The Evolution Equations for Dirac-harmonic Maps

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1. Introduction and Outline

Before introducing the notion of *Dirac-harmonic maps*, we want to give a short overview of both *harmonic maps* and *harmonic spinors*.

Harmonic maps

Among variational problems in Riemannian geometry, *harmonic maps* are probably the richest ones. Harmonic maps are critical points of the energy functional

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dM, \tag{1.1}$$

where $\phi\colon M \to N$ is a smooth map from a closed Riemannian manifold M to another Riemannian manifold N. Critical points of $E(\phi)$ satisfy

$$\tau(\phi) = \nabla_{e_\alpha} d\phi(e_\alpha) = 0, \qquad \tau(\phi) \in \Gamma(\phi^{-1}TN). \tag{1.2}$$

The operator $\tau(\phi)$ is called the *tension field* of the map ϕ. Often, one likes to apply the Nash embedding theorem to isometrically embed the manifold N into some $\mathbb{R}^q$ of sufficiently large dimension q. Then, the Euler-Lagrange equation for ϕ acquires the form

$$\Delta\phi = \mathrm{I\!I}(\phi)(d\phi, d\phi), \tag{1.3}$$

where $\mathrm{I\!I}$ denotes the second fundamental form of the embedding. Harmonic maps play an important role in

1. *Differential geometry*
   Minimal immersions from surfaces are harmonic maps which are also conformal. In this context, we have to mention the work of Sacks and Uhlenbeck [SU81], who used harmonic maps to establish the existence of a minimal immersion from S^2 to another Riemannian manifold. However, also such ordinary objects like geodesics are captured by the theory of harmonic maps.

2. *Partial differential equations*
   The Euler-Lagrange equation for harmonic maps (1.3) is of the form Laplacian of ϕ is equal to the square of the gradient of ϕ. Although this equation is non-linear and hence standard methods for linear partial differential equations cannot be applied, many results about the existence of solutions of the harmonic map equation and their properties have been obtained.

3. *Theoretical physics*
   In theoretical physics harmonic maps appear on several occasions, for example in the context of σ-*models* in particle physics. In the context of string theory, harmonic maps arise as the Polyakov action.

In the case that M is two-dimensional, the functional $E(\phi)$ is conformally invariant, such that harmonic maps from Riemann surfaces share special properties. For more details about harmonic maps, see the classical survey papers [EL95a] and [EL95b].

A natural question arising is the question concerning the existence of harmonic maps. We will briefly present two different approaches. The first one is due to Sacks-Uhlenbeck and uses the following regularization of the energy functional

$$E_\alpha(\phi) = \frac{1}{2}\int_M (1 + |d\phi|^2)^\alpha dM.$$

For $\alpha = 1$, the regularized functional has the same critical points as $E(\phi)$. For $\alpha > 1$, the functional E_α satisfies the Palais-Smale condition, which guarantees the existence of critical points. Consequently, one has to carefully study the limit $\alpha \to 1$, which was done in

Theorem 1.1 (Sacks-Uhlenbeck(1981)). *Suppose that M is a compact Riemann surface without boundary and N a closed Riemannian manifold. Let $\phi_\alpha \in C^\infty(M, N)$ be critical points of E_α with $E_\alpha(\phi_\alpha) < K$ for $\alpha > 1$ and $\phi_\alpha \to \phi$ weakly in $H^{1,2}(M, N)$ as $\alpha \to 1$. Then there exists a subsequence $\{\beta\} \subset \{\alpha\}$ and a finite number of points $\{x_1, \dots, x_k\} \subset M$ such that $\phi_\beta \to \phi$ in $C^2_{loc}(M \setminus \{x_1, \dots, x_k\}, N)$. Moreover, $\phi \in C^\infty(M, N)$ is a smooth harmonic map.*

Closely related to the work of Sacks-Uhlenbeck is the following

Theorem 1.2 (Lemaire(1978), Schoen-Yau(1979)). *Assume that M is a closed Riemann surface and $\pi_2(N) = 0$. Then any map $\phi_0 \in C^\infty(M, N)$ is homotopic to a smooth harmonic map.*

This result was proven independently by Lemaire [Lem78] and Schoen, Yau [SY79].

The second approach due to Eells and Sampson [ES64] uses the L^2 gradient flow of the energy $E(\phi)$. More precisely, the gradient flow is given by the following parabolic partial differential equation

$$\frac{\partial\phi}{\partial t} = \tau(\phi), \qquad \phi(\cdot, 0) = \phi_0. \tag{1.4}$$

In the case that the target manifold N has non-positive curvature, Eells and Sampson proved

Theorem 1.3 (Eells-Sampson(1964)). *Suppose M and N are compact Riemannian manifolds without boundary and the sectional curvature K^N of N is non-positive. Then for any $\phi_0 \in C^\infty(M, N)$ the evolution problem (1.4) admits a unique smooth solution $\phi \in C^\infty(M \times [0, \infty), N)$, which, as $t \to \infty$ suitably, converges to a harmonic map $\phi \in C^\infty(M, N)$ in $C^2(M, N)$.*

Of course one may be tempted to try to relax the curvature condition on N. But on the other hand, Eells and Wood proved that there does not exist a harmonic map $\phi : T^2 \to S^2$ with $\deg \phi = \pm 1$ [EW76]. Thus, the question comes up what happens if one tries to deform a given map ϕ_0 by the gradient flow in this case. The first answer to this question was given by Struwe [Str85], he showed that the flow has to become singular in this case.

Theorem 1.4 (Struwe(1985))**.** *Suppose M is a compact Riemannian surface without boundary and N is a compact Riemannian manifold without boundary. Then for any smooth map $\phi_0 : M \to N$ there exists a global distribution solution $\phi : M \times [0, \infty) \to N$ with finite energy $E(\phi_t) \leq E(\phi_0)$, which is regular on $M \times [0, \infty)$ with exception of at most finitely many singular points $(x_k, t_k), 1 \leq k \leq K$. The solution is unique in this class.*
At each singular point (x_k, t_k) a non-constant, smooth harmonic map $\bar{\phi} : S^2 \to N$ separates in the sense that for sequences $R_m \to 0,\ t_m \to t,\ x_m \to x$ as $m \to \infty$,

$$\phi_m(x) = \phi(exp_{x_m}(R_m x), t_m) \to \bar{\phi} \qquad in\ \ H^{2,2}_{loc}(\mathbb{R}^2, N).$$

Finally, $\phi(\cdot, t)$ converges weakly in $H^{1,2}(M, N)$ to a smooth harmonic map $\phi_\infty : M \to N$ as $t \to \infty$ suitably.

For the current known results about harmonic maps and their heat flows see the book [LW08].

Harmonic spinors

Spinors and especially *harmonic spinors* are rather different objects than harmonic maps. Spinors are sections in the spinor bundle ΣM, which is a vector bundle over a Riemannian spin manifold M. In contrast to harmonic maps, we thus require more structure on the manifold M, but we do not need a target manifold N. The definition of spinors involves the Riemannian metric of the manifold M and the choice of a spin structure. The natural operator acting on spinors is the Dirac operator $\not{\partial}: \Gamma(\Sigma M) \to \Gamma(\Sigma M)$. The Dirac operator is of first order, weakly elliptic, and is self-adjoint with respect to the L^2 norm. One calls ψ a *harmonic spinor* if it satisfies

$$\not{\partial}\psi = 0. \tag{1.5}$$

Spinors and especially harmonic spinors appear in

1. *Differential geometry*
   The Atiyah-Singer index theorem links topological data of the manifold M with the index of elliptic differential operators, like for example the Dirac operator. In addition, the Dirac operator can be used to study the existence of metrics with positive scalar curvature.

2. *Theoretical physics*
   In particle physics fermions, for example electrons, are described by spinors. The mass of these fermions is given by the eigenvalues of the Dirac operator $\not{\partial}$. Consequently, massless fermions are characterized by harmonic spinors.

The methods to study the existence of harmonic spinors are quite different from the ones used to study the existence of harmonic maps. The results about the existence of harmonic spinors differ depending on the dimension of the manifold M. We want to give a brief summary of the known results, for more details see [Bär98].

Starting in dimension one, we note that the only compact manifold is the unit circle S^1. There exist two spin structures on S^1, but only one of them admits harmonic spinors.

In the case that M is a closed Riemannian surface, the existence of harmonic spinors depends on the genus of the surface in the following way

- $g_M = 0$: No harmonic spinors exist [Bär92].
- $g_M = 1, 2$: The existence of harmonic spinors is independent of the metric, but depends on the spin structure.
- $g_M = 3, 4$: In general, the existence of harmonic spinors depends on both spin structure and metric.
- $g_M \geq 5$: The existence of harmonic spinors varies with the choice of metric.

In addition, we would like to point out that on any closed surface of genus $g_M \geq 1$, one can always choose the metric and the spin structure such that there exist harmonic spinors. For a detailed discussion see [Hit74] and [BS92]. For $\dim M \geq 3$, we have the following

Theorem 1.5 (Hitchin(1974), Bär(1996))**.** *Let (M, h) be a closed Riemannian spin manifold of dimension $m = 0, 1, -1 \mod 8$ or $m = 3 \mod 4$ with fixed spin structure. Then there exists a Riemannian metric h on M such that the Dirac operator $\not{\partial}$ has a non-trivial kernel, i.e. there exist non-trivial harmonic spinors.*

This theorem was proven by Hitchin for $m = 0, 1, -1 \mod 8$ using the Atiyah-Singer index theorem [Hit74]. For $m = 3 \mod 4$, the theorem was shown by Bär [Bär96] by considering the Dirac spectrum of the connected sum of two manifolds.
For the current known results about the existence of harmonic spinors, the reader may take a look at [Gin09], p. 94, section 6.2. The latest developments regarding the existence of harmonic spinors are covered in [ADH11].

Dirac-harmonic maps

Dirac-harmonic maps are a combination of harmonic maps and harmonic spinors. They are critical points of the functional

$$E(\phi,\psi)=\frac{1}{2}\int_M(|d\phi|^2+\langle\psi,\not{D}\psi\rangle)dM$$

and satisfy the Euler-Lagrange equations

$$\begin{aligned}\tau(\phi) &= \mathcal{R}(\phi,\psi), \\ \not{D}\psi &= 0.\end{aligned}\tag{1.6}$$

Here, ψ is a spinor along the map ϕ (we will later give a precise definition of the functional) and the curvature term $\mathcal{R}(\phi,\psi)$ depends on $\psi, d\phi$, and the curvature of the manifold N. In addition, we assume that M is a compact Riemannian spin manifold without boundary. The study of this functional is motivated by what physicists call *non-linear supersymmetric σ-models*, see, e.g. [Del99], chapter 3. For the sake of completeness, we want to mention that when talking about *non-linear supersymmetric σ-models*, physicists usually refer to the functional

$$E_c(\phi,\psi)=\frac{1}{2}\int_M|d\phi|^2+\langle\psi,\not{D}\psi\rangle\ -\frac{1}{6}R_{ikjl}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle dM,$$

involving an additional curvature term. This functional has also been investigated from the perspective of geometric analysis, see [CJW07]. Both functionals $E(\phi,\psi)$ and $E_c(\phi,\psi)$ are especially interesting if M is a compact Riemann surface, since they are conformally invariant in that case.
We would also like to point out that harmonic maps and harmonic spinors can be thought of as limiting cases of Dirac-harmonic maps.

Dirac-harmonic maps were introduced in [CJLW06] together with a first analysis including a removable singularity theorem. Most of the subsequent publications analyzed the regularity of Dirac-harmonic maps. First of all, it was shown that Dirac-harmonic maps from $S^2\to S^n$ are smooth [CJLW05], which was later extended to Dirac-harmonic maps from Riemann surfaces to hypersurfaces [Zhu09b]. The regularity of Dirac-harmonic maps in general was studied in [WX09]. All of these results made use of the fact that the inhomogeneity $\mathcal{R}(\phi,\psi)$ has a Jacobian determinant structure, such that the powerful tools from Helein [Hél02] and Riviere [Riv07] are applicable.

An important tool in the analysis of harmonic maps is the so called energy identity. The energy identity for Dirac-harmonic maps from $S^2\to S^n$ was proven in [CJLW05] and generalized to Dirac-harmonic maps from arbitrary Riemannian spin surfaces in [Zha07b]. In [Zhu09a], the energy identity was established for a sequence of Dirac-harmonic maps from Riemann surfaces, where the surface M is allowed to vary as well.

Concerning the existence of Dirac-harmonic maps much less is known. Some explicit solutions of the Euler-Lagrange equations for Dirac-harmonic maps have been constructed in [JMZ09]. General existence results could be obtained in the case that M is two-dimensional Minkowski space, see [Han05] and [Zha07a].

Recently, several publications regarding the existence of Dirac-harmonic maps appeared. In [Iso12], the author studies the case that $M = S^1$ and adds a non-linear interaction term $F(\phi, \psi)$ to the functional originally considered, namely

$$E_F(\phi, \psi) = \frac{1}{2} \int_{S^1} (|d\phi|^2 + \langle \psi, \not{D}\psi \rangle - F(\phi, \psi)) dM. \tag{1.7}$$

Using Hilbert space methods, the author could establish existence results for the functional $E_F(\phi, \psi)$ under assumptions on the interaction term $F(\phi, \psi)$.
A different approach was pursued by Ammann et.al. Using index-theoretical methods it was shown that for a given harmonic map ϕ one can construct a spinor ψ such that one gets a Dirac-harmonic map [AG11], more precisely:

Theorem 1.6. *Assume that M is a closed Riemannian spin manifold and N a closed Riemannian manifold. Consider the homotopy class $[\phi]$ of maps $\phi\colon M \to N$ such that the index $\alpha(M, [\phi])$ is non-trivial. Furthermore, assume that $\phi_0 \in [\phi]$ is a harmonic map. Then there is a linear space V such that all $(\phi_0, \psi), \psi \in V$ are Dirac-harmonic maps.*

The solutions constructed above are uncoupled in the sense that they satisfy

$$\tau(\phi) = 0 = \mathcal{R}(\phi, \psi), \qquad \not{D}\psi = 0.$$

A criterion for the decoupling of Dirac-harmonic maps between surfaces was given in [Yan09]. The decoupling of the Euler-Lagrange equations for Dirac-harmonic maps seems to be a general phenomena. Almost all explicit solutions of the Euler-Lagrange equations share this property. Nevertheless, a few coupled solutions were constructed in [JMZ09].

Shortly before this thesis was completed, the boundary value problem for Dirac-harmonic maps was successfully treated in [CJW].

Finally, let us mention that in his survey paper "Perspectives on geometric analysis", [Yau06], p. 31, Yau suggests to study supersymmetric σ-models from the point of view of geometric analysis.

This thesis investigates the existence of Dirac-harmonic maps by using a combination of the heat flow method from Eells-Sampson and a regularization in the spirit of Sacks-Uhlenbeck. Let us mention the following preprint aiming in a similar direction as this thesis: In [HY10], the gradient flow for the Sacks-Uhlenbeck functional $E_\alpha(\phi)$ was investigated.

1.1. Outline

In *Chapter two* we introduce the notion of Dirac-harmonic maps and present a regularization prescription for the energy functional $E(\phi,\psi)$, which will be denoted as $E_\varepsilon(\phi,\psi)$. We compute the critical points of $E_\varepsilon(\phi,\psi)$. In addition, we derive the second variation of the energy functional $E(\phi,\psi)$.

In the *third Chapter*, we introduce the evolution equations associated to the regularized functional $E_\varepsilon(\phi,\psi)$. With the help of the evolution equations, we compute the behaviour of certain "energies" under the evolution of ϕ and ψ. This allows us to compare the evolution equations for regularized Dirac-harmonic maps with the well-studied evolution equation for harmonic maps. For the further analysis, we deal with the situation that the target manifold N is isometrically embedded in some $\mathbb{R}^q$ of sufficiently large dimension. Moreover, we derive the evolution equations for N being embedded in some $\mathbb{R}^q$. In the last part of the chapter we prove the existence of a short-time solution.

The *fourth Chapter* analyzes the evolution equations in the case that $M = S^1$. First of all, we study some simple examples, for which the evolution equations can be solved explicitly. As a next step, we use the scalar maximum principle to derive energy estimates. With the help of these energy estimates we can then establish the long-time existence of the evolution equations. We finish this chapter by exploring the convergence of the evolution equations. We also address the question if we can remove the regularization.

Chapter five investigates the evolution equations for M being a closed Riemannian spin surface. Similar to the previous chapter, we first of all derive energy estimates. However, we cannot apply the scalar maximum principle any longer due to the presence of multiple non-linearities. Consequently, we are forced to derive integral estimates. The most important tool here is a local Sobolev inequality. By application of these estimates we can guarantee the existence of a long-time solution, but only up to a finite number of singular points. Afterwards, we sketch how one can perform a blowup analysis of the singular points. Moreover, we show that the evolution equations weakly converge to a limiting map.
In addition, we study the structure of Dirac-harmonic maps between some surfaces of lower genus.
Finally, we discuss the removal of the regularization for the limiting map constructed before.

2. Dirac-harmonic Maps and Regularization

2.1. Dirac-harmonic Maps

Throughout this thesis, $(M, h_{\alpha\beta})$ and (N, g_{ij}) are compact, smooth Riemannian manifolds without boundary. In addition, we assume that the manifold M admits a spin structure. Coordinates on M are denoted by x, whereas coordinates on the target manifold N are denoted by y. Indices on M are labeled by Greek letters, whereas indices on N are labeled by Latin letters. We use the Einstein summation convention, which means that we will sum over repeated indices.
Given a map $\phi : M \to N$, we consider the pull-back bundle $\phi^{-1}TN$ of TN. Since M admits a spin structure by assumption, we can twist the spinor bundle ΣM with the pull-back bundle $\phi^{-1}TN$. On this twisted bundle $\Sigma M \otimes \phi^{-1}TN$ there is a metric induced from the metrics on ΣM and $\phi^{-1}TN$. The induced connection on $\Sigma M \otimes \phi^{-1}TN$ will be denoted by $\tilde{\nabla}$. We will always assume that all connections are metric and free of torsion. Locally, sections of $\Sigma M \otimes \phi^{-1}TN$ can be expressed as

$$\psi(x) = \psi^i(x) \otimes \frac{\partial}{\partial y^i}(\phi(x)).$$

We denote the Dirac operator on ΣM by $\partial\!\!\!/$ and the Dirac operator on the twisted bundle by $D\!\!\!\!/$. In terms of local coordinates $D\!\!\!\!/\psi$ can be expressed as

$$D\!\!\!\!/\psi = \partial\!\!\!/\psi^i \otimes \frac{\partial}{\partial y^i}(\phi(x)) + \Gamma^i_{jk}\frac{\partial\phi^j}{\partial x_\alpha}e_\alpha \cdot \psi^k(x) \otimes \frac{\partial}{\partial y^i}(\phi(x)).$$

It is easy to see that $D\!\!\!\!/$ is self-adjoint with respect to the L^2 norm. After these preliminary definitions we study the following energy functional:

$$E(\phi, \psi) = \frac{1}{2}\int_M (|d\phi|^2 + \langle\psi, D\!\!\!\!/\psi\rangle)dM.$$

Concerning the first term, the scalar product is taken on the bundle $T^*M \otimes \phi^{-1}TN$. For the second term we use the metric on $\Sigma M \otimes \phi^{-1}TN$. The critical points of $E(\phi, \psi)$ were calculated in [CJLW06], p. 413, Prop. 2.1:

Proposition 2.1. *The Euler-Lagrange equations for the functional $E(\phi, \psi)$ are given by*

$$\tau(\phi) = \mathcal{R}(\phi, \psi), \tag{2.1}$$
$$D\!\!\!\!/\psi = 0, \tag{2.2}$$

where $\tau(\phi)$ is the tension field of the map ϕ and the right hand side $\mathcal{R}(\phi,\psi)$ is explicitly given by

$$\mathcal{R}(\phi,\psi) = \frac{1}{2}R^N(e_\alpha \cdot \psi, \psi)d\phi(e_\alpha).$$

Written in coordinates, the Euler-Lagrange equations acquire the form

$$\begin{aligned} \tau^m(\phi) - \frac{1}{2}R^m_{lij}(\phi)\langle\psi^i, \nabla\phi^l \cdot \psi^j\rangle_{\Sigma M} &= 0, \\ \partial\!\!\!/\psi^i + \Gamma^i_{jk}(\phi)\frac{\partial\phi^j}{\partial x_\alpha}e_\alpha \cdot \psi^k &= 0. \end{aligned}$$

Solutions of the system (2.1), (2.2) are called *Dirac-harmonic maps* from $M \to N$.

Proposition 2.2 (Second Variation of $E(\phi,\psi)$)**.** *Assume that (ϕ,ψ) is a smooth Dirac-harmonic map. Then the second variation of the energy functional $E(\phi,\psi)$ is given by*

$$\begin{aligned} \frac{\delta^2}{\delta\phi^2}E(\phi,\psi) &= \int_M \Big(|\nabla\eta|^2 - \langle R^N(\eta, d\phi(e_\alpha))\eta, d\phi(e_\alpha)\rangle \qquad (2.3) \\ &\quad +\frac{1}{2}\langle(\nabla_\eta R^N)(e_\alpha\cdot\psi,\psi)d\phi(e_\alpha),\eta\rangle + \frac{1}{2}\langle\eta, R^N(e_\alpha\cdot\psi,\psi)\nabla\eta\rangle \\ &\quad +\langle R^N(e_\alpha\cdot\psi, \psi^i\otimes\nabla_\eta\frac{\partial}{\partial y^i})d\phi(e_\alpha),\eta\rangle\Big)dM, \\ \frac{\delta^2}{\delta\psi^2}E(\phi,\psi) &= \int_M \langle\xi, D\!\!\!\!/\,\xi\rangle dM. \qquad (2.4) \end{aligned}$$

Proof. We choose a local orthonormal basis $\{e_\alpha\}$ on M such that $[e_\alpha, \partial_t] = 0$ and also $\nabla_{\partial_t}e_\alpha = 0$ at a considered point. First, we compute the second variation of $E(\phi,\psi)$ with respect to ϕ. Therefore, consider a family of smooth variations of ϕ satisfying $\frac{\partial\phi_t}{\partial t}\big|_{t=0} = \eta$, while keeping the ψ^i in $\psi(x) = \psi^i(x)\otimes\frac{\partial}{\partial y^i}(\phi(x))$ fixed. It is well known that the second variation of the Dirichlet energy is given by

$$\frac{\partial^2}{\partial t^2}\Big|_{t=0}\frac{1}{2}\int_M |d\phi_t|^2 dM = \int_M (|\nabla\eta|^2 - \langle R^N(\eta, d\phi(e_\alpha))\eta, d\phi(e_\alpha)\rangle + \langle\nabla_\eta\eta, \tau(\phi)\rangle)dM$$

([LW08], p. 8, Prop.1.6.2). In addition, we find

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\Big|_{t=0}\frac{1}{2}\int_M\langle\psi, D\!\!\!\!/\,\psi\rangle dM &= \frac{\partial}{\partial t}\Big|_{t=0}\int_M\langle\frac{\partial\phi_t}{\partial t}, \mathcal{R}(\phi_t,\psi)\rangle dM \\ &= \int_M(\langle\nabla_\eta\eta, \mathcal{R}(\phi,\psi)\rangle + \langle\eta, \frac{\nabla}{\partial t}\big|_{t=0}\mathcal{R}(\phi_t,\psi)\rangle)dM. \end{aligned}$$

Differentiating $\mathcal{R}(\phi_t,\psi)$ with respect to t yields

$$\begin{aligned} \frac{\nabla}{\partial t}\mathcal{R}(\phi_t,\psi) &= \frac{\nabla}{\partial t}\frac{1}{2}R^N(e_\alpha\cdot\psi,\psi)d\phi_t(e_\alpha) \\ &= \frac{1}{2}(\nabla_{d\phi_t(\partial_t)}R^N)(e_\alpha\cdot\psi,\psi)d\phi_t(e_\alpha) + R^N(e_\alpha\cdot\psi, \frac{\nabla}{\partial t}\psi)d\phi_t(e_\alpha) \\ &\quad +\frac{1}{2}R^N(e_\alpha\cdot\psi,\psi)\nabla\frac{\partial\phi_t}{\partial t}. \end{aligned}$$

The second term involving the derivative of ψ with respect to t should be understood as follows:

$$\frac{\nabla}{\partial t}\psi = \psi^i \otimes \nabla_{d\phi_t(\partial_t)}\frac{\partial}{\partial y^i}(\phi_t(x)).$$

Summing up the different contributions, evaluating at $t = 0$, and using the fact that ϕ solves the Euler-Lagrange equation for Dirac-harmonic maps, the second variation formula for ϕ follows.

