcal{P}_5$ is: the correspondence $U \mapsto \mathcal{S}(U)$ is a presheaf, over $\mathbb{R}^n$, of mappings from open sets in $\mathbb{R}^n$ into the space $\mathbb{R}^k$. Then, $\mathcal{P}_5$ and $\mathcal{P}_6$ just amount to saying that the presheaf $\mathcal{S}$ is actually a sheaf.

A presheaf $\mathcal{S}$ with properties $\mathcal{P}_1$ and $\mathcal{P}_3$ is easily verified to consist of continuous mappings.

1.2. **Examples.** Let us give several examples of classes possessing the properties $\mathcal{P}_1$-$\mathcal{P}_6$.

Example 1.1. The class of locally constant mappings from open sets in $\mathbb{R}^n$ into the space $\mathbb{R}^k$ satisfies $\mathcal{P}_1$-$\mathcal{P}_6$.

Example 1.2. The class of holomorphic mappings from open sets in $\mathbb{C}^n$ into the space $\mathbb{C}^k$ possesses all the properties $\mathcal{P}_1$-$\mathcal{P}_6$. In this case we identify the mappings with those of domains in $\mathbb{R}^{2n}$ into the space $\mathbb{R}^{2k}$.

Example 1.3. Let $\mathcal{S}$ be the class of locally conformal mappings from open sets in $\mathbb{R}^n$, $n \geq 3$, into the same space. Here, by a conformal mapping of a domain $U \subset \mathbb{R}^n$ is meant the restriction, to U, of some Möbius transformation of $\mathbb{R}^n$ which takes finite values on U. A locally conformal mapping is either orientation preserving or not. For the sake of definiteness we shall assume that the mappings of $\mathcal{S}$ are orientation preserving. Then, the class $\mathcal{S}$ is easily seen to satisfy all the properties except $\mathcal{P}_4$, for the uniform limit of a sequence of locally conformal mappings may be a locally constant one.

Example 1.4. The class of harmonic mappings from open sets $\mathbb{R}^n$ into the space $\mathbb{R}^k$ possesses the properties $\mathcal{P}_1$-$\mathcal{P}_6$. By a harmonic mapping we mean any mapping $u = (u_1, \ldots, u_k)$, whose components u_j are harmonic functions.

Example 1.5. The classes of Examples 1.1 1.2 and 1.4 can be characterised from a unique point of view. Namely, let A be an $(l \times k)$-matrix of homogeneous scalar differential operators of order m with constant coefficients in $\mathbb{R}^n$. Assume moreover that the rank of $A(\xi)$ is equal to k for all $\xi \in \mathbb{R}^n \setminus \{0\}$. We denote by $\mathcal{S}$ the sheaf $U \mapsto \mathrm{Sol}\,(U)$ of solutions to the system $Au = 0$ on open sets in $\mathbb{R}^n$. Then, $\mathcal{S}$ have all the properties $\mathcal{P}_1$-$\mathcal{P}_6$. Indeed, $\mathcal{P}_3$ and $\mathcal{P}_4$ follow from the Stieltjes-Vitali Theorem. All the other properties are obvious.

The latter class plays a special role: the principal aim of this chapter is to construct the foundation of a stability theory for this class.

1.3. **Closeness functionals.** Returning to the general situation, consider a locally bounded mapping $f : U \to \mathbb{R}^k$ of a domain U in the space $\mathbb{R}^n$. For a number $\theta \in (0, 1)$ and an arbitrary ball $B \subset U$, construct the following functional:

$$\mathfrak{d}_{\theta,B}(f, \mathcal{S}) = \begin{cases} \inf\limits_{u \in \mathcal{S}(B)} \Big(\sup\limits_{y \in \theta B} \dfrac{|f(y) - u(y)|}{\operatorname{diam} f(B)} \Big), & \text{if } \operatorname{diam} f(B) \neq 0, \infty; \\ 0, & \text{in the opposite case.} \end{cases} \tag{1.1}$$

where θB stands for the smaller concentric ball whose radius amounts to θ times that of B.

This functional is a measure of how close the mapping f is to the class $\mathcal{S}$ inside the ball θB, in the uniform metric and relative to the linear size of the image of the ball B under f. It can take both finite and $+\infty$ values.

If one assumes that the class $\mathcal{S}$ satisfies $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_4$, then, as a rule, we shall consider that this class contains all constant mappings. Indeed, let $u \in \mathcal{S}(U)$, where $U \subset \mathbb{R}^n$ is open, and express U as a union of an increasing sequence $\{U_\nu\}_{\nu=1,2,\ldots}$ of relatively compact open subsets of U. By condition $\mathcal{P}_1$, there is a sequence $\{c_\nu\}$ of positive numbers, such that, for each $\nu = 1, 2, \ldots$, we have $|u(x)| \leq c_\nu$ for all $x \in U_\nu$. Condition $\mathcal{P}_2$ shows that $u_\nu = (1/\nu c_\nu)u \in \mathcal{S}(U)$, for each $\nu = 1, 2, \ldots$. Since the sequence $\{u_\nu\}$ converges uniformly on compact subsets of U to the mapping $0\colon U \to \mathbb{R}^k$, identically equal to $0 \in \mathbb{R}^k$, property $\mathcal{P}_4$ guarantees that $0 \in \mathcal{S}$. Using property $\mathcal{P}_2$ once again, we reach the desired conclusion.

According to the property of the class $\mathcal{S}$ just proved, it is natural to require the value $\mathfrak{d}_{\theta,B}(f,\mathcal{S})$ to be equal to 0 whenever $\operatorname{diam} f(B) = 0$. The same equality is even more natural when $\operatorname{diam} f(B) = \infty$.

Lemma 1.6. *If the class $\mathcal{S}$ has properties $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_5$, then, for a fixed mapping $f\colon U \to \mathbb{R}^k$ and ball B in U, the value $\mathfrak{d}_{\theta,B}(f,\mathcal{S})$ is a non-decreasing, non-negative function of the parameter θ in the interval $(0,1)$, bounded above by 1.*

Proof. The fact that the function $\phi(\theta) = \mathfrak{d}_{\theta,B}(f,\mathcal{S})$ is non-negative follows from the definition.

Now let $0 < \theta_1 < \theta_2 < 1$. If the diameter of $f(B)$ is either 0 or ∞, then we have $\phi(\theta_1) = \phi(\theta_2) = 0$. If not, consider an arbitrary positive number ϵ and a mapping $u_\epsilon\colon B \to \mathbb{R}^k$, such that

$$\sup_{y \in \theta_2 B} |f(y) - u_\epsilon(y)| < (\phi(\theta_2) + \epsilon) \operatorname{diam} f(B).$$

Since

$$\begin{aligned} \sup_{y \in \theta_2 B} |f(y) - u_\epsilon(y)| &\geq \sup_{y \in \theta_1 B} |f(y) - u_\epsilon(y)| \\ &\geq \phi(\theta_1) \operatorname{diam} f(B) \end{aligned}$$

and ϵ is arbitrary, we get $\phi(\theta_1) \leq \phi(\theta_2)$.

Therefore, we need only to prove the last claim. Let ϵ be an arbitrary positive number. Properties $\mathcal{P}_1$ and $\mathcal{P}_5$ for the class $\mathcal{S}$ show immediately that there exists a bounded mapping $u_1 : B(x,R) \to \mathbb{R}^k$ of class $\mathcal{S}$. Now property $\mathcal{P}_2$ guarantees that there exists, in $\mathcal{S}$, a mapping $u_2 : B \to \mathbb{R}^k$ bounded by the number ϵ. Using property $\mathcal{P}_2$, again, we obtain in $\mathcal{S}(B)$ the mapping $u_3(y) = f(x) + u_2(y)$, $y \in B$, satisfying

$$\frac{|f(y) - u_3(y)|}{\operatorname{diam} f(B)} \leq 1 + \frac{\epsilon}{\operatorname{diam} f(B)}$$

for all $y \in B$. Consequently,

$$\phi(\theta) \leq 1 + \frac{\epsilon}{\operatorname{diam} f(B)},$$

and passing to the limit, as $\epsilon \to 0$, we arrive at the required estimate.

If $\operatorname{diam} f(B)$ is either 0 or ∞, then $\phi(\theta) = 0$. This completes the proof of the lemma. □

Now using the auxiliary functional $\mathfrak{d}_{\theta,B}(f,\mathcal{S})$, we construct a closeness functional (measuring distance between a mapping f and the class $\mathcal{S}$ inside the domain U), setting

$$\mathfrak{d}_\theta(f,\mathcal{S}) = \sup_{B \subset U} \mathfrak{d}_{\theta,B}(f,\mathcal{S}).$$

Let us list some properties of the functional $\mathfrak{d}_\theta$.

Lemma 1.7. *If the class $\mathcal{S}$ satisfies condition $\mathcal{P}_2$, then the functional $\mathfrak{d}_\theta$ is invariant under the simple transformations appearing in condition $\mathcal{P}_2$.*

Proof. This is a direct consequence of the definition of the functional $\mathfrak{d}_\theta$ and of property $\mathcal{P}_2$ for the class $\mathcal{S}$. □

Lemma 1.8. *If the class $\mathcal{S}$ satisfies conditions $\mathcal{P}_1$ and $\mathcal{P}_5$, then, for each $u \in \mathcal{S}$, we have $\mathfrak{d}_\theta(u, \mathcal{S}) = 0$, $\theta \in (0,1)$.*

Proof. The proof is obvious. □

Lemma 1.9. *Suppose the class $\mathcal{S}$ satisfies the conditions $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_4$, $\mathcal{P}_5$, and $\mathcal{P}_6$. Let $\theta \in (0,1)$. Then, for a locally bounded mapping $f : U \to \mathbb{R}^k$ from a domain $U \subset \mathbb{R}^n$, the equality $\mathfrak{d}_\theta(f, \mathcal{S}) = 0$ implies that $f \in \mathcal{S}$.*

Proof. Pick $x \in U$. Consider a ball $B = B(x, R)$ contained, together with its closure, in U. Since f is locally bounded, the restriction of f to B is bounded, too.

Assume first that $\operatorname{diam} f(B) \neq 0$. Since $\mathfrak{d}_\theta(f, \mathcal{S}) = 0$, for each $\nu = 1, 2, \ldots$ there exists a mapping $u_\nu \in \mathcal{S}(B)$ satisfying

$$|f(y) - u_\nu(y)| \leq \frac{1}{\nu} \operatorname{diam} f(B)$$

for all $y \in \theta B$. Consequently, the sequence $u_\nu|_B$, $\nu = 1, 2, \ldots$, converges to $f|_B$ uniformly in the ball θB. By condition $\mathcal{P}_5$, we have $u_\nu|_B \in \mathcal{S}$ for all ν. Then condition $\mathcal{P}_4$ yields $f|_{\theta B} \in \mathcal{S}$.

Now let $\operatorname{diam} f(B) = 0$, i.e., the restriction of f to B be a constant mapping. As described above, we see that $f|_B \in \mathcal{S}$. Then condition $\mathcal{P}_5$ shows that $f|_{\theta B} \in \mathcal{S}$.

Therefore, we have exhibited, in both cases, a neighborhood θB of the centre of B, such that the restriction $f|_{\theta B}$ is of class $\mathcal{S}$. Finally, condition $\mathcal{P}_6$ shows that $f \in \mathcal{S}$, and the proof is complete. □

Lemma 1.6 enables us to introduce the functional measuring the closeness of f to the class $\mathcal{S}$ which is basic to our theory. This functional is equivalent, in a certain sense, to each of the functionals $\mathfrak{d}_\theta$, $0 < \theta < 1$, and is independent of the actual values of the parameter θ.

Assume that the class $\mathcal{S}$ enjoys properties $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_5$. Proceeding in the same way as we did for the functional $\mathfrak{d}_\theta$, we construct the new closeness functionals in two steps. First, for a locally bounded mapping $f : U \to \mathbb{R}^k$ of a domain $U \subset \mathbb{R}^n$ and an arbitrary ball $B \subset U$, we set

$$\mathfrak{d}_B(f, \mathcal{S}) = \int_0^1 \mathfrak{d}_{\theta,B}(f, \mathcal{S})\, d\theta$$

(the existence of the integral is guaranteed by Lemma 1.6). Secondly, using this auxiliary functional, we construct the functional

$$\mathfrak{d}(f, \mathcal{S}) = \sup_{B \subset U} \mathfrak{d}_B(f, \mathcal{S}). \tag{1.2}$$

The functional $\mathfrak{d}_B$ measures how close f is to the class $\mathcal{S}$ inside the ball $B \subset U$, while $\mathfrak{d}$ does the same for all such balls.

As a direct consequence of its definition, the functional $\mathfrak{d}$ has properties similar to those of functional $\mathfrak{d}_\theta$, given in Lemmas 1.7, 1.8 and 1.9. Moreover, Lemma 1.6 shows that the values of $\mathfrak{d}$ are bounded by 1.

The following theorem of [Kop82] gives the asymptotic equivalence of the functional $\mathfrak{d}$ with each functional $\mathfrak{d}_\theta$, where $\theta \in (0,1)$.

Theorem 1.10. *Let class $\mathcal{S}$ satisfy conditions $\mathcal{P}_1 - \mathcal{P}_6$. Then, for each pair of real numbers $\varepsilon > 0$ and $\theta \in (0,1)$, there exists a positive number $\delta = \delta(\varepsilon, \theta)$ such that, for each locally bounded mapping $f : U \to \mathbb{R}^k$ from a domain U in $\mathbb{R}^n$, we have:*

1) if $\mathfrak{d}(f, \mathcal{S}) \leq \delta$, then $\mathfrak{d}_\theta(f, \mathcal{S}) \leq \varepsilon$;

2) if $\mathfrak{d}_\theta(f,\mathcal{S}) \le \delta$*, then* $\mathfrak{d}(f,\mathcal{S}) \le \varepsilon$*.*

We understand the asymptotic equivalence of closeness functionals to be precisely the fact reflected in this theorem: if the values of any of these functionals at f is small, then so is the value of the other functional at f.

