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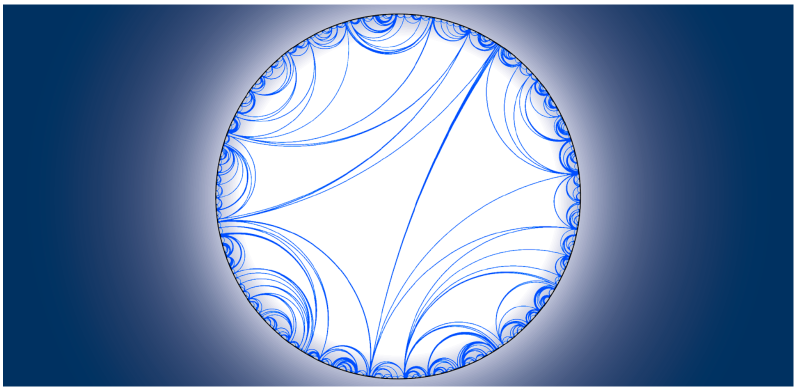

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Bibliografische Information der Deutschen Nationalbibliothek
Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über http://dnb.de abrufbar.

Universitätsverlag Potsdam 2013
http://info.ub.uni-potsdam.de/verlag.htm

Am Neuen Palais 10, 14469 Potsdam
Tel.: +49 (0)331 977 2533 / Fax: 2292
E-Mail: verlag@uni-potsdam.de

Die Schriftenreihe **Preprints des Instituts für Mathematik der Universität Potsdam** wird herausgegeben vom Institut für Mathematik der Universität Potsdam.

ISSN (online) 2193-6943

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Titelabbildungen:
1. Karla Fritze | Institutsgebäude auf dem Campus Neues Palais
2. Nicolas Curien, Wendelin Werner | Random hyperbolic triangulation
Published at: http://arxiv.org/abs/1105.5089

Online veröffentlicht auf dem Publikationsserver der Universität Potsdam
URL http://pub.ub.uni-potsdam.de/volltexte/2013/6472/
URN urn:nbn:de:kobv:517-opus-64725
http://nbn-resolving.de/urn:nbn:de:kobv:517-opus-64725

The Capture of a Particle into Resonance at Potential Hole with Dissipative Perturbation

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March 7, 2013

Abstract

We study the capture of a particle into resonance at a potential hole with dissipative perturbation and periodic outside force. The measure of resonance solutions is evaluated. We also derive an asymptotic formula for the parameter range of those solutions which are captured into resonance.

1 Introduction

Second order differential equations are of fundamental importance for applications in physics. This is an obvious consequence of Newton's second law. In the case of one and a half freedom degrees this is a recognized testing area for nonlinear equations. At the end of 19 th century Poincaré called the study of such systems the most important problem of dynamics. In spite of progress both in qualitative comprehension of processes in systems with one and a half degrees of freedom and in quantitative investigation, analytic theory is by no means completed.

The equation of motion for a particle at a potential hole with small almost periodic outside force and dissipation

$$u'' + g(u) = \epsilon f(t) - \epsilon \mathit{\Gamma} u' \tag{1.1}$$

is an example of dissipative perturbed system with one and a half degrees of freedom. Here ϵ is a small parameter, $f(t)$ a smooth almost periodic function, and $\mathit{\Gamma} > 0$ the parameter of dissipation.

This equation is a model for mathematical investigation of nonlinear oscillatory systems with dissipation. Dissipation leads to a decrease of energy and so it results in a change of the oscillation period. The period of oscillations of a system goes through resonance values under an outside force. Locally in a neighbourhood of resonance the solution is determined by the equation of nonlinear resonance. In the dissipationless case, if moreover $f(t) = \cos(\omega t)$, then (1.1) is the equation of mathematical pendulum with outer momentum [1].

When passing through resonance without capture, the solution is known to undergo a change as large as the square root of the perturbation parameter [2].

Because of dissipation the phase portrait of the equation of nonlinear resonance changes essentially and there appears a region of trajectories which are captured into resonance. Estimates for the measure of the region of captured solutions are obtained in [3]. The papers [4, 5, 6] give the complete qualitative investigation of solutions to Duffing's equation with dissipation in a neighbourhood of resonance level. In [7], a qualitative approach to the study of resonances in nonlinear systems is presented.

In spite of achievements in qualitative theory and the understanding of local analysis, the capture into resonance is interpreted from the viewpoint of probability theory [9, 10, 11, 12] and symbolic dynamics [13, 14]. In problems on perturbations near a separatrix there appear asymptotics on Cantor sets [15]. Hence the change to a mathematical tool which allows one to investigate the measure of trajectory regions might be of use. However, this does not cancel the importance of investigations of the behaviour of single trajectories with characteristic properties.

From the viewpoint of perturbation theory, in the problem of capture into resonance one can distinguish three time intervals with distinctive behavior of solution. Firstly, this is a time interval far off the resonance. Secondly, this is a time interval near the resonance which is, however, outside of capture region. Thirdly, this is the capture region. It turns out that for each of the intervals there are specific small parameters which enable to construct an asymptotic solution suitable for the region. Formulas for asymptotic solutions in these regions represent intermediate asymptotics [16]. The regions of applicability of the intermediate asymptotics meet each other. Hence, one can match the parameters of constructed asymptotic in much the same way as one does in the method of matching of asymptotic expansions [17]. As a result one manages to construct an asymptotic solution of the original problem which is fit for all concerned regions.

Away from resonance one uses the Krylov-Bogolyubov method to construct asymptotics of solutions, see [20, 18, 19, 21]. In order to find the connection of parameters, one exploits the Whitham method, see [22, 23, 21]. However, the results of [22, 23, 21] do not apply directly in our case, for we deal with non-periodical perturbations. We thus had to develop this approach by using averaging over all fast time rather than averaging over a period.

Capture into resonance corresponds to crossing the separatrix in the perturbed equation of nonlinear resonance. Under perturbation the separatrices of non-perturbed equation split. The value of the splitting is determined by Mel'nikov's integral [8]. Under crossing the separatrix, the values of action [9] and phase [10] related to the non-perturbed equation of nonlinear resonance change. Equations and their solutions under crossing the separatrix were considered in [24, 25]. Formulas for solutions which are fit for all the regions enable one to investigate the properties of single trajectories. The papers [26, 27, 28, 29] contain such formulas for autoresonance problems, constructed by the method of matching of asymptotic expansions [17].

In the present paper we obtain the following new results. First, we derive a connection formula for asymptotics away from the region of capture into nonlinear resonance and the parameters of trajectories captured into resonance in the capture region. In this formula the value of the phase shift of oscillations is of crucial importance. It turns out that this value is singular in the parameter of perturbation. Second, in order to compute connection formulas we ought to develop perturbation theory for trajectories which are similar to the trajectories of angular motion of asymptotic pendulum away from the separatrix.

The setting of the problem and results of the paper are presented in Section 2. In Section 3 we construct an asymptotic solution outside the resonance. In Section 4 we obtain an asymptotic solution in a neighbourhood of resonance. Finally, in Section 5 we adjust the constructed asymptotics to each other and obtain the desired formula for the parameter values of those solutions which will be captured into resonance.

2 Formal setting and results

Before we formulate the problem, let us determine the properties of solutions of the non-perturbed equation and admissible perturbations. By the non-perturbed equation we mean

$$U'' + g(U) = 0. \tag{2.1}$$

Let $g(U)$ be a smooth function corresponding to a potential hole. We introduce

$$G(U) = \int_{U*}^{U} g(u)du,$$

where $\min\limits_{U} g(U) = g(U_*)$.

Denote by $U(t - t_0, E)$ the periodical general solution of the non-perturbed equation. Here, $t_0 \in \mathbb{R}$ and

$$E = \frac{1}{2}(U')^2 + G(U)$$

are parameters of the solution. We will confine our consideration to those $g(U)$, for which the period of the solution of (2.1) depends monotonically on the parameter E, i.e.

$$T(E) := \int_L \frac{dU}{U'} < \infty, \quad \frac{dT}{dE} \neq 0, \tag{2.2}$$

where L is the closed curve in the space of parameters (U, U') given implicitly by the equation $\frac{1}{2}(U')^2 + G(U) = E$.

In the paper we consider an interval of those values of E for which there are periodical solutions of the form $U(t - t_0, E)$ real analytic in $t \in \mathbb{R}$. As a typical example one can consider the equation of mathematical pendulum. In this latter case we have $g(U) = \sin(U)$, $U_* = -\pi/2$, $G(U) = -\cos(U)$, and $E \in (-1, 1)$.