As a second step, we derive the second variation of $E(\phi, \psi)$ with respect to ψ. Therefore, consider a family of smooth variations of ψ satisfying $\frac{\tilde{\nabla}\psi_t}{\partial t}\big|_{t=0} = \xi$ and keeping ϕ fixed. We know that

$$\frac{\partial}{\partial t}\bigg|_{t=0}\frac{1}{2}\int_M \langle\psi_t, \not{D}\psi_t\rangle dM = \int_M \langle\xi, \not{D}\psi\rangle dM.$$

Differentiating again with respect to t and evaluating at $t = 0$, we get

$$\frac{\partial^2}{\partial t^2}\bigg|_{t=0}\frac{1}{2}\int_M \langle\psi_t, \not{D}\psi_t\rangle dM = \int_M (\langle\not{D}\psi, \frac{\tilde{\nabla}^2}{\partial t^2}\psi_t\bigg|_{t=0}\rangle + \langle\xi, \not{D}\xi\rangle)dM.$$

The claim now follows since ψ solves $\not{D}\psi = 0$ by assumption. □

We denote the curvature operator on $\Sigma M \otimes \phi^{-1}TN$ by $\tilde{R}(\cdot,\cdot)$. It can naturally be decomposed into

$$\tilde{R}(e_\alpha, e_\beta) = R^{\Sigma M}(e_\alpha, e_\beta) \otimes \mathbb{1}_{\phi^{-1}TN} + \mathbb{1}_{\Sigma M} \otimes R^N(d\phi(e_\alpha), d\phi(e_\beta)).$$

Lemma 2.3 (Weitzenböck formula for the twisted Dirac operator $\not{D}$)**.** *Assume that $\psi \in C^2(M, \Sigma M \otimes \phi^{-1}TN)$. Then the square of the twisted Dirac operator $\not{D}$ satisfies*

$$\not{D}^2\psi = -\tilde{\Delta}\psi + \frac{1}{4}R\psi + \frac{1}{2}e_\alpha \cdot e_\beta \cdot R^N(d\phi(e_\alpha), d\phi(e_\beta))\psi. \tag{2.5}$$

Proof. We choose a local orthonormal basis $\{e_\alpha\}$ on M such that $\nabla_{e_\alpha}e_\beta = 0$ at a considered point. We compute

$$\begin{aligned}
\not{D}^2\psi &= e_\alpha \cdot e_\beta \cdot \tilde{\nabla}_{e_\alpha}\tilde{\nabla}_{e_\beta}\psi \\
&= -\tilde{\Delta}\psi + \sum_{e_\alpha < e_\beta} e_\alpha \cdot e_\beta \cdot \tilde{R}(e_\alpha, e_\beta)\psi \\
&= -\tilde{\Delta}\psi + \frac{1}{2}e_\alpha \cdot e_\beta \cdot \tilde{R}(e_\alpha, e_\beta)\psi \\
&= -\tilde{\Delta}\psi + \frac{1}{2}e_\alpha \cdot e_\beta \cdot R^{\Sigma M}(e_\alpha, e_\beta)\psi + \frac{1}{2}e_\alpha \cdot e_\beta \cdot R^N(d\phi(e_\alpha), d\phi(e_\beta))\psi \\
&= -\tilde{\Delta}\psi + \frac{1}{4}R\psi + \frac{1}{2}e_\alpha \cdot e_\beta \cdot R^N(d\phi(e_\alpha), d\phi(e_\beta))\psi.
\end{aligned}$$

This could of course also be deduced from the general Weitzenböck formula for twisted Dirac operators, see for example [LM89], p. 164, Theorem 8.17. □

We will often encounter the situation that the Euler-Lagrange equations for Dirac-harmonic maps decouple. Therefore, we make the following

Definition 2.4. A Dirac-harmonic map (ϕ, ψ) is called *uncoupled* if ϕ is a harmonic map, otherwise it is called *coupled*.

2.2. Regularization of the Energy Functional

In the analysis of the energy functional $E(\phi, \psi)$ one often faces the problem that the energy is unbounded from below. This problem originates in the fact that the Dirac-operator is unbounded. To overcome these analytical difficulties, we propose a method to "improve" the energy functional by adding a small regularizing term. More precisely, we consider

$$E_\varepsilon(\phi, \psi) = \frac{1}{2}\int_M (|d\phi|^2 + \langle \psi, \not{D}\psi\rangle + \varepsilon|\tilde{\nabla}\psi|^2)dM$$

with $\varepsilon > 0$. Unless stated otherwise, we will always assume that $\varepsilon < 1$. Note that for $\varepsilon \to 0$ the regularized functional coincides with $E(\phi, \psi)$. Before deriving the Euler-Lagrange equations for $E_\varepsilon(\phi, \psi)$, we make the following

Remark 2.5. The functional $E_\varepsilon(\phi, \psi)$ satisfies

$$-\frac{m}{8\varepsilon}\int_M |\psi|^2 dM \leq E_\varepsilon(\phi, \psi) \leq \infty,$$

where m is the dimension of the manifold M.

Proof. The estimate follows from combining the inequalities

$$\langle \psi, \not{D}\psi\rangle \geq -\frac{m}{4\varepsilon}|\psi|^2 - \frac{\varepsilon}{m}|\not{D}\psi|^2, \qquad |\tilde{\nabla}\psi|^2 \geq \frac{1}{m}|\not{D}\psi|^2.$$

□

As a next step we derive the Euler-Lagrange equations for $E_\varepsilon(\phi, \psi)$.

Proposition 2.6 (Euler-Lagrange equations of $E_\varepsilon(\phi, \psi)$)**.** *The critical points of the functional $E_\varepsilon(\phi, \psi)$ are given by*

$$\tau(\phi) = \mathcal{R}(\phi, \psi) + \varepsilon\mathcal{R}_c(\phi, \psi), \tag{2.6}$$

$$\varepsilon\tilde{\Delta}\psi = \not{D}\psi \tag{2.7}$$

with the vector fields

$$\begin{aligned}\mathcal{R}(\phi, \psi) &= \frac{1}{2}R^N(e_\alpha \cdot \psi, \psi)d\phi(e_\alpha) \in \Gamma(\phi^{-1}TN),\\ \mathcal{R}_c(\phi, \psi) &= R^N(\tilde{\nabla}_{e_\alpha}\psi, \psi)d\phi(e_\alpha) \in \Gamma(\phi^{-1}TN)\end{aligned}$$

and $\tilde{\Delta}$ denoting the connection Laplacian on the bundle $\Sigma M \otimes \phi^{-1}TN$.

Proof. We choose a local orthonormal basis $\{e_\alpha\}$ on M such that $[e_\alpha, \partial_t] = 0$ and also $\nabla_{\partial_t} e_\alpha = 0$ at a considered point. We start by deriving the Euler-Lagrange equation for the spinor ψ. Therefore, we consider a variation of ψ with ϕ fixed and $\frac{\tilde{\nabla}\psi_t}{\partial t}\big|_{t=0} = \chi$. We find

$$\begin{aligned}
\frac{\delta}{\delta\psi} E_\varepsilon(\phi, \psi_t) &= \frac{1}{2}\int_M (\langle \chi, \not{D}\psi\rangle + \langle \psi, \not{D}\chi\rangle + 2\varepsilon\langle \chi, \tilde{\nabla}^*_{e_\alpha}\tilde{\nabla}_{e_\alpha}\psi\rangle) dM \\
&= \int_M \langle \chi, \not{D}\psi - \varepsilon\tilde{\Delta}\psi\rangle dM.
\end{aligned}$$

To derive the Euler-Lagrange equation for ϕ, consider a family of smooth variations of ϕ satisfying $\frac{\partial\phi_t}{\partial t}\big|_{t=0} = \eta$, while keeping the ψ^i in $\psi(x) = \psi^i(x) \otimes \frac{\partial}{\partial y^i}(\phi_t(x))$ fixed. The variation with respect to ϕ of the following terms has already been computed in [CJLW06], p. 413, Prop. 2.1.:

$$\begin{aligned}
\frac{\delta}{\delta\phi}\frac{1}{2}\int_M |d\phi_t|^2 dM &= -\int_M \langle \tau(\phi), \eta\rangle dM, \\
\frac{\delta}{\delta\phi}\frac{1}{2}\int_M \langle \psi, \not{D}\psi\rangle dM &= \int_M (\langle \frac{\nabla\psi}{\partial t}, \not{D}\psi\rangle + \langle \mathcal{R}(\phi,\psi), \eta\rangle) dM.
\end{aligned}$$

Finally, we compute the variation of the regularizing term, namely

$$\frac{\partial}{\partial t}\Big|_{t=0} \frac{\varepsilon}{2}\int_M |\tilde{\nabla}\psi|^2 dM = \varepsilon\int_M \big(\langle \frac{\nabla\psi}{\partial t}, \tilde{\nabla}^*_{e_\alpha}\tilde{\nabla}_{e_\alpha}\psi\rangle + \langle R^E(\partial_t, e_\alpha)\psi, \tilde{\nabla}_{e_\alpha}\psi\rangle\big)\big|_{t=0} dM,$$

where R^E denotes the curvature tensor on the the bundle $E = T^*M \otimes \Sigma M \otimes \phi_t^{-1}TN$. The only curvature contribution arises from the pull-back bundle $\phi_t^{-1}TN$ and we compute

$$\begin{aligned}
\langle R^{\phi_t^{-1}TN}(\partial_t, e_\alpha)\psi, \tilde{\nabla}_{e_\alpha}\psi\rangle\big|_{t=0} &= \langle R^N(d\phi_t(\partial_t), d\phi(e_\alpha))\psi, \tilde{\nabla}_{e_\alpha}\psi\rangle\big|_{t=0} \\
&= \langle R^N(\tilde{\nabla}_{e_\alpha}\psi, \psi) d\phi(e_\alpha), \frac{\partial\phi_t}{\partial t}\rangle\big|_{t=0} \\
&= \langle \mathcal{R}_c(\phi,\psi), \eta\rangle.
\end{aligned}$$

Adding up the different contributions, we get

$$\frac{\delta}{\delta\phi} E_\varepsilon(\phi_t, \psi) = \int_M \big(\langle -\tau(\phi) + \mathcal{R}(\phi,\psi) + \varepsilon\mathcal{R}_c(\phi,\psi), \eta\rangle + \langle \not{D}\psi + \varepsilon\tilde{\nabla}^*_{e_\alpha}\tilde{\nabla}_{e_\alpha}\psi, \frac{\nabla\psi}{\partial t}\big|_{t=0}\rangle\big) dM.$$

Using the Euler-Lagrange equation for ψ, which was deduced before, the result follows. □

Written in local coordinates, the new terms arising from the variation of $E_\varepsilon(\phi,\psi)$ acquire the following form:

$$\begin{aligned}
\mathcal{R}_c(\phi,\psi) &= R^m_{\ lij}\frac{\partial}{\partial y^m}\frac{\partial\phi^l}{\partial x_\alpha}\langle\nabla^{\Sigma M}_{e_\alpha}\psi^i,\psi^j\rangle_{\Sigma M} \\
&\quad + R^m_{\ lij}\frac{\partial}{\partial y^m}\Gamma^j_{rs}\frac{\partial\phi^l}{\partial x_\alpha}\langle\psi^i,\psi^r\rangle_{\Sigma M}\frac{\partial\phi^s}{\partial x_\alpha}, \\
\tilde{\Delta}\psi &= \Delta^{\Sigma M}\psi^i\otimes\frac{\partial}{\partial y^i} + 2\nabla^{\Sigma M}_{e_\alpha}\psi^i\otimes\Gamma^k_{ij}\frac{\partial\phi^j}{\partial x_\alpha}\frac{\partial}{\partial y^k} \\
&\quad + \psi^i\otimes\Gamma^k_{ij,p}\frac{\partial\phi^p}{\partial x_\alpha}\frac{\partial\phi^j}{\partial x_\alpha}\frac{\partial}{\partial y^k} + \psi^i\otimes\Gamma^k_{ij}\frac{\partial^2\phi^j}{\partial x_\alpha^2}\frac{\partial}{\partial y^k} \\
&\quad + \psi^i\otimes\Gamma^k_{ij}\Gamma^r_{ks}\frac{\partial\phi^j}{\partial x_\alpha}\frac{\partial\phi^s}{\partial x_\alpha}\frac{\partial}{\partial y^r}.
\end{aligned}$$

Solutions of the system (2.6), (2.7) will be called *regularized Dirac-harmonic maps* from $M \to N$.

3. Dirac-harmonic Maps and Gradient Flows

3.1. Introduction and Overview

In this chapter we want to introduce the L^2-gradient flow for regularized Dirac-harmonic maps. Since (regularized) Dirac-harmonic maps form a pair of a map ϕ and a spinor ψ along that map, the gradient flow is given by a system of two coupled evolution equations. Ultimately, we want to achieve that we can deform given initial data (ϕ_0, ψ_0) into a Dirac-harmonic map. Here, $\phi_0 \colon M \to N$ and $\psi_0 \in \Gamma(\Sigma M \otimes \phi_0^{-1}TN)$ is defined along the map ϕ_0. More precisely, the evolution equations we want to use are

$$\begin{cases} \frac{\partial \phi_t}{\partial t} = \tau(\phi_t) - \mathcal{R}(\phi_t, \psi_t) - \varepsilon \mathcal{R}_c(\phi_t, \psi_t), \\ \phi(x, 0) = \phi_0(x), \end{cases} \tag{3.1}$$

$$\begin{cases} \frac{\tilde{\nabla} \psi_t}{\partial t} = \varepsilon \tilde{\Delta} \psi_t - \not{D} \psi_t, \\ \psi(x, 0) = \psi_0(x). \end{cases} \tag{3.2}$$

Throughout this thesis, we will refer to the system (3.1), (3.2) as *evolution equations for regularized Dirac-harmonic maps* or *regularized Dirac-harmonic map heat flow.*

The study of the evolution equations involves the following steps:

1. *Short-time existence*
   Roughly speaking, the existence of a short-time solution of the evolution equations guarantees that we can start deforming. More precisely, we expect that for $M, N, \varepsilon, \phi_0, \psi_0$ arbitrary, there exists a small time $T_{max} > 0$ such that the evolution equations admit a solution for $0 \leq t < T_{max}$.

2. *Long-time existence*
   We cannot expect that we can solve the evolution equations for $0 \leq t < \infty$ in the most general situation. In particular, it could happen that the evolution equations blow-up after a finite time. Consequently, we have to find conditions on M, N and the initial data that guarantee the existence of a long-time solution.

3. *Convergence*
   After having established the existence of a long-time solution, we can study the limit $t \to \infty$. The natural question arising is, if the evolution equations converge to a limiting map $(\phi_\infty, \psi_\infty)$.

4. *Removing the regularization*
   So far, our analysis is based on the evolution equations for the regularized functional $E_\varepsilon(\phi,\psi)$, but we finally want to find critical points of $E(\phi,\psi)$. Hence, we have to study the limit $\varepsilon \to 0$ after letting $t \to \infty$.

We will see that the more steps we want to establish, the more restrictions we get.

Remark 3.1. Of course, we could also try to apply the L^2-gradient flow to the functional $E(\phi,\psi)$, which would lead to the set of equations

$$\begin{aligned}\frac{\partial \phi_t}{\partial t} &= \tau(\phi_t) - \mathcal{R}(\phi_t,\psi_t), \qquad \phi(x,0)=\phi(x),\\ \frac{\tilde{\nabla}\psi_t}{\partial t} &= -\not{D}\psi_t, \qquad \psi(x,0)=\psi_0(x).\end{aligned}$$

Let us make some comments about this system of evolution equations: The evolution equation for the map ϕ_t is *non-linear*, but *parabolic*. Hence, for this equation, analytical tools like the maximum principle are applicable. The evolution equation for the spinor ψ_t is of a different nature. In contrast to the equation for ϕ_t, it is a first order evolution equation. An evolution equation of this type was already considered in [Che73], p. 403. It was pointed out that such an equation is a first-order *symmetric hyperbolic system*. Consequently, one cannot expect that such an equation tends to an equilibrium state as $t \to \infty$. This behaviour is also reflected in the fact that the energy functional $E(\phi,\psi)$ is unbounded from below. Since the gradient flow tries to decrease the energy, we cannot expect that the evolution equation for the spinor ψ_t will converge to a limiting spinor ψ_∞.

Another approach one could pursue is to try to deform only one of the fields (ϕ,ψ) by an evolution equation and use a different method for the other one, like for example index theory. On the other hand, we note that our ansatz, the simultaneous deformation of ϕ and ψ, seems to be the most general one.
Almost all estimates derived later on depend on the regularization parameter ε in a non-trivial fashion. We will make the dependence on ε explicit only when it is necessary. Moreover, C will denote a generic constant changing from line to line.

Lemma 3.2. *Let $(\phi_t,\psi_t) \in C^\infty(M\times[0,T),N)\times C^\infty(M\times[0,T),\Sigma M\otimes\phi_t^{-1}TN)$ be a solution of (3.1) and (3.2). Then we have for all $t\in[0,T)$*

$$E_\varepsilon(\phi_t,\psi_t) + \int_0^t\int_M \left(|\frac{\partial\phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla}\psi_t}{\partial t}|^2\right) dM dt = E_\varepsilon(\phi_0,\psi_0)$$

and also

$$-\frac{m}{8\varepsilon}\int_M |\psi_t|^2 dM \leq E_\varepsilon(\phi_t,\psi_t) \leq E_\varepsilon(\phi_0,\psi_0),$$

where m is the dimension of M.

Proof. This is a direct consequence of the gradient flow. $\square$

Remark 3.3. In the last Lemma, we saw that the regularized energy $E_\varepsilon(\phi,\psi)$ is bounded from below by the L^2-norm of the spinor ψ_t. Hence, we may expect that this L^2-norm will play an important role whenever we will discuss the convergence of the gradient flow. Moreover, the inequality tells us that the energy is decreasing, which is of course a general feature of the gradient flow.

Remark 3.4. For simplicity we will mostly assume that the initial data (ϕ_0,ψ_0) is smooth. Of course, we could also admit initial data of lower regularity. Since the evolution equations (3.1) and (3.2) form a parabolic system, they will have a smoothing effect on the solution anyway.

3.2. Evolution of Energies

In this section we analyze how the norms of $\psi_t, d\phi_t$ and $\tilde{\nabla}\psi_t$ behave under the evolution of ϕ_t and ψ_t. Later, we will use these equations to derive energy estimates.

Lemma 3.5. *Let $\psi_t \in C^\infty(M\times[0,T),\Sigma M\otimes\phi_t^{-1}TN)$ be a solution of (3.2). Then we have for all $t\in[0,T)$*

$$\frac{\partial}{\partial t}\frac{1}{2}|\psi_t|^2=\varepsilon\Delta\frac{1}{2}|\psi_t|^2-\varepsilon|\tilde{\nabla}\psi_t|^2-\langle\psi_t,\not{D}\psi_t\rangle.$$

Proof. The statement follows directly from the evolution equation (3.2). $\square$

Lemma 3.6. *Let $\phi_t\in C^\infty(M\times[0,T),N)$ be a solution of (3.1). Then we have for all $t\in[0,T)$*

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{2}|d\phi_t|^2 \;=\;& \Delta\frac{1}{2}|d\phi_t|^2-|\nabla d\phi_t|^2+\langle R^N(d\phi_t(e_\alpha),d\phi_t(e_\beta))d\phi_t(e_\alpha),d\phi_t(e_\beta)\rangle\\ &-\langle d\phi_t(Ric^M(e_\alpha)),d\phi_t(e_\alpha)\rangle-\langle\nabla_{e_\alpha}\mathcal{R}(\phi_t,\psi_t),d\phi_t(e_\alpha)\rangle\\ &-\varepsilon\langle\nabla_{e_\alpha}\mathcal{R}_c(\phi_t,\psi_t),d\phi_t(e_\alpha)\rangle. \end{aligned}\tag{3.3}$$

Proof. We calculate

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{2}|d\phi_t|^2 &= \langle\frac{\nabla}{\partial t}d\phi_t(e_\alpha),d\phi(e_\alpha)\rangle\\ &=\langle\nabla_{e_\alpha}\frac{\partial\phi_t}{\partial t},d\phi_t(e_\alpha)\rangle\\ &=\langle\nabla_{e_\alpha}\tau(\phi_t),d\phi_t(e_\alpha)\rangle-\langle\nabla_{e_\alpha}\mathcal{R}(\phi_t,\psi_t),d\phi_t(e_\alpha)\rangle\\ &\quad-\varepsilon\langle\nabla_{e_\alpha}\mathcal{R}_c(\phi_t,\psi_t),d\phi_t(e_\alpha)\rangle,\end{aligned}$$

where we used that $\nabla\frac{\partial\phi_t}{\partial t}=\frac{\nabla}{\partial t}d\phi_t$, which is due to the torsion freeness of the connection. In order to manipulate the first term, we use the formula for the curvature on the bundle $T^*M\otimes\phi^{-1}TN$. Namely,

$$\nabla_{e_\beta}\nabla_{e_\alpha}d\phi(e_\alpha)=\nabla_{e_\alpha}\nabla_{e_\beta}d\phi(e_\alpha)+R^N(d\phi(e_\alpha),d\phi(e_\beta))d\phi(e_\alpha)-d\phi(\mathrm{Ric}^M(e_\beta)).$$

Since $\tau(\phi) = \nabla_{e_\alpha} d\phi(e_\alpha)$ and $\nabla_{e_\beta} d\phi(e_\alpha) = \nabla_{e_\alpha} d\phi(e_\beta)$, we find that

$$\begin{aligned}
\Delta\frac{1}{2}|d\phi_t|^2 &= |\nabla d\phi_t|^2 + \langle \Delta d\phi_t(e_\alpha), d\phi(e_\alpha)\rangle \\
&= -\langle R^N(d\phi_t(e_\alpha), d\phi_t(e_\beta))d\phi_t(e_\beta), d\phi_t(e_\alpha)\rangle + \langle d\phi_t(Ric^M(e_\alpha)), d\phi_t(e_\alpha)\rangle \\
&\quad + \langle \nabla_{e_\alpha}\tau(\phi_t), d\phi_t(e_\alpha)\rangle,
\end{aligned}$$

which finally proves the assertion. □

Lemma 3.7. *Let $\psi_t \in C^\infty(M\times[0,T), \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (3.2). Then we have for all $t\in[0,T)$*

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{2}|\tilde\nabla\psi_t|^2 &= \varepsilon\Delta\frac{1}{2}|\tilde\nabla\psi_t|^2 - \varepsilon|\tilde\nabla^2\psi_t|^2 - \langle\tilde\nabla\psi_t, \tilde\nabla \not{D}\psi_t\rangle + \langle\frac{\partial\phi_t}{\partial t}, \mathcal{R}_c(\phi_t,\psi_t)\rangle \qquad (3.4)\\
&\quad -\varepsilon\langle\tilde\nabla^*_{e_\beta}R^{E_2}(e_\alpha,e_\beta)\psi_t, \tilde\nabla_{e_\alpha}\psi_t\rangle - \varepsilon\langle R^{E_1}(e_\alpha,e_\beta)\tilde\nabla_{e_\beta}\psi_t, \tilde\nabla_{e_\alpha}\psi_t\rangle
\end{aligned}$$

with the bundles $E_1 = T^*M\otimes\Sigma M\otimes\phi_t^{-1}TN$ and $E_2 = T^*M\otimes E_1$.