Proof. The first statement of the theorem is an immediate consequence of the definition of $\mathfrak{d}$. Indeed,

$$\begin{aligned}\mathfrak{d}_B(f,\mathcal{S}) &\ge \int_\theta^1 \mathfrak{d}_{t,B}(f,\mathcal{S})\,dt \\ &\ge (1-\theta)\,\mathfrak{d}_{\theta,B}(f,\mathcal{S}) \qquad (1.3)\end{aligned}$$

for each ball $B \subset U$. The last inequality in (1.2) follows from the fact that $\mathfrak{d}_\theta(f,\mathcal{S})$ is a monotonic function of θ (cf. Lemma 1.6). Since B is arbitrary, we obtain

$$\mathfrak{d}_\theta(f,\mathcal{S}) \le \frac{1}{1-\theta}\,\mathfrak{d}(f,\mathcal{S}),$$

which gives 1).

The proof of the second part of the theorem requires most effort than the proof of the first part. We refer the reader to the original paper [Kop82]. □

To conclude this subsection, we consider yet another closeness functional $\mathcal{D}$ defined by

$$\mathfrak{D}(f,\mathcal{S}) = \sup_{x\subset U}\Big(\limsup_{R\to 0}\,\mathfrak{d}_{B(x,R)}(f,\mathcal{S})\Big), \qquad (1.4)$$

for each locally bounded mapping $f\colon U \to \mathbb{R}^k$ of a domain $U \subset \mathbb{R}^n$.

The functional $\mathfrak{D}$ measures how close f is to the class $\mathcal{S}$ in the uniform norm and in each of the infinitesimal balls contained in the domain of definition of f.

Remark 1.11. It follows immediately from (1.2) and (1.4) that $\mathfrak{D}(f,\mathcal{S}) \le \mathfrak{d}(f,\mathcal{S})$, for each locally bounded mapping $f\colon U \to \mathbb{R}^k$ from a domain $U \subset \mathbb{R}^n$.

1.4. **Stability.** Suppose $\mathcal{S}$ is a class of mappings from open sets in the space $\mathbb{R}^n$ into the space $\mathbb{R}^k$, satisfying conditions $\mathcal{P}_1$-$\mathcal{P}_6$ of Subsection 1.1. Further, let ε be a non-negative real number.

Definition 1.12. A locally bounded mapping $f\colon U \to \mathbb{R}^k$ from a domain U of the space $\mathbb{R}^n$ is said to be globally ε-close to the class $\mathcal{S}$ if $\mathfrak{d}(f,\mathcal{S}) \le \varepsilon$.

We obtain a concept of local ε-closeness to the class $\mathcal{S}$ by replacing the functional of global closeness $\mathfrak{d}$ with the functional of local closeness $\mathfrak{D}$.

Remark 1.11 shows that if f is globally ε-close to $\mathcal{S}$, then it is locally ε-close to this class. This gives rise to the following definition which is basic for our theory.

Definition 1.13. Let $\mathcal{C}$ be some class of locally bounded mappings from domains in the space $\mathbb{R}^n$ into the space $\mathbb{R}^k$. The class $\mathcal{S}$ is called stable relative to the class $\mathcal{C}$ if there exists a non-negative function $\delta = \delta(\varepsilon)$, defined on some interval $[0,\varepsilon_0)$, such that:

1) $\delta(\varepsilon) \to \delta(0) = 0$ as $\varepsilon \to 0$;

2) if $f \in \mathcal{C}$ is locally ε-close to $\mathcal{S}$, then f is globally $\delta(\varepsilon)$-close to $\mathcal{S}$, for each $\varepsilon \in [0,\varepsilon_0)$.

Taking into account the asymptotic character of the notion of stability, we need not specify the particular semiinterval $[0,\varepsilon_0)$ where ε takes its values.

In terms of the notions introduced above, the fundamental problem in the stability theory, that we are discussing, can be formulated as follows: Considering a class $\mathcal{S}$, satisfying $\mathcal{P}_1$-$\mathcal{P}_6$, and a class $\mathcal{C}$ rich enough, determine whether the class $\mathcal{S}$ is stable relative to the class $\mathcal{C}$.

When giving an affirmative answer, we get the strongest form of the theorem concerning stability when $\mathcal{C}$ is the class of all locally bounded mappings from domains in the space $\mathbb{R}^n$ into the space $\mathbb{R}^k$. In this case, we shall simply say that the class $\mathcal{S}$ is stable.

1.5. **Problems of the theory of stability.** Now we shall clarify how the basic problem of the stability theory can be answered for the classes considered in Examples 1.1-1.5.

The role of the class $\mathcal{C}$, relative to whom we establish the stability of the particular classes of mappings, is played by the class $W^{1,n+0}_{\mathrm{loc}}$ of mappings $f : U \to \mathbb{R}^k$ from open sets in $\mathbb{R}^n$ into the space $\mathbb{R}^k$, having first-order generalised derivatives in U locally summable at a power $q > n$. [1]

Since each mapping of class $W^{1,n+0}_{\mathrm{loc}}$ becomes continuous, when one changes, if necessary, its values on a set of measure zero, we shall assume from now on that the mappings of class $W^{1,n+0}_{\mathrm{loc}}$ are continuous.

Theorem 1.14. *The class of locally constant mappings from open sets in $\mathbb{R}^n$ into the space $\mathbb{R}^k$ is stable relative to the class $W^{1,n+0}_{\mathrm{loc}}$.*

Proof. Denote the class in question by $\mathcal{S}$. Let $f : U \to \mathbb{R}^k$ be a locally bounded mapping of a domain $U \subset \mathbb{R}^n$, with $\mathfrak{D}(f,\mathcal{S}) < 1/16$, and let $\epsilon > 0$ satisfy the inequality $\mathfrak{D}(f,\mathcal{S}) + \epsilon < 1/16$.

Pick an arbitrary point $x^0 \in U$. By the definition of $\mathfrak{D}$, there is a positive number R such that the ball $B(x^0,R)$ lies, together with its closure, in U and $\mathfrak{d}_{B(x^0,r)}(f,\mathcal{S}) < \mathfrak{D}(f,\mathcal{S}) + \epsilon/2$ for all $r \in (0,R]$. Inequality (1.3) implies

$$\begin{aligned} \mathfrak{d}_{\frac{1}{2},B(x^0,r)}(f,\mathcal{S}) &< 2\Big(\mathfrak{D}(f,\mathcal{S}) + \frac{\epsilon}{2}\Big) \\ &< \frac{1}{8} - \epsilon, \end{aligned}$$

for each $r \in (0,R]$.

There are two possibilities: 1) there exists a number $r \in (0,R]$ such that $\operatorname{diam} f(B(x^0,r)) = 0$, or 2) $\operatorname{diam} f(B(x^0,r)) \neq 0$ for all $r \in (0,R]$. In case 1), the restriction $f|_{B(x^0,r)}$ is constant, and so the differential $df(x^0)$ of the mapping f at the point x^0 is identically equal to $0 \in \mathbb{R}^k$. We show now that the same conclusion holds for case 2).

Indeed, let x be a point in the ball $B(x^0,R/2)$, different from x^0. Choose a natural number ν_x such that

$$2^{-(\nu_x+1)}\, R < |x - x^0| \le 2^{-\nu_x}\, R. \tag{1.5}$$

Since the restriction $f|_{B(x^0,R)}$ is bounded, the definition of $\mathfrak{d}_{\theta,B(x^0,2^{-\nu}R)}(f,\mathcal{S})$ shows (cf. (1.1)) that for each $\nu = 1,2,\ldots$ there exists a mapping $u_\nu \in \mathcal{S}(B(x^0,2^{-\nu}R))$

[1] In other words, $W^{1,n+0}_{\mathrm{loc}} = \cup W^{1,q}_{\mathrm{loc}}(U,\mathbb{R}^k)$, where the union is taken over all open sets U in $\mathbb{R}^n$ and all values $q > n$ of the parameter q.

such that

$$|f(y)-u_\nu(y)| < \Big(\mathfrak{d}_{\frac{1}{2},B(x^0,2^{-\nu}R)}(f,\mathcal{S}) + \frac{\epsilon}{2}\Big)\operatorname{diam} f(B(x^0,2^{-\nu}R))$$

for all $y \in B(x^0, 2^{-(\nu+1)}R)$.

We now use the fact that the mapping u_ν is constant. From this we obtain the inequality

$$|f(y)-f(x^0)| < 2\Big(\mathfrak{d}_{\frac{1}{2},B(x^0,2^{-\nu}R)}(f,\mathcal{S}) + \frac{\epsilon}{2}\Big)\operatorname{diam} f(B(x^0,2^{-\nu}R)) \qquad (1.6)$$

holding for the same values of y. According the choice of ϵ, we have $\epsilon < 1/16$ and $2\Big(\mathfrak{d}_{1/2,B(x^0,2^{-\nu}R)}(f,\mathcal{S}) + \epsilon/2\Big) < 1/4 - \epsilon$. Since

$$\operatorname{diam} f(B(x^0,2^{-\nu}R)) \le 2 \sup_{y\in B(x^0,2^{-\nu}R)} |f(y)-f(x^0)|,$$

estimate (1.6) yields

$$|f(y)-f(x^0)| < \Big(\frac{1}{2} - 2\epsilon\Big) \sup_{y\in B(x^0,2^{-\nu}R)} |f(y)-f(x^0)|$$

for all $y \in B(x^0, 2^{-(\nu+1)}R)$.

Continuing, we get

$$\sup_{y\in B(x^0,2^{-\nu}R)} |f(y)-f(x^0)| < \Big(\frac{1}{2} - 2\epsilon\Big)^{\nu} \sup_{y\in B(x^0,R)} |f(y)-f(x^0)|, \qquad (1.7)$$

and, combining (1.5) and (1.7),

$$|f(x)-f(x^0)| < \Big(\frac{1}{2} - 2\epsilon\Big)^{\nu_x} \sup_{y\in B(x^0,R)} |f(y)-f(x^0)|.$$

Again, using inequality (1.5), we find that

$$\begin{aligned}\Big(\frac{1}{2} - 2\epsilon\Big)^{\nu_x} &= (1-4\epsilon)^{\nu_x}\, 2^{-\nu_x} \\ &< 2\,\frac{1}{R}\,|x-x^0|\,(1-4\varepsilon)^{\frac{\log(R|x-x^0|^{-1})}{\log 2}-1}.\end{aligned}$$

Consequently,

$$\frac{|f(x)-f(x^0)|}{|x-x^0|} < 2\,\frac{1}{R}\,(1-4\varepsilon)^{\frac{\log(R|x-x^0|^{-1})}{\log 2}-1} \sup_{y\in B(x^0,R)} |f(y)-f(x^0)|.$$

As the right-hand side of the last inequality tends to zero as $x \to x^0$, we see that $df(x^0)$ vanishes.

Since x^0 was chosen arbitrarily, we conclude that the mapping f is locally constant. Therefore, for each $\varepsilon \in (0, 1/16]$, the class of mappings $f \in W^{1,n+0}_{\mathrm{loc}}$ which are ε-close to the class $\mathcal{S}$ coincides with $\mathcal{S}$. Hence, the class $\mathcal{S}$ meets Definition 1.13 with $\delta(\varepsilon) \equiv 0$, which, in turn, implies that $\mathcal{S}$ is stable relative to $W^{1,n+0}_{\mathrm{loc}}$. This proves the theorem. $\square$

Using Theorem 1.10, it is not hard to see that Theorems 0.1 and 0.2 are equivalent to the following two statements:

1) The class $\mathcal{S}$ of holomorphic mappings in the plane is stable relative to the class C^1_{loc} of all continuously differentiable mappings from domains in $\mathbb{C}$ into $\mathbb{C}$.

2) The mappings $f \in C^1_{\mathrm{loc}}$ which are locally close to $\mathcal{S}$ can be characterised as solutions of Beltrami's systems with "small" coefficients Q.

As mentioned, Kopylov [Kop82] carried over these results to the higher-dimensional case in the following strengthened form.

Theorem 1.15. *The class of holomorphic mappings from open sets in $\mathbb{C}^n$ into the space $\mathbb{C}^k$ is stable relative to the class $W^{1,2n+0}_{\mathrm{loc}}$.*

In [Kop82] it is also shown that the class of locally conformal mappings from open sets in $\mathbb{R}^n$, $n > 2$, into the same space is stable relative to the class $W^{1,n+0}_{\mathrm{loc}}$.

On the other hand, the class of Example 1.4 is not stable even in the class of analytic mappings. More precisely, the following holds true.

Theorem 1.16. *Suppose $n > 1$. Then, the class $\mathcal{S}$ of harmonic mappings from open sets in $\mathbb{R}^n$ into the space $\mathbb{R}^k$ is not stable relative to the class $\mathcal{A}$ of real analytic mappings.*

Proof. Let us consider the mapping $f : \mathbb{R}^n \to \mathbb{R}^k$, whose first component is $f_1(x_1, \ldots, x_n) = \arctan x_1$ and whose other components are all zero. Taking into account that the differential of a mapping is linear, and thus harmonic, we get easily that $\mathfrak{D}(f, \mathcal{S}) = 0$ (see [Kop82] for more details). Since f is not a harmonic mapping, Lemma 1.9 and the definition of $\mathfrak{d}$ show that $\mathfrak{d}(f, \mathcal{S}) > 0$. Hence the desired conclusion follows. □

Theorem 1.16 suggests that, when discussing the stability theory for the sheaf of solutions of an elliptic system, one should require the order of the system to be one.

1.6. **Liouville's theorem.** For mappings $f : \mathbb{R}^n \to \mathbb{R}^k$ globally close to a class $\mathcal{S}$ with properties $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_5$, there is an analog of the classical Liouville Theorem. The proof of this is based on some technical lemma which is of independent interest. It gives us an estimate of the modulus of continuity of f in a ball B, i.e.,

$$m_B(f)\,(\delta) = \sup_{\substack{x,y \in B \\ |y-x|<\delta}} |u(y) - u(x)|,$$

via the closeness functional $\mathfrak{d}_\theta(f, \mathcal{S})$ and a measure of equicontinuity of uniformly bounded families in $\mathcal{S}$.

