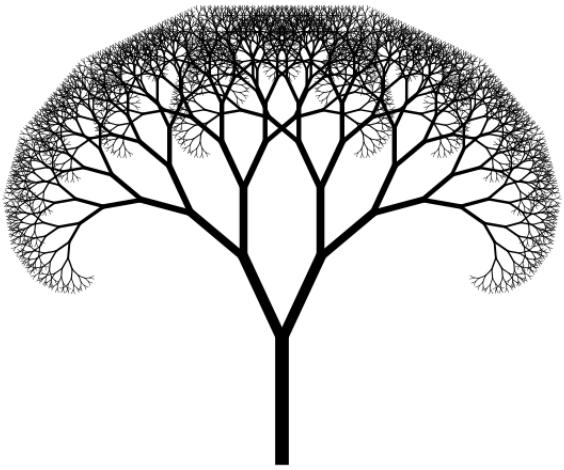
# Hardy Inequalities on Graphs



**Univ.-Diss.**

zur Erlangung des akademischen Grades
„doctor rerum naturalium“
(Dr. rer. nat.)
in der Wissenschaftsdisziplin Mathematik
eingereicht von


**Florian Fischer**

an der
Mathematisch-Naturwissenschaftlichen Fakultät
Institut für Mathematik
der Universität Potsdam


Tag der Disputation: 31. Mai 2024

| | |
|---|---|
| Hauptbetreuer: | Prof. Dr. Matthias Keller |
| Zweitbetreuer: | Prof. Dr. Yehuda Pinchover |
| Mentor: | Prof. Dr. Markus Klein |
| | |
| Gutachter: | Prof. Dr. Matthias Keller |
| | Prof. Dr. Yehuda Pinchover |
| | Prof. Dr. Ralph Chill |

Weitere Mitglieder der Prüfungskommission:

| | |
|---|---|
| | Prof. Dr. Sven Raum (Vorsitz) |
| | Prof. Dr. Christian Bär |
| | Prof. Dr. Maximilian Lein |
| | Prof. Dr. Jan Metzger |

Published online on the
Publication Server of the University of Potsdam:
https://doi.org/10.25932/publishup-64773
https://nbn-resolving.org/urn:nbn:de:kobv:517-opus4-647730

# Acknowledgements


> It was a perfectly ordinary night at Christ's high table, except that Hardy was dining as a guest. He had just returned to Cambridge as Sadleirian professor, and I had heard something of him from young Cambridge mathematicians. They were delighted to have him back: he was a *real* mathematician, they said, not like those Diracs and Bohrs the physicists were talking about: he was the purest of the pure.
>
> C. P. Snow in the foreword to G. H. Hardy's *A Mathematician's Apology*

My first thanks go undoubtedly to my supervisor Matthias Keller whose influence goes back to the very beginning of my master studies with a fascinating lecture series on the analysis on graphs. Since then he supported me in almost uncountably many ways: he gave me many interesting problems to deal with, supported all my scientific travels, initiated many first contacts and supported all my applications with thorough proofreading – to name a few. But beyond all of that, he explained mathematics to me hundreds of hours, and sharpened my way of thinking and writing mathematics.

Secondly, I also want to thank my second supervisor Yehuda Pinchover deeply. The time shortly before until shortly after the annual visits where usually my most productive weeks, and I typically went home after a meeting with a long list of new research problem for future investigations. These lists will guide me in the next years.

During the past years I enjoyed the hospitality of various institutions. In particular, I want to thank Krzysztof Bogdan (Wrocław University of Science and Technology) Oleksey Kostenko (University of Vienna), Norbert Peyerimhoff (University of Durham), Yehuda Pinchover (Technion Haifa), Alberto Setti (University of Insubria) and Karl-Theodor Sturm (University of Bonn) for kind invitations and the nice atmosphere they created during my stays. For some of these trips I got additional financial support by the Heinrich Böll foundation which I gratefully acknowledge.

In the first year of my studies I was supported by the DFG. Thereafter, from January 2020 until September 2023, I was supported by the Heinrich Böll foundation (P139140), and during the last weeks via DFG SPP 2265. I am very thankful for these supports which enabled me to focus purely on my research.

Moreover, I want to thank the organisers of the conferences and workshops 'Geometric aspects of harmonic analysis and spectral theory' at Technion, 'Analysis on graphs' at University of Jena, 'QGraph 2021' at University of Stockholm, and 'Geometric Aspects of Evolution and Control' at FernUniversität Hagen for the financial support for travel and/or accommodation.

I am grateful to my (future) co-authors Andrea Adriani, Matthias Keller, Anna Muranova, Noema Nicolussi, Norbert Peyerimhoff, Felix Pogorzelski and Alberto Setti for extensive scientific discussions that have not (yet) lead to joint articles. Working with them has broadened my mathematical knowledge and strongly influenced my mathematical thinking.

Special thanks go also to my colleagues in Potsdam. In particular, my time in Potsdam would not have been as memorable without my office mates Noema and Siegfried. Moreover, I want to thank Alberto, Christian R., Christian S., Lior, Matti, Michael, Noema, Philipp, Ram, Siegfried and Yannik for many fruitful discussions during lunch breaks and common room gatherings.

Since the last year of my master's I was the PhD representative of the institute and organised numerous social and scientific events. That would not have been half that fun without the help of Alberto, Diego, Franziska, Giuseppe, Josie, Tobias, and in particular, Undine. Moreover, I want to thank Melina for the initiating idea and her tireless support in founding and having a lively and lovely SIAM Chapter.

Especially, I want to thank the Heinrich Böll foundation also for the personal support. Via various seminars and gatherings during my time as a local coordinator for Potsdam I met many nice people with fascinating ideas. Special thanks go to Friederike with whom I organised a lovely two day online workshop.

Moreover, I want to thank the institute of mathematics in Potsdam for supporting me during all these years and giving me the opportunities to present the mathematics that I love in front of high school students in Frankfurt (Oder), Ludwigsfelde and Potsdam. I learned a lot from these experiences.

Finally, I want to express my love and gratitude to my wife Friederike and my daughters Fia and Femke. Without Friederike's constant support this thesis would not have been possible. She always covered my back and made it possible for me to see the world even though it often meant a lot of additional care work for her. Main parts of this thesis have been worked out during various lock-downs and isolations. I am very happy that Friederike made it possible for me to concentrate on mathematics in these times. Fia and Femke grounded me and showed me constantly that there is so much more than mathematics. I am very thankful to have such a wonderful family.

# Contents


**1 Introduction** **1**

**2 Preliminaries** **7**
- 2.1 Graphs and Schrödinger Operators . . . . . . . . . . . . . . . . . . . . . . 7
- 2.2 Energy Functionals Associated with Graphs . . . . . . . . . . . . . . . . 12

**3 Examples** **19**
- 3.1 Examples of Quasi-linear Schrödinger Operators . . . . . . . . . . . . . . . 19
- 3.2 Examples of Graphs . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20

**4 Ground State Representation** **23**
- 4.1 The Formula . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
- 4.2 Some Remarks on the Representation . . . . . . . . . . . . . . . . . . . 25
- 4.3 Elementary Inequalities and Equivalences . . . . . . . . . . . . . . . . . . 29
- 4.4 Proof of Theorem 4.1 . . . . . . . . . . . . . . . . . . . . . . . . . . . 30

**5 General Principles and Inequalities** **31**
- 5.1 Local Harnack Inequality and Harnack Principle . . . . . . . . . . . . . . 31
- 5.2 Picone and Anane-Díaz-Saá Inequalities . . . . . . . . . . . . . . . . . . 36
- 5.3 Principal Eigenvalues on Finite Subsets . . . . . . . . . . . . . . . . . . . 40
- 5.4 Poisson-Dirichlet Problems on Finite Subsets . . . . . . . . . . . . . . . . 43
- 5.5 Characterisations of the Maximum Principle on Finite Subsets . . . . . . 45

**6 The Agmon-Allegretto-Piepenbrink Theorem** **49**

**7 Comparison Principles** **53**
- 7.1 A Weak Comparison Principle for Non-Negative Potentials . . . . . . . . 53
- 7.2 A Weak Comparison Principle on Finite Subsets . . . . . . . . . . . . . . 54

**8 The Variational Capacity** **59**
- 8.1 Basic Properties . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 59
- 8.2 Consequences of the Comparison Principle . . . . . . . . . . . . . . . . . 62

**9 Global Results for Subcritical and Critical Energy Functionals** **65**
- 9.1 Minimal Growth At Infinity . . . . . . . . . . . . . . . . . . . . . . . . 66
- 9.2 Existence and Properties of Global Green's Functions . . . . . . . . . . . 67
- 9.3 Examples of Green's functions . . . . . . . . . . . . . . . . . . . . . . . 70
- 9.4 Bounds for the Principal Eigenvalue . . . . . . . . . . . . . . . . . . . . 72

**10 Characterisations of Criticality** 75
10.1 Subcriticality on Subsets . . . 77
10.2 Proof of Theorem 10.1 . . . 80
10.3 Examples of Subcritical and Critical Energy Functionals . . . 84
10.4 Liouville Comparison Principle . . . 86

**11 Decay of Hardy Weights** 89
11.1 Convexity-type Results . . . 90
11.2 Proof of Theorem 11.1 . . . 94

**12 Optimal Hardy Weights** 97
12.1 Supersolution Constructions . . . 99
12.2 Coarea Formula . . . 102
12.3 Proof of Theorem 12.1 . . . 106
12.3.1 Criticality . . . 106
12.3.2 Null-criticality . . . 110
12.3.3 Optimality Near Infinity . . . 112
12.4 Examples of Optimal Hardy Weights . . . 112
12.5 Applications . . . 115
12.5.1 Heisenberg-Pauli-Weyl-type Inequality . . . 116
12.5.2 A First Rellich-type Inequality . . . 117

**13 Rellich-Type Inequalities** 119

**A Elementary Estimates** 125
A.1 Proof of Theorem 1.1 . . . 125
A.2 Proof of Lemma 4.8 . . . 133
A.3 Estimates for the Simplified Energy . . . 144

**Bibliography** 149

# 1. Introduction

> [T]he study of mathematics is, if an unprofitable, a perfectly harmless and innocent occupation.
>
> G. H. Hardy, *A Mathematician's Apology*, p. 74

This thesis is a study of existence and optimality of the Hardy inequality on graphs which can be seen as a part of discrete quasi-linear potential theory. The first analysis of such an inequality goes back a little bit more than hundred years when Hardy found an elegant and simple proof of Hilbert's double series theorem, see [Har20]. Although it is not mentioned explicitly, the paper contains the essential argument for his then famous inequality, see [KMP06] for a detailed historical survey about the origins of Hardy's inequality.

It was then Landau who proved (in a letter to Hardy, [Lan21]) that the following inequality is true and the constant is sharp: For all $p \in (1, \infty)$, and all compactly supported functions $\varphi \in C_c(\mathbb{N})$ with $\varphi(0) = 0$ we have

$$\sum_{n=1}^{\infty} |\varphi(n) - \varphi(n-1)|^p \geq \sum_{n=1}^{\infty} w^H(n) |\varphi(n)|^p,$$

where

$$w^H(n) = \left(\frac{p-1}{p}\right)^p \frac{1}{n^p}, \qquad n \in \mathbb{N}.$$

This inequality was first highlighted in [HLP34] and is referred to as a *p-Hardy inequality* on $\mathbb{N}$ with $p$-Hardy weight $w^H$. Since then various proofs of this inequality were given, where short and elegant ones are due to Elliott [Ell26] and Ingham, see [HLP34, p. 243] and by Lefèvre [Lef19; Lef20] (see also [Hua23b] for an improvement of the latter). For $p = 2$, other nice proofs (and, in fact, improvements) were found recently by Huang in [Hua21] and by Keller, Pinchover, and Pogorzelski in [KPP18a].

As a consequence of one of the main results in thesis, we can show that the classical $p$-Hardy weight can be improved. To be more precise, we can prove the following:

**Theorem 1.1** *Let $p \in (1, \infty)$. Then, for all $\varphi \in C_c(\mathbb{N})$ with $\varphi(0) = 0$,*

$$\sum_{n=1}^{\infty} |\varphi(n) - \varphi(n-1)|^p \geq \sum_{n=1}^{\infty} w(n) \, |\varphi(n)|^p \,,$$

*where $w_p$ is a strictly positive function given by*

$$w(n) = \left(1 - \left(1 - \frac{1}{n}\right)^{\frac{p-1}{p}}\right)^{p-1} - \left(\left(1 + \frac{1}{n}\right)^{\frac{p-1}{p}} - 1\right)^{p-1}.$$

*Furthermore, $w_p$ is optimal (see Chapter 2 for the definition), and we have for all $n \in \mathbb{N}$*

$$w(n) > w^H(n).$$

*Moreover, for integer $p \geq 2$, we have $w(n) = \sum_{k \in 2\mathbb{N}_0} c_k n^{-k-p}$ with $c_k > 0$.*

A proof is given in Appendix A.1. Note that $c_0 = ((p-1)/p)^p$ is the famous $p$-Hardy constant.

The original Hardy inequality was generalised to various contexts. In an abstract way, it can be stated as

$$E \geq W,$$

where $E$ is an energy functional associated with a non-negative (non-linear) operator, and $W$ is the canonically obtained non-negative functional from a positive weight function $w$. The classical choice for $W$ is the $p$-th power of the $\ell^p$-norm with weight $w$. For a detailed analysis in the continuum, we refer to the monographs [BEL15; KPS17; OK90; RS19], and references therein. In the continuum, the domain of $E$ and $W$ is usually the set of smooth and compactly supported functions, and in the discrete, the domain is the set of finitely supported functions.

The two leading questions in this thesis are the following.

- When do we have such an abstract inequality for a sufficiently large class of functions? This leads to various characterisations and is known as *criticality theory*, see Chapter 9 and Chapter 10.
- In the case of existence of a Hardy inequality: How "large" can we make $w$, i.e., how do we get an optimal $w$? This question was proposed first in [Agm82, page 6], and an answer is given in Chapter 12. The corresponding theory is called *optimality theory*.

This thesis is by far not the first paper addressing these two questions, and the main novelty is the quite general non-linear and non-local setting together with Schrödinger operators with arbitrary potentials. On general graphs beyond the locally finite setting, we are only aware of results on the free $p$-Laplacian in [GHJ21; HM15; KM16; MS23; Mug13; SY93a], and we are not aware of any result on general $p$-Schrödinger operators. Moreover, the cited papers do not deal with optimality theory, and only [SY93a] contributed to criticality theory.

Especially, the first question is rather old and many people have contributed to its solution in various underlying settings, see here a discussion of linear settings in the notes of [KLW21, Chapter 6]. In the last two decades, criticality theory of local energy functionals associated with quasi-linear Schrödinger operators with not necessarily non-negative potential part was studied and many characterisations of criticality in this local but non-linear setting where shown, see [DD14; DP16; HPR24; PP16; PR15; PT08; PT09; PTT08]. In this thesis, we show the non-local counterparts to these results, see Theorem 10.1. It generalises the known results associated with quasi-linear standard Laplacians on locally finite graphs and networks in [KY84; NY76; MY92; Pra04; SY93b;

Yam77; Yam86] and on almost locally finite graphs and networks in [SY93a]. The proof of Theorem 10.1 can be seen as a profound application of the ground state representation formula (see Chapter 4) and comparison principles (see Chapter 7).

Answers to the second question are relatively new and rare in comparison. One reason is that for a long time only the constant in front of the classical Hardy weight was studied in detail. The most important contributions here are [DFP14] for the local linear setting, and [DP16; Ver23] for the local quasi-linear setting. Our main result on optimal $p$-Hardy weights can be seen as a counterpart of the main result in [Ver23], see Theorem 12.1. Moreover, it is a quasi-linear generalisation of the seminal work [KPP18b] in the linear setting on locally summable graphs.

In [DFP14], an *optimal Hardy weight* $w$ associated with a linear Schrödinger operator on domains in $\mathbb{R}^d$ (or on non-compact Riemannian manifolds) was defined first. Roughly speaking, $w$ is optimal if

- $w$ is critical, i.e., for all $\tilde{w} \gneq w$ the Hardy inequality fails,
- $w$ is null-critical, i.e., the corresponding ground state is not an eigenfunction, and
- $w$ is optimal near infinity, i.e., for any $\lambda > 0$, the Hardy inequality outside of any compact set fails for the weight $(1+\lambda)w$.

In [DFP14] also a way of obtaining optimal Hardy weights was given. Using a different approach, the main result and definition of optimality of [DFP14] were generalised to $p$-Laplacians, $p \in (1, \infty)$, on Riemannian manifolds in [DP16]. Recently, for a large class of potentials (including non-positive potentials), optimal Hardy weights for $p$-Schrödinger operators were constructed in [Ver23].

On weighted locally finite graphs, inspired by the approaches from [DFP14; DP16], a way of obtaining optimal Hardy weights for linear Schrödinger operators with arbitrary potential parts was given in [KPP18b].

In the present thesis, we show in Theorem 12.1 how to obtain optimal Hardy weights for $p$-Schrödinger operators with arbitrary potential term on weighted locally finite graphs with locally summable boundary, following partially the approaches in [DP16; KPP18b; Ver23]. To be more specific, we evolve discrete quasi-linear versions of them, to prove criticality and null-criticality. The main tools in these proofs in [DP16; KPP18b; Ver23] were a coarea formula and the ground state representation formula. Corresponding discrete quasi-linear versions of both will also be of fundamental importance here.

For proving optimality near infinity, we establish instead discrete versions of results in [KP20], Theorem 11.1 and Theorem 11.2. Specifically, Theorem 11.1 states a necessary decay condition of Hardy weights in terms of a certain $\ell^1$-summability with respect to weights which are related to positive superharmonic functions. These results are also valid for arbitrary potential parts.

By a supersolution construction technique, we show how to obtain optimal Hardy weights for $p$-Schrödinger operators. This can be seen as a discrete version of results in [Ver23]. The results in the continuum made use of the chain rule, which does not hold on graphs in general. We use instead the mean value theorem to circumvent the problem.

The main tools towards the main results, Theorem 10.1 and Theorem 12.1, are

- the ground state representation formula (Theorem 4.1),
- the Agmon-Allegretto-Piepenbrink theorem (Theorem 6.1),
- the weak comparison principle (Proposition 7.3), and
- the coarea formula (Proposition 12.10).

In the local theory, the ground state representation is of fundamental importance, see e.g. [BEL15; DP16; HPR24; PP16; PR15; PT09; PTT08] for applications. This representation is an equivalence between functionals. It states that the $p$-energy functional associated with a $p$-Schrödinger operator is equivalent to a simplified energy functional consisting of non-negative terms only. For non-local $p$-Schrödinger operators in the Euclidean space, which includes graphs as a special case, only a one-sided inequality for $p \geq 2$ was known, see [FS08].

The novelty here is that we show a ground state representation formula for non-local $p$-Schrödinger operators for all $p > 1$ in terms of an equivalence between the corresponding $p$-energy functional and the simplified energy. We show this statement on graphs in Theorem 4.1 and Corollary 4.2.

In the statement and corresponding proof, we focus on graphs but since the fundamental methods are based on pointwise estimates, also corresponding results in other non-local settings are valid. To be more specific, our results can also be extended to non-local $p$-Schrödinger operators on the Euclidean space in the spirit of [FS08], confer also with [And+08; And+09; BF14; CMS18; DKP16; KMS15]. We chose graphs because they do not have local regularity issues and therefore the presentation is less technical than, e.g., for non-local $p$-Schrödinger operators on $\mathbb{R}^d$, and fits in the remaining setting of the thesis.

As a second main tool, we prove an Agmon-Allegretto-Piepenbrink-type theorem, see Theorem 6.1. It states that the energy functional is non-negative if and only if there is a positive superharmonic function. Since our $p$-Hardy weights should be positive weights, the non-negativity of the energy functional is a natural standing assumption. See [All74; Pie74; Sul87] for a linear version in the continuum, [Dod84; KPP20b] for a linear version in the discrete setting, [PP16] for a recent non-linear version in the continuum, and [LSV09] for a corresponding result on strongly local Dirichlet forms.

For the proof of the Agmon-Allegretto-Piepenbrink theorem, we need to show the following basic results on finite subsets of the infinite graph: a local Harnack principle, a Picone-type inequality, an Anane-Díaz-Saá-type inequality, the existence of a principal eigenvalue, the existence and uniqueness of solutions to the Poisson-Dirichlet problem, characterisations of the maximum principle. While these results are known on finite graphs (see [Amg08; BH09; CL11; HS97a; HS97b; PC11; PKC09]), some adaption is needed to deal with the possibly infinite boundaries, and so we included these methods for convenience. These tools are folklore in the linear case, and they are also well understood in the quasi-linear but local case (see here [HPR24; PP16; PR15; PT08; PT09]).

The third main toolbox is the weak comparison principle, meaning that for two nice functions $u, v$, we have $u \leq v$ outside and $Hu \leq Hv$ inside a given set $V$ implies that also $u \leq v$ inside $V$. Here $H$ denotes the operator of interest. In the linear setting, this is equivalent to the weak maximum principle. For non-negative potentials, the weak comparison principle holds under very mild assumptions on arbitrary subsets of the graph, see Lemma 7.1, which can be seen as a generalisation of results in [HS97a; KLW21; Pra04]. However, relaxing the assumption on the potential to possibly negative or sign-changing potentials seems to tighten the assumption on subsets to compact subsets. This, however, has also been observed in the local case, see [GS98; PP16].

The fourth main tool is the coarea formula which is a standard tool from geometric analysis. It relates the weighted $p$-energy of a function to a weighted integral over the boundary of the level sets, see [KLW21; KPP18b; KPP20a] for linear Schrödinger operators on graphs. It seems to be new on graphs in the non-linear case $p \neq 2$.

The main parts of the thesis can also be found in [Fis22; Fis23; Fis24; FKP23]. More specifically,

- in [FKP23], an optimal $p$-Hardy weight on the line graph $\mathbb{N}$ is computed and it is shown that it is strictly larger than the classical $p$-Hardy weight. This corresponds mainly to Theorem 1.1, and Appendix A.1;
- in [Fis23], the ground state representation formula and some characterisations of $p$-criticality are obtained. This corresponds mainly to Chapter 4 and Chapter 10;
- in [Fis22], an Agmon-Allegretto-Piepenbrink-type theorem, comparison principles and the (non-)existence of Green's functions are discussed. This corresponds mainly to Chapter 5, Chapter 7, Chapter 9 and Chapter 10;
- in [Fis24], the decay and optimality of $p$-Hardy weights are discussed. This corresponds mainly to Chapter 11 and Chapter 12.

We also want to highlight that this thesis is not a simple collection of the four paper [Fis22; Fis23; Fis24; FKP23]. In fact the possibility of rewriting and reorganising is used in several ways. The main changes and additions can be summarised as follows:

- Whereas the mentioned paper state results for the whole graph, we usually show here the results on subgraphs.
- To make the ideas and statements more accessible and vivid we included a large number of examples, most of them have not been discussed before in the non-linear setting. For comparison, we often state the corresponding results from the continuum.
- Whereas capacities have only be studied for singletons, we state the results here for arbitrary sets.

- Whereas Green's potentials have only be studied for delta charges, that is, Green's functions, we state the results here for arbitrary compactly supported charges.
- We added some more characterisations of criticality in Theorem 10.1. This lead also to an improvement of Theorem 11.1.
- Chapter 13 about Rellich-type inequalities is new.

Moreover, for some examples we make use of very recent results in [AFS24]. This text, however, is only focussing on non-negative or trivial potentials, and is not included in this thesis, which exclusively discusses the possibility of having sign changing or non-positive potentials. A lot more can be derived (and usually in a much simpler way) if one is only interested in non-negative potentials. A good example for the arising difficulties are the comparison principles in Chapter 7.

Every author has a typical reader in mind when preparing and writing a manuscript. This thesis is written for graduate students who are new to the subject and have only basic knowledge of functional analysis. Some standard techniques (as e.g. the application of the comparison principle) are worked out in detail multiple times. The phrases "here are the details" and "indeed" are used to signalise experts that the techniques in the corresponding proof have been used before and remaining part of the proof might be skipped.

The thesis is organised as follows: In Chapter 2, we introduce all necessary definitions to follow the remaining parts. Thereafter, we explain briefly the leading examples of quasi-linear Schrödinger operators and graphs. The next four chapters can be interpreted as the toolbox chapters. Some of the results might be seen as folklore but all of them are new in our setting and the proofs usually differ from those in the classical linear case. The toolbox chapters include the ground state representation formula (Chapter 4), the Harnack inequality and principle, existence and uniqueness of solution to certain Poisson problem, characterisations of the maximum principle on finite subsets (all Chapter 5), the Agmon-Allegretto-Piepenbrink theorem (Chapter 6), and weak comparison principles (Chapter 7). Thereafter, we study the variational $p$-capacity in detail in Chapter 8. Then, Green's functions are constructed in Chapter 9 which is the last step before we proof our first main result about characterisation of criticality in Chapter 10. In Chapter 11, we have a closer look on decay properties of $p$-Hardy weights. Using the results of the two previous chapters, we are finally in a position to show the second main result about optimal $p$-Hardy weights in Chapter 12. In Chapter 13, an application in terms of a Rellich inequality is given. In an appendix we prove a number of elementary estimates which would have disturbed the flow of reading in the main part. Some open problem are attached in the main part as well.

# 2. Preliminaries

> *Is* mathematics 'unprofitable'? In some ways, plainly, it is not; for example, it gives great pleasure to quite a large number of people. [...] Is mathematics 'useful', *directly* useful, as other sciences [...] are? [...] I shall ultimately say No, though some mathematicians and most outsiders, would no doubt say Yes. And is mathematics 'harmless'? Again the answer is not obvious, and the question is one which I should have in some ways preferred to avoid.
>
> G. H. Hardy, *A Mathematician's Apology*, p. 75

In this chapter, we start by introducing graphs. Thereafter, we define quasi-linear Schrödinger operators on graphs. We end this part by introducing $p$-energy functionals and showing a connection to $p$-Schrödinger operators via Green's formula.

This chapter contains all definitions which are necessary to understand the main results of this thesis. However, non-standard definitions will be recalled before use in later chapters.

## 2.1 Graphs and Schrödinger Operators

Let an infinite set $X$ equipped with the discrete topology and a symmetric function $b\colon X\times X\to[0,\infty)$ with zero diagonal be given such that $b$ is locally summable, i.e., the vertex degree $\deg\colon X\to[0,\infty]$ satisfies

$$\deg(x) := \sum_{y\in X} b(x,y) < \infty, \qquad x\in X.$$

We refer to $b$ as a *graph* over $X$ and elements of $X$ are called *vertices*.

Two vertices $x,y$ are called *connected* (or neighbours) with respect to the graph $b$ if $b(x,y)>0$, in terms $x\sim y$. A subset $V\subseteq X$ is called *connected* with respect to $b$, if for every two vertices $x,y\in V$ there are vertices $x_0,\ldots,x_n\in V$, such that $x=x_0$, $y=x_n$ and $x_{i-1}\sim x_i$ for all $i\in\{1,\ldots,n-1\}$. Let denote by $\partial V=\{y\in X\setminus V\colon y\sim z\in V\}$ the exterior boundary of $V$. Throughout this paper we will always assume that

$$X \text{ is connected with respect to the graph } b.$$

An important class of examples are the locally finite graphs. Here, a graph $b$ on $X$ is called *locally finite* on $V$ if for all $x\in V$

$$\#\{y\in X : y\sim x\} < \infty.$$

We now turn to functions: Let $S$ be some arbitrary set. A function $f\colon S\to\mathbb{R}$ is called *non-negative*, *positive*, or *strictly positive* on $I\subseteq S$, if $f\geq 0$, $f\gneq 0$, $f>0$ on $I$, respectively. We denote the positive and negative part of $f$ by $f_+$ and $f_-$, respectively; i.e., $f_+ = f \vee 0$ and $f_- = -f \vee 0$. If for two non-negative functions $f_1, f_2 \colon S \to \mathbb{R}$ there exists a constant $C > 0$ such that $C^{-1} f_1 \leq f_2 \leq C f_1$ on $I \subset S$, we write

$$f_1 \asymp f_2 \quad \text{on } I,$$

and call them *equivalent* on $I$. Equivalent functions play an important role in Chapter 4.

A function $f \in C(X)$ that is positive on $V \subseteq X$ is called *almost proper* on $V$ if $f^{-1}(I) \cap V$ is a finite set for any compact set $I \subseteq (0, \infty)$. Note that this definition differs slightly from standard definitions of a proper function as we allow $u$ to vanish infinitely often on $V$. Moreover, a strictly positive and almost proper function on $V$ is called *proper* on $V$. If $f$ is a proper function on $V$, then $0$ and $\infty$ are the only possible accumulation points of $f|_V$, and if $V$ is also infinite at least one of them is always an accumulation point.

Moreover, $f \in C(X)$ is called *of bounded oscillation* on $V$ if

$$\sup_{x,y \in V, x \sim y} |f(x)/f(y)| < \infty.$$

In particular, such a function cannot vanish on $V$.

Proper functions of bounded oscillation play an important role in Chapter 11 and Chapter 12.

**Remark 2.1 (Locally finiteness)** If there exists an almost proper function of bounded oscillation on $V \cup \partial V \subseteq X$ (where $\partial V$ might be empty), then the graph is locally finite on $V$.

This can be seen as follows: Assume that $f$ is such a function. First note that $f > 0$ on $V \cup \partial V$ since $f$ is of bounded oscillation. Then, being proper on $V \cup \partial V$ implies that $0$ or $\infty$ are the only accumulation points of $f|_{V \cup \partial V}$. Being of bounded oscillation implies that neighbouring vertices never reach an accumulation point. Thus, there exists a compact set in $(0, \infty)$ containing all the images of $f|_{V \cup \partial V}$ of neighbours of a vertex and by the properness this implies that any vertex can only have finitely many neighbours, i.e., the graph is locally finite on $V$.

The space of real valued functions on $V \subseteq X$ is denoted by $C(V)$ and is a subspace of $C(X)$ by extending the functions of $C(V)$ by zero on $X \setminus V$. The space of functions with compact support in $V$ is denoted by $C_c(V)$. Sometimes it is convenient to speak of the support of a function $f \in C(X)$; this is defined via

$$\operatorname{supp}(f) := \{x \in X : f(x) \neq 0\}.$$

Note that $\operatorname{supp}(f) \subseteq V$ if and only if $f \in C(V)$.

A strictly positive function $m \in C(X)$ extends to a measure with full support via $m(V) = \sum_{x \in V} m(x)$ for $V \subseteq X$. To emphasise the measure on the vertices, we will sometimes also speak of a graph as a triple $(X, b, m)$, or as a graph $b$ over $(X, m)$.

The next fundamental definition is the one of the $p$-Laplacian. But first, we have to introduce some notation. For showing the connection to the counterpart in the continuum, we introduce the difference operator $\nabla$ on $C(X \times X)$ via

$$\nabla_{x,y} f := f(x) - f(y), \qquad x, y \in X.$$

Let $p \in [1, \infty)$. For $V \subseteq X$, let the *formal space* $F(V) = F_{b,p}(V)$ be given by

$$F(V) := \{f \in C(X) : \sum_{y \in X} b(x, y) \left|\nabla_{x,y} f\right|^{p-1} < \infty \text{ for all } x \in V\}.$$

Note that $F(V) \subseteq F(W)$ for $W \subseteq V \subseteq X$, but $C(W) \subseteq C(V)$. If $V = X$ we also write $F = F(X)$.

For $1 \leq p < 2$ we make the convention that $|t|^{p-2}\, t = 0$ if $t = 0$, i.e., $\infty \cdot 0 = 0$. Then, we can write for all $p \geq 1$,

$$(t)^{\langle p-1 \rangle} := |t|^{p-1} \operatorname{sgn}(t) = |t|^{p-2} t, \qquad t \in \mathbb{R}.$$

Here, $\operatorname{sgn}\colon \mathbb{R} \to \{-1, 0, 1\}$ is the sign function, that is $\operatorname{sgn}(t) = 1$ for all $t > 0$, $\operatorname{sgn}(t) = -1$ for all $t < 0$, and $\operatorname{sgn}(0) = 0$. We remark that $F(V) = C(X)$ if $p = 1$, by the local summability assumption on the graph.

Next, we show a basic lemma, which states an alternative representation for the formal space. There (and for various other estimates in the preceding chapters), we need the following *elementary inequality*: we have for all $p \geq 0$ that

$$|\alpha + \beta|^p \leq 2^p (|\alpha|^p + |\beta|^p), \qquad \alpha, \beta \in \mathbb{R}. \tag{2.1}$$

This follows from $|\alpha + \beta|^p \leq (2 \max\{|\alpha|, |\beta|\})^p \leq 2^p (|\alpha|^p + |\beta|^p)$. The constant can be improved to $2^{p-1}$ if $p \geq 1$.

We set

$$\ell^\infty(V) := \left\{f \in C(X) : \sup_{x \in V} |f(x)| < \infty\right\}.$$

The last two statements of the following lemma appeared first in [MS23, Lemma 4.1].

**Lemma 2.2 (Properties of the formal space)** *Let $V \subseteq X$ and $p \geq 1$. Moreover, set $F(V) = F_{b,p}(V)$. Then,*

$$F(V) = \{f \in C(X) : \sum_{y \in X} b(x, y) \left|f(y)\right|^{p-1} < \infty \textit{ for all } x \in V\}.$$

*In particular, $C_c(X) \subseteq \ell^\infty(V) \subseteq F(V)$. If $f \in F(V)$ and $q \in [1, \infty)$ then $f^{1/q} \in F(V)$. Moreover, if $q \in [p, \infty)$, then $F_{b,q}(V) \subseteq F(V)$. If $f \in C(X)$ is of bounded oscillation on $V$, then $f \in F(V)$.*

*Proof.* The case $p = 1$ is trivial. Let $p > 1$, and denote the set on the right-hand side by $\hat{F}(V)$. We obviously have that $C_c(V) \subseteq \ell^\infty(V) \subseteq \hat{F}(V)$. Furthermore, let $f \in \hat{F}(V)$. Then, using the elementary inequality (2.1), we get for any $x \in V$ that

$$\sum_{y\in X} b(x,y)\,|\nabla_{x,y} f|^{p-1} \leq 2^{p-1}\Big(|f(x)|^{p-1}\sum_{y\in X} b(x,y) + \sum_{y\in X} b(x,y)\,|f(y)|^{p-1}\Big).$$

The first sum on the right-hand side is finite by the local summability property of the graph $b$. The second sum is finite since $f \in \hat{F}(V)$. This shows $f \in F(V)$.

Moreover, if $f \in F(V)$ we obtain $f \in \hat{F}(V)$ since for all $x \in V$

$$\sum_{y\in X} b(x,y)\,|f(y)|^{p-1} \leq 2^{p-1}\Big(|f(x)|^{p-1}\sum_{y\in X} b(x,y) + \sum_{y\in X} b(x,y)\,|\nabla_{x,y} f|^{p-1}\Big) < \infty.$$

The last three statements follow from Hölder's inequality and the summability condition on the graph $b$. □

Now, we are in a position to define the Laplacian: Let $m$ be a measure on $X$. Then, the *p-Laplace operator* $L = L_{b,m,p} \colon F(V) \to C(V)$ is defined via

$$Lf(x) := \frac{1}{m(x)}\sum_{y\in X} b(x,y)\,(\nabla_{x,y} f)^{\langle p-1\rangle}, \qquad x \in V.$$

Let $p \geq 1$. If we have additionally $m = 1$, $b(X \times X) \subseteq \{0,1\}$, then $L$ is called *standard p-Laplacian*.

**Remark 2.3** Following [Mug13; Pra04; Tak03], there is the following analogy to $p$-Laplacians in the continuum: A vector field (or flow) $v$ is a function in $C(X \times X)$ such that $v(x,y) = -v(y,x)$, $x, y \in X$. Moreover, define div on the space of absolutely summable vector fields in the second entry via

$$(\operatorname{div} v)(x) = \frac{1}{m(x)}\sum_{y\in X} v(x,y).$$

Then, for all $f \in F$, and $p \geq 1$,

$$Lf(x) = \operatorname{div}(b\,|\nabla f|^{p-2}\,\nabla f)(x), \qquad x \in X.$$

This shows that our Laplacian is a discrete analogue to weighted pseudo $p$-Laplace-type operators on manifolds (see e.g. [BK04]). Note that for $p = 2$, this is just a weighted Laplace-Beltrami operator. However, unless otherwise stated, we will usually compare our theory with results concerning the classical $p$-Laplacian (see e.g. [Lin19]). See also [TZ24] for a recent discussion on the two operators on locally finite graphs.

**Remark 2.4** The definition of the action of the $p$-Laplacian to vanish outside of $V \subseteq X$ is an arbitrary choice. It evolves from Theorem 4.1, where we are taking test functions only from $C_c(V)$. This choice implies that the functions $u$ in Theorem 4.1 can be taken from the largest possible set, namely $F(V)$, and that the action of the $p$-Laplacian outside of $V$ does not matter. If we enlarge the set of test functions then we need to be more careful with the definition of the action of the $p$-Laplacian outside of $V$. Then, we might also reconsider to define the $p$-Laplacian only weakly. A larger set of test functions would on the other hand result in a smaller set of functions $u$ to which we can apply our main results. Since Theorem 4.1 is proven by summing up pointwise estimates, similar results hold mutatis mutandis when considering other sets of test functions and related $p$-Laplacians, see [HM15; Mug13] for definitions of Dirichlet and Neumann $p$-Laplacians.

Finally, we can define Schrödinger operators as follows: Let $c \in C(X)$. Then the *$p$-Schrödinger operator* $H = H_{b,c,m,p} \colon F(V) \to C(V)$ is given by

$$Hf(x) := Lf(x) + \frac{c(x)}{m(x)} (f(x))^{\langle p-1 \rangle}, \qquad x \in V.$$

The function $c$ is then usually called the *potential* of $H$. If $c$ is non-negative, then $H$ is called *$p$-Laplace-type* operator. If $c = 0$, then $H$ is called *free $p$-Laplacian*.

A function $u \in F(V)$ is said to be a *(p-)solution, ((p-)supersolution, (p-)sub-solution)* on $V \subseteq X$ with respect to $H$ and $g \in C(V)$ if

$$Hu = g \quad (Hu \geq g,\ Hu \leq g) \qquad \text{on } V.$$

If $g = 0$ we speak of *(p-)harmonic, ((p-)superharmonic, (p-)subharmonic)* functions on $V$. If a function is superharmonic but not harmonic in $V$, we call it *strictly superharmonic* in $V$. If $V = X$ we only speak of super-/sub-/harmonic functions, respectively super-/sub-/solutions with respect to $g$.

A function $u \in F(V)$, $u \neq 0$, is said to be a *(generalised $p$-)eigenfunction* to the *(generalised $p$-)eigenvalue* $\lambda \in \mathbb{R}$ on $V$ with respect to $H$ if

$$Hu = \lambda\, (u)^{\langle p-1 \rangle} \qquad \text{on } V.$$

If $u > 0$ is an eigenfunction to $\lambda \in \mathbb{R}$ on $V$, then $\lambda$ is called *(generalised) principal (p-)eigenvalue* on $V$ with respect to $H$.

We also need the following definitions which seem to be new on graphs in this generality but have a long history in the continuum: Let $V \subseteq X$ be connected and $K \subseteq V$ be finite. By $\mathcal{M}(V \setminus K) \subseteq F(V \setminus K)$, we denote the set of strictly positive functions $u$ which are $p$-harmonic on $V \setminus K$, and which have the following minimal growth property: for any finite and connected subset $\mathcal{K} \subseteq V$ with $K \subseteq \mathcal{K}$, and any positive function $v \in F(V \setminus \mathcal{K})$ which is $p$-superharmonic in $V \setminus \mathcal{K}$, we have

$$u \leq v \text{ on } \mathcal{K} \quad \text{implies} \quad u \leq v \text{ in } V \setminus \mathcal{K}.$$

A function $u \in \mathcal{M}(V \setminus K)$ is called *positive $p$-harmonic function of minimal growth at infinity* in $V$ with respect to $K$.

If $u \in \mathcal{M}(V)$, then $u$ is called a *global minimal positive $p$-harmonic function* in $V$, or an *Agmon ground state*. We will see in Chapter 10 that if we have an Agmon ground state then it is unique up to multiplication with a positive constant, and thus we sometimes speak of *the* Agmon ground state.

A function $g_o \in \mathcal{M}(V \setminus \{o\}) \cap F(V)$ which is not $p$-harmonic in some fixed vertex $o \in V$ is called a *(minimal positive) Green's function* in $V$ at $o$. If, in addition to that, $g_o$ satisfies $Hg_o = 1_o$ on $V$, then a Green's function $g_o$ in $V$ at $o$ is called *normalised*.

Positive harmonic function of minimal growth at infinity play an important role in Chapter 9 and Chapter 10. For $p = 2$, Agmon ground states and minimal positive Green's functions for Schrödinger operators on locally summable graphs have been discussed in [KPP20b].

**Remark 2.5 ($\varphi$-Laplacians and other generalisations)** The here presented $p$-Schrödinger operator is a natural quasi-linear generalisation of the classical linear Schrödinger operator on graphs. However, this is of course not the only interesting choice of a generalisation. The most popular alternative generalisation might be the fractional Schrödinger operator, see e.g. [KN23].

We also want to mention, that generalisations of the $p$-Laplacian have been studied in the continuum in the past years. Here, the so-called $\varphi$-Laplacian seems to be the most popular. Here, $\varphi$ is the derivative of a Young function, and introductions to this topic can be found in [CSS21; HS01; PRS02; PRS05; PS14]. It would be very interesting to see if similar results can also be obtained for $\varphi$-Laplacians on graphs. Our unweighted case corresponds to the choice $\varphi(t) = (t)^{\langle p-1 \rangle}$, $p > 1$.

Another popular generalisation is the $\mathcal{A}$-Laplacian as defined in the monograph [HKM06], see also [HPR24]. This operator has also not been touched in the discrete setting. Here, for our setting, $\mathcal{A}([x, y], t) \asymp b(x, y)\,(t)^{\langle p-1 \rangle}$, $p > 1$. To consider the $\mathcal{A}$-Laplacian might be a first step towards the weighted $\varphi$-Laplacian.

## 2.2 Energy Functionals Associated with Graphs

The *$p$-energy functional* $h = h_{b,c,p} \colon C_c(X) \to \mathbb{R}$ is defined via

$$h(f) := \frac{1}{2} \sum_{x,y \in X} b(x, y)\, |\nabla_{x,y} f|^p + \sum_{x \in X} c(x)\, |f(x)|^p \,.$$

If $p = 2$, then the corresponding energy functional is a quadratic form, and called Schrödinger form.

**Remark 2.6 ($D$)** Another natural choice of a domain for $h$ is the set of functions of finite energy $D = D_{b,c,p}$, which is given by

$$D := \Big\{ f \in C(X) : \frac{1}{2} \sum_{x,y \in X} b(x, y)\, |\nabla_{x,y} f|^p + \sum_{x \in X} |c(x)|\, |f(x)|^p < \infty \Big\}.$$

Note that $C_c(X) \subseteq D_{b,c,p} \subseteq D_{b,c_+,p} \subseteq D_{b,0,p} \subseteq F$, where the last inclusion follows from Hölder's inequality and the local summability of the graph.

We are mainly interested in estimating $h$ on the set of compactly supported functions, since the theory presented here does not require specific Banach space techniques and $C_c(X)$ is dense in $C(X)$. This gives also more flexibility for future applications. Nevertheless, if $h$ is non-negative on $C_c(X)$, then a Fatou-type argument shows that $h(f)$ consists of absolutely converging sums also for $f \in D_{b,c_+,p}$. The space $D_{b,c_+,p}$ can be equipped with a norm naturally, see also Remark 2.13. A first analysis of this space on graphs can be found in [DKP24]. In the continuum, it has been studied in [DP23].

However, we want to remark that some of the results for $h$ do not only hold in $C_c(X)$ but also for larger domains like $D$. For instance, in Lemma 2.8, $\varphi$ can also be in $D$.

As in the continuum or the linear case on graphs, there exists a so-called Green's formula (or integration by parts formula) which shows a connection between $H$ and $h$ on $C_c(X)$. The Green's formula seems to be folklore in both worlds. However, for the convenience of the reader we include a proof here. A similar proof of the Green's formula for the normalised $p$-Laplacian, that is $m = \deg$ and $c = 0$, is given in [Tak03].

Let $V \subseteq X$. To shorten notation, we define a weighted bracket $\langle \cdot, \cdot \rangle_V$ on $C(X) \times C_c(X)$ via

$$\langle f, \varphi \rangle_V := \sum_{x \in V} f(x)\varphi(x)m(x), \qquad f \in C(X), \varphi \in C_c(X).$$

**Lemma 2.7 (Green's formula)** *Let $p \geq 1$ and $V \subseteq X$. Let $f \in F(V)$ and $\varphi \in C_c(X)$. Then all of the following sums converge absolutely and*

$$\langle Hf, \varphi \rangle_V = \frac{1}{2} \sum_{x,y \in V} b(x,y) \left(\nabla_{x,y} f\right)^{\langle p-1 \rangle} \left(\nabla_{x,y} \varphi\right) + \sum_{x \in V} c(x) \left(f(x)\right)^{\langle p-1 \rangle} \varphi(x)$$
$$+ \sum_{x \in V, y \in \partial V} b(x,y) \left(\nabla_{x,y} f\right)^{\langle p-1 \rangle} \varphi(x).$$

*In particular, the formula can be applied to $f \in C_c(X)$, and*

$$h(\varphi) = \langle H\varphi, \varphi \rangle_V, \qquad \varphi \in C_c(V).$$

*Proof.* Since $\varphi \in C_c(X)$, the absolute convergence follows from

$$\sum_{x \in V} |Lf(x)\varphi(x)|\, m(x) \leq \sum_{x \in V} |\varphi(x)| \sum_{y \in X} b(x,y) \left|\nabla_{x,y} f\right|^{p-1} < \infty,$$

for any $f \in F(V)$. Applying Fubini's theorem, using the absolute convergence of the sums and the symmetry of $b$, we get

$$\begin{aligned}\sum_{x\in V} Lf(x)\varphi(x)m(x) &= \sum_{x\in V, y\in X} b(x,y)\,(\nabla_{x,y}f)^{\langle p-1\rangle}\,\varphi(x)\\ &= \frac{1}{2}\sum_{x,y\in V} b(x,y)\,(\nabla_{x,y}f)^{\langle p-1\rangle}\,\varphi(x) - \frac{1}{2}\sum_{\hat{x},\hat{y}\in V} b(\hat{x},\hat{y})\left(\nabla_{\hat{x},\hat{y}}f\right)^{\langle p-1\rangle}\varphi(\hat{y})\\ &\qquad + \sum_{x\in V, y\in\partial V} b(x,y)\,(\nabla_{x,y}f)^{\langle p-1\rangle}\,\varphi(x)\\ &= \frac{1}{2}\sum_{x,y\in V} b(x,y)\,(\nabla_{x,y}f)^{\langle p-1\rangle}\,\nabla_{x,y}\varphi + \sum_{x\in V, y\in\partial V} b(x,y)\,(\nabla_{x,y}f)^{\langle p-1\rangle}\,\varphi(x).\end{aligned}$$

The assertions for the Schrödinger operator $H$ follow now easily. □

Green's formula is sometimes taken to define a $p$-Schrödinger operator weakly. By taking the test function $\varphi = 1_z$ for $z \in V \subseteq X$, we see that the action of a weakly defined operator agrees with the action of our (strongly) defined $p$-Schrödinger operator on $V$.

Moreover, another nice connection between $h$ and $H$ is that $p\cdot H$ is the Gâteaux derivative of $h$ on $C_c(X)$, and since we will use this observation later several times, we include the proof here.

**Lemma 2.8 (Gâteaux derivative)** *Let $p \geq 1$. For all $\varphi, \psi \in C_c(X)$ we have that $\varphi + \psi \in C_c(X)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}\,t} h(\varphi + t\,\psi)\Big|_{t=0} = p\,\langle H\varphi, \psi\rangle_X.$$

*Proof.* The formula follows easily via Green's formula, Lemma 2.7,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t} h(\varphi + t\,\psi)\Big|_{t=0} &= \frac{p}{2}\sum_{x,y\in X} b(x,y)\,(\nabla_{x,y}\varphi)^{\langle p-1\rangle}\left(\nabla_{x,y}\psi\right) + p\langle c/m, (\varphi)^{\langle p-1\rangle}\,\psi\rangle_X\\ &= p\sum_{x\in X} H\varphi(x)\psi(x)m(x) = p\langle H\varphi, \psi\rangle_X.\end{aligned}$$ □

Because of Lemma 2.7 (or Lemma 2.8), it is convenient to define

$$h(\varphi,\psi) := \langle H\varphi, \psi\rangle_X, \qquad \varphi,\psi \in C_c(X),$$

see e.g. [SW04] for a discussion for the $p$-Laplacian and associated energy on the Sierpinski gasket. However, we will not explicitly need off-diagonal entries of the energy, and thus, stay in the following with the one-entry definition.

If the functional $h$ is non-negative on $C_c(V)$, $V \subseteq X$, then $h$ is called *(p-)subcritical* in $V$ if the *(p-)Hardy inequality* holds true, that is, there exists a positive function $w \in C(V)$ such that

$$h \geq \|\cdot\|_{p,w}^p \qquad \text{on } C_c(V).$$

Here,

$$\|\varphi\|_{p,w} := \left( \sum_{x \in X} |\varphi(x)|^p \, w(x) \right)^{1/p}, \qquad \varphi \in C_c(X).$$

If such a $w$ does not exist, then $h$ is called *(p-)critical* in $V$. Moreover, $h$ is called *(p-)supercritical* in $V$ if $h$ is not non-negative on $V$.

**Remark 2.9 (History)** This classification of energy functionals in terms of sub-/super-/critical goes back to [Sim80], see also [Mur84; Mur86; Pin88], and is motivated from the analysis of the energy functional $h$.

In the special case of non-negative potentials and $p = 2$, a graph is usually called *transient* if $h$ is subcritical in $X$, and *recurrent* otherwise, see e.g. [KLW21]. This notation has its origin in probability theory, see e.g. [Woe09], and goes back at least to [Pól21]. It is also common when studying Dirichlet forms, see e.g. [FOT11; Stu94], whereas subcritical and critical is used for Schrödinger forms, see [Tak14; Tak16; Tak23; TU23; Miu23; Sch22]. In a hand waving way, recurrence means that the associated random walker returns almost surely, and transience means that the random walker escapes any finite set with positive probability.

Moreover, if $c = 0$, then a graph is sometimes called *p-hyperbolic* if $h$ is $p$-subcritical in $X$, and *p-parabolic* otherwise. This notation has its origin in the geometry of surfaces, see here e.g. [AFS24; Gri99; KY84; MR22; MR24a; MR24b; PRS05; PS14; Shi21; SY93a; SY93b; Tro00; Tro99; Yam77]. By the celebrated uniformisation theorem of Klein, Koebe and Poincaré in the linear ($p = 2$)-case, any simply connected Riemann surface is conformally equivalent to either the sphere (surface of elliptic type), the Euclidean plane (surface of parabolic type) or the hyperbolic plane (surface of hyperbolic type). Since the sphere is the only one with compact surface, the *type problem* is to decide whether a surface is of hyperbolic or of parabolic type. Surprisingly, it could be shown in the linear case that the recurrence of Brownian motion is equivalent to the parabolicity of the Riemann surface. Important contributions are here [Ahl52; AS60; CY75; Den70; Hun54; Kak53; Nev40; Roy52; Var83].

**Remark 2.10** In the literature, there are many closely related Hardy-type inequalities. We will name two prominent ones.

The first are so-called *weighted* $p$-Hardy-type inequalities where one is typically interested in optimising weights on the $p$-energy functional and $p$-norm simultaneously, see here e.g. [Mic99; SRA21] for results on $\mathbb{N}$ and Muckenhoupt bounds, or to visualise the impact of a weight on the $p$-energy functional, see here [HY24] for locally finite graphs with a special focus on $\mathbb{Z}^d$. A future goal might be to generalise the main results of this thesis in the weighted setting of [HY24] for locally summable graphs.

Secondly, there are also alternative generalisations of the ($p = 2$)-energy functional. Important here are the *Sobolev-Bregman forms*, also known as $p$-forms. For $p$-Hardy-type inequalities in this context see [Bog+22] and references therein. There, the divergence part of the $p$-energy functional is replaced by

$$\frac{1}{2}\sum_{x,y\in X} b(x,y)\,(\nabla_{x,y}\varphi)\left(\nabla_{x,y}\,(\varphi)^{\langle p-1\rangle}\right), \qquad \varphi\in C_c(X).$$

We continue with definitions. Closely connected to criticality is the following definition: A sequence $(e_n)$ in $C_c(V)$, $V\subseteq X$, of non-negative functions is called *null-sequence* in $V$ if there exists $o\in V$ and $\alpha>0$ such that $e_n(o)=\alpha$ and $h(e_n)\to 0$ as $n\to\infty$.

It will be shown in Chapter 10, that $h$ is $p$-critical on connected $V\subseteq X$ if and only if there exists a unique positive $p$-superharmonic function (up to multiplies) on $V$. This function is $p$-harmonic on $V$ and the Agmon ground state of $h$.

Let $h$ be a $p$-critical energy functional. We call $h$ *(p-)null-critical* on $V$ with respect to the non-negative function $w$ if the Agmon ground state is not in $\ell^p(V,w)$, and otherwise we call it *(p-)positive-critical* on $V$ with respect to $w$. Clearly, if $V$ is finite, than any $p$-critical energy functional is $p$-positive critical on $V$ with respect to any $w\geq 0$.

Here, for all $1\leq p<\infty$, $V\subseteq X$ and non-negative functions $w$ on $V$, we define

$$\ell^p(V,w):=\Big\{f\in C(X):\sum_{x\in V}|f(x)|^p\,w(x)<\infty\Big\}.$$

Note that $(\ell^p(X,w),\|\cdot\|_{p,w})$ is a reflexive Banach space for $p\in(1,\infty)$, and a Banach space for $p=1$.

During the thesis the following two quantities will be of fundamental interest: Let $V\subseteq X$, and define $\lambda_0(V)=\lambda_0(V,H)$ via

$$\lambda_0(V):=\inf_{\varphi\in C_c(V),\varphi\neq 0}\frac{h(\varphi)}{\|\varphi\|_{p,m}^p}=\inf_{\varphi\in C_c(V),\|\varphi\|_{p,m}^p=1}h(\varphi), \tag{2.2}$$

furthermore, the (variational $p$-)capacity is defined as follows: For all $K\subseteq V$, $K$ finite, we set

$$\mathrm{cap}_h(K,V):=\inf_{0\leq\varphi\in C_c(V),\varphi=1 \text{ on } K}h(\varphi).$$

These numbers play an important role in Chapters 5, 8 and 10. With a slight abuse of notation $\lambda_0(V)$ is sometimes called (generalised) principal eigenvalue even though we have not shown explicitly the existence of a non-trivial positive generalised $p$-eigenfunction. However, in Proposition 5.17 and Theorem 10.1, we will see conditions when this is indeed the case.

The focus of this thesis is on estimates of functionals. It is therefore comfortable to use the following notation: Any function $w\in C(X)$ gives rise to a canonical $p$-functional $w_p$ on $C_c(X)$ via

$$w_p(\varphi):=\sum_{x\in X}|\varphi(x)|^p\,w(x), \qquad \varphi\in C_c(X).$$

The $p$-Hardy inequality then reads as $h-w_p\geq 0$ on $C_c(X)$ for some $w\gneq 0$ on $X$. If it is clear that we mean the functional $w_p$ and not the function $w$, we sometimes simply write $w$ instead of $w_p$.

We also need the definition of an optimal $p$-Hardy weight, confer [KPP18b] in the discrete ($p = 2$)-setting and [DP16] in the continuum.

The function $w \gneq 0$ is called an *optimal $p$-Hardy weight* on $V$ for the $p$-energy functional $h$ in $X$ if

(i) $h - w_p$ is $p$-critical in $V$,

(ii) $h - w_p$ is $p$-null-critical with respect to $w$ in $V$,

(iii) $h - w_p \geq \lambda w_p$ fails to hold on $C_c(V \setminus K)$ for all $\lambda > 0$ and finite $K \subseteq V$, in which case we say $w$ is *optimal near infinity* for $h$.

Optimal $p$-Hardy weights play an important role in Chapter 12.

**Remark 2.11** The definition of optimality evolved historically, see [DFP14; DP16]. It is natural to ask whether there is a connection between optimality near infinity and null-criticality. In the continuum, it is shown in [KP20, Corollary 3.4], that indeed, (ii) implies (iii). Moreover, [DP16, Remark 1.3] states an example that the other implication '(iii) $\Longrightarrow$ (ii)' fails in general.

On graphs associated with linear Schrödinger operators with compactly supported potential part it is shown in [KPP18b], that (ii) implies (iii) for a special Hardy weight.

Here, we will show in Theorem 11.2, that (ii) implies (iii) for all $p > 1$, and all possible potentials if an Agmon ground state is of bounded oscillation and in $\ell^p(V, b(x, \cdot))$ for all $x \in V$.

If not stated otherwise, we will always assume that

$$p \in (1, \infty).$$

**Remark 2.12 (The borderline cases)** The cases $p \in \{1, \infty\}$ are usually of very different nature in comparison to $p \in (1, \infty)$, and mainly not covered in this thesis. Sometimes we explicitly mention the difficulties which usually arise from the fact that $|\cdot|^p$ is not strictly convex for $p \in \{1, \infty\}$. To understand the two borderline cases will be a future investigation.

By an *exhaustion* of $V \subseteq X$, we mean a sequence $(V_n)$ of subsets of $X$ such that $V_n \subseteq V_{n+1} \subseteq V$, $n \in \mathbb{N}$, and $\cup_{n \in \mathbb{N}} V_n = V$.

**Remark 2.13 (Potentials)** Our main goal is to allow negative values for $c$. As a byproduct we are usually not able to use standard convexity arguments or methods for monotone operators. A workaround will very often be the method of approximating the graph by an exhaustion of $X$ with finite sets and an analysis of the corresponding limit.

Note that assuming $c \geq 0$ allows the usage of standard functional analytic tools which are very similar to the continuous case. For example, if $c \geq 0$, then we can use [Sho97, Proposition 7.6] and get that the subdifferential of $h$ on $W^{1,p}(X)$ is a singelton containing only the $p$-Laplace-type operator. Here,

$$W^{1,p}(X) = \ell^p(X, m) \cap D$$

is equipped with the norm

$$\|f\|_{1,p,m} = \left(\|f\|_{p,m}^{p} + h(f)\right)^{1/p}.$$

For more information on such spaces on locally summable or locally finite graphs, see the recent preprints [MS23; SYZ23; TZ24].

We always explicitly mention the extra assumption that the underlying graph is locally finite on some $V \subseteq X$.

# 3. Examples

> What we do may be small, but it has a certain character of permanence; and to have produced anything of slightest permanent interest, whether it be a copy of verses or a geometrical theorem, is to have done something utterly beyond the powers of the vast majority of men.
>
> G. H. Hardy, *A Mathematician's Apology*, p. 76

The main concepts and results of this thesis will be made approachable via examples. Most of the examples are not part of the underlying papers [Fis23; Fis22; Fis24].

We separate the examples into two categories: examples of Schrödinger operators, and examples of graphs. The most fundamental example will be the free $p$-Laplacian on the line graph $\mathbb{N}_0$.

## 3.1 Examples of Quasi-linear Schrödinger Operators

Here we list three examples of Schrödinger operators – all of them have a significantly different potential part.

**Example 3.1 (Free $p$-Laplacian)** The most important example of a quasi-linear Schrödinger operator is the *free $p$-Laplacian*, meaning that $c = 0$. In other words,

$$Hu(x) = Lu(x) = \frac{1}{m(x)} \sum_{x,y \in X} b(x,y) \left(\nabla_{x,y} u\right)^{\langle p-1 \rangle}, \qquad u \in F, x \in X.$$

If we take $b(X \times X) \subseteq \{0,1\}$ and $m = 1$, we call it *standard $p$-Laplacian*. Explicit calculations for this operator will appear on various graphs.

Next, we state an example of a $p$-Laplace-type operator.

**Example 3.2 (Generalised harmonic oscillator)** One of the most prominent examples is the model of a harmonic oscillator which has many applications in mechanics. A possible generalisation to the quasi-linear setting is given by the Schrödinger operator $H$ with potential part given by $c(x) = C_p d^p(x,o) \geq 0$, $x \in X$, where $d(x,o)$ denotes some distance between $x$ and $o$, and $C_p > 0$ is some positive constant depending only on $p$. Hence,

$$Hu(x) = \frac{1}{m(x)} \sum_{x,y \in X} b(x,y) \left(\nabla_{x,y} u\right)^{\langle p-1 \rangle} + C_p \frac{d^p(x,o)}{m(x)} \left(u(x)\right)^{\langle p-1 \rangle}, \quad u \in F, x \in X.$$

The following example is about a Schrödinger operator with a negative potential.