Proof. Using the evolution equation (3.2), we compute

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{2}|\tilde\nabla\psi_t|^2 &= \langle\frac{\tilde\nabla}{\partial t}\tilde\nabla_{e_\alpha}\psi_t, \tilde\nabla_{e_\alpha}\psi_t\rangle \\
&= \langle R^{E_1}(\partial_t, e_\alpha)\psi_t, \tilde\nabla_{e_\alpha}\psi_t\rangle + \langle\tilde\nabla_{e_\alpha}\frac{\tilde\nabla}{\partial t}\psi_t, \tilde\nabla_{e_\alpha}\psi_t\rangle \\
&= \langle\frac{\partial\phi_t}{\partial t}, \mathcal{R}_c(\phi_t,\psi_t)\rangle - \langle\tilde\nabla_{e_\alpha}\psi_t, \tilde\nabla_{e_\alpha}\not{D}\psi_t\rangle + \varepsilon\langle\tilde\nabla_{e_\alpha}\psi_t, \tilde\nabla_{e_\alpha}\tilde\Delta\psi_t\rangle,
\end{aligned}$$

where $R^{E_1}(\cdot,\cdot)$ denotes the curvature on the bundle $E_1 = T^*M\otimes\Sigma M\otimes\phi_t^{-1}TN$. Since only the bundle $\phi_t^{-1}TN$ is t-dependent, we only get one curvature term. In addition, we compute

$$\Delta\frac{1}{2}|\tilde\nabla\psi|^2 = |\tilde\nabla^2\psi|^2 + \langle\tilde\Delta\tilde\nabla\psi, \tilde\nabla\psi\rangle.$$

To proceed, we have to interchange the connection Laplacian on $T^*M\otimes\Sigma M\otimes\phi_t^{-1}TN$ with the covariant derivative $\tilde\nabla$. Namely,

$$\begin{aligned}
\tilde\Delta\tilde\nabla_{e_\alpha}\psi_t &= \tilde\nabla^*_{e_\beta}\tilde\nabla_{e_\beta}\tilde\nabla_{e_\alpha}\psi_t \\
&= \tilde\nabla^*_{e_\beta}R^{E_2}(e_\alpha,e_\beta)\psi_t + \tilde\nabla^*_{e_\beta}\tilde\nabla_{e_\alpha}\tilde\nabla_{e_\beta}\psi_t \\
&= \tilde\nabla^*_{e_\beta}R^{E_2}(e_\alpha,e_\beta)\psi_t + R^{E_1}(e_\alpha,e_\beta)\tilde\nabla_{e_\beta}\psi_t + \tilde\nabla_{e_\beta}\tilde\Delta\psi_t
\end{aligned}$$

with the bundles $E_1 = T^*M\otimes\Sigma M\otimes\phi_t^{-1}TN$ and $E_2 = T^*M\otimes E_1$. Combining the three equations proves the assertion. □

Lemma 3.8. *Let $\phi_t\in C^\infty(M\times[0,T),N)$ be a solution of (3.1). Then we have for all $t\in[0,T)$*

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{2}|\frac{\partial\phi_t}{\partial t}|^2 &= \Delta\frac{1}{2}|\frac{\partial\phi_t}{\partial t}|^2 - |\nabla\frac{\partial\phi_t}{\partial t}|^2 + \langle R^N(d\phi_t(e_\alpha), \frac{\partial\phi_t}{\partial t})d\phi_t(e_\alpha), \frac{\partial\phi_t}{\partial t}\rangle \qquad (3.5)\\
&\quad -\langle\frac{\nabla}{\partial t}\mathcal{R}(\psi_t,\phi_t), \frac{\partial\phi_t}{\partial t}\rangle - \varepsilon\langle\frac{\nabla}{\partial t}\mathcal{R}_c(\psi_t,\phi_t), \frac{\partial\phi_t}{\partial t}\rangle.
\end{aligned}$$

Proof. We compute

$$\frac{\partial}{\partial t}\frac{1}{2}|\frac{\partial\phi_t}{\partial t}|^2 = \langle\frac{\nabla}{\partial t}\left(\tau(\phi_t)-\mathcal{R}(\phi_t,\psi_t)-\varepsilon\mathcal{R}_c(\phi_t,\psi_t)\right),\frac{\partial\phi_t}{\partial t}\rangle.$$

On the other hand, we have

$$\begin{aligned}\Delta\frac{1}{2}|\frac{\partial\phi_t}{\partial t}|^2 &= |\nabla\frac{\partial\phi_t}{\partial t}|^2+\langle\Delta\frac{\partial\phi_t}{\partial t},\frac{\partial\phi_t}{\partial t}\rangle\\ &= |\nabla\frac{\partial\phi_t}{\partial t}|^2+\langle\frac{\nabla}{\partial t}\tau(\phi_t),\frac{\partial\phi_t}{\partial t}\rangle+\langle R^N(d\phi_t(e_\alpha),\frac{\partial\phi_t}{\partial t})d\phi_t(e_\alpha),\frac{\partial\phi_t}{\partial t}\rangle,\end{aligned}$$

where we used the same Bochner technique as in the evolution equation for $|d\phi_t|^2$. Combining both equations yields the result. □

Lemma 3.9. *Let $\psi_t\in C^\infty(M\times[0,T),\Sigma M\otimes\phi_t^{-1}TN)$ be a solution of (3.2). Then we have for all $t\in[0,T)$*

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{2}|\frac{\tilde\nabla\psi_t}{\partial t}|^2 =& \varepsilon\Delta\frac{1}{2}|\frac{\tilde\nabla\psi_t}{\partial t}|^2-\varepsilon|\tilde\nabla\frac{\tilde\nabla\psi_t}{\partial t}|^2-\langle e_\alpha\cdot R^N(d\phi_t(\partial_t),d\phi_t(e_\alpha))\psi_t,\frac{\tilde\nabla\psi_t}{\partial t}\rangle\\ &-\langle\frac{\tilde\nabla\psi_t}{\partial t},\not{D}\frac{\tilde\nabla\psi_t}{\partial t}\rangle-\varepsilon\langle R^N(d\phi_t(\partial_t),d\phi_t(e_\alpha))\tilde\nabla_{e_\alpha}\psi_t,\frac{\tilde\nabla\psi_t}{\partial t}\rangle\\ &-\varepsilon\langle\tilde\nabla_{e_\alpha}(R^N(d\phi_t(\partial_t),d\phi_t(e_\alpha))\psi_t),\frac{\tilde\nabla\psi_t}{\partial t}\rangle.\end{aligned}\tag{3.6}$$

Proof. We choose a local orthonormal basis $\{e_\alpha\}$ on M such that $[e_\alpha,\partial_t]=0$ and also $\nabla_{\partial_t}e_\alpha=0$ at a considered point. Moreover, we compute

$$\frac{\partial}{\partial t}\frac{1}{2}|\frac{\tilde\nabla\psi_t}{\partial t}|^2 = -\langle\frac{\tilde\nabla}{\partial t}\not{D}\psi,\frac{\tilde\nabla\psi}{\partial t}\rangle+\varepsilon\langle\frac{\tilde\nabla}{\partial t}\tilde\Delta\psi,\frac{\tilde\nabla\psi}{\partial t}\rangle.$$

To interchange the covariant derivative with respect to t with the spatial derivatives, we need the following formulas:

$$\begin{aligned}\frac{\tilde\nabla}{\partial t}\not{D}\psi_t &= e_\alpha\cdot\tilde R(\partial_t,e_\alpha)\psi_t+\not{D}\frac{\tilde\nabla}{\partial t}\psi_t,\\ \frac{\tilde\nabla}{\partial t}\tilde\Delta\psi_t &= R^{\phi_t^{-1}TN}(\partial_t,e_\alpha)\tilde\nabla_{e_\alpha}\psi_t+\tilde\nabla^*_{e_\alpha}(R^{\phi_t^{-1}TN}(\partial_t,e_\alpha)\psi_t)+\tilde\Delta\frac{\tilde\nabla}{\partial t}\psi_t,\end{aligned}$$

where we used that only the bundle $\phi_t^{-1}TN$ depends on t. Computing the Laplacian

$$\Delta\frac{1}{2}|\frac{\tilde\nabla\psi_t}{\partial t}|^2=|\tilde\nabla\frac{\tilde\nabla\psi_t}{\partial t}|^2+\langle\tilde\Delta\frac{\tilde\nabla\psi_t}{\partial t},\frac{\tilde\nabla\psi_t}{\partial t}\rangle$$

and applying the curvature formulas from above, we get the result. □

Remark 3.10. Some of the non-linear terms in the evolution equations for $|d\phi_t|^2$ and $|\frac{\partial\phi_t}{\partial t}|^2$ can be controlled by curvature assumptions on the target manifold N. If we assume that N has negative sectional curvature, the terms

$$\langle R^N(d\phi_t(e_\alpha), d\phi_t(e_\beta))d\phi_t(e_\alpha), d\phi_t(e_\beta)\rangle, \qquad \langle R^N(d\phi_t(e_\alpha), \frac{\partial\phi_t}{\partial t})d\phi_t(e_\alpha), \frac{\partial\phi_t}{\partial t}\rangle$$

can be estimated by zero. This is the reason why Eells and Sampson [ES64] succeeded in their work on the existence of harmonic maps.

Unfortunately, the curvature terms appearing in the evolution equation for $\frac{1}{2}|\tilde{\nabla}\psi_t|^2$, as for example

$$\langle R^{E_1}(e_\alpha, e_\beta)\tilde{\nabla}_{e_\beta}\psi_t, \tilde{\nabla}_{e_\alpha}\psi_t\rangle,$$

cannot be related to geometric properties of the manifolds M and N. Here, E_1 denotes the vector bundle $\Sigma M \otimes \phi_t^{-1}TN \otimes T^*M$. The curvature $R^{\Sigma M}$ on the spinor bundle is related to the curvature tensor on M by the following formula

$$R^{\Sigma M}(e_\alpha, e_\beta)\psi = \frac{1}{4}h(R^M(e_\alpha, e_\beta)e_\gamma, e_\delta)e_\delta \cdot e_\gamma \cdot \psi = \frac{1}{4}R_{\alpha\beta\gamma\delta}\, e_\delta \cdot e_\gamma \cdot \psi.$$

In the case that M is a Riemannian surface, we find

$$\langle R^{\Sigma M}(e_\alpha, e_\beta)\tilde{\nabla}_{e_\beta}\psi_t, \tilde{\nabla}_{e_\alpha}\psi_t\rangle = \frac{R}{2}(|\not{D}\psi_t|^2 - |\tilde{\nabla}\psi_t|^2),$$

where R is the scalar curvature of M. The other two curvature contributions from $R^{\phi_t^{-1}TN}$ and R^{T^*M} cannot be transformed into a nicer form.

3.3. A first Estimate

In this section we want to analyze the maximal time of existence of the evolution equations (3.1), (3.2) by application of the pointwise maximum principle. In other words, we want to determine how far we can go beyond the short-time solution. From the abstract theory of semi-linear parabolic partial differential equations one knows that the lower order terms on the right hand side determine T_{max}. In the case of the regularized Dirac-harmonic map heat flow this means that we have to derive estimates of $\psi_t, d\phi_t$ and $\tilde{\nabla}\psi_t$.
For the rest of this section, we will rescale the variable t in the spinor ψ_t, by $t \to \varepsilon t$. Since the evolution equation for the spinor ψ_t is linear, we can directly apply the maximum principle to get a first estimate.

Lemma 3.11. *Let $\psi_t \in C^\infty(M\times[0,T), \Sigma M\otimes\phi_t^{-1}TN)$ be a solution of (3.2) and perform the rescaling $t \to \varepsilon t$. Then the norm of the spinor ψ_t satisfies the following estimate:*

$$|\psi_t|^2 \leq e^{\frac{m}{2\varepsilon^2}t}|\psi_0|^2. \tag{3.7}$$

Proof. Taking into account the rescaling of t in the evolution equation for ψ_t, we directly find

$$\frac{\partial}{\partial t}\frac{1}{2}|\psi_t|^2 = \Delta\frac{1}{2}|\psi_t|^2 - |\tilde{\nabla}\psi_t|^2 - \frac{1}{\varepsilon}\langle\psi_t, \not{D}\psi_t\rangle.$$

Applying the Cauchy-Schwarz inequality ($|\not{D}\psi|^2 \leq m|\tilde{\nabla}\psi|^2$, where m is the dimension of the manifold M) and in addition Young's inequality, we get the estimate

$$\frac{\partial}{\partial t}\frac{1}{2}|\psi_t|^2 \leq \Delta\frac{1}{2}|\psi_t|^2 + \frac{m}{4\varepsilon^2}|\psi_t|^2.$$

Finally, we apply the maximum principle to the function $e^{-\frac{m}{2\varepsilon^2}t}|\psi_t|^2$. $\square$

With the help of the estimate on ψ_t, we now estimate $d\phi_t$ and $\tilde{\nabla}\psi_t$.

Lemma 3.12. *Let $(\phi_t, \psi_t) \in C^\infty(M \times [0,T), N) \times C^\infty(M \times [0,T), \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (3.1) and (3.2). Assume that we have performed the rescaling $t \to \varepsilon t$ in the evolution equation for ψ_t. Then for all $t \in [0,T)$ the quantity*

$$F_t := \frac{1}{2}(|d\phi_t|^2 + \varepsilon|\tilde{\nabla}\psi_t|^2) \tag{3.8}$$

satisfies the following inequality:

$$\frac{\partial F_t}{\partial t} \leq \Delta F_t + (C_1 + C_2 e^{C_3 t})(F_t + F_t^2), \tag{3.9}$$

where the constant C_1 depends on M, N and ε, the constant C_2 depends on M, N, ε and ψ_0 and the constant C_3 depends on M, ε. In particular, $C_2 = 0$ for $\psi = 0$.

Proof. Using the evolution equations for $|d\phi_t|^2$ and $|\tilde{\nabla}\psi_t|^2$, taking into account the rescaling, we find that the quantity F_t defined above, satisfies

$$\begin{aligned}
\frac{\partial}{\partial t}F_t \;=\; & \Delta F_t - |\nabla d\phi_t|^2 - |\tilde{\nabla}^2\psi_t|^2 \\
& + \langle R^N(d\phi_t(e_\alpha), d\phi_t(e_\beta))d\phi_t(e_\alpha), d\phi_t(e_\beta)\rangle - \langle d\phi_t(Ric^M(e_\alpha)), d\phi_t(e_\alpha)\rangle \\
& - \langle\nabla_{e_\beta}\mathcal{R}(\phi_t, \psi_t), d\phi_t(e_\beta)\rangle - \varepsilon\langle\nabla_{e_\beta}\mathcal{R}_c(\phi_t, \psi_t), d\phi_t(e_\beta)\rangle \\
& - \langle\tilde{\nabla}\psi_t, \tilde{\nabla}\not{D}\psi_t\rangle + \varepsilon\langle R^{E_1}(e_\alpha, e_\beta)\tilde{\nabla}_{e_\alpha}\psi_t, \tilde{\nabla}_{e_\beta}\psi_t\rangle + \varepsilon\langle\tilde{\nabla}_{e_\alpha}R^{E_2}(e_\alpha, e_\beta)\psi_t, \tilde{\nabla}_{e_\beta}\psi_t\rangle \\
& + \varepsilon\langle\mathcal{R}_c(\phi_t, \psi_t), \tau(\phi_t)\rangle - \varepsilon\langle\mathcal{R}(\phi_t, \psi_t), \mathcal{R}_c(\phi_t, \psi_t)\rangle - \varepsilon^2|\mathcal{R}_c(\phi_t, \psi_t)|^2
\end{aligned}$$

with the vector bundles $E_1 = T^*M \otimes \Sigma M \otimes \phi_t^{-1}TN$ and $E_2 = T^*M \otimes E_1$. First of all, we estimate

$$\begin{aligned}
|\langle R^N(d\phi_t(e_\alpha), d\phi_t(e_\beta))d\phi_t(e_\alpha), d\phi_t(e_\beta)\rangle| &\leq C|d\phi_t|^4, \\
|\langle d\phi_t(Ric^M(e_\alpha)), d\phi_t(e_\alpha)\rangle| &\leq C|d\phi_t|^2.
\end{aligned}$$

We choose a local orthonormal basis $\{e_\alpha\}$ on M such that $\nabla_{e_\alpha} e_\beta = 0$ at a considered point and compute

$$\begin{aligned}
\nabla_{e_\beta}\mathcal{R}(\phi,\psi) &= \frac{1}{2}(\nabla_{d\phi(e_\beta)}R^N)(e_\alpha\cdot\psi,\psi)d\phi(e_\alpha) + R^N(e_\alpha\cdot\psi,\tilde{\nabla}_{e_\beta}\psi)d\phi(e_\alpha) \\
&\quad +\frac{1}{2}R^N(e_\alpha\cdot\psi,\psi)\nabla_{e_\beta}d\phi(e_\alpha), \\
\nabla_{e_\beta}\mathcal{R}_c(\phi,\psi) &= (\nabla_{d\phi(e_\beta)}R^N)(\tilde{\nabla}_{e_\alpha}\psi,\psi)d\phi(e_\alpha) + R^N(\tilde{\nabla}_{e_\beta}\tilde{\nabla}_{e_\alpha}\psi,\psi)d\phi(e_\alpha) \\
&\quad +R^N(\tilde{\nabla}_{e_\alpha}\psi,\tilde{\nabla}_{e_\beta}\psi)d\phi(e_\alpha) + R^N(\tilde{\nabla}_{e_\alpha}\psi,\psi)\nabla_{e_\beta}d\phi(e_\alpha)
\end{aligned}$$

and therefore we get the estimates

$$\begin{aligned}
|\langle\nabla_{e_\beta}\mathcal{R}(\phi_t,\psi_t),d\phi_t(e_\beta)\rangle| &\leq C(|\psi_t|^2|d\phi_t|^3 + |\psi_t||\tilde{\nabla}\psi_t||d\phi_t|^2 + |\psi_t|^2|\nabla d\phi_t||d\phi_t|), \\
|\langle\nabla_{e_\beta}\mathcal{R}(\phi_t,\psi_t),d\phi_t(e_\beta)\rangle| &\leq C(|\psi_t||\tilde{\nabla}\psi_t||d\phi_t|^3 + |\psi_t||\tilde{\nabla}^2\psi_t||d\phi_t|^2 \\
&\quad +|\tilde{\nabla}\psi_t|^2|d\phi_t|^2 + |\psi_t||\tilde{\nabla}\psi_t||\nabla d\phi_t||d\phi_t|).
\end{aligned}$$

The terms involving the curvature of the vector bundles E_1 and E_2 can be estimated by

$$\begin{aligned}
|\langle R^{E_1}(e_\alpha,e_\beta)\tilde{\nabla}_{e_\alpha}\psi_t,\tilde{\nabla}_{e_\beta}\psi_t\rangle| &\leq C(|\tilde{\nabla}\psi_t|^2 + |\tilde{\nabla}\psi_t|^2|d\phi_t|^2), \\
|\langle\tilde{\nabla}_{e_\alpha}R^{E_2}(e_\alpha,e_\beta)\psi_t,\tilde{\nabla}_{e_\beta}\psi_t\rangle| &\leq C(|\psi||\tilde{\nabla}\psi_t| + |\psi_t||\tilde{\nabla}\psi_t||d\phi_t|^3 + |\tilde{\nabla}\psi_t|^2 \\
&\quad +|\psi_t||\tilde{\nabla}\psi_t||\nabla d\phi_t||d\phi_t| + |d\phi_t|^2|\tilde{\nabla}\psi_t|^2).
\end{aligned}$$

Again, by Young's inequality, we find

$$\begin{aligned}
\varepsilon\langle\mathcal{R}_c(\phi_t,\psi_t),\tau(\phi_t)\rangle - \varepsilon\langle\mathcal{R}(\phi_t,\psi_t),\mathcal{R}_c(\phi_t,\psi_t)\rangle - \varepsilon^2|\mathcal{R}(\phi_t,\psi_t)|^2 \\
\leq \frac{1}{8}|\tau(\phi_t)|^2 + C|\psi_t|^4|d\phi_t|^2.
\end{aligned}$$

Estimating very roughly, using Young's inequality once more to get rid of the second order derivatives on the right hand side, and applying the estimate on the norm of ψ, we finally get

$$\frac{\partial F_t}{\partial t} \leq \Delta F_t + (C_1 + C_2 e^{C_3 t})(F_t + F_t^2),$$

which proves the assertion. □

The next Lemma is the analogue of [CD90], p. 570, Lemma 2.1 for the regularized Dirac-harmonic map heat flow. It provides an estimate on the maximal time of existence in terms of $\varepsilon, M, N, \psi_0, d\phi_0$ and $\tilde{\nabla}\psi_0$.

Lemma 3.13. *Let $(\phi_t,\psi_t) \in C^\infty(M\times[0,T),N)\times C^\infty(M\times[0,T),\Sigma M\otimes\phi_t^{-1}TN)$ be a solution of (3.1) and (3.2). As long as*

$$t \leq \frac{1}{C_1 + C_2 e^{C_3 t}}\log\left(\frac{1+F_0}{F_0}\right), \tag{3.10}$$

the evolution equations (3.1) and (3.2) do not blow up. The constant C_1 depends on M, N and ε, the constant C_2 on M, N, ε and ψ_0. The constant C_3 depends on M, ε and we abbreviate $F_0 = F(\phi_0,\psi_0)$. In particular, $C_2 = 0$ for $\psi = 0$.

Proof. We use the inequality (3.9) derived in the last Lemma to obtain a first estimate for the function F_t. Therefore, we drop the term involving the Laplacian and consider the ordinary differential equation

$$\frac{\partial F_t}{\partial t} \leq (C_1 + C_2 e^{C_3 t})(F_t + F_t^2), \qquad F_0 = F(\phi_0, \psi_0),$$

which can be integrated as

$$F_t \leq \frac{F_0}{(1+F_0)e^{-C_1 t - C_2 e^{C_3 t} t} - F_0},$$

giving the result. □

In principle, we can now estimate the maximal time of existence T_{max} of the regularized Dirac-harmonic map heat flow by solving (3.10).

Remark 3.14. With the help of the last Lemma we can draw a comparison between the usual harmonic map heat flow and the regularized Dirac-harmonic map heat flow. Instead of the quantity F_t, the behaviour of the harmonic map heat flow is governed by the evolution of $e(\phi) = \frac{1}{2}|d\phi_t|^2$. Namely,

$$\frac{\partial e(\phi)}{\partial t} \leq \Delta e(\phi) + C(e(\phi) + e(\phi)^2).$$

This corresponds to the inequality (3.9) for the function F_t with $C_2 = 0$. Comparing the inequality for $e(\phi)$ with the evolution equation (3.9) for the function F_t, we realize that in the inequality for F_t there is an additional exponential factor on the right hand side. Consequently, we expect that the regularized Dirac-harmonic map heat flow behaves worse than the usual harmonic map heat flow.
Another difference between the two heat flows, we would like to mention, is that the non-positive curvature condition on N controls all non-linear terms in the harmonic map heat flow. On the contrary the worst non-linearities in the regularized Dirac-harmonic map heat flow originate in the curvature terms that couple the fields ϕ and ψ.
Note, that by the above argument, the evolution equations for (ϕ_t, ψ_t) do not necessarily have to blow-up in finite time. We have only seen that we cannot expect to establish a global existence result by the pointwise maximum principle.

3.4. Some Differential Geometry

The aim of this section is to develop formulas for composite Dirac-harmonic maps. In the end, we want to apply the Nash embedding theorem to isometrically embed the manifold N in some $\mathbb{R}^q$ of sufficiently large dimension via the map ι.
To this end, let N' be another compact Riemannian manifold. Assume that $f: N \to N'$ such that we can consider the composite function $\phi' = f \circ \phi : M \to N'$.

Lemma 3.15 (Composite tension fields)**.** *Let M, N, N' be Riemannian manifolds. Assume that $\phi\colon M \to N$ and $f : N \to N'$. Then the tension field τ applied to the composite map $f \circ \phi$ satisfies*

$$\tau(f \circ \phi) = \operatorname{Tr} \nabla df(d\phi, d\phi) + df(\tau(\phi)). \tag{3.11}$$

Proof. For $X, Y \in \Gamma(TM)$, we compute

$$\begin{aligned}
\nabla d(f \circ \phi)(X, Y) &= \nabla_X (df \circ d\phi(Y)) - d(f \circ \phi)(\nabla_X Y) \\
&= (\nabla_{d\phi(X)} df)(df(Y)) + df(\nabla_X d\phi(Y)) - df \circ d\phi(\nabla_X Y) \\
&= \nabla df(d\phi(X), d\phi(Y)) + df(\nabla d\phi(X, Y)).
\end{aligned}$$

Taking the trace, yields the result. □

From the point of view of the manifold N, the spinor ψ behaves like a tangent vector. This is the reason why we have to use the differential of the map f if we want to define a spinor along the composite map ϕ'. More precisely, if ψ is a spinor along ϕ, then ψ' is a spinor along ϕ'. Both are related by

$$\psi' = df(\psi). \tag{3.12}$$

We now derive the relations between the Dirac operator and the connection Laplacian acting on ψ and ψ'.

