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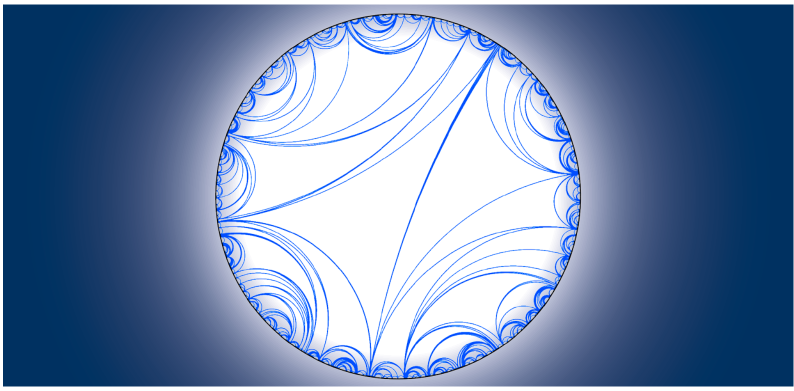

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NORMALLY SOLVABLE NONLINEAR BOUNDARY VALUE PROBLEMS

AMMAR ALSAEDY AND NIKOLAI TARKHANOV

Abstract. We study a boundary value problem for an overdetermined elliptic system of nonlinear first order differential equations with linear boundary operators. Such a problem is solvable for a small set of data, and so we pass to its variational formulation which consists in minimising the discrepancy. The Euler-Lagrange equations for the variational problem are far-reaching analogues of the classical Laplace equation. Within the framework of Euler-Lagrange equations we specify an operator on the boundary whose zero set consists precisely of those boundary data for which the initial problem is solvable. The construction of such operator has much in common with that of the familiar Dirichlet to Neumann operator. In the case of linear problems we establish complete results.

Contents

Introduction

A single nonlinear partial differential equation is as inexhaustible as the whole mathematics. Nevertheless informal theory is still possible for some general classes of nonlinear partial differential equations. As but one example we mention the theory of boundary value problems for higher order elliptic quasilinear equations of divergence type developed in the 1960s by Browder (see [Bro63] and the references given there).

The boundary value problems of [Bro63] can be specified within Euler-Lagrange equations for variational problems with one unknown function lying in a certain vector space. Thus, nonlinear conditions on the boundary are not permitted by the

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very setting, which restricts essentially applications of the theory. The boundary value problem for an elliptic quasilinear equation of divergence type with homogeneous Dirichlet conditions on the boundary is a typical example of problems treated by Browder.

The class of boundary value problems we study in the present paper is well motivated by applications to overdetermined elliptic systems. As usual such a system possesses no solution, let alone the boundary value problems for solutions of the system. If the boundary conditions may be satisfied by functions leaving out the system then one looks for a function which fulfills the boundary conditions and minimises the discrepancy in the system. This leads to a variational problem for a functional whose critical points are solutions of the so-called Euler-Lagrange equations. These latter can therefore be thought of as relaxation of the original boundary value problem.

The Euler-Lagrange equations represent a boundary value problem for an elliptic system of partial differential equations. This is what can actually be called the nonlinear Laplacian associated with the original problem. If the differential equations of the original system are nonhomogeneous then those of Euler-Lagrange equations are nonlinear, too. Moreover, if the original boundary conditions are of mixed type then the boundary conditions of Euler-Lagrange equations bear essential nonlinearities.

A classical reference on variational calculus still unexcelled is the monograph [Mor66]. Some developments along classical lines are presented in [Tay96]. We also mention the paper [LT09] where a variational approach was developed to study the Cauchy problem for nonlinear elliptic equations with data on a part of the boundary.

The relaxation of boundary value problems to Euler-Lagrange equations has evident advantages. If the original problem possesses a solution then this is among the solutions of Euler-Lagrange equations. Variational problems are amenable to efficient numerical methods, just recall the classical Ritz method [Rit09]. The Euler-Lagrange equations are endowed with weak formulations by the very nature. As but one typical feature of these equations we mention that they are of generic position, i.e., the number of equations just amounts to the number of unknown functions.

By the above, the Euler-Lagrange equations constitute a broad class of nonlinear boundary value problems whose study is well motivated both by internal applications in mathematics and by natural sciences. With this as our main purpose we elaborate in the present paper a variational approach to boundary value problems for systems of nonlinear partial differential equations. The ellipticity appears quite naturally in the study.

In Section 1 we discuss the relaxation of the initial problem for a nonlinear ordinary differential equation on a finite interval. We are looking for a global solution of the problem and so minimise the discrepancy in a class of functions satisfying the initial condition. By this example we demonstrate how the so-called Dirichlet to Neumann operator appears to describe the set of all initial data for which the original problem has a solution. In Section 2 we study boundary value problems for a nonlinear elliptic system of first order partial differential equations. The boundary conditions are assumed to be of the form $Bu = u_0$, where B is a right invertible matrix of smooth functions and u_0 a given function on the boundary, cf. [Agr69].

We look for a function which minimises the discrepancy of the system under precisely keeping boundary conditions. We compute the Euler-Lagrange equations of the variational problem. They constitute a boundary value problem for solutions of a second order elliptic system satisfying both $Bu = u_0$ and a suitable trace of the original system on the boundary. The problem is of stable character in the sense that the number of equations just amounts to the number of search-for functions. In this way we obtain what is called a relaxation of the original problem. Yet another designation of this boundary value problem is the nonlinear Laplacian of the original problem, for the way it shows up looks like that of the classical Laplace operator. In Section 3 we examine the algebraic structure of Euler-Lagrange equations and trace out techniques to construct a weak solution. In Section 4 we study the generalised Laplacian of boundary value problems for overdetermined elliptic systems of linear partial differential equations of higher order. If the discrepancy of the system is evaluated in the Hilbert space L^2 in the domain, then the Euler-Lagrange equations represent a boundary value problem for the generalised Laplacian of the original system in the domain with linear boundary conditions. A direct application of Green formula shows that any sufficiently smooth solution of the Euler-Lagrange equations is actually a solution of the original problem. This raises the problem of regularity of weak solutions of Euler-Lagrange equations, as but one example we remind of the $\bar{\partial}$-Neuman problem in complex analysis which initiated ample investigations of subelliptic operators. In fact the situation is much the same for nonlinear boundary value problems. Using generalised Laplacians allows one to reduce nonlinear boundary value problems to a nonlinear integro-differential problem on the boundary. The Dirichlet to Neumann operator treated in Section 5 is a key tool of this reduction. In Section 6 we develop a theory of nonlinear mappings of Banach spaces modelled on elliptic nonlinear boundary value problems investigated in the previous sections. They can be referred to as nonlinear Fredholm mappings and their use goes far beyond boundary value problems, see for instance [Sma65], [Pok69], [Bab74]. Finally, in Section 7 we study in detail the Cauchy problem for the derivative operator with data at a boundary piece. This problem is overdetermined, and so we consider a suitable variational relaxation of the problem and prove that it has a unique solution. This allows one to explicitly construct a nonlinear Dirichlet to Neumann operator.

1. A leading example

Let $\mathcal{X} = [a, b]$ be a bounded interval in $\mathbb{R}$ and f a continuous function on $[a, b] \times \mathbb{R}$ with real values. We assume that $f(x, u)$ satisfies a Lipschitz condition in u uniformly in $x \in [a, b]$. Consider the problem of finding a function $u \in H^1[a, b]$ satisfying

$$\left\{ \begin{array}{rcl} u'(x) & = & f(x,u) \quad \text{for} \quad x \in (a,b), \\ u(a) & = & u_0, \end{array} \right. \tag{1.1}$$

where u_0 is a given number.

If $u \in H^1[a, b]$ is a solution of (1.1), then u is continuous on $[a, b]$, which is due to the Sobolev embedding theorem. The differential equation in (1.1) implies that u is actually continuously differentiable on $[a, b]$. By the Picard-Lindelöf theorem, problem (1.1) has a unique local solution for all data $u_0 \in \mathbb{R}$. The domain of the maximal solution depends essentially on u_0 and need not coincide with entire (a, b). Hence, (1.1) is solvable not for all initial data u_0. We relax the equation in

(a,b) and look for a function $u \in H^1[a,b]$ which satisfies $u(a) = u_0$ and for which the discrepancy $u' - f(\cdot, u)$ is minimal. In terms of [IVT78], such an approximate solution of the problem is called a quasisolution. The most suitable norm to evaluate the discrepancy is the $L^2[a,b]$-norm. We thus minimise the integral

$$I(u) = \int_a^b |u'(x) - f(x,u)|^2 \, dx$$

over the set $\mathcal{A}$ of all $u \in H^1[a,b]$ satisfying $u(a) = u_0$.

A familiar argument of calculus of variations leads to a necessary condition for a function $u \in H^1[a,b]$ to be a local extremum of the functional I on $\mathcal{A}$. Namely, u should satisfy

$$\int_a^b (u'(x) - f(x,u))(v'(x) - f'_u(x,u)v(x))dx = 0$$

for all $v \in H^1[a,b]$ vanishing at the point a. Notice that the derivative f'_u exists almost everywhere on $[a,b] \times \mathbb{R}$ and it is essentially bounded, which is due to Rademacher's theorem. Using the main lemma of calculus of variations we rewrite the integral condition as boundary value problem on u. We thus get what is known as Euler-Lagrange equations

$$\left\{ \begin{array}{rcll} (-\dfrac{d}{dx} - f'_u(x,u))(u'(x) - f(x,u)) & = & 0 & \text{for} \quad x \in (a,b), \\ u(a) & = & u_0, & \\ u'(b) - f(b,u(b)) & = & 0, & \end{array} \right. \tag{1.2}$$

where the differential equation and the condition at b are understood in the sense of distributions.