Lemma 1.17. *Suppose that the class $\mathcal{S}$ meets the conditions $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_5$. Let $f : B(x^0, R) \to \mathbb{R}^k$ be a bounded mapping from a ball in $\mathbb{R}^n$, satisfying $\mathfrak{d}_\theta(f, \mathcal{S}) < 1/2$ for some $\theta \in (0, 1)$. Then, for each numbers $t \in (0, 1)$ and $\delta \in (0, (1-t)\varepsilon\theta R)$, one has*

$$\begin{aligned} & m_{B(x^0,tR)}(f)\,(\delta) \\ & \quad \le ((1+2\mathfrak{d}_\theta(f, \mathcal{S})) \sup_{\substack{u \in \mathcal{S}(B(0,1)) \\ |u| \le 1}} m_{B(0,1/2)}(u)\,(\varepsilon) + 2\mathfrak{d}_\theta(f, \mathcal{S}))^\nu \operatorname{diam} f(B(x^0, R)), \end{aligned}$$

where

$$\nu = \frac{\log(1 + 2\frac{1}{\delta}(1-t)(1-(1/2)\varepsilon\theta)R)}{-\log((1/2)\varepsilon\theta)} - 1.$$

Proof. Cf. Theorem 4 in [Kop82]. □

The lemma becomes more interesting when we realise that the number $\varepsilon \in (0, 1/2)$ can be chosen so that the expression under the power ν is less that 1. Indeed, condition $\mathcal{P}_3$ guarantees that, when $\varepsilon \to 0$, the function $m_{B(0,1/2)}(u)\,(\varepsilon)$ is infinitely small, uniformly in $u \in \mathcal{S}(B(0,1))$ with $|u| \le 1$, and $2\,\mathfrak{d}_\theta(f,\mathcal{S}) < 1$ by hypothesis.

Theorem 1.18. *Assume that a given class $\mathcal{S}$ of mappings from domains in the space $\mathbb{R}^n$ into the space $\mathbb{R}^k$ has properties $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_5$. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a bounded mapping such that $\mathfrak{d}_\theta(f,\mathcal{S}) < 1/2$ for some number $\theta \in (0,1]$. Then f is a constant mapping.*

Proof. Since $\mathfrak{d}_\theta(f,\mathcal{S}) \le \mathfrak{d}_1(f,\mathcal{S})$, for each number $\theta \in (0,1)$ and each mapping $f : U \to \mathbb{R}^k$, it suffices to consider the case when $0 < \theta < 1$. Suppose that $|f(x)| \le C$ for all $x \in \mathbb{R}^n$. Let $t = 1/2$ and let the number $\varepsilon \in (0,1/2)$ be so chosen that

$$\begin{aligned} \Delta &= (1+2\mathfrak{d}_\theta(f,\mathcal{S})) \sup_{\substack{u\in\mathcal{S}(B(0,1))\\ |u|\le 1}} m_{B(0,1/2)}(u)\,(\varepsilon) + 2\mathfrak{d}_\theta(f,\mathcal{S}) \\ &< 1. \end{aligned}$$

Given two arbitrary points $x, y \in \mathbb{R}^n$, fix a natural number $N_0 > 2(1/\varepsilon\theta)\,|y-x|$ and consider the sequence of balls $B(x,N)$, $N = N_0, N_0+1, \ldots$. Obviously, each of the balls $B(x,(1/2)N)$, $N \ge N_0$, contains the point y. Applying Lemma 1.17 to the restriction of the mapping f to $B(x,N)$, for $N \ge N_0$, and the values t and ε chosen above, we get

$$\begin{aligned} |f(y)-f(x)| &\le m_{B(x,(1/2)N)}(f)\,(|y-x|) \\ &\le \Delta^\nu \operatorname{diam} f(B(x,N)), \end{aligned} \tag{1.8}$$

where

$$\nu = \frac{\log(1+|y-x|^{-1}(1-(1/2)\varepsilon\theta)N)}{-\log((1/2)\varepsilon\theta)} - 1.$$

However, we have $\operatorname{diam} f(B(x,N)) \le 2C$ for all N. Consequently, as $N \to \infty$, the right-hand side of inequality (1.8) tends to 0, and hence $f(x) = f(y)$, as required. □

Corollary 1.19. *Suppose $\mathcal{S}$ meets conditions $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_5$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a bounded mapping of class $\mathcal{S}$, then f is constant.*

Corollary 1.20. *If one replaces $\mathfrak{d}_\theta$ by $\mathfrak{d}$ in the statement of Theorem 1.18, then the theorem remains valid.*

Proof. Indeed, the inequalities $\mathfrak{d}(f,\mathcal{S}) < 1/2$ and (1.3) imply the existence of a number $\theta \in (0,1)$, such that $\mathfrak{d}_\theta(f,\mathcal{S}) < 1/2$, and we still satisfy the hypothesis of Theorem 1.18. □

Note that the constant 1/2 in Corollary 1.20 is sharp. More precisely, given any pair of natural numbers n and k, there exists a bounded mapping $f : \mathbb{R}^n \to \mathbb{R}^k$, such that f is not constant while $\mathfrak{d}(f,\mathcal{S}) = 1/2$ for each class $\mathcal{S}$ of mappings from domains in $\mathbb{R}^n$ into the space $\mathbb{R}^k$ with properties $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_5$ (cf. [Bez83]).

2. First order elliptic systems

We now turn to the case when $\mathcal{S}$ is the sheaf of solutions of a first order overdetermined elliptic system $Au = 0$ over $\mathbb{R}^n$. We write $\mathrm{Sol}(A)$ for this sheaf, or simply Sol when no confusion can arise. One may assume, by separating the real and imaginary parts of solutions if necessary, that the coefficients of the differential operator A are real-valued. Moreover, in order that the sheaf Sol may inherit the property $\mathcal{P}_2$, it is necessary to require A to have constant coefficients and to be homogeneous. Thus,

$$A(\partial) = \sum_{j=1}^{n} A_j \partial_j,$$

where A_j, $j = 1, \ldots, n$, are $(l \times k)$-matrices of real numbers, and $\partial_j = \partial / \partial x_j$. The ellipticity condition implies, in particular, that $l \geq k$.

We begin with general results on such systems. For more details, the reader may consult e.g. the book [Tar90] and elsewhere.

2.1. **Cauchy's theorem.** For holomorphic functions of a single variable, the following result is known as Cauchy's theorem. For the convenience of references, we retain this designation also in the case of first order overdetermined elliptic systems.

Lemma 2.1. *Let $\mathcal{D} \subset\subset \mathbb{R}^n$ be a domain with piecewise smooth boundary. Then, for each solution $u \in \mathrm{Sol}\,(\mathcal{D})$ continuous up to the boundary of $\mathcal{D}$, we have*

$$\int_{\partial \mathcal{D}} A(\nu(y)) u(y)\, ds(y) = 0. \tag{2.1}$$

Recall that $\nu(y)$ stands for the unit outward normal vector to the boundary of $\mathcal{D}$ at a point y.

Proof. Since the (unique) Green operator for the first order differential operator A is given by

$$G_A(g, u) = \sum_{j=1}^{n} g\, A_j u \, \star dx_j,$$

where $\star$ stands for the Hodge star operator, and since $\star dx_j \,|_{\partial \mathcal{D}} = \nu_j(x)\, ds(x)$, equality (2.1) follows from Stokes' formula. □

2.2. **Morera's theorem.** The following auxiliary result is an analog of the classical Morera theorem for holomorphic functions of a single variable.

Lemma 2.2. *Let $u \in C_{\mathrm{loc}}(U)^k$, where U is an open set in $\mathbb{R}^n$. Then, in order that u satisfy $Au = 0$ in U, it is sufficient that*

$$\int_{\partial B} A(\nu(y)) u(y)\, ds(y) = 0 \tag{2.2}$$

for each ball $B \subset\subset U$.

Note that the necessity of condition (2.2) follows from Cauchy's theorem (cf. Lemma 2.1).

Proof. We make use of standard regularisation

$$R^{(\varepsilon)} u = \frac{1}{\varepsilon^n}\, \omega\Big(\frac{\cdot}{\varepsilon}\Big) * u$$

in $\mathbb{R}^n$, where $\omega \in C^\infty_{\mathrm{comp}}(B(0,1))$ is a non-negative function depending only on $|x|$ and normalised by the condition $\int \omega(x)\, dx = 1$. For each ball B lying, along with its ε-neighborhood, in U, we have

$$\begin{aligned}\int_{\partial B} A(\nu(x))\, R^{(\varepsilon)}u(x)\, ds(x) &= \sum_{j=1}^{n} \int_{\partial B} A_j \Big(\int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} \omega\Big(\frac{z}{\varepsilon}\Big) u(x-z)dz \Big) \star dx_j \\ &= \int_{\mathbb{R}^n} \Big(\int_{\partial(B-z)} A(\nu(y))u(y)ds(y) \Big) \frac{1}{\varepsilon^n} \omega\Big(\frac{z}{\varepsilon}\Big) dz \\ &= 0,\end{aligned}$$

the last equality being a consequence of condition (2.2). Since $R^{(\varepsilon)}u$ is smooth, it follows that $A\, R^{(\varepsilon)}u = 0$ at each point of U, whose distance from the boundary is at least ε.

On the other hand, when $\varepsilon \to 0$, we have $R^{(\varepsilon)}u \to u$, uniformly on compact subsets of U. By the Stieltjes-Vitali Theorem, we deduce that $Au = 0$ in U, as desired. $\square$

2.3. **Cauchy's formula.** Let Φ be the standard left fundamental solution of convolution type for the differential operator A. We have

$$\Phi(x) = \int_{\mathbb{S}^{n-1}} w(\langle x, \xi \rangle)\, A_{\text{left}}^{-1}(\xi)\, ds(\xi), \tag{2.3}$$

where ds is the standard area form on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ and

$$w(\theta) = \begin{cases} \dfrac{(-1)^{(n/2)+1}(n-2)!}{(2\pi)^n} \dfrac{1}{\theta^{n-1}}, & \text{if } n \text{ is even,} \\ \dfrac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}}\, \delta^{(n-2)}(\theta), & \text{if } n \text{ is odd} \end{cases}$$

(cf. [Shi84]).

Lemma 2.3. *Suppose $\mathcal{D} \subset \mathbb{R}^n$ is a bounded domain with piecewise smooth boundary. Then, for each solution $u \in \mathrm{Sol}\,(\mathcal{D})$ continuous up to the boundary of $\mathcal{D}$, we have*

$$-\int_{\partial\mathcal{D}} \Phi(x-y)\, A(\nu(y))u(y)\, ds(y) = \begin{cases} u(x), & \text{if} \quad x \in \mathcal{D}, \\ 0, & \text{if} \quad x \in \mathbb{R}^n \setminus \overline{\mathcal{D}}. \end{cases} \tag{2.4}$$

Proof. This is a very particular case of Green's formula [Tar95, (8.3.16)], for $B_0 = I_k$ and $C_0 = (A(\nu))^T$. $\square$

2.4. **Some singular integral operators.** From (2.3) it follows that each derivative $(\partial/\partial x_j)\Phi(x)$, $j = 1, \ldots, n$, is homogeneous of degree $-n$ in $\mathbb{R}^n \setminus 0$.

Lemma 2.4. *For $j = 1, \ldots, n$, the Calderon-Zygmund kernel $(\partial/\partial x_j)\,\Phi\,(x-y)$ is regular, i.e.,*

$$\int_{\mathbb{S}^{n-1}} (\partial/\partial x_j)\, \Phi\,(\xi)\, ds(\xi) = 0.$$

Proof. It is sufficient to use (2.3). $\square$

Lemma 2.4 makes it obvious that the principal value integral in our next lemma exists at each Lebesgue point of the section f.

Lemma 2.5. *Suppose $f \in L^q_{\mathrm{comp}}(\mathbb{R}^n)^l$, where $q > 1$. Then, for each $j = 1, \ldots, n$, we have*

$$\frac{\partial}{\partial x_j} \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} \Phi(x-y) f(y) dy + \Big(\int_{\mathbb{S}^{n-1}} \Phi(\xi) \star d\xi_j \Big) f(x) \tag{2.5}$$

almost everywhere on $\mathbb{R}^n$.

Proof. Since smooth functions of compact support are dense in the Lebesgue spaces $L^q_{\mathrm{comp}}(\mathbb{R}^n)$, $q < \infty$, and these spaces behave properly under action of pseudodifferential operators, it suffices to prove (2.5) for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)^l$.

For such an f, we have $(\partial/\partial x_j) \Phi * f = \Phi * (\partial/\partial x_j) f$, so the left-hand side of (2.5) is equal to the integral

$$\int_{\mathbb{R}^n} \Phi(x-y)\, \partial_j f(y)\, dy.$$

Moreover, since the kernel $\Phi(x - y)$ is locally summable in $\mathbb{R}^n$, for fixed x, we see that

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x-y)\, \partial_j f(y)\, dy \; &= \; \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \Phi(x-y)\, \partial_j f(y)\, dy \\ &= \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{\partial}{\partial x_j} \Phi(x-y)\, f(y)\, dy - \int_{\partial B(x,\varepsilon)} \Phi(x-y) f(y) \star dy_j, \end{aligned} \tag{2.6}$$

the second equality being a consequence of Stokes' formula.

In the integral over the sphere $\partial B(x, \varepsilon)$, we write $f(y) = (f(y) - f(x)) + f(x)$ and change the variables by $x - y = \varepsilon \xi$, where $\xi \in \mathbb{S}^{n-1}$. As $\det dy(\xi) = (-\varepsilon)^n$ and $\Phi(x)$ is homogeneous of degree $1 - n$, we arrive at the equality

$$\begin{aligned} &\int_{\partial B(x,\varepsilon)} \Phi(x-y) f(y) \star dy_j \\ &\quad = - \int_{\mathbb{S}^{n-1}} \Phi(\xi)\, (f(x-\varepsilon\xi) - f(x)) \star d\xi_j - \Big(\int_{\mathbb{S}^{n-1}} \Phi(\xi) \star d\xi_j \Big) f(x). \end{aligned}$$

The first integral in the right-hand side is $O(\varepsilon)$, for f is smooth. Thus, letting $\varepsilon \to 0$ in (2.6), we get (2.5), as required. □

2.5. **The ellipticity is a necessary condition for the stability.** The ellipticity condition on the differential operator A in question is in fact necessary if we want to retain the property $\mathcal{P}_3$ for the sheaf of C^∞ solutions to the system $Au = 0$.