Lemma 3.16 (Composite operators on $\Sigma M \otimes \phi^{-1}TN$)**.** *Let M, N, N' be Riemannian manifolds, where $N \subset N'$ is a Riemannian submanifold. Suppose that $\phi\colon M \to N$, $f\colon N \to N'$ and set $\phi' = f \circ \phi : M \to N'$. Let ψ be a spinor along ϕ and $\psi' = df(\psi)$ a spinor along ϕ'. Then the following relations hold*

$$\begin{aligned}
\frac{\tilde{\nabla}'\psi'}{\partial t} &= df(\frac{\tilde{\nabla}\psi}{\partial t}) + A(\frac{\partial \phi}{\partial t}, \psi), && (3.13)\\
\not{D}'\psi' &= df(\not{D}\psi) + A(d\phi(e_\alpha), e_\alpha \cdot \psi), && (3.14)\\
\tilde{\Delta}'\psi' &= df(\tilde{\Delta}\psi) + 2A(d\phi(e_\alpha), \tilde{\nabla}_{e_\alpha}\psi) \\
&\quad + (\nabla_{e_\alpha} A)(d\phi(e_\alpha), \psi) + A(\tau(\phi), \psi), && (3.15)
\end{aligned}$$

where A denotes the second fundamental form of f in N'.

Proof. We choose a local orthonormal basis $\{e_\alpha\}$ on M such that $\nabla_{e_\alpha} e_\beta = 0$ at a considered point. To prove the first statement, we compute

$$\begin{aligned}
\frac{\tilde{\nabla}'\psi'}{\partial t} &= (\nabla_{d\phi(\partial_t)} df)(\psi) + df(\frac{\tilde{\nabla}\psi}{\partial t}) \\
&= A(\frac{\partial \phi}{\partial t}, \psi) + df(\frac{\tilde{\nabla}\psi}{\partial t}),
\end{aligned}$$

the second one follows similarly (see [CJLW05], p. 64):

$$\begin{aligned}
\not{D}'\psi' &= e_\alpha \cdot \tilde{\nabla}'_{e_\alpha}(df(\psi)) \\
&= (\nabla_{d\phi(e_\alpha)} df)(e_\alpha \cdot \psi) + df(\not{D}\psi) \\
&= A(d\phi(e_\alpha), e_\alpha \cdot \psi) + df(\not{D}\psi).
\end{aligned}$$

Finally, we compute

$$\begin{aligned}\tilde{\nabla}'^*_{e_\alpha}\tilde{\nabla}'_{e_\alpha}\psi' &= \tilde{\nabla}'^*_{e_\alpha}\Big(A(d\phi(e_\alpha),\psi)+df(\tilde{\nabla}_{e_\alpha}\psi)\Big)\\ &= df(\tilde{\nabla}^*_{e_\alpha}\tilde{\nabla}_{e_\alpha}\psi)+2A(d\phi(e_\alpha),\tilde{\nabla}_{e_\alpha}\psi)+(\nabla_{e_\alpha}A)(d\phi(e_\alpha),\psi)+A(\tau(\phi),\psi),\end{aligned}$$

which completes the proof. ☐

Lemma 3.17 (Composite curvature terms)**.** *Let M, N, N' be Riemannian manifolds, where $N \subset N'$ is a Riemannian submanifold. Suppose that $\phi\colon M \to N$, $f : N \to N'$ and set $\phi' = f \circ \phi : M \to N'$. Let ψ be a spinor along ϕ and $\psi' = df(\psi)$ a spinor along ϕ'. The curvature terms in the Euler-Lagrange equations for ϕ then satisfy*

$$\mathcal{R}(\phi,\psi) = P(A(d\phi(e_\alpha),e_\alpha\cdot\psi),\psi)+\mathcal{R}'(\phi,\psi), \tag{3.16}$$

$$\mathcal{R}_c(\phi,\psi) = P(A(d\phi(e_\alpha),\tilde{\nabla}_{e_\alpha}\psi),\psi)-P(A(d\phi(e_\alpha),\psi),\tilde{\nabla}_{e_\alpha}\psi) +\mathcal{R}'_c(\phi,\psi), \tag{3.17}$$

where A denotes the second fundamental form of f in N' and P the shape operator.

Proof. The first formula was already computed in [CJLW05], p. 65. For $X \in \Gamma(TN)$, we calculate, using the equation of Gauss,

$$\begin{aligned}\langle\mathcal{R}(\phi,\psi)\rangle,X\rangle &= \frac{1}{2}\langle R^N(e_\alpha\cdot\psi,\psi)d\phi(e_\alpha),X\rangle\\ &= \frac{1}{2}\langle R^{N'}(e_\alpha\cdot\psi,\psi)d\phi(e_\alpha),X\rangle+\frac{1}{2}\langle A(e_\alpha\cdot\psi,d\phi(e_\alpha)),A(\psi,X)\rangle\\ &\quad -\frac{1}{2}\langle A(\psi,d\phi(e_\alpha)),A(e_\alpha\cdot\psi,X)\rangle\\ &= \frac{1}{2}\langle R^{N'}(e_\alpha\cdot\psi,\psi)d\phi(e_\alpha),X\rangle+\langle A(d\phi(e_\alpha),e_\alpha\cdot\psi),A(\psi,X)\rangle.\end{aligned}$$

We apply the fundamental relation of the shape operator

$$\langle P(\xi,X),Y\rangle_{TN} = \langle A(X,Y),\xi)\rangle_{TN'}$$

with $X, Y \in \Gamma(TN), \xi \in \Gamma(T^\perp N)$ and find

$$\langle A(d\phi(e_\alpha),e_\alpha\cdot\psi),A(\psi,X)\rangle_{TN'} = \langle P(A(d\phi(e_\alpha),e_\alpha\cdot\psi),\psi),X\rangle_{TN}.$$

Since X was arbitrary, we conclude

$$\mathcal{R}(\phi,\psi) = P(A(d\phi(e_\alpha),e_\alpha\cdot\psi),\psi)+\mathcal{R}'(\phi,\psi).$$

Regarding the second equation, we again use the equation of Gauss with $X \in \Gamma(TN)$

$$\begin{aligned}\langle\mathcal{R}_c(\phi,\psi)\rangle,X\rangle &= \langle R^{N'}(\tilde{\nabla}_{e_\alpha}\psi,\psi,)d\phi(e_\alpha),X\rangle+\langle A(\tilde{\nabla}_{e_\alpha}\psi,d\phi(e_\alpha)),A(\psi,X)\rangle\\ &\quad -\langle A(\psi,d\phi(e_\alpha)),A(\tilde{\nabla}_{e_\alpha}\psi,X)\rangle.\end{aligned}$$

Again, X being arbitrary and utilizing the fundamental relation of the shape operator once more, we find

$$\mathcal{R}_c(\phi,\psi) = P(A(d\phi(e_\alpha),\tilde{\nabla}_{e_\alpha}\psi),\psi) - P(A(d\phi(e_\alpha),\psi),\tilde{\nabla}_{e_\alpha}\psi) + \mathcal{R}'_c(\phi,\psi),$$

which completes the proof. □

In the following, we apply the embedding theorem of Nash. We consider the case that $N' = \mathbb{R}^q$ and $f = \iota$, where ι denotes the isometric embedding. Then, $u\colon M \to \mathbb{R}^q$ can be thought of as a vector-valued function. The spinor ψ turns into a vector of usual spinors $\psi = (\psi^1,\dots,\psi^q)$ with $\psi^i \in \Gamma(\Sigma M),\ i = 1,\dots,q$. The condition that ψ is along the map ϕ is encoded by

$$\sum_{i=1}^{q} \nu_i\psi^i = 0 \qquad \text{for a normal vector } \nu_i \in \mathbb{R}^q \text{ at } \phi(x).$$

With the help of these preparations, we now have the following

Corollary 3.18 (Applying the embedding theorem of Nash). *Assume that the manifold N' is isometrically embedded in some $\mathbb{R}^q$ via the map ι. For $u = \iota\circ\phi\colon M \to \mathbb{R}^q$ and $\psi' = d\iota(\psi)\colon M \to \Sigma M\otimes T\mathbb{R}^q$, the terms appearing in the evolution equation for ϕ acquire the form*

$$\begin{aligned}
d\iota(\tau(\phi)) &= \Delta u - \mathrm{I\!I}(du,du), && (3.18)\\
d\iota(\mathcal{R}(\phi,\psi) &= P(\mathrm{I\!I}(du(e_\alpha),e_\alpha\cdot\psi'),\psi'), && (3.19)\\
d\iota(\mathcal{R}_c(\phi,\psi) &= P(\mathrm{I\!I}(du(e_\alpha),\nabla_{e_\alpha}\psi'),\psi') - P(\mathrm{I\!I}(du(e_\alpha),\psi'),\nabla_{e_\alpha}\psi') && (3.20)\\
&\quad + B_u(du,\psi',du,\psi').
\end{aligned}$$

with

$$B_u(du,\psi,du,\psi) = \big(P(\mathrm{I\!I}_u(\partial_{y^i},\partial_{y^m}),\partial_{y^j})\Gamma^m_{kl} - P(\mathrm{I\!I}_u(\partial_{y^i},\partial_{y^j}),\partial_{y^m})\Gamma^m_{kl}\big)\,\frac{\partial u^i}{\partial x_\alpha}\frac{\partial u^k}{\partial x_\alpha}\psi'^l\psi'^j.$$

Similarly, we have for the terms in the evolution equation of ψ

$$\begin{aligned}
d\iota(\frac{\tilde{\nabla}\psi}{\partial t}) &= \frac{\nabla\psi'}{\partial t} - \mathrm{I\!I}(\frac{\partial u}{\partial t},\psi'), && (3.21)\\
d\iota(\tilde{\not{D}}\psi) &= \not{\partial}\psi' - \mathrm{I\!I}(du(e_\alpha),e_\alpha\cdot\psi'), && (3.22)\\
d\iota(\tilde{\Delta}\psi) &= \Delta\psi' - 2\mathrm{I\!I}(du(e_\alpha),\tilde{\nabla}_{e_\alpha}\psi') - (\nabla_{e_\alpha}\mathrm{I\!I})(du(e_\alpha),\psi') - \mathrm{I\!I}(\tau(u),\psi'). && (3.23)
\end{aligned}$$

Here, $\mathrm{I\!I}$ denotes the second fundamental form of N in $\mathbb{R}^q$. In addition, we identified ϕ with u.

Proof. The first two formulas for ϕ follow directly from the formulas for composite maps applied to $u = \iota\circ\phi$. Concerning the third formula, we use

$$\tilde{\nabla}_{e_\alpha}\psi = \nabla^{\Sigma M}_{e_\alpha}\psi^i\otimes\partial_{y^i} + \psi^k\otimes\frac{\partial\phi^j}{\partial x_\alpha}\Gamma^i_{jk}\partial_{y^i}$$

and the formula for composite maps.
The statements regarding the spinor ψ follow directly from the formulas for spinors along composite maps. $\square$

To simplify the notation, we will drop the superscript at the spinors from now on.
With the help of the last corollary, we can write down the evolution equations for the case that the target manifold N is embedded in some $\mathbb{R}^q$. Then, $u\colon M\times[0,T)\to\mathbb{R}^q$ and ψ is a spinor along u. More precisely, $\psi\in\Gamma(\Sigma M\otimes T\mathbb{R}^q)$. Now, the function u satisfies the following equation:

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)u=&-\mathrm{I\!I}_u(du,du)-P(\mathrm{I\!I}(du(e_\alpha),e_\alpha\cdot\psi),\psi)+\varepsilon P(\mathrm{I\!I}(du(e_\alpha),\psi),\nabla_{e_\alpha}\psi)\\&-\varepsilon P(\mathrm{I\!I}(du(e_\alpha),\nabla_{e_\alpha}\psi),\psi)-\varepsilon B(du,\psi,du,\psi)\end{aligned}\tag{3.24}$$

with the initial condition $u_0=\iota(\phi_0)$. For the spinor $\psi\in\Gamma(\Sigma M\otimes T\mathbb{R}^q)$, we get

$$\begin{aligned}\left(\frac{\nabla}{\partial t}-\varepsilon\Delta\right)\psi=&-\not{\partial}\psi+\mathrm{I\!I}(du(e_\alpha),e_\alpha\cdot\psi)+\mathrm{I\!I}(\frac{\partial u}{\partial t},\psi)-2\varepsilon\mathrm{I\!I}(du(e_\alpha),\tilde{\nabla}_{e_\alpha}\psi)\\&-\varepsilon(\nabla_{e_\alpha}\mathrm{I\!I})(du(e_\alpha),\psi))-\varepsilon\mathrm{I\!I}(\tau(u),\psi)\end{aligned}\tag{3.25}$$

with the initial condition $\psi_0'=d\iota(\psi_0)$.

Remark 3.19. It is not hard to establish the equivalence of the initial value problems (3.1), (3.2) and (3.24), (3.25). This can be achieved by the methods presented in [Nis02], Prop. 4.6, p. 133. The first part of the equivalence follows directly by considering the formulas for composite maps and spinors along them. The other part of the equivalence uses the canonical projection into a tubular neighbourhood.

In the following, we will address the problem how to compare spinors in a reasonable way. More precisely, we want to get an estimate on expressions like $\psi-\chi$, where ψ and χ are sections in different vector bundles. The classical references dealing with this issue are [BG92] and [BGM05], but we will resolve the problem by introducing a suitable notion of parallel transport. For two points $y_1,y_2\in N$ with $d^N(y_1,y_2)<i_N$, where $d^N(y_1,y_2)$ denotes the Riemannian distance between y_1 and y_2 and i_N the injectivity radius of N, there exists a unique length minimizing geodesic, which depends smoothly on y_1 and y_2. By

$$P_{y_1,y_2}\colon T_{y_1}N\to T_{y_2}N$$

we denote the parallel transport along this geodesic. The map P_{y_1,y_2} also depends smoothly on y_1 and y_2. For $\phi_1,\phi_2\in C^\infty(M,N)$, we define the operator

$$T_{\phi_1,\phi_2}(x)\colon E\otimes\phi_1^{-1}TN\to E\otimes\phi_2^{-1}TN$$

by

$$T_{\phi_1,\phi_2}(x):=\mathbb{1}_E\otimes P_{\phi_1(x),\phi_2(x)}.\tag{3.26}$$

Here, E is a vector bundle over the manifold M. Later, E will mostly be ΣM or T^*M. The map $T_{\phi_1,\phi_2}(x)$ acts trivially on E and by parallel transport on the tangent bundle in N. The next Lemma contains some important properties of the map $T_{\phi_1,\phi_2}(x)$. Some of these were already proven in [Iso12], Lemma 7.4.

Lemma 3.20 (Properties of the parallel transporter)**.** *Assume that M and N are compact Riemannian manifolds, $\phi_1, \phi_2 \in C^\infty(M,N)$, and let E be a vector bundle over M. Furthermore, let $\psi_1 \in \Gamma(\Sigma M \otimes \phi_1^{-1}TN)$ and $\psi_2 \in \Gamma(\Sigma M \otimes \phi_2^{-1}TN)$. Then the operator $T_{\phi_1,\phi_2}(x)$ defined in (3.26) has the following properties:*

1. *$T_{\phi_1,\phi_2}(x)$ is an isometry.*
2. *Interchanging covariant derivatives and the parallel transporter gives*
$$\begin{aligned}&\tilde{\nabla}_{e_\alpha}(T_{\phi_1,\phi_2}\psi_2)(x) - (T_{\phi_1,\phi_2}\tilde{\nabla}_{e_\alpha}\psi_2)(x)\\ &\quad = \mathbb{1}_E \otimes (\nabla_{d\phi_1(e_\alpha)}P_{\phi_1(x),\phi_2(x)})\psi_1(x) + \mathbb{1}_E \otimes (\nabla_{d\phi_2(e_\alpha)}P_{\phi_1(x),\phi_2(x)})\psi_1(x).\end{aligned} \tag{3.27}$$
3. *We have the following estimate*
$$\begin{aligned}|\tilde{\nabla}_{e_\alpha}\psi_1 - T_{\phi_1,\phi_2}\tilde{\nabla}_{e_\alpha}\psi_2| \leq & |\tilde{\nabla}_{e_\alpha}(\psi_1 - T_{\phi_1,\phi_2}\psi_2)|\\ &+ C|\phi_1(x)-\phi_2(x)|(|d\phi_1(e_\alpha)| + |d\phi_2(e_\alpha)|)|\psi(x)|\\ &+ C|d\phi_1(e_\alpha) - T_{\phi_1,\phi_2}d\phi_2(e_\alpha)||\psi(x)|.\end{aligned} \tag{3.28}$$

Proof. Parallel translation defines an isometry. Hence, the operator T_{ϕ_1,ϕ_2} also does. The second statement follows by a direct computation. Namely,

$$\tilde{\nabla}_{e_\alpha}(T_{\phi_1,\phi_2}\psi_1)(x) \quad = \quad (\nabla_{e_\alpha}T_{\phi_1,\phi_2})\psi_1(x) + (T_{\phi_1,\phi_2}\tilde{\nabla}_{e_\alpha}\psi_1)(x).$$

Concerning the third claim, we use the second statement and rewrite

$$\begin{aligned}&\nabla_{d\phi_1(e_\alpha)}P_{\phi_1(x),\phi_2(x)} + \nabla_{d\phi_2(e_\alpha)}P_{\phi_1(x),\phi_2(x)}\\ &\qquad = \nabla_{d\phi_1(e_\alpha)}P_{\phi_1(x),\phi_2(x)-\phi_1(x)} + \nabla_{d\phi_1(e_\alpha)}P_{\phi_1(x),\phi_1(x)}\\ &\qquad\quad + \nabla_{d\phi_2(e_\alpha)}P_{\phi_1(x),\phi_1(x)-\phi_2(x)} + \nabla_{d\phi_1(e_\alpha)}P_{\phi_1(x),\phi_1(x)}.\end{aligned}$$

Estimating the right hand side and using $\nabla_{d\phi_2(e_\alpha)}P_{\phi_1(x),\phi_1(x)} = -\nabla_{d\phi_1(e_\alpha)}P_{\phi_1(x),\phi_1(x)}$, the result follows. ∎

3.5. Short-time Existence

In this section we will set up the short-time existence for the regularized Dirac-harmonic map heat flow. We start by analyzing the general structure of the evolution equations

$$\frac{\partial\phi}{\partial t}(x,t) = \tau(\phi)(x,t) - \mathcal{R}(\phi,\psi)(x,t) - \varepsilon\mathcal{R}_c(\phi,\psi)(x,t), \tag{3.29}$$

$$\frac{\tilde{\nabla}\psi}{\partial t}(x,t) = \varepsilon\tilde{\Delta}\psi(x,t) - \not{D}\psi(x,t) \tag{3.30}$$

with $(x,t) \in M \times [0,T)$ and initial data (ϕ_0,ψ_0). To get a better understanding of the equations above, we compute the principal symbols of the differential operators involved

$$\sigma_2(\tau,\xi) = -|\xi|^2, \qquad \sigma_2(\varepsilon\tilde{\Delta},\xi) = -\varepsilon|\xi|^2, \qquad \sigma_1(\not{D},\xi) = \cdot\xi.$$

We conclude that the operators τ and $\tilde{\Delta}$ are uniformly elliptic, whereas the twisted Dirac operator $\not{D}$ is weakly elliptic. The connection Laplacian on the bundle $\Sigma M \otimes \phi^{-1}TN$ is a linear operator, whereas the tension field τ is a non-linear operator. In addition, we note that the principal symbols do not depend on t. Hence, the evolution equations do not change the defining structure of the operators $\tau, \tilde{\Delta}$ and $\not{D}$.
As soon as we start evolving, the connections on the bundles $\Sigma M \otimes \phi^{-1}TN$ and $\phi^{-1}TN$ become t-dependent. Thus, the operators $\tau, \tilde{\Delta}$ and $\not{D}$ also become t-dependent. But since the principal symbols do not depend on t, the t-dependence of, for example the Dirac-operator $\not{D}$, can be extracted in an endomorphism $V(t)$, such that we have the following splitting

$$\not{D}_t\psi_t = (\not{D}_0 + V(t))\psi_t.$$

Unfortunately, we cannot formulate any statement about the endomorphism $V(t)$, apart from the fact that $V(t)$ is small for t sufficiently small due to continuity.
The twisted spinors ψ take their values in the vector bundle $\Sigma M \otimes \phi^{-1}TN$. As long as we have enough control over ϕ_t, we can always use parallel transport in N defined before to identify

$$\Sigma M \otimes \phi_t^{-1}TN \cong \Sigma M \otimes \phi_0^{-1}TN.$$

Remark 3.21. When considering the operator $P_t := \varepsilon\tilde{\Delta}_t - \not{D}_t$, we remember that we can always choose the connection on $\Sigma M \otimes \phi_t^{-1}TN$ in such a way that the differential operator P is of the form

$$P_t = \tilde{\Delta}_t + K_t$$

with an endomorphism K_t, see [BGV92], p. 64, Prop. 2.5. But on the other hand, the connection varies as t varies. Thus, we cannot guarantee the splitting from above for all t without further assumptions. In addition, this splitting is not useful to derive estimates.

We want to establish the short-time existence of the coupled system (3.29), (3.30) via the Banach fixed-point theorem. Since the evolution equation for ϕ depends on ψ, and the evolution equation for ψ depends on ϕ, we have to solve the coupled system (ϕ,ψ) simultaneously.

Theorem 3.22 (Short-time existence). *Suppose that both (M,h) and (N,g) are compact Riemannian manifolds without boundary. Furthermore, assume that M is a Riemannian spin manifold with fixed spin structure. Then, for any*

$$(\phi_0,\psi_0)\in C^{2+\alpha}(M,N)\times C^{2+\alpha}(M,\Sigma M\otimes\phi_0^{-1}TN)$$

there exists a maximal time

$$0<T_{max}=T_{max}(M,N,\phi_0,\psi_0,\varepsilon,\alpha)<\infty,$$

such that the system (3.29), (3.30) admits a unique solution

$$(\phi_t,\psi_t)\in C^{2+\alpha,1+\alpha/2}(M\times[0,T_{\max}),N)\times C^{2+\alpha,1+\alpha/2}(M\times[0,T_{\max}),\Sigma M\otimes\phi_t^{-1}TN).$$

Proof. First of all, we rescale the time parameter t in the evolution equation for the spinor ψ to get rid of the ε in front of the Laplacian. More precisely, we rescale $t\to\varepsilon t$, such that ψ solves

$$\frac{\tilde\nabla}{\partial t}\psi(x,\varepsilon t)=\tilde\Delta\psi(x,\varepsilon t)-\frac{1}{\varepsilon}\not{D}\psi(x,\varepsilon t).$$

For the further analysis, we write the tension field $\tau(\phi)$ in the following form (see for example [Str88a], p. 294):

$$\tau(\phi)=\Delta\phi+\Gamma(\phi)(d\phi,d\phi).$$

We set $Q=M\times[0,T)$, $E=\Sigma M\otimes\phi^{-1}TN$ and define the spaces

$$\begin{aligned}X&=\{(\phi,\psi)\in C^{1+\alpha}(Q,N)\times C^{1+\alpha}(Q,E)\mid\phi(x,0)=\phi_0(x),\psi(x,0)=\psi_0(x)\},\\Y&=\{(\phi,\psi)\in C^{\alpha}(Q,N)\times C^{\alpha}(Q,E)\mid\phi(x,0)=\phi_0(x),\psi(x,0)=\psi_0(x)\}\end{aligned}$$

with $0<\alpha<1$. With these preparations, the combined system (3.29), (3.30) can be written as

$$\frac{\nabla}{\partial t}(\phi_t,\psi_t)=\Delta(\phi_t,\psi_t)+P(\phi_t,\psi_t)\tag{3.31}$$

with the map $P(\phi,\psi)\colon X\to Y$ defined by

$$P(\phi,\psi)=\big(\Gamma(\phi)(d\phi,d\phi)-\mathcal{R}(\phi,\psi)-\varepsilon\mathcal{R}_c(\phi,\psi),-\frac{1}{\varepsilon}\not{D}\psi\big).$$

Moreover, for $\delta>0$ we define the space

$$\begin{aligned}X_\delta=\{&(\phi_t,\psi_t)\in C^0(M\times[0,\delta),N)\times C^0(M\times[0,\delta),E_t)\\&\mid(\phi(\cdot,t),\psi(\cdot,t))\in C^1(M\times[0,\delta),N)\times C^1(M\times[0,\delta),E_t),\\&(\phi(\cdot,0),\psi(\cdot,0))=(\phi_0,\psi_0),\ 0\le t\le\delta\}.\end{aligned}$$

We equip the space X_δ with the following norm

$$\begin{aligned}|(\phi_t,\psi_t)|_{X_\delta}\ :=\ &\sup_{t\in[0,\delta)}|\phi(\cdot,t)|_{C^0(M\times[0,\delta),N)}+\sup_{t\in[0,\delta)}|d\phi(\cdot,t)|_{C^0(M\times[0,\delta),N)}\\&+\sup_{t\in[0,\delta)}|\psi(\cdot,t)|_{C^0(M\times[0,\delta),E_t)}+\sup_{t\in[0,\delta)}|\tilde\nabla\psi(\cdot,t)|_{C^0(M\times[0,\delta),E_t)}\end{aligned}$$

with the vector bundle $E_t := \Sigma M \otimes \phi_t^{-1} TN$. In addition, we define the integral operator $W : X_\delta \to X_\delta$ by

$$(\phi,\psi)(x,t) = e^{t\Delta}(\phi_0,\psi_0)(x) + \int_0^t e^{(t-s)\Delta} P(\phi,\psi)(x,s)ds := W(\phi,\psi)(x,t) \tag{3.32}$$

with the operator $P(\phi,\psi)$ from above. If one wants to be more precise, one would have to use an appropriate heat kernel in the integral operator above. Since we are only interested in the short-time asymptotic, the above definition suffices for our needs.
Consequently, finding a fixed-point for the operator $W(\phi,\psi)$ is equivalent to establishing a short-time solution to the system (3.29), (3.30). For $\gamma > 0$ fixed, we set

$$B_\gamma = \{(\phi,\psi) \in X_\delta \mid |(\phi,\psi)(x,t) - (\phi_0,\psi_0)(x)|_{X_\delta} \leq \gamma\},$$

where we use the isometric embedding ι and the parallel transporter T_{ϕ_1,ϕ_2} to compare $(\phi,\psi)(x,t)$ with $(\phi_0,\psi_0)(x)$. The operator norm of the Laplacian can be estimated as, (see [Tay11], p. 274)

$$|e^{t\Delta}|_{\mathcal{L}(C^\alpha, C^{\alpha+s})} \leq C_s t^{-\frac{s}{2}}, \qquad 0 < t \leq 1$$

for $s > 0$. In the following, we want to show that for any $\gamma > 0$, there exists a $\delta > 0$ such that

1. $W(\phi_t,\psi_t)\colon B_\gamma \to B_\gamma$,
2. $W(\phi_t,\psi_t)$ is a contraction, i.e. there exists a $\beta \in (0,1)$ such that

$$|W(\phi_1,\psi_1) - W(\phi_2,\psi_2)|_{X_\delta} \leq \beta|(\phi_1,\psi_1) - (\psi_2,\psi_2)|_{X_\delta},$$

where we again use the parallel transporter T_{ϕ_1,ϕ_2} and the isometric embedding ι.