Each solution of original initial problem (1.1) is automatically a solution of boundary value problem (1.2). This latter can be specified in the class of Sturm-Liouville nonlinear boundary value problems. If the original problem is nonlinear then both the differential equation and the added boundary condition of (1.2) are nonlinear. A weak formulation of Euler-Lagrange equations is given by their origin. The Sturm-Liouville problems behave much better under perturbations than the initial problems. The differential equation in (1.2) reduces to $-u'' + f'_x + f'_u f = 0$ in (a,b) and represents a Laplace-type equation. Notice that if (1.2) has a global solution $u \in H^1[a,b]$ then u is also a solution to (1.1). Indeed, the function $y = u' - f$ satisfies the linear differential equation $-y' - f'_u y = 0$ in (a,b). Hence it follows that

$$y(x) = c \, \exp\Big(- \int_a^x f'_u(\vartheta, u(\vartheta)) \, d\vartheta \Big)$$

for all $x \in (a,b)$. Since y vanishes at the point b, it vanishes identically in (a,b), as desired.

We now introduce an operator Ψ in the set of data u_0 of problem (1.1) in the following way. Let the domain $\mathcal{D}_\Psi$ of Ψ consists of all $u_0 \in \mathbb{R}$ with the property that problem (1.2) has a solution $u \in H^1[a,b]$. Then $u' - f \in H^1[a,b]$ is continuous on $[a,b]$ and we set

$$\Psi(u_0) := u'(a) - f(a,u_0), \tag{1.3}$$

which is well defined. The operator Ψ represents a nonlinear analogue of the Dirichlet to Neumann operator, see [LT11]. By the above, Ψ vanishes on all of its domain, for any solution $u \in H^1[a,b]$ to (1.2) fulfills also (1.1). However, this proves to be purely one-dimensional effect, i.e., the Dirichlet to Neumann operator need not be zero in general. Thus, for the solvability of initial problem (1.1) it is necessary and sufficient that $\Psi(u_0) = 0$.

We call problem (1.1) normally solvable if the set of those $u_0 \in \mathbb{R}$, for which there is at least one $u \in H^1[a,b]$ satisfying (1.1), is closed. If the Dirichlet to Neumann operator is continuous and its domain closed, then problem (1.1) is normally solvable.

2. Relaxations of boundary value problems

Consider a (possibly, overdetermined) elliptic system of first order quasilinear partial differential equations in a neighbourhood of a closed bounded domain $\mathcal{X}$ in $\mathbb{R}^n$ with smooth boundary. We write it in the form $Au = f(x,u)$, where A is a (possibly, overdetermined) elliptic linear partial differential operator of the first order near $\mathcal{X}$ and $f(x,u)$ a continuous function on $\mathcal{X} \times \mathbb{R}^\ell$ with values in $\mathbb{R}^m$. The operator A is given by an $(m \times \ell)$-matrix of scalar differential operators in a neighbourhood of $\mathcal{X}$.

Let B be an $(\ell' \times \ell)$-matrix of smooth functions on the boundary $\partial\mathcal{X}$ of $\mathcal{X}$, such that $\operatorname{rank} B(x) = \ell'$ for all $x \in \partial\mathcal{X}$. We are interested in the boundary value problem

$$\left\{ \begin{array}{rcll} Au & = & f(x,u) & \text{in } \ \mathcal{X}, \\ Bu & = & u_0 & \text{at } \ \partial\mathcal{X} \end{array} \right. \tag{2.1}$$

with data u_0 on $\partial\mathcal{X}$. The most conventional Hilbert space setting of this problem is $H^1 := W^{1,2}$, hence we choose u_0 in $H^{1/2}(\partial\mathcal{X}, \mathbb{R}^\ell)$ and look for a $u \in H^1(\mathcal{X}, \mathbb{R}^\ell)$ satisfying (2.1).

Lemma 2.1. *Let C be an $((\ell - \ell') \times \ell)$-matrix C of smooth functions on $\partial\mathcal{X}$, such that*

$$\operatorname{rank} \begin{pmatrix} B(x) \\ C(x) \end{pmatrix} = \ell$$

for all $x \in \partial\mathcal{X}$. Then there are unique matrices B^* and C^* of continuous functions on $\partial\mathcal{X}$ with the property that

$$\int_{\partial\mathcal{X}} \left((Bu, C^* g)_x - (Cu, B^* g)_x \right) ds = \int_{\mathcal{X}} \left((Au, g)_x - (u, A^* g)_x \right) dx \tag{2.2}$$

for all $u \in H^1(\mathcal{X}, \mathbb{R}^\ell)$ and $g \in H^1(\mathcal{X}, \mathbb{R}^m)$, where ds is the surface measure on the boundary.

As usual, we write A^* for the formal adjoint of the differential operator A in a neighbourhood of $\mathcal{X}$.

Proof. By assumption, the $(\ell \times \ell)$-matrix

$$T(x) = \begin{pmatrix} B(x) \\ C(x) \end{pmatrix}$$

is invertible for all $x \in \partial\mathcal{X}$. Write $(T(x))^{-1} = (T_1(x), T_2(x))$ where T_1 and T_2 are $(\ell \times \ell')$- and $(\ell \times (\ell - \ell'))$-matrices of smooth functions on $\partial\mathcal{D}$, respectively. The equalities $T^{-1}T = E_\ell$ and $TT^{-1} = E_\ell$ amount to $T_1 B + T_2 C = E_\ell$ and

$$\begin{array}{rclcrcl} BT_1 & = & E_{\ell'}, & & BT_2 & = & 0, \\ CT_1 & = & 0, & & CT_2 & = & E_{\ell - \ell'}, \end{array} \tag{2.3}$$

where E_ℓ stands for the unity $(\ell \times \ell)$-matrix.

Given any $u \in H^1(\mathcal{X}, \mathbb{R}^\ell)$ and $g \in H^1(\mathcal{X}, \mathbb{R}^m)$, the Green formula of [Tar95, 2.4.2] shows that

$$\int_{\partial \mathcal{X}} (\sigma(x)u, g)_x \, ds = \int_{\mathcal{X}} ((Au, g)_x - (u, A^* g)_x) \, dx$$

where $\sigma(x)$ is the principal symbol of A evaluted at the point $(x, -\imath \nu(x))$ of the complexified cotangential bundle of $\mathcal{X}$, $\nu(x)$ being the outward normal unit vector of the boundary at $x \in \partial \mathcal{X}$. Substitung $u = (T_1 B + T_2 C)u$ into this formula yields (2.2) with

$$\begin{array}{rcl} C^* & = & (\sigma T_1)^*, \\ B^* & = & -(\sigma T_2)^*, \end{array} \tag{2.4}$$

as desired. □

From (2.4) it follows immediately that the rank of C^* is equal to ℓ' and the rank of B^* is $\ell - \ell'$.

Formula (2.2) is implicitly contained in the study of boundary value problems for first order systems of pseudodifferential operators, see [Agr69] and the references given there.

Our standing requirement on f is that $u \mapsto f(x, u)$ be a continuous map of $H^1(\mathcal{X}, \mathbb{R}^\ell)$ to $L^2(\mathcal{X}, \mathbb{R}^m)$. Since problem (2.1) fails to have a solution for most data u_0 on the boundary, we look for a solution of the variational problem $I(u) \to \min$ for the functional

$$I(u) = \int_{\mathcal{X}} |Au - f(x, u)|^2 \, dx \tag{2.5}$$

over the affine subspace $\mathcal{A}$ of $H^1(\mathcal{X}, \mathbb{R}^\ell)$ that consists of all u satisfying $Bu = u_0$ on $\partial \mathcal{X}$. Obviously, every solution of (2.1) minimises (2.5). The converse assertion is not true.

Write m for the infimum of $I(u)$ over $u \in \mathcal{A}$. In order that $u \in \mathcal{A}$ may satisfy $I(u) = m$ it is necessary that u would fulfill the so-called Euler-Lagrange equations. We now describe these.

Let $\delta \in C^\infty(\mathcal{X}, \mathbb{R}^\ell)$ be an arbitrary function satisfying $B\delta = 0$ on $\partial \mathcal{D}$. For each $\varepsilon \in \mathbb{R}^\ell$, the variation $u + \varepsilon \delta$ does not go beyond $\mathcal{A}$. (Here, we identify $\mathbb{R}^\ell$ with the algebra of all real-valued functions on the set $\{1, \ldots, \ell\}$.) Therefore, if $I(u) = m$, then the function $F(\varepsilon) = I(u + \varepsilon \delta)$ takes on its minimum at $\varepsilon = 0$. It follows that $\varepsilon = 0$ is a critical point of F.

Given any $i = 1, \ldots, \ell$, an easy computation shows that

$$F'_{\varepsilon_i}(0) = 2 \int_{\mathcal{X}} (A(\delta_i e_i) - \delta_i f'_{u_i}, Au - f)_x \, dx,$$

where e_i is the ℓ-column whose entries are all zero except for the i th entry which is 1, and f'_{u_i} is the partial derivative of f in u_i, if there is any. Summing up over $i = 1, \ldots, \ell$ yields

$$\int_{\mathcal{X}} (A\delta - f'_u \delta, Au - f)_x \, dx = 0 \tag{2.6}$$

for all $\delta \in C^\infty(\mathcal{X}, \mathbb{R}^\ell)$ such that $B\delta = 0$ on $\partial \mathcal{X}$. If $g = Au - f$ is of class $H^1(\mathcal{X}, \mathbb{R}^m)$, then we can apply Green formula (2.2) on the left-hand side and move A from δ to $Au - f$, thus obtaining

$$\int_{\partial \mathcal{X}} (C\delta, B^* g)_x \, ds + \int_{\mathcal{X}} (\delta, (A^* - (f'_u)^*) g)_x \, dx = 0$$

for all $\delta \in C^\infty(\mathcal{X}, \mathbb{R}^\ell)$ satisfying $B\delta = 0$ on the boundary. We first choose δ to be arbitrary with compact support in the interior of $\mathcal{X}$ and so conclude by the main lemma of variational calculus that $(A^* - (f'_u)^*)g$ vanishes almost everywhere in $\mathcal{X}$. Hence, the boundary integral is equal to zero for all $\delta \in C^\infty(\mathcal{X}, \mathbb{R}^\ell)$, such that $B\delta = 0$ on $\partial\mathcal{X}$. On the other hand, if δ is an arbitrary function on $\partial\mathcal{X}$ with values in $\mathbb{R}^\ell$, then

$$\begin{aligned} C\delta &= C(T_1 B + T_2 C)\delta \\ &= C(T_2 C)\delta \end{aligned}$$

and $B(T_2 C)\delta = 0$, which is due to (2.3). Therefore, the boundary integral actually vanishes for all function $\delta \in C^\infty(\mathcal{X}, \mathbb{R}^\ell)$, not only for those satisfying $B\delta = 0$ on $\partial\mathcal{X}$. Hence, $B^* g = 0$ on $\partial\mathcal{X}$.