An external perturbation is an almost periodic smooth real-valued function [30] with Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} f_k \, e^{\imath \omega_k t}. \tag{2.3}$$

We arrange for a special decomposition of $f(t)$ into summands of different orders in ϵ. For $n = 1, 2, \ldots$, denote by $\mathcal{K}_n(\epsilon)$ the set of those indices k, for which

$\epsilon^{n+1} < |f_k| \leq \epsilon^n$. For $n = 0$, we define $k \in \mathcal{K}_0(\epsilon)$ if $f_k > \epsilon$. Set $\mathfrak{f}_k = \epsilon^{-n} f_k$ for $k \in \mathcal{K}_n(\epsilon)$. Then

$$f(t) = \sum_{\mathcal{K}_0(\epsilon)} \mathfrak{f}_k e^{\imath \omega_k t} + \sum_{n=1}^{\infty} \epsilon^n \sum_{\mathcal{K}_n(\epsilon)} \mathfrak{f}_k e^{\imath \omega_k t}.$$

We introduce

$$f_n(t) = \sum_{k \in \mathcal{K}_n(\epsilon)} \mathfrak{f}_k e^{\imath \omega_k t}$$

for $n = 0, 1, \ldots$.

Denote by $\Omega(E)$ the frequency of oscillations of solution of the non-perturbed equation. A parameter value $E = E^n_{m,k}$ satisfying

$$m\, \Omega(E^n_{m,k}) = \omega_k,$$

for $m \in \mathbb{Z}$ and $k \in \mathcal{K}_n(\epsilon)$, is called a resonant level at order n.

Since $T(E)$ is monotonic, one can order the resonant levels according to the increase of E. In this paper we consider functions $g(U)$ and $f(t)$ with the property that the neighbouring resonant levels of E at order 0 differ from each other by a quantity much larger than $\sqrt{\epsilon}$, i.e.

$$\min |E^0_{m_1,k_1} - E^0_{m_2,k_2}| \gg \sqrt{\epsilon} \tag{2.4}$$

provided $(m_1, k_1) \neq (m_2, k_2)$. This condition is called the asymptotic condition of non-overlap of nonlinear resonances.

Remark 2.1. Condition (2.4) is wittingly weaker than the well-known non-overlap condition for resonances by Chirikov [31].

Remark 2.2. Condition (2.4) may be violated close to separatrices. Consider e.g. $g(U) \equiv \sin U$ in a neighbourhood of $E = 1$.

In this paper we evaluate, away from the resonance region, the parameters of those asymptotic solutions of equation (1.1) which will be captured into resonance. To this end, we construct an asymptotic solution which fits both away from and nearby the resonance.

In order to formulate the main result it is convenient to use the following designations related to asymptotics outside of resonance and in a neighbourhood of resonance. Let $u_0(S, E)$ be an even periodic function of zero mean value in S satisfying the equation

$$(\sigma')^2\, \partial_S^2 u_0 + g(u_0) = 0,$$

and

$$E = \frac{1}{2}(\sigma')^2\ (\partial_S u_0)^2 + G(u_0).$$

Here, the variable S stands for the fast time while σ is a function of slow time $\theta = \epsilon t$. As is usual in the method of multiple scales, S and θ are thought of as independent variables.

More precisely, the function $\sigma(\theta)$ is defined to be a solution of the Cauchy problem

$$\begin{array}{rcl} \Big(\displaystyle\int_L \frac{du_0}{\partial_S u_0} \Big) \sigma' & = & 2\pi, \\ \sigma(\theta_{m,k}) & = & 0, \end{array}$$

L being the closed curve in the space of parameters $(u_0, \partial_S u_0)$ given implicitly by the equation $\dfrac{1}{2}(\partial_S u_0)^2 + G(u_0) = E$.

We introduce also functions $I_0(\theta)$ and $\alpha(\theta)$ as solutions of the Cauchy problems

$$\begin{aligned} I_0' + \mathit{\Gamma} I_0 &= 0, \\ I_0(\theta_{m,k}) &= \mathcal{I} := \sigma' \int_L \partial_S u_0 \, du_0, \end{aligned}$$

with $E = E^0_{m,k}$, and

$$\begin{aligned} \alpha' &= \sigma', \\ \alpha(\theta_{m,k}) &= \mathcal{A}. \end{aligned}$$

Set

$$\begin{aligned} \mathit{\Omega}(E) &= \int_L \partial_S u_0 \, du_0, \\ q(\phi) &= \frac{\mathit{\Omega}}{2\mathcal{I}} \lim_{s\to\infty} \frac{1}{s} \int_0^s f(t) \, \partial_S u_0 \Big(t + \frac{\phi}{\mathit{\Omega}} \Big) dt, \end{aligned}$$

where $\mathit{\Omega} := \mathit{\Omega}(E^0_{m,k})$. Let $\{\phi_J\}$ be the sequence of roots of the transcendental equation $q(\phi) + 2\mathit{\Omega}\mathit{\Gamma} = 0$, such that $q'(\phi_{2j+1}) < 0$ and $q'(\phi_{2j}) > 0$. We now define

$$\begin{aligned} \mathcal{E}^-_{2j+1} &= Q(\phi_{2j+1}) + \mathit{\Omega}\mathit{\Gamma}\phi_{2j+1}, \\ \mathcal{E}^+_{2j+1} &= \mathcal{E}^-_{2j+1} + \sqrt{\epsilon}\mathit{\Gamma} \int_{\mathcal{L}} \phi' d\phi, \end{aligned}$$

where Q is a primitive of q, i.e. $Q'(\phi) = q(\phi)$, and $\mathcal{L}$ is the closed curve in the space of parameters (ϕ, ϕ') defined implicitly by

$$\frac{1}{2}(\phi')^2 + Q(\phi) + \mathit{\Omega}\mathit{\Gamma}\phi = \mathcal{E}_{2j+1}.$$

Under this notation, the main result of this paper reads as follows.

Theorem 2.3. *The asymptotic solution of equation (1.1) on an interval of length of order ϵ^{-1} for $t < \theta_{k,m}/\epsilon$ has the form $u \sim u_0(S, E)$, where the variable S and parameter $E(\theta)$ are determined from the equalities $\theta = \epsilon t$, $S = \sigma(\theta)/\epsilon + \alpha$ and*

$$I_0(\theta) = \sigma' \int_L \partial_S u_0 \, du_0.$$

This solution is captured into resonance in a neighbourhood of $t = \theta_{m,k}/\epsilon$ if, for any j, we get

$$\frac{\mathcal{E}^-_{2j+1}}{\mathit{\Omega}(E^0_{m,k})} + \frac{\theta_{m,k}}{\epsilon} < \mathcal{A} < \frac{\mathcal{E}^+_{2j+1}}{\mathit{\Omega}(E^0_{m,k})} + \frac{\theta_{m,k}}{\epsilon}.$$

3 Non-resonant regions of parameter

In this section we construct an asymptotic solution of equation (1.1) far away from the resonant levels $E^0_{m,k}$. The solution is built by the method of two scales combined with the Whitham averaging method and Krylov-Bogolyubov expansions for the parameters of solution. However, in contrast to the standard approach, we trace accurately beginnings of singularities in a neighbourhood of resonant levels. Intimate knowledge of singularities allows one to determine the applicability region of Krylov-Bogolyubov asymptotics when approaching resonant levels of the parameter E.

3.1 An equation for averaged action

Here we introduce slow and fast variables and derive an integrate equation for averaged action.

Outside of the resonant levels of the parameter E the asymptotic solution of equation (1.1) is built by the method of two scales. We change the independent variable by introducing the fast time variable

$$S = \frac{\sigma(\theta)}{\epsilon} + \alpha(\theta, \epsilon)$$

and the slow time variable $\theta = \epsilon t$. In the new variables the original variable t is given by

$$t = \frac{\theta}{\sigma(\theta)} S - \frac{\theta}{\sigma(\theta)} \alpha(\theta, \epsilon).$$

The differential equation (1.1) takes the form

$$(\sigma')^2 \partial_S^2 u + g(u) = \epsilon\, f(t) - \epsilon\, D_1 u - \epsilon^2\, D_2 u, \tag{3.1}$$

where

$$\begin{aligned} D_1 &= 2\sigma'\alpha'\partial_S^2 + \sigma''\partial_S + 2\sigma'\partial_\theta\partial_S - \Gamma\sigma'\partial_S, \\ D_2 &= (\alpha')^2\partial_S^2 + \alpha''\partial_S + 2\alpha'\partial_S\partial_\theta + \partial_\theta^2 + \Gamma\alpha'\partial_S + \Gamma\partial_\theta. \end{aligned}$$

Let $G(y)$ be a primitive of $g(y)$. On multiplying equation (3.1) by $\partial_S u$ and integrating in S we get

$$(\sigma')^2 \frac{(\partial_S u)^2}{2} + G(u) = E + \epsilon \int_{S_0}^{S} (f(t) - D_1 u - \epsilon D_2 u)\partial_S u\, dS,$$

where $E = E(\theta)$ is a parameter of the solution.