To prove the first assertion, we choose the time T_1 such that

$$|e^{t\Delta}(\phi_0,\psi_0) - (\phi_0,\psi_0)|_{X_\delta} \leq \frac{\gamma}{2}, \qquad t \in [0,T_1).$$

For $(\phi_0,\psi_0) \in B_\gamma$ we find that $|P(\phi,\psi)(s)|_{X_\delta} \leq C$ for $s \in [0,T_1)$. Hence, we deduce that

$$\left| \int_0^t e^{(t-s)\Delta} P(\phi,\psi)(x,s)ds \right|_{X_\delta} \leq Ct|e^{t\Delta}| \leq Ct^{1-\frac{s}{2}}.$$

We choose $T_2 \leq T_1$ small enough, such that the above expression is bounded by $\frac{\gamma}{2}$ for $t \in [0,T_1)$. But then for $T \leq T_2$, we have $W(\phi,\psi)\colon B_\gamma \to B_\gamma$.
Proving the second statement is more involved. We assume both $(\phi_1,\psi_1),(\phi_2,\psi_2) \in B_\gamma$, where ψ_1 is a spinor along ϕ_1 and ψ_2 is a spinor along ϕ_2. To show that the operator $W(\phi,\psi)$ defines a contraction, we calculate

$$(W(\phi_1,\psi_1) - W(\phi_2,\psi_2))(x,t) = \int_0^t e^{(t-s)\Delta}(P(\phi_1,\psi_1) - P(\phi_2,\psi_2))(x,s)ds$$

and, therefore, we can estimate

$$|W(\phi_1,\psi_1)-W(\phi_2,\psi_2)|_{X_\delta}\leq t|e^{t\Delta}|_{\mathcal{L}(C^\alpha,C^{\alpha+s})}|P(\phi_1,\psi_1)-T_{\phi_1,\phi_2}(P(\phi_2,\psi_2))|_{X_\delta}.$$

To estimate $|P(\phi_1,\psi_1)-T_{\phi_1,\phi_2}(P(\phi_2,\psi_2))|_{X_\delta}$, we have to deal with the following expressions:

$$\begin{aligned}
I_1 &:= \Gamma(\phi_1)(d\phi_1,d\phi_1)-T_{\phi_1,\phi_2}(\Gamma(\phi_2)(d\phi_2,d\phi_2)),\\
I_2 &:= \mathcal{R}(\phi_1,\psi_1)-T_{\phi_1,\phi_2}(\mathcal{R}(\phi_2,\psi_2)),\\
I_3 &:= \mathcal{R}_c(\phi_1,\psi_1)-T_{\phi_1,\phi_2}(\mathcal{R}_c(\phi_2,\psi_2)),\\
I_4 &:= \not{D}\psi_1-T_{\phi_1,\phi_2}(\not{D}\psi_2).
\end{aligned}$$

We start by rewriting the I_1 term

$$\begin{aligned}
&\Gamma(\phi_1)(d\phi_1,d\phi_1)-T_{\phi_1,\phi_2}\Gamma(\phi_2)(d\phi_2,d\phi_2)\\
&\qquad=(\Gamma(\phi_1)-T_{\phi_1,\phi_2}\Gamma(\phi_2))(d\phi_1,d\phi_1)+\Gamma(\phi_1)(T_{\phi_1,\phi_2}d\phi_2-d\phi_1,d\phi_1)\\
&\qquad\quad+\Gamma(\phi_1)(d\phi_1-T_{\phi_1,\phi_2}d\phi_2,T_{\phi_1,\phi_2}d\phi_2).
\end{aligned}$$

Applying the mean value theorem to the first contribution, we find

$$\begin{aligned}
&|\Gamma(\phi_1)(d\phi,d\phi)-T_{\phi_1,\phi_2}\Gamma(\phi_2)(d\phi_2,d\phi_2)|_{X_\delta}\\
&\qquad\leq C(|d\phi_1|^2|\phi_1-\phi_2|+(|d\phi_1|+|d\phi_2|)|d\phi_1-T_{\phi_1,\phi_2}d\phi_2|)\\
&\qquad\leq C|\phi_1-\phi_2|_{X_\delta}.
\end{aligned}$$

To estimate I_2, we rewrite

$$\begin{aligned}
&\mathcal{R}(\phi_1,\psi_1)-T_{\phi_1,\phi_2}\mathcal{R}(\phi_2,\psi_2)\\
&\qquad=\frac{1}{2}\left(R^N(e_\alpha\cdot\psi_1,\psi_1)d\phi_1(e_\alpha)-T_{\phi_1,\phi_2}(R^N(e_\alpha\cdot\psi_2,\psi_2)d\phi_2(e_\alpha))\right)\\
&\qquad=\frac{1}{2}\big(R^N(e_\alpha\cdot(\psi_1-T_{\phi_1,\phi_2}\psi_2),\psi_1)d\phi_1(e_\alpha)\\
&\qquad\quad+R^N(e_\alpha\cdot T_{\phi_1,\phi_2}\psi_2,T_{\phi_1,\phi_2}\psi_2)(d\phi_1(e_\alpha)-T_{\phi_1,\phi_2}d\phi_2(e_\alpha))\\
&\qquad\quad+R^N(e_\alpha\cdot T_{\phi_1,\phi_2}\psi_2,\psi_1-T_{\phi_1,\phi_2}\psi_2)d\phi_1(e_\alpha)\big),
\end{aligned}$$

and hence we may estimate

$$\begin{aligned}
&|\mathcal{R}(\phi_1,\psi_1)-T_{\phi_1,\phi_2}\mathcal{R}(\phi_2,\psi_2)|_{X_\delta}\\
&\qquad\leq C(|d\phi_1||\psi_1-T_{\phi_1,\phi_2}\psi_2||\psi_1|+|d\phi_1(e_\alpha)-T_{\phi_1,\phi_2}d\phi_2(e_\alpha)||\psi_2|^2\\
&\qquad\quad+|\psi_2||d\phi_1||\psi_1-T_{\phi_1,\phi_2}\psi_2|)\\
&\qquad\leq C|(\phi_1,\psi_1)-T_{\phi_1,\phi_2}(\phi_2,\psi_2)|_{X_\delta}.
\end{aligned}$$

To estimate I_3, we rewrite

$$\begin{aligned}
&\mathcal{R}_c(\phi_1,\psi_1) - T_{\phi_1,\phi_2}\mathcal{R}_c(\phi_2,\psi_2) \\
&\qquad = R^N(\psi_1,\tilde{\nabla}_{e_\alpha}\psi_1)d\phi_1(e_\alpha) - T_{\phi_1,\phi_2}(R^N(\psi_2,\tilde{\nabla}_{e_\alpha}\psi_2)d\phi_2(e_\alpha)) \\
&\qquad = R^N(\psi_1 - T_{\phi_1,\phi_2}\psi_2,\tilde{\nabla}_{e_\alpha}\psi_1)d\phi_1(e_\alpha) \\
&\qquad\quad + R^N(T_{\phi_1,\phi_2}\psi_2,\tilde{\nabla}_{e_\alpha}\psi_1)(d\phi_1(e_\alpha) - T_{\phi_1\phi_2}d\phi_2(e_\alpha)) \\
&\qquad\quad + R^N(T_{\phi_1,\phi_2}\psi_2,(\tilde{\nabla}_{e_\alpha}\psi_1 - T_{\phi_1,\phi_2}\tilde{\nabla}_{e_\alpha}\psi_2))d\phi_2(e_\alpha),
\end{aligned}$$

and again we estimate

$$\begin{aligned}
&|\mathcal{R}_c(\phi_1,\psi_1) - T_{\phi_1,\phi_2}\mathcal{R}_c(\phi_2,\psi_2)|_{X_\delta} \\
&\qquad \leq C(|d\phi_1||\psi_1 - T_{\phi_1\phi_2}\psi_2||\tilde{\nabla}\psi_1| + |d\phi_1(e_\alpha) - T_{\phi_1\phi_2}d\phi_2(e_\alpha)||\psi_2||\tilde{\nabla}\psi_1| \\
&\qquad\quad + |\psi_2||d\phi_2||\tilde{\nabla}\psi_1 - T_{\phi_1,\phi_2}(\tilde{\nabla}\psi_2)|) \\
&\qquad \leq C|(\phi_1,\psi_1) - T_{\phi_1,\phi_2}(\phi_2,\psi_2)|_{X_\delta},
\end{aligned}$$

where we interchanged the covariant derivative with the parallel transporter in the last step, applying Lemma 3.20.
To estimate the I_4 term, we note that

$$\not{D}\psi_1 - T_{\phi_1,\phi_2}(\not{D}\psi_2) = e_\alpha\cdot(\tilde{\nabla}_{e_\alpha}\psi_1 - T_{\phi_1,\phi_2}\tilde{\nabla}_{e_\alpha}\psi_2).$$

Using Lemma 3.20 again, we find

$$|\not{D}\psi_1 - T_{\phi_1,\phi_2}(\not{D}\psi_2)|_{X_\delta} \leq |(\phi_1,\psi_1) - T_{\phi_1,\psi_2}(\phi_2,\psi_2)|_{X_\delta}.$$

Adding up I_1, I_2, I_3, I_4, we obtain

$$|P(\phi_1,\psi_1) - T_{\phi_1,\phi_2}(P(\phi_2,\psi_2))|_{X_\delta} \leq C|(\phi_1,\psi_1) - T_{\phi_1,\phi_2}(\phi_2,\psi_2)|_{X_\delta}.$$

Applying the bound on the operator norm of the Laplacian from above, we find

$$|W(\phi_1,\psi_1) - T_{\phi_1,\phi_2}(W(\phi_2,\psi_2))|_{X_\delta} \leq Ct^{1-\frac{s}{2}}|(\phi_1,\psi_1) - T_{\phi_1,\phi_2}(\phi_2,\psi_2)|_{X_\delta}.$$

Choosing t sufficiently small, we realize that the map $W(\phi,\psi)$ is a contraction mapping and thus has a unique fixed-point. The regularity of the solution is determined by (B.11). □

Remark 3.23. Now, that the existence of a short-time solution is guaranteed, we can scale back the time parameter in the evolution equation for ψ_t such that the ε is in front of the connection Laplacian $\tilde{\Delta}$. This of course also changes the value of T_{max}.

Theorem 3.24 (Regularity of short-time solution). *Let (M,h) and (N,g) be compact, oriented Riemannian manifolds. Moreover, we assume that M is a spin manifold with fixed spin structure. For $(\phi_0,\psi_0)\in C^{2+\alpha}(M,N)\times C^{2+\alpha}(M,\Sigma M\otimes\phi_0^{-1}TN)$ there exists a positive number $T_{max}=T_{max}(M,N,\varepsilon,\phi_0,\psi_0,\alpha)>0$ such that the system*

$$\begin{aligned}\frac{\partial\phi}{\partial t}(x,t) &= \tau(\phi)(x,t)-\mathcal{R}(\phi,\psi)(x,t)-\varepsilon\mathcal{R}_c(\phi,\psi)(x,t), \qquad (3.33)\\ \frac{\tilde\nabla\psi}{\partial t}(x,t) &= \varepsilon\tilde\Delta\psi(x,t)-\not{D}\psi(x,t)\end{aligned}$$

admits a solution, which is

$$\begin{aligned}\phi_t &\in C^{2+\alpha,1+\alpha/2}(M\times[0,T),N)\cap C^\infty(M\times(0,T),N),\\ \psi_t &\in C^{2+\alpha,1+\alpha/2}(M\times[0,T),\Sigma M\otimes\phi_t^{-1}TN)\cap C^\infty(M\times(0,T),\Sigma M\otimes\phi_t^{-1}TN).\end{aligned}$$

Proof. We choose local coordinates to analyze the regularity of the solution (ϕ_t,ψ_t), leading to the following system:

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\phi^m &= \Gamma^m_{jk}\frac{\partial\phi^j}{\partial x_\alpha}\frac{\partial\phi^k}{\partial x_\alpha}+R^m_{lij}\frac{\partial\phi^l}{\partial x_\alpha}\langle\psi^i,e_\alpha\cdot\psi^j\rangle_{\Sigma M} \qquad (3.34)\\ &\quad +\varepsilon R^m_{lij}\frac{\partial\phi^l}{\partial x_\alpha}\langle\nabla^{\Sigma M}_{e_\alpha}\psi^i,\psi^j\rangle_{\Sigma M}+\varepsilon R^m_{lij}\Gamma^j_{rs}\frac{\partial\phi^l}{\partial x_\alpha}\langle\psi^i,\psi^r\rangle_{\Sigma M}\frac{\partial\phi^s}{\partial x_\alpha},\\ \left(\frac{\nabla}{\partial t}-\varepsilon\Delta\right)\psi^m &= 2\varepsilon\Gamma^m_{ij}\nabla^{\Sigma M}_{e_\alpha}\psi^i\frac{\partial\phi^j}{\partial x_\alpha}+\varepsilon\Gamma^m_{ij,p}\psi^i\frac{\partial\phi^p}{\partial x_\alpha}\frac{\partial\phi^j}{\partial x_\alpha}+\varepsilon\Gamma^m_{ij}\psi^i\frac{\partial^2\phi^j}{\partial x_\alpha^2} \qquad (3.35)\\ &\quad +\varepsilon\Gamma^k_{ij}\Gamma^m_{ks}\psi^i\frac{\partial\phi^j}{\partial x_\alpha}\frac{\partial\phi^s}{\partial x_\alpha}-\not{\partial}\psi^m-\Gamma^m_{jk}\frac{\partial\phi^j}{\partial x_\alpha}e_\alpha\cdot\psi^k-\Gamma^m_{jk}\frac{\partial\phi^j}{\partial t}\psi^k.\end{aligned}$$

To establish the regularity statement, we apply estimates concerning the differentiability for solutions of parabolic partial differential equations (B.7). By assumption, we have that $\phi\in C^{2+\alpha,1+\alpha/2}(M\times[0,T),N)$ and also $\psi^m\in C^{2+\alpha,1+\alpha/2}(M\times[0,T),\Sigma M)$. Hence, the right hand side of (3.34) is in $C^{1+\alpha,\alpha/2}$, and by B.7, we get that $\phi\in C^{3+\alpha,1+\alpha/2}$. Using this estimate on ϕ, we can improve the regularity of ψ^m. Since the right hand side of (3.35) is in $C^{1+\alpha,\alpha/2}$, we get, again by B.7, that $\psi^m\in C^{3+\alpha,1+\alpha/2}$. Then, this estimate can be used to improve the regularity of ϕ. By iteration of this procedure, we find that the pair (ϕ_t,ψ_t) is smooth. □

Remark 3.25. Looking back at the the proof leading to the short-time existence, we clearly see that the existence interval $[0,T_{max})$ depends crucially on the regularizing parameter ε. Consequently, in general we have

$$\lim_{\varepsilon\to 0}T_{max}=0. \qquad (3.36)$$

In addition, it is not hard to see that we can only apply the estimates ensuring the regularity of ψ^m in (3.35) as long as $\varepsilon\neq 0$.

4. Dirac-harmonic Maps from Curves

4.1. Introduction and Results

Throughout this chapter, we assume that the manifold M is one-dimensional. The only compact one-dimensional manifold is the circle S^1. It is well known that Clifford multiplication on S^1 is given by multiplication with the imaginary unit i.
On S^1 there are two spin structures, which will be abbreviated by σ_1 and σ_2. In the case of σ_1, spinors can be identified as periodic complex-valued functions on S^1 satisfying $\psi(x+2\pi)=\psi(x)$. Regarding the other spin structure σ_2, spinors can be identified as antiperiodic complex-valued functions on S^1 satisfying $\psi(x+2\pi)=-\psi(x)$.
We will study the following set of equations

$$\frac{\partial \phi_t}{\partial t} = \tau(\phi_t)-\mathcal{R}(\phi_t,\psi_t)-\varepsilon\mathcal{R}_c(\phi_t,\psi_t), \tag{4.1}$$

$$\frac{\tilde{\nabla}\psi_t}{\partial t} = \varepsilon\tilde{\Delta}\psi_t-\not{D}\psi_t \tag{4.2}$$

together with the initial data $\phi(x,0)=\phi_0(x)$ and $\psi(x,0)=\psi_0(x)$.
The final purpose of this chapter is to prove the following

Theorem 4.1. *Assume that $M=S^1$ with fixed spin structure and N is a compact Riemannian manifold without boundary. Then for any smooth initial data (ϕ_0,ψ_0) and ε small, there exists a unique smooth solution of (4.1) and (4.2) for all $t\in[0,\infty)$.
If $\varepsilon\geq 1$, the evolution equations converge in $C^2(S^1,N)\times C^2(S^1,\Sigma S^1\otimes\phi_t^{-1}TN)$ to a regularized Dirac-harmonic map from $S^1\to N$.
Finally, the limit $\varepsilon\to 0$ exists and we obtain a smooth Dirac-harmonic map.*

Before addressing the general case, let us discuss some illuminating examples.

Example 4.2. Assume that $M=N=S^1$. Then the evolution equations for (ϕ_t,ψ_t) are given by

$$\begin{cases}\partial_t\phi(x,t)=\partial_x^2\phi(x,t),\\ \phi(x,0)=\phi_0(x),\\ \phi(0,t)=\phi(2\pi,t),\end{cases}\qquad\begin{cases}\partial_t\psi(x,t)=\varepsilon\partial_x^2\psi(x,t)-i\partial_x\psi(x,t),\\ \psi(x,0)=\psi_0(x),\\ \psi(0,t)=\pm\psi(2\pi,t).\end{cases}$$

The sign in the boundary condition for the spinor depends on the chosen spin structure. The fundamental solution for the heat equation on S^1 can be obtained by a Fourier decomposition and is given by

$$\xi(x,t)=\sum_{k=-\infty}^{\infty}a_k e^{-k^2t}e^{ikx},$$

with coefficients a_k. To incorporate the initial condition $\phi_0(x)$, we have to calculate the convolution of $\xi(x,t)$ with $\phi_0(x)$.

To derive the fundamental solution for the evolution equation for $\psi(x,t)$, we make a separation ansatz of the form $\chi(x,t) = A(x)B(t)$ leading to

$$\frac{\dot{B}(t)}{B(t)} = C = \frac{-iA'(x) + \varepsilon A''(x)}{A(x)}$$

with a constant C. By λ_k we denote the k-th eigenvalue of the Dirac operator on S^1, such that we get

$$\chi(x,t)_{\sigma_i} = \sum_{k=-\infty}^{\infty} b_k e^{i\lambda_k x} e^{-(\lambda_k + \varepsilon\lambda_k^2)t}, \qquad i = 1,2$$

with coefficients b_k. It is known, that for the spin structure σ_1, the eigenvalues of the Dirac operator are all integer numbers $\lambda_k = k$ whereas for the second spin structure σ_2 the eigenvalues are given by $\lambda_k = k + \frac{1}{2}$ with $k \in \mathbb{Z}$. Consequently, only σ_1 admits harmonic spinors. To incorporate the initial condition at $t = 0$, we again consider the convolution

$$\psi(x,t) = \frac{1}{2\pi}\int_0^{2\pi} \psi_0(y)\chi_{\sigma_i}(x-y,t)dy, \qquad i = 1,2.$$

We will analyze the evolution equation for different spin structures and different choices of $\psi_0(x)$. To this end, we first fix σ_1.

- If $\psi_0(x) = \sum_{k=0}^{\infty} b_k e^{ikx}$, then

  $$\psi(x,t) = \sum_{k=0}^{\infty} b_k e^{ikx} e^{(k-k^2\varepsilon)t}.$$

  Without any further restriction on b_k, ε or k, we cannot make any statement about $\psi(x,t)$ as $t \to \infty$.

- Since $-\frac{1}{4\varepsilon} \leq \lambda_k + \varepsilon\lambda_k^2 \leq \infty$, there exists a k_0 such that for all $k > k_0$ the expression $\lambda_k + \varepsilon\lambda_k^2 > 0$. If we now choose $\psi_0(x) = \sum_{k>k_0}^{\infty} b_k e^{-ikx}$, then

  $$\psi(x,t) = \sum_{k>k_0}^{\infty} b_k e^{ikx} e^{-(k+k^2\varepsilon)t}.$$

  In this case the limit $t \to \infty$ exists and we find

  $$\psi_\infty(x) = b_0.$$

As a second step we fix the other spin structure σ_2.

- If $\psi_0(x) = \sum_{k=0}^{\infty} b_k e^{i(k+\frac{1}{2})x}$, then again we cannot make a general statement as $t \to \infty$.
- If $\psi_0(x) = \sum_{k>k_0}^{\infty} b_k e^{-i(k+\frac{1}{2})x}$ with k_0 as before, then

$$\psi(x,t) = \sum_{k>k_0}^{\infty} b_k e^{-i(k+\frac{1}{2})x} e^{-((k+\frac{1}{2})+\varepsilon(k+\frac{1}{2})^2)t}.$$

  Again, for this special initial data, the limit $t \to \infty$ exists, but the limiting spinor ψ_∞ will vanish.

From the simple example above, we learned that the convergence of the evolution equation for the spinor $\psi(x,t)$ will depend both on the initial spinor $\psi_0(x)$ and the spin structure σ_i. Note that the above example could also be studied for $\varepsilon = 0$. The condition on the initial data can be thought of as an APS type condition, as it appears in the context of boundary value problem for Dirac operators.
If we drop the compactness assumption on M, we can find another example in which the evolution equations for ϕ and ψ can be solved explicitly.