**Example 3.3 (Generalised hydrogen atom)** There are models of the hydrogen atom via (linear) Schrödinger operators, see e.g. [RS72, p. 304] or [Cyc+87, p. 41]. A possible generalisation to the quasi-linear setting is given by the potential $c(x) = -C_p/d^{p-1}(x, o)$ for $x \in X \setminus \{o\}$, where $d$ denotes some distance function and $C_p > 0$ is a positive constant. This defines a Schrödinger operator on $X \setminus \{o\}$, i.e., for all $x \in X \setminus \{o\}$ and $u \in F(X \setminus \{o\})$

$$Hu(x) = \frac{1}{m(x)} \sum_{x,y \in X} b(x, y) \left(\nabla_{x,y} u\right)^{\langle p-1 \rangle} - \frac{C_p}{m(x) d^{p-1}(x, o)} \left(u(x)\right)^{\langle p-1 \rangle}.$$

There are many interesting examples left, like having a-/periodic potentials or a quasi-linear version of the Laguerre operator (see [Kos21] for the linear Laguerre operator). Unfortunately, we cannot treat all of them here.

## 3.2 Examples of Graphs

Here, we state our leading examples of graphs.

**Example 3.4 (Standard line graph $\mathbb{N}_0$)** The line graph on $X = \mathbb{N}_0 = \{0, 1, \dots\}$ is defined for all $x, y \in \mathbb{N}_0$ via $b(x, y) = 1$ if $\|x - y\|_1 = 1$ and $b(x, y) = 0$ otherwise. The $p$-Laplacian is then given by

$$Lf(n) = \begin{cases} \frac{(\nabla_{0,1} f)^{\langle p-1 \rangle}}{m(0)}, & n = 0 \\ \frac{1}{m(n)} \left( (\nabla_{n,n-1} f)^{\langle p-1 \rangle} + (\nabla_{n,n+1} f)^{\langle p-1 \rangle} \right), & n > 0 \end{cases},$$

for every $f \in F = C(\mathbb{N}_0)$, and the corresponding $p$-energy functional is given by

$$h(\varphi) = \sum_{n \in \mathbb{N}_0} |\nabla_{n,n+1} f|^p + \sum_{n \in \mathbb{N}_0} c(n) \, |f(n)|^p, \qquad \varphi \in C_c(\mathbb{N}_0).$$

Moreover, we often set $m = 1$.

The standard counterpart in the continuum to the line graph on $\mathbb{N}_0$ is obviously the interval $(0, \infty)$.

**Example 3.5 (Euclidean lattice $\mathbb{Z}^d$, $d \geq 1$)** The $d$-dimensional Euclidean lattice is given by $X = \mathbb{Z}^d$ with $b(x, y) = 1$ if $\|x - y\|_1 = 1$ and $b(x, y) = 0$ otherwise for all $x, y \in \mathbb{Z}^d$, $d \in \mathbb{N}$. If $(e_i)_{i=1}^d$ is the standard base of $\mathbb{Z}^d$, then

$$Lu(x) = \frac{1}{m(x)} \sum_{i=1}^{d} \left( \nabla_{x, x \pm e_i} f \right)^{\langle p-1 \rangle}, \qquad f \in F.$$

The standard counterpart in the continuum to the Euclidean lattice is the Euclidean space $\mathbb{R}^d$. However, keep in mind that $\mathbb{R}^d$ is radial but $\mathbb{Z}^d$ is not. This makes computation on $\mathbb{Z}^d$ often much more technical.

We turn to a large class of examples which share some radial symmetry as well, the model graphs. Thereafter, we pick two important representatives of this class: homogeneous trees and anti-trees. Model graphs are special cases of weakly spherically symmetric graphs (where the reference set is just a singleton) which are discussed in detail in the linear setting in [KLW21].

We need some notation first: The function $d\colon X \times X \to [0,\infty)$ is called *combinatorial graph distance*, if its value is the least number of edges of a path connecting two given vertices. On model graphs, the underlying distance function will always be the combinatorial graph distance. Moreover, for some fixed vertex $o \in X$ we set

$$B_r(o) := \{x \in X : d(x,o) \leq r\}, \quad \text{and} \quad S_r(o) := \{x \in X : d(x,o) = r\}.$$

The *inner curvature* $k_-\colon X \to [0,\infty)$ and *outer curvature* $k_+\colon X \to [0,\infty)$ are defined via

$$k_\pm(x) = \frac{1}{m(x)} \sum_{y \in S_{r\pm 1}(o)} b(x,y), \quad x \in S_r(o), \qquad k_-(o) = 0.$$

**Example 3.6 (Model graph)** Let us fix $o \in X$. A graph $(X,b,m)$ with potential $c$ is called *model graph* with respect to $o$ if $k_\pm$ and $c/m$ are spherically symmetric functions, i.e., $k_\pm(x) = k_\pm(y)$ and $c(x)/m(x) = c(y)/m(y)$ for all $x,y \in S_r(o)$ and all $r \geq 0$. The vertex $o$ is also called *root*. The $p$-Laplacian of spherically symmetric function $f = f(r)$ is given by

$$Lf(0) = k_+(0)\,(\nabla_{0,1} f)^{\langle p-1\rangle},$$

and

$$Lf(r) = k_+(r)\,(\nabla_{r,r+1} f)^{\langle p-1\rangle} + k_-(r)\,(\nabla_{r,r-1} f)^{\langle p-1\rangle}, \qquad r \geq 1.$$

We remark that if $m \geq C_r > 0$ on $S_r(o)$ for all $r \geq 0$, then the graph is locally finite, see [KLW21, p. 380].

The standard counterpart in the continuum to model graphs are model manifolds or also harmonic manifolds, see [FP23] for details on harmonic manifolds.

**Example 3.7 (Homogeneous regular tree $\mathbb{T}_{d+1}$, $d \geq 2$)** Let us fix $o \in X$. A model graph is a homogeneous $(d+1)$-regular trees, denoted by $\mathbb{T}_{d+1}$, $d \geq 2$, if $b(X \times X) = \{0,1\}$, $m = 1$, $k_+(x) = d$ for all $x \in X$, $k_-(x) = 1$ for all $x \neq o$, and $b(S_r(o) \times S_r(o)) = \{0\}$ for all $r \geq 0$.

The standard counterpart in the continuum to these trees are hyperbolic spaces or also Damek-Ricci spaces, see [FP23] for details on Damek-Ricci spaces.

**Example 3.8 (Anti-tree)** Let us fix $o \in X$, and set $s(r) = \#S_r(o)$ for all $r \geq 0$. A model graph is an anti-tree of sphere size $s$, if $b(X \times X) = \{0,1\}$, $m = 1$, and $k_\pm(x) = s(r)$ for all $x \in S_{r\mp 1}(o)$.

For anti-trees, we do not know of a standard example in the continuum but as they are special model graphs, the suggestions above should fit here as well.

The following graph is the standard example of a locally infinite graph.

**Example 3.9 (Star graph)** A graph $(\mathbb{N}_0, b, m)$ such that for all $n, k \in \mathbb{N}_0$ we have $b(n,k) > 0$ if and only if either $n = 0$ or $k = 0$, is called *star graph* with centre 0. Then, for all $u \in F$ and $n \in \mathbb{N}$ we have

$$Lu(n) = \frac{b(n,0)}{m(n)} \left(\nabla_{n,0} u\right)^{\langle p-1 \rangle}, \text{ as well as} \qquad Lu(0) = \frac{1}{m(0)} \sum_{n=1}^{\infty} b(0,n) \left(\nabla_{0,n} u\right)^{\langle p-1 \rangle}.$$

Hence, for any $k \in \mathbb{N}_0$,

$$Lu(k) = -\sum_{n \neq k}^{\infty} \frac{m(n)}{m(k)} Lu(n).$$

To the best of our knowledge, locally infinite graphs do not have standard counterparts in the continuum, at least not in the Riemannian sense.

We stick to these examples with the knowledge that many remain untouched in this work (bipartite graphs, latter graphs, ...). However, too many examples would take the focus from the general theory, and are left for the interested reader.

# 4. Ground State Representation

Here, on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night?

Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine.

A. E. Housman cited by Hardy, *A Mathematician's Apology*, p. 77

In the classical linear case, ground state representations are transformations which use a superharmonic function to turn a quadratic energy form associated with a linear Schrödinger operator into a quadratic energy form associated with a linear Laplace operator, see e.g. [KPP20b, Proposition 4.8] for such a statement on graphs (or also e.g. [FLW14; HK11; KLW21]), and e.g. [Dav89, p. 109] for a counterpart in the continuum. In the linear non-local case, the ground state representation is basically a smart rearrangement of summands. It turns out that the situation is way more complicated in the non-linear case.

In the non-linear ($p \neq 2$)-case, we do not have an equality via a transformation between functionals anymore. But instead, we achieve an equivalence between functionals, providing that a positive $p$-superharmonic function exists. The equivalent functional has the property that it consists of non-negative terms only.

Our representations in Theorem 4.1 and Corollary 4.2 can be seen as the non-local analogues to the local and non-linear representations in [PR15; PTT08], where $p$-Schrödinger operators on domains in $\mathbb{R}^d$ are discussed.

Applications of our representations are given in Chapter 6, Chapter 10, Chapter 11, Chapter 12 and Chapter 13.

## 4.1 The Formula

Let $p > 1$, and $0 \leq u \in F(V)$ for some $V \subseteq X$. The *simplified energy (functional)* $h_u$ of $h$ with respect to $u$ on $C_c(V)$ be given by

$$\begin{aligned} h_u(\varphi) :=& \sum_{x,y\in X} b(x,y)u(x)u(y)(\nabla_{x,y}\varphi)^2 \\ &\qquad \cdot \left( \big(u(x)u(y)\big)^{1/2} |\nabla_{x,y}\varphi| + \frac{|\varphi(x)| + |\varphi(y)|}{2} |\nabla_{x,y} u| \right)^{p-2} \\ =& \sum_{x,y\in X} b(x,y)\big(u(x)u(y)\big)^{p/2} |\nabla_{x,y}\varphi|^p \cdot \left( 1 + \frac{|\varphi(x)| + |\varphi(y)|}{2\,|\nabla_{x,y}\varphi|} \frac{|\nabla_{x,y} u|}{\big(u(x)u(y)\big)^{1/2}} \right)^{p-2}, \end{aligned}$$

where we set $0 \cdot \infty = 0$ if $1 < p < 2$.

We state now the main result of this chapter. We call Equation 4.1 *ground state representation formula*, even though it is actually a two-sided estimate. However, this name was used before in the quasi-linear literature, see [FS08], and shows the connection to the linear $(p = 2)$-case, where the representation can be interpreted as a transformation, see [KPP20b, Section 4.2].

**Theorem 4.1 (Ground state representation)** *Let $p > 1$ and $0 \leq u \in F(V)$ for some $V \subseteq X$. Then, we have*

$$h(u\varphi) - \langle Hu, u\,|\varphi|^p\rangle_V \asymp h_u(\varphi), \qquad \varphi \in C_c(V). \tag{4.1}$$

*The constants in the equivalence only depend on $p$. Furthermore, the equivalence becomes an equality if $p = 2$.*

Later, if $u$ is $p$-superharmonic we also often use the functional notation, i.e., we write $(muHu)_p(\varphi) = \langle Hu, u\,|\varphi|^p\rangle_V$.

In many applications the function $u$ is assumed to be $p$-harmonic in $V \subseteq X$. In this case the representation in (4.1) reduces to

$$h(u\varphi) \asymp h_u(\varphi), \qquad \varphi \in C_c(V).$$

A further consequence of (4.1) is, that the corresponding left-hand side is non-negative, i.e,

$$h(u\varphi) \geq \langle Hu, u\,|\varphi|^p\rangle_V, \qquad \varphi \in C_c(V).$$

This inequality is known as *Picone's inequality*, see [AH98; AM16; Amg08; BF14; Fis22; FS08; MS23; PKC09; Pic10; PTT08] for applications of this inequality in various contexts. Picone's inequality is relevant in Chapter 5 and Chapter 6.

From the inequalities in Theorem 4.1, we get as consequences estimates between the energy associated with the Schrödinger operator and other functionals, which are usually also referred to as *simplified energies* (see e.g. [DP16; PTT08]). They all are called simplified, because they consist of non-negative terms only, and the difference operator $\nabla$ applies either to $u$ or $\varphi$ but not to the product $u \cdot \varphi$.

We set on $C_c(V)$,

$$h_{u,1}(\varphi) := \sum_{x,y \in X} b(x,y)(u(x)u(y))^{p/2} \left|\nabla_{x,y}\varphi\right|^p,$$

and for $p \geq 2$, we define on $C_c(V)$

$$h_{u,2}(\varphi) := \sum_{x,y \in X} b(x,y)u(x)u(y) \left|\nabla_{x,y} u\right|^{p-2} \left(\frac{|\varphi(x)| + |\varphi(y)|}{2}\right)^{p-2} \left|\nabla_{x,y}\varphi\right|^2.$$

Note that $h_{u,2}$ is not a $p$-energy functional. We discuss this more in detail in Remark 4.7.

The following corollary is an immediate consequence of Theorem 4.1.

**Corollary 4.2** *Let $p > 1$. If $1 < p \leq 2$, then there is a positive constant $c_p$ such that for all $0 \leq u \in F(V)$*

$$h(u\varphi) - \langle Hu, u\,|\varphi|^p\rangle_V \leq c_p h_{u,1}(\varphi), \qquad \varphi \in C_c(V), \tag{4.2}$$

*and if $p \geq 2$ the reversed inequality in (4.2) holds true, i.e.,*

$$h(u\varphi) - \langle Hu, u\,|\varphi|^p\rangle_V \geq c_p h_{u,1}(\varphi), \qquad \varphi \in C_c(V). \tag{4.3}$$

*Furthermore, both inequalities become equalities if $p = 2$.*

*Moreover, if $p \geq 2$, we have for all $0 \leq u \in F(V)$,*

$$h(u\varphi) - \langle Hu, u\,|\varphi|^p\rangle_V \asymp h_{u,1}(\varphi) + h_{u,2}(\varphi), \qquad \varphi \in C_c(V). \tag{4.4}$$

The statements in Theorem 4.1 and Corollary 4.2 will follow mainly by pointwise inequalities without summation. Then, we will sum over $X \times X$ and use Green's formula to obtain the results. The elementary inequalities are basically given in the upcoming lemma, Lemma 4.8.

The proof does not include the case $p = 1$. This is because we use a quantification of the strict convexity of the mapping $x \mapsto |x|^p$, $p > 1$.

## 4.2 Some Remarks on the Representation

**Remark 4.3 (Comparison with a local non-linear analogue)** Let us compare our ground state representation formula with results in [PTT08]. Similar results associated with weighted $p$-Schrödinger operators can be found in [PR15].

Fix $p \in (1,\infty)$ and a domain $\Omega \subseteq \mathbb{R}^d$. Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and let $\Delta(u) := -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ be the $p$-Laplacian on $\Omega$. Furthermore, let $V \in L^\infty_{\mathrm{loc}}(\Omega)$. The corresponding energy functional to the Schrödinger operator $\Delta + V$ is given by

$$Q(\varphi) := \int_\Omega |\nabla\varphi|^p + V\,|\varphi|^p \,\mathrm{d}x, \qquad \varphi \in C_c^\infty(\Omega).$$

Then, by [PTT08, Lemma 2.2], we have the following: If $u$ is a positive $p$-harmonic function of $\Delta + V$ in the weak sense, i.e., $\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla\varphi + V |u|^{p-2} u\varphi \mathrm{d}x = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then

$$Q(u\varphi) \asymp \int_\Omega u^2 |\nabla\varphi|^2 \left(u |\nabla\varphi| + \varphi |\nabla u|\right)^{p-2} \mathrm{d}x, \qquad 0 \le \varphi \in C_c^1(\Omega). \tag{4.5}$$

In particular, for $p > 2$, we have

$$Q(u\varphi) \asymp \int_\Omega u^p |\nabla\varphi|^p + u^2 |\nabla u|^{p-2} \varphi^{p-2} |\nabla\varphi|^2 \mathrm{d}x, \qquad 0 \le \varphi \in C_c^1(\Omega). \tag{4.6}$$

In the case of $1 < p < 2$, we have by [PTT08, Remark 1.12] that

$$\int_\Omega u^2 |\nabla\varphi|^2 \left(u |\nabla\varphi| + \varphi |\nabla u|\right)^{p-2} \mathrm{d}x \le \int_\Omega u^p |\nabla\varphi|^p \mathrm{d}x. \tag{4.7}$$

Now, we do the comparison: In the continuum, domains of $\mathbb{R}^d$ are considered. On graphs, we can take any subset of the graph.

Recall that $u$ is $p$-harmonic. It is very easy to compare $h_u(\varphi)$ with the right-hand side in (4.5), see Table 4.1.

**Table 4.1:** Comparison of the terms in the right-hand side (RHS) of (4.5) with $h_u(\varphi)$.

| RHS of (4.5) | $h_u(\varphi)$ |
|---|---|
| $u^2 \|\nabla\varphi\|^2$ | $u(x)u(y) \|\nabla_{x,y}\varphi\|^2$ |
| $u \|\nabla\varphi\| + \varphi \|\nabla u\|$ | $(u(x)u(y))^{1/2} \|\nabla_{x,y}\varphi\| + \frac{1}{2}(\|\varphi(x)\| + \|\varphi(y)\|) \|\nabla_{x,y} u\|$ |

This motivates to call the simplified energy $h_u$ the analogue to the simplified energy in the local non-linear case. Note that in the continuum, we only consider non-negative compactly supported functions $\varphi$, whereas on graphs, we allow $\varphi$ to take negative values. Thus, the version in the continuum contains hidden moduli of $\varphi$.

Furthermore, we see that the equivalence (4.6) has the same structure as the equivalence (4.4). For a comparison of $h_{u,1}(\varphi) + h_{u,2}(\varphi)$ with the right-hand side in (4.6) see Table 4.2.

**Table 4.2:** Comparison of the terms in the right-hand side (RHS) of (4.6) with $h_{u,1}(\varphi) + h_{u,2}(\varphi)$.

| RHS of (4.6) | $h_{u,1}(\varphi) + h_{u,2}(\varphi)$ |
|---|---|
| $u^p \|\nabla\varphi\|^p$ | $(u(x)u(y))^{p/2} \|\nabla_{x,y}\varphi\|^p$ |
| $u^2 \|\nabla u\|^{p-2} \varphi^{p-2} \|\nabla\varphi\|^2$ | $u(x)u(y) \|\nabla_{x,y} u\|^{p-2} \left(\frac{1}{2}(\|\varphi(x)\| + \|\varphi(y)\|)\right)^{p-2} \|\nabla_{x,y}\varphi\|^2$ |

Furthermore, we see that the estimate in (4.7) together with (4.5) has the same structure as the upper bound (4.2).

It should be mentioned that the strategy to prove the ground state representation in [PTT08] and here are similar. There, an elementary equivalence is the key ingredient and then a Picone identity is used. Here, we use different elementary equivalences and the Green's formula. However, the proof of the elementary equivalences in the discrete is technically much harder than the proof of the corresponding one in the continuum. Thus, the differences above might come from the fact that in the continuum we have a Picone identity (see [PTT08, Section 2]) which is established via the chain rule. Whereas in the discrete, we only have a one-sided Picone inequality. A general version of this one-sided Picone inequality is discussed in the next chapter, see also [BF14; Fis22].

Moreover, in [PTT08, Proposition 5.1] it was shown that for $p > 2$ both summands in the integral in (4.6) are needed in general for an upper bound. We expect that the same holds true on graphs, i.e., we expect that both $h_{u,1}$ and $h_{u,2}$ are needed in general for an upper bound of $h$.

**Remark 4.4 (Discussion of the constants)** First of all, let us mention that the constants in Theorem 4.1 and Corollary 4.2 only depend on $p$.

By comparing Theorem 4.1 with [FS08, Proposition 2.3] and Lemma 4.8 (the lemma below) with [FS08, Lemma 2.6], we see that $c_p$ in (4.3) can be stated explicitly as a minimiser, i.e., for $p \geq 2$

$$c_p = \min_{t\in(0,1/2)} \big((1-t)^p - t^p + pt^{p-1}\big) \in (0,1].$$

Note that $c_2 = 1$. By comparison, the constant $C_p = \max_{t\in(0,1/2)}\big((1-t)^p - t^p + pt^{p-1}\big) \in [1,2)$ is also the best upper bound in (4.2). Moreover, we expect that the best constants in Theorem 4.1 are between 0 and 2.

**Example 4.5 (Free $p$-Laplacian on $\mathbb{N}$)** Here, we calculate the representation for one of the simplest cases: for the line graph on $\mathbb{N}$ from Example 3.4.

It is not difficult to see that $u \in F$ defined via $u(n) = n^{(p-1)/p}$ is a positive $p$-superharmonic function such that $Lu = wu^{p-1}$, where $w$ is the improved $p$-Hardy weight from Theorem 1.1. Let $q := p/(p-1)$ and $\alpha(n) := (1-1/n)^{1/q}$, $n \in \mathbb{N}$. Then, the equivalence (4.1) reads as follows: for all $\varphi \in C_c(\mathbb{N})$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} |\nabla_{n,n-1}\varphi|^p - w(n)\,|\varphi(n)|^p \\
&\asymp \sum_{n=2}^{\infty} \frac{1}{\alpha^{p-1}(n)} \big(\alpha(n)\varphi(n) - \varphi(n-1)\big)^2 \\
&\quad\cdot \left(\alpha^{1/2}(n)\,|\alpha(n)\varphi(n) - \varphi(n-1)| + \frac{\alpha(n)\,|\varphi(n)| + |\varphi(n-1)|}{2}\big(1-\alpha(n)\big)\right)^{p-2}.
\end{aligned}$$

If $p = 2$, then the equivalence is an equality and gives exactly the result of [KŠ22, Theorem 1].

Moreover, Inequality (4.3) ($p \geq 2$) in Corollary 4.2 is here

$$\sum_{n=1}^{\infty} |\nabla_{n,n-1}\varphi|^p - w(n)\,|\varphi(n)|^p \geq c_p \sum_{n=2}^{\infty} \frac{1}{\alpha^{p/2}(n)}\, |\alpha(n)\varphi(n) - \varphi(n-1)|^p$$

for all $\varphi \in C_c(\mathbb{N})$. By (4.2), the reversed inequality holds for $1 < p \leq 2$.

**Remark 4.6 (Alternative Representation)** By redoing the proof of Theorem 4.1, we see that also the following alternative representation is true: Let $p > 1$, and $f > 0$ in some $V \subseteq X$. Then, we have

$$h(\psi f) - \langle H\psi, \psi f^p \rangle_V \asymp h_\psi(f), \qquad 0 \leq \psi \in C_c(V). \tag{4.8}$$

This will be used in Chapter 13.

**Remark 4.7 (Simplified Energy Functionals)** In the linear case, an application of the ground state representation formula is to get from Schrödinger forms $h$ with arbitrary potential part via an equality to a new Schrödinger form $h_u + \langle uHu, |\cdot|^2 \rangle$ with non-negative potential part. This new form corresponds then to the graph $b_u$ given by $b_u(x,y) := b(x,y)u(x)u(y)$, $x, y \in X$, where $u$ is a positive superharmonic function with respect to the Schrödinger operator associated with $h$, and has the potential $c_u := uHu$, see [KPP18b; KPP21]. An advantage of Schrödinger forms with non-negative potential part is that they are Markovian.

For $p \neq 2$, the simplified energy $h_u$ is not a $p$-energy functional anymore and the described method from the linear situation cannot be applied directly. However, there is a partial workaround via $h_{u,1}$. By applying Corollary 4.2 for some fixed positive superharmonic $u$ in $X$, we see that there is a positive constant $c_p$ such that

$$h(u\varphi) \leq c_p h_{u,1}(\varphi) + \langle uHu, |\varphi|^p \rangle \qquad \varphi \in C_c(X), p \in (1,2].$$

The right-hand side is a $p$-energy functional associated with the graph $b_u$ given by $b_u(x,y) := c_p b(x,y)(u(x)u(y))^{p/2}$, $x, y \in X$, and has the non-negative potential $c_u := uHu$.

In the case of $p > 2$, the situation is more complicated because the right-hand side gets additionally the functional $h_{u,2}$, which is not a $p$-energy functional. Nevertheless, the following observation has proven to be useful (see [DP16; Fis24]): applying Hölder's inequality to $h_{u,2}$, we get the existence of a positive constant $C_p$ such that

$$h(u\varphi) \leq C_p\Big(h_{u,1}(\varphi) + \left(\frac{h_{u,1}(\varphi)}{h_{u,3}(\varphi)}\right)^{2/p} h_{u,3}(\varphi)\Big) + \langle uHu, |\varphi|^p \rangle, \qquad p > 2. \tag{4.9}$$

where

$$h_{u,3}(\varphi) := \sum_{x,y \in X} b(x,y) \left(\frac{|\varphi(x)| + |\varphi(y)|}{2}\right)^p |\nabla_{x,y} u|^p,$$

which can be interpreted as a $p$-energy functional on $F$ for some fixed $\varphi \in C_c(X)$. We will use this estimate frequently in Chapter 12.

Let us close this section with an open problem: It would be very interesting to see if also other means than our particular choice in the definition of $h_u$ of geometric and arithmetic mean are possible. In particular, having twice the same mean in the formula would be a possibly future goal.

## 4.3 Elementary Inequalities and Equivalences

The next lemma is the most important tool in order to derive the ground state representations, Theorem 4.1 and Corollary 4.2. It provides us with pointwise estimates which result in the desired estimates of the energy functionals by summing over all vertices.

**Lemma 4.8 (Fundamental inequalities and equivalences)** *Let $a \in \mathbb{R}$, $0 \leq t \leq 1$, and $p > 1$. Then we have*

$$|a-t|^p - (1-t)^{p-1}(|a|^p - t) \asymp t\,|a-1|^2\,(|a-t|+1-t)^{p-2}, \tag{4.10}$$

*where the right-hand side is understood to be zero if $1 < p < 2$ and $a = t = 1$.*

*Moreover, we have*

$$|a-t|+1-t \asymp t^{1/2}\,|a-1| + (1-t)\frac{|a|+1}{2}, \tag{4.11}$$

*where the right-hand side is an upper bound with optimal constant $c = 2$, and it is a lower bound with optimal constant $c = 1/2$.*

*Furthermore, if $1 < p \leq 2$, then*

$$t\,|a-1|^2 \leq t^{p/2}\,|a-1|^p\,(|a-t|+1-t)^{2-p}, \tag{4.12}$$

*and for $p \geq 2$, the reserved inequality holds, i.e.,*

$$t\,|a-1|^2\,(|a-t|+1-t)^{p-2} \geq t^{p/2}\,|a-1|^p\,. \tag{4.13}$$

*Moreover, we have the following refinement of the elementary inequality (2.1): for all $p \geq 0$, we have*

$$\alpha^p + \beta^p \asymp (\alpha+\beta)^p, \qquad \alpha,\beta \geq 0, \tag{4.14}$$

*where the right-hand side is an upper bound with optimal constant $c_p = 2^{1-p}$ if $0 \leq p \leq 1$ and $c_p = 1$ if $p \geq 1$, and it is a lower bound with optimal constant $c_p = 1$ for $0 \leq p \leq 1$ and $c_p = 2^{1-p}$ for $p \geq 1$.*

A proof is given in Appendix A.2.

We do not claim that the constants we get in (4.10) are optimal. We expect that they can be improved and that the best constants should be either on the boundary of $[0,1] \times \mathbb{R}$, or at $(t,0), (t,t), (t,1)$, $t \in [0,1]$. Moreover, we expect that the optimal constants are between 0 and 2.

Also note that the inequalities (4.10) and (4.13) show that we improved an elementary one-sided result in [FS08] for $p > 2$.

Moreover, in the case of $1 < p < 2$, the "$\geq$"-inequality in (4.10) was proven in [AM16, Lemma 3.3]. However, the basic strategy to prove the remaining inequalities in (4.10) up to a certain point will be the similar, i.e., we start the proof with the same substitution and then use the same Taylor-Maclaurin formula (confer this also with the proof of [Lin90, Lemma 4.2]).

Furthermore, note that (4.10) is false for $p = 1$ as the left-hand side vanishes for $a > 1 \geq t > 0$ but the right-hand side does not. A similar argument can also be made for (4.12).

## 4.4 Proof of Theorem 4.1

Now we prove our main results of this chapter, Theorem 4.1 and Corollary 4.2. The basic strategy is to use the pointwise equivalences and estimates from Lemma 4.8 and then sum over all vertices.

*Proof (of Theorem 4.1).* Let $\varphi \in C_c(V)$ and $0 \le u \in F(V)$ for some $V \subseteq X$. If either $u(x) = 0$ or $u(y) = 0$ for some $x, y \in V \cup \partial V$, then

$$|\nabla_{x,y}(u\varphi)|^p - (\nabla_{x,y}u)^{\langle p-1\rangle}\, \nabla_{x,y}(u\,|\varphi|^p) = 0.$$

Moreover, $u(x)u(y)(\nabla_{x,y}\varphi)^2 = 0$. Thus, it remains in the following to consider the case $u(x), u(y) > 0$.

Firstly, let $u(x) \ge u(y) > 0$ for some fixed $x, y \in V \cup \partial V$. Moreover, assume that $\varphi(y) \neq 0$. Then, setting $t = u(y)/u(x)$ and $a = \varphi(x)/\varphi(y)$ in (4.10) combined with (4.11) results in

$$\begin{aligned}&|\nabla_{x,y}(u\varphi)|^p - (\nabla_{x,y}u)^{p-1}\nabla_{x,y}(u\,|\varphi|^p)\\ &\qquad \asymp u(x)u(y)(\nabla_{x,y}\varphi)^2\Big((u(x)u(y))^{1/2}\,|\nabla_{x,y}\varphi| + \frac{|\varphi(x)|+|\varphi(y)|}{2}\nabla_{x,y}u\Big)^{p-2}.\end{aligned}$$

If $\varphi(y) = 0$, then we get the equivalence above if we can show that

$$1-(1-t)^{p-1} \asymp t(t^{1/2} + (1-t)/2)^{p-2}, \qquad t = u(y)/u(x) \in (0,1].$$

Using (4.11) with $a = 0$, we see that $t^{1/2} + (1-t)/2 \asymp 1$. Moreover, the left-hand side lies between $t \mapsto (p-1)t$ and the identity on $(0,1]$. This shows the claim.

By a symmetry argument, we also get for all $x, y \in V \cup \partial V$ such that $u(y) \ge u(x) > 0$,

$$\begin{aligned}&|\nabla_{x,y}(u\varphi)|^p - (\nabla_{y,x}u)^{p-1}\nabla_{y,x}(u\,|\varphi|^p)\\ &\qquad \asymp u(x)u(y)(\nabla_{x,y}\varphi)^2\Big((u(x)u(y))^{1/2}\,|\nabla_{x,y}\varphi| + \frac{|\varphi(x)|+|\varphi(y)|}{2}\nabla_{y,x}u\Big)^{p-2}.\end{aligned}$$

Note that by Green's formula, Lemma 2.7 for the $p$-Laplacian $L$,

$$\sum_{x,y\in V\cup\partial V} b(x,y)\,(\nabla_{x,y}u)^{\langle p-1\rangle}\,\nabla_{x,y}(u\,|\varphi|^p) = 2\sum_{x\in V} u(x)Lu(x)\,|\varphi(x)|^p\,m(x).$$

Summing over all $x, y \in X$ with respect to $b$ and using the calculation above yields then (4.1). □

Now, we can directly continue with proving Corollary 4.2.

*Proof (of Corollary 4.2).* The proof of (4.2) and (4.3) can simply be read off (4.1).

The inequalities in (4.4) follow easily from (4.1) and (4.14). □

Alternatively, one can also deduce (4.2) and (4.3) from (4.13), (4.12) and (4.10). The proof can then be mimicked from the proof of Theorem 4.1.

# 5. General Principles and Inequalities

> There are many highly respectable motives which may lead men to prosecute research but three which are much more important than the rest. The first [...] is intellectual curiosity, desire to know the truth. Then, professional pride, anxiety to be satisfied with one's performance, the shame that overcomes any self-respecting craftsman when his work is unworthy of his talent. Finally, ambition, desire for reputation, even the power or the money, which it brings. [...] So if a mathematician [...] were to tell me that the driving force in his work had been the desire to benefit humanity, then I should not believe him.
>
> G. H. Hardy, *A Mathematician's Apology*, p. 79

Here, we introduce the necessary toolbox to achieve global results as e.g. the Agmon-Allegretto-Piepenbrink theorem. This will be done by showing a variety of local results. The actual proofs of the main results will then follow often by a limiting process. To be more specific, in this chapter we show

- a local Harnack-type inequality and principle,
- a Picone-type inequality,
- an Anane-Díaz-Saá-type inequality,
- the existence of a principal eigenvalue on finite subsets,
- the existence and uniqueness of solutions to the Poisson-Dirichlet problem on finite subsets,
- characterisations of the maximum principle on finite subsets.

## 5.1 Local Harnack Inequality and Harnack Principle

Next we show that locally, i.e., on finite and connected sets, our graphs fulfil a so-called Harnack inequality for non-negative supersolutions. This inequality implies that non-negative supersolutions are either strictly positive or the zero function, and they do not tend to infinity in the interior of the graph.

There is a long list of proofs of various Harnack-type inequalities for the $p$-Laplacian, see for instance for metric spaces [BB11, Theorem 8.12] where a $p$-Poincaré inequality is assumed. The corresponding analogue for linear Schrödinger operators on locally summable graphs can be found in [KPP20b]. The basic idea of the following proof of the local Harnack inequality can also be found in [Pra04], where the standard $p$-Laplacian on locally finite graphs without potential (i.e., $c = 0$) is considered, and [HS97a], where the standard $p$-Laplacian on finite graphs without potential, is considered.

Thereafter, we show a Harnack-type principle, which is a consequence of the local Harnack inequality. Note that here, we explicitly use that $p \neq 1$. The application of the Harnack inequality and principle in the main results is also a reason for an exclusion of the case $p = 1$.

**Lemma 5.1 (Local Harnack inequality)** *Let $p > 1$, $V \subseteq X$ be connected and $f \in C(X)$. Let $u \in F(V)$ be non-negative on $V \cup \partial V$ such that $Hu \geq f u^{p-1}$ on $V$. Then, for any $x \in V$ and $y \sim x$ we have*

$$u(y) \leq \left( \left( \frac{\deg(x) + c(x) - f(x)m(x)}{b(x,y)} \right)^{\frac{1}{p-1}} + 1 \right) u(x).$$

*In particular, we have $\deg + c \geq fm$ on $V$. Furthermore, if $u(x) = 0$ for some $x \in V$, then $u(x) = 0$ for all $x \in V \cup \partial V$. In other words, any function which is positive on $V \cup \partial V$ and $p$-superharmonic on $V$ is strictly positive on $V$.*

*If $V$ is also finite, then there exists a positive constant $C_{V,H,f}$ depending only on $V$, $H$ and $f$, such that*

$$\max_V u \leq C_{V,H,f} \min_V u.$$

*The constant $C_{V,H,f}$ can be chosen to be monotonous in the sense that if $f \leq g \in C(X)$ then $C_{V,H,f} \geq C_{V,H,g}$.*

*Proof.* Let $V \subseteq X$ be connected and let $u \in F(V)$ be such that $u \geq 0$ on $V \cup \partial V$ and $Hu \geq f u^{p-1}$ on $V$ for some $f \in C(X)$.

If $u(x_0) = 0$ for some $x_0 \in V$, then we have

$$0 = f(x_0)u^{p-1}(x_0)m(x_0) \leq Hu(x_0)m(x_0) = -\sum_{y \in X} b(x_0, y)u(y)^{p-1} \leq 0.$$

Thus, for all $x \sim x_0$, we have $u(x) = 0$ and since $V$ is connected we infer by induction that $u(x) = 0$ for all $x \in V \cup \partial V$.

Hence, we can assume that $u > 0$ on $V$. Because of $Hu \geq f u^{p-1}$ on $V$, we have

$$\sum_{y \in X, \nabla_{x,y} u \leq 0} b(x,y) \left( \frac{u(y)}{u(x)} - 1 \right)^{p-1}$$
$$\leq \sum_{y \in X, \nabla_{x,y} u > 0} b(x,y) \left( 1 - \frac{u(y)}{u(x)} \right)^{p-1} + c(x) - f(x)m(x)$$

for any $x \in V$. The right-hand side can be estimated as follows:

$$\ldots \leq \sum_{y \in X, \nabla_{x,y} u > 0} b(x,y) + c(x) - f(x)m(x) \leq \deg(x) + c(x) - f(x)m(x).$$

Let $d_f := \deg + c - fm$. The previous calculations imply that $d_f \geq 0$ on $V$. Now assume that there is a vertex $y_0 \sim x$ such that $u(x) \leq u(y_0)$. Then the previous calculations also imply

$$b(x, y_0)\Big(\frac{u(y_0)}{u(x)} - 1\Big)^{p-1} \leq d_f(x), \text{ i.e.,} \qquad u(y_0) \leq \bigg(\Big(\frac{d_f(x)}{b(x, y_0)}\Big)^{\frac{1}{p-1}} + 1\bigg)u(x).$$

Hence, for all $y \sim x$ we have

$$u(y) \leq \bigg(\Big(\frac{d_f(x)}{b(x, y)}\Big)^{\frac{1}{p-1}} + 1\bigg)u(x).$$

Let $x \in V$ and $y \in V \cup \partial V$. Since $V$ is connected, there is a path $x_1 \sim \ldots \sim x_n$ in $V$ such that $x_1 \sim x_0 := y$ and $x_n = x$. Then we derive

$$u(y) \leq \prod_{i=0}^{n-1}\bigg(\Big(\frac{d_f(x_i)}{b(x_i, x_{i+1})}\Big)^{\frac{1}{p-1}} + 1\bigg)u(x).$$

Note that the obtained product does not only depend on $V$, $H$ and $f$ but also on $x$ and $y$, and the chosen path. The dependence on the path can be overcome by considering all possible paths in $V$ such that the starting vertex is connected to $y$. Taking then the infimum of all resulting products, we get a function $C_{V,H,f} \colon V \times V \cup \partial V \to [1, \infty)$ which is still dependent on the vertices but independent of a specific path. As we would like to get a unifying constant for all vertices, one possibility is to simply take the supremum of $C_{V,H,f}(x, y)$ over all $x \in V, y \in V \cup \partial V$. However, this supremum might not be finite. A way to ensure finiteness is to assume additionally that $V$ is finite and $x, y \in V$. Then, $C_{V,H,f} := \max_{x,y\in V} C_{V,H,f}(x, y) < \infty$ which yields the desired statement. Moreover, if $f \leq g \in C(X)$ then $d_f \geq d_g$ and hence $C_{K,H,f} \geq C_{K,H,g}$. $\square$

Now, we want to prove the Harnack principle. We do this by dividing the statement into two partial results, Lemma 5.2 and Proposition 5.5. The technical part is extracted in the following lemma. This lemma has many analogues in other settings, see e.g. [BB11; GP23; KLW21; KPP20b; KMM07], and is a standard statement in potential theory.

Let $V \subseteq X$ and $o \in V$ be a fixed reference point. Then, define $S_o^+(V) = S_o^+(V, H)$ as follows

$$S_o^+(V) := \{u \in F(V) : u(o) = 1, Hu \geq 0 \text{ on } V, u \geq 0 \text{ on } V \cup \partial V\}.$$

**Lemma 5.2 (Harnack Principle)** *Let $V \subseteq X$ be connected, and $(V_n)$ be an increasing exhaustion of $V$ with connected subsets. Let $f_n \in C(V_n)$ such that $f_n \to f \in C(V)$ pointwise. Let $(u_n)$ be a sequence of non-negative functions such that $Hu_n \geq f_n u_n^{p-1}$ on $V_n$, and which converges pointwise on $X$ to some extended function $u$. Then either $u(x) = \infty$ for all $x \in V$ or $u$ is non-negative on $X$ and $Hu \geq f u^{p-1}$ on $V$.*

*Moreover, the set $S_o^+(V)$ is compact with respect to the topology of pointwise convergence.*

*Proof.* Let $(u_n)$ be a sequence of non-negative functions such that $Hu_n = f_n u_n^{p-1}$ in $V_n$. We divide the proof into several cases.

If $\lim_{n\to\infty} u_n(x) = \infty$ for some $x \in V$. Then, by the Harnack inequality, Lemma 5.1, we have $\lim_{n\to\infty} u_n(x) = \infty$ for all $x \in V \cup \partial V$.

If $\lim_{n\to\infty} u_n(x) = 0$ for some $x \in V$. Then, again by the Harnack inequality, Lemma 5.1, we have $\lim_{n\to\infty} u_n(x) = 0$ for all $x \in V \cup \partial V$. This is, however, a non-negative harmonic function on $V$.

Now, let us assume that there exists $u \in C(X)$ such that $\lim_{n\to\infty} u_n(x) = u(x) \in (0,\infty)$ for all $x \in V$. By the Harnack inequality, we have $u(x) \in [0,\infty)$ for all $x \in \partial V$. Without loss of generality, we can assume that $u_n > 0$ on $V_n$.

Moreover, assuming $Hu_n \geq f_n u_n^{p-1}$ on $V_n$ is equivalent to

$$\sum_{y\in X, \nabla_{x,y} u_n < 0} b(x,y)(\nabla_{y,x} u_n)^{p-1} \\ \leq \sum_{y\in X, \nabla_{x,y} u_n > 0} b(x,y)(\nabla_{x,y} u_n)^{p-1} + \big(c(x) - f_n(x)m(x)\big)u_n(x)^{p-1}$$

for any $x \in V_n$. Furthermore, since $u_n > 0$ on $V_n$

$$\begin{aligned} 0 &\leq \sum_{y\in X, \nabla_{x,y} u_n < 0} b(x,y)\Big(\frac{u_n(y)}{u_n(x)} - 1\Big)^{p-1} \\ &\leq \sum_{y\in X, \nabla_{x,y} u_n > 0} b(x,y)\Big(1 - \frac{u_n(y)}{u_n(x)}\Big)^{p-1} + c(x) - f_n(x)m(x) \\ &\leq \sum_{y\in X, \nabla_{x,y} u_n > 0} b(x,y) + c(x) - f_n(x)m(x) \\ &\leq \deg(x) + c(x) - f_n(x)m(x) \\ &\to \deg(x) + c(x) - f(x)m(x) < \infty. \end{aligned}$$

Since $(1 - u_n(y)/u_n(x))^{p-1} 1_{\{y: \nabla_{x,y} u_n > 0\}}$ is dominated by the constant and $b(x,\cdot)$-integrable function 1, we can use the lim inf-lim sup formula of Fatou's lemma for the divergence part and get for any $x \in V_n$

$$\begin{aligned} \sum_{y\in X, \nabla_{x,y} u < 0} b(x,y)\Big(\frac{u(y)}{u(x)} - 1\Big)^{p-1} &\leq \liminf_{n\in\mathbb{N}} \sum_{y\in X, \nabla_{x,y} u_n < 0} b(x,y)\Big(\frac{u_n(y)}{u_n(x)} - 1\Big)^{p-1} \\ &\leq \limsup_{n\in\mathbb{N}} \sum_{y\in X, \nabla_{x,y} u_n > 0} b(x,y)\Big(1 - \frac{u_n(y)}{u_n(x)}\Big)^{p-1} + c(x) - f_n(x)m(x) \\ &\leq \sum_{y\in X, \nabla_{x,y} u > 0} b(x,y)\Big(1 - \frac{u(y)}{u(x)}\Big)^{p-1} + c(x) - f(x)m(x). \end{aligned}$$

Multiplying both sides with $u^{p-1}(x)$, dividing by $m(x)$ and rearranging yields in a non-negative function such that $Hu \geq f u^{p-1}$ on any $V_n$ for $n$ large enough. Since $(V_n)$ is an increasing exhaustion of connected sets, we get $Hu = f u^{p-1}$ on $V$.

We now turn to the statements for $S = S_o^+(V)$. By the pointwise convergence it follows from the previous investigations that we get $u \in S$ if even $(u_n)_n$ is in $S$.

Furthermore, note that $V$ is connected, so for all $x \in V$ there exists a path $x_0 \sim \ldots \sim x_n$ such that $x_0 = o$ and $x_n = x$. Let $V = \{x_0, \ldots, x_n\}$ then we can apply Lemma 5.1 to some $u \in S$ and get that there exists a constant $C_x > 0$ such that $C_x^{-1} \leq u(x) \leq C_x$. Hence $S$ is included in the product space $\prod_{x\in X}[C_x^{-1}, C_x]$ which is compact due to Tychonoff's theorem. By the previous part, $S$ is closed and thus, it is also compact. □

Next, we show that one cannot expect in general that the pointwise limit of positive $p$-harmonic functions is also $p$-harmonic, confer with Lemma 5.2.

**Example 5.3 (Harmonic sequence but strictly superharmonic limit)** Let us take a star graph $(\mathbb{N}_0, b, m)$, see Example 3.9, and let us assume that $c > 0$ on $\mathbb{N}_0$. We will see later that this implies the existence of strictly positive $p$-harmonic functions on all finite and connected subsets of $\mathbb{N}_0$, confer Proposition 5.17, and the existence of a $p$-superharmonic function on the whole graph, confer Theorem 6.1 and the examples thereafter.

Hence take a sequence $(u_n)$ of positive functions with $u_n(0) = 1$, and $Hu_n = 0$ on $\{0, \ldots, n\}$. By the Harnack inequality, Lemma 5.1, this sequence has a convergent subsequence with limit $u > 0$. But if $u$ would be $p$-harmonic, then for any $k \in \mathbb{N}_0$

$$0 > -\frac{c(k)}{m(k)} u^{p-1}(k) = Lu(k) = -\sum_{n\neq k} \frac{m(n)}{m(k)} Lu(n) = \sum_{n\neq k} \frac{c(n)}{m(k)} u^{p-1}(n) > 0.$$

This is a contradiction, and thus $u$ has to be strictly $p$-superharmonic.

We continue with a result which seems to be true in every quasi-linear potential theory. It states that the set of non-negative supersolutions is downwards directed, i.e., it is $\wedge$-stable.

**Corollary 5.4** *Let $p \geq 1$, $V \subseteq X$ be connected and let $S$ be a family of functions $s \in F(V)$ which are non-negative on $V \cup \partial V$ and $Hs \geq f\,(s)^{\langle p-1\rangle}$ on $V$ for some $f \in C(X)$. Then, the pointwise infimum $u$ of functions in $S$ is also in $F(V)$, non-negative on $V \cup \partial V$, and $Hu \geq f\,(u)^{\langle p-1\rangle}$ on $V$. Furthermore, if $u(x) = 0$ for some $x \in V$, then $u = 0$ on $V \cup \partial V$.*

*Proof.* Apply Lemma 5.2. For a direct proof see [Fis24]. □

From Lemma 5.2 another principle will follow easily (which is sometimes also called Harnack principle). The corresponding analogue for linear Schrödinger operators on our general graphs can again be found in [KPP20b].

**Proposition 5.5 (Convergence of Solutions)** *Let $V \subseteq X$ be connected, $C > 0$ and $o \in V$. Assume that we have a sequence $(u_n)_n$ in $S_o^+(V)$. Then there exists a subsequence $(u_{n_k})_k$ that converges pointwise to a function $u \in S_o^+(V)$. Furthermore, assume that either*

(a) *the graph is locally finite on $V \cup \partial V$, or*

(b) *the subsequence $(\nabla_{x,y} u_{n_k})_k$ is monotone for all $x \in V$ and $y \sim x$, or*

(c) *there exists a function $f \in F(V)$ such that for all $k \in \mathbb{N}$ we have $\nabla_{x,y} u_{n_k} \leq \nabla_{x,y} f$ for all $x \in V$ and $y \sim x$, or*

(d) *we have $u_{n_k} \leq u$ in $V \cup \partial V$.*

*Then $Hu_{n_k} \to Hu$ pointwise on $V$ as $n_k \to \infty$ .*

*Proof.* By the previous lemma, Lemma 5.2, the first statement follows easily. Now let $(u_{n_k})$ be a subsequence that converges pointwise to a function $u \in S_o^+(V)$.

Ad (a): If the graph is locally finite we have

$$\begin{aligned} \lim_{k\to\infty} Lu_{n_k}(x) &= \lim_{k\to\infty} \frac{1}{m(x)} \sum_{y\in X} b(x,y) \left(\nabla_{x,y} u_{n_k}\right)^{\langle p-1\rangle} \\ &= \frac{1}{m(x)} \sum_{y\in X} b(x,y) \lim_{k\to\infty} \left(\nabla_{x,y} u_{n_k}\right)^{\langle p-1\rangle} = Lu(x), \end{aligned}$$

since we sum over a finite number of elements. The assertion for the Schrödinger operator follows now easily.

Ad (b): If $(\nabla_{x,y} u_{n_k})_k$ is monotone for all $x \in V$ and $y \sim x$, then we can use the dominated convergence theorem to interchange summation and limit as above. Note that $(\cdot)^{\langle p-1\rangle} : \mathbb{R} \to \mathbb{R}$ is monotone increasing.

Ad (c): If $\nabla_{x,y} u_{n_k} \leq \nabla_{x,y} f$ for all $x \in V$, $y \sim x$, and for all $k \in \mathbb{N}$ for some $f \in F$. Then we can use again the dominated convergence theorem to interchange summation and limit.

Ad (d): Set $v_k := u - u_{n_k}$, $k \in \mathbb{N}$. Since $u, u_{n_k} \in S_o^+(V)$, we have $v_k \in F(V)$. Moreover, $u_{n_k} \leq u$ in $V \cup \partial V$ implies $\nabla_{x,y} u_{n_k} \leq \nabla_{x,y} u + v_k(y)$ for all $x \in V, y \sim x$. Since $v_k \to 0$, we can use dominated convergence and get $\lim_k Hu_{n_k}(x) \leq Hu(x)$ for all $x \in V$. For the other inequality, note that $\nabla_{x,y} u \leq \nabla_{x,y} u_{n_k} + v_k(x)$ for all $x \in V, y \sim x$. Similarly as before, we get $Hu(x) \leq \lim_k Hu_{n_k}(x)$. □

## 5.2 Picone and Anane-Díaz-Saá Inequalities

Here, we show an inequality which has many applications, one of them will lead the way to the desired Agmon-Allegretto-Piepenbrink theorem.

For non-local $p$-Laplacians on graphs and on $\mathbb{R}^d$, Picone's *inequality* is a consequence of the ground state representation, see Theorem 4.1. However, this inequality is only a special case of the representation and and can also be achieved more directly, see [FS08, Proof of Proposition 2.2] or [AM16, Lemma 2.3]. For the sake of being self-contained, we show an alternative proof here.

For $p$-Schrödinger operators on finite graphs a corresponding inequality is given in [PKC09, Theorem 4.1] or see [Amg08, Lemma 6.2] for the case of the standard $p$-Laplacian. Here, we show a Picone-type inequality generalising the techniques on finite graphs to infinite graphs.

In the continuum, there exists a so-called (pointwise) Picone *identity* for the $p$-Laplacian, see e.g. [AH98; BF14; PTT08]. Both proofs of Picone's, the local and the non-local case, use a pointwise identity resp. inequality. Here, we employ the following result which also shows, why we cannot hope for an identity in the non-local case in general (without adding a remainder term). In the local case, one has an identity because of an application of the chain rule.

**Lemma 5.6 (Lemma 4.1 in [PKC09])** *Let* $f\colon \mathbb{R}^3 \to \mathbb{R}$ *be such that*

$$f(a,b,c) = |a-b|^p + |a|^p\,(c-1)^{\langle p-1\rangle} + |b|^p\,(1/c-1)^{\langle p-1\rangle}.$$

*Then* $f \geq 0$ *on* $\left\{(a,b,c) \in \mathbb{R}^3 : a,b \geq 0, c>0\right\}$*. Furthermore,* $f=0$ *if and only if* $b = ac$*.*

Now we can show easily the following statement, the Picone inequality.

**Lemma 5.7 (Picone-type inequality)** *Let* $u,v \in C(X)$ *such that* $v(x), v(y) > 0$ *for some* $x,y \in X$*. Then, the following pointwise Picone-type inequality holds,*

$$|\nabla_{x,y} u|^p \geq (\nabla_{x,y} v)^{\langle p-1\rangle}\left(\nabla_{x,y}\frac{|u|^p}{v^{p-1}}\right).$$

*Let* $V \subseteq X$*, and assume that* $v>0$ *on* $V$*. Then for all* $x \in V$ *the following integrated Picone-type inequality holds,*

$$\sum_{y\in V} b(x,y)\left(|\nabla_{x,y} u|^p - (\nabla_{x,y} v)^{\langle p-1\rangle}\left(\nabla_{x,y}\frac{|u|^p}{v^{p-1}}\right)\right) \geq 0,$$

*where we allow the sum to be* $\infty$*.*

*If* $V$ *is connected and* $u \geq 0$*, then equality in the inequality above implies* $u = C\,v$ *on* $V$ *for some constant* $C>0$*.*

*Moreover,* $u = C\,v$ *on* $V$ *for some constant* $C>0$ *implies that we have equality in the inequality above.*

*In particular, we get for all* $\varphi \in C_c(V)$ *and* $0 < v \in F(V)$ *the following Picone-type inequality for the divergence part of the* $p$*-energy functional,*

$$\frac{1}{2}\sum_{x,y\in X} b(x,y)\,|\nabla_{x,y}\varphi|^p \geq \langle L v, \frac{|\varphi|^p}{v^{p-1}}\rangle_V.$$

*Proof.* Firstly, we consider $u \geq 0$: Applying Lemma 5.6 with $a = u(x)$, $b = u(y)$ and $c = v(y)/v(x)$ yields the first pointwise inequality. Hence, every summand is non-negative which gives the second inequality.

Moreover, if we assume equality and $u \geq 0$, then again by Lemma 5.6 this implies that every summand vanishes. Thus, for a fixed $o \in V$ we get for all $y \in V, y \sim o$, that $u(y) = u(o)v(y)/v(o)$. Set $C = u(o)/v(o)$. Since $V$ is connected we get $u = C\,v$ on $V$.

Furthermore, if $u = C\,v$ for some $C > 0$ then we clearly have equality.

For arbitrary $u \in C(X)$, substitute $u$ by $|u|$ and use the reverse triangle inequality.

The latter statement of the lemma follows from Green's formula, Lemma 2.7. $\square$

**Remark 5.8** Another proof of Lemma 5.7 can be obtained as follows. By [FS08, Lemma 2.6] (or using the refinement (4.10)), we have for all $p \in [1, \infty)$, $t \in [0, 1]$ and $a \in \mathbb{R}$ that

$$|a - t|^p \geq (1 - t)^{p-1}(|a|^p - t).$$

For $p > 1$, this inequality is strict unless $a = 1$ or $t = 0$.

Using this inequality and defining $a$ and $t$ properly, we get the desired pointwise estimate to prove Lemma 5.7. Indeed, first of all note that if $u(y) = 0$ for some $y \in V$, then without loss of generality, we can assume that $u(x) > 0$, $x \in X$, and we get the pointwise estimate using the inequality

$$1 \geq |1 - \alpha|^{p-2}\,(1 - \alpha), \quad \alpha \geq 0.$$

We turn to $u(y) > 0$ for all $y \in V$. Then, assume that $v(x) \geq v(y)$ (and by a symmetry argument we get the other case). Then, set $t = v(y)/v(x)$ and $a = u(x)/u(y)$ and apply the inequality. This yields in

$$|\nabla_{x,y}(uv)|^p \geq |\nabla_{x,y}v|^{p-1}\,\nabla_{x,y}(|u|^p\,v).$$

Setting $\psi = uv$, we get the Picone inequality.

The following first consequence of Picone's inequality is a discrete version of [PP16, Lemma 3.3], see also [DS87, Lemme 2] and [Ana87]. See also [GS98, Lemma 4] and [PR15, Lemma 3.5] for special cases of this inequality in the continuum. It is an extension of [PKC09, Corollary 4.1] to infinite graphs.

The following Anane-Díaz-Saá-type inequality will be used to prove characterisations of the maximum principle on finite subsets, Proposition 5.17.

**Proposition 5.9 (Anane-Díaz-Saá-type inequality)** *Let $K \subseteq X$ be finite. Let $u_i \in F(K)$, $u_i > 0$ on $K$, $i = 1, 2$. Then,*

$$\begin{aligned}&\langle \frac{Lu_1}{u_1^{p-1}} - \frac{Lu_2}{u_2^{p-1}}, u_1^p - u_2^p \rangle_K \geq \\ &\sum_{x \in K, y \in \partial K} b(x,y)\left(u_1^p(x) - u_2^p(x)\right)\left(\left(1 - \frac{u_1(y)}{u_1(x)}\right)^{\langle p-1\rangle} - \left(1 - \frac{u_2(y)}{u_2(x)}\right)^{\langle p-1\rangle}\right).\end{aligned} \tag{5.1}$$

*Furthermore, if $K$ is connected, then equality implies $u_1 = Cu_2$ on $K$ for some constant $C > 0$. Moreover, if $u_1 = Cu_2$ on $K \cup \partial K$, then we have equality, and the sums are non-negative.*

*The right-hand side in (5.1) is non-negative if for each pair $(x, y) \in K \times \partial K$ one of the following holds true:*

*(a)* $u_1(x) = u_2(x)$, *or*

*(b)* $u_1(x)u_2(y) = u_1(y)u_2(x)$, *or*

*(c)* $u_1(x) > u_2(x)$, *and* $u_1(x)u_2(y) < u_1(y)u_2(x)$, *or*

*(d)* $u_1(x) < u_2(x)$, *and* $u_1(x)u_2(y) > u_1(y)u_2(x)$.

*In particular, if $\varphi, \psi \in C(K)$, $\varphi, \psi > 0$ on $K$, i.e., $\varphi = \psi = 0$ on $X \setminus K$. Then,*

$$\langle L\varphi, \frac{\varphi^p - \psi^p}{\varphi^{p-1}}\rangle_K + \langle L\psi, \frac{\psi^p - \varphi^p}{\psi^{p-1}}\rangle_K \geq 0. \tag{5.2}$$

*If $K$ is connected, then equality implies $\varphi = C\psi$ for some $C > 0$. Moreover, if $\varphi = C\psi$ for some $C > 0$, then we have equality.*

*Proof.* Set for all $x \in X$

$$\psi_1(x) := 1_K(x)\frac{u_1^p(x) - u_2^p(x)}{u_1^{p-1}(x)}, \qquad \psi_2(x) := 1_K(x)\frac{u_2^p(x) - u_1^p(x)}{u_2^{p-1}(x)}.$$

Note that the finiteness of $K$ implies that the following sums are all absolutely converging. Thus, we calculate using Green's formula, Lemma 2.7,

$$\begin{aligned}
&\sum_{x\in K}\big(Lu_1(x)\psi_1(x) + Lu_2(x)\psi_2(x)\big)m(x)\\
&\quad= \frac{1}{2}\sum_{x,y\in K} b(x,y)\left((\nabla_{x,y}u_1)^{\langle p-1\rangle}\nabla_{x,y}\psi_1 + (\nabla_{x,y}u_2)^{\langle p-1\rangle}\nabla_{x,y}\psi_2\right)\\
&\qquad+ \sum_{x\in K, y\in\partial K} b(x,y)\left((\nabla_{x,y}u_1)^{\langle p-1\rangle}\psi_1(x) + (\nabla_{x,y}u_2)^{\langle p-1\rangle}\psi_2(x)\right)\\
&\quad= \frac{1}{2}\sum_{x,y\in K} b(x,y)\left(|\nabla_{x,y}u_1|^p - (\nabla_{x,y}u_2)^{\langle p-1\rangle}\nabla_{x,y}\frac{u_1^p}{u_2^{p-1}}\right)\\
&\qquad+ \frac{1}{2}\sum_{x,y\in K} b(x,y)\left(|\nabla_{x,y}u_2|^p - (\nabla_{x,y}u_1)^{\langle p-1\rangle}\nabla_{x,y}\frac{u_2^p}{u_1^{p-1}}\right)\\
&\qquad+ \sum_{x\in K, y\in\partial K} b(x,y)\left((\nabla_{x,y}u_1)^{\langle p-1\rangle}\psi_1(x) + (\nabla_{x,y}u_2)^{\langle p-1\rangle}\psi_2(x)\right).
\end{aligned}$$

Using Picone's inequality, Lemma 5.7, yields in

$$\begin{aligned}
\ldots &\geq \sum_{x\in K, y\in\partial K} b(x,y)\left((\nabla_{x,y}u_1)^{\langle p-1\rangle}\psi_1(x) + (\nabla_{x,y}u_2)^{\langle p-1\rangle}\psi_2(x)\right)\\
&= \sum_{x\in K, y\in\partial K} b(x,y)\left(u_1^p(x) - u_2^p(x)\right)\left(\left(1 - \frac{u_1(y)}{u_1(x)}\right)^{\langle p-1\rangle} - \left(1 - \frac{u_2(y)}{u_2(x)}\right)^{\langle p-1\rangle}\right).
\end{aligned}$$

This shows the first statement. Now we turn to the equality: We have used an inequality, which might be an equality, Picone's. Thus, we can read of Lemma 5.7 that if $K$ is connected, we have $u_1 = Cu_2$ on $K$ for some $C > 0$.

Since $x \mapsto x^p, x \geq 0$, is strictly monotone increasing and $x \mapsto (1-x)^{\langle p-1 \rangle}, x \in \mathbb{R}$, is strictly monotone decreasing for $p > 1$, we get the desired non-negativity of the sum in the left-hand side in (5.1) if (a)-(d) are fulfilled. Especially, the left-hand side is non-negative if (a) and (b) are satisfied, which is fulfilled for $u_1 = Cu_2$ on $K \cup \partial K$.