Suppose

$$\lim_{S\to\infty} \frac{1}{S} \int_{S_0}^{S} (f(t) - D_1 u - \epsilon D_2 u)\partial_S u\, dS = O(\epsilon^2), \tag{3.2}$$

then the principal part in ϵ of the solution u of equation (3.1) is given by a function u_0 satisfying

$$(\sigma')^2 \left(\partial_S u_0\right)^2 = 2E - 2G(u_0).$$

This equation for u_0 can be integrated in S by quadratures, namely

$$S = \sigma' \int_{y_0}^{u_0} \frac{dy}{\sqrt{2E - 2G(y)}}$$

where y_0 is a constant in the interval $u_- \le y_0 \le u_+$ whose bounds u_+ and u_- are roots of the equation $2E - 2G(y) = 0$, such that $2E - 2G(y) > 0$ for all

$u_- < y < u_+$. Following [25] we assume for definiteness that $u_0(S, E)$ is an even function of S with zero mean value over the period. We require the period of the function u_0 in the variable S to be equal to one, i.e

$$\sigma' \int \frac{du_0}{\partial_S u_0} = 2\pi. \tag{3.3}$$

In the Krylov-Bogolyubov method formula (3.3) is regarded as differential equation for the unknown function $\sigma(\theta)$. The left-hand side of this equation is not yet defined, for the function $E(\theta)$ has remained indetermined.

Define the averaged action by the formula

$$I(\theta) = (\sigma' + \epsilon\alpha') \lim_{S\to\infty} \frac{1}{S} \int_{S_0}^{S} (\partial_S u)^2 \, dS. \tag{3.4}$$

Then equation (3.2) can be essentially simplified to

$$I' + \Gamma I + F(\theta, I) = 0, \tag{3.5}$$

where

$$F(\theta, I) \equiv \lim_{S\to\infty} \frac{1}{S} \int_{S_0}^{S} f(t) \partial_S u \, dS.$$

The function $E(\theta)$ is determined uniquely through $I(\theta)$. We have thus proved the following lemma.

Lemma 3.1. *Assume that $E(\theta)$, $\sigma(\theta)$ and $\alpha(\theta, \epsilon)$ satisfy equation (3.5). Then*

$$u(t, \epsilon) \sim u_0(S, E).$$

This assertion is a generalisation of the well-known Whitham method for periodical solutions to the non-periodical case. Equation (3.2) is neither linear nor autonomic. To study this equation we develop a perturbation theory below. The properties of solutions to this equation differ essentially from those of known solutions obtained in [25]. This distinction is due to the non-periodicity in S of the solution of equation (3.1).

3.2 The substitution of Krylov-Bogolyubov

We look for a solution to (3.1) by the Krylov-Bogolyubov method in the form of series in the powers of small parameter ϵ. That is,

$$\begin{aligned} u(t, \epsilon) &= \sum_{n=0}^{\infty} \epsilon^n u_n(S, \theta, \epsilon), \\ \alpha(\theta, \epsilon) &= \sum_{n=0}^{\infty} \epsilon^n \alpha_n(\theta). \end{aligned} \tag{3.6}$$

Substitute (3.6) into equation (3.1) and equate the coefficients of the same powers of ϵ. As a result we get a recurrent system of equation for determining the coefficients u_n.

The equation for u_0 is

$$(\sigma')^2 \partial_S^2 u_0 + g(u_0) = 0, \tag{3.7}$$

the equation for u_1

$$(\sigma')^2\partial_S^2 u_1 + g'(u_0)u_1 = f_0(t) - D_1 u_0, \tag{3.8}$$

and the equation for u_2

$$(\sigma')^2\partial_S^2 u_2 + g'(u_0)u_2 = f_1(t) - \frac{1}{2}g''(u_0)u_1^2 - D_1 u_1 - D_2 u_0. \tag{3.9}$$

The techniques of solving these equations for amendments is well understood. The case $f(t) \equiv 0$ is treated in detail, e.g., in the paper [21]. The problem under study does not fit into this theory, for the function $f(t)$ on the right-hand side of (1.1) need not be zero.

By parameters of asymptotic solution (3.6) we will mean an "initial" value of slow time $\theta = \theta_0$ and the values $E_0 = E(\theta_0)$ and $a = \alpha(\theta_0)$ at θ_0. For definiteness we also assume that $u'(\theta_0) > 0$.

3.3 Linearised equation

The equations for the amendments u_n, with $n > 0$, are linear. Their solutions can be obtained by the variation of constants method, when one starts with two linearly independent solutions

$$\begin{aligned} v_1 &= \partial_S u_0, \\ v_2 &= \frac{\Omega'(E)}{\Omega(E)}\, S\partial_S u_0 + \partial_E u_0 \end{aligned}$$

of the corresponding homogeneous linear equation. To check that both v_1 and v_2 satisfy the homogeneous linear equation, one differentiates immediately the nonlinear equation (3.7) in S (for v_1) and E (for v_2).

We proceed by evaluating the Wronskian for v_1, v_2, namely

$$\begin{aligned} W &= v_1\partial_S v_2 - v_2\partial_S v_1 \\ &= \partial_S u_0\Big(\frac{\Omega'(E)}{\Omega(E)}(\partial_S u_0 + S\partial_S^2 u_0) + \partial_S\partial_E u_0\Big) - \partial_S^2 u_0\Big(\frac{\Omega'(E)}{\Omega(E)}\, S\partial_S u_0 + \partial_E u_0\Big) \\ &= \frac{\Omega'(E)}{\Omega(E)}\,(\partial_S u_0)^2 + \partial_S u_0\partial_S\partial_E u_0 - \partial_S^2 u_0\partial_E u_0. \end{aligned}$$

Using the equation for u_0 yields

$$W = \frac{\Omega'(E)}{\Omega(E)}\,(\partial_S u_0)^2 + \partial_E\Big(\frac{1}{2}(\partial_S u_0)^2\Big) + \frac{1}{(\Omega(E))^2}\,\partial_E G(u_0)$$

whence

$$(\Omega(E))^2 W = \partial_E\Big(\frac{1}{2}(\Omega(E))^2(\partial_S u_0)^2 + G(u_0)\Big).$$

In the final shape the Wronskian is

$$W = \frac{1}{(\Omega(E))^2}.$$

The general solution of the linearised equation

$$(\Omega(E))^2\,\partial_S^2 u_n + g'(u_0)u_n = F_n \tag{3.10}$$

for u_n can be written in the form

$$\begin{aligned} u_n(S,\theta) &= c_1(\theta)\, v_1(S,\theta) + c_2(\theta)\, v_2(S,\theta) \\ &+ v_1(S,\theta) \int_0^S F_n v_2(s,\theta)\, ds - v_2(S,\theta) \int_0^S F_n v_1(s,\theta)\, ds. \end{aligned} \tag{3.11}$$

Set

$$\partial_S J_n(S,\theta) = F_n\, \partial_S u_0(S,\theta). \tag{3.12}$$

Consider separately

$$\begin{aligned} &\int_0^S F_n(s,\theta) v_2(s,\theta) ds \\ &= \int_0^S F_n(s,\theta) \Big(\frac{\Omega'(E)}{\Omega(E)}\, s\partial_s u_0 + \partial_E u_0 \Big) ds \\ &= \frac{\Omega'(E)}{\Omega(E)} \int_0^S F_n(s,\theta) s\partial_s u_0\, ds + \int_0^S F_n(s,\theta)\partial_E u_0\, ds \\ &= \frac{\Omega'(E)}{\Omega(E)} S J_n(S,\theta) - \frac{\Omega'(E)}{\Omega(E)} \int_0^S J_n(s,\theta)\, ds + \int_0^S F_n(s,\theta)\partial_E u_0\, ds. \end{aligned}$$

Introduce

$$\partial_S a_n(S,\theta) = \frac{\Omega'(E)}{\Omega(E)} J_n(S,\theta) - F_n(S,\theta)\partial_E u_0. \tag{3.13}$$

Using formulas for solutions v_1 and v_2 we can now transform equality (3.11) to

$$\begin{aligned} u_n(S,\theta) &= c_1(\theta) v_1 + c_2(\theta) v_2 \\ &- \partial_S u_0 a_n(S,\theta) + \partial_S u_0 \frac{\Omega'(E)}{\Omega(E)} S J_n(S,\theta) - \Big(\frac{\Omega'(E)}{\Omega(E)} S\partial_S u_0 + \partial_E u_0 \Big) J_n(S,\theta). \end{aligned}$$

The terms containing the factor S cancel and we arrive at

$$u_n(S,\theta) = c_1(\theta) v_1 + c_2(\theta) v_2 - \partial_S u_0 a_n(S,\theta) - \partial_E u_0 J_n(S,\theta), \tag{3.14}$$

where a_n and J_n are solutions of equations (3.13) and (3.12). The coefficients $c_1(\theta)$ and $c_2(\theta)$ are so far not determined. We have thus proved the following assertion.