Example 4.3. Assume that $M = N = \mathbb{R}$. In this case the evolution equations acquire the form

$$\begin{cases} \partial_t \phi(x,t) = \partial_x^2 \phi(x,t), \\ \phi(x,0) = \phi_0(x), \end{cases} \qquad \begin{cases} \partial_t \psi(x,t) = \varepsilon \partial_x^2 \psi(x,t) - i\partial_x \psi(x,t), \\ \psi(x,0) = \psi_0(x). \end{cases}$$

These equations can be integrated directly. For $\phi(x,t)$, we get the solution to the one-dimensional heat equation

$$\phi(x,t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}},$$

whereas for $\psi(x,t)$, we find the formal solution

$$\psi(x,t) = \frac{1}{\sqrt{t}} e^{\frac{1}{4\varepsilon}t + \frac{i}{2\varepsilon}x - \frac{x^2}{4\varepsilon t}}.$$

Concerning the solution $\psi(x,t)$, we realize that both limits $t \to \infty$ and $\varepsilon \to 0$ are not well defined. This is not surprising since M is non-compact.

Following the ideas from [CJLW06], p. 415, Prop. 2.2, we can give a construction for Dirac-harmonic maps from a closed curve to a Riemannian manifold N.

Proposition 4.4. *Assume that $M = S^1$ and N a compact Riemannian manifold. If ϕ is a harmonic map and the spinor ψ is of the form*

$$\psi = e_\alpha \cdot \chi \otimes d\phi(e_\alpha) \tag{4.3}$$

with χ being a harmonic spinor, then the pair (ϕ, ψ) is a Dirac-harmonic map.

Proof. By assumption ϕ is harmonic. Hence, we have $\tau(\phi) = 0$. By a direct computation, we find that $\mathcal{R}(\phi, \psi)$ is real. Inserting ψ as defined above into $\mathcal{R}(\phi, \psi)$ and using that the expression

$$\overline{\langle e_\alpha \cdot \chi, \chi\rangle} = \langle \chi, e_\alpha \cdot \chi\rangle = -\langle e_\alpha \cdot \chi, \chi\rangle$$

is purely imaginary, we conclude that $\mathcal{R}(\phi, \psi)$ vanishes.
On the other hand, applying the twisted Dirac operator $\not{D}$ to the spinor ψ yields

$$\begin{aligned} \not{D}\psi &= e_\alpha \cdot \tilde{\nabla}_{e_\alpha}(e_\alpha \cdot \chi \otimes d\phi(e_\alpha)) \\ &= e_\alpha \cdot \not{\partial}\chi \otimes d\phi(e_\alpha) - \chi \otimes \tau(\phi) \\ &= 0 \end{aligned}$$

by assumption, which concludes the proof. □

Let us make two comments about the solution constructed above. First of all, it is uncoupled. Secondly, a harmonic spinor on S^1 only exists for the trivial spin structure σ_1 and is given by a constant.

Lemma 4.5. *The Dirac-harmonic maps constructed above are not stable.*

Proof. To check if the Dirac-harmonic maps constructed above are stable, we have to insert the solution into the second variation of the energy functional (2.3), (2.4) and check if it is positive. It is known that for harmonic maps the second variation of the energy functional is positive if the target manifold N has non-positive curvature. In that case, the second term in (2.3) is positive. Let us evaluate the other terms in (2.3). We have seen that the expression $\langle e_\alpha \cdot \chi, \chi\rangle$ is purely imaginary. On the other hand, it is easy to check that

$$\begin{aligned} \langle \eta, R^N(e_\alpha \cdot \psi, \psi)\nabla_{e_\alpha}\eta\rangle &= R_{mlij}\eta^m \frac{\partial \eta^l}{\partial x_\alpha}\langle e_\alpha \cdot \psi^i, \psi^j\rangle, \\ \langle \eta, (\nabla_\eta R^N)(e_\alpha \cdot \psi, \psi)d\phi(e_\alpha)\rangle, &= R_{mlij;k}\eta^m\eta^k \frac{\partial \phi^l}{\partial x_\alpha}\langle e_\alpha \cdot \psi^i, \psi^j\rangle \end{aligned}$$

are both real. Consequently, both terms vanish when inserting the spinor ψ from above. Unfortunately, the term

$$\langle \eta, R^N(e_\alpha \cdot \psi, \psi^i \otimes \nabla_\eta \frac{\partial}{\partial y^i})d\phi(e_\alpha), \eta\rangle = R_{mlij}\Gamma^i_{rk}\eta^m \frac{\partial \phi^l}{\partial x_\alpha}\eta^k\langle e_\alpha \cdot \psi^j, \psi^r\rangle$$

does not vanish on the solution (ϕ, ψ) from above. The contribution from the second variation of $E(\phi, \psi)$ with respect to ψ, (2.4), does not have a definite sign either. □

One should expect that the second variation of the Dirac-energy $\int_M \langle \psi, \not{D}\psi\rangle dM$ is indefinite since the Dirac-energy itself is unbounded from below.

4.2. Energy Estimates by the Maximum Principle

From now on, we study the evolution equations in general. Therefore, we fix one of the two spin structures. Since for $M = S^1$ we have $\not{D}^2 = -\tilde{\nabla}^*\tilde{\nabla}$, we do not need to distinguish between the connection Laplacian and the square of the twisted Dirac operator $\not{D}$.
It turns out to be useful to rescale the t-parameter in the spinor $\psi(x,t)$ in the following way

$$\psi(x,t) \to \psi(x,\varepsilon t),$$

such that the pair (ϕ_t, ψ_t) solves the following set of equations

$$\begin{cases} \frac{\partial \phi_t}{\partial t} = \tau(\phi_t) - \mathcal{R}(\phi_t, \psi_t) - \varepsilon \mathcal{R}_c(\phi_t, \psi_t), \\ \phi(x,0) = \phi_0(x). \end{cases} \tag{4.4}$$

$$\begin{cases} \frac{\tilde{\nabla}\psi_t}{\partial t} = -\not{D}^2\psi_t - \frac{1}{\varepsilon}\not{D}\psi_t, \\ \psi(x,0) = \psi_0(x), \end{cases} \tag{4.5}$$

We will now use the maximum principle (B.12) to establish pointwise energy estimates.

Lemma 4.6. *Assume that $M = S^1$ and let $\psi_t \in C^\infty(M \times [0,T), \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (4.5). Then the norm of the spinor ψ_t satisfies the following estimate:*

$$|\psi_t|^2 \leq e^{\frac{1}{2\varepsilon^2}t}|\psi_0|^2. \tag{4.6}$$

Proof. Due to the rescaling of t in (4.5), the norm of ψ_t satisfies the following evolution equation

$$\frac{\partial}{\partial t}\frac{1}{2}|\psi_t|^2 = \Delta\frac{1}{2}|\psi_t|^2 - \frac{1}{\varepsilon}\langle\psi_t, \not{D}\psi_t\rangle - |\tilde{\nabla}\psi_t|^2.$$

For $M = S^1$ we have $|\not{D}\psi|^2 = |\tilde{\nabla}\psi|^2$. By Young's inequality we get the estimate

$$\frac{\partial}{\partial t}\frac{1}{2}|\psi_t|^2 \leq \Delta\frac{1}{2}|\psi_t|^2 + \frac{1}{4\varepsilon^2}|\psi_t|^2.$$

Now, apply the maximum principle to the function $e^{-\frac{1}{2\varepsilon^2}t}|\psi_t|^2$. □

As a next step, we want to derive pointwise estimates on the norms of $d\phi_t$ and $\not{D}\psi_t$.

Theorem 4.7. *Suppose that $M = S^1$ with fixed spin structure. Moreover, let the pair $(\phi_t, \psi_t) \in C^\infty(M \times [0,T), N) \times C^\infty(M \times [0,T), \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (4.4) and (4.5). The function F_t defined by*

$$F_t := \frac{1}{2}(|d\phi_t|^2 + \varepsilon|\not{D}\psi_t|^2) \tag{4.7}$$

satisfies the following evolution equation:

$$\frac{\partial F_t}{\partial t} \leq \Delta F_t + \frac{C}{\varepsilon^2}e^{\frac{t}{\varepsilon^2}}F_t. \tag{4.8}$$

The constant C depends on N and ψ_0.

Proof. We use the equations derived in (3.3) and (3.4), remembering the rescaling of t in the equation for ψ_t. Since $M = S^1$ most of the curvature terms drop out and we find

$$\begin{aligned}\frac{\partial F_t}{\partial t} &= \Delta F_t - |\nabla d\phi_t|^2 - \langle \nabla_{e_\beta} \mathcal{R}(\phi_t, \psi_t), d\phi_t(e_\beta)\rangle - \varepsilon \langle \nabla_{e_\beta} \mathcal{R}_c(\phi_t, \psi_t), d\phi_t(e_\beta)\rangle \\ &\quad + \varepsilon \langle \frac{\partial \phi_t}{\partial t}, \mathcal{R}_c(\phi_t, \psi_t)\rangle - \langle \not{D}^2 \psi_t, \not{D}\psi_t\rangle - \varepsilon |\tilde{\nabla} \not{D}\psi_t|^2.\end{aligned}$$

A direct computation yields

$$\langle \mathcal{R}(\phi, \psi), d\phi(e_\beta)\rangle_{\phi^{-1}TN} = \frac{1}{2}\langle e_\alpha \cdot R^N(d\phi(e_\alpha), d\phi(e_\beta))\psi, \psi\rangle_{\phi^{-1}TN}.$$

By assumption, M is a curve. Therefore, we have

$$-\langle \nabla_{e_\beta} \mathcal{R}(\phi_t, \psi_t), d\phi_t(e_\beta)\rangle = \langle \mathcal{R}(\phi_t, \psi_t), \tau(\phi_t)\rangle$$

and similarly for $\mathcal{R}_c(\phi_t, \psi_t)$. Applying this identity we find

$$\begin{aligned}\frac{\partial F_t}{\partial t} &= \Delta F_t - |\nabla d\phi_t|^2 - \varepsilon^2 |\mathcal{R}_c(\phi_t, \psi_t)|^2 - \varepsilon \langle \mathcal{R}_c(\phi_t, \psi_t), \mathcal{R}(\phi_t, \psi_t)\rangle \\ &\quad + \langle 2\varepsilon \mathcal{R}_c(\phi_t, \psi_t) + \mathcal{R}(\phi_t, \psi_t), \tau(\phi_t)\rangle - \langle \not{D}^2 \psi_t, \not{D}\psi_t\rangle - \varepsilon |\tilde{\nabla} \not{D}\psi_t|^2.\end{aligned}$$

As a next step, we use the estimate

$$\begin{aligned}&-|\nabla d\phi_t|^2 - \varepsilon^2 |\mathcal{R}(\phi_t, \psi_t)|^2 - \varepsilon \langle \mathcal{R}_c(\phi_t, \psi_t), \mathcal{R}(\phi_t, \psi_t)\rangle \\ &\quad + \langle 2\varepsilon \mathcal{R}_c(\phi_t, \psi_t) + \mathcal{R}(\phi_t, \psi_t), \tau(\phi_t)\rangle \\ &\quad \leq \frac{1}{4}|\mathcal{R}(\phi_t, \psi_t) + 2\varepsilon \mathcal{R}_c(\phi_t, \psi_t)|^2 - \varepsilon^2 |\mathcal{R}_c(\phi_t, \psi_t)|^2 - \varepsilon \langle \mathcal{R}_c(\phi_t, \psi_t), \mathcal{R}(\phi_t, \psi_t)\rangle \\ &\quad = \frac{1}{4}|\mathcal{R}(\phi_t, \psi_t)|^2\end{aligned}$$

and apply Young's inequality again

$$-\langle \not{D}^2 \psi_t, \not{D}\psi_t\rangle - \varepsilon |\tilde{\nabla} \not{D}\psi_t|^2 \leq \frac{1}{4\varepsilon}|\not{D}\psi_t|^2.$$

Finally, we calculate

$$\begin{aligned}\frac{\partial F_t}{\partial t} &\leq \Delta F_t + \frac{1}{4}|\mathcal{R}(\phi_t, \psi_t)|^2 + \frac{1}{4\varepsilon}|\not{D}\psi_t|^2 \\ &\leq \Delta F_t + C\left(|\psi_t|^4 |d\phi_t|^2 + \frac{1}{\varepsilon}|\not{D}\psi_t|^2\right) \\ &\leq \Delta F_t + C\left(e^{\frac{1}{\varepsilon^2}t} |d\phi_t|^2 + \frac{1}{\varepsilon}|\not{D}\psi_t|^2\right) \\ &\leq \Delta F_t + Ce^{\frac{1}{\varepsilon^2}t}\left(|d\phi_t|^2 + \frac{1}{\varepsilon}|\not{D}\psi_t|^2\right) \\ &\leq \Delta F_t + Ce^{\frac{1}{\varepsilon^2}t}\frac{1}{\varepsilon^2}\left(|d\phi_t|^2 + \varepsilon|\not{D}\psi_t|^2\right) \\ &\leq \Delta F_t + \frac{C}{\varepsilon^2}e^{\frac{1}{\varepsilon^2}t} F_t,\end{aligned}$$

where we used the fact that $\varepsilon < 1$. $\square$

Corollary 4.8. *Suppose that $M = S^1$ with fixed spin structure. Moreover, let the pair $(\phi_t, \psi_t) \in C^\infty(M \times [0,T), N) \times C^\infty(M \times [0,T), \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (4.4) and (4.5). Then we have for all $t \in [0,T)$*

$$F_t = \frac{1}{2}(|d\phi_t|^2 + \varepsilon|\not{D}\psi_t|^2) \leq Ce^{Ce^{\frac{t}{\varepsilon^2}}}. \tag{4.9}$$

The constant C depends on $N, \varepsilon, \psi_0, d\phi_0$ and $\not{D}\psi_0$.

Proof. The evolution equation for F_t (4.8) can be rewritten as

$$\frac{\partial}{\partial t}(e^{-Ce^{\frac{t}{\varepsilon^2}}} F_t) \leq \Delta(e^{-Ce^{\frac{t}{\varepsilon^2}}} F_t).$$

Applying the maximum principle, we find

$$F_t \leq e^{-C}e^{Ce^{\frac{t}{\varepsilon^2}}} F_0.$$

Rearranging the constants yields the result. □

Having obtained pointwise estimates for $|d\phi_t|^2$ and $|\not{D}\psi_t|^2$, we can now derive estimates on the t-derivatives of ϕ_t and ψ_t.

Theorem 4.9. *Suppose that $M = S^1$ with fixed spin structure. Moreover, let the pair $(\phi_t, \psi_t) \in C^\infty(M \times [0,T), N) \times C^\infty(M \times [0,T), \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (4.4) and (4.5). The quantity defined by*

$$G_t := \frac{1}{2}\left(|\frac{\partial\phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla}\psi_t}{\partial t}|^2\right) \tag{4.10}$$

satisfies

$$\frac{\partial G_t}{\partial t} \leq \Delta G_t + Z(t)G_t \tag{4.11}$$

with the function $Z(t)$ depending on $|\psi_t|, |d\phi_t|$, and $|\not{D}\psi_t|$.

Proof. Using the equations derived in (3.5), (3.6), taking into account the rescaling of t in the evolution equation for ψ_t and the fact that $M = S^1$, we find

$$\begin{aligned}
\frac{\partial G_t}{\partial t} =& \Delta G_t - |\nabla\frac{\partial\phi_t}{\partial t}|^2 - |\tilde{\nabla}\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 + \langle R^N(d\phi_t(e_\alpha), \frac{\partial\phi_t}{\partial t})d\phi_t(e_\alpha), \frac{\partial\phi_t}{\partial t}\rangle \\
&- \langle\frac{\nabla}{\partial t}\mathcal{R}(\phi_t,\psi_t), \frac{\partial\phi_t}{\partial t}\rangle - \varepsilon\langle\frac{\nabla}{\partial t}\mathcal{R}_c(\phi_t,\psi_t), \frac{\partial\phi_t}{\partial t}\rangle \\
&- \frac{1}{\varepsilon}\langle e_\alpha \cdot R^N(d\phi_t(\partial_t), d\phi_t(e_\alpha))\psi_t, \frac{\tilde{\nabla}\psi_t}{\partial t}\rangle - \frac{1}{\varepsilon}\langle\frac{\tilde{\nabla}\psi_t}{\partial t}, \not{D}\frac{\tilde{\nabla}\psi_t}{\partial t}\rangle \\
&- \langle e_\alpha \cdot R^N(d\phi_t(\partial_t), d\phi_t(e_\alpha))\not{D}\psi_t, \frac{\tilde{\nabla}\psi_t}{\partial t}\rangle - \langle\not{D}(e_\alpha \cdot R^N(d\phi_t(\partial_t), d\phi_t(e_\alpha))\psi_t), \frac{\tilde{\nabla}\psi_t}{\partial t}\rangle.
\end{aligned}$$

Again, we have to estimate all terms on the right hand side. First of all, we estimate the term containing the curvature of $\phi^{-1}TN$ as

$$\langle R^N(d\phi_t(e_\alpha), \frac{\partial \phi_t}{\partial t})d\phi_t(e_\alpha), \frac{\partial \phi_t}{\partial t}\rangle \leq C|d\phi_t|^2|\frac{\partial \phi_t}{\partial t}|^2 := I_1.$$

For the next term we calculate

$$\begin{aligned}\frac{\nabla}{\partial t}\mathcal{R}(\phi_t,\psi_t) &= \frac{1}{2}(\nabla_{d\phi_t(\partial_t)}R^N)(e_\alpha\cdot\psi_t,\psi_t)d\phi_t(e_\alpha) + R^N(e_\alpha\cdot\psi_t, \frac{\tilde{\nabla}\psi_t}{\partial t})d\phi_t(e_\alpha) \\ &\quad + \frac{1}{2}R^N(e_\alpha\cdot\psi_t,\psi_t)\frac{\nabla}{\partial t}d\phi_t(e_\alpha)\end{aligned}$$

and make the following estimate

$$\begin{aligned}|\langle \frac{\nabla}{\partial t}\mathcal{R}(\phi_t,\psi_t), \frac{\partial \phi_t}{\partial t}\rangle| &\leq C\Big(|\frac{\partial \phi_t}{\partial t}|^2|d\phi_t||\psi_t|^2 + |\psi_t||d\phi_t||\frac{\partial \phi_t}{\partial t}||\frac{\tilde{\nabla}\psi_t}{\partial t}| + |\psi_t|^2|\frac{\nabla}{\partial t}d\phi_t||\frac{\partial \phi_t}{\partial t}|\Big) \\ &:= I_2 + I_3 + I_4.\end{aligned}$$

To take care of the next term, we first compute

$$\begin{aligned}\frac{\nabla}{\partial t}\mathcal{R}_c(\phi_t,\psi_t) &= (\nabla_{d\phi_t(\partial_t)}R^N)(\tilde{\nabla}_{e_\alpha}\psi_t,\psi_t)d\phi_t(e_\alpha) + R^N(\frac{\tilde{\nabla}}{\partial t}\tilde{\nabla}_{e_\alpha}\psi_t,\psi_t)d\phi_t(e_\alpha) \\ &\quad + R^N(\tilde{\nabla}_{e_\alpha}\psi_t, \frac{\tilde{\nabla}\psi_t}{\partial t})d\phi_t(e_\alpha) + R^N(\psi_t,\tilde{\nabla}_{e_\alpha}\psi_t)\frac{\nabla}{\partial t}d\phi_t(e_\alpha)\end{aligned}$$

and then estimate

$$\begin{aligned}|\langle \frac{\nabla}{\partial t}\mathcal{R}_c(\phi_t,\psi_t), \frac{\partial \phi_t}{\partial t}\rangle| &\leq C\Big(|\psi_t||\frac{\partial \phi_t}{\partial t}|^2|d\phi_t||\not{D}\psi_t| + |\not{D}\psi_t||d\phi_t||\frac{\partial \phi_t}{\partial t}||\frac{\tilde{\nabla}\psi_t}{\partial t}| \\ &\quad + |\psi_t||\frac{\tilde{\nabla}}{\partial t}\not{D}\psi_t||d\phi_t||\frac{\partial \phi_t}{\partial t}| + |\psi_t||\not{D}\psi_t||\frac{\nabla}{\partial t}d\phi_t||\frac{\partial \phi_t}{\partial t}|\Big) \\ &:= I_5 + I_6 + I_7 + I_8.\end{aligned}$$

Having estimated the terms from the evolution equation for ϕ_t, we now deal with the contributions originating from the evolution equation for ψ_t

$$\begin{aligned}\frac{1}{\varepsilon}|\langle \frac{\tilde{\nabla}\psi_t}{\partial t}, \not{D}\frac{\tilde{\nabla}\psi_t}{\partial t}\rangle| &\leq \frac{2}{\varepsilon^2}|\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 + \frac{1}{8}|\tilde{\nabla}\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 := I_9 + I_{10}, \\ \big|\langle e_\alpha\cdot R^N(d\phi_t(\partial_t), d\phi_t(e_\alpha))\psi_t, \frac{\tilde{\nabla}\psi_t}{\partial t}\rangle\big| &\leq C|\frac{\partial \phi_t}{\partial t}||\frac{\tilde{\nabla}\psi_t}{\partial t}||\psi_t||d\phi_t| := I_{11}, \\ |\langle e_\alpha\cdot R^N(d\phi_t(\partial_t), d\phi_t(e_\alpha))\not{D}\psi_t, \frac{\tilde{\nabla}\psi_t}{\partial t}\rangle| &\leq C|\frac{\partial \phi_t}{\partial t}||\frac{\tilde{\nabla}\psi_t}{\partial t}||\not{D}\psi_t||d\phi_t| := I_{12}.\end{aligned}$$

To estimate the last term, we first of all note that since $M = S^1$, we have

$$\not{D}(e_\alpha\cdot R^N(d\phi_t(\partial_t), d\phi_t(e_\alpha))\psi_t) = -\tilde{\nabla}_{e_\alpha}(R^N(d\phi_t(\partial_t), d\phi_t(e_\alpha))\psi_t)$$

such that we can differentiate

$$\begin{aligned}\tilde{\nabla}_{e_\alpha}(R^N(d\phi_t(\partial_t),d\phi_t(e_\alpha))\psi_t) &= ((\nabla_{d\phi_t(e_\alpha)}R^N)(d\phi_t(\partial_t),d\phi_t(e_\alpha))\psi_t \\ &\quad +R^N(\nabla_{e_\alpha}d\phi_t(\partial_t),d\phi_t(e_\alpha))\psi_t+R^N(d\phi_t(\partial_t),\tau(\phi_t))\psi_t \\ &\quad +R^N(d\phi_t(\partial_t),d\phi_t(e_\alpha))\tilde{\nabla}_{e_\alpha}\psi_t.\end{aligned}$$

The first, second and fourth term can easily be estimated as

$$\begin{aligned}|\langle(\nabla_{d\phi_t(e_\alpha)}R^N)(d\phi_t(\partial_t),d\phi_t(e_\alpha))\psi_t,\frac{\tilde{\nabla}\psi_t}{\partial t}\rangle| &\leq C|d\phi_t|^2|\frac{\partial\phi_t}{\partial t}||\psi_t||\frac{\tilde{\nabla}\psi_t}{\partial t}|, \\ |\langle R^N(\nabla_{e_\alpha}d\phi_t(\partial_t),d\phi_t(e_\alpha))\psi_t,\frac{\tilde{\nabla}\psi_t}{\partial t}\rangle| &\leq C|\nabla\frac{\partial\phi_t}{\partial t}||d\phi_t||\psi_t||\frac{\tilde{\nabla}\psi_t}{\partial t}|, \\ |\langle R^N(d\phi_t(\partial_t),d\phi_t(e_\alpha))\tilde{\nabla}_{e_\alpha}\psi_t,\frac{\tilde{\nabla}\psi_t}{\partial t}\rangle| &\leq C|\frac{\partial\phi_t}{\partial t}||d\phi_t||\tilde{\nabla}\psi_t||\frac{\tilde{\nabla}\psi_t}{\partial t}|.\end{aligned}$$

To take care of the third term, we use the evolution equation for ϕ_t and obtain

$$R^N(d\phi_t(\partial_t),\tau(\phi_t))\psi_t=R^N(d\phi_t(\partial_t),\mathcal{R}(\phi_t,\psi_t))\psi_t+\varepsilon R^N(d\phi_t(\partial_t),\mathcal{R}_c(\phi_t,\psi_t))\psi_t.$$