Now we turn to the last assertion of the statement. Since $\varphi = 0 = \psi$ on $\partial K$, (b) is fulfilled on $K \cup \partial K$. Thus, we have an equality in (5.1). This gives the desired result.□

**Remark 5.10** The statement before can be generalised in the flavour of Lemma 3.3 in [PP16] for shifts of $u_i$, i.e., for functions $u_i + \alpha$ where $\alpha$ is a fixed constant. Since we do not need this generalisation here, we omit it.

## 5.3 Principal Eigenvalues on Finite Subsets

In this section, we have a closer look on finite subsets of $X$. Since $X$ can be exhausted by increasing but finite subsets, the following results can be seen as a toolbox.

Let $V \subseteq X$, and recall that $\lambda_0(V) = \lambda_0(V, H)$ is defined via

$$\lambda_0(V) := \inf_{\varphi \in C_c(V), \varphi \neq 0} \frac{h(\varphi)}{\|\varphi\|_{p,m}^p} = \inf_{\varphi \in C_c(V), \|\varphi\|_{p,m}^p = 1} h(\varphi). \tag{5.3}$$

The following proposition collects and slightly generalises various results in [PKC09, Section 3] and [PC11, Section 4]. There, only finite graphs are considered. Confer [PP16, Theorem 3.9] for an analogue in the continuum. On finite graphs associated with linear Laplace-type operator such a result is also known as a Perron-Frobenius-type theorem, see [KLW21, Theorem 0.55].

**Proposition 5.11 (Variational characterisation of the principal eigenvalue)** *Let the set $K \subseteq X$ be finite. Then there exists a positive function $\varphi_0 \in C(K)$, i.e., $\varphi_0 = 0$ on $X \setminus K$ and $\varphi_0 \gneq 0$ on $K$, such that*

$$H\varphi_0 = \lambda_0(K) \, (\varphi_0)^{\langle p-1 \rangle} \qquad \text{on } K.$$

*The function $\varphi_0$ is a minimiser of (5.3).*

*Moreover, assume additionally that $K$ is connected. Then, $\lambda_0(K)$ is a generalised principal eigenvalue on $K$, that is, there exists a strictly positive generalised $p$-eigenfunction $\varphi_0 \in C(K)$ to $\lambda_0(K)$ on $K$. Furthermore, $\lambda_0(K)$ is simple and any generalised $p$-eigenvalue $\lambda > \lambda_0(K)$ of $H$ on $K$ does only have eigenfunctions which change sign.*

*Proof.* Let $\lambda_0 = \lambda_0(K)$. The set $\{\varphi \in C(K) : \|\varphi\|_{p,m}^p = 1\}$ is compact since $C(K)$ is finite dimensional. Hence, there exists a non-trivial minimiser $\varphi_0 \in C(K)$ of $h$ with

$\|\varphi_0\|_{p,m} = 1$. Let $t \in (-1,1)$ and consider $\varphi_{t,z} = \varphi_0 + t\,1_z$ for some fixed $z \in K$. Then $\|\varphi_{t,z}\|_{p,m} \neq 0$ and $\lambda_0 \leq h(\varphi_{t,z})/\,\|\varphi_{t,z}\|_{p,m}^p$, i.e.,

$$0 \leq h(\varphi_{t,z}) - \lambda_0 \, \|\varphi_{t,z}\|_{p,m}^p \,,$$

where the right-hand side has a minimum in $t = 0$. Hence, using Lemma 2.8,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} h(\varphi_{t,z}) - \lambda_0 \, \|\varphi_{t,z}\|_{p,m}^p \Big|_{t=0} = pH\varphi_0(z)m(z) - \lambda_0 p \, (\varphi_0(z))^{\langle p-1\rangle} \, m(z).$$

Thus, $H\varphi_0 = \lambda_0 \, (\varphi_0)^{\langle p-1\rangle}$ on $K$.

Now we show that $\varphi_0$ can in fact be chosen to be non-negative on $K$. By the reverse triangle inequality, $h(\varphi_0) \geq h(|\varphi_0|)$, and $\|\varphi_0\|_{p,m} = \||\varphi_0|\|_{p,m}$. Hence, $\lambda_0 \geq h(|\varphi_0|)/\,\||\varphi_0|\|_{p,m}^p$. Clearly, we have by the definition of $\lambda_0$ that $\lambda_0 \leq h(|\varphi_0|)/\,\||\varphi_0|\|_{p,m}^p$. Thus, the non-negative function $|\varphi_0| \in C(K)$ solves the desired equation.

Now, we assume for the rest of the proof that $K$ is connected. Since $\varphi_0$ is non-trivial there exists $o \in K$ such that $|\varphi_0(o)| > 0$. Thus, by the Harnack inequality, Lemma 5.1, we get that $|\varphi_0| > 0$ on $K$.

We show that $\lambda_0$ is simple: We have seen that if $\varphi_0$ is an eigenfunction to $\lambda_0$ on $K$, then so is $|\varphi_0|$. Hence, $h(\varphi_0) = h(|\varphi_0|)$, which is equivalent to

$$\sum_{x,y\in K} b(x,y)\,|\nabla_{x,y}\varphi_0|^p = \sum_{x,y\in K} b(x,y)\,|\nabla_{x,y}\,|\varphi_0||^p \,.$$

By the reverse triangle inequality we have $|\nabla_{x,y}\varphi_0| \geq |\nabla_{x,y}\,|\varphi_0||$ for all $x \sim y$ in $K$. This implies that

$$|\nabla_{x,y}\varphi_0| = |\nabla_{x,y}\,|\varphi_0||\,, \qquad x \sim y \text{ in } K,$$

which yields either $\varphi_0 = |\varphi_0|$ on $K$ or $\varphi_0 = -\,|\varphi_0|$ on $K$. Altogether, any eigenfunction to $\lambda_0$ has constant sign.

Let $\varphi_1$ be another eigenfunction to $\lambda_0$ on $K$. By the previous consideration, we can assume without loss of generality that $\varphi_0, \varphi_1 > 0$ on $K$. Then,

$$\begin{aligned}
&\langle L\varphi_0, \frac{\varphi_0^p - \varphi_1^p}{\varphi_0^{p-1}}\rangle_K + \langle L\varphi_1, \frac{\varphi_1^p - \varphi_0^p}{\varphi_1^{p-1}}\rangle_K \\
&\qquad\qquad = \langle(\lambda_0 - \frac{c}{m})\varphi_0^{p-1}, \frac{\varphi_0^p - \varphi_1^p}{\varphi_0^{p-1}}\rangle_K + \langle(\lambda_0 - \frac{c}{m})\varphi_1^{p-1}, \frac{\varphi_1^p - \varphi_0^p}{\varphi_1^{p-1}}\rangle_K \\
&\qquad\qquad = 0.
\end{aligned}$$

By the Anane-Díaz-Saá inequality (5.2), this implies that $\varphi_0 = C\,\varphi_1$ for some $C > 0$ and hence, $\lambda_0(K)$ is simple.

We still have to show that any eigenvalue $\lambda > \lambda_0(K)$ of $H$ on $C(K)$ can only have eigenfunctions which switch sign. Let $\varepsilon > 0$ and assume that $0 < \varphi_\lambda \in C(K)$ is an eigenfunction to $\lambda$. Then by the Anane-Díaz-Saá inequality (5.2), we have

$$0 \leq \langle L\varphi_0, \frac{\varphi_0^p - \varepsilon^p\varphi_\lambda^p}{\varphi_0^{p-1}}\rangle_K + \langle L\varphi_\lambda, \frac{\varepsilon^p\varphi_\lambda^p - \varphi_0^p}{\varphi_\lambda^{p-1}}\rangle_K = \langle \lambda_0 - \lambda, \varphi_0^p - \varepsilon^p\varphi_\lambda^p\rangle_K.$$

Choosing $\varepsilon$ small enough leads to a contradiction. $\square$

A consequence of the previous proposition is the following statement. For a counterpart in the continuum see [PR15, Lemma 5.1].

**Corollary 5.12** *Let $V \subseteq X$ be connected and $K \subsetneq V$ be finite and connected. If $\lambda_0(V) \geq 0$, then $\lambda_0(K) > 0$.*

*In particular, if $h$ is non-negative on $C_c(V)$, then the principal eigenvalue is strictly positive on any finite and connected proper subset of $V$.*

*Proof.* Since $V \neq K$ there exists a finite and connected subset $\mathcal{K} \subseteq V$ such that $K \subsetneq \mathcal{K}$. We clearly have $\lambda_0(K) \geq \lambda_0(\mathcal{K}) \geq \lambda_0(V)$. So, it remains to show that the first inequality is strict.

By the previous statement, Proposition 5.11, there exist strictly positive eigenfunctions $\varphi \in C(K)$ and $\tilde{\varphi} \in C(\mathcal{K})$ to the principal eigenvalues $\lambda_0(K)$ and $\lambda_0(\mathcal{K})$, respectively. Furthermore, we have

$$
\begin{aligned}
(\lambda_0(K) - \lambda_0(\mathcal{K}))\, \|\varphi\|_{p,m}^p &= h(\varphi) - \lambda_0(\mathcal{K})\, \|\varphi\|_{p,m}^p \\
&= \frac{1}{2} \sum_{x,y \in X} b(x,y)\, |\nabla_{x,y}\varphi|^p + \langle \Big(\frac{c}{m} - \lambda_0(\mathcal{K})\Big)\, \tilde{\varphi}^{p-1}, \frac{\varphi^p}{\tilde{\varphi}^{p-1}} \rangle_K \\
&= \frac{1}{2} \sum_{x,y \in X} b(x,y)\, |\nabla_{x,y}\varphi|^p - \langle L\tilde{\varphi}, \frac{\varphi^p}{\tilde{\varphi}^{p-1}} \rangle_K .
\end{aligned}
$$

Assume that $\lambda_0(K) = \lambda_0(\mathcal{K})$, then the calculation above yields

$$
\frac{1}{2} \sum_{x,y \in X} b(x,y)\, |\nabla_{x,y}\varphi|^p = \langle L\tilde{\varphi}, \frac{\varphi^p}{\tilde{\varphi}^{p-1}} \rangle_K . \tag{5.4}
$$

By Green's formula, Lemma 2.7, we have for the right-hand side

$$
\begin{aligned}
\langle L\tilde{\varphi}, \frac{\varphi^p}{\tilde{\varphi}^{p-1}} \rangle_K = \frac{1}{2} \sum_{x,y \in K} b(x,y)\, (\nabla_{x,y}\tilde{\varphi})^{\langle p-1 \rangle}\, \nabla_{x,y} \frac{\varphi^p}{\tilde{\varphi}^{p-1}} \\
+ \sum_{x \in K, y \in \partial K} b(x,y)\, (\nabla_{x,y}\tilde{\varphi})^{\langle p-1 \rangle}\, \frac{\varphi^p(x)}{\tilde{\varphi}^{p-1}(x)} .
\end{aligned}
$$

For all $y \in \partial\mathcal{K}$ and $x \in \mathcal{K}$, we have

$$
|\nabla_{x,y}\varphi|^p - (\nabla_{x,y}\tilde{\varphi})^{\langle p-1 \rangle}\, \frac{\varphi^p(x)}{\tilde{\varphi}^{p-1}(x)} = \varphi(x)^p - \tilde{\varphi}(x)^{p-1} \frac{\varphi^p(x)}{\tilde{\varphi}^{p-1}(x)} = 0. \tag{5.5}
$$

Thus, we have equality in (5.4) for the summands outside of $\mathcal{K} \times \mathcal{K}$.

By the pointwise Picone inequality, Lemma 5.6, with $a = \varphi(x)$, $b = \varphi(y)$ and $c = \tilde{\varphi}(y)/\tilde{\varphi}(x)$, we infer that

$$
|\nabla_{x,y}\varphi|^p - (\nabla_{x,y}\tilde{\varphi})^{\langle p-1 \rangle}\, \nabla_{x,y} \frac{\varphi^p}{\tilde{\varphi}^{p-1}} \geq 0, \qquad x, y \in \mathcal{K}. \tag{5.6}
$$

Hence, by (5.5), we have equality in (5.4) if and only if we have equality in (5.6). By Lemma 5.6, we have equality in (5.6) if and only if $\varphi = C\tilde{\varphi}$ on $\mathcal{K}$ for some constant $C > 0$. But since $\varphi = 0$ on $\mathcal{K} \setminus K \neq \emptyset$, we get a contradiction to the positivity of $\tilde{\varphi}$ in $\mathcal{K}$. Hence, $\lambda_0(K) > \lambda_0(\mathcal{K})$. $\square$

As a simple consequence, we can show an alternative proof of parts of the local Harnack inequality, Lemma 5.1.

**Corollary 5.13** *Let $h$ is non-negative on $C_c(V)$, $V \subseteq X$ connected with at least two elements. Then,* $\deg + c > 0$ *on $V$.*

*Proof.* By Corollary 5.12 and since $V$ is not a singleton, we have $\lambda_0(\{o\}) > \lambda_0(V) \geq 0$ for all vertices $o \in V$. Moreover, $m(o)\lambda_o(\{o\}) = h(1_o) = \deg(o) + c(o)$. $\square$

**Remark 5.14 ($\lambda_1$)** After this short investigation on $\lambda_0$, it is natural to ask for similar results on the next eigenvalue $\lambda_1$. For the free $p$-Laplacian, $\lambda_1$ has been discussed on finite graphs in [BH09; Ber+17b] and on locally summable graphs in [GHJ21; KM16], and it would be very interesting to see if this can be generalised to our quasi-linear Schrödinger operator on locally summable graphs. For results on compact Riemannian manifolds associated with the free $p$-Laplacian confer [Val12a; Val12b].

## 5.4 Poisson-Dirichlet Problems on Finite Subsets

Under the additional assumption that $h$ is positive on specific subsets, we can show the existence of solutions to certain Poisson-Dirichlet problems. This is done next and the following lemma is a discrete analogue of [PP16, Proposition 3.6 and Proposition 3.7]. The following lemma is needed e.g. in characterisations of the maximum principle on finite subsets, Proposition 5.17. See also [PKC09] for finite graphs.

**Lemma 5.15 (Solutions of Poisson-Dirichlet problems)** *Let $K$ be finite. Let $g \in C(K)$, and $f \in C_c(X \setminus K)$. Furthermore, define*

$$K_f = \{\varphi \in C(X) : \varphi = f \text{ on } X \setminus K\} \subseteq C_c(X).$$

*Assume that $h(\varphi) > 0$ for all $\varphi \in K_f$. Then the functional $j = j_g \colon D \to \mathbb{R}$ defined via*

$$j(\varphi) = h(\varphi) - p\langle g, \varphi \rangle_K, \qquad \varphi \in D,$$

*attends a minimum in $K_f$. Moreover, any minimiser of $j$ on $K_f$ solves the Poisson-Dirichlet problem*

$$\begin{cases} Hu &= g \quad \text{on } K \\ \phantom{H}u &= f \quad \text{on } X \setminus K. \end{cases}$$

*Furthermore, if $f \geq 0$ on $\partial K$ or $g \geq 0$ on $K$ then there exist minimiser which are non-negative on $K$.*

*In particular, if also $K$ is connected, then $g \gneq 0$ on $K$ and $f \geq 0$ on $\partial K$, or $g \geq 0$ on $K$,* $\operatorname{supp} f \cap \partial K = \{x_0\}$ *and $f(x_0) > 0$, imply that the minimiser is unique.*

*Proof.* For all $\varphi \in K_f$ with $\|\varphi\|_{p,m} = 1$ we have for any $C > 0$,

$$j(C\,\varphi) = C^p h(\varphi) - C\,p\langle g, \varphi\rangle_K.$$

Since $h(\varphi) > 0$, we have $j(C\,\varphi) \to \infty$ as $C \to \infty$, i.e., $j$ is coercive. In particular, if $(\varphi_n)$ is a minimising sequence of $j$, it is bounded. Thus, on the closed subset $K_f$ of the finite dimensional space $C(K \cup \operatorname{supp} f)$, $(\varphi_n)$ has a convergent subsequence which converges to some $\varphi_0 \in K_f$.

Moreover, since $K$ is finite, $\psi \mapsto p\langle g, \psi\rangle_K$ is a bounded linear functional on $C(K)$ and thus, continuous. Moreover, also $h$ is lower semi-continuous. Altogether, $j$ is lower semi-continuous. Hence, $-\infty < j(\varphi_0) \le \lim_{n_k\to\infty} j(\varphi_{n_k}) = \inf_{\varphi\in K_f} j(\varphi)$.

Now, we show that any minimiser of $j$ on $K_f$ solves the Poisson-Dirichlet problem. Let $\varphi$ be such a minimiser. Since for any $z \in K$ we have $\varphi + t\,1_z \in K_f \subseteq C_c(X)$ where $t \in \mathbb{R}$, we calculate using Lemma 2.8,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} j(\varphi + t\,1_z)\big|_{t=0} = pH\varphi(z)m(z) - pg(z)m(z),$$

which shows that $\varphi$ solves the corresponding Poisson-Dirichlet problem.

Let now assume that $f, g \ge 0$. For all $\psi \in K_f$ we get by the reverse triangle inequality, that $h(\psi) \ge h(|\psi|)$. Since $f$ is non-negative, we have $\psi = |\psi| = f$ on $X \setminus K$, and $|\psi| \in K_f$. Since $g$ is non-negative, we also get $j(\psi) \ge j(|\psi|)$. Thus, there exist minimiser which are non-negative on $K$.

The uniqueness follows from the Anane-Díaz-Saá inequality, Proposition 5.9. Here are the details: Let $u, v$ be such that $Hu = Hv = g \ge 0$ on $K$ and $u = v = f \ge 0$ on $\partial K$. By either $g \gneq 0$ on $K$ or $f(x_0) > 0$ for some $x_0 \in \partial K$, we get $u \ne 0 \ne v$. By the Harnack inequality, we get that $u, v > 0$ on $K$. Then, (5.1) is fulfilled and we can apply Proposition 5.9. This gives

$$\begin{aligned}0 \le \langle g - \frac{c}{m}u^{p-1}, \frac{u^p - v^p}{u^{p-1}}\rangle_K + \langle g - \frac{c}{m}v^{p-1}, \frac{v^p - u^p}{v^{p-1}}\rangle_K \\ = \langle g(v^{p-1} - u^{p-1}), \frac{u^p - v^p}{u^{p-1}v^{p-1}}\rangle_K \le 0.\end{aligned}$$

Again by Proposition 5.9, this implies that $u = C\,v$ on $K$ for some $C > 0$. If $g(x) > 0$ for some $x \in K$, then we get immediately from the calculation above that that $C = 1$ and $u = v$ on $K$. Thus, assume that $g = 0$ on $K$ and $\operatorname{supp} f \cap \partial K = \{x_0\}$. Now $Hv = 0$ on $K$, and $v = f \ge 0$ on $\partial K$, can be rewritten for all $x \in K$ as

$$\begin{aligned}\sum_{y\in K} b(x,y)\,(\nabla_{x,y}v)^{\langle p-1\rangle} + c(x)v^{p-1}(x) &= \sum_{y\in\partial K} b(x,y)\,(v(x) - f(y))^{\langle p-1\rangle} \\ &= b(x,x_0)\,(v(x) - f(x_0))^{\langle p-1\rangle}.\end{aligned}$$

On the other side, $u = Cv$, $Hu = 0$ and $u = f \geq 0$ on $\partial K$, implies

$$\sum_{y \in K} b(x,y)\,(\nabla_{x,y} v)^{\langle p-1\rangle} + c(x) v^{p-1}(x) = \sum_{y \in \partial K} b(x,y) \left( v(x) - \frac{f(y)}{C}\right)^{\langle p-1\rangle}$$
$$= b(x,x_0) \left( v(x) - \frac{f(x_0)}{C}\right)^{\langle p-1\rangle}.$$

By the monotony of $x \mapsto (x)^{\langle p-1\rangle}$ on $\mathbb{R}$, we get $C = 1$. $\square$

**Remark 5.16** If we assume in the statement above that $K$ is infinite but $c \gneq 0$ on $K_f$ and $c \geq 0$ on $X$, then we can mimic the argumentation in Lemma 5.15 on Sobolev-type spaces on graphs. Since they are reflexive Banach spaces, and $j$ is lower semi-continuous and coercive on $K_f$, we can use [Str08, Theorem 1.2] which yields the existence of a minimiser.

In the case $p \geq 2$, the existence of a solution of the Poisson-Dirichlet problem in Lemma 5.15 can also be proven differently: One might use the non-linear Fredholm alternative, see [ADV04, Theorem 12.10], which yields that the restriction of $H$ to any compact set is surjective. Hence, the Poisson-Dirchlet problem can be solved.

Moreover, note that the lemma can be generalised slightly by considering the set

$$K_{f,\geq} = \{\varphi \in C_c(X) : \varphi \geq f \text{ on } X \setminus K\}$$

instead of $K_f$. Since we do not need it in the following, we stay with the special case.

## 5.5 Characterisations of the Maximum Principle on Finite Subsets

The basic strategy to prove the Agmon-Allegretto-Piepenbrink theorem, Theorem 6.1, the existence of Green's functions and potentials, Theorem 9.4 and Theorem 10.1 or the existence of *increasing* null-sequences, also part of Theorem 10.1, will be to analyse increasing exhaustions of finite and connected subsets of $X$. Adding on every such subset a potential that decreases as the exhaustion increases, and analysing the corresponding limit will then yield the proof. Thus, we have to study properties of energy functionals which are not only non-negative, but strictly positive on a finite and connected subset $K$, i.e., $\lambda_0(K) > 0$. We show here that this is equivalent to the existence of non-negative superharmonic functions in $C(K)$. Moreover, $\lambda_0(K) > 0$ is also equivalent to the validity of the maximum principle on $K$.

For the following proposition confer also [PKC09, Theorem 4.2] for finite graphs. Confer [GS98, Theorem 5] and [PP16, Theorem 3.10] for analogue results in the continuum. Moreover, we refer to the monograph [PS07] for details and history of the maximum principle in the continuum.

Let us define what we actually mean by the maximum principle: Let $V \subseteq X$. We say that $H$ satisfies

- the *weak maximum principle* on $V$ if for any function $s \in F(V)$ such that $Hs \geq 0$ on $V$ and $s \geq 0$ on $\partial V$ we have $s \geq 0$ on $V$, and
- the *strong maximum principle* on $V$ if for any function $s \in F(V)$ such that $Hs \geq 0$ on $V$ and $s \geq 0$ on $\partial V$ we have either $s > 0$ or $s = 0$ on $V$.

**Proposition 5.17 (Characterisations of the maximum principle)** *Let $K \subseteq X$ be finite. Consider the following assertions:*

*(i) $H$ satisfies the weak maximum principle on $K$.*

*(ii) $H$ satisfies the strong maximum principle on $K$.*

*(iii) The principal eigenvalue on $K$ is positive, i.e., $\lambda_0(K) > 0$.*

*(iv) For any non-negative function $g \in C(K)$ there exists a non-negative function $u \in C(K)$ such that $Hu = g$ on $K$. This function is a minimiser of the functional $j_g$ defined in Lemma 5.15 on $C(K)$. The minimiser is unique for $g = 0$, and can be chosen to be strictly positive on $K$ if $g \gneq 0$.*

*(v) For any non-negative function $g \in C(K)$ there exists a unique non-negative function $u \in C(K)$ such that $Hu = g$ on $K$, which is strictly positive on $K$ if $g \neq 0$.*

*If $K$ is additionally connected, then all assertions are equivalent.*

*If $K$ is only finite, then (ii) $\iff$ (iii) $\implies$ (iv), (ii) $\implies$ (i), and (v) $\implies$ (iv).*

*Proof.* We set $\lambda_0 = \lambda_0(K)$.

(ii) $\implies$ (i), and (v) $\implies$ (iv) are trivial.

(ii) $\implies$ (iii): We prove the contraposition, therefore let us assume that $\lambda_0 \leq 0$. By Proposition 5.11, $\lambda_0$ is an eigenvalue of $H$ on $K$ with positive eigenfunction $\varphi_0 \in C(K)$. Let $\psi_0 = -\varphi_0$. Then,

$$H\psi_0 = \lambda_0 \left(\psi_0\right)^{\langle p-1\rangle} \geq 0 \qquad \text{on } K,$$

and $\psi_0 \leq 0$ on $K$. This contradicts (ii).

(iii) $\implies$ (ii): We prove the contraposition. Therefore let $v \in F(K)$ such that $Hv \geq 0$ on $K$, $v \geq 0$ on $\partial K$ and $v(x_m) \leq 0$ for some $x_m \in K$. Let $\varphi \in C_c(K) = C(K)$ be defined via $\varphi = v \wedge 0$ on $K$. Then, $\varphi \leq 0$ and $\varphi H v \leq 0$ on $X$. Moreover,

$$\sum_{x \in X} c(x) \left(v(x)\right)^{\langle p-1\rangle} \varphi(x) = \sum_{x \in K} c(x) \left|\varphi(x)\right|^{p}.$$

Furthermore, for all $x \in X$ and $y \sim x$ we have

$$\left(\nabla_{x,y} v\right)^{\langle p-1\rangle} \varphi(x) \geq \left(\nabla_{x,y} \varphi\right)^{\langle p-1\rangle} \varphi(x),$$

which implies that

$$\sum_{x \in X} Hv(x)\varphi(x) \geq \sum_{x \in X} H\varphi(x)\varphi(x).$$

Altogether, $h(\varphi) \leq 0$ which gives $\lambda_0 \leq 0$.

(ii) & (iii) $\implies$ (iv): Firstly, the existence follows from Lemma 5.15 which is applicable due to (iii) (note that here $f = 0$).

Secondly, we show the uniqueness for $g = 0$: Let $0 \leq u, v \in C(K)$ be such that $Hu = Hv = g$ on $K$. By (ii), we get that these solutions are either strictly positive or zero on $K$. If $u = 0$ then we must have $g = 0$. If there would be $v \neq 0$ such that $Hv = 0$, then this would imply $\lambda_0 \leq 0$ which contradicts the assumption $\lambda_0 > 0$, i.e., (iii). Hence, we have uniqueness for $g = 0$.

Thirdly, if $g \gneq 0$, then by the discussion before and (ii), we get $u > 0$ on $K$ for any $0 \leq u \in C(K)$ such that $Hu = g$ on $K$.

Now, we additionally assume that $K$ is connected.

(i) $\implies$ (ii): This is a direct consequence of the Harnack inequality, Lemma 5.1.

(iii) & (iv) $\implies$ (v): The existence is ensured by (iv) as well as the uniqueness for $g = 0$. Since $K$ is connected, we get the uniqueness for $g \gneq 0$ by Lemma 5.15 which is applicable because of (iii).

(v) $\implies$ (iii): By (v), we can assume that $0 < u \in C(K)$ and $0 \lneq g \in C(K)$ be such that $Hu = g$ on $K$. Let $\varphi_0$ be an eigenfunction to $\lambda_0$ on $K$. Since $K$ is connected, we have by Lemma 5.11 that $\varphi_0 > 0$ on $K$. Let $C = \max_{x \in K} u(x)/\varphi_0(x)$, then $u \leq C\varphi_0$ on $K$ and by Proposition 5.9

$$\begin{aligned}
0 &\leq \langle Lu, \frac{u^p - C^p\varphi_0^p}{u^{p-1}}\rangle_K + \langle L(C\varphi_0), \frac{C^p\varphi_0^p - u^p}{C^{p-1}\varphi_0^{p-1}}\rangle_K \\
&\leq \langle \frac{c}{m}, C^p\varphi_0^p - u^p\rangle_K + \langle \lambda_0 - \frac{c}{m}, C^p\varphi_0^p - u^p\rangle_K \\
&= \langle \lambda_0, C^p\varphi_0^p - u^p\rangle_K.
\end{aligned}$$

Assume that $\lambda_0 \leq 0$. This implies that we have equality in the calculation before and by Proposition 5.9, we get that $\varphi_0 = \tilde{C}u$ for some $\tilde{C} > 0$. Hence,

$$0 \geq \lambda_0 \tilde{C}^{p-1}\varphi_0^{p-1} = H(\tilde{C}\varphi_0) = Hu = g \geq 0 \quad \text{on } K.$$

This contradicts the assumption $g \neq 0$ and thus $\lambda_0 > 0$. $\square$

If $\lambda_0(K) > 0$ for a connected and finite set $K \subseteq X$, then Proposition 5.17 implies the existence of local Green's functions on $K$, i.e., if $\lambda_0(K) > 0$, then for any $y \in K$ there exists a function $G_y^K \colon K \to (0, \infty)$ such that

$$HG_y^K = 1_y \qquad \text{on } K.$$

One of the next tasks is to give a criterion for having a Green's function globally on the whole graph. We will later see that a global Green's function exists at all $x \in X$ if and only if $h$ is a subcritical energy functional in $X$.

However, in the next section we do not turn to subcriticality but to non-negativity and show that the non-negativity of energy functionals is equivalent to the existence of a globally positive $p$-superharmonic function.

# 6. The Agmon-Allegretto-Piepenbrink Theorem

> A mathematician, on the other hand, has no material to work with but ideas, and so his patterns are likely to last longer, since ideas wear less with time than words.
> The mathematician's pattern, like the painter's or poet's, must be *beautiful*; the ideas, like the colours or the words, must fit together in a harmonious way.
>
> G. H. Hardy, *A Mathematician's Apology*, p. 84

In this chapter, we turn again to global results and show our next main result (after Theorem 4.1): a discrete non-linear version of the Agmon-Allegretto-Piepenbrink theorem for non-negative $p$-energy functionals associated with $p$-Schrödinger operators on $X$.

Agmon-Allegretto-Piepenbrink-type theorems usually state that the non-negativity of an energy functional is equivalent to the existence of a strictly positive superharmonic function with respect to the corresponding Schrödinger operator.

Since in [All74; Pie74] such results were proven first, many versions and applications of this theorem have been established. We note [PP16, Theorem 4.3] for a recent generalisation in the continuum, [LSV09] for a corresponding result on strongly local Dirichlet forms and [KPP20b, Theorem 4.2] for a corresponding version for linear $(p = 2)$-Schrödinger operators on graphs. We generalise the result in [KPP20b] to $p \in (1, \infty)$ and to subsets of $X$.

**Theorem 6.1 (Agmon-Allegretto-Piepenbrink-type theorem)** *Let $p > 1$ and $V \subseteq X$. Then the following assertions are equivalent:*

*(i) The $p$-energy functional $h$ is non-negative on $C_c(V)$;*

*(ii) there exists a function which is strictly positive in $V$, vanishes in $X \setminus V$, and is $p$-superharmonic in $V$;*

*(iii) there exists a function which is strictly positive in $V$ and is $p$-superharmonic in $V$.*

*Moreover, if the graph is locally finite on the infinite set $V$, then the above is also equivalent to following assertions:*

*(iv) there exists a function which is strictly positive in $V$, vanishes in $X \setminus V$, and is $p$-harmonic in $V$;*

*(v) there exists a function which is strictly positive in $V$ and is $p$-harmonic in $V$.*

Clearly, if the graph is finite, that is, $X$ is finite, then the non-negativity of the energy does not imply the existence of a positive harmonic function, see [KLW21, Corollary 0.56] for $p = 2$ and confer also Proposition 5.17. It is natural to ask whether the implication (i) $\implies$ (v) in Theorem 6.1 does also hold for infinite graphs which are not locally finite. The following two examples show that this is in general not the case. The examples are motivated by an example for $p = 2$ in [HK11, p. 185].

**Example 6.2 (Star graph and free $p$-Laplacian)** Recall the star graph introduced in Example 3.9: Let $(X, b, m)$ be a graph on $X = \mathbb{N}_0$ such that for all $n, k \in \mathbb{N}_0$ we have $b(n, k) > 0$ if and only if either $n = 0$ or $k = 0$. Moreover, set $c = 0$. Thus, $h$ is non-negative on $C_c(\mathbb{N}_0)$. For any function $u \in F$, we have for all $k \in \mathbb{N}_0$

$$Lu(k) = -\sum_{n \neq k}^{\infty} \frac{m(n)}{m(k)} Lu(n).$$

Hence, every $p$-superharmonic function is $p$-harmonic. This shows that there are graphs which are not locally finite and for which we have (i) $\iff$ (v).

**Example 6.3 (Star graph and strictly positive or negative potentials)** Consider again the star graph from Example 3.9, but this time we add a potential $c \in C(\mathbb{N}_0)$ with fixed sign, i.e. $c$ is strictly positive or strictly negative on $\mathbb{N}_0$. For any non-negative $p$-harmonic function $u \in F$, we have for any $k \in \mathbb{N}_0$

$$u^{p-1}(k) = -\sum_{n \neq k}^{\infty} \frac{c(n)}{c(k)} u^{p-1}(n),$$

which implies that $u = 0$. In particular, the generalised hydrogen atom and the generalised harmonic oscillator on a star graph can only have non-negative strictly $p$-superharmonic functions apart from the trivial function. Moreover, in the case of the generalised hydrogen atom, we do not have a non-negative $p$-energy functional and thus, no positive $p$-superharmonic function at all. This shows, in the case of positive potentials, that there are graphs which are not locally finite and for which we do not have (i) $\implies$ (v).

It would be very interesting to have a characterisation of graphs for which the non-negativity of the $p$-energy functional is equivalent to the existence of a positive $p$-harmonic function. Even in the linear ($p = 2$)-case this is an open problem. Note that on $\mathbb{R}^d$, we actually have (i) $\iff$ (v), see [PT07; PP16].

The following lemma in combination with the Agmon-Allegretto-Piepenbrink theorem shows that for any graph with critical $p$-energy functional we have that (i) $\iff$ (v). So, further investigations are needed on non-locally finite graphs associated with subcritical $p$-energy functionals.

**Lemma 6.4** *Let $V \subseteq X$ and $u \in F(V)$ be strictly positive on $V$ such that $Hu \geq gu^{p-1} \geq 0$ on $V$ for some non-negative $g \in C(V)$. Then, we have*

$$h(\varphi) \geq \|\varphi\|_{p,gm}^p \geq 0, \qquad \varphi \in C_c(V).$$

*In particular, $h$ is non-negative on $C_c(V)$.*

*If $h$ is critical in $V$, then any strictly positive $p$-superharmonic function in $V$ is $p$-harmonic on $V$.*

*Proof.* The first statement is a direct consequence of Picone's inequality, Lemma 5.7. To be more specific, Picone's inequality implies

$$h(\varphi) \geq \langle u^{1-p}Hu, |\varphi|^p\rangle_V \geq \langle g, |\varphi|^p\rangle_V \geq 0.$$

The second statement is now a direct consequence. Indeed, let $u$ be a strictly positive $p$-superharmonic function in $V$. Set $w = (Hu)/u^{p-1} \geq 0$. Then we can use the previous calculation with $g = w$ to derive $h \geq (wm)_p$ on $C_c(V)$. Because $h$ is $p$-critical we get $w = 0$, and $u$ is $p$-harmonic in $V$. □

**Remark 6.5** The first statement in Lemma 6.4 can be sharpened and generalised via the ground state representation, Theorem 4.1, but here the approach via Picone's inequality is sufficient.

Note that if $V$ is connected, we get by the Harnack inequality that any positive $p$-superharmonic function is strictly positive, and thus the assumption in Lemma 6.4 can then be weakened softly.

*Proof (of Theorem 6.1).* (i) $\implies$ (ii): Firstly, assume that $V$ is connected. Let $(K_n)$ be an increasing exhaustion of $V \subseteq X$ with finite and connected sets, and let $o \in K_1$. Moreover, let $H_n$ be the $p$-Schrödinger operator we obtain by adding $m/n$ to the potential $c$ of $H$, $n \in \mathbb{N}$. Then by the definition of $\lambda_0$ in (2.2), for all $n \in \mathbb{N}$

$$\lambda_0(K_n, H_n) \geq 1/n > 0.$$

Hence, by Proposition 5.17, for any sequence $(g_n)$ of positive functions on $K_n$ there exists a unique positive function $u_n \in C(K_n)$ such that $H_n u_n = g_n$ on $K_n$.

Fix $n_0 \in \mathbb{N}$. Then, for all $n \geq n_0$

$$H_{n_0} u_n = \big(1/n_0 - 1/n\big) u_n^{p-1} + g_n \geq 0 \quad \text{on } K_{n_0}.$$

Hence, $(u_n)_{n\geq n_0}$ is a sequence of $p$-superharmonic functions on $K_{n_0}$ with respect to $H_{n_0}$. Without loss of generality, we choose $g_n$ such that $u_n(o) = 1$ for all $n \in \mathbb{N}$ (take e.g. $g_n = C_n \cdot 1_{x_n}$, $x_n \in K_n$, and specify the positive constant $C_n$ accordingly). Thus, we have that $(u_n)_{n\geq n_0}$ is in $S_o^+(K_{n_0}, H_{n_0})$. Applying the convergence of solutions principle, Proposition 5.5, we get the existence of a pointwise converging subsequence $(u_{n_i})$ to some $u \in S_o^+(K_{n_0}, H_{n_0})$ for all $n_0 \in \mathbb{N}$. In particular, $u$ is $p$-superharmonic on $K_{n_0}$ with respect to $H_{n_0}$ and by the Harnack inequality, Lemma 5.1, we have that $u > 0$ on $K_{n_0}$.

Furthermore, we notice that for all $x \in K_{n_0}$,

$$0 \leq H_{n_0} u(x) = Hu(x) + \frac{1}{n_0} u^{p-1}(x),$$

i.e., $-u^{p-1}/n_0 \leq Hu$ on $K_{n_0}$. Letting $n_0 \to \infty$, we get the desired positive $p$-superharmonic function on $V$ with respect $H$. Since $V$ is connected, we get by the Harnack inequality, Lemma 5.1, that this positive $p$-superharmonic function is strictly positive in $V$. By the construction, we get that $u = 0$ on $X \setminus V$, and the first implication is proven for connected $V$.

Secondly, if $V$ is not connected, then we can decompose it in a (possibly finite) sequence of connected components $(V_i)$. By the construction above, we get in every $V_i$, that there is a function $u_i \in F(V_i)$ such that $u_i$ is strictly positive and $p$-superharmonic on $V_i$, and vanishes on $X \setminus V_i$. Hence, we can simply add all $u_i$, to obtain the desired function. To be more precise, the pointwise defined function $u = \sum_i u_i$ is strictly positive and $p$-superharmonic on $V$, and vanishes on $X \setminus V$.

(ii) $\implies$ (iii): This is trivial.

(iii) $\implies$ (i): This is ensured by a consequence of Picone's inequality, Lemma 6.4.

Now, we assume that the graph is locally finite on $V$.

(iv) $\implies$ (v): This is trivial.

(v) $\implies$ (i): This follows also from Lemma 6.4.

(i) $\implies$ (iv): Here we follow the proof of (i) $\implies$ (ii) verbatim, where we note that $Hu = 0$ follows then from the convergence of solutions principle, Proposition 5.5. $\square$

# 7. Comparison Principles

> The best mathematics is *serious* as well as beautiful – 'important' if you like, but the word is very ambiguous, and 'serious' expresses what I mean much better. [...] The 'seriousness' of a mathematical theorem lies [...] in the *significance* of the mathematical ideas which it connects.
>
> G. H. Hardy, *A Mathematician's Apology*, p. 89

In Chapter 6, we have seen that the existence of a positive $p$-superharmonic function is equivalent to the non-negativity of the $p$-energy functional. Now, we want to show that the existence of a specific positive $p$-superharmonic function – the so-called Green's function – is equivalent to the subcriticality of the $p$-energy functional. Comparison principles will be the toolbox for the proof of the equivalence.

In the linear case, comparison principles and maximum principles are the same, see e.g. [KPP20b]. In the quasi-linear setting, we can think of maximum principles as special comparison principles.

In this section, we show the discrete counterpart to [PP16, Section 5]. Firstly, we show the comparison principle for $p$-Schrödinger operators with non-negative potentials, see [Pra04, Theorem 2.3.2] for the $p$-Laplacian on locally finite graphs and [GS98; PP16] for the continuous case. Secondly, we will use a sub/supersolution technique to allow negative values of the potential terms. Thirdly, we will prove a weak comparison principle for arbitrary $p$-Schrödinger operators on finite subsets which can be seen as a discrete version of [PP16, Theorem 5.3].

## 7.1 A Weak Comparison Principle for Non-Negative Potentials

The following lemma is the non-linear version of [KLW21, Theorem 1.7], see also [HS97a, Theorem 3.14] for the standard $p$-Laplacian on finite graphs, and [Pra04, Theorem 2.3.2].

**Lemma 7.1 (Weak Comparison Principle for $c \geq 0$)** *Let $V \subsetneq X$ and $c \geq 0$ on $V$. Furthermore, let $u, v \in F(V)$ such that*

$$\begin{cases} Hu \leq Hv & \text{on} \quad V, \\ \phantom{H}u \leq \phantom{H}v & \text{on} \quad \partial V. \end{cases}$$

*Assume that $(v-u) \wedge 0$ attains a minimum in $V$. Then, $u \leq v$ on $V$.*

*Moreover, in each connected component of $V$ we have either $u = v$, or $u < v$.*

*Proof.* Without loss of generality, we can assume that $V$ is connected. Otherwise, we do the following proof in every connected component of $V$.

Assume that there exists a $x \in V$ such that $v(x) \leq u(x)$. Since $(v-u) \wedge 0$ attains a minimum in $V$, there exists $x_0 \in V$ such that $v(x_0) - u(x_0) \leq 0$, and $v(x_0) - u(x_0) \leq v(y) - u(y)$ for all $y \in V$. Since $u \leq v$ on $\partial V$, we get $\nabla_{x_0,y} v \leq \nabla_{x_0,y} u$ for all $y \in V \cup \partial V$. Furthermore, we have

$$
\begin{aligned}
0 &\leq m(x_0)\big(Hv(x_0) - Hu(x_0)\big) \\
&= \sum_{y \in V \cup \partial V} b(x_0, y) \left( (\nabla_{x_0,y} v)^{\langle p-1 \rangle} - (\nabla_{x_0,y} u)^{\langle p-1 \rangle} \right) \\
&\qquad + c(x_0) \left( (v(x_0))^{\langle p-1 \rangle} - (u(x_0))^{\langle p-1 \rangle} \right) \\
&\leq 0,
\end{aligned}
$$

where the second inequality follows from the monotony of $(\cdot)^{\langle p-1 \rangle}$ on $\mathbb{R}$, and $c(x_0) \geq 0$. Thus, we have in fact equality above.

Since $c(x_0) \geq 0$, then we get that $u(y) - v(y)$ is a non-negative constant for all $y \sim x$. By iterating this argument and using that $V$ is connected, we get that $u(y) - v(y)$ is a non-negative constant for all $y \in V \cup \partial V$. Since $u \leq v$ on $\partial V$, we conclude that $u = v$ on $V$. $\square$

## 7.2 A Weak Comparison Principle on Finite Subsets

The goal of this section is to derive a similar statement as in the previous lemma for arbitrary potentials (see Proposition 7.3).

The following lemma is the discrete analogue of [PP16, Proposition 5.2]. The strategy of its proof is to use Lemma 7.1 for the absolute value of the potential part.

**Lemma 7.2 (Sandwiching lemma)** *Let $K \subseteq X$ be finite. Let $0 \leq g \in C(K)$, $0 \leq f \in C_c(X \setminus K)$ such that $\lambda_0(\operatorname{supp} f \cup K) > 0$. Moreover, let $u, v \in F(K)$, such that*

$$
\begin{cases}
Hu \leq g \leq Hv & \text{on } K, \\
u \leq f \leq v & \text{on } \partial K \cup \operatorname{supp} f, \\
0 \leq u \leq v & \text{on } K.
\end{cases}
$$

*Then there exists $0 \leq w \in C(K \cup \operatorname{supp} f)$ such that*

$$
\begin{cases}
Hw = g & \text{on } K, \\
w = f & \text{on } \partial K \cup \operatorname{supp} f, \\
u \leq w \leq v & \text{on } K.
\end{cases}
$$

*Moreover, assume that $K$ is connected. Then, $g \gneq 0$ on $K$, or $\operatorname{supp} f \cap \partial K = \{x_0\}$ and $f(x_0) > 0$, implies that $w$ is unique.*

*Proof.* Let $\mathcal{K} := K \cup \operatorname{supp} f$, and

$$\mathcal{V} = \{w \in C(\mathcal{K}) : 0 \le u \le w \le v \text{ in } K\},$$

and consider $G\colon K \times \mathcal{V} \to \mathbb{R}$ defined via

$$G(x,w) := g(x) + 2 \cdot \frac{c_-(x)}{m(x)} \cdot w^{p-1}(x) \ge 0, \qquad x \in K, w \in \mathcal{V},$$

where $c_-(x) = 0 \vee (-c(x)), x \in X$. Since $\lambda_0(\mathcal{K}) > 0$, we can use Lemma 5.15, and get the existence of $\tilde{w} \in C(\mathcal{K})$ such that

$$\begin{cases} H_{|c|}\tilde{w} = G(\cdot, w) & \text{in } K, \\ \tilde{w} = f & \text{on } X \setminus K, \end{cases} \tag{7.1}$$

where $H_{|c|} := H_{b,|c|,p,m}$. Let $T\colon \mathcal{V} \to D$, $Tw = \tilde{w}$. Then $T$ is monotone. Indeed, let $w_1, w_2 \in \mathcal{V}$, $w_1 \le w_2$, then for $x \in K$,

$$H_{|c|}(Tw_1(x)) = G(x, w_1) \le G(x, w_2) = H_{|c|}(Tw_2(x)),$$

and $Tw_1 = f = Tw_2$ on $X \setminus K$. By Lemma 7.1, we get $Tw_1 \le Tw_2$ on $K$.

Moreover, if $w \in F(K)$ is a subsolution of

$$\begin{cases} H\hat{w} = g & \text{in } K, \\ \hat{w} = f & \text{on } X \setminus K, \end{cases} \tag{7.2}$$

then, $H_{|c|}w(x) = Hw(x) + G(x,w) - g(x) \le G(x,w)$ for $x \in K$ and hence, $w$ is a subsolution of (7.1). Furthermore, $Tw$ is a solution of (7.1) and by Lemma 7.1 we get $w \le Tw$ on $K$. Hence,

$$HTw = g + 2 \cdot \frac{c_-}{m}\big((w)^{\langle p-1\rangle} - (Tw)^{\langle p-1\rangle}\big) \le g, \quad \text{on } K,$$

and $Tw$ is a subsolution of (7.2).

Analogously, we get that if $w$ is a supersolution of (7.2), then $Tw$ is a supersolution of (7.2) and $Tw \le w$ on $K$.

Define the sequences $(u_n), (v_n)$ as follows: $u_1 = u$, $u_n = T(u_{n-1}) = T^n u$ and $v_1 = v$, $v_n = T(v_{n-1}) = T^n v$. Then $u \le u_n \le v_n \le v$ for all $n \in \mathbb{N}$, i.e., both sequences are monotone and bounded, and thus they converge pointwise monotonously on $X$, say to $u_\infty$ and $v_\infty$, respectively. Using the Harnack principle, Proposition 5.5, for monotone and dominated convergence we infer

$$Hu_\infty = \lim_{n\to\infty} Hu_n = g + 2\frac{c_-}{m}\lim_{n\to\infty}\big((u_{n-1})^{\langle p-1\rangle} - (u_n)^{\langle p-1\rangle}\big) = g \quad \text{on } K,$$

and analogously, $Hv_\infty = g$ on $K$. Thus, $u_\infty$ and $v_\infty$ are candidates for $w$.

The uniqueness follows now from Lemma 5.15. □

Note that in the case of $c \geq 0$, the local statement of Lemma 7.2 can be obtained globally using properties of Sobolev-type spaces, i.e., of reflexive Banach spaces.

The following proposition is a discrete analogue of [PP16, Theorem 5.3], confer also [GS98]. Recall that by Corollary 5.12, we get from $h \geq 0$ on $C_c(X)$ that $\lambda_0(\mathcal{K}) > 0$ for every *connected* and finite subset of $X$. We highlight also that the proof of the proposition needs that the Harnack inequality can be applied to the smaller set, and thus, we first have to consider connected components of our finite set $K$.

**Proposition 7.3 (Weak comparison principle for finite subsets)** *Let $K \subseteq \mathcal{K} \subseteq X$, where $K$ and $\mathcal{K}$ are finite, and $\lambda_0(\mathcal{K}) > 0$. Moreover, let $v \in F(K)$ be such that $Hv \geq 0$ on $K$ and $v \geq 0$ on $\partial K \cup \mathcal{K} \setminus K$. Let $u \in F(K)$ such that*

$$\begin{cases} Hu \leq Hv & \text{on } K, \\ u \leq v & \text{on } \partial K \cup \mathcal{K} \setminus K. \end{cases}$$

*If either*

*(a) $v \in C(\mathcal{K}) = C_c(\mathcal{K})$, i.e., $\operatorname{supp}(v) \subseteq \mathcal{K}$, or*

*(b) $u \in C(\mathcal{K})$, $Hu \geq 0$ on $K$, and $u \geq 0$ on $\partial K \cup \mathcal{K} \setminus K$,*

*then $u \leq v$ on $K$.*

*Proof.* If $c \geq 0$ on $K$ and arbitrary on $X \setminus K$ then the statement follows from Lemma 7.1. Thus, we can assume without loss of generality, that $c \neq 0$ on $K$.

Assume, initially, that $K$ is also connected.

Note that $\lambda_0(K) \geq \lambda_0(\mathcal{K}) > 0$. By the strong maximum principle, Proposition 5.17 ((iii) $\implies$ (ii)), we conclude that either $v = 0$ or $v > 0$ on $K$. If $v = 0$ on $K$, then by the connectedness of $K$, we can apply the Harnack inequality, Lemma 5.1, and get $v = 0$ on $K \cup \partial K$, and thus, $Hv = 0$ on $K$. Hence, $Hu \leq 0$ on $K$ and $u \leq 0$ on $\partial K$. Applying the weak maximum principle to $-u$, we get that $u \leq 0$ on $K$.

Now assume that $v > 0$ on $K$ and define $C = 1 \vee (\max_K u / \min_K v)$, then using the assumptions on $u$ and $v$, we see that $u \leq C\, v$ and $C^{-1}u \leq v$ in $K \cup \partial K$. Moreover, by Proposition 5.17, we can assume that $Hv \gneq 0$.

Firstly, assume that (a) holds. Furthermore, let $0 \lneq g := Hv$ and $f := v$ on $X$, and consider for a function $\tilde{v} \in F(K)$ the problem

$$\begin{cases} H\tilde{v} = g & \text{in } K, \\ \tilde{v} = f & \text{on } X \setminus K. \end{cases} \tag{7.3}$$

Then $C\, v$ is a supersolution of (7.3). By Lemma 7.2, there exists a unique solution $w \in C(\mathcal{K})$ of (7.3) such that $u \leq w \leq C\, v$ on $K$ and $w = v$ on $X \setminus K$. Again by the strong maximum principle, $w = 0$ or $w > 0$ on $K$. If $w = 0$ on $K$, then arguing as above, we get that $w = v = 0$ on $\partial K$ and also $u \leq 0$ on $K \cup \partial K$. If $w > 0$ on $K$ and $w = v = 0$ on $\partial K$, then we have uniqueness of the solutions by Proposition 5.17, i.e.,

$w = v$ and hence, $u \le v$ in $K$. If $w > 0$ on $K$ and $w = v \gneq 0$ on $\partial K$, then we have uniqueness of the solutions by Lemma 7.2, i.e., $w = v$ and hence, $u \le v$ in $K$.

Secondly, assume that (b) holds. The proof is similar to (a), but here are the details: Let $g := Hu$ and $f := u$ on $X$, and consider for a function $\tilde{u} \in F(K)$ the problem

$$\begin{cases} H\tilde{u} = g & \text{in } K, \\ \tilde{u} = f & \text{on } X \setminus K. \end{cases} \tag{7.4}$$

Then $C^{-1}u$ is a subsolution of (7.4). By Lemma 7.2, there exists a unique solution $w \in C(\mathcal{K})$ of (7.3) such that $C^{-1}u \le w \le v$ on $K$ and $w = u$ on $X \setminus K$. By the strong maximum principle, $w = 0$ or $w > 0$ on $K$. If $w = 0$ on $K$, then arguing as above, we get that $w = u = 0$ on $\partial K$ and also $u \le 0$ on $K \cup \partial K$ which is a contradiction to (b). If $w > 0$ on $K$ and $w = u = 0$ on $\partial K$, then we have uniqueness of the solutions by Proposition 5.17, i.e., $w = u$ and hence, $u \le v$ in $K$. If $w > 0$ on $K$ and $w = u \gneq 0$ on $\partial K$, then we have uniqueness of the solutions by Lemma 7.2, i.e., $w = u$ and hence, $u \le v$ in $K$.

Now, let $K$ be possibly disconnected. Then, we can apply the previous consideration to every connected component of $K$. This yields the result. $\square$

**Remark 7.4** We say that the *strong comparison principle* holds true for $h$, if the conditions in Proposition 7.3 imply $u < v$ on $K$ unless $u = v$ on $K$.

In the linear case, i.e., $p = 2$, it is shown in [KPP20b, Lemma 5.14], that the strong comparison ($=$ maximum) principle holds true for non-negative $h$ on any finite subset (confer with Proposition 5.17). For $p \neq 2$ it is not known if the strong and the weak comparison principle are equivalent (apart from the trivial case $u = 0$ where it is a consequence of the Harnack inequality). In the continuum, a very nice discussion is given in [FP11, Section 3].

However, if $c \ge 0$ on a connected and finite $K$, then Lemma 7.1, says that the strong comparison principle holds true for $h$. So further investigations are needed on not non-negative potentials and $u \neq 0$.

We come back to this notion of strong comparison in Proposition 9.3 in the context of minimal growth.

# 8. The Variational Capacity

> What parts of mathematics are useful? [...] Euclidean geometry, for example, is useful in so far as it is dull – we do not want the axiomatic of parallels, or the theory of proportion, or the construction of the regular pentagon.
> One rather curious conclusion emerges, that pure mathematics is on the whole distinctly more useful than applied. A pure mathematician seems to have the advantage on the practical as well as on the aesthetic side. For what is useful above all is *technique*, and mathematical technique is taught mainly through pure mathematics.
>
> G. H. Hardy, *A Mathematician's Apology*, p. 133

As the name suggests, there are many different capacities. A good overview is the monograph [BB11], where many of them are discussed in detail in the quasi-linear free Lapacian setting on specific metric spaces which include also metric graphs; see also [Maz11] for various $p$-capacities on $\mathbb{R}^d$. First results go at least back to Choquet. The variational $p$-capacity has been studied intensively for the free $p$-Laplacian on Riemannian manifolds, see e.g. [HKM06; Tro99; Tro00]. For results on local $p$-Schrödinger operators connecting variational capacity and criticality see [PT07; PT08; PT09]. Results for the standard $p$-Laplacian on locally finite graphs can be found in [Pra04]. The capacity on finite graphs has also been investigated in detail, see e.g. [HS97a] and for a recent interpretation as a curvature in the linear and finite setting, see [DL22]. The here presented results are new for $p$-Schrödinger operators on general graphs. The capacity of singletons, however, has been studied briefly by the author in [Fis22].

The weak comparison principle allows us to prove the statement that if the capacity vanishes at some vertex, it vanishes at all vertices. Using this result, we can obtain a Green's function globally in the next chapter.

## 8.1 Basic Properties

Here we show a detailed analysis of the variational capacity. Most of the results seem to be folklore for the free $p$-Laplacian, but are new for arbitrary $p$-Schrödinger operators on graphs; especially there are new on not locally finite graphs.

Recall from Section 2.2 that the (variational $p$-)capacity is defined as follows: For all $K \subseteq V \subseteq X$, $K$ finite, we set

$$\operatorname{cap}_h(K, V) = \inf_{\substack{\varphi \in C_c(V), \\ \varphi = 1 \text{ on } K}} h(\varphi) = \inf_{\substack{0 \leq \varphi \in C_c(V), \\ \varphi = 1 \text{ on } K}} h(\varphi).$$

where the second equality follows from the reversed triangle inequality. Let us set

$\operatorname{cap}_h(x, V) = \operatorname{cap}_h(\{x\}, V)$ for $x \in V$. Note that

$$\operatorname{cap}_h(K, V) \geq \inf_{\substack{\varphi \in C_c(V), \\ \varphi \geq 1 \text{ on } K}} h(\varphi)$$

with equality if $c \geq 0$ which follows from $h(0 \vee \varphi \wedge 1) \leq h(\varphi)$ for all $\varphi \in C_c(V)$ in this case. However, the right hand-side is often used as a definition of capacity, see e.g. [Tro99]. Taking the left-hand side as a definition yields many small necessary changes in auxiliary lemmata as e.g. Lemma 5.15 (confer Remark 5.16), but adds no insight on the theory.

Moreover, for all $W \subseteq V \subseteq X$ we set

$$\operatorname{cap}_h(W, V) := \sup_{K \subseteq W, K \text{ finite}} \operatorname{cap}_h(K, V).$$

The following simple properties show, in particular, that the capacity is an outer measure.

**Lemma 8.1 (Properties of the capacity)** *Let $p \in (1, \infty)$. Let $h \geq 0$ on $C_c(V)$ for some $V \subseteq X$ and let $W \subseteq V_i \subseteq V$, $W_i \subseteq V$, $i \in \mathbb{N}$. Then the following properties hold:*

*(a) the inner capacity equals the outer capacity, i.e.,*

$$\operatorname{cap}_h(W, V) = \inf_{W \subseteq Y \subseteq V} \operatorname{cap}_h(Y, V);$$

*(b)* $\operatorname{cap}_h(\emptyset, V) = 0$;

*(c) the capacity is monotone, i.e.,*

*(c a) if $V_1 \subseteq V_2$ then $\operatorname{cap}_h(W, V_1) \geq \operatorname{cap}_h(W, V_2)$,*
*(c b) if $W_1 \subseteq W_2$ then $\operatorname{cap}_h(W_1, V) \leq \operatorname{cap}_h(W_2, V)$,*
*(c c) if $h_0 \geq h$ on $C_c(V)$ then $\operatorname{cap}_{h_0}(W, V) \geq \operatorname{cap}_h(W, V)$;*

*(d) the capacity is subadditive, i.e.,*

$$\operatorname{cap}_h(W_1 \cup W_2, V) \leq \operatorname{cap}_h(W_1, V) + \operatorname{cap}_h(W_2, V);$$

*(e) the capacity is countably subadditive, i.e.,*

$$\operatorname{cap}_h\Big(\bigcup_{i=1}^{\infty} W_i, V\Big) \leq \sum_{i=1}^{\infty} \operatorname{cap}_h(W_i, V);$$

*(f) we have equality for monotone limits of exhaustions with respect to the limiting set, i.e., if $(V_i)$ is an increasing exhaustion of $V$, and $(W_i)$ is an increasing exhaustion of $W$, then*

$$\lim_{i \to \infty} \operatorname{cap}_h(W, V_i) = \operatorname{cap}_h(W, V) = \lim_{i \to \infty} \operatorname{cap}_h(W_i, V);$$

(g) *we have equality for pointwise converging potentials with respect to the limiting potential, i.e., if* $(c_n)$ *is a sequence of potentials such that* $c_n \to c$ *pointwise, with corresponding* $p$*-energy functional* $h_n \geq 0$ *on* $C_c(V)$*, then*

$$\lim_{n\to\infty} \operatorname{cap}_{h_n}(W,V) = \operatorname{cap}_h(W,V);$$

(h) *the capacity of non-negative potentials is determined on the boundary, more precisely, for* $K \cup \partial K \subseteq V$ *finite and* $c \geq 0$ *on* $K$*, we have*

$$\operatorname{cap}_h(K \cup \partial K, V) = \operatorname{cap}_h(\partial K, V).$$

*Proof.* Ad (a): This follows as in [Pra04, Proposition 3.1.5]. Here are the details: We clearly have "$\geq$" since the right-hand side is an infimum. For "$\leq$", let $\varepsilon > 0$, and take $\hat{Y} \supseteq W$ such that $\operatorname{cap}_h(\hat{Y}, V) < \inf_{W\subseteq Y\subseteq V} \operatorname{cap}_h(Y,V) + \varepsilon$. Moreover, for all finite $K \subseteq Y$, we have $\operatorname{cap}_h(K,V) \leq \operatorname{cap}_h(\hat{Y},V)$. Since $W \subseteq \hat{Y}$, we get by taking the supremum of all finite $K \subseteq W$, $\operatorname{cap}_h(W,V) \leq \operatorname{cap}_h(\hat{Y},V)$. Taking $\varepsilon \to 0$ yields the desired inequality.

Ad (b) and (c): This is clear since $0 \in C_c(V)$ and by the monotonicity of the infimum and supremum.

Ad (d): Without loss of generality, assume that the right-hand side is finite. Firstly, assume that $W_i$, $i = 1, 2$, are finite. Let $\varepsilon > 0$. Then there are $0 \leq \varphi_i \in C_c(V)$ such that $\varphi_i = 1$ on $W_i$ and $h(\varphi_i) \leq \operatorname{cap}_h(W_i,V) + \varepsilon$, $i = 1, 2$. By a case analysis, we see

$$|\nabla_{x,y}(\varphi_1 \vee \varphi_2)|^p \leq |\nabla_{x,y}\varphi_1|^p + |\nabla_{x,y}\varphi_2|^p, \quad x, y \in X.$$

Hence,

$$\begin{aligned}
\operatorname{cap}_h(W_1 \cup W_2, V) &\leq h(\varphi_1 \vee \varphi_2) + h(\varphi_1 \wedge \varphi_2) \\
&\leq h(\varphi_1) + h(\varphi_2) \\
&\leq \operatorname{cap}_h(W_1,V) + \operatorname{cap}_h(W_2,V) + 2\varepsilon.
\end{aligned}$$

Taking $\varepsilon \to 0$ finishes the proof for finite $W_i$, $i = 1, 2$.