Lemma 3.2. *The solution of equation (3.10) can be represented in the form (3.14), where $a_n(S,\theta)$ and $J_n(S,\theta)$ are solutions of system (3.13), (3.12) and $c_1(\theta)$ and $c_2(\theta)$ arbitrary functions of θ.*

Our next goal is to derive a boundedness condition for the solution of equation (3.10) for the almost periodic in S right-hand side $F_n(S,\theta)$. If

$$\lim_{S\to\infty} \frac{1}{S} \int_0^S F_n(s,\theta)\partial_S u_0(s,\theta)\, ds = 0, \tag{3.15}$$

then $J_n(S,\theta)$ is almost periodic in S. Under condition (3.15) one can always choose $c_2(\theta)$ in such a way that u_n is bounded. Namely,

$$c_2(\theta) = \lim_{S\to\infty} \frac{1}{S} \int_0^S F_n(s,\theta)\partial_E u_0(s,\theta)\, ds. \tag{3.16}$$

As a result we deduce that u_n is an almost periodic function of S. We have thus proved

Theorem 3.3. *Let (3.15) be fulfilled. Then the solution of equation (3.10) is an almost periodic function, where $a_n(S,\theta)$ and $J_n(S,\theta)$ are solutions of triangular system (3.13), (3.12) and $c_2(\theta)$ is given by equality (3.16).*

Corollary 3.4. *Suppose*

$$F_n(S,\theta) = \sum_{k=-\infty}^{\infty} \mathcal{F}_{n,k} e^{\imath \nu_k S}$$

is an almost periodic function and there are integers m and k, such that $m\sigma/\theta \to \nu_k$ as $\theta \to \theta_{m,k}$. Then

$$u_k = O\Big(\frac{\mathcal{F}_{n,k}}{(m-\nu_k\theta/\sigma)^2}\Big)$$

as $\theta \to \theta_{m,k}$.

To prove this assertion it suffices to represent $\partial_S u_0$ and $\partial_E u_0$ as Fourier series and integrate explicitly equations (3.15) and (3.16). Then, the result should be substituted into (3.14).

3.4 Construction of the first amendment

In this section we compute the first amendment and derive an equation for the principal term of averaged action.

Write the solution of equation (3.8) in the form

$$u_1(S,E) = \mathcal{U}_{1,f}(S,E) + \mathcal{U}_{1,\Gamma}(S,E), \tag{3.17}$$

where $\mathcal{U}_{1,f}(S,E)$ and $\mathcal{U}_{1,\Gamma}(S,E)$ are solutions of the equations

$$\begin{aligned} (\sigma')^2\partial_S^2\mathcal{U}_{1,f} + g'(u_0)\mathcal{U}_{1,f} &= f_0(t), && (3.18)\\ (\sigma')^2\partial_S^2\mathcal{U}_{1,\Gamma} + g'(u_0)\mathcal{U}_{1,\Gamma} &= -D_1 u_0, && (3.19)\end{aligned}$$

respectively. Here, we study the non-resonant case of outside force $f_0(t)$. To pass to the fast variable S, set $t = (S-\alpha)\theta/\sigma$. The independent variable θ is considered in an interval of the real axis where $m \neq \omega_k\theta/\sigma$ for all $m \in \mathbf{Z}$ and $k \in \mathcal{K}_0$.

In this case the boundedness condition for the solution of equation (3.18) is fulfilled identically in θ. To construct an explicit formula for $\mathcal{U}_{1,f}$ one can exploit Corollary 3.4. Let there be integers m and k with the property that $m\sigma(\theta) \to \omega_k$ as $\theta \to \theta_{m,k}$. Then

$$\mathcal{U}_{1,f} = O\Big(\frac{1}{(m+\omega_k\theta/\sigma)^2}\Big). \tag{3.20}$$

The right-hand side of (3.19) is periodic in S, hence the averaging of it over an unbounded interval can be replaced by averaging over the period. In this way the equality

$$\lim_{S\to\infty}\frac{1}{S}\int_0^S \Big(2\sigma'\alpha'\partial_S^2 u_0 + \sigma''\partial_S u_0 + 2\sigma'\partial_\theta\partial_S u_0 - \Gamma\sigma'\partial_S u_0\Big)\partial_S u_0 ds = 0$$

can be rewritten as ordinary differential equation for the function

$$I_0 = \sigma' \int_0^1 \left(\partial_S u_0(S, E)\right)^2 dS$$

of θ. More precisely, we get

$$I_0' + \Gamma I_0 = 0.$$

Hence it follows that

$$I_0 = \mathcal{I} e^{-\Gamma\theta}, \tag{3.21}$$

where $\mathcal{I} = I_0(\theta_0)$ is an arbitrary constant. When assuming the function $E(I)$ to be invertible, we determine in this way the dependence of the parameter E on θ.

Let us sum up what we obtained in this section.

Lemma 3.5. *Equation (3.8) has a bounded solution if the parameter E evolves in accordance with equation (3.21) for $E \not\equiv E_{m,k}$.*

Obviously, the solution of the equation for the first amendment fails to work for the zeroes of the denominator in (3.20), which are the resonant values $E^0_{m,k}$ of the parameter E. For each ϵ, the set of values of k is finite. Hence, the resonant values are bounded away from each other. The distance between two neighbouring resonances depends on the behavior of the Fourier series of $f(t)$ and is determined by the asymptotic condition of non-overlap of resonances, see (2.4).

We now ascertain the applicability domain of asymptotic expansion (3.6). To this end we consider higher-order amendments.

3.5 Construction of the second amendment

In this section we derive an evolution equation for the phase shift and construct a two-parameter asymptotic solution which suits well outside of the resonant levels of the parameter E.

Let us discuss the influence of outside force on solutions of the equation for the second amendment. In general, the formula for $f_1(t)$ may contain resonant summands in the fast variable S at some point θ. Since the dependence $\sigma(\theta)$ has been determined at the previous step when constructing u_1, the resonances in the second amendment are local with respect to the slow variable. For passing through a local resonance it is necessary to change the averaging operator.

It is well understood that passing through local resonances leads in generic situation to a change in amendments as large as $\epsilon^{-1/2}$, see [2]. From the formal view point this means that in generic situation one can consider the averaging operator of the form

$$\begin{aligned}
&\lim_{S\to\infty} \frac{1}{S} \int^S f_1\Big(\frac{\theta}{\sigma}(s-\alpha)\Big) \partial_S u_0(s,\theta)\, ds \\
\sim\ & \frac{1}{\sigma+\epsilon\alpha} \int_{\sigma_0}^{\alpha+\sigma/\epsilon} f_1\Big(\frac{\theta}{\epsilon}\Big) u_0'(\alpha+\sigma/\epsilon)(\sigma'+\epsilon\alpha')d\theta \\
=\ & O(\sqrt{\epsilon}).
\end{aligned}$$

In order to establish this formula one substitutes Fourier series for the integrands which converge absolutely. Then, one passes on to termwise integration and applies the stationary phase method for evaluating rapidly oscillating integrals.