This allows us to derive the following estimate

$$\begin{aligned}&|\langle R^N(d\phi_t(\partial_t),\tau(\phi_t))\psi_t,\frac{\tilde{\nabla}\psi_t}{\partial t}\rangle| \\ &\quad\leq|\langle R^N(d\phi_t(\partial_t),\mathcal{R}(\phi_t,\psi_t))\psi_t,\frac{\tilde{\nabla}\psi_t}{\partial t}\rangle|+\varepsilon|\langle R^N(d\phi_t(\partial_t),\mathcal{R}_c(\phi_t,\psi_t))\psi_t,\frac{\tilde{\nabla}\psi_t}{\partial t}\rangle| \\ &\quad\leq C\left(|d\phi_t||\psi_t|^3|\frac{\tilde{\nabla}\psi_t}{\partial t}||\frac{\partial\phi_t}{\partial t}|+\varepsilon|d\phi_t||\psi_t|^2|\frac{\tilde{\nabla}\psi_t}{\partial t}||\frac{\partial\phi_t}{\partial t}||\not{D}\psi_t|\right)\end{aligned}$$

and, finally, we have

$$\begin{aligned}&|\langle\not{D}(e_\alpha\cdot R^N(d\phi_t(\partial_t),d\phi_t(e_\alpha))\psi_t,\frac{\tilde{\nabla}\psi_t}{\partial t})\rangle| \\ &\quad\leq C\Big(|d\phi_t|^2|\frac{\partial\phi_t}{\partial t}||\psi_t||\frac{\tilde{\nabla}\psi_t}{\partial t}|+|\nabla\frac{\partial\phi_t}{\partial t}||d\phi_t||\psi_t||\frac{\tilde{\nabla}\psi_t}{\partial t}|+|\frac{\partial\phi_t}{\partial t}||d\phi_t||\psi_t|^3|\frac{\tilde{\nabla}\psi_t}{\partial t}| \\ &\qquad+\varepsilon|\frac{\partial\phi_t}{\partial t}||d\phi_t||\psi_t|^2|\not{D}\psi_t||\frac{\tilde{\nabla}\psi_t}{\partial t}|+|\frac{\partial\phi_t}{\partial t}||d\phi_t||\not{D}\psi_t||\frac{\tilde{\nabla}\psi_t}{\partial t}|\Big) \\ &\quad:=I_{13}+I_{14}+I_{15}+I_{16}+I_{17}.\end{aligned}$$

Collecting all the estimates, the function G_t satisfies

$$\frac{\partial G_t}{\partial t}\leq\Delta G_t+\sum_{j=1}^{17}I_j-|\nabla\frac{\partial\phi_t}{\partial t}|^2-|\tilde{\nabla}\frac{\tilde{\nabla}\psi_t}{\partial t}|^2.$$

We want to use $-|\nabla\frac{\partial\phi_t}{\partial t}|^2-|\tilde{\nabla}\frac{\tilde{\nabla}\psi_t}{\partial t}|^2$ in order to control $I_4, I_7, I_8, I_{10}, I_{14}$. Namely,

$$\begin{aligned}
&-|\nabla\frac{\partial\phi_t}{\partial t}|^2-|\nabla\frac{\tilde{\nabla}\psi_t}{\partial t}|^2+I_4+I_7+I_8+I_{10}+I_{14}\\
&\leq C\left(|\psi_t|^4|\frac{\partial\phi_t}{\partial t}|^2+\varepsilon^2|\frac{\partial\phi_t}{\partial t}|^2|\not{D}\psi_t|^2|\psi_t|^2+|d\phi_t|^2|\psi_t|^2|\frac{\tilde{\nabla}\psi_t}{\partial t}|^2+\varepsilon|\frac{\partial\phi_t}{\partial t}|^2|\psi_t|^2|d\phi_t|^2\right).
\end{aligned}$$

To estimate the I_7 contribution we interchanged covariant derivatives and estimated the resulting terms. Hence, we can write

$$\sum_{j=1}^{17}I_j-|\nabla\frac{\partial\phi_t}{\partial t}|^2-|\tilde{\nabla}\frac{\tilde{\nabla}\psi_t}{\partial t}|^2\leq A_t|\frac{\partial\phi_t}{\partial t}|^2+K_t|\frac{\tilde{\nabla}\psi_t}{\partial t}|^2+W_t|\frac{\partial\phi_t}{\partial t}||\frac{\tilde{\nabla}\psi_t}{\partial t}|$$

with the terms

$$\begin{aligned}
A_t &= C\left(|\psi_t|^2|d\phi_t|^2+\varepsilon|\not{D}\psi_t|d\phi_t||\psi_t|+|\psi_t|^2+\varepsilon|\not{D}\psi_t|^2|\psi_t|^2+|\psi_t|^4+|d\phi_t|^2\right),\\
K_t &= C\left(|\psi_t|^2|d\phi_t|^2+\frac{1}{\varepsilon^2}\right),\\
W_t &= C\Big(|\psi_t||d\phi_t|+\varepsilon|\not{D}\psi_t||d\phi_t|+\frac{1}{\varepsilon}|d\phi_t||\psi_t|+|d\phi_t||\not{D}\psi_t|+|d\phi_t|^2|\psi_t|\\
&\qquad+|\psi_t|^3|d\phi_t|+\varepsilon|d\phi_t||\psi_t|^2|\not{D}\psi_t|\Big).
\end{aligned}$$

By the bounds on $\psi_t, d\phi_t$, and $\tilde{\nabla}\psi_t$ all terms appearing in A_t, K_t and W_t can be controlled. We combine them into one function $Z(t)$, which completes the proof. $\square$

Corollary 4.10. *Suppose that $M=S^1$ with fixed spin structure. Moreover, let the pair $(\phi_t,\psi_t)\in C^\infty(M\times[0,T),N)\times C^\infty(M\times[0,T),\Sigma M\otimes\phi_t^{-1}TN)$ be a solution of (4.4) and (4.5). For all $(x,t)\in S^1\times[0,T)$, we have the following estimate*

$$G_t=\frac{1}{2}\left(|\frac{\partial\phi_t}{\partial t}|^2+|\frac{\tilde{\nabla}\psi_t}{\partial t}|^2\right)\leq Cf(t), \tag{4.12}$$

where $f(t)$ only depends on t and is finite for finite values of t. The constant C depends on $N,\varepsilon,\psi_0,d\phi_0$ and $\tilde{\nabla}\psi_0$.

Proof. We use the inequality for G_t and apply the maximum principle to $e^{-\int_0^T Z(\tau)d\tau}G_t$. The function $Z(t)$ can be explicitly expressed in terms of exponentials and double exponentials. These are bounded for finite t and, consequently, the integral $\int_0^T Z(\tau)d\tau$ is also finite. $\square$

4.3. Long-time Existence

The goal of this section is to establish the existence of a long-time solution to the evolution equations (4.5), (4.4) for all $t \in [0, \infty)$. We will use a standard technique for parabolic equations to show this result. Since this involves choosing a subsequence, we begin by proving a uniqueness statement.

Theorem 4.11 (Stability and uniqueness)**.** *Assume that $M = S^1$ and N compact. Furthermore, let (ϕ_t, ψ_t) and (ξ_t, χ_t) be smooth solutions of (4.4) and (4.5). The spinor $\psi_t \in \Gamma(\Sigma M \otimes \phi_t^{-1} TN)$ is defined along the map ϕ_t and the spinor $\chi_t \in \Gamma(\Sigma M \otimes \xi_t^{-1} TN)$ along the map ξ. If the initial data coincide, i.e. $(\phi_0, \psi_0) = (\xi_0, \chi_0)$, then we have $(\phi_t, \psi_t) = (\xi_t, \chi_t)$ throughout $M \times [0, T)$.*

Proof. We use the Nash embedding theorem to isometrically embed the manifold N into some $\mathbb{R}^q$ via the map ι. We set $u := \iota \circ \phi, v := \iota \circ \xi$ and $\psi' = d\iota(\psi), \xi' = d\iota(\xi)$. To simplify the notation, we omit the superscript at the spinors. We regard u, v as vector valued functions in $\mathbb{R}^q$, i.e. $u, v \colon S^1 \times [0, T) \to \iota(N) \subset \mathbb{R}^q$, and the spinors as $\psi, \chi \colon S^1 \times [0, T) \to \Sigma S^1 \otimes T\mathbb{R}^q$. First of all, we define functions h_1, h_2

$$h_1 \colon M \times [0, T) \to \mathbb{R}^q, \qquad h_2 \colon M \times [0, T) \to \Sigma M \otimes T\mathbb{R}^q$$

by

$$h_1 = (u - v), \qquad h_2 = (\psi - \chi).$$

Using the evolution equation for u derived in (3.24), we calculate

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{2}|h_1|^2 = \Delta\frac{1}{2}|h_1|^2 - |dh_1|^2 &+ \langle \mathrm{II}_u(du, du) - \mathrm{II}_v(dv, dv), h_1\rangle \\
&+ \langle h_1, P(\mathrm{II}_u(du(e_\alpha), e_\alpha \cdot \psi), \psi) - P(\mathrm{II}_v(dv(e_\alpha), e_\alpha \cdot \chi), \chi)\rangle \\
&+ \varepsilon\langle h_1, P(\mathrm{II}_u(du(e_\alpha), e_\alpha \cdot \partial\!\!\!/\psi), \psi) - P(\mathrm{II}_v(dv(e_\alpha), e_\alpha \cdot \partial\!\!\!/\chi), \chi)\rangle \\
&+ \varepsilon\langle h_1, P(\mathrm{II}_u(du(e_\alpha), e_\alpha \cdot \psi), \partial\!\!\!/\psi) - P(\mathrm{II}_v(dv(e_\alpha), e_\alpha \cdot \chi), \partial\!\!\!/\chi)\rangle \\
&+ \varepsilon\langle h_1, B_u(du, \psi, du, \psi) - B_v(dv, \chi, dv, \chi)\rangle.
\end{aligned}$$

We want to estimate the right hand side in terms of the functions h_1 and h_2. To this end, we use the bounds on $d\phi_t, D\!\!\!/\psi_t$, and ψ_t derived before. Rearranging the second fundamental forms,

$$\mathrm{II}_u(du, du) - \mathrm{II}_v(dv, dv) = (\mathrm{II}_u - \mathrm{II}_v)(du, du) + \mathrm{II}_v(du - dv, du) + \mathrm{II}_v(dv, du - dv),$$

and applying the mean value theorem, we find

$$|\langle \mathrm{II}_u(du, du) - \mathrm{II}_v(dv, dv), u - v\rangle| \leq C(|u - v|^2 + |du - dv||u - v|).$$

We rewrite

$$\begin{aligned}
&P(\mathrm{II}_u(du(e_\alpha), e_\alpha \cdot \psi), \psi) - P(\mathrm{II}_v(dv(e_\alpha), e_\alpha \cdot \chi), \chi) \\
&\qquad = P(\mathrm{II}_{u-v}(du(e_\alpha), e_\alpha \cdot \psi), \psi) + P(\mathrm{II}_v(du(e_\alpha) - dv(e_\alpha), e_\alpha \cdot \psi), \psi) \\
&\qquad\quad + P(\mathrm{II}_v(dv(e_\alpha), e_\alpha \cdot (\psi - \chi)), \psi) + P(\mathrm{II}_v(dv(e_\alpha), e_\alpha \cdot \psi), \psi - \chi)
\end{aligned}$$

and estimate again

$$
\begin{aligned}
|\langle u-v, & P(\mathrm{II}_u(du(e_\alpha),e_\alpha\cdot\psi),\psi)-P(\mathrm{II}_v(dv(e_\alpha),e_\alpha\cdot\chi),\chi)\rangle| \\
&\leq C(|du||u-v|^2|\psi|^2+|du-dv||\psi|^2|u-v|+|dv||\psi||\psi-\chi||u-v|) \\
&\leq C(||u-v|^2+|du-dv||u-v|+|\psi-\chi||u-v|).
\end{aligned}
$$

Again, we rewrite

$$
\begin{aligned}
P(\mathrm{II}_u&(du(e_\alpha),e_\alpha\cdot\psi),\partial\!\!\!/\psi)-P(\mathrm{II}_v(dv(e_\alpha),e_\alpha\cdot\chi),\partial\!\!\!/\chi) \\
&=P(\mathrm{II}_{u-v}(du(e_\alpha),e_\alpha\cdot\psi),\partial\!\!\!/\psi)+P(\mathrm{II}_v(du(e_\alpha)-dv(e_\alpha),e_\alpha\cdot\psi),\partial\!\!\!/\psi) \\
&\quad+P(\mathrm{II}_v(dv(e_\alpha),e_\alpha\cdot(\psi-\chi)),\partial\!\!\!/\psi)+P(\mathrm{II}_v(dv(e_\alpha),e_\alpha\cdot\psi),\partial\!\!\!/(\psi-\chi))
\end{aligned}
$$

and estimate

$$
\begin{aligned}
|\langle u-v, & P(\mathrm{II}_u(du(e_\alpha),e_\alpha\cdot\psi),\partial\!\!\!/\psi)-P(\mathrm{II}_v(dv(e_\alpha),e_\alpha\cdot\chi),\partial\!\!\!/\chi)\rangle| \\
&\leq C(|du||u-v|^2|\psi||\partial\!\!\!/\psi|+|du-dv||\psi||\partial\!\!\!/\psi||u-v|+|dv||\partial\!\!\!/\psi||\psi-\chi||u-v| \\
&\quad+|dv||\psi||\partial\!\!\!/\psi-\partial\!\!\!/\chi||u-v|) \\
&\leq C(||u-v|^2+|du-dv||u-v|+|\psi-\chi||u-v|+|\partial\!\!\!/\psi-\partial\!\!\!/\chi||u-v|).
\end{aligned}
$$

The term

$$
\langle u-v, P(\mathrm{II}_u(du(e_\alpha),e_\alpha\cdot\partial\!\!\!/\psi),\psi)-P(\mathrm{II}_v(dv(e_\alpha),e_\alpha\cdot\partial\!\!\!/\chi),\chi)\rangle
$$

can be treated by the same methods and estimated like the previous one. Finally, we rewrite

$$
\begin{aligned}
B_u&(du,\psi,du,\psi)-B_v(dv,\chi,dv,\chi) \\
&=B_{u-v}(du,\psi,du,\psi)+B_v(du-dv,\psi,du,\psi)+B_v(dv,\psi-\chi,du,\psi) \\
&\quad+B_v(dv,\chi,du-dv,\psi)+B_v(dv,\chi,dv,\psi-\chi)
\end{aligned}
$$

such that we can estimate

$$
\begin{aligned}
|\langle u-v, & B_u(du,\psi,du,\psi)-B_v(dv,\chi,dv,\chi)\rangle| \\
&\leq C(|u-v|^2|\psi|^2|du|^2+|u-v||du-dv||\psi|^2|du|+|du||\psi||dv||\psi-\chi||u-v| \\
&\quad+|u-v||dv|\chi||\psi||du-dv|+|u-v||dv|^2|\chi||\psi-\chi|) \\
&\leq C(|u-v|^2+|u-v||du-dv|+|\psi-\chi||u-v|).
\end{aligned}
$$

Collecting all the terms and applying Young's inequality, we find that the norm of h_1 satisfies

$$
\frac{\partial}{\partial t}\frac{1}{2}|h_1|^2\leq\Delta\frac{1}{2}|h_1|^2-\frac{1}{2}|dh_1|^2+\frac{1}{2}|\nabla h_2|^2+C(|h_1|^2+|h_2|^2).
$$

Now, we turn to the function h_2. Using the evolution equation (3.25) a direct computation yields

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{2}|h_2|^2=&\Delta\frac{1}{2}|h_2|^2-|\nabla h_2|^2-\frac{1}{\varepsilon}\langle\partial\!\!\!/(\psi-\chi),\psi-\chi\rangle \\
&+\langle\psi-\chi,(\nabla_{e_\alpha}\mathrm{II}_u)(du(e_\alpha),\psi)-(\nabla_{e_\alpha}\mathrm{II}_v)(dv(e_\alpha),\chi)\rangle.
\end{aligned}
$$

The other terms involving the second fundamental form II vanish since $\mathrm{II} \perp \psi$. The last term in the equation for h_2 can be rewritten as

$$\begin{aligned}(\nabla_{e_\alpha}\mathrm{II}_u)(du(e_\alpha),\psi) &- (\nabla_{e_\alpha}\mathrm{II}_v)(dv(e_\alpha),\chi) \\ &= (\nabla_{e_\alpha}\mathrm{II}_{u-v})(du(e_\alpha),\psi) + (\nabla_{e_\alpha}\mathrm{II}_v)(du(e_\alpha) - dv(e_\alpha),\psi) + (\nabla_{e_\alpha}\mathrm{II}_v)(dv(e_\alpha),\psi-\chi)\end{aligned}$$

and we may estimate

$$\begin{aligned}|\langle\psi-\chi,(\nabla_{e_\alpha}\mathrm{II}_u)(du(e_\alpha),\psi) &- (\nabla_{e_\alpha}\mathrm{II}_v)(dv(e_\alpha),\chi)\rangle| \\ &\leq C(|\psi-\chi||du-dv||du||\psi| + |dv||du-dv||\psi||\psi-\chi| + |dv|^2|\psi-\chi|^2) \\ &\leq C(|u-v|^2 + |du-dv||\psi-\chi| + |\psi-\chi|^2).\end{aligned}$$

Hence, after applying Young's inequality we find for the norm of h_2

$$\frac{\partial}{\partial t}\frac{1}{2}|h_2|^2 \leq \Delta\frac{1}{2}|h_2|^2 - \frac{1}{2}|\nabla h_2|^2 + \frac{1}{2}|dh_1|^2 + C(|h_1|^2 + |h_2|^2).$$

Finally, we define the function $h\colon M \times [0,T) \to \mathbb{R}$ as $h := \frac{1}{2}(|h_1|^2 + |h_2|^2)$. One can think of h as the norm of $(u-v,\psi-\chi)$. Clearly, h satisfies the following inequality

$$\frac{\partial h}{\partial t} \leq \Delta h + Ch.$$

By the maximum principle we get

$$\max_{M\times[0,T)} h(x,t) \leq \max_M h(x,0)e^{Ct},$$

but by assumption $h(x,0) = 0$. Thus, we have $u = v$ and also $\psi = \chi$ throughout $M \times [0,T)$. □

In the next Proposition, we improve the regularity of the pair (ϕ_t,ψ_t) by application of the classical Schauder estimates.

Proposition 4.12 (Applying Schauder theory)**.** *Assume that $M = S^1$, N compact and let $(\phi_t,\psi_t) \in C^\infty(M\times[0,T),N) \times C^\infty(M\times[0,T),\Sigma M\otimes\phi_t^{-1}TN)$ be a solution of (4.4) and (4.5). Then for any $0<\alpha<1$, there exists a positive number C such that*

$$|\phi(\cdot,t)|_{C^{2+\alpha}(M,N)} + \big|\frac{\partial\phi}{\partial t}(\cdot,t)\big|_{C^\alpha(M,N)} \leq C, \tag{4.13}$$

$$|\psi(\cdot,t)|_{C^{2+\alpha}(M,\Sigma M\otimes\phi_t^{-1}TN)} + \big|\frac{\tilde\nabla\psi}{\partial t}(\cdot,t)\big|_{C^\alpha(M,\Sigma M\otimes\phi_t^{-1}TN)} \leq C \tag{4.14}$$

hold for all $t\in[0,T)$, where both constants depend on $M,N,\varepsilon,\alpha,T,\psi_t,d\phi_t$ and $\not{D}\psi_t$.

Proof. Again, we assume that N is isometrically embedded in a q-dimensional Euclidean vector space $\mathbb{R}^q$ and that the vector valued function u is a solution of (3.24) and the spinor ψ a solution of (3.25).

Depending on the point of view, the function u and the spinor ψ both satisfy an elliptic and a parabolic partial differential equation. This allows us to apply the classical Schauder estimate for both elliptic and parabolic equations. First of all, u satisfies the elliptic system

$$\begin{aligned}\Delta u &= \mathrm{I\!I}_u(du,du) + P(\mathrm{I\!I}_u(du(e_\alpha),e_\alpha\cdot\psi),\psi) + \varepsilon P(\mathrm{I\!I}_u(du(e_\alpha),e_\alpha\cdot \partial\!\!\!/\psi),\psi)\\ &\quad -\varepsilon P(\mathrm{I\!I}_u(du(e_\alpha),e_\alpha\cdot\psi),\partial\!\!\!/\psi) + \varepsilon B_u(du,\psi,du,\psi) + \frac{\partial u}{\partial t}\end{aligned}$$

with the Laplacian Δ on M. Let us estimate the right hand side

$$\begin{aligned}|\Delta u| &\leq C(|du|^2 + |\psi|^2|du| + |\psi||\partial\!\!\!/\psi||du| + |\frac{\partial u}{\partial t}| + |\psi|^2|du|^2)\\ &\leq C,\end{aligned}$$

where we used the estimates derived before. Using Schauder estimates for solutions to an elliptic partial differential equation, we find

$$\begin{aligned}|u(\cdot,t)|_{C^{1+\alpha}(S^1,\mathbb{R}^q)} &\leq C\left(\sup_{t\in[0,T)}|\Delta u(\cdot,t)|_{L^\infty(S^1,\mathbb{R}^q)} + \sup_{t\in[0,T)}|u(\cdot,t)|_{L^\infty(S^1,\mathbb{R}^q)}\right)\\ &\leq C,\end{aligned}$$

since u takes values in a compact region of $\mathbb{R}^q$. Here, in the one-dimensional case elliptic Schauder theory is of course not more than just integrating the right hand side. We can improve the regularity of the spinor ψ by the same method. Remember that the spinor $\psi\in\Gamma(\Sigma M\otimes T\mathbb{R}^q)$ solves the elliptic equation

$$\begin{aligned}\Delta\psi = \frac{1}{\varepsilon}\partial\!\!\!/\psi - \frac{1}{\varepsilon}\mathrm{I\!I}_u(du(e_\alpha),e_\alpha\cdot\psi) + \mathrm{I\!I}_u(\frac{\partial u}{\partial t},\psi) + \frac{\nabla\psi}{\partial t} + 2\mathrm{I\!I}_u(du(e_\alpha),\nabla_{e_\alpha}\psi)\\ + (\nabla_{e_\alpha}\mathrm{I\!I}_u)(du(e_\alpha),\psi) + \mathrm{I\!I}_u(\tau(u),\psi)).\end{aligned}$$

Again, we estimate the right hand side with the help of the previous estimates

$$\begin{aligned}|\Delta\psi| &\leq C\big(|\partial\!\!\!/\psi| + |du||\psi| + |\frac{\partial u}{\partial t}||\psi| + \big|\frac{\nabla\psi}{\partial t}\big| + |du||\nabla\psi| + |du|^2|\psi| + |\nabla^2 u||\psi|\big)\\ &\leq C\end{aligned}$$

and apply Schauder estimates for elliptic equations

$$\begin{aligned}&|\psi(\cdot,t)|_{C^{1+\alpha}(S^1,\Sigma S^1\otimes T\mathbb{R}^q)}\\ &\qquad\leq C\left(\sup_{t\in[0,T)}|\Delta\psi(\cdot,t)|_{L^\infty(S^1,\Sigma S^1\otimes T\mathbb{R}^q)} + \sup_{t\in[0,T)}|\psi(\cdot,t)|_{L^\infty(S^1,\Sigma S^1\otimes T\mathbb{R}^q)}\right)\leq C.\end{aligned}$$

After exploiting the elliptic nature of the evolution equations for (ϕ_t,ψ_t), we now take the parabolic point of view. Note that u is also a solution of the parabolic system

$$\begin{aligned}Lu &= \mathrm{I\!I}_u(du,du) + P(\mathrm{I\!I}_u(du(e_\alpha),e_\alpha\cdot\psi),\psi) + \varepsilon P(\mathrm{I\!I}_u(du(e_\alpha),e_\alpha\cdot\partial\!\!\!/\psi),\psi)\\ &\quad -\varepsilon P(\mathrm{I\!I}_u(du(e_\alpha),e_\alpha\cdot\psi),\partial\!\!\!/\psi) + \varepsilon B_u(du,\psi,du,\psi)\end{aligned}$$

with $L = \Delta - \frac{\partial}{\partial t}$ denoting the heat operator on S^1. Utilizing the previous estimate, we can bound the right hand side by

$$|\mathrm{II}_u(du, du) + P(\mathrm{II}_u(du(e_\alpha), e_\alpha \cdot \psi), \psi) + \varepsilon P(\mathrm{II}_u(du(e_\alpha), e_\alpha \cdot \partial\!\!\!/\psi), \psi) \\ - \varepsilon P(\mathrm{II}_u(du(e_\alpha), e_\alpha \cdot \psi), \partial\!\!\!/\psi) + \varepsilon B_u(du, \psi, du, \psi)|_{C^\alpha(S^1, \mathbb{R}^q)} \leq C.$$

Finally, we employ Schauder estimates for linear parabolic partial differential equations

$$|u(\cdot, t)|_{C^{2+\alpha}(S^1, \mathbb{R}^q)} + |\frac{\partial u}{\partial t}(\cdot, t)|_{C^\alpha(S^1, \mathbb{R}^q)} \\ \leq C \left(\sup_{t \in [0,t)} |Lu(\cdot, t)|_{C^\alpha(S^1, \mathbb{R}^q)} + \sup_{t \in [0,t)} |u(\cdot, t)|_{L^\infty(S^1, \mathbb{R}^q)} \right) \leq C,$$

which proves the statement concerning the regularity of u.
Again, taking the parabolic point of view, ψ satisfies

$$L\psi = \frac{1}{\varepsilon}\partial\!\!\!/\psi - \frac{1}{\varepsilon}\mathrm{II}_u(du(e_\alpha), e_\alpha \cdot \psi) + \mathrm{II}_u(\frac{\partial u}{\partial t}, \psi) + \mathrm{II}_u(du(e_\alpha), \nabla_{e_\alpha}\psi) + 2\mathrm{II}_u(du(e_\alpha), \nabla_{e_\alpha}\psi) \\ + (\nabla_{e_\alpha}\mathrm{II}_u)(du(e_\alpha), \psi) + \mathrm{II}_u(\tau(u), \psi).$$

with the heat operator $L = \Delta - \frac{\nabla}{\partial t}$. The right hand side is bounded in C^α such that

$$|\psi(\cdot, t)|_{C^{2+\alpha}(S^1, \Sigma S^1 \otimes T\mathbb{R}^q)} + |\frac{\nabla \psi}{\partial t}(\cdot, t)|_{C^\alpha(S^1, \Sigma S^1 \otimes T\mathbb{R}^q)} \\ \leq C \left(\sup_{t \in [0,t)} |L\psi(\cdot, t)|_{C^\alpha(S^1, \Sigma S^1 \otimes T\mathbb{R}^q)} + \sup_{t \in [0,t)} |\psi(\cdot, t)|_{L^\infty(S^1, \Sigma S^1 \otimes T\mathbb{R}^q)} \right) \leq C,$$

which establishes the regularity of the spinor ψ. □

Based on the estimates deduced so far, the uniqueness and stability result and the Schauder theory, we can establish the long-time existence of the evolution equations.