Lemma 2.2. *For the variational problem $I(u) \to \min$ over $u \in \mathcal{A}$, Euler-Lagrange's equations just amount to*

$$\left\{ \begin{array}{rcll} (A^* - (f'_u)^*)(Au - f) & = & 0 & \textit{in} \ \ \mathcal{X}, \\ Bu & = & u_0 & \textit{at} \ \ \partial\mathcal{X}, \\ B^*(Au - f) & = & 0 & \textit{at} \ \ \partial\mathcal{X}. \end{array} \right. \tag{2.7}$$

Proof. If $u \in \mathcal{A}$ and $Au - f$ is of class $H^1(\mathcal{X}, \mathbb{R}^m)$ then this precisely what has been proved above. For general $u \in \mathcal{A}$ equalities (2.7) are understood in the weak sense suggested by (2.6). Thus, the differential equation is satisfied in the sense of distributions in the interior of $\mathcal{X}$. The interpretation of the second boundary condition in (2.7) is more sophisticated. □

It is worth pointing out that f'_u stands for the Jacobi matrix of $f(x, u)$ with respect to $u = (u_1, \ldots, u_\ell)$. Thus, this is an $(m \times \ell)$-matrix of functions depending on x and u. The equation in $\mathcal{X}$ of (2.7) represents a system of ℓ second order partial differential equations for ℓ unknown functions. The number of boundary conditions just amounts to ℓ.

The boundary value problem (2.7) can be thought of as optimality system for the original problem (2.1). By a weak solution $u \in H^1(\mathcal{X}, \mathbb{R}^\ell)$ of (2.7) is meant any solution of the variational problem $I(u) \to \min$ over $u \in \mathcal{A}$, the functional I being given by (2.5). Even if $I(u)$ takes on its minimum m for some $u \in \mathcal{A}$, the function u need not satisfy (2.1) unless $m = 0$. Hence, if (2.7) possesses a weak solution, then for the solvability of (2.1) it is necessary and sufficient that $m = 0$. The equation $(A^* - (f'_u)^*)(Au - f) = 0$ can be thought of as nonlinear Laplace equation related to the nonlinear differential equation $Au = f$. If f is independent of u it readily reduces to $A^* A u = A^* f$.

Formula (2.2) allows one to derive necessary conditions for the solvability of problem (2.1).

Theorem 2.3. *If problem (2.1) possesses a solution $u \in H^1(\mathcal{X}, \mathbb{R}^\ell)$ then*

$$\int_{\partial\mathcal{X}} (u_0, C^* g)_x \, ds = \int_{\mathcal{X}} (f - f'_u u, g)_x \, dx \tag{2.8}$$

holds for all $g \in H^1(\mathcal{X}, \mathbb{R}^m)$ with the property that $(A - f'_u)^* g = 0$ in $\mathcal{X}$ and $B^* g = 0$ on $\partial\mathcal{X}$.

Proof. By (2.2), we get

$$\int_{\partial\mathcal{X}} ((Bu, C^* g)_x - (Cu, B^* g)_x)\, ds = \int_{\mathcal{X}} (((A - f'_u)u, g)_x - (u, (A - f'_u)^* g)_x)\, dx$$

for all $u \in H^1(\mathcal{X}, \mathbb{R}^\ell)$ and $g \in H^1(\mathcal{X}, \mathbb{R}^m)$. Hence the desired assertion follows immediately. □

The principal significance of the theorem is that it allows one to specify the boundary value problem

$$\begin{cases} (A - f'_u)^* g &= 0 \quad \text{in} \quad \mathcal{X}, \\ B^* g &= 0 \quad \text{at} \quad \partial\mathcal{X} \end{cases} \tag{2.9}$$

as formal adjoint of (2.1) with respect to the Green formula of Lemma 2.1. In case f does not depend on u this concept has proven to be efficient in describing solvability conditions for linear boundary value problems, see for instance Chapter 5 of [Roi96].

Corollary 2.4. *Let $u \in H^1(\mathcal{X}, \mathbb{R}^\ell)$ be a solution of Euler-Lagrange equations, such that $Au - f \in H^1(\mathcal{X}, \mathbb{R}^m)$. If*

$$\int_{\partial\mathcal{X}} (u_0, C^*(Au - f))_x\, ds = \int_{\mathcal{X}} (f - f'_u u, Au - f)_x\, dx$$

then u satisfies (2.1).

Proof. Set $g = Au - f$, then $g \in H^1(\mathcal{X}, \mathbb{R}^m)$ satisfies (2.9). Using formula (2.2) we obtain

$$\int_{\partial\mathcal{X}} (u_0, C^* g)_x\, ds = \int_{\mathcal{X}} ((A - f'_u)u, g)_x\, dx$$

whence

$$\int_{\partial\mathcal{X}} (u_0, C^* g)_x\, ds - \int_{\mathcal{X}} (f - f'_u u, g)_x\, dx = \int_{\mathcal{X}} |g|^2\, dx,$$

and the corollary follows. □

If $u \in H^1(\mathcal{X}, \mathbb{R}^\ell)$ is a solution of Euler-Lagrange equations then $g = Au - f$ is a very particular solution of (2.9). By Theorem 2.3, the hypotheses of Corollary 2.4 are necessary for u to be a solution to (2.1). The interest of the corollary lies in the fact that within the solutions of Euler-Lagrange equations the single moment condition is also sufficient.

3. Existence of weak solutions

Here we develop the techniques of [LT11] to construct a weak solution of boundary value problem (2.7). Our study is within the framework of direct methods of variational calculus.

We are aimed at finding those $u \in \mathcal{A}$ at which the functional $I(u)$ takes on the value m. Write

$$Au = \sum_{j=1}^{n} A_j \partial_j u + A_0 u$$

where $A_1, \ldots, A_n$ and A_0 are $(m \times \ell)$-matrices of C^∞ functions in a neighbourhood of $\mathcal{X}$, and $\partial_j = \partial/\partial x_j$. The integrand $L(x,u,u') = |Au - f|^2$ in (2.5) is verified to be

$$L(x,u,p) = \sum_{j,l=1}^{n} p_l^* \, A_l^* A_j \, p_j + |A_0 u - f|^2 + 2\,(A_0 u - f)^* \sum_{j=1}^{n} A_j p_j, \tag{3.1}$$

where $p = (p_1, \ldots, p_n)$ is an $(\ell \times n)$-matrix whose columns are substitutions for $\partial_j u$.

Lemma 3.1. *For each fixed (x,u), the function $L(x,u,p)$ is convex in the entries of $p \in \mathbb{R}^{\ell \times n}$.*

Proof. Write

$$p_j = \begin{pmatrix} p_j^1 \\ \ldots \\ p_j^\ell \end{pmatrix}$$

for $j = 1, \ldots, n$, then Euler's equality for homogeneous functions shows that

$$\begin{aligned} \sum_{\substack{i,k=1,\ldots,\ell \\ j,l=1,\ldots,n}} L''_{p_j^i, p_l^k} w_j^i w_l^k &= 2 \sum_{j,l=1}^{n} w_l^* \, A_l^* A_j \, w_j \\ &= 2 \Big| \sum_{j=1}^{n} A_j w_j \Big|^2 \\ &\geq 0 \end{aligned}$$

for all

$$w_j = \begin{pmatrix} w_j^1 \\ \ldots \\ w_j^\ell \end{pmatrix},$$

$j = 1, \ldots, n$, in $\mathbb{R}^\ell$. Since moreover $L(x,u,p)$ is of class C^2 in $p \in \mathbb{R}^{\ell \times n}$, the nonnegative definiteness of the quadratic form on the left-hand side is equivalent to the convexity of L as a function of the entries of p, see Lemma 1.8.1 of [Mor66] and elsewhere. □

Thus, the general hypothesis of [Mor66, p. 91] concerning the integrand function L are satisfied.

If $u \in \mathcal{A}$ furnishes a local minimum to (2.5), then necessarily

$$\begin{pmatrix} v^1 \\ \ldots \\ v^\ell \end{pmatrix}^* \begin{pmatrix} \displaystyle\sum_{j,l=1}^{n} L''_{p_j^1, p_l^1} \xi_j \xi_l & \ldots & \displaystyle\sum_{j,l=1}^{n} L''_{p_j^1, p_l^\ell} \xi_j \xi_l \\ & \ldots & \\ \displaystyle\sum_{j,l=1}^{n} L''_{p_j^\ell, p_l^1} \xi_j \xi_l & \ldots & \displaystyle\sum_{j,l=1}^{n} L''_{p_j^\ell, p_l^\ell} \xi_j \xi_l \end{pmatrix} \begin{pmatrix} v^1 \\ \ldots \\ v^\ell \end{pmatrix} \geq 0 \tag{3.2}$$

is fulfilled for all $\xi \in \mathbb{R}^n$ and $v \in \mathbb{R}^\ell$, the derivatives of L being evaluated at (x,u,u'). This classical necessary condition is known as the Legendre-Hadamard condition, see [Mor66, 1.5]. In this case, one says that the integrand function L is regular if the inequality holds in (3.2) for all $\xi \in \mathbb{R}^n$ and $v \in \mathbb{R}^\ell$ which are different from zero.