Let us now compute the action of the averaging operator on the remaining summands of the right-hand side of (3.9),

$$\lim_{S\to\infty}\frac{1}{S}\int^{S}\Big(-\frac{1}{2}g''(u_0)u_1^2 - D_1u_1 - D_2u_0\Big)\partial_S u_0(s,\theta)\,ds. \tag{3.22}$$

It is convenient to break down the computation for the averaging operator into single terms. The integral of the first summand is

$$\int^{S} -\frac{1}{2}g''(u_0)u_1^2\partial_S u_0(s,\theta)\,ds = -\frac{1}{2}g'(u_0)\frac{u_1^2}{2}\Big|^{S} + \int^{S} g'(u_0)u_1u_1'\,ds.$$

On evaluating $g'(u)u_1$ from the equation for the first amendment we get

$$\int^{S} g'(u_0)u_1u_1'\,ds = \int^{S}\Big(-(\sigma')^2\partial_S^2u_1 + f_0(t) - D_1u_0\Big)\partial_S u_1\,ds.$$

The first summand under the integral is explicitly integrable, its average just amounts to zero. As a result the average (3.22) reduces to

$$\lim_{S\to\infty}\frac{1}{S}\int^{S}\left(\left(f_0(t)-D_1u_0\right)\partial_S u_1 - D_2u_0\partial_S u_0(s,\theta)\right)ds.$$

The last summand is evaluated immediately, namely

$$\begin{aligned}\lim_{S\to\infty}\frac{1}{S}\int^{S} -D_2u_0\partial_S u_0(s,\theta)\,ds &= (\alpha''+\varGamma\alpha')\frac{I_0}{\sigma'}+\alpha'\Big(\frac{I_0}{\sigma'}\Big)' \\ &= \Big(\frac{\alpha'}{\sigma'}I_0\Big)' + \varGamma\frac{\alpha'}{\sigma'}I_0.\end{aligned}$$

For evaluating other summand we make use of representation (3.17) for the first summand, obtaining

$$\begin{aligned}&\lim_{S\to\infty}\frac{1}{S}\int^{S}\left(f_0(t)-D_1u_0\right)\partial_S u_1\,ds\\ &= \lim_{S\to\infty}\frac{1}{S}\int^{S}\left(f_0(t)-D_1u_0\right)\left(\partial_S\mathcal{U}_{1,f}+\partial_S\mathcal{U}_{1,\varGamma}\right)ds\\ &= \lim_{S\to\infty}\frac{1}{S}\int^{S}\left(f_0(t)-D_1u_0\right)\partial_S\mathcal{U}_{1,f}ds + \lim_{S\to\infty}\frac{1}{S}\int^{S}\left(f_0(t)-D_1u_0\right)\partial_S\mathcal{U}_{1,\varGamma}ds.\end{aligned}$$

We claim that

$$\begin{aligned}&\lim_{S\to\infty}\frac{1}{S}\int^{S}\left(f_0(t)-D_1u_0\right)\partial_S\mathcal{U}_{1,\varGamma}ds\\ &= \lim_{S\to\infty}\frac{1}{S}\int^{S} f_0(t)\,\partial_S\mathcal{U}_{1,\varGamma}ds - \lim_{S\to\infty}\frac{1}{S}\int^{S} D_1u_0\,\partial_S\mathcal{U}_{1,\varGamma}ds\\ &= 0.\end{aligned}$$

Indeed, the integrand in the first summand is a conditionally periodic function of zero mean value, hence, the average vanishes. The second summand coincides with the integral treated in [25]. This integral vanishes, for its integrand is the product of an even function and an odd function of S. Integrating it over the entire period yields zero.

It remains to consider the average of two summands which do not encounter in [25]. These are

$$\lim_{S\to\infty}\frac{1}{S}\int^{S} f_0(t)\partial_S \mathcal{U}_{1,f}\,ds - \lim_{S\to\infty}\frac{1}{S}\int^{S} D_1 u_0 \partial_S \mathcal{U}_{1,f}\,ds.$$

Since the function $D_1 u_0 \partial_S \mathcal{U}_{1,f}$ is conditionally periodic and of mean value zero, we deduce readily that the second limit vanishes.

To compute the remaining average we write

$$u_0(S,E) = \sum_{k=0}^{\infty} u_{0,k}(E)\cos(kS),$$

whence

$$\begin{aligned} -\partial_S u_0 &= \sum_{k=0}^{\infty} k u_{0,k}(E)\sin(kS), \\ \partial_E u_0 &= \sum_{k=0}^{\infty} v_{0,k}(E)\cos(kS). \end{aligned}$$

Besides, represent f_0 in the form

$$f_0(t) = \sum_{k\in\mathcal{K}_0} |\mathfrak{f}_k| \cos\Big(\omega_k \frac{\theta}{\sigma} S + \delta_k\Big),$$

where $\delta_k = \arg(\mathfrak{f}_k) - \dfrac{\theta}{\sigma}\alpha$.

Further computations may be done in explicit form, e.g., if one uses the programme of analytic computations [32]. One substitutes the series for u_0 and f_0 into (3.12) and (3.13), interchanges the integral and infinite sum and then integrates termwise, thus arriving at an explicit expression for $\mathcal{U}_{1,f}$. Then it is straightforward to compute the integral and the limes of the overaging operator. As a result we obtain

$$\lim_{S\to\infty}\frac{1}{S}\int^{S} f_0(t)\partial_S \mathcal{U}_{1,f}\,ds = 0.$$

Thus, the averaging procedure leads to the equation

$$\Big(\frac{\alpha'}{\sigma'} I_0\Big)' + \Gamma \frac{\alpha'}{\sigma'} I_0 = 0. \tag{3.23}$$

Write a particular solution of this equation in the form

$$\alpha_0(\theta) = \sigma(\theta) + \mathcal{A}, \tag{3.24}$$

where $\mathcal{A}$ is a parameter of the solution.

Theorem 3.6. *Suppose that I_0 and α_0 are determined in (3.21) and (3.24), respectively. Then $u \sim u_0(S,E)$ is a two-parameter asymptotic solution of the form (3.6) defined up to $o(1)$ for $\theta \not\equiv \theta_{m,k}$. The quantities $\mathcal{I}$ and $\mathcal{A}$ are parameters of the solution.*

Denote

$$\mathcal{F}_2(S,\theta) = \int_0^S F_2(s,\theta)\partial_S u_0\, ds.$$

By the properties of averaging and integration of summands with local resonance it follows that

$$\mathcal{F}_2(S,\theta) = o(t) + O(1/\sqrt{\epsilon}).$$

Letting $\theta \to \theta_{m,k}$ we deduce from Theorem 3.3 that

$$u_2 = O((m + \omega_k\theta/\sigma)^{-6}).$$

In much the same way we construct higher-order amendments. One can show that the singularity of u_n for $\theta \to \theta_{m,k}$ increases with n, namely

$$u_n = O\Big(\Big(\frac{1}{m + \omega_k\theta/\sigma}\Big)^{4(n-1)+2}\Big)$$

for $k \in \mathcal{K}_0(\epsilon)$. Then the appropriateness domain of asymptotic $u \sim u_0(S,E)$ is described by

$$\frac{\epsilon}{(m + \omega_k\theta/\sigma)^4} \ll 1.$$

This just amounts to saying that in generic situation the condition of applicability of asymptotic (3.6) is

$$|\theta - \theta_{m,k}| \gg \epsilon^{1/4}. \qquad (3.25)$$

3.6 Resonances in higher-order amendments

By the above, the constructed asymptotics no longer work close to the resonant values of θ. Such resonant values of θ are determined starting from the set $\mathcal{K}_0(\epsilon)$ and an integer k. For passing through the resonances related to the set $\mathcal{K}_0(\epsilon)$ one ought to change the structure of asymptotic at the order of smallness $\sqrt{\epsilon}$, see [2]. For higher-order amendments similar resonances appear also for the values of θ, such that $\Omega(E(\theta))/\omega_k \in \mathbb{Z}$ with some k belonging to the union of the sets $\mathcal{K}_0(\epsilon), \dots, \mathcal{K}_{n-1}(\epsilon)$, where n is the amendment number. The passage through a resonant value of θ in higher-order amendments leads to a structure change of asymptotics at the order of smallness $\epsilon^{n-1/2}$. Asymptotic expansions in neighbourhoods of resonant values of θ are studied in the next section.

4 Asymptotics in resonant regions

4.1 Formal derivation of nonlinear resonance equation

In this section we build an asymptotic solution in a neighbourhood of resonant value $E^0_{m,k}$. A particular attention is paid to the applicability intervals of the asymptotic. We compute also an intermediate asymptotic, when approaching the capture region, and derive a formula of connection between this intermediate asymptotic and parameters of captured solutions.

The equation of nonlinear resonance was derived by Chirikov in the dissipationless case for $f(t) \cong \cos(\omega t)$ in [1]. This is the equation of mathematical pendulum with outer momentum. Since then the derivation of this equation was reproduced by diverse methods in a great number of papers. Here the nonlinear resonance equation is obtained by the Whitham method for equation (1.1) with periodic function $f(t)$ in the presence of dissipative summand.

Let us look for an asymptotic (in ϵ) solution of equation (1.1) in a neighbourhood of resonant level $E = E^0_{m,k}$ by the Whitham method. For this purpose we substitute a function u of the form

$$u = u(t + \varphi(\tau))$$

with $\tau = \sqrt{\epsilon}t$ into (1.1). After differentiation we get

$$u'' + g(u) = -\sqrt{\epsilon}\varphi' u'' + \epsilon(f(t) - \varphi'' u' - \Gamma u' - (\varphi')^2 u'') - \epsilon\sqrt{\epsilon}\,\Gamma\varphi' u'. \quad (4.1)$$

We shall build a solution which is periodic in the fast variable with period $T = 2\pi/\Omega(E^0_{m,k})$. Let us multiply the equation by u' and average it in the fast variable S.