Theorem 2.6. *Suppose A is an $(l \times k)$ -matrix of homogeneous scalar differential operators of order m with constant coefficients in $\mathbb{R}^n$. If the sheaf of C^∞ solutions to the system $Au = 0$ has property $\mathcal{P}_3$, then A has injective symbol.*

Proof. Suppose, contrary to our claim, that A is not a differential operator with injective symbol. Then, there exists a vector ξ on the unit sphere in $\mathbb{R}^n$, such that the rank of the matrix $A(\xi)$ over $\mathbb{R}$ is less than k. This just amounts to saying that there is a non-zero vector $u_0 \in \mathbb{R}^k$ satisfying $A(\xi)u_0 = 0$.

Let us consider the family

$$u_\nu = \cos\left(\nu \langle x, \xi \rangle\right) u_0, \tag{2.7}$$

where $\nu = 1, 2, \ldots$, in $\mathcal{E}(\mathbb{R}^n)^k$. Since A is homogeneous, we get

$$\begin{aligned} Au_\nu &= -\sin(\nu\langle x,\xi\rangle)\, A(\nu\xi)u_0, \\ &= -\nu^m \sin(\nu\langle x,\xi\rangle)\, A(\xi)u_0 \\ &= 0, \end{aligned}$$

for each $\nu = 1, 2, \ldots$. Thus, (2.7) is a family of C^∞ solutions to the system $Au = 0$ in $\mathbb{R}^n$.

It is obvious that the family (2.7) is uniformly bounded in $\mathbb{R}^n$. However, when restricted to any ball in $\mathbb{R}^n$, this family is not equicontinuous, since otherwise some subsequence of $\{u_\nu\}$ would converge uniformly on the ball, which is impossible. This contradicts property $\mathcal{P}_3$, and the proof is complete. □

3. Beltrami equation

The ellipticity condition for a first order differential operator A is known to impose hard restrictions on the symbol of A (cf. [Sol63]). From this we conclude that the solutions of the homogeneous system $Au = 0$ may be regarded as analogues of holomorphic functions of one or several variables. As is already described above, the mappings locally close to holomorphic mappings in the plane are solutions of Beltrami systems

$$(\partial/\partial\bar{z})f = Q(z)\,(\partial/\partial z)f, \tag{3.1}$$

with "small" coefficients $Q(z)$. For holomorphic functions of several variables such a system is studied in [Kop82]. In the general case there is also a system of differential equations which is related to the system $Au = 0$ in much the same way as system (3.1) is related to the Cauchy-Riemann system in the plane.

3.1. A decomposition of the differential.

If some equation of the homogeneous system $Au = 0$ is a linear combination of others then it can be removed from the system without changing the space of solutions to this system. The new system obtained in this way still remains to be overdetermined elliptic. Hence, there is no restriction of generality in assuming that all equations in the system $Au = 0$ are linearly independent. It is easy to see that algebraically this condition just amounts to saying that the rank of the matrix of coefficients of A is equal to the number of equations, i.e.

$$\operatorname{rank}(A_1, \ldots, A_n) = l. \tag{3.2}$$

Lemma 3.1. *Suppose that $A_1, \ldots, A_n$ are $(l \times k)$-matrices. In order that there might exist $(k \times l)$-matrices $C_1, \ldots, C_n$, such that $A_1C_1 + \ldots + A_nC_n = E_l$ (the identity matrix of size $l \times l$) it is necessary and sufficient that condition (3.2) would be fulfilled.*

If the matrices $A_1, \ldots, A_n$ are square and commuting, the lemma can be interpreted within the framework of the weak Nullstellensatz of Hilbert which establishes a fundamental relationship between geometry and algebra, cf. for instance [Eis99] and elsewhere.

Proof. We are thus looking for $(k \times l)$-matrices $X^1, \ldots, X^n$ of real numbers satisfying $A_1X^1 + \ldots + A_nX^n = E_l$. Write A_j^i for the ith row of the matrix A_j and X_m^j for the mth column of the matrix X^j, where $i, m = 1, \ldots, l$. Then the matrix equation $A_1 X^1 + \ldots + A_n X^n = E_l$ reduces immediately to the system of l^2 linear equations

$$
\begin{array}{ccccccccc}
A_1^1 X_i^1 & + & A_2^1 X_i^2 & + & \ldots & + & A_n^1 X_i^n & = & 0, \\
 & & & & \ldots & & & & \\
A_1^i X_i^1 & + & A_2^i X_i^2 & + & \ldots & + & A_n^i X_i^n & = & 1, \\
 & & & & \ldots & & & & \\
A_1^l X_i^1 & + & A_2^l X_i^2 & + & \ldots & + & A_n^l X_i^n & = & 0
\end{array}
\tag{3.3}
$$

for $l\,n$ unknown columns $X_i^1, \ldots, X_i^n$, where $i = 1, \ldots, l$. The matrix of each inhomogeneous system (3.3) is the block matrix $(A_1, \ldots, A_n)$. The solvability of (3.3) for $i = 1, \ldots, l$ just amounts to the fact that the rank of $(A_1, \ldots, A_n)$ is l, as desired. □

Example 3.2. If $Au := du$ is the gradient operator then $A_j = e_j$, for $j = 1, \ldots, n$, where e_j is the j th vector of the canonical basis of $\mathbb{R}^n$. We specify e_j within n-columns, and so the matrix $(A_1, \ldots, A_n)$ is equal to the identity matrix E_n. Hence, the inhomogeneous linear system $A_1 C_1 + \ldots + A_n C_n = E_n$ has unique solution $C_j = e_j^T$, where $j = 1, \ldots, n$.

In the general case the solution of the system $A_1 C_1 + \ldots + A_n C_n = E_l$ is by no means unique, for the number of equations is l^2 while the number of unknowns is $l\,n\,k$.

Assume that $f \in W^{1,n+0}_{\mathrm{loc}}(U)^k$ is a mapping from a domain U in $\mathbb{R}^n$ into the space $\mathbb{R}^k$. It is well-known that such a mapping is differentiable almost everywhere in U (cf. [Ste70, p.286]).

Let $x \in U$ be a point of differentiability of the mapping f and let $df(x)$ denote the differential of f at x. For a fixed x, the differential $df(x)$ is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$, satisfying $|(f(y) - f(x)) - df(x)\,(y - x)| = o(|y - x|)$ as $y \to x$.

To each linear mapping $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)$ there corresponds a unique $(k \times n)$-matrix which represents T in the coordinates with respect to the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^k$. One can identify this matrix with a multivector in $\mathbb{R}^{kn}$ which is formed from the columns of the matrix.

Since A is an overdetermined elliptic operator, we deduce that $l \geq k$ and the matrix

$$A(\xi) = \sum_{j=1}^{n} A_j \xi_j$$

possesses a left inverse matrix $A^{-1}_{\mathrm{left}}(\xi)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. By Lemma 3.1, there are $(k \times l)$-matrices $C_1, \ldots, C_n$ of real numbers, such that $A_1 C_1 + \ldots + A_n C_n = E_l$. We make use these matrices C_j to construct a decomposition of the differential of f almost everywhere in $\mathbb{R}^n$.

Let us define the mappings

$$
\begin{array}{rcl}
\partial'_A f & : & U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k), \\
\partial''_A f & : & U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)
\end{array}
$$

by

$$
\begin{array}{rcl}
\partial''_A f(x)\,(\eta) & = & \displaystyle\sum_{j=1}^{n} (C_j\, A f(x)) \eta_j, \\
\partial'_A f(x)\,(\eta) & = & df(x)\,(\eta) - \partial''_A f(x)\,(\eta),
\end{array}
\tag{3.4}
$$

for a tangent vector $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$. By the above, the mappings $\partial'_A f(x)$ and $\partial''_A f(x)$ are defined almost everywhere on U.

Lemma 3.3. *When regarded as differential operators of type $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^{k\times n}$, the mappings ∂'_A and ∂''_A are given by*

$$\begin{array}{rcl} \partial'_A f &=& \Big(\partial_1 f - C_1\, Af, \ldots, \partial_n f - C_n\, Af\Big), \\ \partial''_A f &=& \Big(C_1\, Af, \ldots, C_n\, Af\Big). \end{array}$$

Proof. This follows immediately from (3.4). □

The definition of $\partial''_A f$ makes it obvious that $\partial''_A f = 0$ whenever $Af = 0$. On the other hand, the linear function $\partial'_A f(x)\,(\eta)$ satisfies the system $Au = 0$.

Lemma 3.4. *Suppose $f \in W^{1,n+0}_{\mathrm{loc}}(U)^k$. Then, for each differentiability point $x \in U$ of f, we have*

$$A(\partial_\eta)\, \partial'_A f(x)\,(\eta) = 0$$

for all $\eta \in \mathbb{R}^n$.

Proof. Indeed,

$$\begin{array}{rcl} A(\partial_\eta)\, \partial'_A f(x)\,(\eta) &=& A(\partial_\eta)\,(df(x)\,(\eta) - \partial''_A f(x)\,(\eta)) \\ &=& \displaystyle\sum_{\iota=1}^{n} A_\iota \frac{\partial}{\partial \eta_\iota} \sum_{j=1}^{n} (\partial_j f(x)\, \eta_j - C_j\, Af(x)\, \eta_j) \\ &=& \displaystyle\sum_{\iota=1}^{n} A_\iota \partial_\iota f(x) - \Big(\sum_{\iota=1}^{n} A_\iota C_\iota\Big) Af(x) \\ &=& Af(x) - Af(x) \\ &=& 0 \end{array}$$

for all $\eta \in \mathbb{R}^n$, as required. □

Thus, given any mapping $f\colon U \to \mathbb{R}^k$ of class $W^{1,n+0}_{\mathrm{loc}}(U)$, we derive the decomposition $df = \partial'_A f + \partial''_A f$ for the differential of f, analogous to the decomposition $df = \partial f + \bar{\partial} f$ in the case of functions of a single complex variable. Moreover, we construct first order differential operators $\partial'_A f$ and $\partial''_A f$ with constant coefficients in $\mathbb{R}^n$, which are of the same nature as the operators $\partial/\partial z$ and $\partial/\partial \bar{z}$ in the complex plane.

3.2. **The Beltrami equation.** Consider the system of differential equations

$$\partial''_A f = Q(x)\, \partial'_A f, \tag{3.5}$$

for a mapping $f\colon U \to \mathbb{R}^k$ of a domain $U \subset \mathbb{R}^n$, where Q is a measurable function on U with values in $\mathcal{L}(\mathbb{R}^{k\times n}, \mathbb{R}^{k\times n})$.

If $Q \equiv 0$, then system (3.5) is equivalent to the system $Af = 0$, which is clear from Lemma 3.3.

Lemma 3.5. *If $\sup\limits_{x\in U} \|Q(x)\|$ is small enough, then (3.5) is a system with injective symbol.*

Here, $\|Q(x)\|$ is the norm of the linear mapping $Q(x)\colon \mathbb{R}^{k\times n} \to \mathbb{R}^{k\times n}$ under the standard Euclidean structure in $\mathbb{R}^{k\times n}$.

Proof. Indeed, if $Q \equiv 0$, then (3.5) is a system with injective symbol. From this the lemma follows by a familiar argument of topology. $\square$

Similarly to (3.1), system (3.1) is referred to as the Beltrami system related to the operator A.

By a solution of system (3.5) is meant any mapping $f \in W^{1,n+0}_{\mathrm{loc}}(U)^k$ satisfying (3.5) almost everywhere in U.

For an $\varepsilon > 0$, we denote by SBS (ε) the class of solutions to all possible Beltrami systems (3.5) with

$$\sup_{x \in U} \|Q(x)\| \leq \varepsilon.$$

3.3. **Local closeness to the sheaf of solutions and the Beltrami equation.** The connection of system (3.5) with the mappings locally close to the sheaf of solutions Sol is expressed in the following theorem (cf. [Tar85]).

Theorem 3.6. *There exists a constant $c > 0$, depending only on A, such that if $f \in W^{1,n+0}_{\mathrm{loc}}$ and $\mathfrak{D}(f, \mathrm{Sol}) \leq \varepsilon$ for some $0 \leq \varepsilon < c$, then $f \in \mathrm{SBS}(\varepsilon/c)$.*

Proof. Let f be a mapping of class $W^{1,n+0}_{\mathrm{loc}}$ from a domain $U \subset \mathbb{R}^n$ into $\mathbb{R}^k$ and let $x \in U$ be a point of differentiability of f.

We first assume that the differential $df(x)$ of f at x is non-zero. From the condition $\mathfrak{D}(f, \mathrm{Sol}) \leq \varepsilon$ it follows that, for each sufficiently small $R > 0$, there is a solution $u_R \in \mathrm{Sol}\,(B(0,2))$ such that

$$\partial'_A f(x)\,(\eta) + \partial''_A f(x)\,(\eta) - \|df(x)\|\, u_R(\eta) = \|df(x)\|\, \varrho_R(\eta) \tag{3.6}$$

for all $\eta \in B(0,1)$, where $\varrho_R(\eta) \in \mathbb{R}^k$ satisfies the condition $\sup\limits_{|\eta| \leq 1} |\varrho_R(\eta)| \to 8\varepsilon$, as $R \to 0$.

Indeed, fix an $R_0 > 0$ such that the closure of the ball $B(x, R_0)$ lies in U, and let $0 < R \leq R_0$. By the definition of $\mathfrak{D}(f, \mathrm{Sol})$, we get $\mathfrak{d}_{B(x,R)}(f, \mathrm{Sol}) \leq \varepsilon + O(R)$, where $O(R) > 0$ and $O(R) \to 0$ when $R \to 0$. Estimate (1.3) now shows that $\mathfrak{d}_{1/2, B(x,R)}(f, \mathrm{Sol}) \leq 2\,(\varepsilon + O(R))$. By assumption, we have $\|df(x)\| > 0$, whence $\mathrm{diam}\, f(B(x,R)) > 0$ for all $0 < R \leq R_0$. From the definition of the functional $\mathfrak{d}_{1/2, B(x,R)}(f, \mathrm{Sol})$ it is clear that, for each $R \in (0, R_0]$, there exists a solution $\tilde{u}_R \in \mathrm{Sol}\,(B(x,R))$ such that

$$f(y) - \tilde{u}_R(y) = 2\,(\varepsilon + O(R))\, \mathrm{diam}\, f(B(x,R))\, \tilde{\phi}_R(y)$$

for all $y \in B(x, R/2)$, where $|\tilde{\phi}_R(y)| \leq 1$.