Lemma 3.2. *The integrand function $L(x,u,u')$ is regular if and only if the system $Au = f(x,u)$ has injective symbol.*

Proof. On arguing as in the proof of Lemma 3.1 we deduce that the left-hand side in (3.2) is equal to

$$\begin{aligned}\sum_{\substack{i,k=1,\ldots,\ell\\ j,l=1,\ldots,n}} L''_{p^i_j,p^k_l}(\upsilon^i\xi_j)(\upsilon^k\xi_l) &= 2\sum_{j,l=1}^{n} \upsilon^*\,\xi_l A^*_l A_j\,\xi_j\upsilon\\ &= 2\,|\sigma(A)(x,\xi)\upsilon|^2,\end{aligned}$$

where $\sigma(A)$ stands for the principal symbol of A. From this the lemma follows immediately. □

By the definition of infimum, there is a sequence $\{u_\nu\}$ in $\mathcal{A}$, such that $I(u_\nu) \searrow m$. Any such sequence is called minimising. Each subsequence of a minimising sequence is also a minimising sequence. Were it possible to extract a subsequence $\{u_{\nu_\iota}\}$ converging to an element $u \in \mathcal{A}$ in the $H^1(\mathcal{X},\mathbb{R}^\ell)$ norm, then $I(u_{\nu_\iota})$ would converge to $I(u) = m$, and so u would be a desired solution of our variational problem. It is possible to require the convergence of a minimising sequence in a weaker topology than that of $H^1(\mathcal{X},\mathbb{R}^\ell)$. However, the functional I should be lower semicontinuous with respect to correspondingly more general types of convergence. In order to find a convergent subsequence of a minimising sequence, one uses a compactness argument. The space $H^1(\mathcal{X},\mathbb{R}^\ell)$ is reflexive. Hence, each bounded sequence in $H^1(\mathcal{X},\mathbb{R}^\ell)$ has a weakly convergent subsequence. Thus, any bounded minimising sequence $\{u_\nu\}$ has a subsequence $\{u_{\nu_\iota}\}$ which converges weakly in $H^1(\mathcal{X},\mathbb{R}^\ell)$ to some function u. By a theorem of Mazur, see [Yos65] and elsewhere, any convex closed subset of a reflexive Banach space is actually weakly closed. It follows that the limit function u satisfies $Bu = u_0$ on $\partial\mathcal{X}$, i.e., it belongs to $\mathcal{A}$. Moreover, Theorem 3.4.4 of [Mor66] says that the subsequence $\{u_{\nu_\iota}\}$ converges also strongly in $L^2(\mathcal{X},\mathbb{R}^\ell)$ to u. Note that the Lagrangian $L(x,u,p)$ is neither normal nor strictly convex in p in general, as one can see by example of the Cauchy-Riemann system. However, Theorem 4.1.1 of [Mor66] still applies to the functional $I(u)$, for L is polynomial in p.

Lemma 3.3. *If u_ν and u lie in $H^1(\mathcal{X},\mathbb{R}^\ell)$ and $u_\nu \to u$ in $L^1(K,\mathbb{R}^\ell)$ for each compact set K interior to $\mathcal{X}$, then*

$$I(u) \le \liminf I(u_\nu).$$

Proof. See Theorem 4.1.1 of [Mor66]. □

We have thus proved that if there is a bounded minimising sequence $\{u_\nu\}$ and u is a weak limit point of this sequence in $H^1(\mathcal{X},\mathbb{R}^\ell)$, then $u \in \mathcal{A}$ and $I(u) = m$, i.e., u is a minimiser. It is clear that any minimising sequence is bounded in $H^1(\mathcal{X},\mathbb{R}^\ell)$ if the functional $I(u)$ majorises the norm of u in $\mathcal{A}$ in the sense that

$$\int_{\mathcal{X}} L(x,u,u')\,dx \ge c\,\|u\|^2_{H^1(\mathcal{X},\mathbb{R}^\ell)} - Q \tag{3.3}$$

for all $u \in \mathcal{A}$, with c and Q constants independent of u. This is obviously a far reaching generalisation of A. Korn's (1908) inequality for the case of nonlinear problems.

Since the boundary operator B has a right inverse operator, there is a function $U_0 \in H^1(\mathcal{X}, \mathbb{R}^\ell)$ satisfying $BU_0 = u_0$ at $\partial\mathcal{X}$. Change the dependent variable by $u = U_0 + U$. Then the variational problem $I(u) \to \min$ over $u \in \mathcal{A}$ reduces to the problem $I(U_0 + U) \to \min$ over the set $\mathcal{A} - U_0$ which consists of all $U \in H^1(\mathcal{X}, \mathbb{R}^\ell)$ satisfying $BU = 0$ at $\partial\mathcal{X}$.

A priori estimate (3.3) distinguishes those boundary operators B, for which the mixed boundary value problem (2.7) is normally solvable. On the other hand, Corollary (2.4) provides a moment condition on a solution u of (2.7) which guarantees that u satisfies original problem (2.1). This gives an insight into the problem of normal solvability of (2.1).

4. Generalised Laplacians of linear problems

In this section we consider in detail the boundary value problem (2.1) linearised at some function $U_0 \in H^1(\mathcal{X}, \mathbb{C}^\ell)$. By abuse of notation, we use the same letter $f(x)$ for $f(x, U_0(x))$, thus restricting our discussion to an inhomogeneous boundary value problem

$$\left\{ \begin{array}{rclcl} Au & = & f(x) & \text{in} & \mathcal{X}, \\ Bu & = & u_0 & \text{at} & \partial\mathcal{X} \end{array} \right. \tag{4.1}$$

in the domain $\mathcal{X}$.

Neither the arguments nor the conclusion is affected if we allow both A and B to be of more general form. Namely, let A be an $(m \times \ell)$-matrix of linear partial differential operators of order r with smooth coefficients in a neighbourhood of $\mathcal{X}$. We assume that A is (possibly, overdetermined) elliptic operator, i.e., the rank of the principal symbol $\sigma(A)(x, \xi)$ just amounts to ℓ for all $x \in \mathcal{X}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. Furthermore, let B be an $(\ell' \times \ell)$-matrix of linear partial differential operators with smooth coefficients in a neighbourhood of $\partial\mathcal{X}$. We assume that the i-th row of B consists of operators of order r_i, where $i = 1, \ldots, \ell'$, and that $0 \leq \ell' \leq r\ell$. Moreover, we require B to satisfy the condition of complementarity with respect to A which means that

$$\left\{ \begin{array}{rclcl} \sigma(A)(x', \xi', D_t)u(t) & = & 0 & \text{for} & t > 0, \\ \sigma(B)(x', \xi', D_t)u(t) & = & 0 & \text{for} & t = 0 \end{array} \right. \tag{4.2}$$

has only trivial solution in $L^2(\mathbb{R}_{>0}, \mathbb{C}^\ell)$ for all $(x', \xi') \in T^*(\partial\mathcal{X})$ with ξ' different from zero. Here, (x', t) are local coordinates in a neighbourhood U of arbitrary boundary point x, such that the portion of $\mathcal{X}$ in U is described in these coordinates by $t \geq 0$.

In order to introduce a formal adjoint boundary value problem for (4.1) which is of the same kind, one singles out the so-called normal boundary operators B. More precisely, B is called normal if $r_i < r$ for all $i = 1, \ldots, \ell'$ and there exists an $((r\ell - \ell') \times \ell)$-matrix C of linear partial differential operators with smooth coefficients in a neighbourhood of $\partial\mathcal{X}$, the i-th row of C consisting of operators of order $s_i < r$, such that

$$\sum_{i=1}^{\ell'} r_i + \sum_{i=1}^{r\ell - \ell'} s_i = \frac{1}{2}\,(r-1)r\ell$$

and the rows of the matrix

$$\left(\begin{array}{c} \sigma(B)(x', \xi', \tau) \\ \sigma(C)(x', \xi', \tau) \end{array} \right)$$

are linearly independent for all $(x', \xi') \in T^*(\partial \mathcal{X})$ with $\xi' \neq 0$.

Remark 4.1. Under the above assumptions, boundary value problem (4.1) is elliptic if $m = \ell$, the operator A is properly elliptic and $\ell' = r\ell/2$.

In the general case problem (4.1) is overdetermined, and so solvable not for all f. Its cokernel is, generally speaking, of infinite dimension.

By an approximate solution of problem (4.1) in $H^r(\mathcal{X}, \mathbb{C}^\ell)$ is meant any function $u \in H^r(\mathcal{X}, \mathbb{C}^\ell)$, such that $Bu = u_0$ at $\partial \mathcal{X}$ and Au is the best approximation of f in the $L^2(\mathcal{X}, \mathbb{C}^m)$-norm. Another way of stating this is to say that u is a solution of the variational problem

$$\int_{\mathcal{X}} |Au - f|^2 dx \mapsto \min \tag{4.3}$$

over the set $\mathcal{A}_{u_0}$ of all functions $u \in H^r(\mathcal{X}, \mathbb{C}^\ell)$ satisfying $Bu = u_0$ at the boundary. (After V.K. Ivanov, approximate solution are called quasisolutions, see [IVT78].) Clearly, the usual solution of problem (4.1) in $H^r(\mathcal{X}, \mathbb{C}^\ell)$, if there is any, is also an approximate solution.

Under the above assumptions, for all $u \in H^r(\mathcal{X}, \mathbb{C}^\ell)$ and $g \in H^r(\mathcal{X}, \mathbb{C}^m)$, the Green formula

$$(Au, g)_{L^2(\mathcal{X},\mathbb{C}^m)} - (Bu, C^*g)_{L^2(\partial\mathcal{X},\mathbb{C}^{\ell'})} = (u, A^*g)_{L^2(\mathcal{X},\mathbb{C}^\ell)} - (Cu, B^*g)_{L^2(\partial\mathcal{X},\mathbb{C}^{r\ell-\ell'})} \tag{4.4}$$

holds, cf. (2.2). Here, A^* is the formal adjoint of A, and B^* and C^* are matrices of partial differential operators with smooth coefficients near $\partial \mathcal{X}$, whose sizes are $(r\ell - \ell') \times m$ and $\ell' \times m$, respectively. For a more general formula, we refer the reader to [L'v78].