Now assume that $W_i$, $i = 1, 2$, are arbitrary subsets of $V$. Let $\varepsilon > 0$. Then there are $K_\cup \subseteq W_1 \cup W_2$, and $K_\cap \subseteq W_1 \cap W_2$ such that

$$\operatorname{cap}_h(K_\cup, V) > \operatorname{cap}_h(W_1 \cup W_2, V) - \varepsilon, \qquad \operatorname{cap}_h(K_\cap, V) > \operatorname{cap}_h(W_1 \cap W_2, V) - \varepsilon.$$

Moreover, set $K_1 = (W_1 \cap K_\cup) \cup K_\cap$ and $K_2 = W_2 \cap K_\cup$. Using 8.1 (c b), we infer

$$\begin{aligned}
\operatorname{cap}_h(W_1 \cup W_2, V) + \operatorname{cap}_h(W_1 \cap W_2, V) &< \operatorname{cap}_h(K_\cup, V) + \operatorname{cap}_h(K_\cap, V) + 2\varepsilon \\
&\leq \operatorname{cap}_h(K_1, V) + \operatorname{cap}_h(K_2, V) + 2\varepsilon \\
&\leq \operatorname{cap}_h(W_1, V) + \operatorname{cap}_h(W_2, V) + 2\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ finishes the proof.

Ad (e): This is a consequence of (d) and follows via induction. A direct alternative proof can be adapted from [Pra04, Proposition 3.2.1].

Ad (f): Let us focus on the first equality. By (c a), we have for all $i \in \mathbb{N}$

$$\mathrm{cap}_h(W, V_i) \geq \mathrm{cap}_h(W, V_{i+1}) \geq \mathrm{cap}_h(W, V) \geq 0.$$

Thus, $(\mathrm{cap}_h(W, V_i))$ is a decreasing sequence bounded from below. Let $\varepsilon > 0$ and choose a finite set $K \subseteq W$. Moreover, there is $\varphi \in C_c(V)$ such that $\varphi = 1$ on $K$ and $h(\varphi) < \mathrm{cap}_h(K, V) + \varepsilon$. Since $\varphi$ is compactly supported and $(V_i)$ is an increasing exhaustion of $V$ there is $i_0 \in \mathbb{N}$ such that for all $j \geq i_0$, we have $\varphi \in C_c(V_j)$. Hence, $\mathrm{cap}_h(K, V_j) < h(\varphi) < \mathrm{cap}_h(K, V) + \varepsilon$. Letting $\varepsilon \to 0$, taking the supremum and the limit yields the result.

Very similarly to the first equality one can prove the second one. The proof is therefore omitted.

Ad (g): This follows from the Harnack principle, Lemma 5.2.

Ad (h): By (c b), "$\geq$" holds. Let $\varepsilon > 0$, then we can find $\varphi \in C_c(V)$ such that $\varphi = 1$ on $\partial K$ and $h(\varphi) < \mathrm{cap}_h(\partial K, V) + \varepsilon$. Consider $\psi \in C_c(V)$ defined via $\psi = \varphi$ on $X \setminus K$ and $\psi = 1$ on $K$. In particular, $\psi = 1$ on $K \cup \partial K$. Observe that $|\nabla_{x,y}\psi|^p \leq |\nabla_{x,y}\varphi|^p$ for all $x, y \in X$. Since $c \geq 0$ on $K$, we get

$$\mathrm{cap}_h(K \cup \partial K, V) \leq h(\psi) \leq h(\varphi) < \mathrm{cap}_h(\partial K, V) + \varepsilon.$$

Letting $\varepsilon \to 0$ yields the result. □

We belief that (d) can be improved as follows: the capacity might be strongly subadditive (also known as submodular), i.e.,

$$\mathrm{cap}_h(W_1 \cup W_2, V) + \mathrm{cap}_h(W_1 \cap W_2, V) \leq \mathrm{cap}_h(W_1, V) + \mathrm{cap}_h(W_2, V).$$

## 8.2 Consequences of the Comparison Principle

Here, we prove the main result of this chapter. For capacities of singletons with respect to the whole graph and without the connection to the generalised principal eigenvalue, the following has been proven in [Fis22].

**Proposition 8.2** *Let $V \subseteq X$ be connected and non-empty, and let $h \geq 0$ on $C_c(V)$. If there is some non-empty set $W_0 \subseteq V$ such that $\mathrm{cap}_h(W_0, V) = 0$ then $\mathrm{cap}_h(W, V) = 0$ for all $W \subseteq V$, and also $\lambda_0(V) = 0$.*

*In particular, if $h$ is subcritical in $V$, then $\mathrm{cap}_h(W, V) > 0$ for all non-empty $W \subseteq V$.*

*Proof.* We divide the proof into several steps.

We start with showing that $\mathrm{cap}_h(W_0, V) = 0$ for some non-empty $W_0 \subseteq V$ implies $\lambda_0(V) = 0$. The preamble ensures, per definitionem, that for all finite and non-empty $K_0 \subseteq W_0$, we find a sequence $(\varphi_n)$ such that $0 \leq \varphi_n \in C_c(V)$, $\varphi_n = 1$ on $K_0$

and $h(\varphi_n) \to 0$. Thus, $\varphi_n \neq 0$, and we can consider $\psi_n = \varphi_n / \|\varphi_n\|_{p,m}^p$. Hence, $\|\psi_n\|_{p,m}^p = 1$, $\psi_n \in C_c(V)$ and $h(\psi_n) \to 0$, which implies $\lambda_0(V) = 0$.

Now we show in two steps that $\mathrm{cap}_h(W_0, V) = 0$ for some $W_0 \subseteq V$ implies $\mathrm{cap}_h(W, V) = 0$ for all $W \subseteq V$.

Firstly, we turn to the case where $V$ is infinite. By Lemma 8.1 (c b), we have for any finite subset $K_0 \subseteq W_0$ that $\mathrm{cap}_h(K_0, V) = 0$. Let $(K_n)$ be an increasing exhaustion of $V$ with finite and connected sets starting with $K_0$. Since $h \geq 0$ on $C_c(V)$, we have using Corollary 5.12 that $\lambda_0(K_n) > 0$ for all $n \in \mathbb{N}_0$. Thus, we can use Lemma 5.15 and get the existence of a function $\varphi_n \in C(K_n)$ which minimises $h$ on $K_{n,0} := \{\varphi \in C(K_n) : \varphi = 1 \text{ on } K_0\}$. By Proposition 5.17, we get that $\varphi_n > 0$ on $K_n$. By the weak comparison principle, Proposition 7.3, we get that $(\varphi_n)$ is increasing. Moreover, using Lemma 8.1 (f) we have

$$0 = \mathrm{cap}_h(K_0, V) = \lim_{n\to\infty} \mathrm{cap}_h(K_0, K_n) = \lim_{n\to\infty} h(\varphi_n).$$

Furthermore, for any $y \in V$, there exists $n_0$ such that $y \in K_n$ for all $n \geq n_0$. Then, using that $(1/\varphi_n(y))_{n\geq n_0}$ is bounded and decreasing, we compute using Lemma 8.1 (f),

$$0 \leq \mathrm{cap}_h(y, V) = \lim_{n\to\infty} \mathrm{cap}_h(y, K_n) \leq \lim_{n\to\infty} \frac{h(\varphi_n)}{\varphi_n^p(y)} = 0.$$

Thus, $\mathrm{cap}_h(y, V) = 0$ for any $y \in V$. Moreover, for any set $W \subseteq V$, we obtain via Lemma 8.1 (e) using that $V$ is countable,

$$0 \leq \mathrm{cap}_h(W, V) \leq \sum_{x\in V} \mathrm{cap}_h(x, V) = 0.$$

Secondly, assume that $V$ is finite, and $\lambda_0(V) = 0$. If $V$ is a singleton, then there is nothing to prove. Hence assume that $V$ has at least two elements $x \neq y$ such that $V \setminus \{x\}$ and $V \setminus \{y\}$ are connected. Thus, $\lambda_0(V \setminus \{x\}) > 0$ and $\lambda_0(V \setminus \{y\}) > 0$ by Corollary 5.12. Now set $K_n = V \setminus \{z\}$ for $z \in \{x, y\}$ and $n \geq 1$ as in the case of infinite $V$. Thus, we get by the monotonicity of the capacity that $\mathrm{cap}_h(o, V) = 0$ for all $o \in V$. Since the capacity is countably subadditive, we get $\mathrm{cap}_h(W, V) = 0$ for all $W \subseteq V$.

We are left to show that subcriticality of the $p$-energy functional implies positivity of the capacity. Let now $h$ be subcritical in $V$, then there exists $o \in V$ such that $w(o) > 0$ for some non-negative function $w \in C(V)$, and $h \geq w_p$ on $C_c(V)$. Thus, $\mathrm{cap}_h(o, V) \geq w(o) > 0$. Assume that there is non-empty set $W_0 \subseteq V$ such that $\mathrm{cap}_h(W_0, V) = 0$. Then, by the first part, $\mathrm{cap}_h(o, V) = 0$, which is a contradiction. Hence, $\mathrm{cap}_h(W, V) > 0$ for all $W \subseteq V$. □

**Remark 8.3 (Alternative proof)** Here we want to show an alternative proof without a case analysis but which is more technical. The proof is inspired by the proof of the Agmon-Allegretto-Piepenbrink theorem, Theorem 6.1, and the proof before in Proposition 8.2. Here are the details:

If $V$ is a singleton, then there is nothing to prove. By Lemma 8.1 (c b), we have for any finite non-empty subset $K_0 \subseteq W_0$ that $\mathrm{cap}_h(K_0, V) = 0$. Let $(K_n)$ be an increasing exhaustion of $V$ with finite and connected sets, and let $K_0 \subseteq K_1$. Moreover, let $H_n$ be the $p$-Schrödinger operator we obtain by adding $m/n$ to the potential $c$ of $H$, $n \in \mathbb{N}$, with $p$-energy functional $h_n$. Then, for all $n \in \mathbb{N}$

$$\lambda_0(K_n, H_n) \geq 1/n > 0, \qquad \operatorname{cap}_{h_n}(K_0, K_n) \geq m(K_0)/n > 0.$$

In particular, $h_n > 0$ on $K_{n,0} := \{\varphi \in C(K_n) : \varphi = 1 \text{ on } K_0\}$, and by Lemma 5.15, we get the existence of a positive function $\varphi_n \in C(K_n)$ which minimises $h_n$ on $K_{n,0}$, i.e., $h_n(\varphi_n) = \operatorname{cap}_{h_n}(K_0, K_n)$. Furthermore, this function solves $H_n\varphi_n = 0$ on $K_n \setminus K_0$, $\varphi_n = 0$ on $\partial K_n$ and $\varphi_n = 1$ on $K_0$. Fix $n_0 \in \mathbb{N}$. Then, for all $n \geq n_0$

$$H_{n_0}\varphi_n = \big(1/n_0 - 1/n\big)\varphi_n^{p-1} \geq 0 \quad \text{on } K_{n_0} \setminus K_0.$$

Hence, $(\varphi_n)_{n\geq n_0}$ is a sequence of $p$-superharmonic functions on $K_{n_0} \setminus K_0$ with respect to $H_{n_0}$. By the weak comparison principle, Proposition 7.3, we get that $(\varphi_n)$ is increasing. Moreover, using Lemma 8.1 (f) and (g), we have

$$0 = \operatorname{cap}_h(K_0, V) = \lim_{n\to\infty} \operatorname{cap}_{h_n}(K_0, K_n) = \lim_{n\to\infty} h_n(\varphi_n).$$

Furthermore, for any $y \in V$, there exists $n_0$ such that $y \in K_n$ for all $n \geq n_0$. Then, using that $(1/\varphi_n(y))_{n\geq n_0}$ is bounded and decreasing, we compute using Lemma 8.1 (f) and (g),

$$0 \leq \operatorname{cap}_h(y, V) = \lim_{n\to\infty} \operatorname{cap}_{h_n}(y, K_n) \leq \lim_{n\to\infty} \frac{h_n(\varphi_n)}{\varphi_n^p(y)} = 0.$$

Thus, $\operatorname{cap}_h(y, V) = 0$ for any $y \in V$. Moreover, for any set $W \subseteq V$, we obtain via Lemma 8.1 (e) using that $V$ is countable,

$$0 \leq \operatorname{cap}_h(W, V) \leq \sum_{x\in V} \operatorname{cap}_h(x, V) = 0.$$

This shows the claim.

**Remark 8.4 ($\lambda_0 = 0$ but subcritical)** Inspired by Proposition 8.2, one might wonder if a vanishing principal eigenvalue implies criticality. This is not the case as the following example shows: It is well-known (at least for $p = 2$) that on the Euclidean lattice $\mathbb{Z}^d$, $d \geq 2$, we have $\lambda_0(\mathbb{Z}^d) = 0$. But $\mathbb{Z}^d$ is $p$-critical if and only if $d \leq p$. Confer also with Corollary 9.13 and Example 10.9.

A consequence of the previous proposition is the following statement.

**Corollary 8.5** *Let $V \subseteq X$ be connected. If $h$ is subcritical in $V$ with corresponding positive Hardy weight $w \in C(V)$, then $w$ can be chosen to be strictly positive on $V$.*

*Proof.* By Proposition 8.2, we have that $\operatorname{cap}_h(x, V) > 0$ for all $x \in V$. Moreover, $\operatorname{cap}_h(x, V)\cdot 1_x$, $x \in V$, is a possible $w$ since $\operatorname{cap}_h(x, V)\,|\varphi(x)|^p \leq h(\varphi)$ for all $\varphi \in C_c(V)$. Furthermore, let $\alpha_x > 0$ such that $\sum_{x\in V} \alpha_x = 1$, then also $\sum_{x\in V} \alpha_x \operatorname{cap}_h(x, V) \cdot 1_x$ is a possible $w$ (which is bounded from above pointwise by the first possible weight and thus the sum is convergent), and thus $w$ can chosen to be strictly positive on $V$. □

# 9. Global Results for Subcritical and Critical Energy Functionals

> I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world. [...] I have just one chance of escaping a verdict of complete triviality, that I may judged to have created something worth creating. And that I have created something is undeniable: the question is about its value.
>
> G. H. Hardy, *A Mathematician's Apology*, p. 150

The main result of this chapter deals with the (non-)existence of a particular $p$-superharmonic function, the normalised Green's functions. The definition is recalled from Chapter 2 next. On Euclidean spaces, the following notion of positive harmonic functions of minimal growth was introduced in the linear case in [Agm82], and was then extended to weighted $p$-Laplace-type and weighted $p$-Schrödinger equations in [PP16; PR15; PT07; PT08]. This notion is new on graphs for $p \neq 2$. In the linear $(p = 2)$-case on graphs, Agmon ground states and Green's function have been discussed for Schrödinger operators in [KPP20b].

Let $V \subseteq X$ be connected and $K \subseteq V$ be finite. A function $u$ which is $p$-harmonic on $V \setminus K$ and strictly positive on $V \cup \partial V$ is called *positive $p$-harmonic function of minimal growth at infinity* in $V$ with respect to $K$, if for any finite and connected subset $\mathcal{K} \subseteq V$ with $K \subseteq \mathcal{K}$, and any positive function $v \in F(V \setminus \mathcal{K})$ which is $p$-superharmonic in $V \setminus \mathcal{K}$, we have

$$u \leq v \text{ on } \mathcal{K} \quad \text{implies} \quad u \leq v \text{ in } V \setminus \mathcal{K}.$$

The corresponding set of positive $p$-harmonic functions of minimal growth at infinity in $V$ with respect to $K$ is denoted by $\mathcal{M}(V \setminus K)$.

If $u \in \mathcal{M}(V)$, then $u$ is called a *global minimal positive $p$-harmonic function* in $V$.

If $u \in \mathcal{M}(V \setminus \{o\}) \cap F(V)$ for some $o \in V$ and $u$ is not $p$-harmonic in $o$, then $u$ is called a *(global minimal positive) Green's function* in $V$ at $o$. If, moreover, $Hu = 1_o$ on $V$, then the Green's function $u$ at $o$ is called *normalised*.

Let $0 \lneq \varphi \in C_c(V)$. If $u \in \mathcal{M}(V \setminus \operatorname{supp} \varphi) \cap F(V)$ and $u$ is not $p$-harmonic in $\operatorname{supp} \varphi$, then $u$ is called a *(global minimal positive) Green's potential* in $V$ with charge $\varphi$. If, moreover, $Hu = \varphi$ on $V$, then the Green's potential $u$ is called *normalised*.

## 9.1 Minimal Growth At Infinity

We will discuss some properties of functions of minimal growth at infinity here. In this section, let $V \subseteq X$ be connected, and $K \subseteq V$ be finite and non-empty.

Note that $\mathcal{M}(V \setminus K) \subseteq \mathcal{M}(V \setminus \mathcal{K})$ for all finite $K \subseteq \mathcal{K} \subseteq V$. On the other hand, the inverse inclusion seems to depend on the strong comparison principle, confer Subsection 7.2. To show this, is one goal of this section.

We start with showing that the non-negativity of $h$ implies the existence of a function of minimal growth at infinity, Lemma 9.1. This lemma needs some notation, which is also needed in the lemmata thereafter. Thus, while introducing the notation, we show here the proof of it before stating the result.

Without loss of generality we only need to consider the case $K \subsetneq V$. Firstly, assume that $V$ is infinite. Let $h$ be non-negative on $C_c(V)$, and let $v$ be a positive $p$-superharmonic function in $V$ which exists by the Agmon-Allegretto-Piepenbrink theorem, Theorem 6.1. Corollary 5.12 implies that $\lambda_0(\mathcal{K}) > 0$ for any finite and connected $\mathcal{K} \subsetneq V$. Let $(K_n)$ be an increasing exhaustion of $V$ with finite and connected sets such that $K \subseteq K_0$. Let $u \in C(K)$ be an arbitrary positive function, i.e., $\mathrm{supp}(u) \subseteq K$ and $u > 0$ on $K$. By Lemma 5.15 there exists a positive solution $u_n \in C(K_n)$ of the following Dirichlet problem

$$\begin{cases} Hw = 0 & \text{in } K_n \setminus K, \\ w = u & \text{in } K. \end{cases}$$

By the weak comparison principle, Proposition 7.3, and the strong maximum principle, Proposition 5.17, $(u_n)$ is a monotone increasing sequence. Let us set $C := \max_{x\in K}(u(x)/v(x)) > 0$, then again by the weak comparison principle, $u_n \le C\, v$ on $V$. Define the pointwise limit

$$u^K := \lim_{n\to\infty} u_n \ge 0.$$

Applying the weak comparison principle once more, we see that $u^K$ does not depend on the choice of the exhaustion. By the convergence of solution principle, Proposition 5.5, we get $Hu^K = 0$ on $V \setminus K$.

If $v \ge u$ on $K$, then by the weak comparison principle, Proposition 7.3, $v \ge u_n$ on any $K_n$, $n \in \mathbb{N}$. Thus, $v \ge u^K$, and $u^K \in \mathcal{M}(V \setminus K)$.

Secondly, if $V$ is finite then on any connected component $C$ of $V \setminus K$, we have $\lambda_0(C) > 0$. Hence, we apply Lemma 5.15, and can define $u^K$ directly without the limiting process from the first case.

In total, we have shown the following (confer with corresponding statement in the continuum in [PP16, Theorem 5.7]).

**Lemma 9.1** *Let $h \ge 0$ on $C_c(V)$ where $V \subseteq X$ is connected. Then, $\mathcal{M}(V \setminus K)$ is non-empty for all finite and non-empty $K \subseteq V$. Specifically, $u^K \in \mathcal{M}(V \setminus K)$.*

The converse statement (that is, if there is a positive function of minimal growth at infinity (with respect to all non-empty $K$), then the $p$-energy functional has to be non-negative), seems to depend on the spectral gap phenomenon, see [DP23; LP19] for results in the continuum.

Similarly, one shows the following lemma. Confer [PR15, Lemma 9.4] for similar statements in the continuum.

**Lemma 9.2** *Let $h \geq 0$ on $C_c(V)$ where $V \subseteq X$ is connected. Let $K \subseteq V$ be finite and non-empty. Let $u \in C(V)$ be a positive function that is $p$-harmonic on $V \setminus K$. Then $u \in \mathcal{M}(V \setminus K)$ if and only if $u = u^{\mathcal{K}}$ for any finite and connected $K \subseteq \mathcal{K} \subseteq V$.*

The following result is the discrete analogue to [PT08, Proposition 5.2]. Recall the definition of the strong comparison principle from the remark in Section 7.2: $h$ fulfils the strong comparison principle on a finite and connected $K \subseteq X$ if and only if the conditions in Proposition 7.3 imply $u < v$ on $K$ unless $u = v$ on $K$.

**Proposition 9.3** *Let $h \geq 0$ on $C_c(V)$ where $V \subseteq X$ is connected and infinite. Assume that the strong comparison principle holds true for $h$ on any finite, non-empty and connected subset. Let $K \subseteq \mathcal{K} \subseteq V$ be two finite and non-empty sets such that there is $\mathcal{K} \subseteq \mathcal{W} \subsetneq V$ connected and finite. Assume that there exists a positive function $u \in \mathcal{M}(V \setminus \mathcal{K}) \cap C(V)$ which is $p$-harmonic in $V \setminus K$. Then $u \in \mathcal{M}(V \setminus K)$.*

*Proof.* The case $K = \mathcal{K}$ is evident. Thus, assume that $K \subsetneq \mathcal{K}$. Moreover, let $W \subseteq \mathcal{W} \subsetneq V$ be two finite and connected sets such that $K \subseteq W$ and $\mathcal{K} \subseteq \mathcal{W}$. Since $u \in \mathcal{M}(V \setminus \mathcal{K})$, we have using Lemma 9.2, that $u = u^{\mathcal{W}}$. Hence, in particular, $u = u^{\mathcal{W}}$ on $V \setminus \mathcal{W}$. Since $W$ is finite, we also have $u \asymp u^{W}$ in $W$. Using the weak comparison principle, Proposition 7.3, and an exhaustion argument, it follows that $u \asymp u^{W}$ in $V \setminus W$.

Set

$$\varepsilon^{W} := \max \left\{ \varepsilon > 0 : \varepsilon u \leq u^{W} \text{ in } V \setminus W \right\}.$$

Then, since $K \subseteq \mathcal{K}$ and $u \in \mathcal{M}(V \setminus \mathcal{K})$, we have $0 < \varepsilon^{W} \leq 1$. Assume that we do not have equality, i.e., assume that $\varepsilon^{W} < 1$. Then, since $\varepsilon^{W} u \lneq u^{W}$ in $V \setminus W$ and $\varepsilon^{W} u < u^{W}$ in $W$, we get by the strong comparison principle that $\varepsilon^{W} u < u^{W}$ in $W_n \setminus W$ for every increasing exhaustion $(W_n)$ of $V$ with finite and connected sets. Therefore, there exists $\tilde{\varepsilon} > 0$ such that $(1 + \tilde{\varepsilon})\varepsilon^{W} < 1$ and $(1 + \tilde{\varepsilon})\varepsilon^{W} u \leq u^{W}$ on $\mathcal{W} \setminus W$, and thus on $V \setminus \mathcal{W}$. Hence, $(1 + \tilde{\varepsilon})\varepsilon^{W} u \leq u^{W}$ on $V \setminus W$, but this a contradiction to the definition of $\varepsilon^{W}$. Thus, $u = u^{W}$ in $V \setminus W$, and therefore, $u \in \mathcal{M}(V \setminus K)$. $\square$

## 9.2 Existence and Properties of Global Green's Functions

We want to show that $h$ is subcritical in a connected set $V \subseteq X$, then a normalised minimal positive Green's function exists in $V$. Recall that a $p$-energy functional is called subcritical if the $p$-Hardy inequality holds. This implication is actually an equivalence, and the missing implication will be shown in Chapter 10. Also, we will see later in Corollary 10.3, that $h$ is subcritical in every proper subset of $X$ if it is non-negative on $C_c(X)$. Thus, on proper and connected subsets, we show that we always have a Green's function. Recall the notation $w_p = \|\cdot\|_{p,w}^{p}$. Moreover, the definition of a Green's function is recalled at the beginning of this chapter.

**Theorem 9.4 (Green's Functions)** *Let $p > 1$ and $V \subseteq X$ be connected and non-empty. If $h$ is subcritical in $V$, then a normalised Green's function $G_o$ exists at all $o \in V$, is unique, is given by*

$$G_o(y) = \left( \frac{m(o)}{\mathrm{cap}_h(o, V)} \right)^{\frac{1}{p-1}} u^{\{o\}}(y),$$

*where $u^{\{o\}}$ is the pointwise limit of solutions $u_n$ of the Dirichlet problem defined in Section 9.1 with value 1 at $o$, and for all $o \in V$ we have*

$$\lim_{n\to\infty} h(u_n) = \frac{\mathrm{cap}_h(o, V)}{m(o)}.$$

*Furthermore, if $c(o) = 0$, then $G_o$ is not constant.*

*Proof.* By Corollary 8.5 and since $h$ is subcritical in $V$, there exists a strictly positive function $w \in C(V)$ such that $h \geq (wm)_p$ on $C_c(V)$, i.e., $w \cdot m$ is a Hardy weight. Let $(K_n)$ be an increasing exhaustion of $V$ with finite and connected sets, and take $K_0 \subseteq K_1$. We get from Corollary 5.12 that $\lambda_0(K_n) > 0$ for any $K_n \subseteq V$. By the lemma about the solutions of Poisson-Dirichlet problems, Lemma 5.15, we get the existence of a positive function $u_n \in C(K_n)$ which is harmonic on $K_n \setminus K_0$, $u_n = 1$ on $K_0$, and which minimises $h$ on $K_{n,0} := \{\varphi \in C(K_n) : \varphi = 1 \text{ on } K_0\}$.

Note that for all $t \in \mathbb{R}$, we have $(1-t)u_n + t1_{K_0} \in K_{n,0}$. By the definition of being a minimiser, the function $t \mapsto h((1-t)u_n + t1_{K_0})$ has derivative zero at $t = 0$. Thus,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} h((1-t)u_n + t1_{K_0})\Big|_{t=0} = -ph(u_n) + p \sum_{x\in K_0} Hu_n(x)m(x).$$

Rearranging and using that $h$ is subcritical in $V$ yields

$$\sum_{x\in K_0} Hu_n(x)m(x) = h(u_n) \geq w(o)m(o) > 0.$$

Here we need that $K_0$ is a singleton to continue. Thus, in the following, we assume that $K_0 = \{o\}$ for some $o \in V$. Hence, $u_n$ is strictly superharmonic on $K_0 = \{o\}$, and in particular superharmonic on $K_n$. Because of $u_n(o) = 1$ , we get by the Harnack inequality that $u_n$ is strictly positive on $K_n$. By the characterisations of the maximum principle on finite subsets, Proposition 5.17 '(iii) $\implies$ (iv)', we have the existence of a unique positive solution $v_n \in C(K_n)$ such that $Hv_n = C_n \cdot 1_o$, where the constant is given by

$$C_n := \mathrm{cap}_h(o, K_n)/m(o) \geq 0.$$

Hence, $u_n = v_n$ and $u_n$ uniquely minimises $j_{C_n \cdot 1_o}$ on $C(K_n)$. Clearly, $(C_n)$ is a decreasing sequence. Since obviously

$$\mathrm{cap}_h(o, V) = \inf_{\varphi\in C_c(V), \varphi(o)=1} h(\varphi) \geq \inf_{\varphi\in C_c(V), \varphi(o)=1} \|\varphi\|_{p,wm}^p \geq w(o)m(o) > 0,$$

we get

$$C_n \geq \frac{\mathrm{cap}_h(o,V)}{m(o)} \geq w(o) > 0.$$

Furthermore, note that

$$\begin{cases} Hu_n = Hu_{n+1} = 0 & \text{on } K_n \setminus \{o\}, \\ u_n \leq u_{n+1} & \text{on } (K_{n+1} \setminus K_n) \cup \{o\}. \end{cases}$$

Since $\lambda_0(K_n \setminus \{o\}) \geq \lambda_0(K_n) > 0$, we can apply the weak comparison principle, Proposition 7.3, and get that $(u_n)$ is increasing pointwise on $V$.

Since for all $n \in \mathbb{N}$, we have $u_n(o) = 1$, we can apply the convergence of solutions principle, Proposition 5.5, and get that the pointwise limit $u$ exists, and $Hu_n \to Hu$ on $V$ pointwise as $n \to \infty$.

Another application of the weak comparison principle shows that $u$ is independent of the choice of the sequence $(K_n)$, i.e., $u$ is uniquely determined.

Since $Hu_n = C_n 1_o$ on $K_n$, we infer using Lemma 8.1 (f) that $Hu = \lim_{n\to\infty} C_n 1_o = \mathrm{cap}_h(o,V)1_o/m(o)$ on $V$. By Proposition 8.2, we get from $\mathrm{cap}_h(o,V) > 0$, that $\mathrm{cap}_h(x,V) > 0$ for all $x \in V$. Thus, we can do this construction for all $x \in V$.

We define for every $o \in V$ the function $G_o \colon V \to (0,\infty)$ via

$$G_o(y) = \left(\frac{m(o)}{\mathrm{cap}_h(o,V)}\right)^{\frac{1}{p-1}} u(y),$$

and have a function which satisfies $HG_o = 1_o$ and thus, a candidate for the desired normalised Green's function.

We show now that $G_o \in \mathcal{M}(V \setminus \{o\})$: Let $o \in K \subseteq V$, where $K$ is finite and connected. Let $0 \leq v \in F(V \setminus K)$ be $p$-superharmonic on $V \setminus K$ and $v(o) \geq u(o) = 1$. Since $u$ is independent of the choice of the exhaustion, we can assume that there is an $n \in \mathbb{N}$ such that $K_n = K$. Then by the weak comparison principle, Proposition 7.3, $v \geq u_n$ on every connected component of $K_n \setminus \{o\}$, and thus, on $K_n$ for every $n \in \mathbb{N}$. Hence, $v \geq u$ on $V$. Using the $(p-1)$-homogeneity, we get that $G_o \in \mathcal{M}(V \setminus \{o\})$ is a normalised Green's function.

The last statement can be seen as follows: It is obvious, that a function $f$ which is constant for all $x \sim o \in V$, is $L$-harmonic in $\{o\}$. Since $LG_o(o) = HG_o(o) = 1$, we conclude that $G_o$ is not constant. □

**Remark 9.5** In the linear case, it is known that a Green's function can not only be obtained via an exhaustion argument as presented here, but also via certain limits of semigroups and resolvents, see [KPP20b]. It is an open question if a similar result can be obtained for $p \neq 2$ (and at least $c \geq 0$). For an introduction to non-linear semigroups associated with $p$-Laplacians on graphs see [Mug13].

## 9.3 Examples of Green's functions

Here we show some examples of Green's functions with respect to free $p$-Laplacians on various graphs. We also state the corresponding Green's function for the counterpart in the continuum before the example. The calculations show that finding an explicit Green's function on a graph is highly non-trivial in general, in particular if one is interested in $p$-Schrödinger operators in general.

Let us start with the most simple example: the free $p$-Laplacian on $\mathbb{N}$.

**Example 9.6 (Free $p$-Laplacian on $\mathbb{N}_0$ and $\mathbb{N}$)** Recall the standard line graph on $\mathbb{N}_0$ from Example 3.4 with $c = 0$. Firstly, observe that on this graph a function $f \in F = C(\mathbb{N}_0)$ is $p$-(super-)harmonic if and only if $f$ is 2-(super-)harmonic by the monotonicity of $(\cdot)^{\langle p-1\rangle}$. In other words, if $f$ is a $p$-superharmonic function on $\mathbb{N}$. Then, for all $n > 0$, we have

$$\nabla_{n,n-1} f \geq \nabla_{n+1,n} f,$$

with equality if $f$ is $p$-harmonic in $n$. This can be interpreted as a discrete concavity-type inequality. Since a necessary condition for a Green's function on 0 is to be positive and $p$-harmonic on $\mathbb{N}$, and motivated by the inequality above, a first candidate is $u(n) = c_1 n + c_2$ for some constants $c_1, c_2 \geq 0$. Then, $u$ is clearly $p$-harmonic on $\mathbb{N}$ but $p$-subharmonic at 0. Thus, it is not possible to normalise $u$ such that $Lu = 1_0$, and $u > 0$.

We will see in the next chapter that the specific choice of $u$ was actually not so important: a normalised Green's function cannot exist on the whole standard line graph. However, the situation changes if we consider different weights (see Example 9.7), or subsets. The latter will be done next.

Let us consider a second example: Take $v_k \in C(\mathbb{N}_0)$ defined pointwise via $v_k = 1_{\mathbb{N}_0 \setminus \{0,\dots,k\}}$ for fixed $k \in \mathbb{N}_0$. Then, $Lv_k(n) = 1_{k+1}(n)$ for all $n > k$. By the weak comparison principle, Lemma 7.1, it follows that $v_k$ is a normalised Green's function at $k+1$ on $\mathbb{N}_0 \setminus \{0,\dots,k\} \subseteq \mathbb{N}_0$.

On $(0,\infty)$, the $p$-superharmonic function are exactly the concave functions (independently of $p$), and therefore also here a candidate for a Green's function with respect to the origin 0 is given by $u(x) = c_1 x + c_2$ for some constants $c_1, c_2 \geq 0$, but this function does not seem to be of minimal growth at infinity.

Recall, for comparison, that the positive minimal Green's function of the free $p$-Laplacian on $\mathbb{R}^d$, $d > p$, with respect to the origin 0 is radial and is given by

$$G_0(|x|) = \frac{p-1}{(d-p)\omega_d^{1/(p-1)}} \, |x|^{(p-d)/(p-1)},$$

where $\omega_d$ is the volume of the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, see [Lin19, p. 39] or [FP23, p. 24].

Next we show how to obtain a Green's function on model graphs. For comparison, the Green's function of the $p$-Laplacian of a $d$-dimensional non-compact harmonic manifolds with respect to the origin 0 is given by

$$G_0(d(x,0)) = \int_{d(x,0)}^{\infty} \frac{\mathrm{d}t}{(\omega_d f(t))^{1/(p-1)}},$$

where $f$ denotes the volume density, $d(x,0)$ is the Riemannian distance from $x$ to $0$ and $\omega_d$ is the volume of the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, see [FP23, p. 25].

We also use the following boundary notation: for every $r \geq 0$ and $o \in X$, we set

$$\partial_b B_r(o) := \sum_{x \in S_r(o), y \in S_{r+1}(o)} b(x,y),$$

i.e., $\partial_b B_r(o)$ denotes the total edge weight between spheres of radius $r$ and spheres of radius $r+1$ with respect to the root $o \in X$. It should not be mixed up with $\partial B_r(o)$ which is the set of all vertices outside of $B_r(o)$ which are connected to a vertex inside of $B_r(o)$.

**Example 9.7 (Free $p$-Laplacian on model graphs)** The following can also be found in [AFS24]. Recall the definition of a model graph from Example 3.6. We show that, if

$$\sum_{r=1}^{\infty} \left( \frac{m(o)}{\partial_b B_r(o)} \right)^{1/(p-1)} < \infty, \qquad p > 1,$$

then a normalised Green's function to the free $p$-Laplacian exists and is given by

$$G_o(x) = G_0(r) = \sum_{k=r}^{\infty} \left( \frac{m(o)}{\partial_b B_k(o)} \right)^{1/(p-1)}, \qquad x \in S_r(o), r \geq 0.$$

Indeed, since $G_o$ is a spherically symmetric function and the graph is a model we get that

$$LG_o(0) = m(o) \frac{\partial_b B_0(o)}{m(o)} \frac{1}{\partial_b B_0(o)} = 1$$

and, for $r > 0$,

$$LG_o(r) = \frac{k_+(r)}{\partial_b B_r(o)} - \frac{k_-(r)}{\partial_b B_{r-1}(o)} = 0.$$

The minimal growth near infinity follows from the weak comparison principle for non-negative potentials, Lemma 7.1.

**Example 9.8 (Free $p$-Laplacian on homogeneous trees $\mathbb{T}_{d+1}$, $d \geq 2$)** Recall the definition of homogeneous trees in Example 3.7. Note that $m = 1$ in this example. Here, we have $\partial_b B_r(o) = d^r$. Hence, a Green's function at $o$ is given by

$$G_o(r) = \sum_{k=r}^{\infty} \left( \frac{1}{d} \right)^{k/(p-1)} = \frac{d^{-r/(p-1)}}{d^{1/(p-1)} - 1}, \qquad p > 1.$$

**Example 9.9 (Free $p$-Laplacian on anti-trees)** Recall the definition of an anti-tree with sphere size $s$ in Example 3.8. Note that $m = 1$ in this example. It is not difficult to see that $\partial_b B_r(o) = s(r)s(r+1)$, confer e.g. [KLW21]. Hence, if it exists, a Green's function at $o$ is given by

$$G_o(r) = \sum_{k=r}^{\infty} \left( \frac{1}{s(k)s(k+1)} \right)^{1/(p-1)} < \infty, \qquad p > 1.$$

It is not difficult to see that the series converges for $p<2$ if $s(k)s(k+1)\geq k$ for $k\in\mathbb{N}$. Furthermore, if $s(k)=d^k$ if $k$ is even and $2\leq d\in\mathbb{N}$, and $s(k)=1$ if $k$ is odd, then for all $p>1$,

$$G_o(r)=\begin{cases}\frac{1}{d^r}+2\sum_{k=r+1}^{\infty}\left(\frac{1}{d}\right)^{2k/(p-1)}=\frac{1}{d^r}+2\frac{d^{(2r+2)/(p-1)}}{1-d^{-2/(p-1)}}, & r\in 2\mathbb{N}_0\\ 2\frac{d^{2r/(p-1)}}{1-d^{-2/(p-1)}}, & r\in 2\mathbb{N}_0+1\end{cases}$$

We also want to give examples for not having a Green's function. We will see in Chapter 10 that this is equivalent to the criticality of the associated $p$-energy functional. There, some additional examples are given. Nevertheless, let us mention the following.

**Example 9.10 (Free $p$-Laplacian on star graphs)** By Example 6.2, we know that on a star graph with the free $p$-Laplacian, every $p$-superharmonic function is $p$-harmonic. Hence, there cannot exist a normalised Green's function on the whole graph.

**Example 9.11 ($p$-Laplace-type operators on star graphs)** Consider a star graph. If we add a potential $c$ such that $h$ is subcritical (e.g. any positive potential $c\gneq 0$), then a normalised Green's function $G_0$ with $HG_0=1_0$ exists by Theorem 9.4. Then, similar as in Example 6.3, we calculate

$$m(0)=\sum_{n=0}^{\infty}c(n)G_0^{p-1}(n).$$

Moreover, by direct calculation, we also see that $HG_0(n)=0$ for $n\geq 1$ is equivalent to

$$c(n)=b(0,n)\left(\frac{G_0(0)}{G_0(n)}-1\right)^{\langle p-1\rangle}.$$

## 9.4 Bounds for the Principal Eigenvalue

We close this chapter by showing upper and lower bounds for $\lambda_0(V), V\subseteq X$. These results go under the name Barta's inequality, see [Bar37] for the original paper by Barta, [Amg08, Theorem 7.1] for the corresponding version for standard $p$-Laplacians on finite graphs or [AH98, Section 2.2] for $p$-Laplacians on the Euclidean space. A linear version of Barta's theorem for finite graphs can be found in [Ura99, Theorem 2.1]. For the linear case in the continuum see e.g. [NP92] and references therein.

**Proposition 9.12 (Barta-type inequality)** *Let $V\subseteq X$.*

*(a) If $h\geq 0$ on $C_c(V)$ and $u\in F(V)$ such that $u>0$ on $V$, then,*

$$\inf_{x\in V}\frac{Hu(x)}{u^{p-1}(x)}\leq\lambda_0(V).$$

*(b) If $\varphi\in C_c(V)$ such that $\varphi>0$ on $V$, then*

$$\lambda_0(V)\leq\sup_{x\in V}\frac{H\varphi(x)}{\varphi^{p-1}(x)}.$$

*Proof.* Ad (a): Let $I = \inf_{x \in V} Hu(x)/u^{p-1}(x)$. Since $h \geq 0$ on $C_c(V)$, we have $\lambda_0(V) \geq 0$. Hence, the case $I \leq 0$ is trivial. If $I > 0$, then we can use Picone's inequality or Lemma 6.4 with $g = I$ to get

$$h(\varphi) \geq I \cdot \|\varphi\|_{p,m}^p, \qquad \varphi \in C_c(V).$$

This implies $\lambda_0 \geq I$.

Ad (b): Let us set $S = \sup_{x \in V} H\varphi(x)/\varphi^{p-1}(x)$. Then, $H\varphi \leq S\,\varphi^{p-1}$ on $V$. Hence,

$$h(\varphi) = \langle H\varphi, \varphi \rangle_V \leq S\,\|\varphi\|_{p,m}^p.$$

Thus, we get $\lambda_0(V) \leq \inf_{\varphi \in C_c(V), \varphi > 0 \text{ on } V} h(\varphi)/\|\varphi\|_{p,m}^p \leq S$. □

Barta's inequality has the following simple consequence.

**Corollary 9.13** *Let $V \subseteq X$ be connected and let $h$ be non-negative on $C_c(V)$. Then we have*

$$0 \leq \lambda_0(V) \leq \inf_{x \in V} \frac{\mathrm{cap}_h(x, V)}{m(x)}.$$

*Proof.* If $h$ is critical in $V$, then clearly $\lambda_0(V) = 0 = \mathrm{cap}_h(x, V)$ for all $x \in V$.

Let $h$ be subcritical in $V$, and let $K_n$ be an increasing exhaustion of $V$ with finite subsets. Furthermore, let $u_n$ be the positive solution of the Dirichlet problem on $K_n$ with value 1 at $o \in V$ as defined in the proof of Theorem 9.4 with limit $u$. Then, we can apply Proposition 9.12 (b), and get

$$\lambda_0(X) \leq \sup_{x \in K_n} \frac{Hu_n(x)}{u_n^{p-1}(x)} = \frac{Hu_n(o)}{u_n^{p-1}(o)} \to \frac{Hu(o)}{u^{p-1}(o)} = \frac{\mathrm{cap}_h(o, V)}{m(o)},$$

since $Hu_n$ is only positive at $o$ in $K_n$. □

**Remark 9.14 (Cheng's Eigenvalue Comparison Theorem)** Since Cheng published a comparison theorem for the principal eigenvalue of the linear $(p = 2)$-Laplacian on a ball in a manifold in [Che75], many improvements and generalisations followed. The proof strongly used curvature bounds and Barta's inequality. On infinite graphs an analogue result is given in [Ura99] where the curvature bounds are interpreted as bounds on the vertex degree. We believe that this result for $p = 2$ and $m = \deg$, $b(X \times X) \subseteq \{0, 1\}$, can be extended to all $p > 1$ and all measures, at least for model graphs.

# 10. Characterisations of Criticality

> [A] chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*.
>
> G. H. Hardy, *A Mathematician's Apology*, p. 94

In this chapter, we will discuss the notion of criticality in detail. For the history of this notion see [Pin07, Remark 2.7] or [KPP20b, Section 5]. There it is stated that in the continuum it goes back to [Sim80] and was then generalised in [Mur86; Pin88]. On locally summable weighted graphs, [KPP20b] is the first paper discussing criticality in the context of linear Schrödinger operators. See also [KLW21, Chapter 6] (and references therein) for corresponding results for linear Laplace-type operators on graphs.

Non-negative energy functionals associated with Schrödinger operators seem to divide naturally into two categories: the ones which are strictly positive, i.e., for which a Hardy inequality holds true, and the ones which are not strictly positive, i.e., for which the Hardy inequality does not hold. In the linear ($p = 2$)-case, there are surprisingly many equivalent formulations to the statement that the Hardy inequality does (not) hold, for graphs confer [KPP20b]. For $p = 2$ and $c = 0$, this is exactly the division of graphs into transient and recurrent graphs. Inspired by the theory of Riemann surfaces, graphs with (sub-)critical $p$-energy functional with respect to the free $p$-Laplacian are also called $p$-parabolic (resp. $p$-hyperbolic), see Remark 2.9 for more details.

Using our developed ground state representation formula and comparison principle, we will see that many of the characterisations in [KPP20b] remain characterisations also if $p \neq 2$.

Let $h$ be a functional which is non-negative on $C_c(V)$, $V \subseteq X$. Recall that, the functional $h$ is called ($p$-)subcritical in $V$ if the $p$-Hardy inequality holds true in $V$, that is, there exists a positive function $w \in C(V)$ such that

$$h(\varphi) \geq w_p(\varphi), \qquad \varphi \in C_c(V).$$

If such a positive $w$ does not exist, then $h$ is ($p$-)critical in $V$. Moreover, the functional $h$ is ($p$-)supercritical in $V$ if $h$ is not non-negative on $C_c(V)$. Recall that we set $w_p(\varphi) = \langle w/m, |\varphi|^p\rangle_V = \|\varphi\|_{p,w}^p$ for $\varphi \in C_c(V)$.

Before we can state the main result of this section, we recall the following definition: A sequence $(e_n)$ in $C_c(V)$, $V \subseteq X$, of non-negative functions is called *null-sequence* in $V$ if there exists $o \in V$ and $\alpha > 0$ such that $e_n(o) = \alpha$ and $h(e_n) \to 0$.

**Theorem 10.1 (Characterisations of criticality)** *Let $p > 1$. Assume that $h$ is non-negative on $C_c(V)$, where $V \subseteq X$ is connected and non-empty. Then the following statements are equivalent:*

*(i) The $p$-energy functional $h$ is critical in $V$.*

(ii) *For any (or equivalently, for some) $o \in V$ and $\alpha > 0$ there is a null-sequence $(e_n)$ in $V$ such that $e_n(o) = \alpha$, $n \in \mathbb{N}$.*

(iii) *The capacity vanishes everywhere (or equivalently, somewhere), i.e., $\mathrm{cap}_h(W, V) = 0$ for all (or, for some) $W \subseteq V$.*

(iv) *For all positive (or equivalently, for some) $p$-harmonic functions $u$ in $V$, the simplified energy $h_u$ is critical in $V$.*

(v) *For any positive $p$-superharmonic function $u \in F(V)$ in $V$ and any null-sequence $(e_n)$ in $V$ there exists a positive constant $\tilde{c}$ such that $e_n(x) \to \tilde{c}\, u(x)$ for all $x \in V$ as $n \to \infty$.*

(vi) *There exists a strictly positive and $p$-harmonic function $u \in F(V)$ in $V$ that vanishes on $X \setminus V$, and a null-sequence $(e_n)$ in $V$ such that $e_n(x) \to u(x)$ for all $x \in V$ as $n \to \infty$.*

(vii) *There exists a strictly positive and $p$-harmonic function $u \in F(V)$ in $V$ that vanishes on $X \setminus V$, and a null-sequence $(e_n)$ in $V$ such that $e_n(x) \nearrow u(x)$ for all $x \in V$ as $n \to \infty$.*

(viii) *there exists a unique function (up to multiplies) which is positive and $p$-superharmonic in $V$, and this function is strictly positive and $p$-harmonic in $V$ and vanishes at $X \setminus V$.*

(ix) *There exists an Agmon ground state, i.e., a global minimal positive $p$-harmonic function, in $V$.*

(x) *There does not exist a normalised Green's function for any (for some) $x \in V$. That is, there does not exists a function $0 < u \in \mathcal{M}(V \setminus \{x\}) \cap F(V)$ such that $Hu = 1_x$.*

(xi) *There does not exist a normalised Green's potential $G_\varphi$ in $V$ for any (for some) charge $0 \lneq \varphi \in C_c(V)$ with $\mathrm{supp}\,\varphi \subsetneq V$. That is, there does not exists a function $0 < u \in \mathcal{M}(V \setminus \mathrm{supp}\,\varphi) \cap F(V)$ such that $Hu = \varphi$.*

Theorem 10.1 is one of the main results of this thesis. It merges results from [Fis22; Fis23; Fis24] and also generalises them to subsets of $X$. Moreover, the characterisation with Green's potentials has only been mentioned in a remark in [Fis22] and the proof of (xi) is therefore new, as well as the consideration of general capacities of sets apart from singletons (iii). Also the statement (vii) about *increasing* null-sequences is new. In [Fis23], this was shown only for $p \geq 2$ (where it follows easily from the Markov property of the simplified energy). For completeness, we show this argument afterwards. That we can always find an increasing null-sequence, also simplifies arguments in the subsequent chapters.

We want to remark that, under the assumption of $c \geq 0$, way more characterisations can be obtained in our quasi-linear locally summable setting, see [AFS24]. It would be very interesting to see if some of them can be generalised to arbitrary potentials.

We divide the proof of this main theorem into two subsections. In the first subsection, we show some more general auxiliary lemmata, and in the second subsection, we show the equivalences.

## 10.1 Subcriticality on Subsets

Before we show the characterisations, we want to proof a list of technical results. First we show that if $h$ is non-negative in $C_c(V \cup \partial V)$ then $h$ cannot be critical on any subset of $V$, see Corollary 10.3. The next statement extracts the technical part.

**Proposition 10.2** *Let $p > 1$, $V \subsetneq X$ be connected, and $h$ be non-negative on $C_c(V)$. Assume that there is a function $u \in F(V)$ which is positive in $V \cup \partial V$, $p$-superharmonic in $V$ and there is a vertex $o \in \partial V$ such that $u(o) > 0$. Then, $h$ is subcritical in $V$.*

*Proof.* Without loss of generality, we can assume that $V \neq \emptyset$. By the Harnack inequality, Lemma 5.1, we have that $u$ is strictly positive in $V$.

Assume that $h$ is critical in $V$. Note that then $u$ is $p$-harmonic in $V$, which is a consequence of Lemma 6.4. Take $x_o \in V$ with $x_o \sim o$, and set $w_n = 1_{x_o}/n$ for $n \in \mathbb{N}$. Then, by the definition of criticality in $V$ we have the existence of a sequence $(\varphi_n)$ in $C_c(V)$ such that

$$0 \leq h(\varphi_n) < |\varphi_n(x_o)|^p / n, \qquad \varphi \in C_c(V). \tag{10.1}$$

By the reversed triangle inequality, we can assume without loss of generality that $\varphi_n \geq 0$ on $V$ for all $n \in \mathbb{N}$. Furthermore, we can normalise $\varphi_n$ such that $\varphi_n(x_o) = 1$ for all $n \in \mathbb{N}$. Then, $(\varphi_n)$ is a null-sequence of $h$ in $V$.

Using the ground state representation, Theorem 4.1, we get

$$h(u\psi) \geq h(u\psi) - (muHu)_p(\psi) \asymp h_u(\psi) \geq 0, \qquad \psi \in C_c(V).$$

Let us define $\psi_n \in C_c(V)$ for all $n \in \mathbb{N}$ via $\psi_n = \varphi_n/u$ on $V$ wherever $u$ is strictly positive and $\psi_n = 0$ otherwise. Then, $(\psi_n)$ is a null-sequence of $h_u$ in $V$.

Firstly, let $p \geq 2$. Since $(\psi_n)$ is a null-sequence of $h_u$ in $V$, we have

$$u(x)u(y) \left|\nabla_{x,y}\psi_n\right|^2 \to 0, \qquad x, y \in V \cup \partial V, x \sim y.$$

In particular, since $u > 0$ on $V \cup \{o\}$, we have

$$\psi_n(y) = |\nabla_{o,y}\psi_n| \to 0, \qquad y \in V, y \sim o.$$

But this is a contradiction, because $\psi_n(x_o) = 1/u(x_o) > 0$ for all $n \in \mathbb{N}$.

Secondly, let $1 < p < 2$. Since $(\psi_n)$ is a null-sequence of $h_u$ in $V$, we have for each $(x, y) \in (V \cup \{o\})^2$, $x \sim y$ either

(i) $|\nabla_{x,y}\psi_n| \to 0$, or

(ii) $|\nabla_{x,y}\psi_n| \to \infty$, or

(iii) $(\psi_n(x) + \psi_n(y))\, |\nabla_{x,y} u| \to \infty$.

Since $|\nabla_{o,x_o}\psi_n| = \psi_n(x_o) = 1/u(x_o) > 0$ for all $n \in \mathbb{N}$ and $o \sim x_o \in V$, we see that neither (i) nor (ii) can apply for the pair $(o, x_o)$. Because of

$$(\psi_n(o) + \psi_n(x_o))\, |\nabla_{o,x_o} u| = \frac{1}{u(x_o)}\, |\nabla_{o,x_o} u| \in [0, \infty),$$

also (iii) cannot apply. Hence, we also have a contradiction in the case of $1 < p < 2$.

Thus, $(\varphi_n)$ cannot be a null-sequence of $h$ in $V$, and therefore the strict inequality in (10.1) does not hold, i.e., $h$ cannot be critical in $V$. □

Comparing Proposition 10.2 and Theorem 6.1, we obtain the following result.

**Corollary 10.3** *Let $V \subsetneq X$ and $o \in \partial V$. If $h$ is non-negative in $C_c(V \cup \{o\})$ then $h$ is subcritical in $V$.*

*Proof.* If $h$ is non-negative in $C_c(V \cup \{o\})$, then by the Agmon-Allegretto-Piepenbrink theorem, Theorem 6.1, this implies the existence of a function which is strictly positive and $p$-superharmonic function on $V \cup \{o\}$. Thus, we can use Proposition 10.2, and get that $h$ is subcritical in $V$. □

**Remark 10.4** Corollary 10.3 has the following interpretation for $p = 2$ and $c = 0$: Given any connected graph, the induced graph on any proper subset is then a graph with boundary and thus transient.

By Corollary 8.5, we know that any subcritical energy functional has a strictly positive $p$-Hardy weight. If we know a little bit more about the lower bound, we get a connection to $\lambda_0(V)$, $V \subseteq X$. This is specified next. In the case of finite $V$, it gives another characterisation of the maximum principle and continues Proposition 5.17. By Proposition 8.2, the statement is not surprising but it gives an example of a $p$-Hardy weight.

**Proposition 10.5** *Let $V \subseteq X$. Then the following holds:*

*(a) If $\lambda_0(V) > 0$, then $h$ is subcritical in $V$ with strictly positive $p$-Hardy weight $w = \lambda_0(V) \cdot m$.*

*(b) If $h$ is subcritical in $V$ with $p$-Hardy weight $w$ such that $\inf_V(w/m) > 0$, then $\lambda_0(V) > 0$. Moreover, if $K \subseteq X$ is finite and $h$ is non-negative in $C_c(K \cup \{o\})$ for some $o \in \partial K$, then $\lambda_0(K) > 0$.*

*In particular, if $h$ is non-negative in $C_c(X)$, then $\lambda_0(K) > 0$ in every finite subset $K \subseteq X$.*

*Proof.* Ad (a): If $\lambda_0(V) > 0$, then for any $\varphi \in C_c(V)$ we get that $h(\varphi) \geq \lambda_0(V) \|\varphi\|^p_{p,m}$. Defining $w = \lambda_0(V) \cdot m$, we have a possible strictly positive $p$-Hardy weight, and $h$ is subcritical in $V$.

Ad (b): Since $\inf_V(w/m) > 0$, we have

$$\lambda_0(V) = \inf_{\varphi \in C_c(X)\setminus\{0\}} \frac{h(\varphi)}{\|\varphi\|^p_{p,m}} \geq \inf_{\varphi \in C_c(X)\setminus\{0\}} \frac{\inf_V(w/m)\,\|\varphi\|^p_{p,m}}{\|\varphi\|^p_{p,m}} = \inf_V(w/m) > 0.$$

The second statement can be seen as follows: Indeed, if $K$ is finite and $h$ is non-negative in $C_c(K \cup \{o\})$, then by Corollary 10.3, $h$ is subcritical in $K$. Thus, by Corollary 8.5, there is a strictly positive $p$-Hardy weight $w$ on $K$. Since $\inf_K(w/m) = \min_K(w/m) > 0$, we can apply the first statement in (b), and get the desired assertion.

The last statement follows also from Corollary 10.3 because if $h$ is non-negative in $C_c(X)$ it is also non-negative in $C_c(K \cup \{o\})$ for any $o \in \partial K$. □

The following lemma is the discrete analogue of [PP16, Proposition 4.11].

**Lemma 10.6** *Let $p > 1$ and $V \subseteq X$ be connected. Assume that there exists a function $u \in F(V)$ that is strictly positive on $V$, vanishes on $X \setminus V$, and is $p$-superharmonic on $V$. Furthermore, assume that there exists a null-sequence $(e_n)$ in $V$ such that $e_n(o) = \alpha$ for some $o \in V$ and $\alpha > 0$. Then, $e_n \to (\alpha/u(o))u$ pointwise on $V$ as $n \to \infty$. In particular, for all $(x, y) \in V \times V$ we have $\nabla_{x,y}(e_n/u) \to 0$ as $n \to \infty$.*

*Proof.* By the Agmon-Allegretto-Piepenbrink theorem, Theorem 6.1, the assumption on $u$ is equivalent to the non-negativity of $h$ on $C_c(V)$.

Let $o \in V$ and $\alpha > 0$ be arbitrary. Set $\varphi_n := e_n/u$. Then, by the ground state representation, Theorem 4.1,

$$0 \leq h_u(\varphi_n) \asymp h(e_n) - (muHu)_p(\varphi_n) \leq h(e_n) \to 0, \qquad n \to \infty. \tag{10.2}$$

Firstly, let $p \geq 2$. Then, (10.2) implies $|\nabla_{x,y}\varphi_n| \to 0$ for all $x, y \in V$, $x \sim y$. Since $V$ is connected, we have for any $x \in V$ an integer $k \in \mathbb{N}$ such that $x = x_1 \sim \ldots \sim x_k = o$. Thus, we obtain

$$\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \Big( \sum_{i=1}^{k-1} \nabla_{x_i,x_{i+1}}\varphi_n + \varphi_n(o) \Big) = \alpha/u(o).$$

Rearranging, yields $e_n \to (\alpha/u(o))u$ pointwise on $V$ as $n \to \infty$.

Secondly, let $1 < p < 2$. Then equation (10.2) implies either

(i) $|\nabla_{x,y}\varphi_n| \to 0$, or

(ii) $|\nabla_{x,y}\varphi_n| \to \infty$, or

(iii) $(\varphi_n(x) + \varphi_n(y))\,|\nabla_{x,y}u| \to \infty$

for each $(x, y) \in V \times V$, $x \sim y$. We show that (ii) and (iii) cannot apply: Using the triangle inequality, it is easy to see that (ii) and (iii) are equivalent for the pair $(x, o) \in V \times V$ with $x \sim o$. They are also equivalent to $e_n(x) \to \infty$ for $x \sim o$. Set

$$\Phi_n(x, o) := (u(x)u(o))^{1/2} |\nabla_{x,o}\varphi_n| + 1/2(|\varphi_n(x)| + |\varphi_n(o)|) |\nabla_{x,o}u| .$$

Then using Hölder's inequality with $\tilde{p} = 2/p > 1$, and $\tilde{q} = 2/(2-p)$, we calculate

$$\begin{aligned}
&b(x, o)(u(x)u(o))^{\frac{p}{2}} |\nabla_{x,o}\varphi_n|^p \\
&\le \Big(b(x, o)(u(x)u(o)) |\nabla_{x,o}\varphi_n|^2 \Phi_n^{p-2}(x, o)\Big)^{\frac{p}{2}} \cdot \big(b(x, o)\Phi_n^p(x, o)\big)^{\frac{2-p}{2}} \\
&\le c_1(p) \cdot h_u^{\frac{p}{2}}(\varphi_n) \\
&\quad \cdot \Big(b(x, o)\big((u(x)u(o))^{\frac{p}{2}} |\nabla_{x,o}\varphi_n|^p + c_2(p)(|\varphi_n(x)|^p + (\alpha^p/u^p(o))) |\nabla_{x,o}u|^p\big)\Big)^{\frac{2-p}{2}} \\
&\le c_1(p) \cdot h_u^{\frac{p}{2}}(\varphi_n) \\
&\quad \cdot \Big(b(x, o)\big((u(x)u(o))^{\frac{p}{2}} |\nabla_{x,o}\varphi_n|^p + c_2(p)(|\varphi_n(x)|^p + (\alpha^p/u^p(o))) |\nabla_{x,o}u|^p\big) + 1\Big) \\
&\le c_1(p) \cdot h_u^{\frac{p}{2}}(\varphi_n) \cdot \Big(b(x, o)\big(((u(x)u(o))^{\frac{p}{2}} + c_3(p)) |\nabla_{x,o}\varphi_n|^p + c_4(p)\big) + 1\Big) ,
\end{aligned}$$

where $c_i(p)$, $i \le 4$, are positive constants depending only on $p$ (and not on $n$). Since $b(x, o)$, $u(x), u(o)$ are also independent of $n$ and strictly positive, we can rewrite the inequality above as

$$|\nabla_{x,o}\varphi_n|^p \le C_1(p) \cdot h_u^{p/2}(\varphi_n) \cdot \big(|\nabla_{x,o}\varphi_n|^p + C_2(p)\big),$$

for some positive constants $C_i(p)$, $i = 1, 2$. Since $h_u(\varphi_n) \to 0$ as $n \to \infty$, we conclude that $|\nabla_{x,o}\varphi_n| \to 0$, and $(e_n(x))$ does not converge to $\infty$ for all $x \sim o$. Hence, (ii) and (iii) cannot apply for all $x \sim o$, and only (i) holds true for all $x \sim o$. Thus, we can continue as in the case $p \ge 2$ to get that $e_n(x) \to (\alpha/u(o))u(x)$ for all $x \sim o$, $x \in V$.