Since the mapping f is differentiable at the point x, we arrive at the equality

$$df(x)\,(y-x) + f(x) - \tilde{u}_R(y) = 2\,(\varepsilon + O(R))\, \mathrm{diam}\, f(B(x,R))\, \tilde{\phi}_R(y) + o(|y-x|), \tag{3.7}$$

where $\dfrac{o(|y-x|)}{|y-x|} \to 0$ as $y \to x$. Setting

$$u_R(\eta) = 2\, \frac{1}{R\|df(x)\|} \Big(\tilde{u} \left(x + \frac{R}{2}\eta \right) - f(x) \Big),$$

for $\eta \in B(0,2)$, and taking into account that $\operatorname{diam} f(B(x,R)) = 2R\|df(x)\| + o(R)$, where $o(R)/R \to 0$ as $R \to 0$, we rewrite (3.7) in the following way:

$$\begin{aligned}&\frac{1}{\|df(x)\|}\, df(x)\,(\eta) - u_R(\eta)\\ &= \;\; 4\,(\varepsilon + O(R))\Big(2 + \frac{1}{\|df(x)\|}\,\frac{o(R)}{R}\Big)\phi_R(\eta) + \frac{2}{R\|df(x)\|}\, o\Big(\frac{R}{2}|\eta|\Big).\end{aligned}$$

Here, we have $\eta \in B(0,1)$ and $|\phi_R(\eta)| = |\tilde{\phi}_R(x + (R/2)\eta)| \le 1$. Denoting the right-hand side of this equality by $\varrho_R(\eta)$ and using the decomposition $df = \partial'_A f + \partial''_A f$, we arrive at (3.6), as desired.

Let us apply the integral operator

$$u \mapsto \int_{\mathbb{S}^{n-1}} A(\nu(\eta))u(\eta)ds(\eta)$$

to both sides of equality (3.6). Since the restrictions of the mappings $\partial'_A f(x)\,(\eta)$ and $u_R(\eta)$ to the ball $B(0,1)$ satisfy the condition of Cauchy's theorem, we get, by (2.1),

$$\int_{\mathbb{S}^{n-1}} A(\nu(\eta))\, \partial''_A f(x)\,(\eta)\, ds(\eta) = \|df(x)\| \int_{\mathbb{S}^{n-1}} A(\nu(\eta))\, \varrho_R(\eta)\, ds(\eta). \tag{3.8}$$

In our case, $\nu(\eta) = \eta$. Hence it follows that

$$\begin{aligned} A(\nu(\eta))\, \partial''_A f(x)\,(\eta) &= \Big(\sum_{\iota=1}^{n} A_\iota \eta_\iota\Big)\Big(\sum_{j=1}^{n} C_j \eta_j\Big)\, Af(x)\\ &= \Big(\sum_{\iota=1}^{n} A_\iota C_\iota \eta_\iota^2 + \sum_{\substack{\iota,j=1,\ldots,n\\ \iota\ne j}} A_\iota C_j \eta_\iota \eta_j\Big)\, Af(x)\\ &= \frac{1}{n}\,|\eta|^2\, Af(x) + \Big(\sum_{\substack{\iota,j=1,\ldots,n\\ \iota\ne j}} A_\iota C_j \eta_\iota \eta_j\Big)\, Af(x)\end{aligned}$$

and, since $\displaystyle\int_{\mathbb{S}^{n-1}} \eta_\iota \eta_j\, ds(\eta) = 0$ for $\iota \ne j$, equality (3.8) becomes

$$\frac{1}{n}\,\sigma_n\, Af(x) = \|df(x)\| \int_{\mathbb{S}^{n-1}} A(\nu(\eta))\, \varrho_R(\eta)\, ds(\eta).$$

Letting $R \to 0$ yields

$$\frac{1}{n}\,|Af(x)| \le 8\varepsilon\,\Big(\sup_{|\xi|=1} \|A(\xi)\|\Big)\, \|df(x)\|. \tag{3.9}$$

Our next objective is to evaluate the "derivative" $\partial''_A f(x)$. Taking into account that $df(x) = \partial'_A f(x) + \partial''_A f(x)$ and

$$\begin{aligned} |\partial''_A f(x)| &= \Big(\sum_{j=1}^{n} |C_j\, Af(x)|^2\Big)^{1/2}\\ &\le \Big(\sum_{j=1}^{n} \|C_j\|^2\Big)^{1/2} |Af(x)|,\end{aligned}$$

we conclude that

$$\begin{aligned} |\partial''_A f(x)| &\leq 8\varepsilon n \Big(\sum_{j=1}^n \|C_j\|^2 \Big)^{1/2} \Big(\sup_{|\xi|=1} \|A(\xi)\| \Big) \|df(x)\| \\ &\leq 8\varepsilon n \Big(\sum_{j=1}^n \|C_j\|^2 \Big)^{1/2} \Big(\sup_{|\xi|=1} \|A(\xi)\| \Big) \left(|\partial'_A f(x)| + |\partial''_A f(x)| \right), \end{aligned}$$

and hence

$$|\partial''_A f(x)| \leq \frac{8\varepsilon n \Big(\sum\limits_{j=1}^n \|C_j\|^2 \Big)^{1/2} \Big(\sup\limits_{|\xi|=1} \|A(\xi)\| \Big)}{1 - 8\varepsilon n \Big(\sum\limits_{j=1}^n \|A_j^{-1}\|^2 \Big)^{1/2} \Big(\sup\limits_{|\xi|=1} \|A(\xi)\| \Big)} |\partial'_A f(x)|.$$

Set

$$c = \frac{1}{16 n \Big(\sum\limits_{j=1}^n \|C_j\|^2 \Big)^{1/2} \Big(\sup\limits_{|\xi|=1} \|A(\xi)\| \Big)},$$

then, for $\varepsilon < c$, we obtain

$$\begin{aligned} |\partial''_A f(x)| &\leq \frac{\varepsilon/(2c)}{1 - \varepsilon/(2c)} |\partial'_A f(x)| \\ &= \frac{\varepsilon}{2c - \varepsilon} |\partial'_A f(x)| \\ &\leq \frac{\varepsilon}{c} |\partial'_A f(x)|. \end{aligned} \tag{3.10}$$

In case $df(x)$ vanishes inequality (3.10) is obvious.

We are left with the task of constructing a measurable function Q on the domain U with values in $\mathcal{L}(\mathbb{R}^{k\times n}, \mathbb{R}^{k\times n})$, defined almost everywhere in the domain U, such that $\partial''_A f = Q(x)\, \partial'_A f$ and

$$\sup_{x\in U} \|Q(x)\| \leq \frac{\varepsilon}{c}.$$

To this end, given a matrix $M = (M_{ij})$ of Euclidean norm $|M| = 1$ in $\mathbb{R}^{k\times n}$, we construct an orthogonal linear transformation O_M of $\mathbb{R}^{k\times n}$, sending M to the first element e_{11} of the canonical basis of $\mathbb{R}^{k\times n}$, in the following way. Setting $M_{ij} = v_{i+k(j-1)}$, for $i = 1, \ldots, k$ and $j = 1, \ldots, n$, we identify the matrix M with the vector $v \in \mathbb{R}^{kn}$ whose coordinates are v_ι. Let ι_0 be the number of the first non-zero coordinate of v, i.e., $v_{\iota_0} \neq 0$ but $v_1 = \ldots = v_{\iota_0 - 1} = 0$. Consider the linearly independent system

$$\begin{array}{rl} v,\, e_2,\, e_3,\, \ldots,\, e_{kn}, & \text{if} \quad \iota_0 = 1; \\ v,\, e_1,\, e_3,\, \ldots,\, e_{kn}, & \text{if} \quad \iota_0 = 2; \\ v,\, e_2,\, \ldots,\, e_{\iota_0-1},\, e_1,\, e_{\iota_0+1},\, \ldots,\, e_{kn}, & \text{if} \quad \iota_0 > 2, \end{array} \tag{3.11}$$

in $\mathbb{R}^{kn}$. With the help of the Gram-Schmidt orthogonalisation, we construct from the system (3.11) a new orthonormal system $e_1, \ldots, e_{kn}$ in $\mathbb{R}^{kn}$. (To shorten notation we use the same letters e_ι for the vectors in the system just obtained.) By construction, $e_1 = v$. The transformation matrix from this new basis $e_1, \ldots, e_{kn}$ to the canonical basis of $\mathbb{R}^{kn}$ is orthogonal and provides the desired transformation O_M.

We are now in a position to complete the proof. Indeed, let $x \in U$ be a point of differentiability of f, such that $|\partial''_A f(x)| > 0$. By (3.10), we can assert that $|\partial'_A f(x)| > 0$. Then, $\frac{\partial'_A f(x)}{|\partial'_A f(x)|}$ and $\frac{\partial''_A f(x)}{|\partial''_A f(x)|}$ are matrices of Euclidean norm 1 in $\mathbb{R}^{k \times n}$. We set

$$Q(x) = \frac{|\partial''_A f(x)|}{|\partial'_A f(x)|} O^{-1}_{\partial''_A f(x)} \circ O_{\partial'_A f(x)}.$$

In case $\partial''_A f(x)$ vanishes we simply set $Q(x) = 0$. The measurability of Q is clear from its construction because the partial derivatives of f are measurable functions. The mapping Q so obtained meets all the required conditions, and the theorem follows. □

4. Stability of the sheaf of solutions

4.1. **Statement of the main theorems.** The sheaf of solutions to the system $Au = 0$ inherits the stability property of holomorphic functions, which is the subject of our next theorem.

Theorem 4.1. *The class of mappings from open sets in $\mathbb{R}^n$ into the space $\mathbb{R}^k$, satisfying the system $Au = 0$, is stable relative to the class $W^{1,n+0}_{\mathrm{loc}}$.*

According to Definition 1.13, this means that if a mapping $f \in W^{1,n+0}_{\mathrm{loc}}$ is locally close to a solution of the system $Au = 0$, then it is globally close, too. In fact, there exists a non-negative function $\delta = \delta(\varepsilon)$, defined on some interval $[0, \varepsilon_0)$, such that:

1) $\delta(\varepsilon) \to \delta(0) = 0$ as $\varepsilon \to 0$;
2) if $f \in W^{1,n+0}_{\mathrm{loc}}$ and $\mathfrak{D}(f, \mathrm{Sol}) \le \varepsilon$ for some $\varepsilon \in [0, \varepsilon_0)$, then $\mathfrak{d}(f, \mathrm{Sol}) \le \delta(\varepsilon)$.

Furthermore, the local closeness of a mapping $f \in W^{1,n+0}_{\mathrm{loc}}$ to the class Sol is asymptotically equivalent to the fact that this mapping is a solution to a Beltrami system (3.5) with a small value of the norm $\sup_{x \in U} \|Q(x)\|$.

Theorem 4.2. *There exists a non-negative function $\delta = \delta(\varepsilon)$, defined on some interval $[0, \varepsilon_0)$, such that:*

1) $\delta(\varepsilon) \to \delta(0) = 0$ as $\varepsilon \to 0$;
2) if $f \in W^{1,n+0}_{\mathrm{loc}}$ and $\mathfrak{D}(f, \mathrm{Sol}) \le \varepsilon$ for some $\varepsilon \in [0, \varepsilon_0)$, then $f \in \mathrm{SBS}\,(\delta(\varepsilon))$, and, conversely, if $f \in \mathrm{SBS}\,(\varepsilon)$ for some $\varepsilon \in [0, \varepsilon_0)$, then $\mathfrak{D}(f, \mathrm{Sol}) \le \delta(\varepsilon)$.

In the case $l = k$ both Theorem 4.1 and Theorem 4.2 are contained in [Tar85], [Tar95].

Since $\mathfrak{D}(f, \mathrm{Sol}) \le \mathfrak{d}(f, \mathrm{Sol})$, Theorems 4.1 and 4.2 are direct consequences of Theorem 3.6 and the following more hard result.

Theorem 4.3. *There exists a non-negative function $\Delta = \Delta(\varepsilon)$, defined on some interval $[0, \varepsilon_0)$, such that:*

1) $\Delta(\varepsilon) \to \Delta(0) = 0$ as $\varepsilon \to 0$;
2) if $f \in \mathrm{SBS}\,(\varepsilon)$ for some $\varepsilon \in [0, \varepsilon_0)$, then $\mathfrak{d}(f, \mathrm{Sol}) \le \Delta(\varepsilon)$.

The remainder of this section will be devoted to the proof of Theorem 4.3. The proof is based on the study of properties of solutions to Beltrami system (3.5). The key result is an L^q-estimate for the derivatives of these solutions. Deriving this estimate is the objective of Subsection 4.5, which builds on the following three subsections.

4.2. **Generalised Cauchy's formula.** As described in Lemma 3.3, the "differential" ∂''_A acts through the differential operator A. Vice versa,

$$Af = \sum_{j=1}^{n} A_j \, \partial''_A f \, e_j, \tag{4.1}$$

as is easy to check.

Lemma 4.4. *Suppose $\mathcal{D} \subset \mathbb{R}^n$ is a bounded domain with piecewise smooth boundary. Then, for each $f \in W^{1,n+0}(\mathcal{D})^k$ continuous up to the boundary of $\mathcal{D}$, we have*

$$\begin{aligned} & - \int_{\partial\mathcal{D}} \Phi(x-y) \, A(\nu(y)) f(y) \, ds(y) \\ & + \int_{\mathcal{D}} \Phi(x-y) \Big(\sum_{j=1}^{n} A_j \, \partial''_A f(y) \, e_j \Big) \, dy = \begin{cases} f(x), & \text{if} \quad x \in \mathcal{D}, \\ 0, & \text{if} \quad x \in \mathbb{R}^n \setminus \overline{\mathcal{D}}. \end{cases} \end{aligned} \tag{4.2}$$

In the case of one complex variable this result coincides with the classical generalised Cauchy (or Cauchy-Green) integral formula for smooth functions due to D. Pompeiu.

Proof. For $f \in C^1(\overline{\mathcal{D}})^k$, formula (4.2) is an easy consequence of Green's formula and equality (4.1). In the general case, it is obtained from the case of smooth mappings.