The necessary condition of boundedness of solution is

$$(-\varphi'' - \Gamma - \sqrt{\epsilon}\Gamma\varphi') \lim_{S\to\infty} \frac{1}{S}\int_0^S (u')^2 ds + \lim_{S\to\infty} \frac{1}{S}\int_0^S f(s)u'(s+\varphi)ds = 0. \quad (4.2)$$

Denote by

$$I = I(T) := \lim_{S\to\infty} \frac{1}{S}\int_0^S (u')^2 ds$$

the action for a given T to be treated as parameter independent of τ.

Represent $u = u(t + \varphi)$ and $f(t)$ by their Fourier series

$$\begin{aligned} u &= \sum_{j=0}^{\infty} \left(a_j \cos(j\Omega(t+\varphi)) + b_j \sin(j\Omega(t+\varphi))\right), \\ f(t) &= \sum_{l=0}^{\infty} \left(g_k \cos(\omega_k t) + h_k \sin(\omega_k t)\right). \end{aligned}$$

Differentiate the series for u termwise in t, multiply the series for u' by the series for $f(t)$ and transform the products of trigonometric functions into trigonometric functions of the sum and difference of arguments. Then apply the averaging operator to the series obtained in this way. As a result we conclude that the action of averaging operator is different from zero only for j and k, such that $j\Omega = \omega_k$. Hence

$$\lim_{S\to\infty} \frac{1}{S}\int_0^S f(s)u'(s+\varphi)\,ds = \sum_{k=0}^{\infty} \left(A_k \cos(k\Omega\varphi) + B_k \sin(k\Omega\varphi)\right).$$

Denoting the right-hand side of this equality by $\Sigma(\varphi)$, we rewrite (4.2) as equation for φ,

$$(-\varphi'' - \Gamma - \sqrt{\epsilon}\Gamma\varphi')I + \Sigma(\varphi) = 0.$$

The change of variables

$$\phi = \varphi/\Omega, \quad \gamma = \Gamma/\Omega, \quad \varepsilon = \sqrt{\epsilon}\Gamma, \quad q(\phi) = -\frac{1}{I\Omega}\Sigma(\Omega\phi)$$

reduces the above equation to

$$\phi'' + q(\phi) + \gamma = -\varepsilon\phi'. \tag{4.3}$$

Here, $q(\phi)$ is a periodic function of period $2\pi/\Omega$ and zero mean value. From the convergence of Fourier series for the functions $\partial_S u_0$ and $f(t)$ it follows that $q(\phi) \to 0$ as $m, k \to \infty$.

Consider for instance $g(u) := \sin u$ and $f(t) := A\cos(\omega t)$. Then, for $\omega < 1$, there may occur resonances on subharmonics, i.e. $m\Omega(E) = \omega$. As a result we get an equation of the form

$$\phi'' + h\sin(\phi + a) + \gamma = -\varepsilon\phi',$$

where

$$\gamma = \frac{\Gamma}{m\Omega}, \quad h = \frac{A}{m\Omega I}\sqrt{a_m^2 + b_m^2}, \quad a = \arctan\frac{a_m}{b_m},$$

a_m and b_m being the Fourier coefficients of the elliptic function $\partial_S u_0$. For $\varepsilon = 0$ this is the well-known equation of nonlinear resonance.

4.2 Inner asymptotic

Equations (4.1) and (4.3) constitute a system. The equations of this system separate asymptotically provided that $\sqrt{\epsilon}\varphi \ll 1$. We'll look for a solution of equation (4.1) in the form of power series in powers of $\sqrt{\epsilon}$, namely

$$u(t,\epsilon) = \sum_{k=0}^{\infty} \epsilon^{k/2} u_k(t + \varphi). \tag{4.4}$$

To obtain equations for u_k one substitutes (4.4) into equation (4.1) and equates the coefficients of the same powers of $\sqrt{\epsilon}$. The equation for the principal term u_0 looks like

$$u_0'' + g(u_0) = 0,$$

the equation for the first amendment is

$$u_1'' + g'(u_0)u_1 = -\varphi' u_0'',$$

the equation for the second amendment is

$$u_2'' + g'(u_0)u_2 = -\frac{1}{2}g_0'(u_0)u_1^2 - \varphi' u_1'' + f(t) - \varphi'' u_0' - \gamma u_0' - (\varphi')^2 u_0'',$$

and similarly for higher order amendments. Notice that the equation for the amendment u_n contains a term proportional to $(\phi')^n$. One can prove that, if secular terms cancel in all amendments, expansion (4.4) is asymptotic for $\sqrt{\epsilon}\varphi' \ll 1$. Equation (4.2) is a necessary condition for the expansion to be uniform.

4.3 Capture into resonance

This section is aimed at evaluating the measure of resonance solutions in the space of parameters of solutions. A solution of equation (4.3), for which there is a constant C with the property that $|\phi| < C$ as $\tau \to \infty$, is called captured into resonance.

The solutions of equation (4.3) can be parametrised with the help of three parameters. These are an initial value $\tau = \tau_0$ and "initial data" $\phi(\tau_0) = v_0$ and $\phi'(\tau_0) = v_1$, all the parameters are real numbers. In the phase plane the trajectories of solutions are characterised by two parameters ϕ' and ϕ.

The existence of equilibrium states of equation (4.3) is determined by the inequality $|\gamma| \le \max |q(\phi)|$. The equilibrium states are saddle and focal points. They are solutions of the transcendent equation

$$\gamma + q(\phi_j) = 0.$$

It is easily seen that the saddles and foci are situated on the axis $\phi' = 0$ and alternate with each other. We enumerate these solutions with index j. Then, the saddles are at the points $(0, \phi_{2j+1})$, such that $q'(\phi_{2j+1}) < 0$, and the foci are at the points $(0, \phi_{2j})$, such that $q'(\phi_{2j}) > 0$.

The capture into resonance is possible not at all resonant levels $E^0_{m,k}$. From the asymptotic property $q(\phi) \to 0$, as $m, k \to \infty$, it follows that for large m and k equation (4.3) has no stationary solutions, and so there might be no capture into resonance. Although one ought to take proper account of equation (4.3) for passing through such resonant levels, no capture into resonance occurs nevertheless. When passing through a resonant level for which $|\gamma| > \max |q(\phi)|$, the solution changes by an amount $O(\sqrt{\epsilon})$.

Write $p = \phi'$, then the equation for phase trajectories is

$$p \, \frac{dp}{d\phi} = -q(\phi) - \gamma - \varepsilon p.$$

For $\varepsilon = 0$ this equation is integrated explicitly, giving

$$\mathcal{E} = (\phi')^2 + Q(\phi) + \gamma \phi, \tag{4.5}$$

where $Q(\phi)$ is a primitive of $q(\phi)$. The parameter $\mathcal{E}$ varies slowly on solutions of equation (4.3), for

$$\frac{d\mathcal{E}}{dt} = \varepsilon \, (\phi')^2.$$

Hence, it is convenient to use $\mathcal{E}$ for parametrising the trajectories. At the phase curve we have

$$\frac{d\mathcal{E}}{d\phi} = \varepsilon \phi'.$$

An equivalent formulation of this equation is

$$\frac{d\mathcal{E}}{d\phi} = \operatorname{sgn}(\phi') \, \varepsilon \sqrt{\mathcal{E} - Q(\phi) - \gamma \phi}. \tag{4.6}$$

The trajectories captured into resonance are located between separatrices which tend to a saddle as $\tau \to \infty$. At the saddle point

$$\mathcal{E}(\phi_{2j+1}) = \mathcal{E}^-_{2j+1} := Q(\phi_{2j+1}) + \gamma \phi_{2j+1}.$$

It should be noted that the separatrix which loops round the focus ϕ_{2j} has actually two different values at $\phi = \phi_{2j+1}$ as $\tau \to \infty$. The corresponding values $\mathcal{E}_{2j+1}^{\pm}$ of $\mathcal{E}$ differ by the value of Mel'nikov's integral over the loop of the separatrix, i.e.

$$\mathcal{E}_{2j+1}^{+} = \mathcal{E}_{2j+1}^{-} + \varepsilon \int_{\mathcal{L}} \phi' d\phi.$$

The trajectories of resonance solutions lie between the separatrices with values $\mathcal{E}(\phi_{2j+1}) = \mathcal{E}_{2j+1}^{+}$ and $\mathcal{E}(\phi_{2j+1}) = \mathcal{E}_{2j+1}^{-}$. One uses these values to approximate the bounds of the region of capture into resonance in the phase plane.