The boundary value problem

$$\left\{ \begin{array}{rclcl} A^*g & = & v & \text{in} & \mathcal{X}, \\ B^*g & = & g_0 & \text{at} & \partial\mathcal{X} \end{array} \right. \tag{4.5}$$

is said to be the formal adjoint of (4.1) with respect to Green's formula (4.4), cf. (2.9).

Given any $u \in H^r(\mathbb{R}_{>0}, \mathbb{C}^\ell)$ and $g \in H^r(\mathbb{R}_{>0}, \mathbb{C}^m)$, Green's (4.4) localises to imply

$$\begin{array}{rl} & (\sigma(A)(x',\xi',D_t)u, g)_{L^2(\mathbb{R}_{>0},\mathbb{C}^m)} - (\sigma(B)(x',\xi',D_t)u\,(0), \sigma(C^*)(x',\xi',D_t)g\,(0))_{\mathbb{C}^{\ell'}} \\ = & (u, \sigma(A^*)(x',\xi',D_t)g)_{L^2(\mathbb{R}_{>0},\mathbb{C}^\ell)} - (\sigma(C)(x',\xi',D_t)u\,(0), \sigma(B^*)(x',\xi',D_t)g\,(0))_{\mathbb{C}^{r\ell-\ell'}} \end{array} \tag{4.6}$$

for all $(x', \xi') \in T^*(\partial \mathcal{X})$.

Denote by $A\mathcal{A}_0$ the set of all functions $f \in L^2(\mathcal{X}, \mathbb{C}^m)$ of the form Au, where $u \in \mathcal{A}_0$. As is shown in [Sol71], the space $A\mathcal{A}_0$ is closed in $L^2(\mathcal{X}, \mathbb{C}^m)$. From Green's formula it follows immediately that each $f \in A\mathcal{A}_0$ is orthogonal to the null-space of problem (4.5).

Lemma 4.2. *Suppose problem (4.1) is elliptic, i.e. $m = \ell$, the operator A is properly elliptic and $\ell' = r\ell/2$. Then, for the solvability of (4.1) it is necessary and sufficient that*

$$(f, g)_{L^2(\mathcal{X},\mathbb{C}^m)} - (u_0, C^*g)_{L^2(\partial\mathcal{X},\mathbb{C}^{\ell'})} = 0 \tag{4.7}$$

for all g in the null-space of formal adjoint problem (4.5).

Proof. This is a well-known result of elliptic boundary value problems. For the Douglis-Nirenberg elliptic boundary value problems, see [L'v78]. □

In much the same way as in Lemma 2.2 we derive Euler-Lagrange's equations for problem (4.3). They read

$$\left\{ \begin{array}{rcll} A^*A\,u & = & v & \text{in} \quad \mathcal{X}, \\ Bu & = & u_0 & \text{at} \quad \partial\mathcal{X}, \\ B^*A\,u & = & u_1 & \text{at} \quad \partial\mathcal{X}, \end{array} \right. \tag{4.8}$$

where $v = A^*f$ and $u_1 = B^*f$.

Lemma 4.3. *Boundary value problem (4.8) is elliptic.*

Proof. Since A is (possibly, overdetermined) elliptic operator of order r, the Laplacian A^*A is a properly elliptic operator of order $2r$ given by a square matrix of size $\ell \times \ell$.

Problem (4.8) contains precisely $r\ell$ boundary conditions. In order to see that this problem is elliptic, it remains to prove that the boundary operator (B, B^*A) satisfies the condition of complementarity with respect to A^*A. This condition just amounts to the fact that

$$\left\{ \begin{array}{rcll} \sigma(A^*)(x',\xi',D_t)\sigma(A)(x',\xi',D_t)\,u(t) & = & 0 & \text{for} \quad t > 0, \\ \sigma(B)(x',\xi',D_t)\,u(t) & = & 0 & \text{for} \quad t = 0, \\ \sigma(B^*)(x',\xi',D_t)\sigma(A)(x',\xi',D_t)\,u(t) & = & 0 & \text{for} \quad t = 0 \end{array} \right. \tag{4.9}$$

has only trivial solution in $L^2(\mathbb{R}_{>0}, \mathbb{C}^\ell)$ for all $(x', \xi') \in T^*(\partial\mathcal{X})$ with ξ' different from zero.

Let $u(t)$ be a solution of the initial problem (4.9). Applying formula (4.6) we readily get

$$\begin{array}{rcl} 0 & = & (\sigma(A^*)(x',\xi',D_t)\sigma(A)(x',\xi',D_t)\,u, u)_{L^2(\mathbb{R}_{>0},\mathbb{C}^\ell)} \\ & = & (\sigma(A)(x',\xi',D_t)\,u, \sigma(A)(x',\xi',D_t)\,u)_{L^2(\mathbb{R}_{>0},\mathbb{C}^m)}, \end{array}$$

which shows that $u(t)$ is actually a solution of initial problem (4.2). Since B satisfies the condition of complementarity with respect to A, it follows that $u(t) \equiv 0$, as desired. □

Applying Green's formula (4.4) twice one sees that boundary value problem (4.8) is selfadjoint. Its null-space coincides with the null-space of problem (4.1). Indeed, the null-space of (4.1) is obviously contained in the null-space of (4.8). Conversely, the elements of the null-space of problem (4.8) are smooth up to the boundary of $\mathcal{X}$, for the problem is elliptic by Lemma 4.3. Hence, they belong to the null-space of problem (4.1), which is due to Green's formula (4.4). The arguments are similar to those in the proof of the fact that each solution of problem (4.9) satisfies actually (4.2).

On applying Lemma 4.2 to problem (4.8) we conclude readily that the moment relations

$$(v, u)_{L^2(\mathcal{X},\mathbb{C}^\ell)} - (u_1, Cu)_{L^2(\partial\mathcal{X},\mathbb{C}^{r\ell-\ell'})} = 0 \tag{4.10}$$

for all functions u in the null-space of problem (4.1) constitute a necessary and sufficient condition for solvability of (4.8).

Having disposed of these preliminary steps we now return to the Euler-Lagrange equations for variational problem (4.3), i.e.

$$\left\{\begin{array}{rcll} A^*A\,u & = & A^*f & \text{in} \quad \mathcal{X}, \\ Bu & = & u_0 & \text{at} \quad \partial\mathcal{X}, \\ B^*A\,u & = & B^*f & \text{at} \quad \partial\mathcal{X}, \end{array}\right. \tag{4.11}$$

cf. (4.8). Notice that solvability conditions (4.10) are automatically fulfilled for problem (4.11).

Theorem 4.4. *Suppose A is a (possibly, overdetermined) elliptic operator of order r and B a normal boundary operator satisfying the complementarity condition relative to A. Then, for any $f \in H^r(\mathcal{X}, \mathbb{C}^m)$ and $u_0 : \partial\mathcal{X} \to \mathbb{C}^{\ell'}$ with components in $H^{2r-r_i-1/2}(\partial\mathcal{X})$, each solution of Euler-Lagrange's equations (4.11) is an approximate solution of problem (4.1).*

This theorem is due to [KL85] who found also an abstract version of the theorem [KL87]. The same proof works for boundary value problems (4.1) where A is (possibly, overdetermined) elliptic in the sense of Petrovskii. In this case Euler-Lagrange's equations constitute a Douglis-Nirenberg elliptic boundary value problem. The theorem is still valid for $f \in L^2(\mathcal{X}, \mathbb{C}^m)$ and $u_0 \in \oplus H^{r-r_i-1/2}(\partial\mathcal{X})$. The proof is based on the theory of elliptic boundary value problems in Sobolev spaces of negative smoothness, see [Roi96]. However, this topic exceeds the scope of this paper.

Proof. Let first $u_0 = 0$. Then, $u \in \mathcal{A}_0$ is an approximate solution of problem (4.1) if Au is the nearest to f in $A\mathcal{A}_0$.

Assume $f \in H^r(\mathcal{X}, \mathbb{C}^m)$. By Green's formula (4.4), the data of problem (4.11) satisfy solvability conditions (4.10). Hence, there is a solution $u \in H^{2r}(\mathcal{X}, \mathbb{C}^\ell)$ of Euler-Lagrange's equations. Then $A^*(Au - f) = 0$ in $\mathcal{X}$ and $B^*(Au - f) = 0$ at $\partial\mathcal{X}$, i.e. $g = Au - f$ belongs to the null-space of formal adjoint problem (4.5). As already mentioned, the functions of the null-space of problem (4.5) are orthogonal to $A\mathcal{A}_0$. Since Au belongs to $A\mathcal{A}_0$, it follows that Au is the projection of f into $A\mathcal{A}_0$, and so Au is the nearest to f in $A\mathcal{A}_0$. We see that the solution u of (4.11) is an approximate solution of problem (4.1).

Let u_0 be a function on the boundary with values in $\mathbb{C}^{\ell'}$ whose components belong to $H^{2r-r_i-1/2}(\partial\mathcal{X})$. Since B is a normal boundary operator, there is a function $U_0 \in H^{2r}(\mathcal{X}, \mathbb{C}^\ell)$ satisfying $BU_0 = u_0$ at $\partial\mathcal{X}$. Each function $u \in H^{2r}(\mathcal{X}, \mathbb{C}^\ell)$ satisfying $Bu = u_0$ at $\partial\mathcal{X}$, has the form $u = U_0 + U$, where $U \in H^{2r}(\mathcal{X}, \mathbb{C}^\ell)$ fulfills the condition $BU = 0$ at the boundary. If u is an approximate solution of problem (4.1), then AU is the nearest to $f - AU_0$ in $A\mathcal{A}_0$, i.e. U is a solution of Euler-Lagrange's equations (4.11), with f replaced by $f - AU_0$ and u_0 replaced by 0. But then $u = U_0 + U$ is precisely a solution of boundary value problem (4.11), which establishes the theorem. □

Note that if the null-space of problem (4.1) is trivial then the operator A is actually a topological isomorphism of $\mathcal{A}_0$ onto $A\mathcal{A}_0$. Assume now that $f \in L^2(\mathcal{X}, \mathbb{C}^m)$ and $\{f_k\}_{k=1,2,\ldots}$ is a sequence in $H^r(\mathcal{X}, \mathbb{C}^m)$ which converges to f in the $L^2(\mathcal{X}, \mathbb{C}^m)$-norm. For each f_k, we construct an approximate solution $u_k \in \mathcal{A}_0$ of problem (4.1). The functions Au_k are the projections of f_k into $\mathcal{A}_0$. Since the inverse of A is continuous, it follows that the sequence $\{u_k\}$ converges to a function $u \in \mathcal{A}_0$. This latter proves immediately to be an approximate solution of problem (4.1) with data

f and $u_0 = 0$. The same arguments still go when we choose as u_k those unique solutions of Euler-Lagrange's equations which are orthogonal to the null-space of problem (4.1). A more complete theory may be obtained by elaborating the Hodge theory for Euler-Lagrange's equations but we will not develop this point here, see [Tar95].