Indeed, let $f \in W^{1,q}(\mathcal{D})^k$, where $q > n$. There exists a sequence $\{f_\nu\}$ in $C^\infty(\overline{\mathcal{D}})^k$, such that $f_\nu \to f$ in the norm of $W^{1,q}(\mathcal{D})^k$ and uniformly on $\overline{\mathcal{D}}$. One can take, for instance, $f_\nu = R^{(1/\nu)} \tilde{f} \,|_{\mathcal{D}}$, where $\tilde{f} \in W^{1,q}_{\mathrm{comp}}(\mathbb{R}^n)^k$ is any extension of f to the whole space.

Consider formula (4.2) for each mapping f_ν, $\nu = 1, 2, \ldots$, and let $\nu \to \infty$. Since $f_\nu \to f$ uniformly on $\partial\mathcal{D}$, the integral of f_ν over the boundary in (4.2) converges to the corresponding integral of f, uniformly in x on compact sets away from $\partial\mathcal{D}$. We shall have established the lemma if we prove that the integral of $\partial''_A f_\nu$ over the domain in (4.2) converges to the corresponding integral of $\partial''_A f$, for each fixed $x \in \mathbb{R}^n$.

To this end, let us denote by $\mathcal{P}_v \partial''_A f$ the integral operator defined by the second summand in the left-hand side of formula (4.2). We have

$$\begin{aligned} |\mathcal{P}_v \partial''_A f\,(x) - \mathcal{P}_v \partial''_A f_\nu\,(x)| &= |\int_{\mathcal{D}} \Phi(x-y) \, (Af(y) - Af_\nu(y)) \, dy| \\ &\leq \|\Phi(x-\cdot)\|_{L^{q'}(\mathcal{D})} \, \|A(f-f_\nu)\|_{L^q(\mathcal{D})}, \end{aligned}$$

by the Hölder inequality. Since $f_\nu \to f$ in the norm of $W^{1,q}(\mathcal{D})^k$, we conclude that $\|A(f-f_\nu)\|_{L^q(\mathcal{D})} \to 0$ as $\nu \to \infty$. Moreover, the norm $\|\Phi(x-\cdot)\|_{L^{q'}(\mathcal{D})}$ is dominated by $c \, \||z|^{1-n}\|_{L^{q'}(x-\mathcal{D})}$, and hence locally bounded in $\mathbb{R}^n$, for $q' < n/(n-1)$. It follows that $\mathcal{P}_v \partial''_A f_\nu \to \mathcal{P}_v \partial''_A f$ uniformly on compact sets in $\mathbb{R}^n$, when $\nu \to \infty$. This is the desired conclusion. $\square$

4.3. **An estimate for the double layer potential.** Let $\mathcal{D}$ be a bounded domain with piecewise smooth boundary in $\mathbb{R}^n$ and let $f \in L^1(\partial\mathcal{D})^k$ be a given vector-valued function on the boundary of $\mathcal{D}$.

We define the Cauchy-type integral (or double layer potential) of f by the first summand in the left-hand side of (4.2), i.e.

$$\mathcal{P}_{\mathrm{dl}} f\,(x) = -\int_{\partial\mathcal{D}} \Phi(x-y)\,A(\nu(y)) f(y)\,ds(y), \tag{4.3}$$

if $x \notin \partial\mathcal{D}$.

In the sequel, we use formula (4.2) and potential (4.3) in the case where $\mathcal{D} = B$ is a ball in $\mathbb{R}^n$.

Lemma 4.5. *Let $B = B(x^0, R)$ be a ball in $\mathbb{R}^n$ and let $f \colon B \to \mathbb{R}^k$ be a mapping continuous in the closure of the ball. Then, there is a constant $c > 0$, depending only on A (but not on B and f), such that*

$$|\partial'_A \mathcal{P}_{\mathrm{dl}} f\,(x)| \leq c \int_{\partial B} \frac{|f(y) - f(x^0)|}{|y-x|^n}\,ds(y) \tag{4.4}$$

for all $x \in B$.

Proof. Since the constant mappings satisfy the conditions of Lemma 2.3, we have, by (2.4),

$$\mathcal{P}_{\mathrm{dl}} f\,(x) - f(x^0) = -\int_{\partial B} \Phi(x-y)\,A(\nu(y))(f(y) - f(x^0))\,ds(y)$$

for all $x \in B$. If $x \in B$, then

$$\begin{aligned} |\partial_j(\mathcal{P}_{\mathrm{dl}} f\,(x) - f(x^0))| &= |-\int_{\partial B} \partial_j \Phi\,(x-y)\,A(\nu(y))(f(y) - f(x^0))\,ds(y)| \\ &\leq (\sup_{|\xi|=1} \|\partial_j \Phi\,(\xi)\|)\,(\sup_{|\xi|=1} \|A(\xi)\|) \int_{\partial B} \frac{|f(y) - f(x^0)|}{|y-x|^n}\,ds(y), \end{aligned}$$

for each $j = 1, \ldots, n$. Here, we used the fact that the derivative $\partial_j \Phi(x)$ is homogeneous of degree $-n$ away from the origin. Returning to $\partial'_A \mathcal{P}_{\mathrm{dl}} f$, we get

$$\begin{aligned} |\partial'_A \mathcal{P}_{\mathrm{dl}} f\,(x)| &= |\partial'_A\,(\mathcal{P}_{\mathrm{dl}} f\,(x) - f(x^0))| \\ &\leq \mathrm{const}\,(A) \int_{\partial B} \frac{|f(y) - f(x^0)|}{|y-x|^n}\,ds(y) \end{aligned}$$

for all $x \in B$, the constant being independent of B and f, as desired. □

4.4. **An estimate for the volume potential.** For a matrix-valued function $F \in L^q(\mathcal{D})^{k\times n}$, $q > 1$, we introduce the volume potential of F by

$$\mathcal{P}_v F\,(x) = \int_{\mathcal{D}} \Phi(x-y) \Big(\sum_{j=1}^{n} A_j F(y) e_j\Big)\,dy, \tag{4.5}$$

for $x \in \mathbb{R}^n$. Unless otherwise stated we identify the density F with its extension as zero on $\mathbb{R}^n \setminus \mathcal{D}$, thus making use of the equality $L^q(\mathcal{D}) = L^q_{\mathcal{D}}(\mathbb{R}^n)$. Then, we need not specify the domain of integration in (4.5).

By Lemma 2.5, the potential $\mathcal{P}_v F$ is differentiable at almost every point $x \in \mathbb{R}^n$. Thus, the "differential" $\partial'_A \mathcal{P}_v F\,(x)$ is defined almost everywhere on $\mathbb{R}^n$. The continuity of the operator $\partial'_A \mathcal{P}_v$ in the spaces $L^q(\mathbb{R}^n)^{k\times n}$ with non-extreme exponents q is established by our next lemma.

Lemma 4.6. *Let B be a ball in $\mathbb{R}^n$ and let $F : B \to \mathbb{R}^{k\times n}$ be a mapping of class $L^q(B)$, $1 < q < \infty$. Then, $\partial'_A \mathcal{P}_v F \in L^q(\mathbb{R}^n)^{k\times n}$ and there is a constant $c > 0$, depending only on q and A (but not on B and F), such that*

$$\|\partial'_A \, \mathcal{P}_v F\|_{L^q(\mathbb{R}^n)} \leq c \, \|F\|_{L^q(\mathbb{R}^n)}. \tag{4.6}$$

Proof. Lemma 2.5 yields

$$\begin{aligned} \frac{\partial}{\partial x_j} \, \mathcal{P}_v F\,(x) \;\; &= \;\; \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} \, \varPhi(x-y) \Big(\sum_{j=1}^n A_j F(y) e_j \Big) \, dy \\ & \quad + \Big(\int_{\mathbb{S}^{n-1}} \varPhi(\xi) \star d\xi_j \Big) \Big(\sum_{j=1}^n A_j F(x) e_j \Big), \end{aligned}$$

for each $j = 1, \ldots, n$.

To estimate the first integral in the right-hand side, we may invoke, by Lemma 2.4, the theory of singular integral operators of Mikhlin-Calderon-Zygmund (cf. for instance [Ste70]). On the other hand, the second term in the right side is in $L^q(\mathbb{R}^n)^k$. From this we conclude that the potential $\mathcal{P}_v F$ has first order derivatives in $L^q(\mathbb{R}^n)^k$ and there is a constant $c > 0$, depending only on q and A (but not on B and F), such that

$$\|(\partial/\partial x_j) \, \mathcal{P}_v F\|_{L^q(\mathbb{R}^n)} \leq c \, \|F\|_{L^q(\mathbb{R}^n)}$$

for all $j = 1, \ldots, n$.

Since ∂'_A is a first order differential operator, we get at once the assertion of the lemma, possibly with a new constant $c(q)$ independent of B and F. □

4.5. **L^q-estimates of the derivatives of solutions to the Beltrami equation.** Using Lemmas 4.4, 4.5 and 4.6, we prove the main technical result of this section. It plays a crucial role in investigations on the stability of classes of solutions to the system $Au = 0$.

Theorem 4.7. *Let the real numbers $\varepsilon \in [0, 1)$, $\theta \in (0, 1)$ and $q > 1$ satisfy the conditions $\varepsilon \, c(q) \, (1 - \theta)^{-n} < 1$ and $\varepsilon \, c(n) < 1$, where $c(q)$ is the constant of inequality (4.6). Then, for each mapping $f : B(x^0, R) \to \mathbb{R}^k$ of class* SBS(ε)*, the following inequality holds:*

$$\|\partial'_A f\|_{L^q(B(x^0,\theta R))} \leq \mathrm{const}\,(A) \, \frac{\sigma_n^{1+1/q} R^{n/q-1} \theta^{-n}}{(1-\theta)^n - \varepsilon \, c(q)} \, \mathrm{diam}\, f(B(x^0, R)). \tag{4.7}$$

Proof. We can certainly assume that $\mathrm{diam}\, f(B(x^0, R)) < \infty$, since otherwise inequality (4.7) is obvious.

Moreover, we shall assume that f is of class $W^{1,q}(B(x^0, R))^k$ for some $q > n$. For arbitrary $f \in W^{1,n+0}_{\mathrm{loc}}(B(x^0, R))^k$, the estimate (4.7) can be derived from this particular case by a passage to the limit.

Fix an extending sequence of balls $B_\nu = B(x^0, (1 - (1-\theta)^\nu)R)$, $\nu = 1, 2, \ldots$. By Lemma 4.4, we have

$$f(x) = \mathcal{P}^{(\nu)}_{\mathrm{dl}} f\,(x) + \mathcal{P}^{(\nu)}_v \partial''_A f\,(x),$$

$x \in B_\nu$, where

$$\begin{aligned} \mathcal{P}^{(\nu)}_{\mathrm{dl}} f\,(x) &= -\int_{\partial B_\nu} \varPhi(x-y)\, A(\nu(y)) f(y)\, ds(y), \\ \mathcal{P}^{(\nu)}_{v} \partial''_A f\,(x) &= \int_{B_\nu} \varPhi(x-y)\, (\sum_{j=1}^{n} A_j\, \partial''_A f(y)\, e_j)\, dy. \end{aligned}$$

From Lemma 4.6 we see that the "differential" $\partial'_A\, \mathcal{P}^{(\nu)}_{v} \partial''_A f$ exists almost everywhere in $\mathbb{R}^n$. Therefore, for each $\nu = 1, 2, \ldots$, we have

$$\partial'_A f\,(x) = \partial'_A\, \mathcal{P}^{(\nu)}_{\mathrm{dl}} f\,(x) + \partial'_A\, \mathcal{P}^{(\nu)}_{v} \partial''_A f\,(x) \tag{4.8}$$

a.e. in $\mathbb{R}^n$.

Let $Q\colon B(x^0, R) \to \mathcal{L}(\mathbb{R}^{k\times n}, \mathbb{R}^{k\times n})$ be the mapping defining system (3.5), and let $F\colon U \to \mathbb{R}^{k\times n}$ be a mapping of an open set $U \subset \mathbb{R}^n$. We define the mapping $QF\colon \mathbb{R}^n \to \mathbb{R}^{k\times n}$ by

$$QF\,(x) = \begin{cases} Q(x)F(x), & \text{if } x \in U \cap B(x^0, R), \\ 0, & \text{in the opposite case.} \end{cases}$$

If $F \in L^q(U)^{k\times n}$, then, from $\sup_{x\in B(x^0,R)} \|Q(x)\| \le \varepsilon$, we deduce that

$$\|QF\|_{L^q(U)} \le \varepsilon\, \|F\|_{L^q(U)}. \tag{4.9}$$

Moreover, we make use of the following estimate for the potential $\partial'_A\, \mathcal{P}^{(\nu)}_{v} F$, which is clear from Lemma 4.6:

$$\|\partial'_A\, \mathcal{P}^{(\nu)}_{v} F\|_{L^q(\mathbb{R}^n)} \le c(q)\, \|F\|_{L^q(B_\nu)}, \tag{4.10}$$

provided that $F \in L^q(B_\nu)^{k\times n}$, $1 < q < \infty$.