Our next objective is to construct an asymptotic solution of the Cauchy problem for the parameter $\mathcal{E}$ with data $\mathcal{E}(\phi_{2j+1}) = \mathcal{E}_0$. Let

$$\mathcal{E} = \mathcal{E}_0 + \sum_{k=1}^{\infty} \mathcal{E}_k(\varphi)\, \varepsilon^k. \tag{4.7}$$

Substitute this expression into differential equation (4.3) and equate the coefficients of the same powers of ε. As a result we get a series of Cauchy problems

$$\begin{array}{lllrcl}
\dfrac{d\mathcal{E}_1}{d\phi} & = & \sqrt{\mathcal{E}_0 - Q(\phi) - \gamma\phi}, & \mathcal{E}_1(\phi_{2j+1}) & = & 0; \\
\dfrac{d\mathcal{E}_2}{d\phi} & = & \dfrac{\mathcal{E}_1(\phi)}{2\sqrt{\mathcal{E}_0 - Q(\phi) - \gamma\phi}}, & \mathcal{E}_2(\phi_{2j+1}) & = & 0; \\
\dfrac{d\mathcal{E}_3}{d\phi} & = & \dfrac{\mathcal{E}_2(\phi)}{2\sqrt{\mathcal{E}_0 - Q(\phi) - \gamma\phi}} - \dfrac{\mathcal{E}_1(\phi)^2}{8(\sqrt{\mathcal{E}_0 - Q(\phi) - \gamma\phi})^3}, & \mathcal{E}_3(\phi_{2j+1}) & = & 0,
\end{array}$$

etc., provided that $\phi' > 0$.

The measure of trajectories in the phase plane, which are captured into resonance, amounts to the sum of the area of separatrix loop

$$S_j = \int_{\mathcal{E}_{2j+1}^{-}}^{\mathcal{E}_{2j+1}^{+}} \phi' d\phi$$

and the area of the region between the separatrices towards the saddle point ϕ_{2j+1}

$$\varDelta_j(\phi) = \int_{\phi_{2j+1}}^{\phi} \big(\phi'(\mathcal{E}_{2j+1}^{+}) - \phi'(\mathcal{E}_{2j+1}^{-})\big)\, d\phi.$$

It follows that the measure of trajectories captured into resonance of level $E_{m,k}^0$ near the focus ϕ_j, for $|\phi| \ll \varepsilon^{-1}$, is

$$\mu_{m,k} \sim S_j + \varDelta_j = O(1).$$

In this way we arrive at

Theorem 4.1. *The measure of trajectories captured into resonance in a neighbourhood of the $2j+1$ th focus has smallness order $\mu_{m,k}$.*

If $\phi \to -\infty$ then $\mathcal{E}_1 = O(\phi^{3/2})$, $\mathcal{E}_2 = O(\phi^2)$ and $\mathcal{E}_3 = O(\phi^{5/2})$. Moreover, one can show that $\mathcal{E}_k = O(\phi^{k/2+1})$ for all integers $k \geq 4$. This estimate allows one to directly evaluate the region of applicability of asymptotic (4.7). More precisely,

$$\frac{\varepsilon^{k+1}\mathcal{E}_{k+1}}{\varepsilon^k \mathcal{E}_k} \ll -1, \tag{4.8}$$

and so $\varepsilon(-\phi)^{1/2} \ll 1$ whence $-\phi \ll \varepsilon^{-2}$.

4.4 Asymptotic solutions of the equation of nonlinear resonance

In this section we build an unbounded asymptotic solution of equation (4.3) for $\tau \to \infty$. Besides, we derive a connection formula for the parameters of the asymptotic and parameters of solution in a small neighbourhood of capture into resonance.

The solution is searched for in the form

$$\phi = s(\tau, \varepsilon) + \psi(s, \varepsilon), \tag{4.9}$$

where

$$s'' + \lambda(s, \varepsilon) = -\varepsilon s'.$$

The summand $\lambda(s, \varepsilon)$ will be determined below under constructing an asymptotic for the function ψ.

Substitute expression (4.9) into equation (4.3), obtaining

$$(-\lambda - \varepsilon s') + (s')^2 \psi'' + (-\lambda - \varepsilon s')\psi' + q(s + \phi) + \gamma = -\varepsilon s' - \varepsilon s' \psi'.$$

Elementary transformations yield an equation for ψ, namely

$$(s')^2 \psi'' - \lambda \psi' + q(s + \psi) + (\gamma - \lambda) = 0.$$

Our concern is the behaviour of ψ as $s' \to \infty$. To this end it is convenient to rewrite the equation in the form

$$\psi'' = \frac{1}{(s')^2} \left(\lambda \psi' - q(s + \psi) - (\gamma - \lambda) \right).$$

Set $z(s) = 1/s'$. We look for a solution of the form $\psi(s, \varepsilon) = \psi(s, z)$ by the method of two scales. Here, s is thought of as fast variable and z as slow variable. The total derivative in s is

$$\frac{d}{ds} = \partial_s + \frac{dz}{ds} \partial_z.$$

To evaluate z', we make use of the equation for s in the form

$$s' \frac{ds'}{ds} = -\lambda - \varepsilon s',$$

implying

$$\frac{dz}{ds} = \frac{-1}{(s')^2} \frac{ds'}{ds} = \frac{-1}{(s')^2} \Big(-\frac{\lambda}{s'} - \varepsilon \Big) = \lambda z^3 + \varepsilon z^2.$$

Hence, the total derivative in s just amounts to

$$\frac{d\psi}{ds} = \partial_s \psi + (\lambda z^3 + \varepsilon z^2)\, \partial_z \psi$$

and the second derivative is

$$\begin{aligned} \frac{d^2\psi}{ds^2} &= \partial_s^2 \psi \\ &+ \quad \left(\lambda z^3 + \varepsilon z^2\right) \left(2\partial_s \partial_z \psi + \left(3\lambda z^2 + z^3 \partial_z \lambda + 2\varepsilon z\right) \partial_z \psi + \left(\lambda z^3 + \varepsilon z^2\right) \partial_z^2 \psi\right). \end{aligned}$$

As a result the equation of the method of two scales for ψ takes the form

$$\begin{aligned}\partial_s^2\psi &= (\lambda z^3+\varepsilon z^2)\left(-2\partial_s\partial_z\psi-(3\lambda z^2+z^3\partial_z\lambda+2\varepsilon z)\partial_z\psi-(\lambda z^3+\varepsilon z^2)\partial_z^2\psi\right)\\ &+ \lambda z^2\partial_s\psi+\lambda(\lambda z^3+\varepsilon z^2)\partial_z\psi-q(s+\psi)-(\gamma-\lambda).\end{aligned} \tag{4.10}$$

We look for a solution of this equation of the form

$$\psi(s,z)=\sum_{k=2}^{\infty}\psi_k(s)\,z^k \tag{4.11}$$

for small values of z. Assume that the parameter λ is represented by a similar series in powers of z, i.e.

$$\lambda(s,\epsilon)=\sum_{k=0}^{\infty}\lambda_k\,z^k, \tag{4.12}$$

λ_k being undetermined constant coefficients. As usual, we substitute expressions (4.10) and (4.12) into equation (4.10), equate the coefficients of the same powers of z and arrive at a recurrent system of second order ordinary differential equations for the unknown functions $\psi_k(s)$. In particular,

$$\psi_2''=-q(s)-(\gamma-\lambda_0).$$

This equation possesses a (bounded) periodic solution if $\lambda_0=\gamma$. We take such a periodic solution as $\psi_2(s)$. The equation for $\psi_3(s)$ is

$$\psi_3''=-4\varepsilon\psi_2'+\lambda_1.$$

This equation has a periodic solution if $\lambda_1=0$. The equation for ψ_4 is in turn

$$\psi_4''=-6\varepsilon^2\psi_2-5\gamma\psi_2'-6\varepsilon\psi_3'-3\varepsilon^2\psi_2-q'(s)\psi_2+\lambda_2.$$

A periodic solution of this equation exists if

$$\frac{1}{T}\int_0^T q'(s)\psi_2(s)\,ds+\lambda_2=0,$$

which is the case for $\lambda_2=0$, as is easy to check.