Arguing similarly, we have for all $y \sim x \sim o$, $y \in V$, that

$$|\nabla_{y,x}\varphi_n|^p \le C_1(p) \cdot h_u^{p/2}(\varphi_n) \cdot \big(|\nabla_{y,x}\varphi_n|^p + C_2(p) |\varphi_n(x)|^p + C_3(p)\big)$$

for some positive constants $C_i(p)$, $i \le 3$. Thus, as before, (ii) and (iii) cannot apply for all $y \sim x \sim o$, $y, x \in V$, which results in $e_n(y) \to (\alpha/u(o))u(y)$. Since $V$ is connected, we get by induction that $e_n(y) \to (\alpha/u(o))u(y)$ for all $y \in V$. This proves the statement for $1 < p < 2$. □

## 10.2 Proof of Theorem 10.1

Here, we prove the characterisations of criticality.

*Proof (of Theorem 10.1).* Ad (i) $\implies$ (ii) 'for all': Let $w_n = 1_o/n$ for $o \in V$ and $n \in \mathbb{N}$. Then by the criticality of $h$ in $V$ we have the existence of a function $e_n \in C_c(V)$ such that $h(e_n) < (w_n)_p(e_n)$. By the reverse triangle inequality, we have $h(|e_n|) \leq h(e_n)$ and thus, we can assume that $e_n \geq 0$. By assumption, we have that $h$ is non-negative in $C_c(V)$, and therefore we get

$$0 \leq h(e_n) < (w_n)_p(e_n) = e_n^p(o)/n.$$

Hence, we can normalise $e_n$ such that $e_n(o) = \alpha$ for any $\alpha > 0$. Altogether, $h(e_n) < \alpha^p/n$ and $(e_n)$ is a null sequence in $V$.

Ad (ii) 'for all' $\implies$ (i): Let $(e_n)$ be a null-sequence in $V$ with $e_n(o) = \alpha > 0$ for some $o \in V$. Let $w \geq 0$ on $X$ such that $h(\varphi) \geq w_p(\varphi)$ for $\varphi \in C_c(V)$. Then,

$$0 = \lim_{n\to\infty} h(e_n) \geq \lim_{n\to\infty} w_p(e_n) \geq \lim_{n\to\infty} w(o)e_n^p(o) = w(o)\alpha^p.$$

Since $o \in V$ is arbitrary and $\alpha > 0$, we get $w = 0$ on $V$.

Ad (iii) 'for all' $\iff$ (iii) 'for some': This is Proposition 8.2.

Ad (ii) $\iff$ (iii): This follows immediately from the definitions. In particular, (ii) 'for all' $\iff$ (ii) 'for some'.

Ad (i) $\iff$ (x) 'for some' $\iff$ (iv) 'for all': This follows from the ground state representation, Theorem 4.1. Note that the existence of such a strictly positive $p$-harmonic function is ensured by the Agmon-Allegretto-Piepenbrink theorem, Theorem 6.1, together with Lemma 6.4.

Ad (ii) 'for some' $\implies$ (v): This is Lemma 10.6.

Ad ((i) & (ii) & (v)) $\implies$ (vi): The assumption ensures the existence of a strictly positive $p$-superharmonic function $u$ in $V$ via the Agmon-Allegretto-Piepenbrink theorem, Theorem 6.1. By Lemma 6.4, the criticality of $h$ in $V$ implies that any strictly positive $p$-superharmonic in $X$ is a strictly positive $p$-harmonic function in $V$.

By (v), any null-sequence converges to a constant multiple of $u$. The existence of a null-sequence is ensured by (ii). This shows (vi).

Ad (vi) $\implies$ (i): By Lemma 6.4, $0 \leq h(e_n) \to 0$. Hence, $h$ is critical.

Ad (v) & (vi) $\implies$ (viii): This is trivial.

Ad (i) $\implies$ (x) & (xi): If $h$ is critical in $V$, then by Lemma 6.4, every positive superharmonic function in $V$ is harmonic in $V$. Hence, there cannot exists a Green's function nor potential in $V$.

Ad (viii) $\implies$ (x) & (xi): By assumption, there cannot exists a Green's function nor potential in $V$.

Ad (ix) $\implies$ (x) & (xi): Assume that $0 < u \in C(V)$ is a global minimal positive $p$-harmonic function in $V$. Furthermore, assume that there exists a normalised Green's function or potential in $V$, that is in particular, a positive function $v \in F(V) \cap C(V)$ such that $Hv \gneq 0$ on $V$.

Let $K \subseteq V$ be finite, non-empty and connected, and set $\varepsilon = \max_{x\in K} \{u(x)/v(x)\}$. Since $u$ is a global minimal positive harmonic function in $V$, and $\varepsilon u \leq v$ on $K$, we have $\varepsilon u \leq v$ on $V \setminus K$. Moreover, we have $\varepsilon u \neq v$, since otherwise $v$ would be harmonic, which is a contradiction. Thus, there exists a finite and connected subset $\mathcal{K}$ of $X$ and $\tilde{\varepsilon} > 0$ such that $(1 + \tilde{\varepsilon})\varepsilon u \leq v$ on $\mathcal{K}$. But since $u$ is a global minimal positive harmonic function, we get $(1+\tilde{\varepsilon})\varepsilon u \leq v$ on $V$, and in particular, on $K$, which is a contradiction to the maximality of $\varepsilon$ on $K$. Hence, we cannot have a positive and strictly $p$-superharmonic function on $V$, and thus, we cannot have a normalised Green's function nor potential.

Ad (viii) $\implies$ (ix) & (vii): Firstly, we show (ix). By (viii), there is nothing to prove for $K = \emptyset$ or $V$ being a singleton. Denote by $u$ the Agmon ground state on $V$. Take an arbitrary increasing exhaustion $(K_n)$ of $V$ with finite, non-empty and connected sets, and some $o \in K_1$. Without loss of generality, we can assume that $u(o) = 1$.

Let us do once again our standard trick: let $H_n$ be the $p$-Schrödinger operator we obtain by adding $m/n$ to the potential $c$ of $H$, $n \in \mathbb{N}$, with $p$-energy functional $h_n$ (which should not be mixed up with the simplified energy $h_u$ – the simplified energy does not appear in this part). Then, for all $n \in \mathbb{N}$,

$$\lambda_0(W, H_n) \geq 1/n > 0, \qquad \emptyset \neq W \subseteq V.$$

By Proposition 5.17, we get the existence of a sequence $(v_n)$ in $C(K_n)$ such that $H_n v_n = g_n$ on $K_n$ for any $0 \leq g_n \in C(K_n)$. Assume that $0 \lneq g_n \to g$ pointwise. Then, by Proposition 5.17, $v_n > 0$ on $K_n$.

If $v_n(o) \to 0$ as $n \to \infty$, then by the Harnack inequality, Lemma 5.1, $v_n \to 0$ on $V$ which is a contradiction unless $g = 0$. However, $0$ is not a positive function. This motivates to consider $w_n := v_n/v_n(o)$ and $\tilde{g}_n := g_n/v_n^{p-1}(o)$ instead.

Note that $(w_n)_{n \geq n_0}$ is in $S_o^+(K_{n_0}, H_{n_0})$ for any $n \geq n_0 \in \mathbb{N}$. This implies by the Harnack inequality also that $w_n$ cannot converge to $\infty$. Thus, by the Harnack principle, Lemma 5.2, $(w_n)$ converges pointwise to some $w \in S_o^+(K_{n_0}, H_{n_0})$ for all $n_0 \in \mathbb{N}$, and $0 \leq H_{n_0} w(x) = Hw(x) + w^{p-1}(x)/n_0$ for all $x \in K_{n_0}$. Letting $n_0 \to \infty$, we see that the limit $w$ is positive and $p$-superharmonic on $V$. By (viii), this is a contradiction unless $w = u$. Note that this implies the convergence of $(\tilde{g}_n)$ to $0$.

Now, let $K \subsetneq V$ be finite, and $\tilde{v} \in F(V \setminus K)$ be a positive $p$-superharmonic function on $V \setminus K$ and $u \leq \tilde{v}$ on $K$. Without loss of generality, we can assume that $g_n$, and thus $\tilde{g}_n$, is only supported in $K_1$, and $K_1 \subseteq K$. Then for any $\varepsilon > 0$ there exists $n_\varepsilon$ such that for all $n \geq n_\varepsilon$, we have $H\tilde{v} \geq Hw_n = 0$ on $K_n \setminus K$, and $0 \leq w_n \leq (1+\varepsilon)\tilde{v}$ on $K \cup V \setminus K_n$. On any connected component of $K_n \setminus K$, we get by the weak comparison principle that $w_n \leq (1+\varepsilon)\tilde{v}$ on $V \setminus K$ for any $n \in \mathbb{N}$. Thus, $u \leq (1+\varepsilon)\tilde{v}$ on $V \setminus K$. Letting $\varepsilon \to 0$ we obtain $u \leq \tilde{v}$ on $V \setminus K$. This shows (ix).

Let us turn to the proof of (vii). Our first candidate of an increasing null-sequence is $(w_n)$. Let us try to apply the weak comparison principle, Proposition 7.3. Obviously, $w_n \leq w_k$ on $X \setminus K_n$, and $k \geq n$. Moreover, for all $x \in K_n$, and $k \geq n$,

$$H_n w_n(x) - H_n w_k(x) = \tilde{g}_n(x) - \tilde{g}_k(x) - \left(\frac{1}{n} - \frac{1}{k}\right) w_k^{p-1}(x)$$

Because of $w_k \in S_o^+(K_n, H_n)$, let us now choose the specific function $g_n = 1_o$. By the weak comparison principle, Proposition 7.3, $(v_n)$ is increasing, and by the first part, it converges pointwise to $\infty$. Then, there is a subsequence $(\tilde{g}_{n_k})$ such that for all $x \in K_n$,

$$\tilde{g}_n(x) - \tilde{g}_k(x) \leq \left(\frac{1}{n} - \frac{1}{k}\right) w_k^{p-1}(x).$$

Thus, $(w_{n_k})$ is increasing by the weak comparison principle.

By construction, $0 \le \tilde{g}_{n_k} w_{n_k} \to 0 \cdot u = 0$ pointwise on $V$. Hence, there is a decreasing sub-subsequence $(\tilde{g}_{n_{k_l}} w_{n_{k_l}})$. Thus, we conclude with the aid of Green's formula and monotone convergence,

$$0 \le h(w_{n_{k_l}}) \le h_{n_{k_l}}(w_{n_{k_l}}) = \langle \tilde{g}_{n_{k_l}}, w_{n_{k_l}} \rangle_V \to 0, \quad n \to \infty.$$

This proves (vii).

Ad (vii) $\implies$ (vi): This is trivial.

Ad (x) $\implies$ (i): This is Theorem 9.4. Since in this theorem a Green's function is constructed via the capacity 'for any' and 'for some' are equivalent.

(x) 'for some' $\implies$ (xi) 'for some' and (xi) 'for all' $\implies$ (xi) 'for some': This is obvious.

Ad (xi) 'for some' $\implies$ (i): The following proof is very similar to '(viii) $\implies$ (ix) & (vii)' but the parts where we get contradictions interchange in some sense. We show the contraposition. Here are the details: We use the same notation for $H_n$ and $h_n$ as above.

Fix $0 \lneq \varphi \in C_c(V)$ with $\operatorname{supp} \varphi \subsetneq V$. Take an arbitrary increasing exhaustion $(K_n)$ of $V$ with finite, non-empty and connected sets, and $\operatorname{supp} \varphi \subsetneq K_1$. Since $h \ge 0$ on $C_c(V)$, we get $\lambda_0(W, H_n) \ge 1/n > 0$ for every $n \in \mathbb{N}$ and non-empty $W \subseteq V$. By Proposition 5.17, we get the existence of a sequence $(v_n)$ in $C(K_n)$ such that $H_n v_n = \varphi$ on $K_n$. Since $\varphi \gneq 0$, we can assume that $v_n > 0$ on $K_n$. Note that $H_n v_k \ge H_n v_n = \varphi$ on $K_n$ for all $k \ge n$. By the weak comparison principle, Proposition 7.3, we get that $(v_n)$ is monotone increasing. Set $w_n := v_n / v_n(o)$ for some $o \in K_1 \setminus \operatorname{supp} \varphi$.

Firstly, if $(v_n(o))$ is bounded, then by the Harnack principle, Lemma 5.2, we get that $v_n$ converges pointwise to a function $v$ and $Hv \ge \varphi$. Since $(v_n)$ is increasing, we can apply the convergence of solutions principle, Proposition 5.5, and get that $H_n v_n \to Hv$, i.e., $Hv = \varphi$.

Now, let $K \subseteq V$ be finite such that $\operatorname{supp} \varphi \subseteq K$, and $\tilde{v} \in F(V \setminus K)$ be a positive $p$-superharmonic function on $V \setminus K$ and $v \le \tilde{v}$ on $K$. Then for any $\varepsilon > 0$ there exists $n_\varepsilon$ such that for all $n \ge n_\varepsilon$, we have $H_n \tilde{v} \ge H_n v_n \ge 0$ on $K_n \setminus K$, and $0 \le v_n \le (1+\varepsilon)\tilde{v}$ on $K \cup V \setminus K_n$. On any connected component of $K_n \setminus K$, we get by the weak comparison principle that $v_n \le (1+\varepsilon)\tilde{v}$ on $V \setminus K$ for any $n \in \mathbb{N}$. Thus, $v \le (1+\varepsilon)\tilde{v}$ on $V \setminus K$. Letting $\varepsilon \to 0$ we obtain $v \le \tilde{v}$ on $V \setminus K$.

Secondly, if $v_n(o) \to \infty$ as $n \to \infty$, we consider instead the sequence $(w_n)$. Then, as in the proof of '(viii) $\implies$ (vii)', we can construct a null-sequence which is a contradiction to the subcriticality of $h$ in $V$.

This finishes the proof. $\square$

**Remark 10.7** We give an alternative proof for having an increasing null-sequence in the case $p \ge 2$ if $h$ is critical in $V$: Let $p \ge 2$, and let $(e_n)$ be a null-sequence such that $e_n(o) = u(o)$ for some $o \in V$, and $e_n \to u$. Consider the sequence $(e_n \wedge u)$, where $\wedge$ denotes the minimum. We show that it is a null-sequence. Indeed, since for all $\alpha, \beta, \gamma \in \mathbb{R}$, we have

$$|\alpha \wedge \gamma - \beta \wedge \gamma| \le |\alpha - \beta|,$$

we conclude,

$$0 \leq h(e_n \wedge u) \asymp h_u(u^{-1}(e_n \wedge u)) = h_u(u^{-1}e_n \wedge 1) \leq h_u(u^{-1}e_n) \asymp h(e_n).$$

Thus, $h(e_n \wedge u) \to 0$, and $e_n(o) = u(o) > 0$, i.e., $(e_n \wedge u)$ is a null-sequence. Since $e_n \to u$, we conclude that $(e_n \wedge u) \nearrow u$. Hence, there exists a monotone increasing subsequence of $(e_n \wedge u)$, and this subsequence is also a null-sequence and converges to $u$.

Sadly, this simple argument does not hold if $p \in (1, 2)$, and thus we did a workaround via the maximum and comparison principles in the proof of (vii).

We end this section by giving a connection between the $p$-energy functional associated with the graph $b$, and the $p$-energy functional associated with the graph $b_u$, where $b_u(x, y) = b(x, y)(u(x)u(y))^{p/2}$ for $0 \leq u \in F$.

**Corollary 10.8** *Let $p > 1$, and $0 \leq u \in F$.*

*(a) If $p > 2$ and $u$ is a ground state of $h$, then $1$ is a ground state of $h_{u,1}$.*

*(b) If $1 < p < 2$ and $h_{u,1}$ is critical in $X$, then $h(\cdot) - \langle u^{1-p}Hu, |\cdot|^p\rangle$ is critical in $X$.*

*Proof.* Ad (a): Recall that a ground state is $p$-harmonic, i.e., $uHu = 0$. Moreover, $h$ is critical. Then, by the ground state representation, Corollary 4.2, we have that $h_{u,1}$ is critical. Since $1$ is a $p$-harmonic function with respect to the Laplace operator associated with $h_{u,1}$, it is a ground state.

Ad (b): This is a direct consequence of the ground state representation. □

## 10.3 Examples of Subcritical and Critical Energy Functionals

In Section 9.3, we have calculated Green's functions on various graphs. By Theorem 10.1, we see that the corresponding $p$-energy functionals are all subcritical. However, we want to have a closer look on some examples.

**Example 10.9 ($\mathbb{Z}^d$)** In [Mae77] it is shown that $\mathbb{Z}^d$ is subcritical for $d > p$ with respect to the free $p$-Laplacian. There, a flow is constructed to get the implication via a Kelvin-Nevanlinna-Royden characterisation of $p$-subcriticality (see [AFS24]). In [Mae77], it is also shown that $\mathbb{Z}^d$ is critical for $d \leq p$ (by explicitly calculating the resistance which is the reciprocal of the capacity).

Obviously, the generalised harmonic oscillator on $\mathbb{Z}^d$ is subcritical for any $d \in \mathbb{N}$ (take the positive potential as a $p$-Hardy weight). Moreover, the generalised hydrogen atom on $\mathbb{Z}^d$ is supercritical for $d \leq p$ (since the potential is non-positive). Confer Example 3.2 and Example 3.3 for the definition of the generalised harmonic oscillator and hydrogen atom, respectively.

**Example 10.10** **($\mathbb{N}$)** The following observation can also been found in [FKP23]. Consider the $p$-energy functional $h$ on $\mathbb{N}$ with potential part $c = -w$, where $w(n) = L(n^{(p-1)/p})/n^{(p-1)^2/p}$. We show the existence of a null-sequence $(\varphi_N)$ with $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi_N$ converging pointwise to $u(n) = n^{(p-1)/p}$, confer Theorem 10.1. We choose

$$\varphi_N = u\psi_N \qquad \text{with} \qquad \psi_N(n) = 0 \vee \left(1 - \frac{\log n}{\log N}\right)$$

for $n, N \in \mathbb{N}$ and $u(n) = n^{1/q}$. By the ground state representation formula, Theorem 4.1, we have

$$h(u\psi) \asymp h_u(\psi) = 2\sum_{n=0}^{\infty} u(n)u(n+1)|\nabla_{n,n+1}\psi|^2 \\ \cdot \left( (u(n)u(n+1))^{\frac{1}{2}} |\nabla_{n,n+1}\psi| + \frac{|\psi(n)| + |\psi(n+1)|}{2} |\nabla_{n,n+1}u| \right)^{p-2}.$$

We employ $u(n) = n^{1/q}$ and the definition of $\varphi_N, \psi_N$ to obtain

$$h(\varphi_N) \asymp h_u(\psi_N) \leq \frac{C}{\log^p N} \left( \sum_{n=1}^{N} n^{p-1} \log^p \left(1 + \frac{1}{n}\right) + \sum_{n=1}^{N} n^{-1} \log^{p-2}\left(\frac{N}{n}\right) \right),$$

where we used $\alpha(\beta+\gamma)^r \leq C(\alpha\beta^r + \alpha\gamma^r)$ for all $\alpha, \beta, \gamma \geq 0$, $r \in \mathbb{R}$ and some $C = C(r)$ to split the sum into two sums, $|(0 \vee \alpha) - (0 \vee \beta)| \leq |\alpha - \beta|$ for all $\alpha, \beta \in \mathbb{R}$ and $|n^r - (n+1)^r| \asymp n^{r-1}$ for $r \in (0, \infty)$, see e.g. [KLW21, Lemma 2.28]. Now, using $\log(1 + 1/n) \leq 1/n$ and again $\alpha(\beta+\gamma)^r \leq C(\alpha\beta^r + \alpha\gamma^r)$, we infer

$$h_u(\psi_N) \leq \frac{C}{\log^p N}\left(\log N + \log^{p-1} N\right) \to 0, \qquad N \to \infty$$

which finishes the proof.

In particular, any $p$-energy functional with pointwise larger potential is subcritical, e.g. the functional associated to the free $p$-Laplacian or the generalised harmonic oscillator are subcritical.

Note that $w$ is a strictly positive function given by

$$w(n) = \left(1 - \left(1 - \frac{1}{n}\right)^{\frac{p-1}{p}}\right)^{p-1} - \left(\left(1 + \frac{1}{n}\right)^{\frac{p-1}{p}} - 1\right)^{p-1} > \left(\frac{p-1}{p}\right)^p \frac{1}{n^p}.$$

Confer the appendix for a proof of the inequality.

**Example 10.11 (Model graphs)** The following can also be found in [AFS24]. Recall the definition of a model graph from Example 3.6, and the existence of a Green's function on a model graph in Example 9.7. We show that, if

$$\sum_{r=1}^{\infty} \frac{1}{(\partial_b B_r(o))^{1/(p-1)}} = \infty,$$

for some $o \in X$, then there exists a null sequence $(e_n)$, and the model graph associated with the free $p$-Laplacian is critical, confer Theorem 10.1. Note that the Agmon ground state is then the function which is 1 everywhere.

Consider a sequence $(\varphi_n)$ of spherically symmetric functions in $C_c(X)$ such that $\varphi_n(x_0) = 1$ and $\varphi_n(x) = 0$ for every $x \in S_r(o)$, $r \geq n+1$. For such a sequence we get

$$h(\varphi_n) = \frac{1}{2}\sum_{x,y\in X} b(x,y)\,|\nabla_{x,y}\varphi_n|^p = \frac{1}{2}\sum_{r=0}^{n}\sum_{\substack{x\in S_r(o)\\ y\in S_{r+1}(o)}} b(x,y)\,|\nabla_{x,y}\varphi_n|^p$$

$$= \frac{1}{2}\sum_{r=0}^{n}\sum_{\substack{x\in S_r(o)\\ y\in S_{r+1}(o)}} b(x,y)\,|\nabla_{r,r+1}\varphi_n|^p = \frac{1}{2}\sum_{r=0}^{n}|\nabla_{r,r+1}\varphi_n|^p\,\partial_b B_r(o).$$

We define $e_n(x) = e_n(r)$ such that $e_n(0) = 1$ and

$$e_n(r) - e_n(r+1) = \frac{c_n}{(\partial_b B_r(o))^{1/(p-1)}},$$

with

$$c_n = \left(\sum_{r=0}^{n}\left(\frac{1}{\partial_b B_r(o)}\right)^{1/(p-1)}\right)^{-1}.$$

It follows that $e_n \nearrow 1$ and, using the previous computation,

$$h(e_n) = \sum_{r=0}^{n}\partial_b B(r)\frac{c_n^p}{(\partial_b B_r(o))^{p/(p-1)}} = c_n^{p-1} \to 0 \qquad n\to\infty.$$

Hence, we have our desired null-sequence, and $h$ is critical.

**Example 10.12 (Star graphs)** We have seen already in Example 9.10, that the $p$-energy functional to the free $p$-Laplacian is critical since all positive $p$-superharmonic functions are $p$-harmonic, and thus, there cannot exist a normalised Green's function.

## 10.4 Liouville Comparison Principle

Here, we show a consequence of the characterisations of criticality and the ground state representation which is usually referred to as a Liouville comparison principle, confer [Pin07, Section 11] and references therein for the linear case. For the counterpart in the continuum see [PR15, Theorem 8.1], or [PTT08, Theorem 1.9], and for applications in the continuum see e.g. [Ber+20; FP23].

**Proposition 10.13 (Liouville comparison principle)** *Let $p > 1$, $q \in \mathbb{R}$, and $V \subseteq X$ be non-empty and connected. Let $b$ and $\tilde{b}$ be two graphs on $X$, and $c, \tilde{c} \in C(X)$ be two potentials. Let denote $h$ and $\tilde{h}$ the energy functionals with corresponding Schrödinger operators $H := H_{b,c,m,p}$ and $\tilde{H} := H_{\tilde{b},\tilde{c},m,p}$, respectively. Assume that the following assumptions hold true:*

(a) *$h$ is critical in $V$ with Agmon ground state $u \in C(V)$.*

(b) *$\tilde{h} \geq 0$ on $C_c(V)$, and there exists a positive $p$-subharmonic function $\tilde{u} \in C(V)$ with respect to $\tilde{H}$ on $V$.*

(c) *There exists a constant $\alpha > 0$, such that for all $x, y \in V$ we have*

$$b^q(x,y)u(x)u(y) \geq \alpha\, \tilde{b}^q(x,y)\tilde{u}(x)\tilde{u}(y).$$

(d) *There exists a constant $\beta > 0$, such that for all $x, y \in V$ we have for $p > 2$,*

$$b^{\frac{2(1-q)}{p-2}}(x,y)u(x)u(y) \geq \beta\, \tilde{b}^{\frac{2(1-q)}{p-2}}(x,y)\tilde{u}(x)\tilde{u}(y),$$

*and the reversed inequality holds for $1 < p < 2$.*

(e) *There exists a constant $\gamma > 0$ such that for all $x, y \in V$ we have for $p > 2$,*

$$b^{\frac{1-q}{p-2}}(x,y)\left|\nabla_{x,y}u\right| \geq \gamma\, \tilde{b}^{\frac{1-q}{p-2}}(x,y)\left|\nabla_{x,y}\tilde{u}\right|,$$

*and the reversed inequality holds for $1 < p < 2$.*

*Then the energy functional $\tilde{h}$ is critical in $V$ with Agmon ground state $\tilde{u}$.*

*Proof.* By Theorem 10.1, there exists a null-sequence $(e_n)$ with respect to $h$ such that $e_n \to u$ pointwise as $n \to \infty$. Denote $\varphi_n = e_n/u$ on $V$, $n \in \mathbb{N}$. From the ground state representation, Theorem 4.1, we get $h(e_n) \asymp h_u(\varphi_n)$ for all $n \in \mathbb{N}$. Hence, using (c) and (e), we infer $h_u(\varphi_n) \geq \gamma_1 \tilde{h}_{\tilde{u}}(\varphi_n)$ for some constant $\gamma_1 > 0$. By (b), the calculation before, and the ground state representation, we get for some constants $\gamma_2, \gamma_3 > 0$ that

$$0 \leq \tilde{h}(\tilde{u}\varphi_n) \leq \gamma_2 \tilde{h}_{\tilde{u}}(\varphi_n) \leq \gamma_3 h(e_n) \to 0, \qquad n \to \infty.$$

Thus, $(\tilde{u}\varphi_n)$ is a null-sequence for $\tilde{h}$ in $V$ and by Theorem 10.1, $\tilde{h}$ is critical in $V$. Since $\varphi_n \to 1$, we get $\tilde{u}\varphi_n \to \tilde{u}$, and by Theorem 10.1, $\tilde{u}$ is the ground state. $\square$

# 11. Decay of Hardy Weights

> There are two things at any rate which seem essential, a certain *generality* and a certain *depth*, but neither quality is easy to define at all precisely.
>
> G. H. Hardy, *A Mathematician's Apology*, p. 103

Let us investigate how large a Hardy weight might be in a neighbourhood of infinity. We will see that the decay of Hardy weights is closely connected with the decay of functions of minimal growth near infinity. The counterpart in the continuum is [KP20, Theorem 3.1 and Theorem 3.2], and [HPR24, Theorem 7.9]. In the following the potential can again be arbitrary. We want to mention that we follow here mainly [Fis24, Section 3].

We set $\deg_W(x) := \sum_{y \in W} b(x, y)$ for $W \subseteq X$ and $x \in X$.

**Theorem 11.1 (Decay of Hardy weights)** *Let $V \subseteq X$ be connected and non-empty, and $h$ be non-negative on $C_c(V)$. Let $W \subseteq V$ be a non-empty set, and $K \subseteq V$ be a finite non-empty set.*

*(a) Assume that $h$ is critical in $V$ with Agmon ground state $u \in C(V) \cap \ell^p(V \setminus W, \deg_W)$ such that $\sup_{x \in W, y \in V \setminus W, x \sim y} u(x)/u(y) < \infty$. Then for any Hardy weight $w$ on $V \setminus W$, we have*

$$w \in \ell^1(V \setminus W, u^p),$$

*i.e., $u \in \ell^p(V \setminus W, w)$.*

*(b) Assume that $h$ is subcritical in $V$, and that there are only finitely many edges between $V \setminus K$ and $K$. Let $v \in \mathcal{M}(V \setminus K)$. Then for any Hardy weight $w$ on $V$, we have*

$$w \in \ell^1(V, v^p).$$

The existence of a Hardy weight on $V \setminus W$ is ensured by Lemma 11.7. Moreover, the technical assumptions in the preamble of (a) are fulfilled if the subgraph $(V, b|_{V \times V})$ is locally finite. Also note that for any finite $K \subseteq X$, $\deg_K$ is a finite measure on $X$ by the local summability condition on $b$.

Recall the definitions of null-criticality and optimality near infinity: If $h - w_p$ is critical in $V$ with Agmon ground state $u$, then $h - w_p$ is null-critical with respect to $w$ if $u \notin \ell^p(V, w)$, and $w$ is optimal near infinity for $h$ if $h - w_p \geq \lambda w_p$ fails to hold on $C_c(V \setminus K)$ for all $\lambda > 0$ and finite $K \subsetneq V$.

Here, we will show in the following theorem, that null-criticality implies optimality near infinity for all $p > 1$, and all possible potentials if the graph is locally finite and the ground state is smooth enough. It can be interpreted as a corollary of Theorem 11.1 (see Section 11.2 for a proof).

Recall that a function $f \in C(X)$ that is strictly positive on $V \subseteq X$ is called of bounded oscillation on $V$ if $\sup_{x,y\in V, x\sim y} f(x)/f(y) < \infty$.

**Theorem 11.2 (Null-criticality implies optimality near infinity)** *Let $V \subseteq X$ be connected and non-empty. Let $h - w_p$ be null-critical with respect to $w \gneq 0$ in $V$ with Agmon ground state $u$ in $C(V) \cap \ell^p(V, b(x,\cdot))$ for all $x \in V$, and of bounded oscillation in $V$. Then $w$ is optimal near infinity for $h$.*

*Proof.* Let $u$ be an Agmon ground state of the critical $p$-energy functional $h - w_p$ in $V$. We show the contraposition: Assume that $w$ is not optimal near infinity in $V$, then there exists $\mu > 1$ and $K \subseteq V$ finite, such that $h - \mu \cdot w_p \geq 0$ on $C_c(V \setminus K)$. Thus, $(h - w_p) - (\mu - 1)w_p \geq 0$ on $C_c(V \setminus K)$, i.e., $(\mu - 1)w$ is a $p$-Hardy weight for $h - w_p$ on $C_c(V \setminus K)$. By the assumptions, we can use Theorem 11.1 (a), and get $w \in \ell^1(V, u^p)$. The latter is equivalent to $u \in \ell^p(V, w)$, and therefore, $h - w_p$ is not null-critical. □

Note that the technical assumptions minimise if the subgraph $(V, b|_{V\times V})$ is locally finite. Moreover, $u \in C(V) \cap \ell^p(V, b(x,\cdot))$ for all $x \in V$ if and only if $u \in C(V) \cap F_{b,p+1}(V)$. Furthermore, $F_{b,p+1}(V) \subseteq F_{b,p}(V)$, see Lemma 2.2.

Next, we will prove Theorem 11.1. But, firstly, we turn to some preliminary convexity-type results which are also of interest in its own.

## 11.1 Convexity-type Results

First, we show a Poincaré-type inequality. This is the discrete counterpart of [PT07, Theorem 1.6.4], see also [HPR24; PR15; PP16].

**Lemma 11.3 (Poincaré inequality)** *Let $V \subseteq X$ be connected and non-empty. Assume that $h$ is critical in $V$ with Agmon ground state $u$. Then, there exists a strictly positive function $w$ on $V$ such that for every $\psi \in C_c(V)$ with $\langle u, \psi\rangle_V \neq 0$ there exists a positive constant $C = C(\psi)$ such that*

$$C^{-1} w_p(\varphi) \leq h(\varphi) + C\,|\langle \varphi, \psi\rangle_V|^p\,, \qquad \varphi \in C_c(V).$$

*Proof.* We divide the proof into two steps.

*1. Claim: For all non-empty $W \subseteq V$ there exists a strictly positive function $w \in C(V)$ such that*

$$w_p(\varphi) \leq h(\varphi) + (m1_W)_p(\varphi), \qquad \varphi \in C_c(V).$$

Note that the right-hand side is a $p$-energy functional which we denote by $\tilde{h}$. Then, clearly, $\tilde{h}$ is non-negative on $C_c(V)$. Moreover, $\tilde{h} \geq (m1_W)_p$ on $C_c(V)$, i.e., $\tilde{h}$ is subcritical (with $p$-Hardy weight $m1_W$). Since $\tilde{h}$ is subcritical in $V$ such a $w$ exists by Corollary 8.5.

*2. Claim: Let $u$ be an Agmon ground state of the critical $p$-energy functional $h$ on $V$. For every $\psi \in C_c(V)$ with $\langle u, \psi\rangle_X \neq 0$, there exists a positive constant $C = C(\psi)$ such that for all finite and non-empty $K \subseteq V$, we have*

$$(m1_K)_p(\varphi) \leq C\big(h(\varphi) + |\langle \varphi, \psi\rangle_V|^p\big), \qquad \varphi \in C_c(V).$$

Assume that the inequality above is wrong. Then there is a sequence $(\varphi_n)$ in $C_c(V)$ such that $h(\varphi_n) \to 0$, $\langle \varphi_n, \psi \rangle_V \to 0$, but $(m1_K)_p(\varphi_n) \to \alpha \in (0, \infty]$. Thus, since $K$ is finite and non-empty, there exists $o \in K$ and $\varepsilon > 0$ such that for all $n_k \geq n_0$, $\varphi_{n_k}(o) \geq \varepsilon$. Set $e_k = \varphi_{n_k}/\varphi_{n_k}(o)$ for all $n_k \geq n_0$. Then, $(e_k)$ is a null sequence of $h$ in $V$. Thus, $e_k \to \tilde{C}\, u$ by Theorem 10.1 for some positive constant $\tilde{C}$. Since $\langle e_k, \psi \rangle_V \to \tilde{C} \langle u, \psi \rangle_V \neq 0$, we have a contradiction, and the second claim is proven.

Now, applying both claims yields the result. □

Next, we prove a convexity-type result. An alternative proof using Lemma 11.3 can be found in a preprint of [Fis24]. The counterpart in the continuum can be found in [PT07, Proposition 4.3].

**Proposition 11.4** *Let $c_0, c_1$ be two potentials such that $c_0 \neq c_1$ on a connected and non-empty subset $V \subseteq X$. Denote the corresponding $p$-energy functionals by $h_0, h_1$, respectively. Furthermore, define for all $t \in [0, 1]$*

$$h_t(\varphi) := t h_1(\varphi) + (1 - t) h_0(\varphi), \qquad \varphi \in C_c(X).$$

*Let $h_i$ be non-negative on $C_c(V)$, $i = 0, 1$. Then $h_t$ is non-negative on $C_c(V)$ for all $t \in [0, 1]$. Moreover, $h_t$ is subcritical in $V$ for all $t \in (0, 1)$.*

*Proof.* The proof follows the ideas from its counterpart in the continuum. Here are the details: Clearly, $h_t$ is non-negative on $C_c(V)$ for all $t \in [0, 1]$.

If $h_0$ is subcritical in $V$ with Hardy weight $w_0$, then, $h_t$, $t \in [0, 1)$, is subcritical in $V$ with Hardy weight $(1 - t)w_0$. Analogously, we can argue if $h_1$ is subcritical in $V$.

It remains the case that both $h_0$ and $h_1$ are critical in $V$. Denote by $u_0$ and $u_1$ the corresponding Agmon ground states normalised at some $o \in V$. Assume that for some $t \in (0, 1)$ also $h_t$ is critical. Then by Theorem 10.1, $h_t$ has an Agmon ground state $u_t$ on $V$ normalised at $o \in V$ and a null-sequence $(e_n)$ on $V$, such that $0 \leq e_n \to u_t$ pointwise on $V$. We show that this results in a contradiction.

We claim that $c_0 \neq c_1$ implies that $u_t$ cannot be a multiple of $u_1$ or $u_2$. Indeed, let $H_t$ be the $p$-Schrödinger operator associated to $h_t$, $t \in [0, 1]$, and assume that $u_t = C u_0$ for some $C > 0$. Since $u_t$ is an Agmon ground state for $t \in [0, 1]$, we have $0 = H_t u_t = t H_1 u_t + (1 - t) H_0 u_t = t C^{p-1} H_1 u_0$ on $V$. Hence, $u_0$ is also a global $p$-harmonic function for $H_1$ on $V$. Since $u_0(o) = 1 = u_1(o)$, we have by the uniqueness of the Agmon ground state (up to multiplication by a constant) that $u_0 = u_1$. But this yields, $c_1 u_1^{p-1} = -mLu_1 = -mLu_0 = c_0 u_0^{p-1} = c_0 u_1^{p-1}$, i.e., $c_0 = c_1$ on $V$. Interchanging the role of $u_1$ and $u_0$ yields the claim.

Let $(e_n^t)$ be a null sequence for $h_t$ converging pointwise to $u_t$, $t \in [0, 1]$. Then $h_t(e_n^t) \to 0$ implies $h_1(e_n^t) \to 0$ and $h_0(e_n^t) \to 0$. Hence, $(e_n^t)$ is a null sequence for both $h_1$ and $h_0$. By construction, this sequence converges pointwise to $u_1$ and $u_0$, and also $u_t$. But since $c_0 \neq c_1$, we have that $u_t$ cannot be a multiple of $u_1$ or $u_2$, and get a contradiction. Therefore, $h_t$ does not admit an Agmon ground state and by Theorem 10.1, it is subcritical in $V$ for $t \in (0, 1)$. □

A consequence of the previous proposition is given next. For the counterpart in the continuum confer [PT07, Proposition 4.4] or [PP16, Proposition 4.19].

**Corollary 11.5** *Let $V \subseteq X$ be connected and non-empty, and let $h$ be subcritical in $V$. Let $\tilde{w} \in C_c(V)$ such that $\tilde{w}(x) < 0$ for some $x \in V$. Then there exists $\tau_+ \in (0, \infty)$ and $\tau_- \in [-\infty, 0)$ such that $h + t \cdot \tilde{w}_p$ is subcritical in $V$ for $t \in (\tau_-, \tau_+)$, and such that $h + \tau_+ \cdot \tilde{w}_p$ is critical in $V$.*

*Proof.* By Corollary 8.5, there is a strictly positive Hardy weight $w$ associated with $h$ on $V$. Thus, $(w + t \cdot \tilde{w})_p(\varphi) \leq h(\varphi) + t\tilde{w}_p(\varphi)$ for all $\varphi \in C_c(V)$. Since $w > 0$ on $V$, and $\tilde{w}$ is compactly supported, we get that $w + t \cdot \tilde{w} > 0$ on $V$ for sufficiently small values of $|t|$. Let

$$S := \{t \in \mathbb{R} : h + t \cdot \tilde{w}_p \text{ is subcritcal in } V\}.$$

By the previous proposition, Proposition 11.4, $S$ is an interval. Let denote by $\tau_+$ and $\tau_-$, the right and left boundary point of this interval, respectively.

We show that $\tau_+ < +\infty$: Since there is $x \in V$ such that $\tilde{w}(x) < 0$, we have $\tilde{w}_p(1_x) < 0$ and thus, $h(1_x) + t \cdot \tilde{w}_p(1_x) < 0$ for sufficiently large $t$.

We show that $\tau_- = -\infty$ can be obtained: Assume that $\tilde{w} \lneq 0$ on $V$, then $\tilde{w}_p(\varphi) \leq 0$ for all $\varphi \in C_c(V)$, and thus for all $t < 0$, we get $t \in S$.

We show that $h + \tau_+ \tilde{w}_p$ is critical: Clearly, $h + \tau_+ \tilde{w}_p$ is non-negative on $C_c(V)$. Assume that $h + \tau_+ \tilde{w}_p$ is subcritical, i.e., $\tau_+ \in S$. Arguing as in the beginning, we see that for sufficiently small $\varepsilon > 0$, we get that $h + (\tau_+ + \varepsilon)\tilde{w}_p$ is subcritical. But this contradicts that $\tau_+$ is the right boundary point of the interval $S$. □

The following result can be seen as the counterpart of Corollary 11.5 for critical $p$-energy functionals. Confer [PT07; PP16; PR15] for the local case.

**Corollary 11.6** *Let $V \subseteq X$ be connected and non-empty, and let $h$ be critical in $V$ with Agmon ground state $u \in C(V)$.*

(a) *Assume that $\tilde{w} \in \ell^\infty(V)$ and there exists $\tau_+ \in (0, \infty]$ such that $h + \tau_+ \cdot \tilde{w}_p$ is non-negative in $C_c(V)$. Then, for any null-sequence $(e_n)$ of $h$ in $V$, we have* $\liminf_{n\to\infty} \tilde{w}_p(e_n) > 0$.

(b) *Assume that $\tilde{w} \in C_c(V)$ and $\tilde{w}_p(u) > 0$. Then there exists $\tau_+ \in (0, \infty]$ such that $h + t \cdot \tilde{w}_p$ is subcritical for all $t \in (0, \tau_+)$.*

*Proof.* Ad (a): By Proposition 11.4, $h + t \cdot \tilde{w}_p$ is subcritical for all $t \in (0, \tau_+)$. Thus, we can use Corollary 8.5, and get the existence of a strictly positive $p$-Hardy weight $w$ associated with the $p$-energy functional $h + t \cdot \tilde{w}_p$. Let $(e_n)$ be an arbitrary null-sequence of $h$ such that $e_n \to C \cdot u > 0$ for some positive constant $C$, which exists by Theorem 10.1. By Fatou's lemma, we infer

$$\begin{aligned} t \cdot \liminf_{n\to\infty} \tilde{w}_p(e_n) &= \liminf_{n\to\infty} h(e_n) + \liminf_{n\to\infty} t \cdot \tilde{w}_p(e_n) \\ &\geq \liminf_{n\to\infty} w_p(e_n) \geq C^p \cdot w_p(u) > 0. \end{aligned}$$

Dividing by $t > 0$ yields (a).

Ad (b): The strategy is as follows: If there is $\tau_+ \in (0, \infty]$ such that $h + \tau_+ \cdot \tilde{w}_p$ is non-negative in $C_c(X)$, then by Proposition 11.4, we get that $h + t \cdot \tilde{w}_p$ is subcritical for all $t \in (0, \tau_+)$.

Thus, let us assume that such a $\tau_+ \in (0, \infty]$ does not exists. We will show that this leads to a contradiction. Hence, we assume that for all $t > 0$, there exists $\varphi_t \in C_c(V)$ such that

$$h(\varphi_t) + t \cdot \tilde{w}_p(\varphi_t) < 0. \tag{11.1}$$

By the reversed triangle inequality, $h(\varphi_t) \geq h(|\varphi_t|)$. Thus, we can assume without loss of generality that $\varphi_t \geq 0$. Since $h$ is non-negative on $C_c(V)$, it follows from (11.1), that

$$\tilde{w}_p(\varphi_t) < 0. \tag{11.2}$$

In particular, by Inequality (11.2), we have $\operatorname{supp}(\varphi_t) \cap \operatorname{supp}(\tilde{w}) \neq \emptyset$.

Let $\psi_t := \varphi_t / \|\varphi_t\|_{p, 1_{\operatorname{supp} \tilde{w}}}$. Then, using that $\tilde{w}$ is bounded on $V$, we get

$$\lim_{t \to 0} t \cdot \tilde{w}_p(\psi_t) \leq \sup_V(\tilde{w}) \cdot \lim_{t \to 0} t \cdot \|\psi_t\|^p_{p, 1_{\operatorname{supp} \tilde{w}}} = 0.$$

Since $h$ is non-negative on $C_c(V)$ we obtain using (11.1) and the calculation above,

$$0 \leq \lim_{t \to 0} h(\psi_t) \leq \lim_{t \to 0} t \cdot \tilde{w}_p(\psi_t) = 0.$$

Thus, $h(\psi_t) \to 0$ as $t \to 0$.

Since $\tilde{w}$ is finitely supported and since $\operatorname{supp}(\varphi_t) \cap \operatorname{supp}(\tilde{w}) \neq \emptyset$ for all $t > 0$, we have that for some $o \in \operatorname{supp}(\tilde{w})$, there exists $\varepsilon > 0$ and a subsequence $(\psi_{t_n})$ of the net $(\psi_t)$ such that $\psi_{t_n}(o) \geq \varepsilon$, and $t_n \to 0$ as $n \to \infty$.

Let $\psi_n := \psi_{t_n} / \psi_{t_n}(o)$, then $h(\psi_n) = (1/\psi_{t_n}(o))^p h(\psi_{t_n}) \to 0$ as $n \to \infty$. Hence, $(\psi_n)$ is a null sequence of $h$, and thus, $\psi_n \to Cu$ for some positive constant $C$. By the assumptions on $\tilde{w}$, we thus have

$$\lim_{n \to \infty} \tilde{w}_p(\psi_n) = C^p \tilde{w}_p(u) > 0.$$

This contradicts (11.2). $\square$

We continue with an observation which is basically a consequence of Lemma 11.3. Confer [PT07, Proposition 4.2], or [PP16, Proposition 4.18] for the counterpart in the continuum.

Using a different and more technical method, it is shown in Corollary 10.3 that if $h$ is non-negative on $C_c(V \cup \{o\})$ for some $V \subsetneq X$ and $o \in \partial V$ then $h$ is subcritical in $V$. This clearly makes following lemma, Lemma 11.7, redundant. However, for convenience, we give a short argument based on the Poincaré inequality, Lemma 11.3, here.

**Lemma 11.7** *Let $W \subsetneq V \subseteq X$ be both connected and non-empty.*

(a) *If $h$ is non-negative on $C_c(V)$, then $h$ is subcritical in $W$.*

(b) *If $h$ is critical in $W$, then $h$ is supercritical in $V$.*

*Proof.* If $h$ is subcritical in $V$, then Corollary 8.5 implies that we have a Hardy weight $w$ which is strictly positive in $V$. Thus, taking $w \cdot 1_W$, we have a Hardy weight on $W$, and $h$ is subcritical in $W$.

Let $h$ be critical in $V$ with Agmon ground state $u \in C(V)$. Take $o \in V \setminus W$. By Lemma 11.3, there exists a strictly positive function $w$ on $V$ and a positive constant $C$ such that

$$C^{-1} w_p(\varphi) \leq h(\varphi) + C\,|\langle \varphi, 1_o \rangle_V|^p = h(\varphi), \qquad \varphi \in C_c(W).$$

Thus, $h$ is subcritical in $W$. This proves (a). The second statement (b) is just the contraposition of the first one if $h$ is non-negative on $C_c(W)$. □

## 11.2 Proof of Theorem 11.1

*Proof (of Theorem 11.1).* Let us mention that there is nothing to prove for $W = V$ or $K = V$. Hence, in the following, we assume that $W \subsetneq V$ or $K \subsetneq V$.

Ad (a): Let $h$ be critical in $V$. Then, by Theorem 10.1, there is a null sequence $(e_n)$ in $V$ such that $e_n \to u$ pointwise, where $u \in C(V)$ is the Agmon ground state of $h$ in $V$. By Lemma 11.7, we have a $p$-Hardy weight $w$ on $V \setminus W$, i.e.,

$$w_p(\varphi) \leq h(\varphi), \qquad \varphi \in C_c(V \setminus W).$$

By Fatou's lemma, we have

$$w_p(u 1_{V\setminus W}) \leq \liminf_{n\to\infty} w_p(e_n 1_{V\setminus W}).$$

Thus, we have to show that the right-hand side is finite. We calculate using Theorem 4.1,

$$w_p(e_n 1_{V\setminus W}) \leq h(e_n 1_{V\setminus W}) \asymp h_u(e_n 1_{V\setminus W}/u).$$

Moreover,

$$\begin{aligned} h_u(e_n 1_{V\setminus W}/u) = &\sum_{x,y\in V\setminus W} b(x,y)u(x)u(y)\left|\nabla_{x,y}\frac{e_n}{u}\right|^2 \\ &\cdot \left( (u(x)u(y))^{1/2}\left|\nabla_{x,y}\frac{e_n}{u}\right| + \frac{\left|\frac{e_n(x)}{u(x)}\right| + \left|\frac{e_n(y)}{u(y)}\right|}{2}\,|\nabla_{x,y} u| \right)^{p-2} \\ &+ 2\cdot \sum_{x\in W, y\in V\setminus W} b(x,y)u(x)u(y)\frac{e_n^2(y)}{u^2(y)} \\ &\cdot \left( \big(u(x)u(y)\big)^{1/2}\cdot\frac{e_n(y)}{u(y)} + \frac{1}{2}\cdot\frac{e_n(y)}{u(y)}\,|\nabla_{x,y} u| \right)^{p-2} \end{aligned}$$

This first sum on the right-hand side can be estimated from above by $h_u(e_n/u)$. Since $h_u(e_n/u) \asymp h(e_n) \to 0$ as $n \to \infty$, we only need to discuss the second sum. Here we have, using that $u$ satisfies $\sup_{x\in W, y\in V\setminus W, x\sim y} u(x)/u(y) < \infty$, the existence of a constant $C_p > 0$ such that

$$\sum_{x\in W, y\in V\setminus W} b(x,y)u(x)u(y)\frac{e_n^2(y)}{u^2(y)}\left(\big(u(x)u(y)\big)^{1/2}\cdot\frac{e_n(y)}{u(y)} + \frac{1}{2}\cdot\frac{e_n(y)}{u(y)}\,|\nabla_{x,y}u|\right)^{p-2}$$
$$= \sum_{x\in W, y\in V\setminus W} b(x,y)\frac{u(x)}{u(y)}\left(\left(\frac{u(x)}{u(y)}\right)^{1/2} + \left|\frac{u(x)}{u(y)} - 1\right|\right)^{p-2} e_n^p(y)$$
$$\le C_p \sum_{x\in W, y\in V\setminus W} b(x,y)e_n^p(y) = C_p \sum_{y\in V\setminus W} e_n^p(y)\deg_W(y).$$

We can assume by Theorem 10.1 (vii) that $e_n \le u$. Since we have by assumption $u \in \ell^p(V\setminus W, \deg_W)$, we proved (a).

Ad (b): Let $h$ be subcritical in $V$ with corresponding Hardy weight $w$, i.e., $h - w_p \ge 0$ on $C_c(V)$. Let $v \in \mathcal{M}(V\setminus K)$ for some finite $K \subsetneq V$.

Using Corollary 11.5, there is a potential $\tilde{w} \in C_c(K) = C(K)$ such that $h - \tilde{w}_p$ is critical in $V$. Let denote the Agmon ground state on $V$ of $h - \tilde{w}_p$ by $u$. Hence, $u \in \mathcal{M}(V) \subseteq \mathcal{M}(V\setminus K)$ with respect to $h$. Furthermore, since $u$ and $v$ are minimal on $V\setminus K$ and on the finite set $K$ there always exists a constants $C > 0$ for the strictly positive functions $u$ and $v$ such that $u \le Cv$, we have $v \asymp u$ on $V \setminus K$.

Since for all $\varphi \in C_c(V\setminus K)$

$$(h - \tilde{w}_p)(\varphi) = h(\varphi) \ge w_p(\varphi) = w_p(1_{V\setminus K}\varphi),$$

$w$ is also a Hardy weight for $h - \tilde{w}_p$ on $C_c(V\setminus K)$. Thus, we can use the first part, and get that $w \in \ell^1(V\setminus K, v^p) = \ell^1(V\setminus K, u^p) = \ell^1(V, u^p)$. $\square$

# 12. Optimal Hardy Weights

> Some measure of generality must be present in any high-class theorem, but *too much* tends inevitably to insipidity. [...] We do not choose our friends because they embody all the pleasant qualities of humanity, but because they are the people that they are. And so in mathematics; [...] Here at any rate I can quote Whitehead on my side: 'it is the large generalization, limited by a happy particularity, which is the fruitful conception.'
>
> G. H. Hardy, *A Mathematician's Apology*, p. 108

Here, we will derive optimal Hardy weights for subcritical $p$-energy functionals with arbitrary potential part. This will be achieved by the virtue of a coarea formula, Proposition 12.10 in Section 12.2. Moreover, we will need the technicality that roots of superharmonic functions are also superharmonic. This is discussed in Section 12.1. The proof of the main result, Theorem 12.1, is given in Section 12.3. We start proving optimality by showing criticality in Section 12.3.1, and show null-criticality and optimality near infinity in the two succeeding sections. The proofs of the criticality and null-criticality work mostly along the lines of the proofs of [KPP18b] and [DP16; Ver23], by either generalising from $p = 2$ to $p \in (1, \infty)$ or by using discrete non-local versions of local methods in [DP16; Ver23]. The proof of the optimality near infinity is instead inspired by results in [KP20], confer also Theorem 11.1 and Theorem 11.2.

Our main result tells us now how to find optimal Hardy weights. An associated version in the continuum is given in [Ver23, Theorem 1.1]. Recall that by the Harnack inequality, Lemma 5.1, a positive $p$-superharmonic function is strictly positive.

Recall that a function $f \in C(X)$ that is positive on $V \subseteq X$ is called almost proper on $V$ if $f^{-1}(I) \cap V$ is a finite set for any compact set $I \subseteq (0, \infty)$; and if $f$ is almost proper and strictly positive on $V$, it is called proper on $V$. Moreover, if $f$ is strictly positive on $V$, then $f$ is of bounded oscillation on $V$ if $\sup_{x,y \in V, x \sim y} f(x)/f(y) < \infty$. Also recall that the existence of a strictly positive proper function of bounded oscillation on $V \cup \partial V$ implies that the underlying graph is locally finite on $V$.

**Theorem 12.1 (Optimal Hardy weights)** *Let $V \subseteq X$ be infinite such that $(V, b|_{V \times V})$ is locally finite on $V$. Let $h$ be a subcritical $p$-energy functional in $V$ with arbitrary potential $c$, corresponding $p$-Schrödinger operator $H := H_{b,c,p,m}$, and $p$-Laplacian $L := L_{b,p,m}$, $p > 1$.*

*Suppose that $0 \lneq u \in F(V) \cap C(V)$ is a proper function of bounded oscillation on $V$ such that $\tilde{H} u \geq 0$ on $V$, where $\tilde{H} := H_{b, C_p \cdot c, p, m}$, and $C_p := (p/(p-1))^{p-1}$.*

*Furthermore, assume that $Lu \in \ell^1(V, m)$, $u \in \ell^{p-1}(V, c_-)$, and*

*(a) $u$ takes its maximum on $V$, or there exists $S > 0$ such that for all $x \in V$ with $u(x) > S$ we have $Lu(x) \leq 0$, and*

*(b) $u$ takes its minimum on $V$, or there exists $I > 0$ such that for all $x \in V$ with $u(x) < I$ we have $Lu(x) \geq 0$.*

*Then,*

$$w := \frac{H(u^{(p-1)/p})}{u^{(p-1)^2/p}},$$

*multiplied with $m$ is an optimal $p$-Hardy weight of $h$ on $V$.*

Note that a proper function of bounded oscillation on an infinite graph cannot have both, a maximum and a minimum.

We remark that a sufficient condition for $Lu \in \ell^1(V, m)$ is $\tilde{H}u \in \ell^1(V, m)$ and $u \in \ell^{p-1}(V, |c|)$ by the triangle inequality. Moreover, if $\tilde{H}u \in C_c(V)$ – in other words, $u$ is $p$-harmonic outside of a finite set with respect to $\tilde{H}$, then the conditions $\tilde{H}u \in \ell^1(X, m)$, (a) and (b) are satisfied. In [KPP18b], where the case $p = 2$ was investigated, the assumption is that $u$ should be harmonic with respect to $H$. Hence, in a certain sense the assumptions here generalise the result in [KPP18b]. Furthermore, if $0 \leq c \in C_c(V)$, then the conditions $u \in \ell^{p-1}(V, c)$ and (b) are satisfied.

In addition to that, we remark that in the linear ($p = 2$)-case no multiplication of the potential with a constant $C_2$ is needed, see [KPP18b]. There, a discrete chain rule of the square root was applied which remains unknown for $p \neq 2$. However, also for a quasi-linear counterpart in the continuum (that is [Ver23, Theorem 1.1]), the same constant is needed.

The assumption $\tilde{H}u \geq 0$ on $V$ is used to get $H(u^{(p-1)/p}) \gneq 0$ on $V$ and thus, $w \gneq 0$ on $V$, see Proposition 12.5. Alternatively, this condition can be replaced by $H(u^{(p-1)/p}) \gneq 0$ on $V$ without any changes in the proof. However, the last condition might be more difficult to verify in concrete examples.

As a corollary, we get optimal Hardy weights to Laplace-type operators. A corresponding version in the continuum is given in [DP16, Theorem 1.5].

**Corollary 12.2 (Optimal Hardy weights for non-negative potentials)** *Let $V \subseteq X$ be infinite such that $(V, b|_{V \times V})$ is locally finite on $V$, $p > 1$. Let $h$ be a subcritical $p$-energy functional in $V$ with non-negative potential $c$ and corresponding $p$-Schrödinger operator $H$, and $p$-Laplacian $L$. Suppose that $0 \lneq u \in F(V) \cap C(V)$ is a proper function of bounded oscillation on $V$ with $Hu \geq 0$ on $V$ and $Lu \in \ell^1(V, m)$ such that*

*(a) $u$ takes its maximum on $V$, or there exists $S > 0$ such that for all $x \in V$ with $u(x) > S$ we have $Lu(x) \leq 0$, and*

*(b) $u$ takes its minimum on $V$, or there exists $I > 0$ such that for all $x \in V$ with $u(x) < I$ we have $Lu(x) \geq 0$.*

*Then,*

$$w := \frac{H(u^{(p-1)/p})}{u^{(p-1)^2/p}},$$

*multiplied with $m$ is an optimal $p$-Hardy weight of $h$ on $V$.*

*Proof.* Note that $C_p > 1$ in Theorem 12.1. Hence, $\tilde{H}u \geq Hu \geq 0$, and we can apply Theorem 12.1. $\square$

**Remark 12.3 (Green's functions)** It is shown in Theorem 9.4 that whenever $h$ is subcritical in $X$ there exists a Green's function $g_o\colon X \to (0, \infty)$ for every $o \in X$ with the property that $Hg_o = 1_o$. Thus, $g_o$ is a strictly positive $p$-superharmonic function that is harmonic outside of a singleton. Hence, Green's functions are natural candidates for the function $u$ in Theorem 12.1 and Corollary 12.2.

Also note that there exist graphs associated with subcritical energy functionals whose corresponding Green's functions do not vanish at every boundary point: Take, for instants, the standard line graph $\mathbb{N}_0$ from Example 3.4 with $c = 0$ and $m = 1$. On $\mathbb{N}$, this graph is subcritical, and a positive minimal Green's function $G_1$ on $1$ is given by $G_1 = 1$ on $\mathbb{N}$ and $G_1(0) = 0$, confer with Example 9.6. Note that $G_1$ is still of bounded oscillation but not proper. Moreover, since $LG_1^{(p-1)/p} = 1_1$ on $\mathbb{N}$, the corresponding formula does not result in an optimal $p$-Hardy weight on $\mathbb{N}$. Another example of a tree with this property is given in [Woe00, p. 240].

But before we start, we note the following discrete version of [Ver23, Lemma 2.35]. Recall from Chapter 11 that $\ell^p(X, b(x, \cdot)) = F_{b,p+1}(\{x\})$ for some fixed $x \in X$, i.e. $u \in C(V) \cap \ell^p(V, b(x, \cdot))$ for all $x \in V$ if and only if $u \in C(V) \cap F_{b,p+1}(V)$ for some $V \subseteq X$.

**Lemma 12.4 (Versano's lemma about adding potentials)** *Let $V \subseteq X$ be connected and non-empty. Assume that $h$ admits an optimal $p$-Hardy weight $w$ on $V$, and $h - w_p$ has an Agmon ground state in $C(V) \cap \ell^p(V, b(x, \cdot))$ for all $x \in V$, which is of bounded oscillation in $V$. Let $\omega \in C(X)$ be such that $\omega \geq -\varepsilon \cdot w$ on $V$ for some $\varepsilon \in [0, 1)$. Then, $w + \omega$ is an optimal $p$-Hardy weight of $h + \omega_p$ in $V$.*

*Proof.* Firstly, $h + \omega_p$ is non-negative on $C_c(V)$ because of $\omega \geq -\varepsilon w$ on $V$, and $w \gneq 0$ is a $p$-Hardy weight of $h$ in $V$, i.e., $h - \omega_p \geq h - \varepsilon w_p \geq h - w_p \geq 0$ on $C_c(V)$. Since $(h + \omega_p) - (w + \omega)_p = h - w_p$ on $C_c(V)$, also the right-hand side is critical in $V$ with the same ground state $u$. Moreover, because of $(w + \omega)_p(u) \geq (1 - \varepsilon)w_p(u) = \infty$, the functional $(h + \omega_p) - (w + \omega)_p$ is null-critical with respect to $w + \omega$. Thus, by Theorem 11.2, $w + \omega \gneq 0$ is an optimal $p$-Hardy weight of $h + \omega_p$ in $V$. $\square$

Morally, this lemma tells us that sign changing and non-positive potentials are of particular interest. And once, we found an optimal $p$-Hardy weight for some $p$-energy functional, we get a family of optimal $p$-Hardy weights by simply adding certain potentials at the weight and the functional.