We complete this section with two explicit examples which range from the Cauchy boundary data to free boundary data.

Example 4.5. Let $f \in H^1(\mathcal{X}, \mathbb{C}^n)$ and $u_0 \in H^{3/2}(\partial\mathcal{X})$. Then the approximate solution of the Cauchy problem

$$\left\{ \begin{array}{rclcl} u' & = & f & \text{in} & \mathcal{X}, \\ u & = & u_0 & \text{at} & \partial\mathcal{X}, \end{array} \right.$$

is given by the unique solution $u \in H^2(\mathcal{X})$ of the Dirichlet problem for Poisson's equation in $\mathcal{X}$,

$$\left\{ \begin{array}{rclcl} \Delta u & = & \operatorname{div} f & \text{in} & \mathcal{X}, \\ u & = & u_0 & \text{at} & \partial\mathcal{X}, \end{array} \right. \tag{4.12}$$

cf. Section 1.

If the operator A is of finite type, i.e. the null-space of A is finite dimensional, then B might be absent.

Example 4.6. Suppose $f \in H^1(\mathcal{X}, \mathbb{C}^n)$. Then the approximate solution of the inhomogeneous equation

$$u' \quad = \quad f \quad \text{in} \quad \mathcal{X}$$

is given by any solution $u \in H^2(\mathcal{X})$ of the Neumann problem for Poisson's equation in $\mathcal{X}$,

$$\left\{ \begin{array}{rclcl} \Delta u & = & \operatorname{div} f & \text{in} & \mathcal{X}, \\ \partial_\nu u & = & (\nu, f) & \text{at} & \partial\mathcal{X}, \end{array} \right.$$

where $\nu(x)$ is the outward normal unit vector of the boundary surface $\partial\mathcal{X}$ at a point x.

5. The Dirichlet to Neumann operator

Let A be a (possibly, overdetermined) elliptic operator of order r and B a normal boundary operator satisfying the complementarity condition with respect to A. Then problem (4.1) is normally solvable. In this section we describe the solvability conditions of this problem using an analogue of the so-called Dirichlet to Neumann operator, cf. [LT11].

The Dirichlet to Neumann operator proves to be an $(\ell' \times \ell')$-matrix of classical pseudodifferential operators on the boundary of $\mathcal{X}$. Hence, it extends to act continuously on the whole scale of Sobolev spaces on $\partial\mathcal{X}$, including those of negative smoothness. In order to not go beyond the classical setting we introduce the Dirichlet to Neumann operator for

$$\begin{array}{rcl} f & \in & H^r(\mathcal{X}, \mathbb{C}^m), \\ u_0 & \in & \oplus H^{2r-r_i-1/2}(\partial\mathcal{X}). \end{array}$$

Lemma 5.1. *In order that there might exist a function $u \in H^{2r}(\mathcal{X}, \mathbb{C}^{\ell'})$ satisfying $Au = f$ it is necessary that*

$$(f, g)_{L^2(\mathcal{X}, \mathbb{C}^m)} = 0 \tag{5.1}$$

for all $g \in H^r(\mathcal{X}, \mathbb{C}^m)$ *satisfying* $A^* g = 0$ *in the domain* $\mathcal{X}$ *and* $B^* g = 0$, $C^* g = 0$ *at its boundary.*

Proof. The assertion follows from Green's formula (4.4) immediately, cf. Lemma 4.2. □

It should be noted that, under additional assumptions, condition (5.1) is not only necessary but also sufficient for the solvability of $Au = f$, see Section 5.1 in [Tar95].

Given any $u_0 \in \oplus H^{2r - r_i - 1/2}(\partial \mathcal{X})$, choose an arbitrary solution $u \in H^{2r}(\mathcal{X}, \mathbb{C}^\ell)$ of Euler-Lagrange's equations (4.11). Set

$$\Psi(u_0) = C^* (Au - f), \tag{5.2}$$

which is a function on $\partial \mathcal{X}$ with values in $\mathbb{C}^{\ell'}$.

Definition 5.2. The operator Ψ on $\partial \mathcal{X}$ is called the Dirichlet to Neumann operator for boundary value problem (4.1).

Since the i-th row of C^* consists of partial differential operators of order $r - r_i - 1$, the i-th component of $\Psi(u_0)$ is of class $H^{r_i + 1/2}(\partial \mathcal{X})$. Thus, the operators in the i-th row of matrix Ψ have order $2(r - r_i) - 1$ relative to the scale of Sobolev spaces on $\partial \mathcal{X}$.

Lemma 5.3. *The definition of* Ψ *is correct, i.e. it does not depend on the particular choice of the approximate solution* u.

Proof. Let $u_1, u_2 \in H^{2r}(\mathcal{X}, \mathbb{C}^\ell)$ be any two solutions of Euler-Lagrange's equations (4.11). Then the difference $u = u_1 - u_2$ belongs to the null-space of boundary value problem (4.8). As mentioned, the null-space of boundary value problem (4.8) coincides with that of problem (4.1). Hence it follows that $Au = 0$, i.e. $Au_1 = Au_2$ in $\mathcal{X}$ whence

$$C^*(Au_1 - f) = C^*(Au_2 - f),$$

as desired. □

The principal significance of the Dirichlet to Neumann operator lies in the fact that it is a crucial ingredient of the compatibility operator for a normally solvable boundary value problem.

Theorem 5.4. *For the existence of a solution* $u \in H^{2r}(\mathcal{X}, \mathbb{C}^\ell)$ *to problem (4.1) it is necessary and sufficient that* f *would satisfy condition (5.1) and* $\Psi(u_0) = 0$.

Proof.

Necessity. Suppose that $u \in H^{2r}(\mathcal{X}, \mathbb{C}^\ell)$ satisfies $Au = f$ in $\mathcal{X}$ and $Bu = u_0$ at the boundary of $\mathcal{X}$. By Lemma 5.1, f satisfies condition (5.1). Furthermore, u is a solution of Euler-Lagrange's equations (4.11), and so $\Psi(u_0) = C^*(Au - f) = 0$, as desired.

Sifficiency. Conversely, assume that the condition (5.1) for f is fulfilled and $\Psi(u_0) = 0$. Pick any solution $u \in H^{2r}(\mathcal{X}, \mathbb{C}^\ell)$ of Euler-Lagrange's equations (4.11). By Lemma 5.3, we get $\Psi(u_0) = C^*(Au - f)$ whence $C^*(Au - f) = 0$ at $\partial \mathcal{X}$. Set $g = Au - f$, then $g \in H^r(\mathcal{X}, \mathbb{C}^m)$ satisfies $A^* g = 0$ in the domain $\mathcal{X}$ and $B^* g = 0$,

$C^* g = 0$ at its boundary. Using condition (5.1), we conclude by Green's formula (4.4) that

$$\begin{array}{rcl} (g,g)_{L^2(\mathcal{X},\mathbb{C}^m)} & = & (Au,g)_{L^2(\mathcal{X},\mathbb{C}^m)} - (f,g)_{L^2(\mathcal{X},\mathbb{C}^m)} \\ & = & (u,A^*g)_{L^2(\mathcal{X},\mathbb{C}^m)} + (Bu,C^*g)_{L^2(\partial\mathcal{X},\mathbb{C}^{\ell'})} - (Cu,B^*g)_{L^2(\partial\mathcal{X},\mathbb{C}^{r\ell-\ell'})} \\ & = & 0 \end{array}$$

whence $g = 0$ in $\mathcal{X}$. Thus, u is actually a solution of boundary value problem (4.1), which establishes the lemma. □

One may conjecture that Ψ is a formally selfadjoint operator in $L^2(\partial\mathcal{X}, \mathbb{C}^{\ell'})$, cf. [Tar10]. However, this topic exceeds the scope of this paper.

Example 5.5. Consider the Cauchy problem

$$\left\{ \begin{array}{rcll} u' & = & f & \text{in} \quad \mathcal{X}, \\ u & = & u_0 & \text{at} \quad \partial\mathcal{X} \end{array} \right.$$

in $\mathcal{X}$ with data $f \in H^1(\mathcal{X}, \mathbb{C}^n)$ and $u_0 \in H^{3/2}(\partial\mathcal{X})$, cf. Example 4.5. For the solvability of the inhomogeneous system $u' = f$ in $\mathcal{X}$ it is necessary that

$$(f,g)_{L^2(\mathcal{X},\mathbb{C}^n)} = 0 \tag{5.3}$$

for all functions $g \in H^1(\mathcal{X}, \mathbb{C}^n)$ satisfying $\operatorname{div} g = 0$ in $\mathcal{X}$ and $(\nu, g) = 0$ at $\partial\mathcal{X}$. The Dirichlet to Neumann operator is defined by

$$\Psi(u_0) := \partial_\nu u - (\nu, f)$$

at $\partial\mathcal{X}$, where $u \in H^2(\mathcal{X})$ is the unique solution of problem (4.12). By Theorem 5.4, the Cauchy problem has a solution $u \in H^2(\mathcal{X})$ if and only if f satisfies condition (5.3) and $\Psi(u_0) = 0$, cf. Section 1.