In this way we determine the periodic coefficients $\psi_k(s)$ and λ_{k-2} one after another. The coefficient $\psi_k(s)$ is evaluated from an equation

$$\psi_k''=F_k(s,\psi_2,\dots,\psi_{k-2},\psi_2',\dots,\psi_{k-1}',\lambda_0,\dots,\lambda_{k-4})+\lambda_{k-2},$$

where $k>2$. The parameter λ_{k-2} is determined from the condition that the mean value over the period of the right-hand side of equation for ψ_k vanishes. To wit,

$$\lambda_{k-2}=-\frac{1}{T}\int_0^T F_k(s,\psi_2,\dots,\psi_{k-2},\psi_2',\dots,\psi_{k-1}',\lambda_0,\dots,\lambda_{k-4})\,ds.$$

On having constructed $\lambda(z,\varepsilon)$ we pass to the study of equation for s. Up to terms of smallness order $o(z^2)$ the equation has the form

$$s''+\gamma\sim-\varepsilon s'.$$

We are now in a position to formulate the result of this section.

Theorem 4.2. *The solution of equation (4.3) behaves like*

$$\begin{aligned} s &\sim \Big(\frac{\gamma}{\varepsilon^2} - \frac{\gamma}{\varepsilon^2} e^{-\varepsilon(\tau-\tau_0)} - \frac{\gamma(\tau-\tau_0)}{\varepsilon}\Big) + s_0 \\ &\sim s_0 + \frac{\gamma(\tau-\tau_0)^2}{2}\Big(1 + \varepsilon\frac{\tau-\tau_0}{3} + O(\varepsilon^2(\tau-\tau_0)^2)\Big) \end{aligned}$$

as $s' \to \infty$. Here, s_0 and τ_0 are constants of integration.

The theorem implies that unbounded solutions of equation (4.3) behave like

$$\phi \sim \frac{1}{2}\gamma_{m,k}(\tau-\tau_0)^2$$

for $1 \ll (\tau-\tau_0) \ll \varepsilon^{-1}$. Substituting this asymptotic in the estimate (4.8) for the appropriateness region of expansion (4.7) yields an estimate for the region of appropriateness of (4.7). Namely,

$$\varepsilon(\tau-\tau_0) \ll 1, \tag{4.13}$$

i.e. $(\tau-\tau_0) \ll \varepsilon^{-1}$.

The connection between the parameters of the asymptotic of ϕ in (4.9), (4.11) and the parameter $\mathcal{E}$ is obtained by substitution of asymptotic (4.9), (4.11) into formula (4.5). We get

$$s_0 \sim \mathcal{E}$$

as $(\tau-\tau_0) \to \infty$ and $\varepsilon \to 0$.

Asymptotic solution (4.9), (4.11) will be captured in the resonance in a neighbourhood of the $2j$ th equilibrium state, if

$$\mathcal{E}^-_{2j+1} < s_0 < \mathcal{E}^+_{2j+1}.$$

for $\tau \to \infty$.

5 Matching of asymptotic expansions

In this section we match the parameters of asymptotic solutions outside of non-linear resonances and in neighbourhoods of them.

We first rewrite the applicability region of the inner asymptotic expansion in terms of t and ϵ. Namely, since $\tau = \sqrt{\varepsilon}t$, from $(\tau-\tau_0) \ll \varepsilon^{-1}$ it follows that $\sqrt{\epsilon}(t-\tau_0/\sqrt{\epsilon}) \ll \epsilon^{-1}$ which is equivalent to $(t-\tau_0/\sqrt{\epsilon}) \ll \epsilon^{-3/2}$.

The outer asymptotic is valid in the region $(\theta-\theta_{m,k}) \gg \epsilon^{1/4}$ which is equivalent to $(t-\theta_{m,k}/\epsilon) \gg \epsilon^{-3/4}$. Hence, if we choose $\tau_0 = \theta_{m,k}/\sqrt{\epsilon}$, the appropriateness regions for the constructed asymptotics meet each other.

Our next goal is to match the formulas for the asymptotic in the non-resonant region (3.6) and asymptotic (4.4) in a neighbourhood of each resonant level $E = E^0_{m,k}$.

Suppose the parameter $E(\theta)$ evaluated from formula (3.21) takes on the value $E^0_{m,k}$ at some point $\theta = \theta_{m,k}$. We determine the function $\sigma(\theta)$ starting from equation (3.3) and condition $\sigma(\theta_{m,k}) = 0$. Let moreover $\alpha(\theta_{m,k}) = \mathcal{A}$. Then asymptotic solution (3.6) has three independent parameters θ_0, $\mathcal{I}_0$, and $\mathcal{A}$.

The asymptotic solution in a neighborhood of resonance level $E = E^0_{m,k}$ has parameters s_0 and τ_0.

From formulas for the applicability intervals of asymptotics (3.25) and (4.13) one sees readily that the regions of appropriateness of these asymptotics overlap. Set $\tau_0 = \theta_{m,k}/\sqrt{\epsilon}$. Since $E(\theta) \to E^0_{m,k}$ as $\theta \to \theta_{m,k}$, we conclude that the principal terms of the asymptotics, which are the function $u_0(S/\sigma', E)$ of (3.6) and $u_0(t+\varphi)$ of (4.4), respectively, coincide. We now match the arguments of these asymptotics which contain fast variables. In these arguments one ought to single out and equate the parameters which are independent of the variables t, θ, τ, for these parameters are precisely the parameters of the asymptotic solution. Let us compute the asymptotic of the fast argument of S/σ' for $\theta \to \theta_{m,k}$. We have

$$
\begin{aligned}
S &\sim \sigma'(\theta_{m,k})\Big(t - \frac{\theta_{m,k}}{\epsilon}\Big) + \epsilon\frac{\sigma''(\theta_{m,k})}{2}\Big(t - \frac{\theta_{m,k}}{\epsilon}\Big)^2 + \sigma'(\theta_{m,k})\mathcal{A} \\
&+ \epsilon\alpha'(\theta_{m,k})\Big(t - \frac{\theta_{m,k}}{\epsilon}\Big), \\
\frac{1}{\sigma'} &\sim \frac{1}{\sigma'(\theta_{m,k})}\Big(1 - \epsilon\frac{\sigma''(\theta_{m,k})}{\sigma'(\theta_{m,k})}\Big(t - \frac{\theta_{m,k}}{\epsilon}\Big)\Big)
\end{aligned}
$$

whence

$$
\begin{aligned}
\frac{S}{\sigma'} &\sim \mathcal{A} + \Big(t - \frac{\theta_{m,k}}{\epsilon}\Big), \\
t + \varphi(\tau) &\sim t + \frac{s_0}{\Omega(E^0_{m,k})} = \frac{s_0}{\Omega(E^0_{m,k})} + \frac{\theta_{m,k}}{\epsilon} + \Big(t - \frac{\theta_{m,k}}{\epsilon}\Big).
\end{aligned}
$$

The matching condition $\dfrac{S}{\sigma'} \sim t + \varphi$ leads to the following assertion.

Theorem 5.1. *The asymptotic solution, which applies uniformly in the parameter θ in each interval including a neighbourhood of nonlinear resonance point $\theta_{m,k}$, has parameters*

$$
\mathcal{A} = \frac{\theta_{m,k}}{\epsilon} + \frac{s_0}{\Omega(E^0_{m,k})}, \qquad \tau_0 = \theta_{m,k}. \tag{5.1}
$$

The principal significance of formula (5.1) is that it provides the matching of phase shifts of fast variables.

Corollary 5.2. *Assume that in a resonant layer at $E^0_{m,j}$ the capture into resonance is possible, i.e. the condition $|\gamma| \le \max|q(\phi)|$ is fulfilled. Then, a trajectory with parameter $\mathcal{A}$ will be captured into resonance if this parameter satisfies*

$$
\frac{\mathcal{E}^-_{2j+1}}{\Omega(E^0_{m,k})} + \frac{\theta_{m,k}}{\epsilon} < \mathcal{A} < \frac{\mathcal{E}^+_{2j+1}}{\Omega(E^0_{m,k})} + \frac{\theta_{m,k}}{\epsilon} \tag{5.2}
$$

for all integers j.

6 Conclusion

In the paper we establish the values of parameters of asymptotic solutions which are captured into nonlinear resonance. We also evaluate the measure of such solutions amongst those which oscillate at a potential hole. The results of the paper, in particular, formula (5.2) show rather strikingly that the parameter set of captured solutions depends on the perturbation parameter ϵ in a singular way.

Acknowledgements. This research was supported by the Russian Fund of Fundamental Investigations, grant 11-01-91330, and German Research Society, grant TA 289/4-2. The first author wishes to express his gratitude to Prof. L.A. Kalyakin for helpful comments and the University of Potsdam for invitation and hospitality. The authors gratefully acknowledge the use of free computer algebra system Maxima.

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