## 12.1 Supersolution Constructions

One particular crucial step in the proof of Theorem 12.1 will be to show that for a given positive and superharmonic function, certain roots of this function is again superharmonic. It turns out that for positive potentials, this can be deduced from standard techniques, see Proposition 12.7. However, for arbitrary potentials a slightly modified Schrödinger operators has to be considered. This is shown next. In this section, we also consider the case $p = 1$.

The following result can be seen as the discrete counterpart to [DP16, Lemma 2.10] and [Ver23, Corollary 3.6] and uses the mean value theorem to circumvent the use of the chain rule, confer also with [HK14, p. 350] and [KLW21, Section 2.3].

**Proposition 12.5 (Supersolutions via the mean value theorem)** *Let $p \geq 1$ and $u \in F(\{x\})$ for some $x \in X$. Let $\varphi \in C^2([\inf_{\{x\}\cup\partial\{x\}} u, \sup_{\{x\}\cup\partial\{x\}} u], \mathbb{R})$ be an increasing and concave function, i.e., $\varphi', -\varphi'' \geq 0$. Then, we have*

$$L(\varphi \circ u)(x) \geq ((\varphi' \circ u)(x))^{p-1} Lu(x). \tag{12.1}$$

*In particular, if $u(x) \neq u(y)$ for some $y \sim x$ and $\varphi$ is strictly increasing, then the inequality in (12.1) is strict.*

*Moreover, if either*

*(a) $u(x) > 0$, and we have that $\varphi(a) \geq \alpha \cdot a \cdot \varphi'(a)$ for some $\alpha > 0$ and all $a \in [\inf_{\{x\}\cup\partial\{x\}} u, \sup_{\{x\}\cup\partial\{x\}} u]$, and also $H_{b,\alpha^{p-1}\cdot c,p,m} u(x) \geq 0$, or*

*(b) $Hu(x) \geq 0$, and we have $c(x) = 0$, or $((\varphi \circ u)(x))^{\langle p-1\rangle} = ((\varphi' \circ u)(x))^{p-1}$, or $\operatorname{sgn} c(x) = \operatorname{sgn}(((\varphi \circ u)(x))^{\langle p-1\rangle} - ((\varphi' \circ u)(x))^{p-1})$.*

*Then, $H(\varphi \circ u)(x) \geq 0$.*

*In particular, if $u(x) > 0$, and also $H_{b,q^{p-1}\cdot c,p,m} u(x) \geq 0$ for some $q \geq 1$, then $H(u^{1/q})(x) \geq 0$. If, in addition to that, $u(x) \neq u(y)$ for some $y \sim x$, then we have $H(u^{1/q})(x) > 0$.*

*Proof.* By the mean value theorem, for all $z, y \in X$, there exists $\xi \in [u(z) \wedge u(y), u(z) \vee u(y)]$ such that

$$\nabla_{z,y}(\varphi \circ u) = \varphi'(\xi) \nabla_{z,y} u.$$

Thus, for this fixed $x \in X$,

$$m(x) L(\varphi \circ u)(x) = \sum_{y \in X} b(x, y) \left(\varphi'(\xi) \nabla_{x,y} u\right)^{\langle p-1\rangle}.$$

Since $\varphi' \geq 0$, we obtain

$$\ldots = \sum_{y : \nabla_{x,y} u > 0} b(x, y)(\varphi'(\xi))^{p-1} (\nabla_{x,y} u)^{p-1} - \sum_{y : \nabla_{x,y} u < 0} b(x, y)(\varphi'(\xi))^{p-1} (\nabla_{y,x} u)^{p-1}.$$

Moreover, because of $-\varphi'' \geq 0$, we can estimate as follows

$$\begin{aligned} \ldots \geq \sum_{y : \nabla_{x,y} u > 0} & b(x, y)(\varphi'(u(x)))^{p-1} (\nabla_{x,y} u)^{p-1} \\ & - \sum_{y : \nabla_{x,y} u < 0} b(x, y)(\varphi'(u(x)))^{p-1} (\nabla_{y,x} u)^{p-1} \\ & = m(x)(\varphi'(u(x)))^{p-1} Lu(x). \end{aligned}$$

This shows (12.1).

Ad (a): Using the extra assumption on $\varphi$ and $u$, and (12.1), we get for all $x \in V$

$$\begin{aligned} H(\varphi \circ u)(x) &\geq ((\varphi' \circ u)(x))^{p-1} \left( Lu(x) + \frac{c(x)}{m(x)} \cdot \left( \frac{(\varphi \circ u)(x)}{(\varphi' \circ u)(x)} \right)^{p-1} \right) \\ &\geq ((\varphi' \circ u)(x))^{p-1} \left( Lu(x) + \frac{c(x)}{m(x)} \cdot (\alpha u(x))^{p-1} \right). \end{aligned}$$

This shows (a).

Ad (b): If we add and subtract instead of multiplying the missing potential part, we get using (12.1),

$$H(\varphi \circ u)(x) \geq ((\varphi' \circ u)(x))^{p-1} Hu(x) + \frac{c(x)}{m(x)} \left( ((\varphi \circ u)(x))^{\langle p-1 \rangle} - ((\varphi' \circ u)(x))^{p-1} \right).$$

By the assumptions on $\varphi$, $u$ and $c$, the right-hand side is non-negative.

The last assertion follows from (a) by setting $\varphi = (\cdot)^{1/q}$ and $\alpha = q$ for some $q \geq 1$. □

Note that in the case of $c \geq 0$ and $\varphi = (\cdot)^{1/q}$ for some $q \geq 1$, it is sufficient to have $Hu \geq 0$ on $V$ instead of $H_{b,q^{p-1}\cdot c,p,m} u \geq 0$ on $V$.

Moreover, a corresponding result for $p$-subharmonic functions goes as follows.

**Corollary 12.6** *Let $p \geq 1$, $V \subseteq X$, and $u \in F(V)$. Let $\varphi \in C^2([\inf_X u, \sup_X u], \mathbb{R})$ be an increasing and convex function, i.e., $\varphi', \varphi'' \geq 0$. Then for all $x \in V$ we have*

$$L(\varphi \circ u)(x) \leq ((\varphi' \circ u)(x))^{p-1} Lu(x).$$

*In particular, if $u$ is $p$-subharmonic with respect to $L$ on $V$, then so is $\varphi \circ u$.*

*Proof.* Mimic the proof of Proposition 12.5. □

Next, we want to show a similar result as Proposition 12.5 for a not necessarily differentiable function $\varphi$. As a downside, we will need to assume that the potential is non-negative. This is a discrete version of [HKM06, Theorem 7.5], see also [BB11, Section 9.8].

**Proposition 12.7** *Let $c \geq 0$ on $V \subseteq X$ and $p \geq 1$. Let $u \in F(V)$ be positive on $X$ and $p$-superharmonic on $V$. Let further $\varphi\colon (0, \infty) \to [0, \infty)$ be an increasing and concave function. Then $\varphi \circ u$ is also $p$-superharmonic in $V$.*

*In particular, the functions $u^{1/q}$, $q \geq 1$, are $p$-superharmonic on $V$.*

*Proof.* By the Harnack inequality, see [Fis23], $u$ is strictly positive on $V$. If $\varphi$ is concave and increasing, then we have the following identity for all $t \in (0, \infty)$

$$\varphi(t) = \inf \{\alpha t + \beta \colon \alpha, \beta \in \mathbb{R}, \alpha \geq 0, \alpha\tau + \beta \geq \varphi(\tau) \text{ for all } \tau \in (0, \infty)\}.$$

Since $\varphi \geq 0$, we have $\beta \geq 0$. Moreover, because of $c \geq 0$ and $u \geq 0$ on $V$, we get $H(\alpha u + \beta) \geq 0$ on $V$. Since $\varphi \circ u$ is the infimum of a set of non-negative $p$-superharmonic functions it is by Corollary 5.4 also non-negative and $p$-superharmonic. □

**Remark 12.8** We note that in the case of $c = 0$ on $V$, the same proof holds for an increasing and convex function $\varphi\colon (0,\infty) \to \mathbb{R}$. Then, in the definition of such a function via supporting lines the number $\beta$ might be negative.

Moreover, in the case $c = 0$ on $V$, we also have that $\log u$ is $p$-superharmonic on $V$ with respect to $L$ if $u > 0$ is $p$-superharmonic on $V$ with respect to $L$.

## 12.2 Coarea Formula

Here we follow the ideas in [KPP18b, Subsection 2.3] (or [KPP20a, Lemma 9.3.5]) for $p = 2$ and extend them to $p \geq 1$. Additionally, we weaken the assumptions from the linear ($p = 2$)-case slightly. This generalised coarea formula can also be seen as the discrete analogue of the coarea formula in [Ver23].

We will also use the following boundary notation: For $V \subseteq X$ we denote

$$\tilde{\partial} V := \{(x,y) \in V \times X \setminus V : b(x,y) > 0\},$$

i.e., $\tilde{\partial} V$ contains all directed edges from the interior boundary of $V$ to the exterior boundary $\partial V$. Furthermore, for any function $u \in C(X)$, we define the *coarea function* $g = g_u\colon (\inf_X u, \sup_X u) \to [0,\infty]$ via

$$g(t) := \sum_{\substack{x,y \in X \\ u(y) < t \leq u(x)}} b(x,y)(\nabla_{x,y} u)^{p-1}. \tag{12.2}$$

Note that a strictly positive proper function on an infinite set cannot take both simultaneously, its maximum and its minimum since either $0$ or $\infty$ is an accumulation point. Moreover, note that on a finite set any strictly positive function on that set is proper.

**Lemma 12.9 (Stokes-type formula)** *Let $p \geq 1$, and $V \subseteq X$ be non-empty. Let $0 \leq u \in F(V)$ be non-constant on $V \cup \partial V$ and almost proper on $X$. Let $g = g_u$ denote the coarea function. Then, for any $t_1, t_2 \in (\inf_X u, \sup_X u)$ such that $t_1 \leq t_2$, the set*

$$W_{t_1,t_2} := \{x \in W : t_1 < u(x) \leq t_2\}$$

*is finite for any $W \subseteq X$, and*

$$g(t_1) - g(t_2) = \sum_{x \in V_{t_1,t_2}} Lu(x)m(x) + \sum_{(x,y) \in \tilde{\partial}((X \setminus V)_{t_1,t_2})} b(x,y)(\nabla_{x,y} u)^{p-1}, \tag{12.3}$$

*where both sides may take the value $+\infty$.*

*In the following assume that $u = 0$ on $X \setminus V$, i.e., $u \in C(V)$. Furthermore, in the case of infinite $V$, assume also*

$$\sup_{x,y \in V, x \sim y} (u(x) - u(y)) < \sup_{x \in V} u(x) - \inf_{x \in V} u(x), \tag{12.4}$$

*then $g((\inf_X u, \sup_X u)) \subseteq (0,\infty)$.*

*Moreover, assume additionally that $Lu \in \ell^1(V,m)$, and*

(a) *$u$ takes its maximum on $V$, or there exists $S > 0$ such that for all $x \in V$ with $u(x) > S$, we have $Lu(x) \leq 0$; and*

(b) *$u$ takes its minimum on $V$, or there exists $I > 0$ such that for all $x \in V$ with $u(x) < I$, we have $Lu(x) \geq 0$.*

*Then, also $g \asymp 1$ on $X$.*

*Proof.* For $t > 0$, define for any $W \subseteq X$, $W_t := \{x \in W : u(x) > t\}$. Let $t_1, t_2 \in (\inf_X u, \sup_X u)$ with $t_1 \leq t_2$. Then, $W_{t_1,t_2} = W_{t_1} \setminus W_{t_2}$. Since $u$ is almost proper on $X$, the set $W_{t_1,t_2}$ is finite for any $W \subseteq X$. Therefore, the characteristic function $1_{V_{t_1,t_2}}$ of $V_{t_1,t_2}$ is in $C_c(V)$.

Let us abbreviate the notation further by writing

$$b_u(x,y) := b(x,y)\left(\nabla_{x,y} u\right)^{\langle p-1 \rangle}, \qquad x, y \in X.$$

Since $u \in F(V)$, we can apply Green's formula, Lemma 2.7, which yields

$$\begin{aligned}\sum_{x \in V_{t_1,t_2}} Lu(x)m(x) &= \sum_{x \in X} 1_{V_{t_1,t_2}}(x) Lu(x)m(x) \\ &= \frac{1}{2} \sum_{x,y \in X} b_u(x,y) \nabla_{x,y} 1_{V_{t_1,t_2}} = \sum_{(x,y) \in \tilde{\partial} V_{t_1,t_2}} b_u(x,y).\end{aligned}$$

Notice that for $t_1 \leq t_2$, both $g(t_1)$ and $g(t_2)$ share the sum over the set

$$\{(x,y) \in X \times X : u(y) < t_1 \leq t_2 \leq u(x)\}.$$

Thus, we conclude

$$\begin{aligned} g(t_1) - g(t_2) &= \sum_{\substack{x,y \in X \\ u(y) < t_1 \leq u(x) < t_2}} b_u(x,y) - \sum_{\substack{x,y \in X \\ t_1 \leq u(y) < t_2 \leq u(x)}} b_u(x,y) \\ &= \sum_{\substack{x,y \in X \\ u(y) < t_1 \leq u(x) < t_2}} b_u(x,y) + \sum_{\substack{x,y \in X \\ t_1 \leq u(x) < t_2 \leq u(y)}} b_u(x,y) \\ &= \sum_{(x,y) \in \tilde{\partial} V_{t_1,t_2}} b_u(x,y) + \sum_{(x,y) \in \tilde{\partial}((X \setminus V)_{t_1,t_2})} b_u(x,y), \end{aligned}$$

together with the observation before, this shows (12.3).

We turn to $g > 0$: Since $u$ is non-constant and $X$ is connected, $\tilde{\partial} X_t$ is non-empty for all $t \in (\inf_X u, \sup_X u)$, i.e., $g > 0$ on $(\inf_X u, \sup_X u)$.

Now assume that $u \in C(V)$, and we show that $g < \infty$: If $V$ is finite, then $u(x) > t$ only finitely many times and only for $x \in V$. Since $u \in F(V)$, $g < \infty$. If $V$ is infinite, then firstly note that $u \in C(V)$ implies $\tilde{\partial}((X \setminus V)_{t_1,t_2}) = \emptyset$. By using (12.3), and since $V_{t_1,t_2}$ is finite for all $t_1, t_2 \in (\inf_X u, \sup_X u)$, we have $g(t_1) < \infty$ if and only if

$g(t_2) < \infty$. Secondly, note that by Inequality (12.4), there are $t_1, t_2 \in (\inf_X u, \sup_X u)$ such that $t_2 - t_1 > \sup_{x,y\in V, x\sim y}(u(x) - u(y))$. For the choice of these $t_1, t_2$, there is no vertex in $V_{t_2} = X_{t_2}$ that is connected to a vertex outside of $V_{t_1} = X_{t_2}$. Hence, $\tilde{\partial} V_{t_2} = \tilde{\partial} V_{t_2} \cap \tilde{\partial}(X \setminus V_{t_1,t_2})$.

Moreover, we observe that for any $W \subseteq X$, we have $(x, y) \in \tilde{\partial} W$ if and only if $(y, x) \in \tilde{\partial}(X \setminus W)$. Hence, we have by the considerations before and the definition of $b_u$ and the assumption $u \in F(V)$ that

$$g(t_2) = \sum_{(x,y)\in\tilde{\partial} V_{t_2}\cap\tilde{\partial}(X\setminus V_{t_1,t_2})} b_u(x, y) \le \sum_{(x,y)\in\tilde{\partial} V_{t_1,t_2}} b(x, y) \,|\nabla_{x,y} u|^{p-1} < \infty.$$

Thus, $g < \infty$ on $(\inf_X u, \sup_X u)$.

Next, we show the boundedness of $g$ under certain additional assumptions. The lower bound follows from (a), (b) and (12.3). Here are the details: Indeed, assume it takes its maximum on $V$, then $u^{-1}([a, \max_V u])$ is finite for all $a > 0$ by the almost properness of $u$. Hence, by (12.3), $g$ changes its value only finitely many times, and thus $g \asymp 1$ in $[a, \max_V u]$. A same argument holds, if $u$ takes its minimum instead. If $u$ does not have a maximum, then $g$ might converge to zero as $t$ goes to $\sup_V u = \sup_X u$. By (a) and (12.3), $g$ is increasing in a neighbourhood of $\sup_V u = \sup_X u$, and thus cannot converge to zero. By using (b) instead, a similar argument applies, when $u$ does not have a minimum.

The upper bound follows if we additionally assume that $Lu \in \ell^1(X, m)$. Then, (12.3) and (a) yields for all $t_1 \le t_2$ with $t_2 \ge S$ if $u$ does not have a maximum and $t_2 = \max_V u$ in the other case,

$$g(t_1) = g(t_2) + \sum_{x\in V_{t_1,t_2}} Lu(x)m(x) \le g(t_2) + \sum_{x\in V} |Lu(x)|\, m(x) < \infty.$$

The calculation for (b) is similar. □

We remark that many of the latter additional assumptions in Lemma 12.9 are satisfied if $Lu \in C_c(X)$, and the existence of a suitable $S$ and $I$ are then trivially satisfied.

Furthermore, if $u > 0$ on $V$, then Inequality (12.4) is equivalent to

$$\sup_{x,y\in V,\, x\sim y} \frac{u(x)}{u(y)} < \sup_{x,y\in V} \frac{u(x)}{u(y)}.$$

Now, we proof a formula to translate calculations and estimates of infinite sums over graphs to one dimensional integrals – the so-called coarea formula. This formula will be of fundamental importance in the proof of Theorem 12.1.

**Proposition 12.10 (Coarea formula)** *Let $p \ge 1$, and $0 \lneq u \in C(V)$. Let the function $f\colon (\inf u, \sup u) \to [0, \infty)$ be Riemann integrable. Then*

$$\frac{1}{2} \sum_{x,y\in X} b(x, y) \,(\nabla_{x,y} u)^{\langle p-1\rangle} \int_{u(y)}^{u(x)} f(t)\,\mathrm{d}t = \int_{\inf u}^{\sup u} f(t)g(t)\,\mathrm{d}t, \tag{12.5}$$

*where both sides can take the value $+\infty$, and $g = g_u$ is the coarea function given by (12.2). Assume further that $Lu \in \ell^1(V, m)$, and*

*(a) $u$ is almost proper and non-constant on $V \cup \partial V$,*

*(b) $\sup_{x,y\in V, x\sim y}(u(x) - u(y)) < \sup_{x\in V} u(x) - \inf_{x\in V} u(x)$ in the case of infinite $V$,*

*(c) $u$ takes its maximum, or there exists $S > 0$ such that for all $x \in V$ with $u(x) > S$, we have $Lu(x) \leq 0$, and*

*(d) $u$ takes its minimum, or there exists $I > 0$ such that for all $x \in V$ with $u(x) < I$, we have $Lu(x) \geq 0$.*

*Then, $g \asymp 1$.*

**Remark 12.11** Let $u$ and $f$ be as in Proposition 12.10 with $f$ being continuous, and assume $u(x) \neq u(y)$ for all $x \sim y$ with either $x \in V$ or $y \in V$. Then, we can use the mean value theorem and get that there is $\theta_{x,y} \in (u(x) \wedge u(y), u(x) \vee u(y))$ such that

$$f(\theta_{x,y}) = \frac{\int_{u(y)}^{u(x)} f(t)\,\mathrm{d}t}{u(x) - u(y)}.$$

Thus, the coarea formula can be reformulated as

$$\frac{1}{2} \sum_{x,y\in X} b(x,y)\,|\nabla_{x,y} u|^p\, f(\theta_{x,y}) = \int_{\inf u}^{\sup u} f(t) g(t)\,\mathrm{d}t.$$

*Proof (of Proposition 12.10).* Let $t > 0$. As in the previous lemma, Lemma 12.9, we define $X_t = \{x \in X : u(x) > t\}$. Let $1_{x,y}$ be the characteristic function of the interval

$$I_{x,y} = (u(x) \wedge u(y), u(x) \vee u(y)].$$

Observe that $(x, y)$ or $(y, x)$ are in $\tilde{\partial} X_t = X_t \times X \setminus X_t$ if and only if $t \in I_{x,y}$. Using this and Tonelli's theorem, we derive

$$\begin{aligned}
\sum_{x,y\in X} b(x,y)\,(\nabla_{x,y} u)^{\langle p-1\rangle} \int_{u(y)}^{u(x)} f(t)\,\mathrm{d}t \\
&= \sum_{x,y\in X} b(x,y)\,|\nabla_{x,y} u|^{p-1} \int_{\inf u}^{\sup u} f(t) 1_{x,y}(t)\,\mathrm{d}t \\
&= \int_{\inf u}^{\sup u} f(t) \sum_{x,y\in X} b(x,y)\,|\nabla_{x,y} u|^{p-1}\, 1_{x,y}(t)\,\mathrm{d}t \\
&= 2\int_{\inf u}^{\sup u} f(t) \sum_{(x,y)\in\tilde{\partial} X_t} b(x,y)\,|\nabla_{x,y} u|^{p-1}\,\mathrm{d}t \\
&= 2\int_{\inf u}^{\sup u} f(t) \sum_{(x,y)\in\tilde{\partial} X_t} b(x,y)\,(\nabla_{x,y} u)^{\langle p-1\rangle}\,\mathrm{d}t
\end{aligned}$$

since $u(x) \geq u(y)$ for $(x, y) \in \tilde{\partial} X_t$. This shows the first part of the theorem. The second part follows from Lemma 12.9. $\square$

## 12.3 Proof of Theorem 12.1

### 12.3.1 Criticality

We start with an auxiliary lemma. The lemma introduces a cut-off function which has its origin probably in [PS05] where it was successfully used for the $p$-Laplacian on $\mathbb{R}^d$. In [DP16], it was used to show criticality of $p$-Schrödinger operators in the continuum, and in [KPP18b] the same function was used to show criticality for linear Schrödinger operators on graphs. However, this cut-off function is one particular choice but others are possible as well to prove the following main results, take e.g. partially linear functions.

Let $n \in \mathbb{N}$ and define the cut-off function $\psi_n \colon [0,\infty) \to [0,1]$ via

$$\psi_n(t) := \left(2 + \frac{\log(t)}{\log n}\right) 1_{[\frac{1}{n^2},\frac{1}{n}]}(t) + 1_{[\frac{1}{n},n]}(t) + \left(2 - \frac{\log(t)}{\log n}\right) 1_{[n,n^2]}(t). \tag{12.6}$$

Clearly, $\psi_n \nearrow 1$ pointwise as $n \to \infty$. Moreover, we have the following estimate.

**Lemma 12.12** *Let $0 < \beta < \alpha < \infty$, and $\psi_n$ be the cut-off function defined in (12.6), $n \in \mathbb{N}$. Then,*

$$\frac{\left|\nabla_{\alpha,\beta}\psi_n\right|^p}{(\alpha-\beta)^{p-1}} \leq \frac{\int_\beta^\alpha t^{p-1}\left|\psi_n'(t)\right|^p \mathrm{d}t}{\beta^{p-1}} \tag{12.7}$$

*and there is a positive constant $C$ such that for all $q > 1$,*

$$\frac{\left(\alpha^{1/q}-\beta^{1/q}\right)^p\left(\frac{1}{2}(|\psi_n(\alpha)|+|\psi_n(\beta)|)\right)^p}{(\alpha-\beta)^{p-1}} \leq C\alpha^{p/q-p+1}\int_\beta^\alpha \frac{|\psi_n(t)|^p}{t}\mathrm{d}t. \tag{12.8}$$

*Proof.* The inequalities are clearly satisfied if $\alpha \leq 1/n^2$ or $\beta \geq n^2$, since the left-hand sides and the right-hand sides vanish then. Thus, we can assume in the following that $\alpha > 1/n^2$ and $\beta < n^2$.

Ad (12.7): Firstly, we briefly show that

$$\nabla_{\alpha,\beta}\psi_n \leq \frac{\nabla_{\alpha,\beta}\log}{\log n}.$$

This can easily be obtained by a case analysis. Note that

$$\psi_n(t) = \frac{\log(n^2t)}{\log n}1_{[\frac{1}{n^2},\frac{1}{n}]}(t) + 1_{[\frac{1}{n},n]}(t) + \frac{\log(n^2/t)}{\log n}1_{[n,n^2]}(t), \qquad t \in \mathbb{R}.$$

The cases $\alpha,\beta \in (0,n]$ and $\alpha,\beta \in [1/n,\infty)$ can also be obtained from [KPP20a]. The remaining cases follow immediately from the formula above.

Secondly, we show that

$$\frac{\nabla_{\alpha,\beta}\psi_n}{\int_\beta^\alpha t^{p-1}\left|\psi_n'(t)\right|^p \mathrm{d}t} \leq \log^{p-1}(n).$$

Indeed, by the fundamental theorem of calculus, we obtain

$$\frac{\nabla_{\alpha,\beta}\psi_n}{\int_\beta^\alpha t^{p-1}\left|\psi_n'(t)\right|^p \mathrm{d}t} = \frac{\int_\beta^\alpha \psi_n'(t)\mathrm{d}t}{\int_\beta^\alpha t^{p-1}\left|\psi_n'(t)\right|^p \mathrm{d}t} = \log^{p-1}(n)\frac{\int_{\beta\vee 1/n^2}^{\alpha\wedge 1/n} 1/t\,\mathrm{d}t - \int_{\beta\vee n}^{\alpha\wedge n^2} 1/t\,\mathrm{d}t}{\int_{\beta\vee 1/n^2}^{\alpha\wedge 1/n} 1/t\,\mathrm{d}t + \int_{\beta\vee n}^{\alpha\wedge n^2} 1/t\,\mathrm{d}t}$$
$$\leq \log^{p-1}(n).$$

Using the first two results together with

$$\frac{\nabla_{\alpha,\beta}\log}{\alpha-\beta} \leq \log'(\beta) = \frac{1}{\beta},$$

we derive at

$$\frac{\left|\nabla_{\alpha,\beta}\psi_n\right|^p}{(\alpha-\beta)^{p-1}\int_\beta^\alpha t^{p-1}\left|\psi_n'(t)\right|^p \mathrm{d}t} = \frac{\left|\nabla_{\alpha,\beta}\psi_n\right|^{p-2}\nabla_{\alpha,\beta}\psi_n}{\left|\alpha-\beta\right|^{p-2}(\alpha-\beta)} \cdot \frac{\nabla_{\alpha,\beta}\psi_n}{\int_\beta^\alpha t^{p-1}\left|\psi_n'(t)\right|^p \mathrm{d}t}$$
$$\leq \frac{\left|\nabla_{\alpha,\beta}\log\right|^{p-2}\nabla_{\alpha,\beta}\log}{\log^{p-1}(n)\left|\alpha-\beta\right|^{p-2}(\alpha-\beta)} \cdot \log^{p-1}(n)$$
$$\leq \frac{1}{\beta^{p-1}}.$$

This proves the first inequality.

Ad (12.8): By substituting $t=\beta/\alpha\leq 1$, and since $t\mapsto(1-t^{1/q})/(1-t)$ is strictly monotonously decreasing as $t\geq 0$ increases, we have

$$\frac{(\alpha^{1/q}-\beta^{1/q})^p}{(\alpha-\beta)^p} = \alpha^{p/q-p}\cdot\left(\frac{1-t^{1/q}}{1-t}\right)^p \leq \alpha^{p/q-p}.$$

Moreover, it is not difficult to see that there is a positive constant $C$ such that

$$\left(\frac{1}{2}(|\psi_n(\alpha)|+|\psi_n(\beta)|)\right)^p \leq C\int_\beta^\alpha t^{-1}\left|\psi_n(t)\right|^p \mathrm{d}t.$$

The key to the above inequality is that the left-hand side is always smaller than 1, and the right-hand side is equivalent to $\log(n)$ in the case of $\alpha>1/n^2$ and $\beta<n^2$, and to zero elsewhere. In the latter case also the left-hand side vanishes.

Using $(\alpha-\beta)\leq\alpha$ (and $\alpha>1/n^2$, $\beta<n^2$, $0<\beta<\alpha$), we derive at

$$\frac{\left(\alpha^{1/q}-\beta^{1/q}\right)^p\left(\frac{1}{2}(|\psi_n(\alpha)|+|\psi_n(\beta)|)\right)^p}{(\alpha-\beta)^{p-1}\int_\beta^\alpha t^{-1}\left|\psi_n(t)\right|^p \mathrm{d}t} \leq C_p\alpha^{p/q-p}(\alpha-\beta) \leq C_p\alpha^{p/q-p+1}.$$

This proves the second inequality. $\square$

Using the latter lemma and the coarea formula, we will show next that certain $p$-superharmonic functions will give us weights $w$ such that the resulting $p$-energy functional $h-w_p$ is critical. Recall that $F(V)\cap C(V)=C(V)$ if the graph is locally finite on $V\subseteq X$.

**Proposition 12.13 (Criticality)** *Let $V \subseteq X$ be connected and non-empty such that $(V, b|_{V\times V})$ is locally finite on $V$, and let $c$ be an arbitrary potential. Define $H := H_{b,c,p,m}$ with corresponding subcritical $p$-energy functional $h$ and $p$-Laplacian $L$. Suppose that $u \in F(V) \cap C(V)$ is a positive function on $V$, proper and of bounded oscillation on $V$ with $\tilde{H}u \geq 0$ on $V$, where $\tilde{H} := H_{b,q^{p-1}\cdot c,p,m}$, and $q := p/(p-1)$. Furthermore, assume that $Lu \in \ell^1(V, m)$, and*

- *(a) $u$ takes its maximum on $V$, or there exists $S > 0$ such that for all $x \in V$ with $u(x) > S$ we have $Lu(x) \leq 0$, and*
- *(b) $u$ takes its minimum on $V$, or there exists $I > 0$ such that for all $x \in V$ with $u(x) < I$ we have $Lu(x) \geq 0$.*

*Then $h - (wm)_p$ is critical in $V$, where $w := H(u^{1/q})/u^{(p-1)/q}$.*

*Proof.* By the Harnack inequality, Lemma 5.1, $u > 0$ on $V$. We set $v := u^{1/q}$. Because of Proposition 12.5 (a), $v$ is strictly $p$-superharmonic with respect to $H$ on $V$. Furthermore, by the definition of $w$, the function $v$ is a positive $p$-harmonic function with respect to $H - w$ in $V$, i.e., $Hv = wv^{p-1}$ on $V$.

The strategy of the proof is to construct a null-sequence on $V$ with respect to $h - (wm)_p$ which converges pointwise to $u$. By Theorem 10.1, this then implies that $h - (wm)_p$ is critical on $V$.

We take the cut-off function $\psi_n \colon [0, \infty) \to [0, 1]$ defined in (12.6) and set $e_n \in C(V)$ via

$$e_n = \psi_n \circ v \text{ on } V.$$

If $V$ is finite then, clearly $e_n \in C_c(V)$. If $V$ is infinite, we also have $e_n \in C_c(V)$, since $\operatorname{supp}(\psi_n) \subseteq (0, \infty)$, and $\sup_V u = \infty$ or $\inf_V u = 0$ by the properness assumption on $u$. Obviously, $e_n \nearrow 1$ pointwise on $V$ as $n \to \infty$. So, we are left to show $(h - (wm)_p)(ve_n) \to 0$ as $n \to \infty$.

Using (4.9), we get for some positive constant $C_p$,

$$(h - (wm)_p)(ve_n) \leq C_p \cdot \begin{cases} h_{v,1}(e_n), & 1 < p \leq 2, \\ h_{v,1}(e_n) + \left(\frac{h_{v,1}(e_n)}{h_{v,3}(e_n)}\right)^{2/p} h_{v,3}(e_n), & p > 2. \end{cases}$$

We will show that the right-hand sides vanish as $n \to \infty$.

We compute

$$\begin{aligned} h_{v,1}(e_n) &= \sum_{x,y\in X} b(x,y)(u(x)u(y))^{p/2q} \left|\nabla_{x,y} e_n\right|^p \\ &= \sum_{x,y\in X} b(x,y) \left|\nabla_{x,y} u\right|^{p-2} (\nabla_{x,y} u) a_n(x,y) \left( \int_{u(y)}^{u(x)} t^{p-1} \left|\varphi_n'(t)\right|^p \mathrm{d}t \right), \end{aligned}$$

where

$$a_n(x,y) := \frac{\big(u(x)u(y)\big)^{(p-1)/2} \left|\nabla_{x,y} e_n\right|^p}{\left|\nabla_{x,y} u\right|^{p-2} (\nabla_{x,y} u) \int_{u(y)}^{u(x)} t^{p-1} \left|\psi_n'(t)\right|^p \, \mathrm{d}t}$$

whenever the denominator does not vanish and $a_n(x,y) = 0$ otherwise.

Using (12.7), we obtain (assuming without loss of generality that $u(x) \geq u(y)$, otherwise we use a symmetry argument)

$$a_n(x,y) \leq \frac{(u(x)u(y))^{(p-1)/2}}{u^{p-1}(y)} \leq \sup_{x \sim y} \left( \frac{u(x)}{u(y)} \right)^{(p-1)/2} =: C_0 < \infty,$$

since $u$ is of bounded oscillation on $V$.

We use this estimate and apply the coarea formula with $f(t) = t^{p-1} \left|\psi_n'(t)\right|^p$. Note that the assumptions of the corresponding proposition, Proposition 12.10, are fulfilled. Therefore, there exist positive constants $C_1$ and $C_2$ such that for all $n \in \mathbb{N}$

$$\begin{aligned} h_{v,1}(e_n) &\leq C_0 \sum_{x,y \in X} b(x,y) \left|\nabla_{x,y} u\right|^{p-2} (\nabla_{x,y} u) \left( \int_{u(y)}^{u(x)} t^{p-1} \left|\psi_n'(t)\right|^p \, \mathrm{d}t \right) \\ &\leq C_1 \int_{\inf u}^{\sup u} t^{p-1} \left|\psi_n'(t)\right|^p \, \mathrm{d}t \\ &\leq C_2 \left( \frac{1}{\log n} \right)^p \left( \int_{\frac{1}{n^2}}^{\frac{1}{n}} \frac{\mathrm{d}t}{t} + \int_n^{n^2} \frac{\mathrm{d}t}{t} \right) = \frac{2C_2}{\log^{p-1} n}. \end{aligned}$$

The term on the right-hand side tends to $0$ as $n \to \infty$. Thus, for $1 < p \leq 2$, $(ve_n)$ is indeed a null-sequence for $(h-(wm)_p)_v$, which implies by the ground state representation formula the criticality of the functional $h - w_p$.

We are left to analyse the case $p \geq 2$. Here we calculate

$$h_{v,3}(e_n) = \sum_{x,y \in X} b(x,y) \left|\nabla_{x,y} u\right|^{p-2} (\nabla_{x,y} u) \tilde{a}(x,y) \left( \int_{u(y)}^{u(x)} t^{-1} \left|\varphi_n(t)\right|^p \, \mathrm{d}t \right),$$

where

$$\tilde{a}_n(x,y) := \frac{\left|\nabla_{x,y} u^{1/q}\right|^p \left(\frac{|e_n(x)|+|e_n(y)|}{2}\right)^p}{\left|\nabla_{x,y} u\right|^{p-2} \nabla_{x,y} u \int_{u(y)}^{u(x)} t^{-1} \left|\psi_n(t)\right|^p \mathrm{d}t}$$

whenever the denominator is non-zero and $\tilde{a}_n(x,y) = 0$ otherwise.

Using (12.8) with $q = p/(p-1)$, we obtain (assuming without loss of generality that $u(x) \geq u(y)$, otherwise we use a symmetry argument) that there is a positive constant $C_3$ such that

$$\tilde{a}_n(x,y) \leq C_3 u^0(x) = C_3 < \infty.$$

Then, we can use again the coarea formula but this time with $f(t) = t^{-1} |\psi_n(t)|^p$. Thus, there exist positive constants $C_2, C_3, C_4$ such that

$$\begin{aligned}
&h_{v,1}(e_n) + (h_{v,1}(e_n))^{\frac{2}{p}} (h_{v,3}(e_n))^{\frac{p-2}{p}} \\
&\leq \frac{2C_2}{\log^{p-1} n} \\
&+ \left(\frac{2C_2}{\log^{p-1} n}\right)^{\frac{2}{p}} \left( C_3 \sum_{x,y\in X} b(x,y) |\nabla_{x,y} u|^{p-2} \nabla_{x,y} u \left( \int_{u(y)}^{u(x)} t^{-1} |\psi_n(t)|^p \, \mathrm{d}t \right)\right)^{\frac{p-2}{p}} \\
&= \frac{2C_2}{\log^{p-1} n} + \left(\frac{2C_2}{\log^{p-1} n}\right)^{\frac{2}{p}} \left( C_3 \int_{\inf u}^{\sup u} t^{-1} |\psi_n(t)|^p \, \mathrm{d}t \right)^{\frac{p-2}{p}}.
\end{aligned}$$

Since

$$\int_{\inf u}^{\sup u} t^{-1} |\psi_n(t)|^p \, \mathrm{d}t \asymp \int_{1/n}^{n} t^{-1} \mathrm{d}t \asymp \log n,$$

we conclude,

$$\ldots \leq \frac{2C_2}{\log^{p-1} n} + C_4 \frac{\log^{(p-2)/p} n}{\log^{(p-1)\cdot 2/p} n} = \frac{2C_2}{\log^{p-1} n} + \frac{C_4}{\log^{(p+1)/p} n} \to 0, \quad n \to \infty.$$

This shows the statement. □

### 12.3.2 Null-criticality

In the present subsection we prove the null-criticality of $h - w_p$ for a specific $w$ under some familiar constraints, confer [KPP18b] for $p = 2$.

We need the following elementary inequality.

**Lemma 12.14** *For all $p, q \geq 1$ and $0 \leq \beta < \alpha < \infty$, we have*

$$\frac{(\alpha^{1/q} - \beta^{1/q})^p}{(\alpha-\beta)^{p-1} \int_\beta^\alpha 1/t \, \mathrm{d}t} \geq \frac{\alpha^{p/q-p}\beta}{q^p}.$$

*Proof.* If $\beta = 0$ there is nothing to prove. Thus, assume $\beta > 0$ in the following. Because of

$$\frac{\log \alpha - \log \beta}{\alpha - \beta} \leq \log'(\beta) = \frac{1}{\beta},$$

we have

$$\frac{\alpha - \beta}{\int_\beta^\alpha 1/t \, \mathrm{d}t} \geq \beta.$$

Moreover, substituting $t = \beta/\alpha \leq 1$ yields

$$\frac{(\alpha^{1/q} - \beta^{1/q})^p}{(\alpha-\beta)^p} = \alpha^{p/q-p} \cdot \left(\frac{1 - t^{1/q}}{1-t}\right)^p.$$

Let $g(t) = (1 - t^{1/q})/(1 - t)$. Then it is easy to see that $g$ is strictly monotonously decreasing for all $t \geq 0$. Moreover, by L'Hôpital's rule it follows that

$$\min_{t\in[0,1]} g(t) = \lim_{t\to 1} g(t) = 1/q.$$

Combining this with the inequalities before yields the result. □

Now we can show the main result of this subsection.

**Proposition 12.15 (Null-criticality)** *Let $V \subseteq X$ be connected and infinite such that $(V, b|_{V\times V})$ is locally finite on $V$. Define $H := H_{b,c,p,m}$ with corresponding subcritical $p$-energy functional $h$ and $p$-Laplacian $L$. Suppose that $0 \lneq u \in F(V) \cap C(V)$ is a proper function of bounded oscillation in $V$ with $\tilde{H}u \geq 0$ on $V$, where $\tilde{H} := H_{b,q^{p-1}\cdot c,p,m}$ and $q := p/(p-1)$. Furthermore, assume that we have $Lu \in \ell^1(V, m)$, $u \in \ell^{p-1}(V, c_-)$, and*

*(a) $u$ takes its maximum on $V$, or there exists $S > 0$ such that for all $x \in V$ with $u(x) > S$ we have $Lu(x) \leq 0$, and*

*(b) $u$ takes its minimum on $V$, or there exists $I > 0$ such that for all $x \in V$ with $u(x) < I$ we have $Lu(x) \geq 0$.*

*Then, the $p$-energy functional $h - (wm)_p$ with $w := H(u^{1/q})/(u^{(p-1)/q})$, is null-critical in $V$ with respect to $wm$.*

*Proof.* By Proposition 12.13, we know that $h - (wm)_p$ is critical on $V$ with Agmon ground state $u^{1/q}$. We have to show that $(wm)_p(u^{1/q}) = \infty$.

Moreover, by Theorem 10.1 (vii), there is a null sequence $(e_n^{1/q})$ to $h - (wm)_p$ on $V$ which converges pointwise and monotone increasing to $u^{1/q}$. Hence, we have $h(e_n^{1/q}) - (wm)_p(e_n^{1/q}) \to 0$ as $n \to \infty$. If we can show that $h(e_n^{1/q}) \to \infty$, then also $(wm)_p(e_n^{1/q}) \to \infty$. Since $0 \leq e_n^{1/q} \leq u^{1/q}$, this would imply $(wm)_p(u^{1/q}) = \infty$.

If $c(x) \geq 0$ for some $x \in V$, then the potential part at $x$ of the $p$-energy functional can be bounded from below by 0. Because of $u \in \ell^{p-1}(V, c_-)$ and $0 \leq e_n^{1/q} \leq u^{1/q}$, the potential of the negative part of $c$ remains finite for all $n \in \mathbb{N}$. Altogether, we only have to consider the divergence part. Denote $K_n := \operatorname{supp} e_n \in C_c(V)$. We have

$$\sum_{x,y\in X} b(x, y) \left|\nabla_{x,y} e_n^{1/q}\right|^p = \sum_{x,y\in K_n\cup\partial K_n} b(x, y) \left|\nabla_{x,y} e_n^{1/q}\right|^p .$$

By Lemma 12.14, we have whenever $e_n(x) \neq e_n(y)$ for $x \in K_n$ with $x \sim y$

$$a(x, y) := \frac{\left|\nabla_{x,y} e_n^{1/q}\right|^p}{\left|\nabla_{x,y} e_n\right|^{p-2} (\nabla_{x,y} e_n) \int_{e_n(y)}^{e_n(x)} 1/t \,\mathrm{d}t} \geq \frac{1}{q^p} \inf_{x\in K_n, x\sim y} \frac{e_n(y)}{e_n(x)} =: C_n.$$

Otherwise, we estimate $a$ by 0 from below. Since $K_n$ is finite, $e_n$ is almost proper on $V$, and non-constant on $K_n \cup \partial K_n$. We apply the coarea formula (Proposition 12.10) with $f(t) = 1/t$. Thus, we get

$$\begin{aligned}\sum_{x,y\in K_n\cup\partial K_n} b(x,y)(\nabla_{x,y} e_n^{1/q})^p &= \sum_{x,y\in K_n\cup\partial K_n} b(x,y)a(x,y)\,(\nabla_{x,y} e_n)^{\langle p-1\rangle} \int_{e_n(y)}^{e_n(x)} 1/t \,\mathrm{d}t \\ &\geq C_n \sum_{x,y\in V} b(x,y)\,(\nabla_{x,y} e_n)^{\langle p-1\rangle} \int_{e_n(y)}^{e_n(x)} 1/t \,\mathrm{d}t \\ &\geq \tilde{C}_n \int_{\min_{K_n} e_n}^{\max_{K_n} e_n} 1/t \,\mathrm{d}t,\end{aligned}$$

where $\tilde{C}_n$ is a positive constant. Since $u$ is of bounded oscillation, $\lim_{n\to\infty} \tilde{C}_n \in (0,\infty)$. By the properness of $u$ and since $V$ is infinite, we have that 0 or $\infty$ are accumulation points of $u$ in $V$, and therefore $\int_{\inf_V u}^{\sup_V u} 1/t \,\mathrm{d}t = \infty$. Consequently, $(wm)_p(u^{1/q}) = \infty$. □

### 12.3.3 Optimality Near Infinity

Here, we finally prove Theorem 12.1. Because of Proposition 12.13 and also of Proposition 12.15, we are only left to show the optimality near infinity. This, however, is a consequence of Theorem 11.2.

*Proof (of Theorem 12.1).* Proposition 12.13 and Proposition 12.15 imply that the function $w := H(u^{1/q})/u^{(p-1)/q}$, $q := p/(p-1)$, multiplied with $m$ is a $p$-Hardy weight such that $h - (wm)_p$ is null-critical with Agmon ground state $u^{1/q}$.

Furthermore, since $u$ is proper and of bounded oscillation, also $u^{1/q}$ is it. Thus, we can apply Theorem 11.2 and get that $wm$ is indeed an optimal $p$-Hardy weight. □

## 12.4 Examples of Optimal Hardy Weights

**Example 12.16 ($\mathbb{N}$)** By Corollary 12.2, we see that the Hardy weight $w$ on $\mathbb{N}$ obtained in Example 10.10 and given by

$$\begin{aligned}w(n) := \left(1-\left(1-\frac{1}{n}\right)^{\frac{p-1}{p}}\right)^{p-1} - \left(\left(1+\frac{1}{n}\right)^{\frac{p-1}{p}} - 1\right)^{p-1} \\ > \left(\frac{p-1}{p}\right)^p \frac{1}{n^p} =: w^H(n),\end{aligned}$$

is not only an improvement of the original $p$-Hardy weight $w^H$ but also optimal. It is shown in [FKP23] that $w^H$ is the leading term of a Taylor expansion of $w$. By Versano's lemma, Lemma 12.4, we also get an optimal $p$-Hardy weight for the generalised harmonic oscillator for free. Moreover, if the potential of the generalised hydrogen atom is close enough in terms of Lemma 12.4, we also get an optimal $p$-Hardy weight in this case.

On $\mathbb{R}^d$, $p \neq d$, with the free $p$-Laplacian, the classical Hardy weight $W(x) = ((p-1)/(p\,|x|))^p$ is an optimal $p$-Hardy weight on $\mathbb{R}^d \setminus \{0\}$, see [DP16, p. 4] which follows by taking simply the Green's function on $\mathbb{R}^d$ as reference function. Hence, for $d = 1$, we have an significant difference between the continuous and discrete model in terms of the $p$-Hardy weights. The optimal $p$-Hardy weight on $(0, \infty)$ is only the first term of the Taylor expansion of the optimal $p$-Hardy weight on $\mathbb{N}$.

**Example 12.17 ($\mathbb{T}_{d+1}$, $d \geq 2$)** Recall the Green's function from Example 9.8 of the homogeneous regular tree $\mathbb{T}_{d+1}$. Here, we have $G_o(r) = C_p \cdot d^{-r/(p-1)}$, $r \geq 0$ for some constant $C_p > 0$. The Green's function $G_o$ is proper and of bounded oscillation. Obviously, also the remaining conditions in Corollary 12.2 are fulfilled. Hence, an optimal $p$-Hardy weight is given by

$$w(r) = \begin{cases} (d+1)(1-d^{-1/p})^{p-1}, & r = 0 \\ d(1-d^{-1/p})^{p-1} - (d^{1/p}-1)^{p-1}, & r > 0. \end{cases}$$

Note that for $p = 2$, this is the result obtained in [BSV21, Eq. (1.3)], confer also with [KPP20a, Example 9.3.11]. Moreover, this optimal weight is constant for $r \geq 1$ and does not converge to zero. In the case of $p = 2$, $w(r)$ for $r \geq 1$ is exactly the bottom of the $\ell^2$-spectrum of the free Laplacian on $\mathbb{T}_{d+1}$.

Real hyperbolic spaces $\mathbb{H}(\mathbb{R}^d)$ are often considered to be a counterpart in the continuum to homogeneous trees. Here, similar results were obtained, see [Ber+17a; BGG17] (or for the closely connected Damek-Ricci spaces see [FP23]). However, using the analogue method from [DP16], one can show that the weight in the continuum is larger than a similar positive constant and converges exponentially fast to it. Hence, this is somehow the reversed observation than between $\mathbb{N}$ and $(0, \infty)$.

Moreover, it would be nice to obtain a $p$-Hardy weight on $\mathbb{T}_{d+1}$ which converges to $d(1-d^{-1/p})^{p-1} - (d^{1/p}-1)^{p-1}$ slower than in the example before (which would necessarily mean that the new weight is smaller at $o$). An approach for $p = 2$ can be found in [BSV21]. We believe that the main idea can be generalised to $p \in (1, \infty)$ but it remains a subject of current research.

The calculation on $\mathbb{T}_{d+1}$ can be generalised easily to all subcritical model graphs. This is done next.

**Example 12.18 (Model graphs)** Recall the $p$-Laplacian of locally finite model graphs from Example 3.6, and the Green's function $G_o$ from Example 9.7, if it exists. If this is the case, Theorem 12.1 can be applied. Setting $g = G_o^{(p-1)/p}$, we calculate

$$w(r) := \frac{Lg(r)}{g^{p-1}(r)} = \begin{cases} k_+(0)\left(1 - \frac{g(1)}{g(0)}\right)^{p-1} & r = 0, \\ k_+(r)\left(1 - \frac{g(r+1)}{g(r)}\right)^{p-1} - k_-(r)\left(\frac{g(r-1)}{g(r)} - 1\right)^{p-1}, & r \geq 1. \end{cases}$$

Hence, the function $wm$ is an optimal $p$-Hardy weight.

**Example 12.19 (Free $p$-Laplacian on $\mathbb{Z}^d$, $d > p$)** The flows constructed in [Mae77; Pra04] implicitly define functions $u$ which satisfy the assumptions in Corollary 12.2. Thus, an optimal $p$-Hardy weight is given by $w := Lu^{(p-1)/p}/u^{(p-1)^2/p}$.

A downside of the main result of this chapter is that it cannot be applied to non-locally finite graphs as e.g. the star graph. It is natural to ask if the formula also yields optimal weights in this case. Next, we shown an example that this not the case if the formula in Theorem 12.1 is used with Green's functions on star graphs.

**Example 12.20 (Star graphs)** Even though, we cannot apply Theorem 12.1 on a star graph, we can calculate the formula and check by hand if it satisfies all desired properties. Recall the star graph from Example 3.9. Let us add any potential $c$ such that $h$ is subcritical (e.g. take a positive potential $c \gneq 0$). We also calculated the corresponding normalised Green's function $G_0$ implicitly in Example 9.11. Recall that $HG_0 = 0$ and $G_0 > 0$ on $\mathbb{N}$ implies

$$c(k) = b(k,0)\left(\frac{G_0(0)}{G_0(k)} - 1\right)^{\langle p-1\rangle}, \qquad k \in \mathbb{N}. \tag{12.9}$$

Hence, by defining $c$ and $b$ properly, we get a proper Green's function of bounded oscillation (take e.g. $c(k) = 1/k^3$, $b(k,0) = 1/k^2$ for $k \in \mathbb{N}$). Furthermore, by the equality above and since $G_0 \in F$, we get $G_0 \in \ell^{p-1}(X, |c|)$ which, in turn, implies $LG_0 \in \ell^1(X,m)$. Moreover, for $k \geq 1$,

$$HG_0^{(p-1)/p}(k) = \frac{G_0^{(p-1)^2/p}(k)}{m(k)}\left(b(k,0)\left(1 - \left(\frac{G_0(0)}{G_0(k)}\right)^{(p-1)/p}\right)^{\langle p-1\rangle} + c(k)\right).$$

Hence, our candidate for a weight $w$ is defined on $\mathbb{N}$ via

$$\begin{aligned} w(k) &:= \frac{HG_0^{(p-1)/p}(k)}{G_0^{(p-1)^2/p}(k)} = \frac{1}{m(k)}\left(b(0,k)\left(1 - \frac{G_0^{(p-1)/p}(0)}{G_0^{(p-1)/p}(k)}\right)^{\langle p-1\rangle} + c(k)\right) \\ &= \frac{b(0,k)}{m(k)}\left(\left(1 - \frac{G_0^{(p-1)/p}(0)}{G_0^{(p-1)/p}(k)}\right)^{\langle p-1\rangle} + \left(\frac{G_0(0)}{G_0(k)} - 1\right)^{\langle p-1\rangle}\right). \end{aligned}$$

Clearly, $w > 0$ on $\mathbb{N}$. Note that this defines also the value at $0$ implicitly.

The natural candidate for a ground state of $h - (wm)_p$ is $u = G_0^{(p-1)/p}$. Recall that $u \in F$ by Lemma 2.2. Let us set $e_n = 1_{[0,n]} \cdot u$ for $n \in \mathbb{N}$. Then $0 \leq e_n \nearrow u$ as $n \to \infty$, and $(e_n)$ is a good candidate for a null-sequence. Note that $Hu(k) = He_n(k) = w(k)e_n^{p-1}(k) = w(k)u^{p-1}(k)$ for all $0 < k \leq n$. Hence,

$$\begin{aligned} (h - (wm)_p)(e_n) &= \langle He_n, e_n\rangle_{\mathbb{N}_0} - (wm)_p(e_n) \\ &= (He_n(0) - Hu(0))u(0)m(0) \\ &= u^p(0)\sum_{k=n+1}^{\infty} b(0,k)\left(1 - \left(1 - \frac{u(k)}{u(0)}\right)^{\langle p-1\rangle}\right). \end{aligned}$$

Let us assume that $c \gneq 0$. By Equation (12.9), we get that $u(0) \geq u(k)$ for all $k \in \mathbb{N}$. Moreover, by the local summability of the graph, we get

$$\sum_{k=n+1}^{\infty} b(0,k) \left(1 - \left(1 - \frac{u(k)}{u(0)}\right)^{p-1}\right) \leq \sum_{k=n+1}^{\infty} b(0,k) < \infty.$$

Since $1 - (1 - u(k)/u(0))^{p-1} \geq 0$, we can use the theorem of dominated convergence, and get $(h - (wm)_p)(e_n) \to 0$ as $n \to \infty$. Hence, $h - (wm)_p$ is critical with Agmon ground state $u$.

Let us turn to the null-criticality: if we show that $u \notin \ell^p(\mathbb{N}_0, wm)$, then $h - (wm)_p$ is null-critical with respect to $wm$. Since $e_n \nearrow u$, we have $(wm)_p(e_n) \nearrow (wm)_p(u)$. Since $(e_n)$ is a null sequence of $h - (wm)_p$, it suffices to have a look at $h(e_n)$. Here, we calculate

$$\begin{aligned} h(e_n) = \langle He_n, e_n \rangle_{\mathbb{N}_0} &= \sum_{k=0}^{n} He_n(k) e_n(k) m(k) \\ &= \sum_{k=0}^{n} b(0,k) \left|\nabla_{0,k} u\right|^p + \sum_{k=n+1}^{\infty} b(0,k) u(0)^p + \sum_{k=0}^{n} c(k) u^p(k). \end{aligned}$$

By Example 9.11, the third sum converges monotonously to $m(0) \in (0, \infty)$. By Lemma 2.2, we have $u \in F = F_{b,p}$ since $G_0 \in F_{b,p}$. Moreover, by the same lemma, $u \in F_{b,p+1}$, which implies that the first sum remains finite as $n \to \infty$. Furthermore, since the graph is locally summable also the second sum stays finite. Hence, $u \in \ell^p(\mathbb{N}_0, wm)$, and the corresponding functional is positive-critical.

Therefore, the formula of Theorem 12.1 applied to a Green's function on a star graph does not result in an optimal weight.

In Example 12.3, we have seen an example of a non-proper Green's function of bounded oscillation on $\mathbb{N}$ which does not yield an optimal $p$-Hardy weight. Hence, also the properness seems to be a natural requirement for obtaining optimal weights via the supersolution construction.

**Remark 12.21** There are also other techniques known to obtain Hardy-type inequalities than via the supersolution construction together with the coarea formula and ground state representation formula as presented here. Two promising methods can be found e.g. in [Bog+22; FP23; BSV21] and references therein. So far these are only applied to settings in the continuum or to homogeneous trees. To find the corresponding versions on general graphs are ongoing research projects by the author. See also [Dav99; Hua23a] for reviews of some (other) methods in the continuum.

## 12.5 Applications

Here, we briefly discuss two simple applications of the Hardy inequality, an uncertainty-type principle and a Rellich-type inequality. Both follow easily by using also Hölder's inequality. In the next chapter, Chapter 13, we continue the discussion of Rellich-type inequalities.

### 12.5.1 Heisenberg-Pauli-Weyl-type Inequality

The famous Heisenberg-Pauli-Weyl uncertainty principle is a direct consequence of the Hardy inequality. It asserts, roughly speaking, that the position and momentum of a particle can not be determined simultaneously. For further information confer e.g. [BEL15, Subsection 1.6] for a detailed discussion in the Euclidean space, [KÖ09; KÖ13; Kri18] for Riemannian manifolds, or [Ber+20, Section 3] for a recent version in the hyperbolic space.

Assume that $h$ is subcritical in $X$ with $p$-Hardy weight $w$. Then by the Hölder and Hardy inequality, we derive for all $\varphi \in C_c(\mathrm{supp}(w))$

$$\begin{aligned}\sum_{x\in \mathrm{supp}(w)} |\varphi(x)|^p &= \sum_{x\in \mathrm{supp}(w)} \big(w^{-1/p}(x)\,|\varphi(x)|^{p-1}\big)\big(w^{1/p}(x)\,|\varphi(x)|\big)\\ &\leq \left(\sum_{x\in \mathrm{supp}(w)} w^{-1/(p-1)}(x)\,|\varphi(x)|^p\right)^{(p-1)/p} \left(\sum_{x\in X} w(x)\,|\varphi(x)|^p\right)^{1/p}\\ &\leq \left(\sum_{x\in \mathrm{supp}(w)} w^{-1/(p-1)}(x)\,|\varphi(x)|^p\right)^{(p-1)/p} \cdot h^{1/p}(\varphi).\end{aligned}$$

This is a quasilinear version of the Heisenberg-Pauli-Weyl inequality on graphs.

For the special case of the line graph on $X = \mathbb{N}_0$ discussed in Example 10.10, we thus obtain by taking the optimal Hardy weight

$$w(n) := \left(1-\left(1-\frac{1}{n}\right)^{\frac{p-1}{p}}\right)^{p-1} - \left(\left(1+\frac{1}{n}\right)^{\frac{p-1}{p}}-1\right)^{p-1}, \qquad n\in\mathbb{N},$$

the following sequence of inequalities on $C_c(\mathbb{N})$

$$\begin{aligned}\sum_{n\in\mathbb{N}} |\varphi(n)|^p &\leq \left(\sum_{n\in\mathbb{N}} w^{-1/(p-1)}(n)\,|\varphi(n)|^p\right)^{(p-1)/p} \cdot \left(\sum_{n=1}^{\infty} |\nabla_{n,n-1}\varphi|^p\right)^{1/p}\\ &\leq \frac{p}{p-1}\cdot \left(\sum_{n\in\mathbb{N}} n^{p/(p-1)}\,|\varphi(n)|^p\right)^{(p-1)/p} \cdot \left(\sum_{n=1}^{\infty} |\nabla_{n,n-1}\varphi|^p\right)^{1/p},\end{aligned}$$

where the second inequality follows from the comparison with the classical $p$-Hardy weight $w^H(n) = (1-(1-p))^p \cdot (1/n^p)$, $n \in \mathbb{N}$.

### 12.5.2 A First Rellich-type Inequality

Another inequality which has attracted a lot of attention is the Rellich inequality. For more details on the history and generalisations of this inequality in different settings, we suggest the papers [Ber+20; KPP21; KÖ09; KÖ13] and the monograph [BEL15], and references therein.

Assume that $h$ is subcritical in $X$ with strictly positive $p$-Hardy weight $w$. Then by the Hardy inequality, Green's formula (Lemma 2.7), and the Hölder inequality, we obtain for all $\varphi \in C_c(\operatorname{supp}(w))$

$$\sum_{x\in X} w(x)\,|\varphi(x)|^p \leq h(\varphi) = \langle H\varphi, \varphi\rangle_X = \langle H\varphi w^{-1/p}, \varphi w^{1/p}\rangle_{\operatorname{supp}(w)}$$
$$\leq \Bigg(\sum_{x\in\operatorname{supp}(w)} w^{-1/(p-1)}(x)\,|H\varphi(x)|^{p/(p-1)}\Bigg)^{(p-1)/p} \cdot \|\varphi\|_{p,w}\,.$$

This implies the Rellich-type inequality

$$\sum_{x\in X} w(x)\,|\varphi(x)|^p \leq \sum_{x\in\operatorname{supp}(w)} w^{-1/(p-1)}(x)\,|H\varphi(x)|^{p/(p-1)}\,, \qquad \varphi\in C_c(\operatorname{supp}(w)).$$

We have to admit that classical Rellich inequalities have different powers of the Hardy weight on both sides of the equation. This, however, needs more effort.