Having disposed of this preliminary step, we consider, in the ball $B_1 = B(x^0, \theta R)$, the series

$$\sum_{\nu=1}^{\infty} F_\nu, \tag{4.11}$$

where $F_\nu\colon B_1 \to \mathbb{R}^{k\times n}$ are defined in the following way:

$$\begin{array}{rcll} F_1 &=& \partial'_A\, \mathcal{P}^{(2)}_{\mathrm{dl}} f\,|_{B_1}, & \text{if } \nu = 1; \\ F_\nu &=& \partial'_A \mathcal{P}^{(2)}_{v} (Q\, \partial'_A \mathcal{P}^{(3)}_{v} \ldots (Q\, \partial'_A \mathcal{P}^{(\nu)}_{v} (Q\, \partial'_A \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f)) \ldots)\,|_{B_1}, & \text{if } \nu \ge 2. \end{array}$$

Since $q_\nu = \|\partial'_A\, \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f\|_{L^q(B_\nu)}$ is finite, for each $q > 1$, we deduce step by step from (4.9) and (4.10) that

$$\begin{aligned} \|Q\, \partial'_A \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f\|_{L^q(B_\nu)} &\le \varepsilon\, \|\partial'_A\, \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f\|_{L^q(B_\nu)} \\ &= \varepsilon\, q_\nu, \\ \|\partial'_A \mathcal{P}^{(\nu)}_{v} (Q\, \partial'_A \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f)\|_{L^q(B_{\nu-1})} &\le c(q)\, \|Q\, \partial'_A \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f\|_{L^q(B_\nu)} \\ &\le \varepsilon\, c(q)\, q_\nu, \\ \|Q\, \partial'_A \mathcal{P}^{(\nu)}_{v} (Q\, \partial'_A \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f)\|_{L^q(B_{\nu-1})} &\le \varepsilon\, \|\partial'_A \mathcal{P}^{(\nu)}_{v} (Q\, \partial'_A \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f)\|_{L^q(B_{\nu-1})} \\ &\le \varepsilon^2\, c(q)\, q_\nu, \\ \|\partial'_A \mathcal{P}^{(\nu-1)}_{v} (Q \partial'_A \mathcal{P}^{(\nu)}_{v} (Q \partial'_A \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f))\|_{L^q(B_{\nu-2})} &\le c(q)\, \|Q \partial'_A \mathcal{P}^{(\nu)}_{v} (Q \partial'_A \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f)\|_{L^q(B_{\nu-1})} \\ &\le \varepsilon^2\, c(q)^2\, q_\nu, \end{aligned}$$

for $\nu \ge 2$. We now proceed by induction obtaining

$$\|F_\nu\|_{L^q(B_1)} \le (\varepsilon c(q))^{\nu-1}\, q_\nu, \tag{4.12}$$

$\nu = 1, 2, \ldots$.

Let us estimate the quantity q_ν. From Lemma 4.5 it follows that

$$|\partial'_A\, \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f\,(x)| \leq \mathrm{const}\,(A) \int_{\partial B_{\nu+1}} \frac{|f(y) - f(x^0)|}{|y-x|^n}\, ds(y)$$

for all $x \in B_{\nu+1}$. If $x \in B_\nu$, then $|y - x| \geq \theta(1-\theta)^\nu R$ for all $y \in \partial B_{\nu+1}$, whence

$$\begin{aligned}
&|\partial'_A\, \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f\,(x)| \\
&\quad \leq \mathrm{const}\,(A)\ \theta^{-n}(1-\theta)^{-\nu n} R^{-n}\ \sigma_n (1-(1-\theta)^{\nu+1}))^{n-1} R^{n-1}\, \mathrm{diam}\, f(B(x^0,R)) \\
&\quad \leq \mathrm{const}\,(A)\ \theta^{-n}(1-\theta)^{-\nu n}\ \sigma_n R^{-1}\, \mathrm{diam}\, f(B(x^0,R)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
q_\nu &= \Big(\int_{B_\nu} |\partial'_A\, \mathcal{P}^{(\nu+1)}_{\mathrm{dl}} f\,(x)|^q\, dx \Big)^{1/q} \\
&\leq \mathrm{const}\,(A)\ \theta^{-n}(1-\theta)^{-\nu n}\ \sigma_n R^{-1}\, \mathrm{diam}\, f(B(x^0,R))\, (\mathrm{meas}\,(B_\nu))^{1/q} \\
&\leq \mathrm{const}\,(A)\ \theta^{-n}(1-\theta)^{-\nu n}\ \sigma_n^{1+1/q} R^{n/q-1}\, \mathrm{diam}\, f(B(x^0,R)).
\end{aligned}$$

Combining these inequalities with estimate (4.12) and taking into account that $\varepsilon\, c(q)\, (1-\theta)^{-n} < 1$, we obtain

$$\begin{aligned}
&\sum_{\nu=1}^{\infty} \|F_\nu\|_{L^q(B_1)} \\
&\leq \mathrm{const}\,(A)\ \theta^{-n}(1-\theta)^{-n}\ \sigma_n^{1+1/q} R^{n/q-1}\, \mathrm{diam}\, f(B(x^0,R)) \sum_{\nu=1}^{\infty} (\varepsilon c(q)(1-\theta)^{-n})^{\nu-1} \\
&= \mathrm{const}\,(A)\ \theta^{-n}(1-\theta)^{-n}\ \sigma_n^{1+1/q} R^{n/q-1}\, \frac{\mathrm{diam}\, f(B(x^0,R))}{1-\varepsilon c(q)(1-\theta)^{-n}},
\end{aligned}$$

i.e. series (4.11) converges normally in $L^q(B_1)^{k\times n}$.

We next claim that series (4.11) converges to $\partial'_A f|_{B_1}$ in the norm of $L^q(B_1)^{k\times n}$. Since $L^q(B_1)^{k\times n}$ is a Banach space, the normal convergence of (4.11) implies the convergence of this series in $L^q(B_1)^{k\times n}$. What is left is to show that the sum of (4.11) is equal to $\partial'_A f$ in the ball B_1. To this end, we make use of (4.8) to obtain

$$\begin{aligned}
&\partial'_A f\,|_{B_1} \\
&= \partial'_A \mathcal{P}^{(2)}_{\mathrm{dl}} f\,|_{B_1} + \partial'_A \mathcal{P}^{(2)}_{v}\, \partial''_A f\,|_{B_1} \\
&= F_1 + \partial'_A \mathcal{P}^{(2)}_{v}\, (Q \partial'_A f)\,|_{B_1} \\
&= F_1 + \partial'_A \mathcal{P}^{(2)}_{v}\, (Q \partial'_A \mathcal{P}^{(3)}_{\mathrm{dl}} f)\,|_{B_1} + \partial'_A \mathcal{P}^{(2)}_{v}\, (Q \partial'_A \mathcal{P}^{(3)}_{v} \partial''_A f)\,|_{B_1} \\
&= F_1 + F_2 + \partial'_A \mathcal{P}^{(2)}_{v}\, (Q \partial'_A \mathcal{P}^{(3)}_{v} (Q \partial'_A f))\,|_{B_1} \\
&= F_1 + F_2 + \ldots + F_\nu + \partial'_A \mathcal{P}^{(2)}_{v} (Q \partial'_A \mathcal{P}^{(3)}_{v} \ldots (Q \partial'_A \mathcal{P}^{(\nu+1)}_{v} (Q \partial'_A f)) \ldots)\,|_{B_1}.
\end{aligned}$$

As $f \in W^{1,q}(B(x^0,R))^k$ for some $q > n$, we have $\partial'_A f \in L^n(B(x^0,R))^{k\times n}$. Analysis similar to that in the proof of inequality (4.12) shows that

$$\|\partial'_A f - \sum_{\nu=1}^{N} F_\nu\|_{L^n(B_1)} \leq (\varepsilon\, c(n))^{N-1}\, \|\partial'_A f\|_{L^n(B(x^0,R))},$$

for $N = 1, 2, \ldots$. By assumption, $\varepsilon\, c(n) < 1$, hence the latter inequality makes it obvious that series (4.11) converges to $\partial'_A f|_{B_1}$ in the norm of $L^n(B_1)^{k\times n}$. Now, the equality

$$\sum_{\nu=1}^{\infty} F_\nu = \partial'_A f$$

in B_1 follows from the fact that, if a series converges to Σ_1 in $L^{q_1}(B)$ and to Σ_2 in $L^{q_2}(B)$, then $\Sigma_1 = \Sigma_2$ almost everywhere on B. The proof is complete. □

This theorem provides also an estimate for the "differential" $\partial''_A f$, for we have $|\partial''_A f(x)| \le \varepsilon |\partial'_A f(x)|$ for all $f \in \mathrm{SBS}(\varepsilon)$. Namely, under the hypotheses of Theorem 4.7, we get

$$\|\partial''_A f\|_{L^q(B(x^0,\theta R))} \le \varepsilon \,\mathrm{const}\,(A)\, \frac{\sigma_n^{1+\frac{1}{q}} R^{\frac{n}{q}-1} \theta^{-n}}{(1-\theta)^n - \varepsilon\, c(q)}\, \mathrm{diam}\, f(B(x^0,R)). \tag{4.13}$$

Theorem 4.7 shows that, if $\varepsilon > 0$ is sufficiently small, then

$$\inf_{f \in \mathrm{SBS}(\varepsilon)} \sup\{q : \partial'_A(f), \partial''_A(f) \in L^q_{\mathrm{loc}}\} > n,$$

i.e. the maximal power at which the derivatives of maps $f \in \mathrm{SBS}(\varepsilon)$ are summable is uniformly bounded away from n.

Corollary 4.8. *Suppose the assumptions of Theorem 4.7 are fulfilled and, in addition, $q > n$. Then, for each map $f : B(x^0,R) \to \mathbb{R}^k$ of class* SBS(ε)*, it follows that*

$$|f(x') - f(x'')| \le c\, |x' - x''|^{1-n/q}\, \mathrm{diam}\, f(B(x^0,R)),$$

for all $x', x'' \in B(x^0, \theta' R)$ with $\theta' < \theta$, where c is a constant depending only on the operator A and parameters q, θ, θ' and ε but not on f.

Proof. The assertion follows immediately from the generalised Cauchy formula of Lemma 4.4, the estimate of Lemma 4.6 and Theorem 4.7 in the same way as Theorem 6 of [Kop82, Part II] was derived from Lemmas 4 and 6 and Theorem 5 of that paper. □

4.6. **Global closeness to the sheaf of solutions and the Beltrami equation.** The aim of this subsection is to prove Theorem 4.3. For this purpose we introduce one more closeness functional.

Namely, for a locally bounded mapping $f : U \to \mathbb{R}^k$ of a domain $U \subset \mathbb{R}^n$, a number $\theta \in (0,1)$ and an arbitrary ball $B \subset U$, we set

$$\tilde{\mathfrak{d}}_{\theta,B}(f,\mathcal{S}) = \begin{cases} \inf\limits_{u \in \mathcal{S}(\theta B)} \Big(\sup\limits_{y \in \theta B} \dfrac{|f(y) - u(y)|}{\mathrm{diam}\, f(B)} \Big), & \text{if } \mathrm{diam}\, f(B) \ne 0, \infty; \\ 0, & \text{in the opposite case,} \end{cases} \tag{4.14}$$

and

$$\tilde{\mathfrak{d}}_\theta(f,\mathcal{S}) = \sup_{B \subset U} \tilde{\mathfrak{d}}_{\theta,B}(f,\mathcal{S}).$$

Note that, in contrast to the definition of $\mathfrak{d}_{\theta,B}(f,\mathcal{S})$ (cf. (1.1)), the infimum in (4.14) is taken over all the mappings $u \in \mathcal{S}$ defined on the smaller ball θB, not just those defined on the whole ball B. From this it is clear that

$$\tilde{\mathfrak{d}}_{\theta,B}(f,\mathcal{S}) \le \mathfrak{d}_{\theta,B}(f,\mathcal{S}). \tag{4.15}$$

Remark 4.9. The properties of the functionals $\tilde{\mathfrak{d}}_{\theta,B}(f,\mathcal{S})$ and $\tilde{\mathfrak{d}}_\theta(f,\mathcal{S})$ are completely analogous to those of the functionals $\mathfrak{d}_{\theta,B}(f,\mathcal{S})$ and $\mathfrak{d}_\theta(f,\mathcal{S})$, given in Lemmas 1.6, 1.7, 1.8 and 1.9.

The following statement, along with inequality (4.15), allows one to establish the asymptotic equivalence of the functionals $\tilde{\mathfrak{d}}_\theta(f,\mathcal{S})$ and $\mathfrak{d}_\theta(f,\mathcal{S})$.

Lemma 4.10. *Let the class $\mathcal{S}$ satisfy conditions $\mathcal{P}_1 - \mathcal{P}_6$. Then, there exists a function $\delta\colon [0,1/2)\times(0,1)\to[0,1)$, such that:*

1) $\delta(\varepsilon,\theta)\to\delta(0,\theta)=0$ as $\varepsilon\to 0$, and for each $\theta\in(0,1)$;

2) if $f: U\to\mathbb{R}^k$ is a mapping of a domain $U\subset\mathbb{R}^n$ and $\tilde{\mathfrak{d}}_\theta(f,\mathcal{S})\le\varepsilon$ for some $\varepsilon<1/2$, then $\mathfrak{d}_\theta(f,\mathcal{S})\le\delta(\varepsilon,\theta)$.

Proof. Set $\delta(\varepsilon,\theta)=\sup\mathfrak{d}_\theta(f,\mathcal{S})$, the supremum being taken over all mappings $f\colon U\to\mathbb{R}^k$ of domains $U\subset\mathbb{R}^n$, satisfying $\tilde{\mathfrak{d}}_\theta(f,\mathcal{S})\le\varepsilon<1/2$.

If $\tilde{\mathfrak{D}}_\theta(f,\mathcal{S})=0$, then $f\in\mathcal{S}$, by Lemma 1.9, and so $\mathfrak{d}_\theta(f,\mathcal{S})=0$, by Lemma 1.8. We have thus proved that $\delta(0,\theta)=0$ for all $\theta\in(0,1)$.

To complete the proof, it suffices to show that $\delta(\varepsilon,\theta)\to 0$ as $\varepsilon\to 0$, for each $\theta\in(0,1)$. Suppose, contrary to our claim, that there is a number $\theta\in(0,1)$ such that $\delta(\varepsilon,\theta)\not\to 0$ as $\varepsilon\to 0$. Then, there exist both a number $\delta_0>0$ and a sequence $\varepsilon_\nu\in(0,1/2)$, $\nu=1,2,\ldots$, such that $\delta(\varepsilon_\nu,\theta)\ge\delta_0$ while $\varepsilon_\nu\to 0$ as $\nu\to\infty$. Hence it follows in turn that there are a sequence of mappings $f_\nu: U_\nu\to\mathbb{R}^n$ of domains $U_\nu\subset\mathbb{R}^n$ and a sequence of balls $B_\nu=B(x_\nu,R_\nu)$ in U_ν, such that $\tilde{\mathfrak{d}}_\theta(f_\nu,\mathcal{S})\le\varepsilon_\nu$ and

$$\sup_{y\in B(x_\nu,\theta R_\nu)}|f_\nu(y)-u(t)|\ge\frac{\delta_0}{2}\operatorname{diam}f_\nu(B_\nu) \tag{4.16}$$

for each mapping $u\in\mathcal{S}(B_\nu)$. By property $\mathcal{P}_2$, we can rewrite inequality (4.16) in the following form:

$$\sup_{y\in B(0,\theta)}|\phi_\nu(y)-u(t)|\ge\frac{\delta_0}{2},\quad \nu=1,2,\ldots, \tag{4.17}$$

for each $u\in\mathcal{S}(B(0,1))$, where

$$\phi_\nu(y)=\frac{f_\nu(x_\nu+R_\nu y)-f_\nu(x_\nu)}{\operatorname{diam}f_\nu(B_\nu)}.$$

As but one consequence of Lemma 1.17, we mention that if $\mathcal{F}$ is a uniformly bounded family of mappings from a ball $B\subset\mathbb{R}^n$ into the space $\mathbb{R}^k$ and there is a number $\varepsilon_0\in(0,1/2)$ such that $\mathfrak{d}_\theta(f,\mathcal{S})\le\varepsilon_0$ for each $f\in\mathcal{F}$, then $\mathcal{F}$ is equicontinuous on compact subsets of B. Remark 4.9 now shows that this assertion is still true if we replace $\mathfrak{d}_\theta(f,\mathcal{S})$ by $\tilde{\mathfrak{d}}_\theta(f,\mathcal{S})$.