The problem of Example 4.6 does not contain any boundary conditions, hence the Dirichlet to Neumann operator for this problem reduces to zero. Theorem 5.4 just amounts to saying that condition (5.3) is necessary and sufficient for the solvability of inhomogeneous system $du = f$.

6. Normally solvable problems

Let V and W be Banach spaces. We assume that W is uniformly convex, which is always the case for Hilbert spaces or for Sobolev spaces with non-extreme exponent $1 < p < \infty$.

Suppose $A : V \to W$ is a continuous (in general, nonlinear) mapping which possesses a Fréchet derivative $A'(v)$ at each point $v \in V$. For any fixed $v_0 \in V$, the derivative $A'(v_0)$ is a bounded linear operator from V to W. Given a $w \in W$, we consider the equation

$$A(v) = w. \tag{6.1}$$

Definition 6.1. Equation (6.1) is called normally solvable if for each $w \in W$ there exists a (possibly, stationary) sequence $\{w_k\}$, such that $w_k \to w$ and the functional $v \mapsto \|A(v) - w_k\|$ takes on its minimum at a point v_k, and for any such a sequence $\{w_k\}$, if moreover $A(v_k) - w_k$ belongs to the annihilator of $\ker(A'(v_k))^t$, then $w \in A(V)$.

Here, by $(A'(v_k))^t$ is meant the transpose of the linear operator $A'(v_k)$, which acts in the dual spaces $W' \to V'$.

Lemma 6.2. *Assume $A : V \to W$ is linear. Then equation (6.1) is normally solvable if and only if it is normally solvable in the sense of Hausdorff, i.e. the range of A coincides with the annihilator of* $\ker A^t$.

Proof.
Necessity. Suppose that equation (6.1) is normally solvable. By Lemma 6.5 below, the range AV of A is closed. Applying a familiar result of linear operators we conclude immediately that A is normally solvable in the sense of Hausdorff, as desired.

Sufficiency. Conversely, let (6.1) be normally solvable in the sense of Hausdorff. Then the range AV of A is closed in W. Since W is a reflexive Banach space, AV is weakly closed, which is due to a theorem of Mazur, see [Yos65]. For each fixed $w_0 \in W$, the functional $v \mapsto \|Av - w_0\|$ takes on its infimum at some point $v_0 \in V$. Choose $w_k = w_0$, then $\{w_k\}$ is a stanionary sequence, such that $w_k \to w_0$ and the functional $v \mapsto \|A(v) - w_k\|$ takes on its minimum at the point $v_k = v_0$. Let $\{w_k\}$ be an arbitrary sequence in W, such that $w_k \to w$, the functional $v \mapsto \|A(v) - w_k\|$ takes on its minimum at a point v_k, and $A(v_k) - w_k$ belongs to the annihilator of $\ker A^t$. Since $A(v_k)$ belongs to the annihilator of $\ker A^t$, we deduce that w_k belongs to the annihilator of $\ker A^t$. This implies readily that w belongs to the annihilator of $\ker A^t$, for the annihilator is a closed subspace of W and $w_k \to w$. Hence it follows that $w \in AV$, for the linear equation $Av = w$ is normally solvable in the sense of Hausdorff. □

The main result of this section reads in the same way as the familiar Hausdorff lemma [Hau32]. It goes back at least as far as [Pok69].

Theorem 6.3. *In order that nonlinear equation (6.1) be solvable it is necessary and sufficient that the range $A(V)$ of A would be closed in W.*

The proof is based on two lemmata and a theorem of [Ede68]. For a fixed $w_0 \in W$, let v_0 be a point of local minimum of the functional $v \mapsto \|A(v) - w_0\|$ on V. Write $A_0 = A'(v_0)$, so that A_0 is a linear operator from V to W with range A_0V.

Lemma 6.4. *Assume that $A(v_0) - w_0$ belongs to the closure of A_0V in W. Then $A(v_0) = w_0$.*

Proof. Suppose $A(v_0) - w_0$ is different from zero. Since this difference belongs to the closure of A_0V, for $\varepsilon = \|A(v_0) - w_0\|/2 > 0$ there is an element $w_1 \in A_0V$, such that $\|w_1 - (A(v_0) - w_0)\| < \varepsilon$. Fix any $v_1 \in V$ with the property that $A_0v_1 = w_1$. Since $w_1 \neq 0$, it follows that $v_1 \neq 0$, i.e. $\|v_1\| > 0$. On setting $v = v_0 + tv_1$, for real t, we get $A_0(v - v_0) = tw_1$.

We now exploit the Fréchet differentiability of the operator A. More explicitly, we get

$$A(v) = A(v_0) + A_0(v - v_0) + o(\|v - v_0\|)$$

for all $v \in V$, where $o(\cdot)$ stands for the Landau symbol depending on v_0. By assumption, $\|A(v_0) - w_0\| \leq \|A(v) - w_0\|$ holds for all v in a neighbourhood of v_0 whence

$$\|A(v_0) - w_0\| \leq \|A(v_0) - w_0 + A_0(v - v_0) + o(\|v - v_0\|)\|$$

for all v in the neighbourhood of v_0. In particular, for each $v = v_0 + tv_1$ with sufficiently small $|t|$, we obtain

$$\begin{aligned}\|A(v_0)-w_0\| &= \|A(v_0)-w_0+tw_1+o(\|tv_1\|)\|\\ &= \|A(v_0)-w_0+t(A(v_0)-w_0)+t(w_1-(A(v_0)-w_0))+o(\|tv_1\|)\|\\ &\le |1+t|\,\|A(v_0)-w_0\|+|t|\,\|w_1-(A(v_0)-w_0)\|+\|o(\|tv_1\|)\|.\end{aligned}$$

Since $\|w_1 - (A(v_0)-w_0)\| \le \dfrac{1}{2}\,\|A(v_0)-w_0\|$ and $\dfrac{\|o(\|tv_1\|)\|}{\|tv_1\|} \to 0$ as $t \to 0$, it follows that

$$\|A(v_0)-w_0\| \le |1+t|\,\|A(v_0)-w_0\| + \frac{1}{2}\,|t|\,\|A(v_0)-w_0\| + o(1)\,|t|\,\|v_1\|$$

as $t \to 0$. The elements $A(v_0)-w_0$ and v_1 are independent of t, and $\|A(v_0)-w_0\|$ is positive. Hence,

$$1 \le |1+t| + |t|\left(\frac{1}{2} + o(1)\,\frac{\|v_1\|}{\|A(v_0)-w_0\|}\right)$$

is fulfilled for all sufficiently small t, where $o(1) \to 0$ as $t \to 0$. Letting $t \to 0-$ we get a contradiction. □

Lemma 6.5. *Assume that equation (6.1) is normally solvable. Then the range $A(V)$ of A is closed in W.*

Proof. Let $w_k \in A(V)$ and $w_k \to w$ as $k \to \infty$. For every k there is an element $v_k \in V$ satisfying $A(v_k) = w_k$. Thus, for each w_k there is a point v_k at which the functional $v \mapsto \|A(v) - w_k\|$ takes on its minimum equal to zero. Obviously, $A(v_k) - w_k = 0$ belongs to the annihilator of $\ker(A'(v_k))^t$ in W, for the operator $A'(v_k)$ is linear. Then the second part of Definition 6.1 implies that $w \in A(V)$, as desired. □

Finally, we formulate the theorem of [Ede68] which is used in the proof of Theorem 6.3.

Theorem 6.6. *Suppose $\mathcal{S}$ is a nonempty closed set in a uniformly convex Banach space W. Then the set of all elements $w \in W$ for which there is a point $s_w \in \mathcal{S}$ satisfying*

$$\|s_w - w\| = \inf_{s\in\mathcal{S}} \|s - w\|$$

is dense in W.

Proof of Theorem 6.3. Let $A(V)$ be closed in W. We apply Theorem 6.6 to the nonempty closed set $\mathcal{S} = A(V)$ in W. According to this theorem, for each $w \in W$ there exists a sequence $\{w_k\}$ in the space W, such that $w_k \to w$ and each functional $v \mapsto \|A(v) - w_k\|$ takes on its infimum at some point $v_k \in V$. This is precisely the first part of Definition 6.1.

Let now $\{w_k\}$ be an arbitrary sequence in W, such that $w_k \to w$, every functional $v \mapsto \|A(v) - w_k\|$ takes on its minimum at a point v_k, and $A(v_k) - w_k$ belongs to the annihilator of $\ker(A'(v_k))^t$. Since the annihilator of $\ker(A'(v_k))^t$ just amounts to the closure of the range of linear operator $A'(v_k)$, it follows from Lemma 6.4 that $A(v_k) - w_k = 0$ or $w_k = A(v_k)$. Hence, for every $k = 1, 2, \ldots$, the element w_k belongs to the range $A(V)$ of the operator A. Since $w_k \to w$, as $k \to \infty$, and the range $A(V)$ is closed in W, we conclude that $w \in A(V)$. This is the second part of Definition 6.1.

We have thus proved that if $A(V)$ is closed in W then the nonlinear equation $A(v) = w$ is normally solvable. Conversely, if the equation $A(v) = w$ is normally solvable, then $A(V)$ is closed in W, which is due to Lemma 6.2. This establishes the theorem. □

As introduced above, the concept of normal solvability is very technical and inefficient to be widely adopted. However, the nonlinear mappings of closed range constitute the most general class of mappings for which a substantial theory is still possible.