For the special case of the line graph on $X = \mathbb{N}_0$ discussed in Example 10.10, we obtain by taking the classical $p$-Hardy weight $w^H$, and optimal $p$-Hardy weight $w$ from the previous subsection,

$$\left(\frac{p-1}{p}\right)^p \sum_{n\in\mathbb{N}} \left(\frac{|\varphi(n)|}{n}\right)^p \leq \sum_{n\in\mathbb{N}} w(n)\,|\varphi(n)|^p$$
$$\leq \sum_{n\in\mathbb{N}} w^{-1/(p-1)}(n)\,|L\varphi(n)|^{p/(p-1)} \leq \left(\frac{p}{p-1}\right)^{1/(p-1)} \sum_{n\in\mathbb{N}} |n\cdot L\varphi(n)|^{p/(p-1)}\,,$$

for all $\varphi\in C_c(\mathbb{N})$, where

$$L\varphi(n) = \sum_{m:|n-m|=1} (\nabla_{n,m}\varphi)^{\langle p-1\rangle}\,, \qquad n\in\mathbb{N}.$$

In the next chapter, we have a closer look on Rellich-type estimates.

# 13. Rellich-Type Inequalities

For my own part I have never once found myself in a position where such scientific knowledge as I possess, outside pure mathematics, has brought me the slightest advantage.
It is indeed rather astonishing how little practical value scientific knowledge has for ordinary men, how dull and commonplace such of it as has value is, and how its value seems almost to vary inversely to its reputed utility.

G. H. Hardy, *A Mathematician's Apology*, p. 117

In 1953, Rellich showed in lectures given at the New York University (which were published posthumously in [Rel69, Theorem II.7.1]) and explained in an ICM talk in 1954 [Rel56, pp. 247-249] that for all smooth and compactly supported functions $\varphi$ which vanish at the origin of $\mathbb{R}^d$, $d \neq 2$, the following is true with sharp constant:

$$\int_{\mathbb{R}^d} |\Delta\varphi(x)|^2 \,\mathrm{d}x \geq \frac{d^2(d-4)^2}{16} \int_{\mathbb{R}^d} \frac{|\varphi(x)|^2}{|x|^4} \mathrm{d}x.$$

Such inequalities were also found in different setting and are usually referred to as *Rellich-type inequalities*, see e.g. [DH98] for linear operators in $L^p$, $p \neq 2$.

In the preceding subsection, Subsection 12.5.2, we achieved a Rellich-type inequality by a simple application of Hölder's and Hardy's inequalities. Also have a look at the references there, for more background information on this inequality. Let us also mention the recent contributions [BGG17; Caz21; DP16; GKS22; Gup23; HY24; Rob18]. We want to highlight that all Rellich-type inequalities we are aware of are associated with *linear* Schrödinger operators or fractional Laplacians.

The goal of this chapter is to give a quasi-linear generalisation of [KPP21]. Note that in the linear case, we have a certain symmetry in Green's formula and an equality in the ground state representation which we lose if $p \neq 2$ and which was used in [KPP21].

The following theorem is the quasi-linear generalisation of [KPP21, Theorems 2.3 and 6.1]. Note that the following specific constant from the ground state representation formula becomes important: In this chapter let us denote by $C_p$ the smallest strictly positive constant such that

$$h(u\varphi) - \langle Hu, u\,|\varphi|^p \rangle_V \leq C_p h_u(\varphi), \qquad \varphi \in C_c(V), 0 \leq u \in F(V), V \subseteq X.$$

Recall from Remark 4.4 that we have explicit bounds for $C_p$ if $p \in (1, 2]$.

**Theorem 13.1 (Rellich-type inequality)** *Let $h$ be subcritical on $V \subseteq X$ with strictly positive $p$-Hardy weight $wm$ on $V$. If there is a function $f > 0$ on $V$ and $\gamma \in (0,1)$ such that for some (all) $0 \leq \varphi \in C_c(V)$, we have the* weak $p$-eikonal inequality

$$C_p\, h_\varphi(f) \leq \gamma(wm)_p(\varphi f).$$

*Then, for this (all)* $0 \leq \varphi \in C_c(V)$ *we have*

$$\sum_{x\in V} |H\varphi(x)1_{\operatorname{supp}\varphi}(x)|^{p/(p-1)} \frac{(f^p m)(x)}{w^{1/(p-1)}(x)} \geq (1-\gamma)^{p/(p-1)} \sum_{x\in V} |(\varphi f)(x)|^p \, (wm)(x).$$

*Proof.* The main idea of this proof is to use the ground state representation formula in an unusual way (see Remark 4.6), then the rest follows naturally. Here are the details:

Applying the alternative ground state representation formula (4.8), we get

$$\langle H\varphi, \varphi f^p\rangle_V \geq h(\varphi f) - C_p h_\varphi(f).$$

By the Hardy inequality and the weak $p$-eikonal inequality, we obtain

$$h(\varphi f) - C_p h_\varphi(f) \geq (1-\gamma) \, \|\varphi f\|_{p,wm}^p$$

On the other side, by Hölder's inequality, we obtain with $q := p/(p-1)$,

$$\begin{aligned} \langle H\varphi, \varphi f^p\rangle_V &= \langle H\varphi 1_{\operatorname{supp}\varphi} f^{p-1} w^{-1/p}, \varphi f w^{1/p}\rangle_V \\ &\leq \|H\varphi 1_{\operatorname{supp}\varphi}\|_{q, mf^p w^{-1/(p-1)}} \cdot \|\varphi f\|_{p,wm}. \end{aligned}$$

Altogether,

$$\|H\varphi 1_{\operatorname{supp}\varphi}\|_{q, mf^p w^{-1/(p-1)}} \geq (1-\gamma) \, \|\varphi f\|_{p,wm}^{p-1},$$

i.e.,

$$\begin{aligned} &\sum_{x\in V} |H\varphi(x)1_{\operatorname{supp}\varphi}(x)|^{p/(p-1)} \frac{m(x) f^p(x)}{w^{1/(p-1)}(x)} \\ &\qquad \geq (1-\gamma)^{p/(p-1)} \sum_{x\in V} |\varphi(x) f(x)|^p \, m(x) w(x). \end{aligned}$$

□

Note that $f = 1$ yields the Rellich-type inequality from Subsection 12.5.2.

An immediate consequence is the following, confer with the linear version in [Rob18] and [KPP21, Corollary 1.3]. Recall that we fixed $C_p$.

**Corollary 13.2 (Robinson-type inequality)** *Let $h$ be subcritical in $V \subseteq X$ with strictly positive $p$-Hardy weight $wm$ on $V$. Set $q := p/(p-1)$. Assume that there exists $\gamma \in (0,1)$ such that for all $0 \leq \varphi \in C_c(V)$, we have*

$$C_p \, h_\varphi(w^{1/q}) \leq \gamma \, \|\varphi\|_{w^p m}^p.$$

*Then, for all* $0 \leq \varphi \in C_c(X)$ *we have*

$$\|H\varphi 1_{\operatorname{supp}\varphi}\|_{q, mw^{q(p-2)}} \geq (1-\gamma) \, \|\varphi\|_{p, w^p m}^{p-1},$$

*Proof.* Take $f = w^{1/q}$ in Theorem 13.1. □

Next, we will discuss a special case of Theorem 13.1 which somehow has a closer connection to the results on manifolds and local operators. However, it excludes many choices of the function $f$, as $f$ needs to be small and without large oscillation in a certain sense. These two restriction arise from the non-locality. On the other hand, we will see afterwards, that if we know somehow simple $p$-Hardy weights, then examples can be deduced easily from it.

For the next result we need the following auxillary lemma. We want to mention that the constant in the lemma is not optimal.

**Lemma 13.3** *For all $p > 2$ and $\alpha, \beta \geq 0$, we have*

$$\alpha\beta\, |\alpha - \beta|^{p-2} \leq 2^{p-3}(\alpha^p + \beta^p).$$

*Proof.* The inequality is fulfilled if $\alpha$ or $\beta$ vanish. Thus, we can assume that $\alpha, \beta > 0$, and we have $|\alpha - \beta| \leq \alpha + \beta$. Dividing by $\alpha^p$, we see that the inequality above holds if

$$f(t) := \frac{t(1+t)^{p-2}}{1+t^p} \leq 2^{p-3}, \qquad t > 0.$$

We show now that $f(1) = 2^{p-3}$ is the maximum. Note that

$$f'(t) = \frac{(1+t)^{p-3}}{(1+t^p)^2}(1 - t^{p+1} + (p-1)(t - t^p)),$$

and thus, $f' > 0$ on $(0,1)$, $f' < 0$ on $(1, \infty)$, and $f'(1) = 0$. Hence, $f$ takes its maximum at 1, and the statement is proven. □

In the next statement, we make use of the following notation: For all $f \in F(V)$, $V \subseteq X$,

$$|\nabla f|_V^p(x) := \frac{1}{2} \sum_{y \in V} b(x,y)\, |\nabla_{x,y} f|^p\,,$$

and for $p > 2$

$$\left|\tilde{\nabla} f\right|_V^p(x) := \frac{1}{2} \sum_{y \in V} b(x,y) \left(\frac{|f(x)| + |f(y)|}{2}\right)^{p-2} |\nabla_{x,y} f|^2\,.$$

By Hölder's inequality, the term $|\nabla f|_V^p$ is finite for all $f \in F(V)$, $V \subseteq X$. Again by Hölder's inequality, the term $\left|\tilde{\nabla} f\right|_V^p$ is finite for $f \in F(V) \cap \ell^p(V, m)$.

The motivation for this notation comes from the counterpart in the continuum. Note that for $c = 0$, we simply have $h(\varphi) = \sum_{x \in X} |\nabla \varphi|_X^p$ for $\varphi \in D$. Recall that $C_p$ is fixed.

**Corollary 13.4** *Let $h$ be subcritical on $V \subseteq X$ with strictly positive $p$-Hardy weight $wm$ on $V$. If there is a function $f > 0$ on $V$ and $\gamma \in (0,1)$ such that $f$ satisfies the two* pointwise $p$-eikonal inequalities *on $V$*

$$C_p\, |\nabla f|_V^p \leq \gamma f^p wm, \qquad \text{on } V \text{ and for } p > 1,$$

*and*

$$C_p \left|\tilde{\nabla} f\right|_V^p \leq \gamma f^p w m, \qquad \text{on } V \text{ and for } p > 2.$$

*Then, for all* $0 \leq \varphi \in C_c(V)$ *we have*

$$\sum_{x \in V} |H\varphi(x) 1_{\operatorname{supp}\varphi}(x)|^{p/(p-1)} \frac{(f^p m)(x)}{w^{1/(p-1)}(x)} \geq (1-\gamma)^{p/(p-1)} \sum_{x \in V} |(\varphi f)(x)|^p \,(wm)(x).$$

*Proof.* By the classical Young inequality, we have $\varphi(x)\varphi(y) \leq 1/2(\varphi^2(x) + \varphi^2(y))$. Using this observation and the elementary inequality (2.1), we get

$$h_{\varphi,1}(f) \leq \sum_{x \in V} \varphi^p(x) \left|\nabla f\right|_V^p(x) \leq \gamma \frac{1}{C_p} \|\varphi f\|_{p,wm}^p .$$

By Corollary 4.2, $C_p h_\varphi(f) \leq C'_p h_{\varphi,1}(f)$ for $p \in (1,2]$ and some positive constant $C'_p$ (recall that $C_p$ is fixed); and for all $p \in (2,\infty)$

$$C_p h_\varphi(g^{1/p}) \leq C'_p(h_{\varphi,1}(g^{1/p}) + h_{\varphi,2}(g^{1/p})).$$

Since $\varphi \geq 0$ and by Lemma 13.3, we obtain for all $x, y \in X$

$$\varphi(x)\varphi(y)\left|\varphi(x) - \varphi(y)\right|^{p-2} \leq 2^{p-3}(\varphi^p(x) + \varphi^p(y)).$$

Hence,

$$h_{\varphi,2}(g^{1/p}) \leq 2^{p-3} \sum_{x \in V} \varphi^p(x) \left|\tilde{\nabla} g^{1/p}\right|_V^p(x) \leq \gamma \frac{2^{p-3}}{C_p} \|\varphi f\|_{p,wm}^p .$$

Putting both together, we see

$$C_p h_\varphi(g^{1/p}) \leq \gamma \frac{(1+2^{p-3})C'_p}{C_p} \|\varphi f\|_{p,wm}^p .$$

Choose $\gamma$ such that $\gamma\frac{(1+2^{p-3})C'_p}{C_p} < 1$. Applying Theorem 13.1 finishes the proof. $\square$

We want to remark that in the linear case we have $C_2 = 1$ and we only have one pointwise eikonal inequality. Hence, Corollary 13.4 reduces then to [KPP21, Theorem 1.1].

Let us close the main part of this thesis by discussing two examples.

**Example 13.5 ($\mathbb{N}$)** Here, we want to show how to obtain a generalisation of the classical Rellich inequality on $\mathbb{N}$. We know from Example 10.10 that the free $p$-Laplacian on the standard line graph is subcritical. We also calculated an optimal $p$-Hardy weight $w$. In the appendix, we show that $w(n) > (p-1)^p/(pn)^p =: w_H$. The latter is known as the classical $p$-Hardy weight on $\mathbb{N}$. Set $f = w^{1/q}$ with $q = p/(p-1)$. Recall that $m = 1$ in this example. Moreover, $(f^p w)(n) = n^{p^2}$, and for $n \in \mathbb{N}$

$$\left|\nabla f\right|_{\mathbb{N}_0}^p(n) = \frac{n^{p^2}}{2}\left(\left|1 - \left(\frac{n-1}{n}\right)^p\right|^p + \left|1 - \left(\frac{n+1}{n}\right)^p\right|^p\right),$$

as well as

$$\left|\tilde{\nabla} f\right|_{\mathbb{N}_0}^{p}(n) = \frac{n^{p^2}}{2} \cdot$$
$$\cdot \left( \left( \frac{1 + (\frac{n-1}{n})^p}{2} \right)^{p-2} \left| 1 - \left( \frac{n-1}{n} \right)^p \right|^2 + \left( \frac{1 + (\frac{n+1}{n})^p}{2} \right)^{p-2} \left| 1 - \left( \frac{n+1}{n} \right)^p \right|^2 \right).$$

Hence, for $n$ large enough, say $n \geq n_0$, we can apply Corollary 13.4 and also Corollary 13.2 to obtain for some constant $\gamma \in (0,1)$ and all $\varphi \in C_c(\mathbb{N})$,

$$\sum_{n \geq n_0} \frac{|L\varphi(n) 1_{\operatorname{supp}\varphi}(n)|^q}{n^{pq(p-2)}} \geq (1-\gamma)^q \sum_{n \geq n_0} \frac{|\varphi(n)|^p}{n^{p^2}}.$$

**Example 13.6 ($\mathbb{T}_{d+1}$, $d \geq 2$)** Recall from Example 12.17 that the optimal $p$-Hardy weight $w$ on $\mathbb{T}_{d+1}$, $d \geq 2$ associated with the Green's function is constant for $r > 0$. Let us choose $f = r^q$ for some $q > 0$. By similar calculations as in the example before, we see that we can apply Corollary 13.4 for $r$ large enough, say $r \geq r_0 > 0$. Hence, here we get for some constant $\gamma \in (0,1)$ and all $\varphi \in C_c(\mathbb{T}_{d+1})$, and $q > 0$,

$$\sum_{r \geq r_0} |L\varphi(r) 1_{\operatorname{supp}\varphi}(r)|^{p/(p-1)}\, r^{pq} \geq (1-\gamma)^{p/(p-1)} w^{p/(p-1)}(1) \sum_{r \geq r_0} |\varphi(r)|^p\, r^{pq}.$$

# A. Elementary Estimates

> Pure mathematics, on the other hand, seems to me a rock on which all idealism founders: 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but *because it is so*, because mathematical reality is built that way.
>
> G. H. Hardy, *A Mathematician's Apology*, p. 130

## A.1 Proof of Theorem 1.1

By Example 12.16, it is obvious that the weight from Theorem 1.1 is optimal. Also the statement for integer $p$ is clear. We only have to show that $w > w^H$. This is shown next and can also been found in [FKP23].

In fact, for fixed $p \in (1, \infty)$, we analyse the function $w\colon [0,1] \to [0, \infty)$

$$w(x) = \left(1 - (1-x)^{1/q}\right)^{p-1} - \left((1+x)^{1/q} - 1\right)^{p-1}$$

for $x \in [0, 1/2]$ and $x = 1$, where $q \in (1, \infty)$ is such that $1/p + 1/q = 1$. Specifically, we show

$$w(x) > \left(\frac{x}{q}\right)^p.$$

The case $x = 1$ is simple and is treated at the end of the section. The proof for $x \le 1/2$ is also elementary but more involved. We proceed by bringing $w_p$ into form for which we then analyse its parts. This will be eventually done by a case distinction depending on $p$.

Recall the binomial theorem for $r \in [0, \infty)$ and $0 \le x \le 1$

$$(1 \pm x)^r = \sum_{k=0}^{\infty} \binom{r}{k} (\pm 1)^k x^k$$

where $\binom{r}{0} = 1$, $\binom{r}{1} = r$ and $\binom{r}{k} = r(r-1)\cdots(r-k+1)/k!$ for $k \ge 2$ which is derived from the Taylor expansion of the function $x \mapsto (1 \pm x)^r$. Applying this formula to the function $w$ from above we obtain

$$\begin{aligned} w(x) &= \left(-\sum_{k=1}^{\infty} \binom{1/q}{k} (-x)^k\right)^{p-1} - \left(\sum_{k=1}^{\infty} \binom{1/q}{k} x^k\right)^{p-1} \\ &= \left(\frac{x}{q}\right)^{p-1} \left(\left(q \sum_{k=0}^{\infty} \binom{1/q}{k+1} (-x)^k\right)^{p-1} - \left(q \sum_{k=0}^{\infty} \binom{1/q}{k+1} x^k\right)^{p-1}\right) \end{aligned}$$

To streamline notation we set

$$g(x) = q \sum_{k=1}^{\infty} \binom{1/q}{k+1} x^k.$$

Note that since $q\binom{1/q}{1} = 1$ and $q\left|\binom{1/q}{k}\right| < 1$ for $k \geq 2$, we have $0 < |g(\pm x)| < 1$ for $0 < x \leq 1/2$. Thus, we can apply the binomial theorem to $\big(1 + g(\pm x)\big)^{p-1}$ in order to get

$$\begin{aligned}
w(x) &= \left(\frac{x}{q}\right)^{p-1} \left(\big(1 + g(-x)\big)^{p-1} - \big(1 + g(x)\big)^{p-1}\right) \\
&= \left(\frac{x}{q}\right)^{p-1} \left(\sum_{n=0}^{\infty} \binom{p-1}{n} \big(g^n(-x) - g^n(x)\big)\right) \\
&= \left(\frac{x}{q}\right)^{p-1} \left(\binom{p-1}{1} \big(g(-x) - g(x)\big) + \sum_{n=2}^{\infty} \binom{p-1}{n} \big(g^n(-x) - g^n(x)\big)\right)
\end{aligned}$$

Thus, we have to show that the second factor on the right hand side is strictly larger than $x/q$. Using $q = p/(p-1)$ we compute the first term in the parenthesis on the left hand side

$$\begin{aligned}
\binom{p-1}{1} \big(g(-x) - g(x)\big) &= q(p-1) \sum_{k=1}^{\infty} \binom{1/q}{k+1} \left((-x)^k - x^k\right) \\
&= \frac{q(p-1)(1/q)(1/q-1)}{2}(-2x) + q(p-1) \sum_{k=2}^{\infty} \binom{1/q}{k+1} \left((-x)^k - x^k\right) \\
&= \frac{x}{q} - 2p \sum_{k \in 2\mathbb{N}+1} \binom{1/q}{k+1} x^k \\
&= \frac{x}{q} + E_p(x),
\end{aligned}$$

with

$$E_p(x) = -2p \sum_{k \in 2\mathbb{N}+1} \binom{1/q}{k+1} x^k > 0$$

since $-2p\binom{1/q}{k+1} > 0$ for odd $k$ and $x > 0$. So, it remains to show that for the term

$$F_p(x) = \sum_{n=2}^{\infty} \binom{p-1}{n} \big(g^n(-x) - g^n(x)\big)$$

we have for $0 < x \leq 1/2$

$$E_p(x) + F_p(x) > 0.$$

Specifically, we then get with the substitution $x = 1/n$

$$w_p(n) = w(1/n) = \left(\frac{1}{nq}\right)^{p-1}\left(\frac{1}{nq} + E_p(1/n) + F_p(1/n)\right) > \frac{1}{(nq)^p} = w_p^H(n)$$

for $n \geq 2$.

**Remark A.1** It is not hard to see that $F_p \geq 0$ whenever $p \in \mathbb{N}$ is integer valued. Indeed, $g(-x) \geq g(x)$ as all terms in the sum $g(-x)$ are positive since $-\binom{1/q}{k+1} \geq 0$ for odd $k$, while the terms in $g(x)$ alternate, (they are positive for even $k$ and negative for odd $k$). Moreover for positive integers $p$ the binomial coefficients $\binom{p-1}{n}$ are positive. Thus, the Hardy weight we computed is larger than the classical one for integer $p$.

Let us now turn to the proof of

$$E_p(x) + F_p(x) > 0$$

for $p \in (1, \infty)$ and $0 < x \leq 1/2$.

We collect the following basic properties of the function $g$ which were partially already discussed above and will be used subsequently.

**Lemma A.2** *For $p \in (1, \infty)$ and $0 < x \leq 1/2$, we have*

$$-1 < g(x) < 0 < -g(x) < g(-x) < 1.$$

*Proof.* The function $g$ is given by $g(x) = q\sum_{k=1}^{\infty}\binom{1/q}{k+1}x^k$. Since $q > 1$, the coefficients $b_k = q\binom{1/q}{k+1}$ are negative for odd $k$ and positive for even $k$. Furthermore, the sequence $(|b_k|)$ takes values strictly less than 1 and decays monotonically. Thus, the asserted inequalities follow easily. □

We distinguish the following three cases depending on $p$ for which the arguments are quite different:

- $p$ lies between an odd and an even number with the subcases:
  - $p \in [3, \infty)$
  - $p \in (1, 2]$
- $p$ lies between an even and an odd number.

Here, for $a, b \in \mathbb{N}$, we say that $p$ is between $a$ and $b$ if $a \leq p \leq b$.

We start with investigating the case of $p$ lying between an odd and an even number. To this end we consider two subsequent summands as they appear in the sum given by $F_p$ and show that they are positive. Indeed, the sum in $F_p$ starts at $n = 2$ but we also consider the corresponding term for $n = 1$.

**Lemma A.3** *Let $p$ be between an odd and an even integer. Then, for all $0 < x \leq 1/2$ and odd $n \in 2\mathbb{N}-1$*

$$\binom{p-1}{n}\big(g^n(-x) - g^n(x)\big) + \binom{p-1}{n+1}\big(g^{n+1}(-x) - g^{n+1}(x)\big) \geq 0.$$

*Proof.* Let $p$ be between an odd and an even integer. We first consider $n < p-1$. In the case $k \leq p-1$, we have $\binom{p-1}{k} \geq 0$. So, the statement for $n < p-1$ follows directly from Lemma A.2 as $|g(x)| < 1$ for $0 < x \leq 1/2$. Observe that $n+1 \leq p-1$ for $n < p-1$ and $n \in 2\mathbb{N}-1$ as $p$ is between an odd and an even integer.

On the other hand, for odd $n \in 2\mathbb{N}-1$ with $n \geq p-1$,

$$\binom{p-1}{n} \geq -\binom{p-1}{n+1} \geq 0.$$

From Lemma A.2 we know that $g^{n+1}(x) \geq 0 \geq g^n(x)$ for odd $n \in 2\mathbb{N}-1$ and $0 \leq x \leq 1/2$. We obtain

$$\begin{aligned}
\binom{p-1}{n}&\big(g^n(-x) - g^n(x)\big) + \binom{p-1}{n+1}\big(g^{n+1}(-x) - g^{n+1}(x)\big) \\
&= \binom{p-1}{n}\big(g^n(-x) - g^n(x)\big) - \left|\binom{p-1}{n+1}\right|\big(g^{n+1}(-x) - g^{n+1}(x)\big) \\
&\geq \binom{p-1}{n} g^n(-x) - \left|\binom{p-1}{n+1}\right| g^{n+1}(-x) \\
&\geq \left|\binom{p-1}{n+1}\right|\big(g^n(-x) - g^{n+1}(-x)\big) \\
&\geq 0,
\end{aligned}$$

where the last inequality follows from $0 \leq g(-x) < 1$ for $0 \leq x \leq 1/2$ thanks to Lemma A.2. ◻

With Lemma A.3 we can treat the case of $p \geq 3$ lying between an odd and an even number. This is done in the next proposition.

**Proposition A.4** *Let $p \geq 3$ be between an odd and an even integer. Then, for all $0 < x \leq 1/2$ we have $F_p(x) \geq 0$ and*

$$E_p(x) + F_p(x) > 0.$$

*In particular, $w(n) > w^H(n)$ for $n \geq 2$.*

*Proof.* We can write $F_p(x) = \sum_{n=2}^{\infty} \binom{p-1}{n}\big(g^n(-x) - g^n(x)\big)$ as

$$\begin{aligned}
F_p(x) = &\binom{p-1}{2}\big(g^2(-x) - g^2(x)\big) \\
&+ \sum_{n\in 2\mathbb{N}+1}^{\infty} \left(\binom{p-1}{n}\big(g^n(-x) - g^n(x)\big) + \binom{p-1}{n+1}\big(g^{n+1}(-x) - g^{n+1}(x)\big)\right)
\end{aligned}$$

By Lemma A.3 the terms in the sum on the right hand side are all positive. Furthermore, $\binom{p-1}{2} \geq 0$ for $p \geq 3$ and $g(-x) \geq |g(x)|$ by Lemma A.2. Thus, also the first term on the right hand side is positive as well and $F_p \geq 0$ follows. From the discussion in the beginning in the section we take $E_p(x) > 0$ for $0 < x \leq 1/2$. The "in particular" follows from the discussion above Lemma A.2. $\square$

Note that we cannot treat the case $1 \leq p \leq 2$ in the same way since the sum in $F_p$ starts at the index $n = 2$. Hence, there is still a negative term $\binom{p-1}{2}\big(g^2(-x) - g^2(x)\big)$. We deal with this case, $1 \leq p \leq 2$, next.

We denote the Taylor coefficients of $x \mapsto g(-x)$ by $a_k$, i.e.,

$$g(-x) = q \sum_{k=1}^{\infty} \binom{1/q}{k+1} (-x)^k = \sum_{k=1}^{\infty} a_k x^k,$$
$$g(x) = q \sum_{k=1}^{\infty} \binom{1/q}{k+1} x^k = \sum_{k=1}^{\infty} a_k (-1)^k x^k.$$

The function $E_p(x) = -2p \sum_{k \in 2\mathbb{N}+1} \binom{1/q}{k+1} x^k$ is odd and, therefore, we have

$$E_p(x) = 2(p-1) \sum_{n=1}^{\infty} a_{2n+1} x^{2n+1}.$$

Furthermore, recall that $E_p(x) > 0$ for $x > 0$, since $-2p\binom{1/q}{k+1} > 0$ for odd $k$.

**Lemma A.5** *Let $p \geq 1$ and $0 \leq x \leq 1/2$. Then,*

$$g(-x) + g(x) \leq \frac{4}{9} \cdot \frac{(p+1)}{p^2} x^2.$$

*Proof.* We calculate using $a_2 \geq a_n$ for $n \geq 2$, the geometric series, $x \leq 1/2$ and the specific value of the Taylor coefficient $a_2 = q\binom{1/q}{3} = \frac{(p+1)}{6p^2}$

$$g(-x) + g(x) = 2 \sum_{k=1}^{\infty} a_{2k} x^{2k} \leq 2a_2 \frac{x^2}{1-x^2} \leq \frac{8}{3} a_2 x^2 = \frac{4}{9} \cdot \frac{(p+1)}{p^2} x^2. \qquad \square$$

With the help of this lemma and Lemma A.3 we can treat the case $p \in (1, 2]$.

**Proposition A.6** *Let $p \in (1, 2]$. Then, for all $0 < x \leq 1/2$, we have*

$$E_p(x) + F_p(x) > 0.$$

*In particular, $w(n) > w^H(n)$ for $n \geq 2$.*

*Proof.* We show $E_p + F_p > 0$ and deduce the "in particular" from the discussion above Lemma A.2. By Lemma A.3 we have for all $0 < x \leq 1/2$

$$\begin{aligned}
F_p(x) =& \binom{p-1}{2}\big(g^2(-x) - g^2(x)\big) \\
&+ \sum_{n \in 2\mathbb{N}+1}^{\infty} \left( \binom{p-1}{n}\big(g^n(-x) - g^n(x)\big) + \binom{p-1}{n+1}\big(g^{n+1}(-x) - g^{n+1}(x)\big) \right) \\
\geq& \binom{p-1}{2}\big(g^2(-x) - g^2(x)\big) \\
=& \frac{p-2}{2}\big(g(-x) + g(x)\big) \left( E_p(x) + \frac{p-1}{p}x \right) \\
\geq& \frac{2}{9} \cdot \frac{(p-2)(p+1)}{p^2} \left( E_p(x) + \frac{p-1}{p}x \right) \cdot x^2 \\
\geq& -\frac{1}{9}E_p(x) + \frac{2}{9} \cdot \frac{(p-2)(p-1)(p+1)}{p^3} \cdot x^3
\end{aligned}$$

where we used the definition of $E_p$, i.e., $(p-1)\big(g(-x) - g(x)\big) = E_p(x) + \frac{p-1}{p}x$ and Lemma A.5 which is justified since $E_p(x) > 0$ and $p - 2 < 0$. Moreover, in the last step we estimated the coefficient of the first term in its minimum in $p = 1$ and $x = 1/2$.

Now, we use the representation of $E_p$ as a power series to estimate

$$E_p(x) = -2p \sum_{k \in 2\mathbb{N}+1} \binom{1/q}{k+1} x^k \geq -2p \binom{1/q}{4} x^3 = \frac{(p-1)(p+1)(2p+1)}{12p^3} x^3.$$

Putting this together with the estimate on $F_p$ above, we arrive at

$$\begin{aligned}
E_p(x) + F_p(x) &\geq \frac{8}{9}E_p(x) + \frac{2}{9} \cdot \frac{(p-2)(p-1)(p+1)}{p^2} \cdot x^3 \\
&\geq \left( \frac{8}{9} \frac{(p-1)(p+1)(2p+1)}{12p^3} + \frac{2}{9} \cdot \frac{(p-2)(p-1)(p+1)}{p^3} \right) \cdot x^3 \\
&= \frac{10(p-1)^2(p+1)}{27p^3} \cdot x^3
\end{aligned}$$

This finishes the proof. ☐

Hence, it remains to consider the case of $p$ between an even and an odd integer for which we need the following three lemmas.

**Lemma A.7** *Let $p, q \geq 1$ such that $1/p + 1/q = 1$ and $k \geq 2$. Then,*

$$a_k = q \left| \binom{1/q}{k+1} \right| \geq \frac{1}{pk(k+1)} = \frac{1}{q(p-1)k(k+1)}.$$

*Proof.* We calculate using $1/p + 1/q = 1$

$$\begin{aligned} q\left|\binom{1/q}{k+1}\right| &= \frac{(1-1/q)(2-1/q)(3-1/q)\cdots(k-1/q)}{(k+1)!} \\ &= \frac{1}{pk(k+1)}\frac{(1+1/p)(2+1/p)\cdots((k-1)+1/p)}{(k-1)!} \\ &= \frac{1}{pk(k+1)}\left(1+\frac{1}{p}\right)\left(1+\frac{1}{2p}\right)\cdots\left(1+\frac{1}{(k-1)p}\right) \geq \frac{1}{pk(k+1)}. \qquad \square \end{aligned}$$

**Lemma A.8** *Let $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$ and $k \in \mathbb{N}, k > p$. Then,*

$$\left|\binom{p-1}{k}\right| \leq \frac{1}{4(p-1)} = \frac{(q-1)}{4}.$$

*Proof.* Let $n \in \mathbb{N}$ be such that $n - 1 \leq p \leq n$. Moreover, let $\gamma = p - (n-1)$, i.e., $1 - \gamma = n - p$, so, $\gamma \in [0, 1]$. Since $k > p$ and $n, k \in \mathbb{N}$, we have that $k \geq n$ and therefore,

$$\begin{aligned} \left|\binom{p-1}{k}\right| &= \left|\frac{(p-1)(p-2)\cdots(p-(n-1))(p-n)\cdots(p-k)}{k!}\right| \\ &= \left|\left(\frac{p-1}{n-1}\right)\left(\frac{p-2}{n-2}\right)\cdots\frac{(p-(n-1))}{1}\left(\frac{p-n}{n}\right)\cdots\left(\frac{p-k}{k}\right)\right| \\ &\leq \left|\frac{(p-(n-1))(p-n)}{n}\right| = \frac{\gamma(1-\gamma)}{n} \leq \frac{1}{4(p-1)} = \frac{(q-1)}{4}. \qquad \square \end{aligned}$$

**Lemma A.9** *For $0 < x \leq 1/2$ and $q > 1$, we get*

$$g(-x) \leq \frac{(q-1)(5q-1)}{6q^2}x.$$

*Proof.* We calculate using $a_2 \geq a_k$ for $k \geq 2$

$$\begin{aligned} g(-x) &= q\left(\left|\binom{1/q}{2}\right| + \sum_{k=1}^{\infty}\left|\binom{1/q}{k+2}\right|x^k\right)x \\ &\leq q\left(\left|\binom{1/q}{2}\right| + \sum_{k=1}^{\infty}\left|\binom{1/q}{k+2}\right|2^{-k}\right)x \\ &\leq q\left(\left|\binom{1/q}{2}\right| + \left|\binom{1/q}{3}\right|\right)x \\ &= \frac{(q-1)(5q-1)}{6q^2}x. \qquad \square \end{aligned}$$

With the help of these lemmas we can finally treat the case where $p$ lies between an even and an odd number.

**Proposition A.10** *Let $p \in [2, \infty)$ be between an even and an odd integer. Then, for all $0 < x \leq 1/2$ we have*

$$E_p(x) + F_p(x) > 0.$$

*In particular, $w(n) > w^H(n)$ for $n \geq 2$.*

*Proof.* Clearly, we have $\binom{p-1}{n} \geq 0$ for $n \leq p$ and for $n \in 2\mathbb{N}$. Since we have $g(-x) \geq |g(x)|$ by Lemma A.2, we obtain for the first $n \leq p$ terms and the terms for even $n$ in $F_p(x)$ that

$$\binom{p-1}{n}\left(g^n(-x) - g^n(x)\right) \geq 0.$$

Note that $E_p(x) = 2(p-1)\sum_{n=1}^{\infty} a_{2n+1}x^{2n+1} > 2(p-1)\sum_{n=k}^{\infty} a_{2n+1}x^{2n+1}$ since the coefficients $a_k$ are positive. With the observation made at the beginning of the proof, this leads to

$$E_p(x) + F_p(x) > \sum_{n \in 2\mathbb{N}+1, n \geq p} \left(2(p-1)a_n x^n + \binom{p-1}{n}\left(g^n(-x) - g^n(x)\right)\right).$$

We continue to show that all the terms in the sum are strictly positive which finishes the proof. To this end note that for $n \geq p$ with $n \in 2\mathbb{N}+1$, we use $g(-x) \geq |g(x)|$, Lemma A.2, as well as $\binom{p-1}{n} \leq 0$ in the first step and the estimate on $\binom{p-1}{n}$, Lemma A.8, and the estimate on $g(-x)$, Lemma A.9 in the second step in order to get

$$\binom{p-1}{n}\left(g^n(-x) - g^n(x)\right) \geq 2\binom{p-1}{n} g^n(-x) \geq -\frac{(q-1)}{2}\left(\frac{(q-1)(5q-1)}{6q^2}\right)^n x^n$$

We use the estimate on $a_n$, Lemma A.7,

$$2(p-1)a_n x^n \geq 2\frac{1}{qn(n+1)}x^n.$$

Next, we put these two estimates together and find that the minimum in the coefficient is clearly assumed at $q = 2$ since $p \geq 2 \geq q$

$$\begin{aligned}
&\left(2(p-1)a_n x^n + \binom{p-1}{n}\left(g^n(-x) - g^n(x)\right)\right) \\
&\qquad \geq 2\left(\frac{1}{qn(n+1)} - \frac{(q-1)}{4}\left(\frac{(q-1)(5q-1)}{6q^2}\right)^n\right)x^n \\
&\qquad \geq \left(\frac{1}{n(n+1)} - \frac{1}{2}\left(\frac{3}{8}\right)^n\right)x^n \geq \left(\frac{1}{n(n+1)} - \frac{1}{2^{n+1}}\right)x^n > 0,
\end{aligned}$$

where the positivity follows by a simple induction argument. This concludes the proof by noticing that the "in particular" part follows from the discussion above Lemma A.2. $\square$

In summary, the above considerations yield

$$E_p(x) + F_p(x) > 0$$

for $p \in (1, \infty)$ and $0 < x \leq 1/2$. By the discussion at the beginning of the section this yields $w_p(n) > w_p^H(n)$ for $n \geq 2$.

We finish the section by treating the case $n = 1$ which corresponds to $x = 1$. With this we finally conclude that $w(n) > w^H(n)$ for all $n \geq 1$.

**Proposition A.11** *Let $p \in (1, \infty)$. Then, $w(1) > w^H(1)$.*

*Proof.* Recall that $w(1) = 1 - (2^{1-1/p} - 1)^{p-1}$ and $w^H = (1 - 1/p)^p$. By the mean value theorem applied to the function $[1, 2] \to [1, 2^{1-1/p}]$, $t \mapsto t^{1-1/p}$ we find

$$2^{1-1/p} - 1 < 1 - \frac{1}{p}.$$

Therefore,

$$w_p(1) - w_p^H(1) > 1 - \left(1 - \frac{1}{p}\right)^{p-1} - \left(1 - \frac{1}{p}\right)^p.$$

Now the function $\psi\colon (1, \infty) \to (0, \infty)$, $p \mapsto (1 - 1/p)^{p-1} + (1 - 1/p)^p$ is strictly monotonically decreasing because

$$\psi'(p) = \frac{1}{p-1}\left(\frac{p-1}{p}\right)^p \left((2p-1)\log\left(\frac{p-1}{p}\right) + 2\right) < 0,$$

since $\theta\colon p \mapsto (2p-1)\log(p-1)/p$ is strictly monotonically increasing and we have $\lim_{p\to\infty} \theta(p) = -2$. Hence, we conclude

$$w(1) - w^H(1) > 1 - \psi(p) > 1 - \lim_{t\to 1} \psi(t) = 0.$$

This finishes the proof. □

In summary, we have shown the desired strict inequality in Theorem 1.1.

## A.2 Proof of Lemma 4.8

This section can also be found in [Fis23]. Before we can proof Lemma 4.8, we need the following quantification of the strict convexity of the mapping $x \mapsto |x|^p$, $p > 1$. In the following lemma, $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$.

**Lemma A.12 (Lindqvist's lemma, Lemma 4.2 in [Lin90])** *Let $a, b \in \mathbb{R}^n$. Then, for all $p \geq 2$ we have*

$$|a|^p - |b|^p \geq p\,|b|^{p-2}\,\langle b, a-b\rangle_{\mathbb{R}^n} + c_p\,|a-b|^p,$$

*where* $c_p = 1/(2^{p-1}-1) > 0$*. If* $1 < p < 2$*, then*

$$|a|^p - |b|^p \geq p\,|b|^{p-2}\,\langle b, a-b\rangle_{\mathbb{R}^n} + c_p \frac{|a-b|^2}{(|a|+|b|)^{2-p}},$$

*where* $c_p = 3p(p-1)/16 > 0$*, and the fraction is interpreted to be zero if* $a = b = 0$*.*

In the previous lemma, the constant $c_p$ does not seem to be optimal. However, this is not important for our further investigations.

*Proof (of Lemma 4.8). Ad (4.10):* Recall that we have to show that for $p > 1$,

$$|a-t|^p - (1-t)^{p-1}(|a|^p - t) \asymp t\,|a-1|^2\,(|a-t|+1-t)^{p-2}, \quad a \in \mathbb{R}, 0 \leq t \leq 1.$$

The strategy of the proof is as follows: We start with some simple special cases for which the equivalence can be shown very easily. Thereafter, we do a substitution to bring the equivalence in a simpler form for the remaining cases. Then, we divide $\mathbb{R}$ into the three intervals $[1,+\infty)$, $(t,1)$, and $(-\infty,t]$ for some $t \in [0,1]$. In the two intervals $[1,+\infty)$ and $(-\infty,t]$, we then distinguish between proving lower bounds and upper bounds, as well as having $p > 2$ or $1 < p < 2$. In the remaining interval $(t,1)$, we show that we can deduce the equivalence from the validity of the equivalence in $[1,+\infty)$.

**1. The three cases** $t \in \{0,1\}$**,** $a = t$**, and** $p = 2$**:** If $p = 2$, then it is obvious that we have equality for all $a \in \mathbb{R}$ and $t \in [0,1]$.

An easy computation shows that we have indeed equality for $t \in \{0,1\}$.

If $a = t$, we have to show that for all $p > 1$

$$-(1-t)^{p-1}(t^p - t) \asymp t(1-t)^p.$$

Thus, let us consider the function

$$q(t) := \frac{(1-t)^{p-1}(t-t^p)}{t(1-t)^p} = \frac{1-t^{p-1}}{1-t}.$$

If $1 < p < 2$, then $t^{p-1} \geq t$ for $t \in (0,1)$, and thus, $q$ is decreasing. If $p > 2$, we have $t^{p-1} \leq t$ for $t \in (0,1)$, and $q$ is increasing. Moreover, by L'Hôpital's rule $q(1) = p-1$. Hence, for $p > 2$ we have $1 = q(0) \leq q(t) \leq q(1) = p-1$ and for $1 < p < 2$, we have $p - 1 = q(1) \leq q(t) \leq q(0) = 1$.

**2. The remaining cases** $t \in (0,1)$**,** $a \neq t$**, and** $p \neq 2$**:** We do the following substitution: Set $\alpha := (a-t)/(1-t)$, then we have to show that

$$|\alpha|^p - \frac{|\alpha(1-t)+t|^p - t}{1-t} \asymp \frac{t(\alpha-1)^2}{(|\alpha|+1)^{2-p}}. \tag{A.1}$$

We will do this by considering the following three cases separately

- $\alpha \geq 1$,
- $1 > \alpha > 0$, and

- $\alpha < 0$.

Furthermore, let

$$f_\alpha(t) := \frac{|\alpha(1-t)+t|^p - t}{1-t} = \frac{|\alpha + t(1-\alpha)|^p - t}{1-t}.$$

Note that $f_\alpha(0) = |\alpha|^p$.

**2.1. The case $\alpha \geq 1$:** The basic strategy is to use the Taylor-Maclaurin formula. Thus, let us calculate the first and the second derivatives with respect to $t$. Note that for $\alpha \geq 1$, we have $|\alpha + t(1-\alpha)| = \alpha + t(1-\alpha)$. Hence, we calculate

$$f'_\alpha(t) = \frac{p(1-\alpha)(\alpha+t(1-\alpha))^{p-1} - 1}{1-t} + \frac{f_\alpha(t)}{1-t},$$

and using $\alpha + t(1-\alpha) = \beta + 1$, where $\beta := (\alpha-1)(1-t) \geq 0$, we get

$$\begin{aligned}
f''_\alpha(t) &= \frac{p(p-1)(1-\alpha)^2(\alpha+t(1-\alpha))^{p-2}}{1-t} + \frac{p(1-\alpha)(\alpha+t(1-\alpha))^{p-1} - 1}{(1-t)^2} \\
&\quad + \frac{f'_\alpha(t)}{1-t} + \frac{f_\alpha(t)}{(1-t)^2} \\
&= \frac{(\alpha+t(1-\alpha))^{p-2}}{(1-t)^3}\Big(p(p-1)(\alpha-1)^2(1-t)^2 \\
&\quad -2p(\alpha-1)(1-t)(\alpha+t(1-\alpha)) + 2(\alpha+t(1-\alpha))^2\Big) - 2\frac{1-t+t}{(1-t)^3} \\
&= \frac{(\beta+1)^{p-2}}{(1-t)^3}\Big(p(p-1)\beta^2 - 2p\beta(\beta+1) + 2(\beta+1)^2\Big) - \frac{2}{(1-t)^3} \\
&= \frac{(\beta+1)^{p-2}}{(1-t)^3}\Big(-(p-1)(2-p)\beta^2 + 2(2-p)\beta + 2\Big) - \frac{2}{(1-t)^3} \qquad \text{(A.2)} \\
&= \frac{g(\beta) - 2}{(1-t)^3},
\end{aligned}$$

where

$$g(\beta) := \big((p-1)(p-2)\beta^2 + 2(2-p)\beta + 2\big)(\beta+1)^{p-2}, \qquad \beta \geq 0.$$

Let us analyse $g(\beta)$ for $\beta \geq 0$. Then, $g'(\beta) = p(p-1)(p-2)(\beta+1)^{p-3}\beta^2$, which is positive for $p > 2$ and negative for $1 < p < 2$. Hence, $g(0) = 2$ is a minimum for $p > 2$ and a maximum for $1 < p < 2$. This implies that for all $t \in (0,1)$

$$f''_\alpha(t) \begin{cases} \leq 0 & \text{if } 1 < p < 2, \\ \geq 0 & \text{if } p > 2. \end{cases}$$

Now, we apply the Taylor-Maclaurin formula

$$f_\alpha(t) = f_\alpha(0) + t f'_\alpha(0) + \int_0^t (t-s) f''_\alpha(s) \mathrm{d}s.$$

Since $f_\alpha(0) = \alpha^p$, we have

$$
\begin{aligned}
|\alpha|^p - \frac{|\alpha(1-t)+t|^p - t}{1-t} &= f_\alpha(0) - f_\alpha(t) \\
&= -tf'_\alpha(0) - \int_0^t (t-s) f''_\alpha(s)\,\mathrm{d}s \\
&= t\big((p-1)\alpha^p - p\alpha^{p-1} + 1\big) - \int_0^t (t-s) f''_\alpha(s)\,\mathrm{d}s.
\end{aligned} \tag{A.3}
$$

This term will be analysed in the following for upper and lower bounds and different values of $p$.

**2.1.1. Lower bound for $1 < p < 2$ and $\alpha \geq 1$:** Then $f''_\alpha \leq 0$ on $(0,1)$. Thus we conclude from (A.3),

$$
|\alpha|^p - \frac{|\alpha(1-t)+t|^p - t}{1-t} \geq t\big((p-1)\alpha^p - p\alpha^{p-1} + 1\big).
$$

Using Lindqvist's lemma, Lemma A.12, with $b = \alpha$ and $a = 1$, we see

$$
t\big((p-1)\alpha^p - p\alpha^{p-1} + 1\big) = t\big(1 - \alpha^p - p\alpha^{p-2}\alpha(1-\alpha)\big) \geq C_p \frac{t(\alpha-1)^2}{(\alpha+1)^{2-p}}.
$$

This is the desired lower bound in (A.1) for $1 < p < 2$ and $\alpha \geq 1$.

**2.1.2. Upper bound for $p > 2$ and $\alpha \geq 1$:** Then $f''_\alpha \geq 0$ on $(0,1)$. Thus we conclude from (A.3),

$$
|\alpha|^p - \frac{|\alpha(1-t)+t|^p - t}{1-t} \leq t\big((p-1)\alpha^p - p\alpha^{p-1} + 1\big).
$$

Hence, it remains to show that there exists $C_p > 0$ such that

$$
\big((p-1)\alpha^p - p\alpha^{p-1} + 1\big) \leq C_p(\alpha-1)^2(\alpha+1)^{p-2}.
$$

For any positive constant $C_p$ we have using $(1+\alpha^{-1})^{p-2} \geq 1$,

$$
\begin{aligned}
j(\alpha) &:= \alpha^{p-2}\Big(\big((p-1)\alpha^2 - p\alpha + \alpha^{2-p}\big) - C_p(\alpha-1)^2(1+\alpha^{-1})^{p-2}\Big) \\
&\leq \alpha^{p-2}\Big(\big((p-1)\alpha^2 - p\alpha + \alpha^{2-p}\big) - C_p(\alpha-1)^2\Big) \\
&= \alpha^{p-2}\Big(\big(p-1-C_p\big)\alpha^2 + \big(2C_p - p\big)\alpha + \alpha^{2-p} - C_p\Big).
\end{aligned}
$$

Let $g(\alpha) := \big(p-1-C_p\big)\alpha^2 + \big(2C_p - p\big)\alpha + \alpha^{2-p} - C_p$ for $\alpha > 0$, then

$$
g'(\alpha) = 2(p-1-C_p)\alpha + (2C_p - p) - (p-2)\alpha^{1-p}
$$

has a root at $\alpha = 1$. If we can show that $g$ is concave on $[1,\infty]$, then $g(1) = 0$ is a maximum. Since

$$
g''(\alpha) = 2(p-1-C_p) + (p-2)(p-1)\alpha^{-p} \leq 2(p-1-C_p) + (p-2)(p-1),
$$

$g$ is concave on $[1, \infty]$ for all $C_p \geq p(p-1)/2$, we found a possible constant such that $j(\alpha) \leq 0$. In other words, we have the desired upper bound for $p > 2$. However, it is obvious that the constant can be improved.

**2.1.3. Upper bound for $1 < p < 2$ and $\alpha \geq 1$:** For $1 \leq p \leq 2$, the function $|\cdot|^{p-1}$ is concave on $(0, \infty)$, thus

$$|\alpha(1-t)+t|^{p-1} \geq (1-t)\alpha^{p-1} + t.$$

Using this estimate in the left-hand side of (A.1), we get

$$|\alpha|^p - \frac{|\alpha(1-t)+t|^p - t}{1-t} \leq t(\alpha^{p-1}-1)(\alpha-1) \tag{A.4}$$

Define for $\alpha \geq 1$,

$$g(\alpha) := (\alpha+1)^{2-p}(\alpha^{p-1}-1) - \alpha + 1,$$

then

$$\begin{aligned} g(\alpha) &= (\alpha^{p-1}-1)\alpha^{2-p}(1+\alpha^{-1})^{2-p} - \alpha + 1 \\ &= (\alpha - \alpha^{2-p}) \sum_{k=0}^{\infty} \binom{2-p}{k} \alpha^{-k} - \alpha + 1. \end{aligned}$$

Since for all $k \in 2\mathbb{N}$, $1 \leq p \leq 2$ and $\alpha \geq 1$, we have

$$\binom{2-p}{k}\alpha^{-k} + \binom{2-p}{k+1}\alpha^{-k-1} \leq 0,$$

we get for all $1 \leq p \leq 2$ and $\alpha \geq 1$,

$$\begin{aligned} g(\alpha) &\leq (\alpha - \alpha^{2-p})\big(1 + \frac{2-p}{\alpha}\big) - \alpha + 1 \\ &= (2-p) + 1 - \alpha^{2-p} - (2-p)\alpha^{1-p} =: l(\alpha). \end{aligned}$$

Since $l'(\alpha) = (2-p)((p-1)-\alpha)\alpha^{-p} \leq 0$ for $\alpha \geq 1$ and $1 \leq p \leq 2$, we get

$$g(\alpha) \leq l(\alpha) \leq l(1) = 0.$$

Thus, using that $g \leq 0$ on $[1, \infty]$ results in (A.4) in

$$(\alpha^{p-1}-1)(\alpha-1) \leq \frac{(\alpha-1)^2}{(\alpha+1)^{2-p}}.$$

This results in the right-hand side of (A.1) with constant 1.

**2.1.4. Lower bound for $p > 2$ and $\alpha \geq 1$:** For $p \geq 2$, the function $|\cdot|^{p-1}$ is convex on $(0, \infty)$, thus

$$|\alpha(1-t)+t|^{p-1} \leq (1-t)\alpha^{p-1} + t.$$

Using this estimate in the left-hand side of (A.1), we get

$$|\alpha|^p - \frac{|\alpha(1-t)+t|^p - t}{1-t} \geq t(\alpha^{p-1}-1)(\alpha-1) \tag{A.5}$$

Define for $\alpha \geq 1$, and some constant $C_p > 0$,

$$g(\alpha) := \alpha^{p-1} - 1 - C_p(\alpha-1)(\alpha+1)^{p-2},$$

then

$$g'(\alpha) = \alpha^{p-2}\big(p-1-C_p\big((1+\frac{1}{\alpha})^{p-2} + (p-2)(1+\frac{1}{\alpha})^{p-3}(1-\frac{1}{\alpha})\big)\big).$$

If $p \geq 3$, then

$$\ldots \geq \alpha^{p-2}\big(p-1-C_p(2^{p-2}+(p-2)2^{p-3})\big).$$

Choosing $C_p = 2^{3-p}(p-1)/p$, we get $g' \geq 0$ on $[1,\infty)$. In particular,

$$g(\alpha) \geq g(1) = 0.$$

Thus, for $p \geq 3$,

$$(\alpha^{p-1}-1)(\alpha-1) \geq C_p(\alpha-1)^2(\alpha+1)^{p-2}. \tag{A.6}$$

If $2 \leq p \leq 3$, then

$$g'(\alpha) \geq \alpha^{p-2}\big(p-1-C_p(2^{p-2}+p-2)\big).$$

Choosing $C_p = (p-1)/(2^{p-2}+p-2)$, we get $g' \geq 0$ on $[1,\infty)$. Thus, for $2 \leq p \leq 3$,

$$(\alpha^{p-1}-1)(\alpha-1) \geq C_p(\alpha-1)^2(\alpha+1)^{p-2}. \tag{A.7}$$

Applying (A.6) and (A.7) to (A.5), results in the right-hand side of (A.1).

Moreover, this was the last puzzle stone to show (A.1) for $\alpha \geq 1$ and all $1 < p < \infty$.

**2.2. The case** $0 < \alpha < 1$**:** We have shown that (A.1) holds for all $\alpha > 1$ and $t \in (0,1)$. Then it holds in particular for $s = 1-t$, i.e.,

$$|\alpha|^p - \frac{|\alpha s + 1 - s|^p - (1-s)}{s} \asymp \frac{(1-s)(\alpha-1)^2}{(|\alpha|+1)^{2-p}}.$$

Now, for any $\alpha > 1$ let $\beta := 1/\alpha \in (0,1)$. Then, we get by multiplying both sides with $\beta^p s/(1-s)$,

$$|\beta|^p - \frac{|\beta(1-s)+s|^p - s}{1-s} \asymp \frac{s(\beta-1)^2}{(|\beta|+1)^{2-p}},$$

which is the desired equivalence.

**2.3. The case $\alpha < 0$:** Set $\beta := -\alpha$. Then, substituting into (A.1), we have to show that for all $\beta > 0$ and $t \in (0,1)$,

$$|\beta|^p - \frac{|\beta(1-t)-t|^p - t}{1-t} \asymp \frac{t(\beta+1)^2}{(|\beta|+1)^{2-p}} = t(\beta+1)^p. \tag{A.8}$$

We have

$$|\beta|^p - \frac{|\beta(1-t)-t|^p - t}{1-t} = |\beta|^p - \frac{|\beta(1-t)+t|^p - t}{1-t} + g_t(\beta),$$

where

$$g_t(\beta) := \frac{1}{1-t}\big((\beta(1-t)+t)^p - |\beta(1-t)-t|^p\big), \qquad \beta > 0,\ t \in (0,1).$$

Before we continue with the estimates, let us note that

$$g_t \geq 0 \quad \text{and} \quad g_t' \geq 0.$$

The first inequality can be seen as follows: let $\gamma > 0$. Firstly assume that $\gamma > t$. Then,

$$(\gamma+t)^p - (\gamma-t)^p = 2\gamma^p \sum_{k\in 2\mathbb{N}-1} \binom{p}{k} \left(\frac{t}{\gamma}\right)^k > 0.$$

Secondly, if $\gamma \leq t$, then a similar calculation can be done to get the desired inequality (factor $t$ out of the sum and use the binomial theorem).

Note that for all $p \geq 1$,

$$g_t'(\beta) = p\big(|\beta(1-t)+t|^{p-1} - |\beta(1-t)-t|^{p-1}\operatorname{sgn}(\beta(1-t)-t)\big) \geq 0.$$

Now we continue with showing (A.8): By the first parts of the proof, i.e., the proof of (A.1), we have that for all $\beta > 0$,

$$|\beta|^p - \frac{|\beta(1-t)+t|^p - t}{1-t} \asymp \frac{t(\beta-1)^2}{(|\beta|+1)^{2-p}}. \tag{A.9}$$

The strategy for the upper bound will be as follows: Clearly, $(\beta-1)^2 \leq (\beta+1)^2$ for all $\beta > 0$. If we apply this estimate to (A.9), we are left to show that also

$$g_t(\beta) \leq C_p t(\beta+1)^p,$$

for some positive constant $C_p$ in order to show the upper bound in (A.8).

Let us turn to the strategy for the lower bound: It is obvious, that there does not exists a positive constant $C_p$ such that $(\beta-1)^2 \geq C_p(\beta+1)^2$ since the left-hand side has a root at $\beta = 1$. However, fix $0 < \varepsilon < 1$, then we clearly have for all $\beta \in (0,\infty)\setminus(1-\varepsilon, 1+\varepsilon)$ that $(\beta-1)^2 \geq C_{p,\varepsilon}(\beta+1)^2$ for some constant $C_{p,\varepsilon} > 0$. Since $g \geq 0$, we have the desired lower bound of (A.8) using (A.9) in $(0,\infty)\setminus(1-\varepsilon, 1+\varepsilon)$.

For the lower bound, we are left to discuss the compact interval $[1-\varepsilon, 1+\varepsilon]$. On this interval, we clearly have $(\beta+1)^p \asymp 1$. The equivalence (A.9) shows in particular that the corresponding left-hand side is positive. Thus, we are left to show that there exists $C_{p,\varepsilon} > 0$ such that

$$g_t \geq C_{p,\varepsilon} t \qquad \text{on } [1-\varepsilon, 1+\varepsilon].$$

**2.3.1. Lower bound for $1 < p < 2$ and $\beta = -\alpha \geq 0$:** By the discussion before we only have to show that $g_t \geq C_{p,\varepsilon} t$ on $[1-\varepsilon, 1+\varepsilon]$. Since $g_t' \geq 0$, we have for all $\beta \in [1-\varepsilon, 1+\varepsilon]$,

$$g_t(\beta) \geq g_t(1-\varepsilon) = \frac{1}{1-t}\big(((1-\varepsilon)(1-t)+t)^p - |(1-\varepsilon)(1-t)-t|^p\big).$$

Using Lindqvist's lemma, Lemma A.12, we get with $a = (1-\varepsilon)(1-t)+t$ and $b = |(1-\varepsilon)(1-t)-t|$ that

$$\begin{aligned} |a|^p - |b|^p \geq p\,|(1-\varepsilon)(1-t)-t|^{p-1}\big((1-\varepsilon)(1-t)+t-|(1-\varepsilon)(1-t)-t|\big) \\ + C_p \frac{\big((1-\varepsilon)(1-t)+t-|(1-\varepsilon)(1-t)-t|\big)^2}{\big((1-\varepsilon)(1-t)+t+|(1-\varepsilon)(1-t)-t|\big)^{2-p}}. \end{aligned} \tag{A.10}$$

If $(1-\varepsilon)(1-t)-t \geq 0$, i.e., $t \in (0, (1-\varepsilon)/(2-\varepsilon))$, the latter reduces to

$$\begin{aligned} \ldots &= p\big((1-\varepsilon)(1-t)-t\big)^{p-1}(2t) + C_p \frac{(2t)^2}{\big(2(1-\varepsilon)(1-t)\big)^{2-p}} \\ &= t\left(2p\big((1-\varepsilon)(1-t)-t\big)^{p-1} + 4C_p \frac{t}{\big(2(1-\varepsilon)(1-t)\big)^{2-p}}\right) \end{aligned}$$

Using this, we get

$$g_t(\beta) \geq g_t(1-\varepsilon) \geq t\left(2p\frac{\big((1-\varepsilon)(1-t)-t\big)^{p-1}}{1-t} + \frac{4C_p}{(2(1-\varepsilon))^{2-p}} \cdot \frac{t}{(1-t)^{3-p}}\right).$$

Since $t \mapsto \big((1-\varepsilon)(1-t)-t\big)^{p-1}/(1-t)$ is continuous on $[0, (1-\varepsilon)/(2-\varepsilon)]$, strictly positive on $[0, (1-\varepsilon)/(2-\varepsilon))$ and has a root at $t = (1-\varepsilon)/(2-\varepsilon)$, and $t \mapsto t/(1-t)^{3-p}$ is continuous and strictly positive on $(0,1)$, has a root at $t = 0$, we conclude that there is a positive constant which bounds the sum from below on $[0, (1-\varepsilon)/(2-\varepsilon)] \subset [0,1]$.

If $(1-\varepsilon)(1-t)-t < 0$, i.e., $t \in ((1-\varepsilon)/(2-\varepsilon), 1)$, then (A.10) reduces instead to

$$\ldots = p\big(-(1-\varepsilon)(1-t)+t\big)^{p-1}(2(1-\varepsilon)(1-t)) + C_p \frac{(2(1-\varepsilon)(1-t))^2}{(2t)^{2-p}}.$$

Using this, we get

$$g_t(\beta) \geq g_t(1-\varepsilon) \geq t\left(2p(1-\varepsilon)\frac{\big(-(1-\varepsilon)(1-t)+t\big)^{p-1}}{t} + 2^p C_p (1-\varepsilon)^2 \frac{1-t}{t^{3-p}}\right).$$

Since $t \mapsto \big(-(1-\varepsilon)(1-t)+t\big)^{p-1}/t$ is continuous and strictly positive on $((1-\varepsilon)/(2-\varepsilon),1]$ and vanishes at $t=(1-\varepsilon)/(2-\varepsilon)$, and $t \mapsto (1-t)/t^{3-p}$ is continuous and strictly positive on $((1-\varepsilon)/(2-\varepsilon),1)$ (and is only zero at $t=1$), we conclude that there is a positive constant which bounds the sum from below.