Pick $\varepsilon_0\in(0,1/2)$ and choose N such that $\varepsilon_\nu\le\varepsilon_0$ for all $\nu\ge N$. Consider the family of mappings

$$\mathcal{F}=\{\phi_\nu|_{B(0,1)}\}_{\nu\ge N}.$$

Since $\phi_\nu(0)=0$ and $\operatorname{diam}\phi_\nu(B(0,1))=1$, for each $\nu=1,2,\ldots$, it follows that the family $\mathcal{F}$ is uniformly bounded. Moreover,

$$\begin{array}{rcl}\tilde{\mathfrak{d}}_\theta(\phi_\nu|_{B(0,1)},\mathcal{S}) & = & \tilde{\mathfrak{d}}_\theta(f_\nu|_{B_\nu},\mathcal{S})\\ & \le & \varepsilon_\nu\\ & \le & \varepsilon_0\end{array}$$

for all $\nu\ge N$. According to the above remark, the family $\mathcal{F}$ is equicontinuous in each ball $B(0,t)$, $t\in(0,1)$. By the Arzelà-Ascoli Theorem, there is a subsequence ϕ_{ν_i}, $i=1,2,\ldots$, and a mapping $u: B(0,1)\to\mathbb{R}^k$ such that ϕ_{ν_i} converges to u pointwise in $B(0,1)$ and uniformly on each ball $B(0,t)$, $t<1$.

We next claim that $u \in \mathcal{S}$. To prove this, let $x \in B(0,1)$ and let $R < 1 - |x|$ be a fixed positive number. Then, $B(x,R) \subset B(0,t)$ for each $t \in (0,1)$ larger than $|x| + R$. From the inequality

$$\begin{aligned} \tilde{\mathfrak{d}}_\theta(\phi_{\nu_i}|_{B(x,R)}, \mathcal{S}) &\leq \tilde{\mathfrak{d}}_\theta(\phi_{\nu_i}|_{B(0,1)}, \mathcal{S}) \\ &\leq \varepsilon_{\nu_i} \end{aligned}$$

we conclude that, for each $i = 1, 2, \ldots$, there is a mapping $u_i \in \mathcal{S}(B(x, \theta R))$ such that

$$\begin{aligned} \sup_{y \in B(x,\theta R)} |\phi_{\nu_i}(y) - u_i(y)| &\leq 2\, \tilde{\mathfrak{d}}_{\theta, B(x,R)}(\phi_{\nu_i}, \mathcal{S})\, \mathrm{diam}\, \phi_{\nu_i}(B(x,R)) \\ &\leq 2\, \varepsilon_{\nu_i}\, \mathrm{diam}\, \phi_{\nu_i}(B(0,1)) \\ &= 2\, \varepsilon_{\nu_i}. \end{aligned}$$

Since $\phi_{\nu_i} \to u$ uniformly in each ball $B(0,t)$, $t < 1$, and $\varepsilon_{\nu_i} \to 0$, as $i \to \infty$, it follows that $u_i \to u$ uniformly in the ball $B(x, \theta R)$. Property $\mathcal{P}_4$ implies that $u|_{B(x,\theta R)}$ is of class $\mathcal{S}$, i.e., the mapping u is locally of class $\mathcal{S}$. By $\mathcal{P}_6$, we get $u \in \mathcal{S}$, as required.

The existence of a sequence $\{\phi_{\nu_i}\}$, converging to a mapping $u \in \mathcal{S}(B(0,1))$ uniformly on each ball $B(0,t)$, $t < 1$ (and, consequently, on the ball $B(0,\theta)$), contradicts (4.17). This proves the lemma. □

Proof of Theorem 4.3. Consider the function

$$\Delta(\varepsilon) = \sup_{\mathrm{SBS}(\varepsilon)} \mathfrak{d}(f, \mathrm{Sol})$$

for ε in some interval $[0, \varepsilon_0)$. Using this function, one can reformulate Theorem 4.3 as $\Delta(\varepsilon) \to 0$ when $\varepsilon \to 0$.

On assuming that the last assertion fails and arguing in the same way as in the proof of the second assertion of Theorem 3 in [Kop82, Part I], we construct a sequence of maps $f_\nu : B(0,1) \to \mathbb{R}^k$, where $\nu = 1, 2, \ldots$, such that $f_\nu \in \mathrm{SBS}(\nu^{-1})$, the diameter of $f_\nu(B(0,1))$ just amounts to 1 and there are numbers $\theta \in (0,1)$ and $\mathfrak{d}_0 > 0$ satisfying

$$\sup_{y \in B(0,\theta)} |f_\nu(y) - u(y)| > \mathfrak{d}_0 \tag{4.18}$$

for all solutions u to the system $Au = 0$ in $B(0,1)$.

Fix a number $q_1 > n$ and a sequence $(r_i)_{i=1,2,\ldots}$ of real numbers with the property that $r_1 = \theta$, $r_{i+1} > r_i$ for all i and $r_i \to 1$ as $i \to \infty$. Consider the ball $B(0,r_1) = B(0,\theta)$ and choose a number $\varepsilon = \varepsilon_1$ satisfying the hypotheses of Corollary 4.8. Since the diameter of $f_\nu(B(0,1))$ is equal to 1 for all ν, we get by Corollary 4.8 that

$$|f_\nu(x') - f_\nu(x'')| \leq c_1\, |x' - x''|^{1-n/q_1}$$

for all $x', x' \in B(0,r_1)$, provided that $\nu^{-1} \leq \varepsilon_1$. Therefore, the sequence (f_ν) represents a uniformly bounded and equicontinuous family of mappings of the closed ball $\overline{B(0,r_1)}$ to $\mathbb{R}^k$. According to the Ascoli-Arzel'a theorem the sequence contains a subsequence (f_{ν_j}) which converges uniformly in the ball $\overline{B(0,r_1)}$ to a continuous mapping u_1.

We claim that u_1 is holomorphic inside the closure of $B(0,r_1)$. Indeed, decompose f_{ν_j} in $B(0,r_1)$ according to the generalised Cauchy formula of Lemma 4.4, obtaining

$$f_{\nu_j}(x) = \mathcal{P}_{\mathrm{dl}} f_{\nu_j}(x) + \mathcal{P}_v \partial''_A f_{\nu_j}(x)$$

for $x \in B(0,r_1)$. From the uniform convergence of the sequence in $\overline{B(0,r_1)}$ it follows that $\mathcal{P}_{\mathrm{dl}} f_{\nu_j}(x) \to \mathcal{P}_{\mathrm{dl}} u_1(x)$ for all $z \in B(0,r_1)$. Further, using Hölder's inequality we get

$$\begin{aligned}
|\mathcal{P}_v \partial''_A f_{\nu_j}(x)| &\leq \sqrt{\sum_{j=1}^{n} \|A_j\|^2} \int_{B(0,r_1)} \Big\| \Phi(\frac{x-y}{|x-y|}) \Big\| |x-y|^{1-n} |\partial''_A f_{\nu_j}(y)| \, dy \\
&\leq \mathrm{const}(A) \int_{B(0,r_1)} |x-y|^{1-n} |\partial''_A f_{\nu_j}(y)| \, dy \\
&\leq \mathrm{const}(A) \, \| |y|^{1-n} \|_{L^{q'_1}(B(0,2r_1))} \, \|\partial''_A f_{\nu_j}\|_{L^{q_1}(B(0,r_1))}
\end{aligned}$$

for $\nu_j^{-1} \leq \varepsilon_1$, where $1/q_1 + 1/q'_1 = 1$. By (4.13), the right-hand side here is majorised by

$$n_j^{-1} \, \mathrm{const}\,(A) \, \| |y|^{1-n} \|_{L^{q'_1}(B(0,2r_1))} \, \frac{\sigma_n^{1+\frac{1}{q_1}} r_1^{-n}}{(1-r_1)^n - n_j^{-1} \, c(q_1)}, \tag{4.19}$$

the norm $\| |y|^{1-n} \|_{L^{q'_1}(B(0,2r_1))}$ is finite because of $q'_1 < n/(n-1)$. It follows that $\mathcal{P}_v \partial''_A f_{\nu_j}(x) \to 0$ as $\nu_j \to \infty$, whence

$$u_1(x) = \mathcal{P}_{\mathrm{dl}} u_1(x)$$

for all $x \in B(0,r_1)$. Therefore, the mapping u_1 is continuous in the closed ball $\overline{B(0,r_1)}$ and is represented in its interior by the generalised Cauchy formula. Using the results of Section 3.4 of [Tar90] we conclude that u_1 satisfies $Au_1 = 0$ in the ball $B(0,r_1)$.

Iterating this argument successively for the balls $B(0,r_2), B(0,r_3), \ldots$ and using considerations from the proof of Theorem 1 in [Kop82, Part I], we derive a subsequence f_{ν_j} of the initial sequence $(f_\nu)_{\nu=1,2,\ldots}$ and a solution $u : B(0,1) \to \mathbb{R}^k$ to the system $Au = 0$, such that $f_{\nu_j} \to u$ uniformly on compact subsets of $B(0,1)$ as $\nu_j \to \infty$. However, this contradicts (4.18), which proves the theorem. $\square$

If $l = k$, i.e. A is a determined elliptic operator, an effective proof of this theorem is given in Section 9.4.6 of [Tar95]. It exploits the generalised Cauchy-type integral $u = \mathcal{P}_{\mathrm{dl}} f$ of Lemma 4.4 which satisfies $Au = 0$ and approximates f with suitable accuracy. For $l > k$, i.e. overdetermined elliptic operators A, the proof no longer works, for the potential $u = \mathcal{P}_{\mathrm{dl}} f$ does not satisfy $Au = 0$. In the general case the proof runs along more abstract lines elaborated in [Kop82], see Theorem 4 of Part II *ibid.*

5. Properties of mappings close to the sheaf of solutions

5.1. **Proximity of the derivatives.** Let ε be a positive number. Set

$$q(\varepsilon) = \inf_{f \in \mathrm{SBS}(\varepsilon)} \sup \{ q \in \mathbb{R} : \partial'_A f, \partial''_A f \in L^q_{\mathrm{loc}} \}.$$

Theorem 5.1. *As defined above, the function $q(\varepsilon)$ satisfies*

$$\lim_{\varepsilon \to 0} q(\varepsilon) = \infty.$$

Proof. It is sufficient to show that, for each $q > 1$, there exists an $\varepsilon_0 > 0$ such that $q(\varepsilon) \geq q$, provided $0 \leq \varepsilon < \varepsilon_0$. To this end, pick $q > 0$ and choose $\varepsilon_0 \in (0, 1)$ such that $\varepsilon_0 c(q) < 1$ and $\varepsilon_0 c(n) < 1$, where $c(q)$ is the constant of (4.6). It is a simple matter to see that there is a number $\theta \in (0, 1)$ satisfying $\varepsilon_0\, c(q)\, (1-\theta)^{-n} < 1$. Now, let $f : U \to \mathbb{R}^k$ be a mapping of class $\mathrm{SBS}(\varepsilon)$, with $0 \leq \varepsilon < \varepsilon_0$. Suppose x is an arbitrary point of the domain U and $B(x, R)$ a ball whose closure lies in U. From Theorem 4.7 it follows that, for the restriction of f to $B(x, R)$, inequalities (4.7) and (4.13) hold. Since $\operatorname{diam} f(B(x, R)) < \infty$, these inequalities imply that both $\partial'_A f$ and $\partial''_A f$ are of class $L^q(B(x, \theta R))^{k\times n}$. Hence $q(\varepsilon) \geq q$, which is the desired conclusion. $\square$

For hypercomplex analogues of the Cauchy-Riemann system, one can prove that $q(\varepsilon) = O(1)\, \frac{1}{\varepsilon}$ as $\varepsilon \to 0$ (cf. [Bez83]).

5.2. **Generalised Liouville's theorem.** Let $\mathcal{S} = \mathrm{Sol}$ be the class of solutions to the system $Au = 0$. As described above, it meets conditions $\mathcal{P}_1$-$\mathcal{P}_6$.

Consider a mapping $f : \mathbb{R}^n \to \mathbb{R}^k$ of class $W^{1,n+0}_{\mathrm{loc}}$, such that $\mathfrak{D}(f, \mathcal{S}) \leq \varepsilon$. According to Theorem 4.1 on the stability of the class $\mathcal{S}$, we have $\mathfrak{d}(f, \mathcal{S}) \leq \delta(\varepsilon)$, where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Choose a number $\varepsilon_0 > 0$ such that $\delta(\varepsilon) < 1/2$, for each $\varepsilon \in [0, \varepsilon_0)$. Then, $\mathfrak{d}(f, \mathcal{S}) < 1/2$ for all $\varepsilon \in [0, \varepsilon_0)$. By Corollary 1.20, we can assert that if f is bounded, then f is constant. We are thus led to the following strengthening of Theorem 1.18.

Theorem 5.2. *There exists a positive number ε_0 such that, if $f : \mathbb{R}^n \to \mathbb{R}^k$ is a bounded mapping of class $W^{1,n+0}_{\mathrm{loc}}$, satisfying $\mathfrak{D}(f, \mathrm{Sol}) \leq \varepsilon$ for some $\varepsilon \in [0, \varepsilon_0)$, then f is constant.*

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