Theorem 6.7. *Suppose the range $A(V)$ of A is closed. If $\ker (A'(v))^t = \{0\}$ for all $v \in V$, then the equation $A(v) = w$ has a solution $v \in V$ for each $w \in W$.*

Proof. From the closedness of the range $A(V)$ we conclude by Theorem 6.3 that the equation $A(v) = w$ is normally solvable. Then, for each $w \in W$ there is a sequence $\{w_k\}$ in W converging to w and such that each functional $v \mapsto \|A(v) - w_k\|$ takes on its infimum at a point $v_k \in V$. Since the null-space of $(A'(v))^t$ is trivial for all $v \in V$, its annihilator coincides with the entire spave W, and so $A(v_k) - w_k$ lies in the annihilator of $\ker(A'(v))^t$. By Lemma 6.4, we get $w_k = A(v_k)$, i.e. w_k belongs to $A(V)$ for all k. Since $w_k \to w$, as $k \to \infty$, and the range of A is closed in W, we see that $w \in A(V)$, as desired. □

Theorem 6.7 gains in interest if we realise that, under its assumptions, the implicit function theory is not applicable, for one does not assume that $\ker A'(v)$ is a direct summand of V and the range of $A'(v)$ is closed in W. As but one application of Theorem 6.7 we consider the equation $A(v) = w$ with nonlinear Fredholm operator.

Definition 6.8. A nonlinear operator $A : V \to W$ of class C^1 is called Fredholm if, for each $v \in V$, the derivative $A'(v)$ is a (linear) Fredholm operator from V to W.

Recall that $A'(v)$ is a Fredholm operator if 1) $\ker A'(v)$ is finite dimensional; 2) the range of $A'(v)$ is closed in W, and 3) $\operatorname{coker} A'(v) := W/A'(v)V$ is finite dimensional.

Definition 6.8 goes back at least as far as [Sma65]. This paper also introduces the index of a nonlinear Fredholm operator A. By this is meant $\operatorname{ind} A := \operatorname{ind} A'(v)$ for some $v \in V$, for the right-hand side is actually independent of the particular choice of v.

Corollary 6.9. *Let $A : V \to W$ be a Fredholm operator of closed range and index zero. If $\ker A'(v) = \{0\}$ for all $v \in V$, then the equation $A(v) = w$ has a solution $v \in V$ for each $w \in W$.*

Proof. For a Fredholm operator $A'(v)$, the spaces $\ker(A'(v))^t$ and $\operatorname{coker} A'(v)$ are dual, hence, their dimensions coincide. Since the index of $A'(v)$ is equal to zero, it follows that the null-space of $(A'(v))^t$ is trivial. To complete the proof it suffices to exploit Theorem 6.7. □

7. The Cauchy problem for the derivative

Our viewpoint sheds some new light on problems which are basic and have attracted considerable attention both in geometry and analysis of the 20 th century.

Namely, let $\mathcal{S}$ be a nonempty open piece of the boundary surface $\partial\mathcal{X}$. Consider the Cauchy problem

$$\left\{ \begin{array}{rcll} u' & = & f & \text{in } \mathcal{X}, \\ u & = & u_0 & \text{at } \mathcal{S} \end{array} \right. \tag{7.1}$$

in $\mathcal{X}$ with data $f \in L^p(\mathcal{X}, \mathbb{C}^n)$ and $u_0 \in W^{1/p',p}(\mathcal{S})$, where $1 < p < \infty$ and $p' = p/(p-1)$, cf. Example 4.5.

As mentioned in Example 5.5, for the solvability of the inhomogeneous system $u' = f$ in $\mathcal{X}$ it is necessary that $(f, g)_{L^2(\mathcal{X},\mathbb{C}^n)} = 0$ for all functions $g \in W^{1,p'}(\mathcal{X}, \mathbb{C}^n)$ satisfying $\operatorname{div} g = 0$ in $\mathcal{X}$ and $(\nu, g) = 0$ at $\partial\mathcal{X}$, cf. (5.3). Choosing $g = d^*v$, where v is a smooth differential form of degree 2 with compact support in the interior of $\mathcal{X}$ and d^* the formal adjoint of the exterior derivative, we conclude that $df = 0$ in $\mathcal{X}$. If the de Rham cohomology of $\mathcal{X}$ at step 1 is zero (e.g., if the domain $\mathcal{X}$ is contractible), then the condition $df = 0$ is also sufficient for the existence of a function $u \in W^{1,p}(\mathcal{X})$ satisfying $u' = f$ in $\mathcal{X}$. This is a consequence of the ellipticity of the Neumann problem for the de Rham complex, see Section 4.1.3 of [Tar95] and elsewhere.

Since problem (7.1) is overdetermined, we study its relaxation which consists in minimising the discrepancy

$$I(u) := \int_{\mathcal{X}} |u' - f|^p \, dx$$

over the set $\mathcal{A}$ of all functions $u \in W^{1,p}(\mathcal{X})$ satisfying $u = u_0$ at $\mathcal{S}$. A direct computation shows that the Euler-Lagrange equations for the variational problem $I(u) \mapsto \min$ are

$$\left\{ \begin{array}{rcll} \operatorname{div}\left(|u' - f|^{p-2}(u' - f)\right) & = & 0 & \text{in } \mathcal{X}, \\ u & = & u_0 & \text{at } \mathcal{S}, \\ |u' - f|^{p-2}\,(u' - f, \nu) & = & 0 & \text{at } \partial\mathcal{X} \setminus \mathcal{S}, \end{array} \right. \tag{7.2}$$

which actually constitute a mixed boundary value problem of Zaremba's type [Zar10].

Lemma 7.1. *Suppose $f \in L^p(\mathcal{X})$ and $u_0 \in W^{1/p',p}(\mathcal{S})$, where $1 < p < \infty$. If $H^1_{\mathrm{dR}}(\mathcal{X}) = 0$ and $df = 0$ in $\mathcal{X}$, then problem (7.2) possesses a unique solution $u \in W^{1,p}(\mathcal{X})$.*

Proof. By the above, there is a function $U_0 \in W^{1,p}(\mathcal{X})$ satisfying $U_0' = f$ in $\mathcal{X}$. Set $u = U_0 + U$. Then $u' - f = U'$, and so problem (7.2) reduces to finding a function $U \in W^{1,p}(\mathcal{X})$, such that

$$\left\{ \begin{array}{rcll} \operatorname{div}\left(|U'|^{p-2}U'\right) & = & 0 & \text{in } \mathcal{X}, \\ U & = & u_0 - U_0 & \text{at } \mathcal{S}, \\ |U|^{p-2}\,U'_\nu & = & 0 & \text{at } \partial\mathcal{X} \setminus \mathcal{S}. \end{array} \right.$$

This latter problem has been studied in [She13] who shows in particular that for each data $u_0 \in W^{1/p',p}(\mathcal{S})$ there is precisely one solution $U \in W^{1,p}(\mathcal{X})$ to the problem. Hence it follows that $u = U_0 + U \in W^{1,p}(\mathcal{X})$ is a unique solution of (7.2), as desired. □

The paper [She13] gives more, namely the function U is a solution of the variational problem

$$\int_{\mathcal{X}} |U'|^p \, dx \mapsto \min$$

over the affine space of all $U \in W^{1,p}(\mathcal{X})$ satisfying $U = u_0 - U_0$ at $\mathcal{S}$. Thus, the only solution $u \in W^{1,p}(\mathcal{X})$ of problem (7.2) is actually a solution of the variational problem $I(u) \mapsto \min$ over $u \in \mathcal{A}$. One may ask whether the variational problem $I(u) \mapsto \min$ has a unique solution for all data $f \in L^p(\mathcal{X}, \mathbb{C}^n)$ and $u_0 \in W^{1/p',p}(\mathcal{S})$, not only for those satisfying $df = 0$ in $\mathcal{X}$. However, we will not develop this point here.

Having granted the unique solvability of inhomogeneous problem (7.2), we are in a position to introduce a Dirichlet to Neumann operator for problem (7.1). Pick $u_0 \in W^{1/p',p}(\mathcal{S})$. By Lemma 7.1, there is a unique function $u \in W^{1,p}(\mathcal{X})$ satisfying (7.2). Set

$$\Psi(u_0) := |u' - f|^{p-2} \, (u' - f, \nu)$$

at $\mathcal{S}$, the restriction on $\mathcal{S}$ being understood in an appropriate sense clarified in [She13]. Then, $\Psi(u_0) \in W^{-1/p',p'}(\mathcal{S})$.

Theorem 7.2. *Let $f \in L^p(\mathcal{X})$ and $u_0 \in W^{1/p',p}(\mathcal{S})$, where $1 < p < \infty$. The Cauchy problem (7.1) has a solution $u \in W^{1,p}(\mathcal{X})$ if and only if f satisfies condition (5.3) and $\Psi(u_0) = 0$.*

Proof. Necessity. If the Cauchy problem has a solution $u \in W^{1,p}(\mathcal{X})$, then u satisfies $u' = f$, and so (5.3) is fulfilled for f. Moreover, u is a solution of problem (7.2) whence $\Psi(u_0) = 0$, as desired.

Sufficiency. Suppose $f \in L^p(\mathcal{X}, \mathbb{C}^n)$ satisfies the condition $(f, g)_{L^2(\mathcal{X}, \mathbb{C}^n)} = 0$ for all $g \in C^\infty(\mathcal{X}, \mathbb{C}^n)$, such that $\operatorname{div} g = 0$ in $\mathcal{X}$ and $(\nu, g) = 0$ at $\partial\mathcal{X}$. Using the Hodge theory, we write

$$f = Hf + (Gd^*)df + d(Gd^*)f$$

in $\mathcal{X}$, where H is the orthogonal projection onto the space of harmonic 1-forms in $\mathcal{X}$ and G is the Green operator in $\mathcal{X}$, see [Tar95, 4.1]. From the assumptions on f it follows immediately that both Hf and df vanish in $\mathcal{X}$. Hence, we get $f = dU_0$, where $U_0 = (Gd^*)f$ belongs to $W^{1,p}(\mathcal{X})$, which is due to the regularity properties of the Green operator G. Analysis similar to that in the proof of Lemma 7.1 shows that there is a unique function $u \in W^{1,p}(\mathcal{X})$ satisfying (7.2). If $\Psi(u_0) = 0$ on $\mathcal{S}$, then $|u' - f|^{p-2} \, (u' - f, \nu)$ vanishes on the whole boundary $\partial\mathcal{X}$. Since the Neumann problem for the p-Laplace equation has only constant solutions, we deduce readily that $u' - f = 0$ in $\mathcal{X}$. $\square$

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