This shows the desired lower bound for $1<p<2$ and $\beta \geq 0$.

**2.3.2. Lower bound for $p>2$ and $\beta=-\alpha \geq 0$:** As in the case for $p<2$, it suffices to show that $g_t \geq t \cdot C_{p,\varepsilon} > 0$ on $[1-\varepsilon, 1+\varepsilon]$. Since $g_t' \geq 0$, we have for all $\beta \in [1-\varepsilon,1+\varepsilon]$,

$$g_t(\beta) \geq g_t(1-\varepsilon) = \frac{1}{1-t}\big(((1-\varepsilon)(1-t)+t)^p - |(1-\varepsilon)(1-t)-t|^p\big).$$

Using Lindqvist's lemma, Lemma A.12, we get with $a=(1-\varepsilon)(1-t)+t$ and $b=|(1-\varepsilon)(1-t)-t|$ that

$$\begin{aligned} |a|^p - |b|^p \geq p\,|(1-\varepsilon)(1-t)-t|^{p-1}\,&\big((1-\varepsilon)(1-t)+t-|(1-\varepsilon)(1-t)-t|\big) \\ &+ C_p\,|(1-\varepsilon)(1-t)+t-|(1-\varepsilon)(1-t)-t||^p\,. \end{aligned} \tag{A.11}$$

If $(1-\varepsilon)(1-t)-t \geq 0$, i.e., $t \in (0,(1-\varepsilon)/(2-\varepsilon))$, the latter reduces to

$$\ldots = 2tp\big((1-\varepsilon)(1-t)-t\big)^{p-1} + C_p(2t)^p.$$

Using this, we get

$$g_t(\beta) \geq g_t(1-\varepsilon) \geq t\left(2p\frac{\big((1-\varepsilon)(1-t)-t\big)^{p-1}}{1-t} + 2^pC_p\frac{t^{p-1}}{1-t}\right).$$

Since $t \mapsto \big((1-\varepsilon)(1-t)-t\big)^{p-1}/(1-t)$ is continuous on $[0,(1-\varepsilon)/(2-\varepsilon)]$, strictly positive on $[0,(1-\varepsilon)/(2-\varepsilon))$ and vanishes at $t=(1-\varepsilon)/(2-\varepsilon)$, and $t \mapsto t^{p-1}/(1-t)$ is continuous and strictly positive on $(0,1)$, and vanishes at $t=0$, we conclude that there is a positive constant which bounds the sum from below on $[0,(1-\varepsilon)/(2-\varepsilon)]$.

If $(1-\varepsilon)(1-t)-t<0$, i.e., $t \in ((1-\varepsilon)/(2-\varepsilon),1)$, then (A.11) reduces instead to

$$\ldots = p\big(-(1-\varepsilon)(1-t)+t\big)^{p-1}(2(1-\varepsilon)(1-t)) + 2^pC_p(1-\varepsilon)^p(1-t)^p.$$

Using this, we get

$$\begin{aligned} g_t(\beta) &\geq g_t(1-\varepsilon) \\ &\geq t\left(2p(1-\varepsilon)\frac{\big(-(1-\varepsilon)(1-t)+t\big)^{p-1}}{t} + 2^pC_p(1-\varepsilon)^p\frac{(1-t)^{p-1}}{t}\right). \end{aligned}$$

Since $t \mapsto \big(-(1-\varepsilon)(1-t)+t\big)^{p-1}/t$ is continuous and strictly positive on $((1-\varepsilon)/(2-\varepsilon),1]$ and vanishes only at $t=(1-\varepsilon)/(2-\varepsilon)$, and $t \mapsto (1-t)^{p-1}/t$ is continuous and strictly positive on $((1-\varepsilon)/(2-\varepsilon),1)$ and vanishes only at $t=1$, we conclude that there is a positive constant which bounds the sum from below.

This shows the desired lower bound for $p>2$ and $\beta \geq 0$, and we are left to show the upper bounds.

**2.3.3. Upper bound for $1<p<2$ and $p>2$, and $\beta=-\alpha \geq 0$:** It remains to show that $g_t(\beta) \leq C_p t(\beta+1)^p$ for all $\beta \geq 0$.

Recall that by the convexity of $|\cdot|^p$, we have

$$|a|^p - |b|^p \leq p\,|a|^{p-2}\,a(a-b), \qquad a,b \in \mathbb{R}.$$

Let $a=\beta(1-t)+t$ and $b=|\beta(1-t)-t|$, then we get by the convexity that

$$g_t(\beta) \leq p(\beta(1-t)+t)^{p-1}(\beta(1-t)+t-|\beta(1-t)-t|).$$

Since $\beta(1-t)+t \leq \beta+1$ and $(\beta+1)^{p-1} \leq (\beta+1)^p$ for all $\beta \geq 0$, $1<p<\infty$ and $t \in [0,1]$, we get

$$\ldots \leq p(\beta+1)^p(\beta(1-t)+t-|\beta(1-t)-t|).$$

If $\beta(1-t) \geq t$, then $\beta(1-t)+t-|\beta(1-t)-t| = 2t$. If $\beta(1-t) \leq t$, then $\beta(1-t)+t-|\beta(1-t)-t| = 2\beta(1-t) \leq 2t$. Thus, we get altogether,

$$g_t(\beta) \leq 2pt(\beta+1)^p.$$

This finishes the proof of (A.8) and moreover, it also finishes the proof of (4.10).

*Ad* (4.11)*:* The assertion follows by a simple case analysis. Here are the details: Let

$$f_{t,C}(a) := C\,t^{1/2}\,|a-1| + (1-t)\left(C\,\frac{|a|+1}{2}-1\right) - |a-t|\,, \qquad a \in \mathbb{R}.$$

We have to show that $f_{t,C} \geq 0$ for all $t \in [0,1]$ and $C \geq 2$, $f_{t,C} \leq 0$ for all $t \in [0,1]$ and $0 \leq C \leq 1/2$, and for every $C \in (1/2,2)$, the function $f_{\cdot,C}(\cdot)$ changes sign.

**1. The cases $t \in \{0,1\}$ and $a=t$:** For $t=0$, we have

$$f_{0,C}(a) = \frac{C-2}{2}(|a|+1),$$

which is non-negative for $C \geq 2$ and strictly negative for $C<2$. If $t=1$, then

$$f_{1,C}(a) = (C-1)\,|a-1|\,,$$

which is non-negative for $C \geq 1$ and strictly negative for $C<1$. If $a=t$, then

$$f_{t,C}(t) = (1-t)\left(C\frac{2t^{1/2}+t+1}{2}-1\right),$$

which is non-negative for $C \geq 2$, non-positive for $C \leq 1/2$ and changes sign from negative to positive as $t$ increases in $1/2<C<2$. Hence, it is easy to see that $f_{\cdot,C}(\cdot)$

changes sign for any $1/2 < C < 2$ and an appropriate choice of $t$ by evaluating $f_{t,C}$ at $0, t$ and $1$.

**2. The remaining cases $t \in (0,1)$, $a \neq t$:** Note that for $a \notin \{0, t, 1\}$, we can calculate the derivative, i.e.,

$$f'_{t,C}(a) = C\, t^{1/2} \operatorname{sgn}(a-1) + \frac{C}{2}(1-t) \operatorname{sgn}(a) - \operatorname{sgn}(a-t).$$

We have for all $t \in (0,1)$ and $C \geq 2$,

$$f'_{t,C}(a) = \begin{cases} -C\, t^{1/2} + \frac{C}{2} t + \frac{2-C}{2} \leq 0, & \text{for } a < 0, \\ -C\, t^{1/2} - \frac{C}{2} t + \frac{2+C}{2} \geq 0, & \text{for } 0 < a < t, \\ -C\, t^{1/2} - \frac{C}{2} t + \frac{C-2}{2} \leq 0, & \text{for } t < a < 1, \\ \;\; C\, t^{1/2} - \frac{C}{2} t + \frac{C-2}{2} \geq 0, & \text{for } a > 1. \end{cases}$$

If $0 \leq C \leq 1/2$, then $f'_{t,C}$ has opposite sign on every subinterval.

Hence, $f_{t,C}$ has two extrema, one at $a = 0$ and one at $a = t$. If $C \geq 2$, the extrema are minima, and if $0 \leq C \leq 1/2$, the extrema are maxima. By the computations in the first case and since

$$f_{t,C}(0) = C\, t^{1/2} - \frac{C}{2} t + \frac{C-2}{2},$$

which is non-negative for $C \geq 2$ and non-positive for $0 \leq C \leq 1/2$, it follows that $f_{t,C}$ is non-negative if $C \geq 2$ and non-positive if $0 \leq C \leq 1/2$ for all $t \in [0,1]$ and we have shown that the right-hand side in (4.11) is an upper bound for every $C \geq 2$, and lower bound if $0 \leq C \leq 1/2$.

*Ad* (4.12) *and* (4.13)*:* We will show these inequalities similarly as we showed (4.10). Recall that we have to show that

$$t\, |a-1|^2 \leq t^{p/2} \, |a-1|^p \, (|a-t| + 1 - t)^{2-p}, \qquad 1 < p \leq 2,$$

and

$$t\, |a-1|^2 \, (|a-t| + 1 - t)^{p-2} \geq t^{p/2} \, |a-1|^p \,, \qquad p \geq 2.$$

Note that the inequalities basically come from the fact that for $t \in [0,1]$, we have $t^{p/2} \geq t$ for $1 < p \leq 2$, whereas $t^{p/2} \leq t$ for $p \geq 2$. Here are the details:

**1. The three cases $t \in \{0,1\}$, $a = t$, and $p = 2$:** If $p = 2$, then it is obvious that we have equality for all $a \in \mathbb{R}$ and $t \in [0,1]$.

An easy computation shows that we indeed have equality for $t \in \{0,1\}$.

If $a = t$, then note that $t \in [0,1]$ implies $t^{p/2} \geq t$ for $1 < p \leq 2$, and $t^{p/2} \leq t$ for $p \geq 2$. This immediately yields the desired inequalities.

**2. The remaining cases $t \in (0,1)$, $a \neq t$, and $p \neq 2$:** We consider the cases $a > t$ and $a < t$ separately.

**2.1. The case $a > t$:** Here, we have to show that

$$t\, |a-1|^2 \leq t^{p/2} \, |a-1|^p \, (a + 1 - 2t)^{2-p}, \qquad 1 < p \leq 2 \tag{A.12}$$

as well as

$$t\,|a-1|^2\,(a+1-2t)^{p-2} \geq t^{p/2}\,|a-1|^p\,, \qquad p \geq 2. \tag{A.13}$$

Firstly, consider the case $1 < p < 2$. We clearly have $a+1-2t \geq a-1$. Thus, $(a+1-2t)^{2-p} \geq (a-1)^{2-p}$. Moreover, $t \leq t^{p/2}$ for $t \in (0,1)$. This shows the inequality (A.12).

Secondly, consider the case $p > 2$. Because $(a+1-2t)^{p-2} \geq (a-1)^{p-2}$ as well as $t \geq t^{p/2}$ for $t \in (0,1)$, we get the desired inequality (A.13).

**2.2. The case** $a < t$**:** Note that $a < t < 1$. Thus, we have to show that

$$t(1-a)^2 \leq t^{p/2}(1-a)^p(1-a)^{2-p}, \qquad 1 < p \leq 2$$

as well as

$$t(1-a)^2(1-a)^{p-2} \geq t^{p/2}(1-a)^p, \qquad p \geq 2.$$

Since $t \leq t^{p/2}$ for $1 < p < 2$ and $t \geq t^{p/2}$ for $p > 2$, we get the desired result.

*Ad* (4.14)*:* Recall that there we assume that $p \geq 0$. The desired inequality is clearly fulfilled if $\alpha = \beta = 0$. Thus, assume that both do not vanish at the same time. Setting $t = \alpha/(\alpha+\beta)$, then (4.14) is equivalent to

$$f(t) := t^p + (1-t)^p \asymp 1, \qquad t \in [0,1].$$

If $0 \leq p < 1$, then $f$ has a minimum at $0$ and $1$, and a maximum at $1/2$. If $p \geq 1$, then $f$ has a maximum at $0$ and $1$, and a minimum at $1/2$. Since $f(0) = f(1) = 1$ and $f(1/2) = 2^{1-p}$, we finished the proof. $\square$

## A.3 Estimates for the Simplified Energy

We have seen that a very useful toolbox in achieving the characterisations of criticality or the optimality of $p$-Hardy weights is the ground state representation formula. Here, we show additional estimates of the simplified energy $h_{u,1}$ which were not used in the main part but are of interest in its own.

The proof of the following statement is motivated by a related estimate for the fractional $p$-Laplacian in [AB17; AM16]. In the two papers, the associated statement is proven in the special case where the function $u$ below is a radial Green's function with respect to $0$ on the punctured space $\mathbb{R}^d \setminus \{0\}$.

**Proposition A.13** *Let $p > 1$, $\varphi \in C_c(X)$, $0 \leq u \in F$, $V := \operatorname{supp} u$, and $\psi = u\varphi$. For any $\varepsilon > 0$ we have*

$$\begin{aligned} h_{u,1}(\varphi) \geq (1-2p\varepsilon) \sum_{x,y\in V} b(x,y)\,|\nabla_{x,y}\psi|^p \\ - 2(q\varepsilon)^{1-p}\langle Lu, u\,|\varphi|^p\rangle_V + 2(q\varepsilon)^{1-p}\,\|\psi 1_V\|^p_{p,\deg_{X\setminus V}} \\ + (q\varepsilon)^{1-p} \sum_{x,y\in V} b(x,y)\,(\nabla_{x,y}u)^{\langle p-1\rangle}\left(u(y)\,|\varphi(x)|^p - u(x)\,|\varphi(y)|^p\right). \end{aligned} \tag{A.14}$$

*In particular, if $c = 0$ and $u > 0$ on $X$, and $u$ is $p$-harmonic on $X$, then (A.14) reduces to*

$$h_{u,1}(\varphi) \geq (1 - 2p\varepsilon)h(u\varphi) + (q\varepsilon)^{1-p} \sum_{x,y\in X} b(x,y)\,(\nabla_{x,y}u)^{\langle p-1\rangle}\left(u(y)\,|\varphi(x)|^p - u(x)\,|\varphi(y)|^p\right). \quad \text{(A.15)}$$

*Moreover, we have the following upper bound for the simplified energy,*

$$h_{u,1}(\varphi) \leq \langle \varphi L\varphi, u^p\rangle_X + \frac{1}{2}\sum_{x,y\in X} b(x,y)\,(\nabla_{x,y}\varphi)^{\langle p-1\rangle}\,(\varphi(x)u^p(y) - \varphi(y)u^p(x)). \quad \text{(A.16)}$$

*Proof.* Let $\psi = u\varphi$, then we can write for all $x, y \in \operatorname{supp} u =: V$,

$$\begin{aligned}(u(x)u(y))^{p/2}\,|\nabla_{x,y}\varphi|^p &= (u(x)u(y))^{p/2}\left|\nabla_{x,y}\frac{\psi}{u}\right|^p \\ &= \frac{|\psi(x)u(y) - \psi(y)u(x)|^p}{(u(x)u(y))^{p/2}} \\ &= \left(\frac{u(y)}{u(x)}\right)^{p/2}|\nabla_{x,y}\psi - \varphi(y)\nabla_{x,y}u|^p =: f(x,y).\end{aligned}$$

Hence, interchanging the role of $x$ and $y$ in the previous calculation results in

$$\begin{aligned}h_{u,1}(\varphi) &= \sum_{x,y\in X} b(x,y)(u(x)u(y))^{p/2}\,|\nabla_{x,y}\varphi|^p \\ &= \frac{1}{2}\sum_{x,y\in V} b(x,y)f(x,y) + \frac{1}{2}\sum_{x,y\in V} b(x,y)f(y,x).\end{aligned}$$

Moreover, define

$$Q(x,y) = \frac{(u(x)u(y))^{p/2}}{u^p(x) + u^p(y)}.$$

Then, $1/Q(x,y) = (u(x)/u(y))^{p/2} + (u(y)/u(x))^{p/2}$, and $Q(x,y) \leq 1/2$. The latter can be seen e.g. by the geometric-arithmetic mean inequality.

Furthermore, we get from the convexity of $x \mapsto |x|^p$, $p \geq 1$, easily that

$$|a|^p \geq |b|^p + p\,|b|^{p-2}\,b(a-b), \qquad a, b \in \mathbb{R}.$$

Setting $a = \nabla_{x,y}\psi - \varphi(y)(\nabla_{x,y}u)$ and $b = \nabla_{x,y}\psi$, we get for all $x, y \in V$,

$$\begin{aligned}f(x,y) &\geq 2Q(x,y)f(x,y) \\ &\geq 2Q(x,y)\left(\frac{u(y)}{u(x)}\right)^{p/2}\left(|\nabla_{x,y}\psi|^p - p\,|\nabla_{x,y}\psi|^{p-2}\left(\nabla_{x,y}\psi\right)\left(\varphi(y)\nabla_{x,y}u\right)\right) \\ &\geq 2Q(x,y)\left(\frac{u(y)}{u(x)}\right)^{p/2}\left(|\nabla_{x,y}\psi|^p - p\,|\nabla_{x,y}\psi|^{p-1}\,|\varphi(y)|\,|\nabla_{x,y}u|\right) \\ &\geq 2Q(x,y)\left(\frac{u(y)}{u(x)}\right)^{p/2}|\nabla_{x,y}\psi|^p - 2p\,|\nabla_{x,y}\psi|^{p-1}\,|\varphi(y)|\,|\nabla_{x,y}u|\,,\end{aligned}$$

where the latter inequality follows from

$$Q(x,y)\Big(\frac{u(y)}{u(x)}\Big)^{p/2}=\frac{u^p(y)}{u^p(x)+u^p(y)}\le 1,$$

and an analogue inequality holds for $f(y,x)$. Let us apply the Young inequality, that is $ab\le a^q/q+b^p/p$ for all $a,b\ge 0$ and $q=p/(p-1)$, to

$$j(x,y):=|\nabla_{x,y}\psi|^{p-1}\,|\varphi(y)|\,|\nabla_{x,y}u|\,,$$

and $j(y,x)$, respectively. Thus, we get for some $\varepsilon>0$, and $q=p/(p-1)$, via $a=(q\varepsilon)^{1/q}\,|\nabla_{x,y}\psi|^{p-1}$ and $b=(q\varepsilon)^{-1/q}\left|\frac{\psi(y)}{u(y)}\right|\,|\nabla_{x,y}u|$ that

$$j(x,y)\le\varepsilon\,|\nabla_{x,y}\psi|^p+\frac{(q\varepsilon)^{1-p}}{p}\,|\varphi(y)|^p\,|\nabla_{x,y}u|^p\,.$$

We get altogether,

$$\begin{aligned}
h_{u,1}(\varphi)&=\frac12\sum_{x,y\in V}b(x,y)f(x,y)+\frac12\sum_{x,y\in V}b(x,y)f(y,x)\\
&\ge\sum_{x,y\in V}b(x,y)\frac{Q(x,y)}{Q(x,y)}\,|\nabla_{x,y}\psi|^p\\
&\quad-p\sum_{x,y\in V}b(x,y)\big(j(x,y)+j(y,x)\big)\\
&\ge\sum_{x,y\in V}b(x,y)\,|\nabla_{x,y}\psi|^p-2p\varepsilon\sum_{x,y\in V}b(x,y)\,|\nabla_{x,y}\psi|^p\\
&\quad-(q\varepsilon)^{1-p}\sum_{x,y\in V}b(x,y)\,|\nabla_{x,y}u|^p\,\big(|\varphi(y)|^p+|\varphi(x)|^p\big).
\end{aligned}$$

Moreover, the latter sum can be written as follows,

$$\begin{aligned}
&\sum_{x,y\in V}b(x,y)\,|\nabla_{x,y}u|^p\,\big(|\varphi(y)|^p+|\varphi(x)|^p\big)\\
&\quad=\sum_{x,y\in V}b(x,y)\,(\nabla_{x,y}u)^{\langle p-1\rangle}\,\big(u(x)\,|\varphi(x)|^p-u(y)\,|\varphi(y)|^p\big)\\
&\qquad+\sum_{x,y\in V}b(x,y)\,(\nabla_{x,y}u)^{\langle p-1\rangle}\,\big(u(x)\,|\varphi(y)|^p-u(y)\,|\varphi(x)|^p\big)\\
&\quad=2\sum_{x\in V}u(x)\,|\varphi(x)|^p\sum_{y\in X}b(x,y)\,|\nabla_{x,y}u|^{p-2}\,\nabla_{x,y}u\\
&\qquad-\sum_{x,y\in X}b(x,y)\,(\nabla_{x,y}u)^{\langle p-1\rangle}\,\big(u(y)\,|\varphi(x)|^p-u(x)\,|\varphi(y)|^p\big)\\
&\quad=2\sum_{x\in V}u(x)\,|\varphi(x)|^p\,Lu(x)m(x)-2\sum_{x\in V}|\psi(x)|^p\deg_{X\setminus V}(x)\\
&\qquad-\sum_{x,y\in V}b(x,y)\,(\nabla_{x,y}u)^{\langle p-1\rangle}\,\big(u(y)\,|\varphi(x)|^p-u(x)\,|\varphi(y)|^p\big)
\end{aligned}$$

This proves (A.14). The second inequality is obvious. To prove the third inequality, that is (A.15), let us use again Young's inequality. Thus, $2(u(x)u(y))^{p/2} \leq u^p(x) + u^p(y)$. Moreover, by similar calculations as before, we see

$$\begin{aligned} h_{u,1}(\varphi) &\leq \frac{1}{2} \sum_{x,y \in X} b(x,y)(u^p(x) + u^p(y)) \, |\nabla_{x,y}\varphi|^p \\ &= \langle \varphi L\varphi, u^p \rangle_X + \frac{1}{2} \sum_{x,y \in X} b(x,y) \, (\nabla_{x,y}\varphi)^{\langle p-1 \rangle} \, (\varphi(x)u^p(y) - \varphi(y)u^p(x)), \end{aligned}$$

which is the desired result. □

By Picone's inequality, Lemma 5.7, or the ground state representation, we know that $2\langle Lu, u\,|\varphi|^p \rangle_V \leq \sum_{x,y\in V} b(x,y)\,|\nabla_{x,y}\psi|^p$ which simplifies formula (A.14) minimally.

Moreover, in the case of $p \geq 2$, we might use Estimate (4.3) with constant $c_p > 0$ in combination with (A.15) to obtain for $0 < \varepsilon \neq -(c_p^{-1} - 1)/2p$,

$$h(u\varphi) \geq \frac{(q\varepsilon)^{1-p}}{2p\varepsilon + c_p^{-1} - 1} \sum_{x,y \in X} b(x,y) \, (\nabla_{x,y}u)^{\langle p-1 \rangle} \left( u(y) \, |\varphi(x)|^p - u(x) \, |\varphi(y)|^p \right).$$

# Bibliography


[AB17] B. Abdellaoui and R. Bentifour. "Caffarelli-Kohn-Nirenberg type inequalities of fractional order with applications". In: *J. Funct. Anal.* 272.10 (2017), pages 3998–4029 (cited on page 144).

[ADV04] J. Appell, E. De Pascale, and A. Vignoli. *Nonlinear spectral theory*. Volume 10. De Gruyter Series in Nonlinear Analysis and Applications. Walter de Gruyter & Co., Berlin, 2004, pages xii+408 (cited on page 45).

[AFS24] A. Adriani, F. Fischer, and A. Setti. *Characterisations of $p$-Parabolicity on Graphs. In preparation*. 2024 (cited on pages 6, 15, 71, 77, 84, 85).

[Agm82] S. Agmon. *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of $N$-body Schrödinger operators*. Volume 29. Mathematical Notes. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982, page 118 (cited on pages 2, 65).

[AH98] W. Allegretto and Y. X. Huang. "A Picone's identity for the $p$-Laplacian and applications". In: *Nonlinear Anal.* 32.7 (1998), pages 819–830 (cited on pages 24, 37, 72).

[Ahl52] L. V. Ahlfors. "On the characterization of hyperbolic Riemann surfaces". In: *Ann. Acad. Sci. Fennicae Ser. A. I. Math.-Phys.* 1952.125 (1952), page 5 (cited on page 15).

[All74] W. Allegretto. "On the equivalence of two types of oscillation for elliptic operators". In: *Pacific J. Math.* 55 (1974), pages 319–328 (cited on pages 4, 49).

[AM16] B. Abdellaoui and F. Mahmoudi. "An improved Hardy inequality for a nonlocal operator". In: *Discrete Contin. Dyn. Syst.* 36.3 (2016), pages 1143–1157 (cited on pages 24, 29, 36, 144).

[Amg08] S. Amghibech. "On the discrete version of Picone's identity". In: *Discrete Appl. Math.* 156.1 (2008), pages 1–10 (cited on pages 4, 24, 37, 72).

[Ana87] A. Anane. "Simplicité et isolation de la première valeur propre du $p$-laplacien avec poids". In: *C. R. Acad. Sci. Paris Sér. I Math.* 305.16 (1987), pages 725–728 (cited on page 38).

[And+08] F. Andreu, J. M. Mazón, J. D. Rossi, and J. Toledo. "A nonlocal $p$-Laplacian evolution equation with Neumann boundary conditions". In: *J. Math. Pures Appl. (9)* 90.2 (2008), pages 201–227 (cited on page 4).

[And+09] F. Andreu, J. M. Mazón, J. D. Rossi, and J. Toledo. "A nonlocal $p$-Laplacian evolution equation with nonhomogeneous Dirichlet boundary conditions". In: *SIAM J. Math. Anal.* 40.5 (2009), pages 1815–1851 (cited on page 4).

[AS60] L. V. Ahlfors and L. Sario. *Riemann surfaces*. Volume No. 26. Princeton University Press, Princeton, N.J., 1960, pages xi+382 (cited on page 15).

[Bar37] J. Barta. "Sur la vibration fondamentale d'une membrane." French. In: *C. R. Acad. Sci., Paris* 204 (1937), pages 472–473 (cited on page 72).

[BB11] A. Björn and J. Björn. *Nonlinear potential theory on metric spaces*. Volume 17. EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2011, pages xii+403 (cited on pages 31, 33, 59, 101).

[BEL15] A. A. Balinsky, W. D. Evans, and R. T. Lewis. *The analysis and geometry of Hardy's inequality*. Universitext. Springer, Cham, 2015, pages xv+263 (cited on pages 2, 4, 116, 117).

[Ber+17a] E. Berchio, L. D'Ambrosio, D. Ganguly, and G. Grillo. "Improved $L^p$-Poincaré inequalities on the hyperbolic space". In: *Nonlinear Anal.* 157 (2017), pages 146–166 (cited on page 113).

[Ber+17b] G. Berkolaiko, J. B. Kennedy, P. Kurasov, and D. Mugnolo. "Edge connectivity and the spectral gap of combinatorial and quantum graphs". In: *J. Phys. A* 50.36 (2017), pages 365201, 29 (cited on page 43).

[Ber+20] E. Berchio, D. Ganguly, G. Grillo, and Y. Pinchover. "An optimal improvement for the Hardy inequality on the hyperbolic space and related manifolds". English. In: *Proc. R. Soc. Edinb., Sect. A, Math.* 150.4 (2020), pages 1699–1736 (cited on pages 86, 116, 117).

[BF14] L. Brasco and G. Franzina. "Convexity properties of Dirichlet integrals and Picone-type inequalities". In: *Kodai Math. J.* 37.3 (2014), pages 769–799 (cited on pages 4, 24, 27, 37).

[BGG17] E. Berchio, D. Ganguly, and G. Grillo. "Sharp Poincaré-Hardy and Poincaré-Rellich inequalities on the hyperbolic space". In: *J. Funct. Anal.* 272.4 (2017), pages 1661–1703 (cited on pages 113, 119).

[BH09] T. Bühler and M. Hein. "Spectral Clustering Based on the Graph P-Laplacian". In: *Proceedings of the 26th Annual International Conference on Machine Learning*. ICML '09. Montreal, Quebec, Canada: Association for Computing Machinery, 2009, pages 81–88 (cited on pages 4, 43).

[BK04] M. Belloni and B. Kawohl. "The pseudo-$p$-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$." English. In: *ESAIM, Control Optim. Calc. Var.* 10 (2004), pages 28–52 (cited on page 10).

[Bog+22] K. Bogdan, T. Jakubowski, J. Lenczewska, and K. Pietruska-Pałuba. "Optimal Hardy inequality for the fractional Laplacian on $L^p$". In: *J. Funct. Anal.* 282.8 (2022), Paper No. 109395, 31 (cited on pages 15, 115).

[BSV21] E. Berchio, F. Santagati, and M. Vallarino. "Poincaré and Hardy inequalities on homogeneous trees". In: *Geometric properties for parabolic and elliptic PDEs*. Volume 47. Springer INdAM Ser. Springer, Cham, 2021, pages 1–22 (cited on pages 113, 115).

[Caz21] C. Cazacu. "The method of super-solutions in Hardy and Rellich type inequalities in the $L^2$ setting: an overview of well-known results and short proofs". In: *Rev. Roumaine Math. Pures Appl.* 66.3-4 (2021), pages 617–638 (cited on page 119).

[Che75] S. Y. Cheng. "Eigenvalue comparison theorems and its geometric applications". In: *Math. Z.* 143.3 (1975), pages 289–297 (cited on page 73).

[CL11] S.-Y. Chung and H. Lee. "The $p$-Laplacian operators with potential terms". In: *Commun. Korean Math. Soc.* 26.4 (2011), pages 591–601 (cited on page 4).

[CMS18] W. Chen, S. Mosconi, and M. Squassina. "Nonlocal problems with critical Hardy nonlinearity". In: *J. Funct. Anal.* 275.11 (2018), pages 3065–3114 (cited on page 4).

[CSS21] N. A. Cantizano, A. M. Salort, and J. F. Spedaletti. "Continuity of solutions for the $\Delta_\varphi$-Laplacian operator". In: *Proc. Roy. Soc. Edinburgh Sect. A* 151.4 (2021), pages 1355–1382 (cited on page 12).

[CY75] S. Y. Cheng and S. T. Yau. "Differential equations on Riemannian manifolds and their geometric applications". In: *Comm. Pure Appl. Math.* 28.3 (1975), pages 333–354 (cited on page 15).

[Cyc+87] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry*. Study. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1987, pages x+319 (cited on page 20).

[Dav89] E. B. Davies. *Heat kernels and spectral theory*. English. Volume 92. Cambridge etc.: Cambridge University Press, 1989, pages ix + 197 (cited on page 23).

[Dav99] E. B. Davies. "A review of Hardy inequalities". English. In: *The Maz'ya anniversary collection. Vol. 2: Rostock conference on functional analysis, partial differential equations and applications, Rostock, Germany, August 31–September 4, 1998*. Basel: Birkhäuser, 1999, pages 55–67 (cited on page 115).

[DD14] L. D'Ambrosio and S. Dipierro. "Hardy inequalities on Riemannian manifolds and applications". English. In: *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 31.3 (2014), pages 449–475 (cited on page 2).

[Den70] J. Deny. "Méthodes hilbertiennes en théorie du potentiel". In: *Potential Theory (C.I.M.E., I Ciclo, Stresa, 1969)*. Edizioni Cremonese, Rome, 1970, pp 121–201 (cited on page 15).

[DFP14] B. Devyver, M. Fraas, and Y. Pinchover. "Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon". In: *J. Funct. Anal.* 266.7 (2014), pages 4422–4489 (cited on pages 3, 17).

[DH98] E. B. Davies and A. M. Hinz. "Explicit constants for Rellich inequalities in $L_p(\Omega)$". In: *Math. Z.* 227.3 (1998), pages 511–523 (cited on page 119).

[DKP16] A. Di Castro, T. Kuusi, and G. Palatucci. "Local behavior of fractional $p$-minimizers". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 33.5 (2016), pages 1279–1299 (cited on page 4).

[DKP24] U. Das, M. Keller, and Y. Pinchover. *The space of Hardy-weights for quasilinear operators on discrete graphs*. 2024. arXiv: `2407.02116 [math.AP]` (cited on page 13).

[DL22] K. Devriendt and R. Lambiotte. "Discrete curvature on graphs from the effective resistance". In: *Journal of Physics: Complexity* 3.2 (2022), page 025008 (cited on page 59).

[Dod84] J. Dodziuk. "Difference equations, isoperimetric inequality and transience of certain random walks". In: *Trans. Amer. Math. Soc.* 284.2 (1984), pages 787–794 (cited on page 4).

[DP16] B. Devyver and Y. Pinchover. "Optimal $L^p$ Hardy-type inequalities". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 33.1 (2016), pages 93–118 (cited on pages 2, 3, 4, 17, 24, 28, 97, 98, 100, 106, 113, 119).

[DP23] U. Das and Y. Pinchover. "The space of Hardy-weights for quasilinear equations: Maz'ya-type characterization and sufficient conditions for existence of minimizers". In: *J. Anal. Math.* tbd (2023), tbd (cited on pages 13, 67).

[DS87] J. I. Díaz and J. E. Saá. "Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires". In: *C. R. Acad. Sci. Paris Sér. I Math.* 305.12 (1987), pages 521–524 (cited on page 38).

[Ell26] E. B. Elliott. "A Simple Exposition of Some Recently Proved Facts as to Convergency". In: *J. London Math. Soc.* 1.2 (1926), pages 93–96 (cited on page 1).

[Fis22] F. Fischer. *Quasi-Linear Criticality Theory and Green's Functions on Graphs*. To appear in Journal d'Analyse Mathematique. 2022. arXiv: 2207.05445 [math-ph] (cited on pages 5, 19, 24, 27, 59, 62, 76).

[Fis23] F. Fischer. "A non-local quasi-linear ground state representation and criticality theory". In: *Calc. Var. Partial Differential Equations* 62.5 (2023), Paper No. 163 (cited on pages 5, 19, 76, 101, 133).

[Fis24] F. Fischer. "On the optimality and decay of $p$-Hardy weights on graphs". In: *Calc. Var. Partial Differential Equations* 63.7 (2024), Paper No. 162 (cited on pages 5, 19, 28, 35, 76, 89, 91).

[FKP23] F. Fischer, M. Keller, and F. Pogorzelski. "An improved discrete $p$-Hardy inequality". In: *Integral Equations Operator Theory* 95.4 (2023), Paper No. 24, 17 (cited on pages 5, 85, 112, 125).

[FLW14] R. L. Frank, D. Lenz, and D. Wingert. "Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory". In: *J. Funct. Anal.* 266.8 (2014), pages 4765–4808 (cited on page 23).

[FOT11] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*. extended. Volume 19. de Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 2011, pages x+489 (cited on page 15).

[FP11] M. Fraas and Y. Pinchover. "Positive Liouville theorems and asymptotic behavior for $p$-Laplacian type elliptic equations with a Fuchsian potential". In: *Confluentes Math.* 3.2 (2011), pages 291–323 (cited on page 57).

[FP23] F. Fischer and N. Peyerimhoff. *Sharp Hardy-type inequalities for non-compact harmonic manifolds and Damek-Ricci spaces*. To appear in Israel Journal of Mathematics. 2023. arXiv: 2305.01288 [math.AP] (cited on pages 21, 70, 71, 86, 113, 115).

[FS08] R. L. Frank and R. Seiringer. "Non-linear ground state representations and sharp Hardy inequalities". In: *J. Funct. Anal.* 255.12 (2008), pages 3407–3430 (cited on pages 4, 24, 27, 29, 36, 38).

[GHJ21] H. Ge, B. Hua, and W. Jiang. "The limit of first eigenfunctions of the $p$-Laplacian on graphs". In: *Pacific J. Math.* 312.1 (2021), pages 103–112 (cited on pages 2, 43).

[GKS22] B. Gerhat, D. Krejcirik, and F. Stampach. *An improved discrete Rellich inequality on the half-line*. 2022. arXiv: 2206.11007 [math.SP] (cited on page 119).

[GP23] R. K. Giri and Y. Pinchover. "Positive solutions of quasilinear elliptic equations with Fuchsian potentials in Wolff class". English. In: *Milan J. Math.* 91.1 (2023), pages 59–96 (cited on page 33).

[Gri99] A. Grigor'yan. "Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds". In: *Bull. Amer. Math. Soc. (N.S.)* 36.2 (1999), pages 135–249 (cited on page 15).

[GS98] J. García-Melián and J. Sabina de Lis. "Maximum and comparison principles for operators involving the $p$-Laplacian". In: *J. Math. Anal. Appl.* 218.1 (1998), pages 49–65 (cited on pages 5, 38, 45, 53, 56).

[Gup23] S. Gupta. "Hardy and Rellich inequality on lattices". In: *Calc. Var. Partial Differential Equations* 62.3 (2023), Paper No. 81, 18 (cited on page 119).

[Har20] G. H. Hardy. "Note on a theorem of Hilbert". In: *Math. Z.* 6.3-4 (1920), pages 314–317 (cited on page 1).

[HK11] S. Haeseler and M. Keller. "Generalized solutions and spectrum for Dirichlet forms on graphs". In: *Random walks, boundaries and spectra*. Volume 64. Progr. Probab. Birkhäuser/Springer Basel AG, Basel, 2011, pages 181–199 (cited on pages 23, 50).

[HK14] B. Hua and M. Keller. "Harmonic functions of general graph Laplacians". In: *Calc. Var. Partial Differential Equations* 51.1-2 (2014), pages 343–362 (cited on page 100).

[HKM06] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY, 2006, pages xii+404 (cited on pages 12, 59, 101).

[HLP34] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge, at the University Press, 1934 (cited on page 1).

[HM15] B. Hua and D. Mugnolo. "Time regularity and long-time behavior of parabolic $p$-Laplace equations on infinite graphs". In: *J. Differential Equations* 259.11 (2015), pages 6162–6190 (cited on pages 2, 11).

[HPR24] Y. Hou, Y. Pinchover, and A. Rasila. "Positive solutions of the $\mathcal{A}$-Laplace equation with a potential". English. In: *Potential Anal.* 60.2 (2024), pages 721–758 (cited on pages 2, 4, 5, 12, 89, 90).

[HS01] S. Heikkilä and S. Seikkala. "Maximum principles and uniqueness results for phi-Laplacian boundary value problems". In: *J. Inequal. Appl.* 6.3 (2001), pages 339–357 (cited on page 12).

[HS97a] I. Holopainen and P. M. Soardi. "$p$-harmonic functions on graphs and manifolds." In: *Manuscripta mathematica* 94.1 (1997), pages 95–110 (cited on pages 4, 5, 32, 53, 59).

[HS97b] I. Holopainen and P. M. Soardi. "A strong Liouville theorem for $p$-harmonic functions on graphs". In: *Ann. Acad. Sci. Fenn. Math.* 22.1 (1997), pages 205–226 (cited on page 4).

[Hua21] Y. C. Huang. "A first order proof of the improved discrete Hardy inequality". In: *Arch. Math. (Basel)* 117.6 (2021), pages 671–674 (cited on page 1).

[Hua23a] Y. C. Huang. "A remark on Davies' Hardy inequality". English. In: *Funct. Anal. Appl.* 57.1 (2023), pages 83–86 (cited on page 115).

[Hua23b] Y. C. Huang. *Change of variable and discrete Hardy inequality*. 2023. arXiv: 2307.04971 [math.CA] (cited on page 1).

[Hun54] G. A. Hunt. "On positive Green's functions". In: *Proc. Nat. Acad. Sci. U.S.A.* 40 (1954), pages 816–818 (cited on page 15).

[HY24] X. Huang and D. Ye. "One-dimensional sharp discrete Hardy-Rellich inequalities". English. In: *J. Lond. Math. Soc., II. Ser.* 109.1 (2024). Id/No e12838, page 26 (cited on pages 15, 119).

[Kak53] S. Kakutani. "Random walk and the type problem of Riemann surfaces". In: *Contributions to the theory of Riemann surfaces*. Volume no. 30. Annals of Mathematics Studies. Princeton University Press, Princeton, N.J., 1953, pp 95–101 (cited on page 15).

[KLW21] M. Keller, D. Lenz, and R. K. Wojciechowski. *Graphs and discrete Dirichlet spaces*. English. Cham: Springer, 2021, pages xv + 668 (cited on pages 2, 5, 15, 21, 23, 33, 40, 50, 53, 71, 75, 85, 100).

[KM16] M. Keller and D. Mugnolo. "General Cheeger inequalities for $p$-Laplacians on graphs". In: *Nonlinear Anal.* 147 (2016), pages 80–95 (cited on pages 2, 43).

[KMM07] J. Kinnunen, N. Marola, and O. Martio. "Harnack's principle for quasiminimizers". English. In: *Ric. Mat.* 56.1 (2007), pages 73–88 (cited on page 33).

[KMP06] A. Kufner, L. Maligranda, and L.-E. Persson. "The prehistory of the Hardy inequality". In: *Amer. Math. Monthly* 113.8 (2006), pages 715–732 (cited on page 1).

[KMS15] T. Kuusi, G. Mingione, and Y. Sire. "Nonlocal equations with measure data". In: *Comm. Math. Phys.* 337.3 (2015), pages 1317–1368 (cited on page 4).

[KN23] M. Keller and M. Nietschmann. "Optimal Hardy Inequality for Fractional Laplacians on the Integers". In: *Ann. Henri Poincaré* 24 (2023), pages 2729–2741 (cited on page 12).

[KÖ09] I. Kombe and M. Özaydin. "Improved Hardy and Rellich inequalities on Riemannian manifolds". In: *Trans. Amer. Math. Soc.* 361.12 (2009), pages 6191–6203 (cited on pages 116, 117).

[KÖ13] I. Kombe and M. Özaydin. "Hardy-Poincaré, Rellich and uncertainty principle inequalities on Riemannian manifolds". In: *Trans. Amer. Math. Soc.* 365.10 (2013), pages 5035–5050 (cited on pages 116, 117).

[Kos21] A. Kostenko. "Heat kernels of the discrete Laguerre operators". English. In: *Lett. Math. Phys.* 111.2 (2021). Id/No 32, page 29 (cited on page 20).

[KP20] H. Kovařík and Y. Pinchover. "On minimal decay at infinity of Hardy-weights". English. In: *Commun. Contemp. Math.* 22.5 (2020). Id/No 1950046, page 18 (cited on pages 3, 17, 89, 97).

[KPP18a] M. Keller, Y. Pinchover, and F. Pogorzelski. "An improved discrete Hardy inequality". In: *Amer. Math. Monthly* 125 (2018), pages 347–350 (cited on page 1).

[KPP18b] M. Keller, Y. Pinchover, and F. Pogorzelski. "Optimal Hardy inequalities for Schrödinger operators on graphs". In: *Commun. Math. Phys.* 358 (2018), pages 767–790 (cited on pages 3, 5, 17, 28, 97, 98, 102, 106, 110).

[KPP20a] M. Keller, Y. Pinchover, and F. Pogorzelski. "Critical Hardy Inequalities on Manifolds and Graphs". In: *Analysis and Geometry on Graphs and Manifolds*. Edited by M. Keller, D. Lenz, and R. K. Wojciechowski. London Mathematical Society Lecture Note Series. Cambridge University Press, 2020, pages 172–202 (cited on pages 5, 102, 106, 113).

[KPP20b] M. Keller, Y. Pinchover, and F. Pogorzelski. "Criticality theory for Schrödinger operators on graphs". In: *J. Spectr. Theory* 10.1 (2020), pages 73–114 (cited on pages 4, 12, 23, 24, 32, 33, 35, 49, 53, 57, 65, 69, 75).

[KPP21] M. Keller, Y. Pinchover, and F. Pogorzelski. "From Hardy to Rellich inequalities on graphs". In: *Proc. Lond. Math. Soc. (3)* 122.3 (2021), pages 458–477 (cited on pages 28, 117, 119, 120, 122).

[KPS17] A. Kufner, L.-E. Persson, and N. Samko. *Weighted inequalities of Hardy type*. Second. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017, pages xx+459 (cited on page 2).

[Kri18] A. Kristály. "Sharp uncertainty principles on Riemannian manifolds: the influence of curvature". In: *J. Math. Pures Appl. (9)* 119 (2018), pages 326–346 (cited on page 116).

[KŠ22] D. Krejčiřík and F. Štampach. "A sharp form of the discrete Hardy inequality and the Keller-Pinchover-Pogorzelski inequality". English. In: *Am. Math. Mon.* 129.3 (2022), pages 281–283 (cited on page 27).

[KY84] T. Kayano and M. Yamasaki. "Boundary limit of discrete Dirichlet potentials". In: *Hiroshima Math. J.* 14.2 (1984), pages 401–406 (cited on pages 2, 15).

[Lan21] E. Landau. *A letter to G. H. Hardy, June 21, 1921*. 1921 (cited on page 1).

[Lef19] P. Lefèvre. "A short direct proof of the discrete Hardy's inequality". In: *Archiv der Mathematik* (2019), pages 1–4 (cited on page 1).

[Lef20] P. Lefèvre. *Weighted Discrete Hardy's Inequalities*. 2020 (cited on page 1).

[Lin19] P. Lindqvist. *Notes on the stationary $p$-Laplace equation*. SpringerBriefs in Mathematics. Springer, Cham, 2019, pages xi+104 (cited on pages 10, 70).

[Lin90] P. Lindqvist. "On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$". In: *Proc. Amer. Math. Soc.* 109.1 (1990), pages 157–164 (cited on pages 29, 133).

[LP19] P. D. Lamberti and Y. Pinchover. "$L^p$ Hardy inequality on $C^{1,\gamma}$ domains". English. In: *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* 19.3 (2019), pages 1135–1159 (cited on page 67).

[LSV09] D. Lenz, P. Stollmann, and I. Veselić. "The Allegretto-Piepenbrink theorem for strongly local Dirichlet forms". In: *Doc. Math.* 14 (2009), pages 167–189 (cited on pages 4, 49).

[Mae77] F.-Y. Maeda. "A remark on parabolic index of infinite networks". In: *Hiroshima Math. J.* 7.1 (1977), pages 147–152 (cited on pages 84, 114).

[Maz11] V. G. Maz'ya. *Sobolev spaces. With applications to elliptic partial differential equations. Transl. from the Russian by T. O. Shaposhnikova*. English. 2nd revised and augmented ed. Volume 342. Grundlehren Math. Wiss. Berlin: Springer, 2011 (cited on page 59).

[Mic99] L. Miclo. "An example of application of discrete Hardy's inequalities". In: *Markov Process. Related Fields* 5.3 (1999), pages 319–330 (cited on page 15).

[Miu23] Y. Miura. "Optimal Hardy inequalities for Schrödinger operators based on symmetric stable processes". In: *J. Theoret. Probab.* 36.1 (2023), pages 134–166 (cited on page 15).

[MR22] Á. Martínez-Pérez and J. M. Rodríguez. "On $p$-parabolicity of Riemannian manifolds and graphs". In: *Rev. Mat. Complut.* 35.1 (2022), pages 179–198 (cited on page 15).

[MR24a] Á. Martínez-Pérez and J. M. Rodríguez. "Parabolicity and Cheeger's constant on graphs". In: *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* 118.4 (2024), Paper No. 139 (cited on page 15).

[MR24b] Á. Martínez-Pérez and J. M. Rodríguez. "Parabolicity on Graphs". In: *Results Math.* 79.2 (2024), Paper no. 70 (cited on page 15).

[MS23] S. Murmann and M. Schmidt. *Harris' criterion and Hardy inequalites on graphs*. 2023. arXiv: `2303.07092 [math.FA]` (cited on pages 2, 9, 18, 24).

[Mug13] D. Mugnolo. "Parabolic theory of the discrete $p$-Laplace operator". In: *Nonlinear Anal.* 87 (2013), pages 33–60 (cited on pages 2, 10, 11, 69).

[Mur84] M. Murata. "Positive solutions and large time behaviors of Schrödinger semigroups, Simon's problem". In: *J. Funct. Anal.* 56.3 (1984), pages 300–310 (cited on page 15).

[Mur86] M. Murata. "Structure of positive solutions to $(-\Delta + V)u = 0$ in $\mathbf{R}^n$". In: *Duke Math. J.* 53.4 (1986), pages 869–943 (cited on pages 15, 75).

[MY92] A. Murakami and M. Yamasaki. "Nonlinear potentials on an infinite network". In: *Mem. Fac. Sci. Shimane Univ.* 26 (1992), pages 15–28 (cited on page 2).

[Nev40] R. Nevanlinna. "Ein Satz über offene Riemannsche Flächen". German. In: *Ann. Acad. Sci. Fenn., Ser. A* 54 (1940), pages 1–18 (cited on page 15).

[NP92] R. D. Nussbaum and Y. Pinchover. "On variational principles for the generalized principal eigenvalue of second order elliptic operators and some applications". In: volume 59. Festschrift on the occasion of the 70th birthday of Shmuel Agmon. 1992, pages 161–177 (cited on page 72).

[NY76] T. Nakamura and M. Yamasaki. "Generalized extremal length of an infinite network". In: *Hiroshima Math. J.* 6.1 (1976), pages 95–111 (cited on page 2).

[OK90] B. Opic and A. Kufner. *Hardy-type inequalities*. Volume 219. Pitman Research Notes in Mathematics Series. Longman Scientific & Technical, Harlow, 1990 (cited on page 2).

[PC11] J.-H. Park and S.-Y. Chung. "The Dirichlet boundary value problems for $p$-Schrödinger operators on finite networks". In: *J. Difference Equ. Appl.* 17.5 (2011), pages 795–811 (cited on pages 4, 40).

[Pic10] M. Picone. "Sui valori eccezionali di un parametro da cui dipende un'equazione differenziale lineare ordinaria del secondo ordine." Italian. In: *Pisa Ann.* 11 (1910), 141 s. (Cited on page 24).

[Pie74] J. Piepenbrink. "Nonoscillatory elliptic equations". In: *J. Differential Equations* 15 (1974), pages 541–550 (cited on pages 4, 49).

[Pin07] Y. Pinchover. "Topics in the theory of positive solutions of second-order elliptic and parabolic partial differential equations". In: *Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday*. Volume 76. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2007, pages 329–355 (cited on pages 75, 86).

[Pin88] Y. Pinchover. "On positive solutions of second-order elliptic equations, stability results, and classification". In: *Duke Math. J.* 57.3 (1988), pages 955–980 (cited on pages 15, 75).

[PKC09] J.-H. Park, J.-H. Kim, and S.-Y. Chung. "The $p$-Schrödinger equations on finite networks". In: *Publ. Res. Inst. Math. Sci.* 45.2 (2009), pages 363–381 (cited on pages 4, 24, 37, 38, 40, 43, 45).

[Pól21] G. Pólya. "Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz." German. In: *Math. Ann.* 84 (1921), pages 149–160 (cited on page 15).

[PP16] Y. Pinchover and G. Psaradakis. "On positive solutions of the $(p, A)$-Laplacian with potential in Morrey space". In: *Anal. PDE* 9.6 (2016), pages 1317–1358 (cited on pages 2, 4, 5, 38, 40, 43, 45, 49, 50, 53, 54, 56, 65, 66, 79, 90, 92, 93).

[PR15] Y. Pinchover and N. Regev. "Criticality theory of half-linear equations with the $(p, A)$-Laplacian". English. In: *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods* 119 (2015), pages 295–314 (cited on pages 2, 4, 5, 23, 25, 38, 42, 65, 67, 86, 90, 92).

[Pra04] L. M.-G. Prado. *p-Potential theory on graphs: p-Parabolicity and p-hyperbolicity*. Thesis (Ph.D.)–City University of New York. ProQuest LLC, Ann Arbor, MI, 2004, page 143 (cited on pages 2, 5, 10, 32, 53, 59, 61, 62, 114).

[PRS02] S. Pigola, M. Rigoli, and A. G. Setti. "Maximum principles and singular elliptic inequalities". In: *J. Funct. Anal.* 193.2 (2002), pages 224–260 (cited on page 12).

[PRS05] S. Pigola, M. Rigoli, and A. G. Setti. "Maximum principles on Riemannian manifolds and applications". In: *Mem. Amer. Math. Soc.* 174.822 (2005), pages x+99 (cited on pages 12, 15).

[PS05] A. Poliakovsky and I. Shafrir. "Uniqueness of positive solutions for singular problems involving the $p$-Laplacian". In: *Proc. Amer. Math. Soc.* 133.9 (2005), pages 2549–2557 (cited on page 106).

[PS07] P. Pucci and J. Serrin. *The maximum principle*. English. Volume 73. Basel: Birkhäuser, 2007, pages x + 235 (cited on page 45).

[PS14] S. Pigola and A. G. Setti. *Global divergence theorems in nonlinear PDEs and geometry*. Volume 26. Ensaios Matemáticos [Mathematical Surveys]. Sociedade Brasileira de Matemática, Rio de Janeiro, 2014, pages ii+77 (cited on pages 12, 15).

[PT07] Y. Pinchover and K. Tintarev. "Ground state alternative for $p$-Laplacian with potential term". English. In: *Calc. Var. Partial Differ. Equ.* 28.2 (2007), pages 179–201 (cited on pages 50, 59, 65, 90, 91, 92, 93).

[PT08] Y. Pinchover and K. Tintarev. "On positive solutions of minimal growth for singular $p$-Laplacian with potential term". English. In: *Adv. Nonlinear Stud.* 8.2 (2008), pages 213–234 (cited on pages 2, 5, 59, 65, 67).

[PT09] Y. Pinchover and K. Tintarev. "On positive solutions of $p$-Laplacian-type equations". In: *Analysis, partial differential equations and applications*. Volume 193. Oper. Theory Adv. Appl. Birkhäuser Verlag, Basel, 2009, pages 245–267 (cited on pages 2, 4, 5, 59).

[PTT08] Y. Pinchover, A. Tertikas, and K. Tintarev. "A Liouville-type theorem for the $p$-Laplacian with potential term". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 25.2 (2008), pages 357–368 (cited on pages 2, 4, 23, 24, 25, 26, 27, 37, 86).

[Rel56] F. Rellich. "Halbbeschränkte Differentialoperatoren höherer Ordnung". In: *Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III*. Erven P. Noordhoff N. V., Groningen; North-Holland Publishing Co., Amsterdam, 1956, pp 243–250 (cited on page 119).

[Rel69] F. Rellich. *Perturbation theory of eigenvalue problems*. Assisted by J. Berkowitz., With a preface by Jacob T. Schwartz. Gordon and Breach Science Publishers, New York-London-Paris, 1969, pages x+127 (cited on page 119).

[Rob18] D. W. Robinson. "Hardy inequalities, Rellich inequalities and local Dirichlet forms". In: *J. Evol. Equ.* 18.3 (2018), pages 1521–1541 (cited on pages 119, 120).

[Roy52] H. L. Royden. "Harmonic functions on open Riemann surfaces". In: *Trans. Amer. Math. Soc.* 73 (1952), pages 40–94 (cited on page 15).

[RS19] M. Ruzhansky and D. Suragan. *Hardy inequalities on homogeneous groups*. Volume 327. Progress in Mathematics. 100 years of Hardy inequalities. Birkhäuser Springer, Cham, 2019, pages xvi+571 (cited on page 2).

[RS72] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis.* Academic Press, New York-London, 1972, pages xvii+325 (cited on page 20).

[Sch22] M. Schmidt. "(Weak) Hardy and Poincaré inequalities and criticality theory". In: *Dirichlet forms and related topics*. Volume 394. Springer Proc. Math. Stat. Springer, Singapore, 2022, pages 421–459 (cited on page 15).

[Shi21] R. Shimizu. "Parabolic index of an infinite graph and Ahlfors regular conformal dimension of a self-similar set". English. In: *Analysis and partial differential equations on manifolds, fractals and graphs. Contributions of the conference, Tianjin, China, September 2019*. Berlin: De Gruyter, 2021, pages 201–274 (cited on page 15).

[Sho97] R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*. Volume 49. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997, pages xiv+278 (cited on page 17).

[Sim80] B. Simon. "Brownian motion, $L^p$ properties of Schrödinger operators and the localization of binding". In: *J. Funct. Anal.* 35.2 (1980), pages 215–229 (cited on pages 15, 75).

[SRA21] S. H. Saker, S. S. Rabie, and R. P. Agarwal. "On discrete weighted Hardy type inequalities and properties of weighted discrete Muckenhoupt classes". In: *J. Inequal. Appl.* (2021), Paper No. 168, 19 (cited on page 15).

[Str08] M. Struwe. *Variational methods*. Fourth. Volume 34. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Applications to nonlinear partial differential equations and Hamiltonian systems. Springer-Verlag, Berlin, 2008, pages xx+302 (cited on page 45).

[Stu94] K.-T. Sturm. "Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and $L^p$-Liouville properties". In: *J. Reine Angew. Math.* 456 (1994), pages 173–196 (cited on page 15).

[Sul87] D. Sullivan. "Related aspects of positivity in Riemannian geometry". In: *J. Differential Geom.* 25.3 (1987), pages 327–351 (cited on page 4).

[SW04] R. S. Strichartz and C. Wong. "The $p$-Laplacian on the Sierpinski gasket". In: *Nonlinearity* 17.2 (2004), pages 595–616 (cited on page 14).

[SY93a] P. M. Soardi and M. Yamasaki. "Classification of infinite networks and its application". English. In: *Circuits Syst. Signal Process.* 12.1 (1993), pages 133–149 (cited on pages 2, 3, 15).

[SY93b] P. M. Soardi and M. Yamasaki. "Parabolic index and rough isometries". In: *Hiroshima Math. J.* 23.2 (1993), pages 333–342 (cited on pages 2, 15).

[SYZ23] M. Shao, Y. Yang, and L. Zhao. *Sobolev spaces on locally finite graphs*. 2023. arXiv: `2306.02262 [math.AP]` (cited on page 18).

[Tak03] H. Takeuchi. "The spectrum of the $p$-Laplacian and $p$-harmonic morphisms on graphs". In: *Illinois J. Math.* 47.3 (2003), pages 939–955 (cited on pages 10, 13).

[Tak14] M. Takeda. "Criticality and subcriticality of generalized Schrödinger forms". In: *Illinois J. Math.* 58.1 (2014), pages 251–277 (cited on page 15).

[Tak16] M. Takeda. "Criticality for Schrödinger type operators based on recurrent symmetric stable processes". In: *Trans. Amer. Math. Soc.* 368.1 (2016), pages 149–167 (cited on page 15).

[Tak23] M. Takeda. "Optimal hardy-type inequalities for Schrödinger forms". In: *Osaka Journal of Mathematics* 60.4 (2023), pages 761–776 (cited on page 15).

[Tro00] M. Troyanov. "Solving the $p$-Laplacian on manifolds". English. In: *Proc. Am. Math. Soc.* 128.2 (2000), pages 541–545 (cited on pages 15, 59).

[Tro99] M. Troyanov. "Parabolicity of manifolds". English. In: *Sib. Adv. Math.* 9.4 (1999), pages 125–150 (cited on pages 15, 59, 60).

[TU23] M. Takeda and T. Uemura. "Criticality of Schrödinger forms and recurrence of Dirichlet forms". In: *Trans. Amer. Math. Soc.* 376.6 (2023), pages 4145–4171 (cited on page 15).

[TZ24] Y. Tian and L. Zhao. *A note for $W^{1,p}(V)$ and $W_0^{1,p}(V)$ on a locally finite graph*. 2024. arXiv: `2406.08053 [math.AP]` (cited on pages 10, 18).

[Ura99] H. Urakawa. "Eigenvalue comparison theorems of the discrete Laplacians for a graph". In: *Geom. Dedicata* 74.1 (1999), pages 95–112 (cited on pages 72, 73).

[Val12a] D. Valtorta. *On the p-Laplace operator on Riemannian manifolds*. 2012. arXiv: `1212.3422` (cited on page 43).

[Val12b] D. Valtorta. "Sharp estimate on the first eigenvalue of the $p$-Laplacian". English. In: *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods* 75.13 (2012), pages 4974–4994 (cited on page 43).

[Var83] N. T. Varopoulos. "Potential theory and diffusion on Riemannian manifolds". In: *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*. Wadsworth Math. Ser. Wadsworth, Belmont, CA, 1983, pages 821–837 (cited on page 15).

[Ver23] I. Versano. "Optimal Hardy-weights for the $(p, A)$-Laplacian with a potential term". In: *Proc. Roy. Soc. Edinburgh Sect. A* 153.1 (2023), pages 289–306 (cited on pages 3, 97, 98, 99, 100, 102).

[Woe00] W. Woess. *Random walks on infinite graphs and groups*. Volume 138. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2000 (cited on page 99).

[Woe09] W. Woess. *Denumerable Markov Chains - Generating Functions, Boundary Theory, Random Walks on Trees*. 1. ed. EMS Textbooks in Mathematics. European Mathematical Society, 2009 (cited on page 15).

[Yam77] M. Yamasaki. "Parabolic and hyperbolic infinite networks". In: *Hiroshima Math. J.* 7.1 (1977), pages 135–146 (cited on pages 3, 15).

[Yam86] M. Yamasaki. "Ideal boundary limit of discrete Dirichlet functions". In: *Hiroshima Math. J.* 16.2 (1986), pages 353–360 (cited on page 3).