Theorem 4.13 (Long-time Existence)**.** *Assume that $M = S^1$, N compact and let $(\phi_t, \psi_t) \in C^\infty(M \times [0,T), N) \times C^\infty(M \times [0,T), \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (4.4) and (4.5).*
Then for any $(\phi_0, \psi_0) \in C^{2+\alpha}(M, N) \times C^{2+\alpha}(M, \Sigma M \otimes \phi_0^{-1}TN)$, there exists a unique

$$\phi_t \in C^{2+\alpha, 1+\alpha/2}(M \times [0,\infty), N) \cup C^\infty(M \times (0,\infty), N), \\ \psi_t \in C^{2+\alpha, 1+\alpha/2}(M \times [0,\infty), \Sigma M \otimes \phi_t^{-1}TN) \cup C^\infty(M \times (0,\infty), \Sigma M \otimes \phi_t^{-1}TN),$$

such that

$$\begin{cases} \frac{\partial \phi_t}{\partial t} = \tau(\phi_t) - \mathcal{R}(\phi_t, \psi_t) - \varepsilon \mathcal{R}_c(\phi_t, \psi_t), & (x,t) \in S^1 \times (0,\infty), \\ \phi(x, 0) = \phi_0(x), \end{cases} \tag{4.15}$$

$$\begin{cases} \frac{\tilde{\nabla} \psi_t}{\partial t} = \tilde{\Delta}\psi_t - \frac{1}{\varepsilon} D\!\!\!\!/\,\psi_t, & (x,t) \in S^1 \times (0,\infty), \\ \psi(x, 0) = \psi_0(x) \end{cases} \tag{4.16}$$

holds.

Proof. The short-time existence of the evolution equations is guaranteed by Theorem 3.22 for a time interval $0 \le t \le T_{max}$. We now demonstrate that, for N being compact, the regularized Dirac-harmonic map heat flow cannot blow up and will exist for all $t \in [0, \infty)$. We set

$$T_0 = \sup\{t \in [0, \infty) \mid (4.4), (4.5) \text{ have a solution in } M \times [0, t)\}$$

and show that $T_0 = \infty$. Let us assume the opposite case. We choose a sequence of numbers $\{t_i\} \subset [0, T_0)$ such that $t_i \to T_0$ as $i \to \infty$ and set $0 < \alpha < \alpha' < 1$. Since S^1 is compact, the embeddings $C^{k+\alpha'}(S^1, N) \hookrightarrow C^{k+\alpha}(S^1, N)$ and in addition $C^{k+\alpha'}(S^1, \Sigma S^1 \otimes \phi^{-1}TN) \hookrightarrow C^{k+\alpha}(S^1, \Sigma S^1 \otimes \phi^{-1}TN)$ are compact. By Proposition 4.12, the sequences

$$\{\phi(\cdot, t_i), \psi(\cdot, t_i)\} \text{ and } \{\partial_t\phi(\cdot, t_i), \tilde{\nabla}_t\psi(\cdot, t)\}$$

are in $C^{2+\alpha'}(S^1, N) \times C^{2+\alpha'}(S^1, \Sigma S^1 \otimes \phi^{-1}TN)$ and $C^{\alpha'}(S^1, N) \times C^{\alpha'}(S^1, \Sigma S^1 \otimes \phi^{-1}TN)$. Hence, there exists a subsequence $\{t_{i_k}\}$ of $\{t_i\}$ with

$$\begin{aligned}(\phi(\cdot, T_0), \psi(\cdot, T_0)) &\in C^{2+\alpha'}(S^1, N) \times C^{2+\alpha'}(S^1, \Sigma S^1 \otimes \phi^{-1}TN),\\ (\partial_t\phi(\cdot, T_0), \tilde{\nabla}_t\psi(\cdot, T_0)) &\in C^{\alpha'}(S^1, N) \times C^{\alpha'}(S^1, \Sigma S^1 \otimes \phi^{-1}TN)\end{aligned}$$

such that

$$\{\phi(\cdot, t_{i_k}), \psi(\cdot, t_{i_k})\} \text{ and } \{\partial_t\phi(\cdot, t_{i_k}), \tilde{\nabla}_t\psi(\cdot, t_{i_k})\}$$

converge uniformly to $(\phi(\cdot, T_0), \psi(\cdot, T_0))$ and $(\partial_t\phi(\cdot, T_0), \tilde{\nabla}_t\psi(\cdot, T_0))$, as $t_{i_k} \to T_0$. This is true for each t_{i_k}

$$\begin{aligned}\frac{\partial\phi}{\partial t}(\cdot, t_{i_k}) &= (\tau(\phi) - \mathcal{R}(\phi, \psi) - \varepsilon\mathcal{R}_c(\phi, \psi))(\cdot, t_{i_k}),\\ \frac{\tilde{\nabla}\psi}{\partial t}(\cdot, t_{i_k}) &= (\tilde{\Delta}\psi - \frac{1}{\varepsilon}\not{D}\psi)(\cdot, t_{i_k})\end{aligned}$$

and, consequently, also at T_0. Hence, (4.15), (4.16) have a solution in $S^1 \times [0, T_0]$. We can now again apply the short-time existence Theorem 3.22 with initial values $(\phi(\cdot, T_0), \psi(\cdot, T_0))$. For $\delta > 0$ we then get a solution

$$\begin{cases}\frac{\partial\phi_t}{\partial t} = \tau(\phi_t) - \mathcal{R}(\phi_t, \psi_t) - \varepsilon\mathcal{R}_c(\phi_t, \psi_t), & (x, t) \in S^1 \times (T_0, T_0 + \delta),\\ \phi(x, T_0) = \phi_0(x),\end{cases}$$
$$\begin{cases}\frac{\tilde{\nabla}\psi_t}{\partial t}(x) = \tilde{\Delta}\psi_t(x) - \frac{1}{\varepsilon}\not{D}\psi_t(x), & (x, t) \in S^1 \times (T_0, T_0 + \delta),\\ \psi(x, T_0) = \psi_0(x)\end{cases}$$

in

$$\begin{aligned}&\phi \in C^{2+\alpha, 1+\alpha/2}(S^1 \times [T_0, T_0 + \delta), N),\\ &\psi \in C^{2+\alpha, 1+\alpha/2}(S^1 \times [T_0, T_0 + \delta), \Sigma S^1 \otimes \phi^{-1}TN).\end{aligned}$$

We realize that both solutions coincide on $S^1 \times \{T_0\}$ and for this reason, we can glue them to a solution with existence interval $[0, T_0 + \delta)$. Moreover, applying the regularity Theorem 3.24, we find that this solution is smooth. As a matter of fact, the system (4.15), (4.16) has a smooth solution in $[0, T_0 + \delta)$ contradicting the definition of T_0. The uniqueness of (ϕ_t, ψ_t) follows from Theorem 4.11. □

4.4. Convergence

In this section, we want to discuss under which assumptions and in which sense the evolution equations for regularized Dirac-harmonic maps converge as $t \to \infty$. To this end, it is necessary to improve the estimates derived in the previous sections. In particular, we need uniform estimates that do not depend on t. We have already seen in the previous analysis that restrictions on the manifolds M and N do not improve the behaviour of the evolution equations. First of all, we will see that the issue of convergence depends crucially on the norm of the spinor ψ_t.

Proposition 4.14. *Assume that $M = S^1$ with fixed spin structure, N compact and let $(\phi_t, \psi_t) \in C^\infty(M \times [0,\infty), N) \times C^\infty(M \times [0,\infty), \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (4.4) and (4.5). If we find a uniform bound on the spinor ψ_t, namely*

$$|\psi_t|^2 \leq C,$$

then we get a uniform bound on

$$|d\phi_t|^2 + \varepsilon |\not{D}\psi_t|^2 \leq C \tag{4.17}$$

for all $t \in [0, \infty)$. The constant C depends on $M, N, \varepsilon, \psi_0, d\phi_0$ and $\not{D}\psi_0$.

Proof. By assumption, we have a uniform bound on $|\psi_t|^2$. To derive the uniform bound on $|d\phi_t|^2$ and $|\not{D}\psi_t|^2$, we go back into the proof of Theorem 4.7. By the bound on $|\psi_t|^2$, we find that the quantity $F_t := \frac{1}{2}(|d\phi_t|^2 + \varepsilon|\not{D}\psi_t|^2)$ satisfies

$$\frac{\partial F_t}{\partial t} \leq \Delta F_t + \frac{C}{\varepsilon^2} F_t.$$

From the inequality $E_\varepsilon(\phi_t, \psi_t) \leq E_\varepsilon(\phi_0, \psi_0)$, the bound on ψ_t and Young's inequality, we deduce

$$\int_M (|d\phi_t|^2 + \varepsilon|\not{D}\psi_t|^2) dM \leq C.$$

Applying (B.13), the estimates on $|d\phi_t|^2$ and $|\not{D}\psi_t|^2$ follow. □

By the estimate just derived, we are now able to bound the t derivatives of ϕ_t and ψ_t uniformly.

Lemma 4.15. *Assume that $M = S^1$ with fixed spin structure, N compact and let $(\phi_t, \psi_t) \in C^\infty(M \times [0,\infty), N) \times C^\infty(M \times [0,\infty), \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (4.4) and (4.5). If we can control the norm of ψ_t uniformly, then we find*

$$|\frac{\partial \phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 \leq C \tag{4.18}$$

for all $t \in [0,\infty)$. The constant C depends on $M, N, \varepsilon, \psi_0, d\phi_0$ and $\not{D}\psi_0$.

Proof. By Theorem 4.9, the quantity $G_t := \frac{1}{2}\left(|\frac{\partial \phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla}\psi_t}{\partial t}|^2\right)$ satisfies the inequality

$$\frac{\partial G_t}{\partial t} \leq \Delta G_t + Z(t)G_t.$$

Applying the bounds on $|\psi_t|^2, |d\phi_t|^2$ and $|\not{D}\psi_t|^2$, we find that $Z(t)$ is uniformly bounded such that G_t satisfies

$$\frac{\partial G_t}{\partial t} \leq \Delta G_t + CG_t.$$

Integrating over M and with respect to t yields

$$\begin{aligned}
\int_M \left(|\frac{\partial \phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla}\psi_t}{\partial t}|^2\right) dM &\leq C\int_0^\infty \int_M \left(|\frac{\partial \phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla}\psi_t}{\partial t}|^2\right) dMdt \\
&\quad + \int_M \left(|\frac{\partial \phi_t}{\partial t}|^2_{t=0} + |\frac{\tilde{\nabla}\psi_t}{\partial t}|^2_{t=0}\right) dM \\
&\leq E_\varepsilon(\phi_0, \psi_0) + C \\
&\leq C.
\end{aligned}$$

The assertion follows from applying (B.13) again. □

We realize that the convergence of the evolution equations depends crucially on the norm of ψ_t. One way of controlling the norm of ψ_t is to choose the parameter ε large enough such that the parabolic nature of the evolution equation for ψ_t dominates, which basically means that the second order term is sufficiently large to control the first order one.

Proposition 4.16. *Assume that $M = S^1$ with fixed spin structure, N compact and let $(\phi_t, \psi_t) \in C^\infty(M \times [0,\infty), N) \times C^\infty(M \times [0,\infty), \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (4.4) and (4.5). For $\varepsilon \geq 1$ we get a uniform bound of $|\psi_t|^2$ for all $t \in [0,\infty)$.*

Proof. Using the evolution equation (4.5), we calculate

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{2}\int_M |\psi_t|^2 dM &= -\int_M |\not{D}\psi_t|^2 dM + \frac{1}{\varepsilon}\int_M \langle \psi_t, \not{D}\psi_t\rangle dM \\
&\leq -\int_M |\tilde{\nabla}\psi_t|^2 dM + \frac{1}{\varepsilon}\int_M |\psi_t||\tilde{\nabla}\psi_t| dM \\
&\leq (\frac{1}{\varepsilon} - 1)\int_M |\tilde{\nabla}\psi_t|^2 dM,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality and the Poincaré inequality on S^1. After integration with respect to t we find for $\varepsilon \geq 1$

$$\int_M |\psi_t|^2 dM \leq \int_M |\psi_0|^2 dM.$$

We have already seen that $|\psi_t|^2$ satisfies the pointwise equation

$$\frac{\partial}{\partial t}\frac{1}{2}|\psi_t|^2 \leq \Delta\frac{1}{2}|\psi_t|^2 + \frac{1}{4\varepsilon^2}|\psi_t|^2,$$

and by (B.13) we get a uniform bound on $|\psi_t|^2$. □

Remark 4.17. Of course, it would be much nicer if one could bound the norm of ψ_t while keeping the regularizing parameter ε small. Unfortunately, this does not seem to be possible.

After having derived pointwise uniform bounds, the regularity of (ϕ_t, ψ_t) can be improved by applying Schauder estimates again. The convergence of the evolution equations can now be derived by standard methods.

Lemma 4.18 (Convergence)**.** *Assume that $M = S^1$, N compact and suppose that $(\phi_t, \psi_t) \in C^\infty(M \times [0,\infty), N) \times C^\infty(M \times [0,\infty), \Sigma M \otimes \phi_t^{-1}TN)$ is a solution of (4.4) and (4.5). If $\varepsilon \geq 1$ then (ϕ_t, ψ_t) converges to a regularized Dirac-harmonic map $(\phi_\infty, \psi_\infty)$ in $C^2(M,N) \times C^2(M, \Sigma M \otimes \phi_t^{-1}TN)$.*

Proof. First of all, we improve the regularity of our estimates with the help of Schauder theory, as in Proposition 4.12 and find

$$\sup_{t\in[0,\infty)} \left(|\phi(\cdot,t)|_{C^{2+\alpha}(M,N)} + |\frac{\partial\phi}{\partial t}(\cdot,t)|_{C^\alpha(M,N)} \right) \leq C,$$
$$\sup_{t\in[0,\infty)} \left(|\psi(\cdot,t)|_{C^{2+\alpha}(M,\Sigma M\otimes\phi_t^{-1}TN)} + |\frac{\tilde{\nabla}\psi}{\partial t}(\cdot,t)|_{C^\alpha(M,\Sigma M\otimes\phi_t^{-1}TN)} \right) \leq C,$$

but now the constants C do not depend on t. In addition, we have the estimate from the inequality for the energy $E_\varepsilon(\phi,\psi)$

$$\int_0^\infty \int_M \left(|\frac{\partial\phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla}\psi_t}{\partial t}| \right) dM dt \leq C.$$

Hence, there exists a subsequence t_k such that as $k \to \infty$ we have

$$|\frac{\partial\phi(\cdot,t_k)}{\partial t}|^2_{L^2(M\times[0,\infty))} \to 0, \qquad |\frac{\tilde{\nabla}\psi(\cdot,t_k)}{\partial t}|^2_{L^2(M\times[0,\infty))} \to 0.$$

Using Schauder estimates again,

$$\sup_{t\in[0,\infty)}\left(|\phi(\cdot,t_k)|_{C^{2+\alpha}(M,N)}+|\frac{\partial\phi}{\partial t}(\cdot,t_k)|_{C^{\alpha}(M,N)}\right)\leq C,$$

$$\sup_{t\in[0,\infty)}\left(|\psi(\cdot,t_k)|_{C^{2+\alpha}(M,\Sigma M\otimes\phi_t^{-1}TN)}+|\frac{\tilde{\nabla}\psi}{\partial t}(\cdot,t_k)|_{C^{\alpha}(M,\Sigma M\otimes\phi_t^{-1}TN)}\right)\leq C,$$

it follows from the Theorem of Arzela Ascoli that there exists a convergent subsequence, which is also denoted by t_k, such that the pair (ϕ_{t_k},ψ_{t_k}) converges in the space $C^2(M,N)\times C^2(M,\Sigma M\otimes\phi_t^{-1}TN)$ to a limiting map $(\phi_\infty,\psi_\infty)$. Since (ϕ_t,ψ_t) is smooth in t, we find that $(\phi_\infty,\psi_\infty)$ is homotopic to (ϕ_0,ψ_0). $\square$

The smoothness of the limiting map $(\phi_\infty,\psi_\infty)$ follows from elliptic estimates, see Theorem 3.24.

Remark 4.19. In the case of the harmonic map heat flow and the assumption $K^N\leq 0$, it is known that the limit $k\to\infty$ is independent of the chosen subsequence. This result is known as Hartmann's theorem [Har67] and makes use of the fact that the second variation of the energy functional is positive. In the case of regularized Dirac-harmonic maps, the second variation of the energy functional $E_\varepsilon(\phi,\psi)$ is not positive and we cannot derive an analogue of Hartmann's theorem.

The next Lemma shows that the critical points of $E(\phi,\psi)$ and $E_\varepsilon(\phi,\psi)$ are related to each other.

Lemma 4.20 (Critical points of $E_\varepsilon(\phi,\psi)$ and $E(\phi,\psi)$)**.** *For $\varepsilon\neq-\frac{1}{\lambda}$, where λ is an eigenvalue of the twisted Dirac-operator $\not{D}$, the regularized functional $E_\varepsilon(\phi,\psi)$ has the same critical points as $E(\phi,\psi)$.*

Proof. The critical points of the functional $E(\phi,\psi)$ are given by

$$\tau(\phi)=\mathcal{R}(\phi,\psi),\qquad \not{D}\psi=0$$

whereas the critical points of the regularized functional $E_\varepsilon(\phi,\psi)$ are given by

$$\tau(\phi)=\mathcal{R}(\phi,\psi)+\varepsilon\mathcal{R}_c(\phi,\psi),\qquad \not{D}\psi+\varepsilon\not{D}^2\psi=0.$$

It is clear, that if (ϕ,ψ) is a Dirac-harmonic map, then it is also a regularized Dirac-harmonic map.
The other direction is slightly more subtle. Assume that (ϕ,ψ) is a regularized Dirac-harmonic map. Using the equation for ψ and integrating over S^1, we obtain

$$\int_M\langle\psi,\not{D}\psi\rangle dM+\varepsilon\int_M|\not{D}\psi|^2dM=0.$$

For this equation to hold, either ψ must be trivial or $\psi+\varepsilon\not{D}\psi=0$. But we have chosen ε in such a way that the second possibility is excluded. We conclude that $\not{D}\psi=0$. Hence, $\mathcal{R}_c(\phi,\psi)=0$ and therefore the pair (ϕ,ψ) is a Dirac-harmonic map. $\square$

4.5. Removing the Regularization

By the considerations we did so far, we have obtained a *regularized Dirac-harmonic map*. Namely, we constructed a smooth solution to the problem

$$\begin{aligned}\Delta\phi_\infty &= \Gamma(\phi_\infty)(d\phi_\infty, d\phi_\infty) + \mathcal{R}(\phi_\infty, \psi_\infty) + \varepsilon\mathcal{R}_c(\phi_\infty, \psi_\infty),\\ \varepsilon\tilde{\Delta}\psi_\infty &= \not{D}\psi_\infty.\end{aligned}$$

To obtain a *Dirac-harmonic map*, we have to remove the regularization and let $\varepsilon \to 0$. It is easy to see that all estimates that were derived when studying the evolution equations for (ϕ, ψ) do not survive the limit $\varepsilon \to 0$. But since we are dealing with a variational problem, we still have an inequality for the energy $E_\varepsilon(\phi, \psi)$ and together with the Euler-Lagrange equations for *Dirac-harmonic maps* we can infer estimates.
After taking the limit $\varepsilon \to 0$, the pair $(\phi_\infty, \psi_\infty)$ solves

$$\Delta\phi_\infty = \mathrm{I\!I}(d\phi_\infty(e_\alpha), d\phi_\infty(e_\alpha)) + P(\mathrm{I\!I}(e_\alpha \cdot \psi_\infty, d\phi_\infty(e_\alpha)), \psi_\infty), \tag{4.19}$$

$$\not{\partial}\psi_\infty = \mathrm{I\!I}(e_\alpha \cdot \psi_\infty, d\phi_\infty(e_\alpha)), \tag{4.20}$$

where we again embedded the target manifold N in some $\mathbb{R}^q$.
An important tool for our analysis will now be Morrey's inequality in one dimension, which states

$$|\phi|_{C^{0,\frac{1}{2}}} \leq C|\phi|_{H^{1,2}}.$$

Plugging the equation for $\not{D}\psi_\infty$ into the inequality for the energy functional $E(\phi, \psi)$, we acquire the uniform bound

$$\int_M |d\phi_\infty|^2 dM \leq E(\phi_0, \psi_0) \leq C,$$

which enables us to deduce

$$\int_M |\Delta\phi_\infty| dM \leq C\int_M (|d\phi_\infty|^2 + |\psi_\infty|^2 |d\phi_\infty|) dM.$$

Since ψ_∞ takes its values on S^1, its L^∞ norm can be bounded by a constant. Hence, we get that $\phi_\infty \in H^{2,1}(S^1, N)$. Applying the Sobolev embedding theorem in one dimension, we find $\phi_\infty \in H^{1,p}(S^1, N)$ for some $p < \infty$. Utilizing the equation for ϕ_∞ again, we conclude that $\phi_\infty \in H^{2,q}(S^1, N)$ for some $q \geq 2$. By the Sobolev embedding theorem we get that $d\phi_\infty \in L^\infty(S^1, N)$, and by the Schauder estimates, we may conclude that $\phi_\infty \in C^{1+\alpha}(S^1, N)$. From the equation for ψ_∞, we deduce

$$\int_M |\not{\partial}\psi_\infty|^2 dM \leq C\int_M |d\phi_\infty|^2 dM \leq C.$$

Using Morrey's inequality, we find $\psi_\infty \in C^{0,\frac{1}{2}}(S^1, \Sigma S^1 \otimes \phi_\infty^{-1} TN)$. Using the equation for ϕ_∞ again, we find $\phi_\infty \in C^{2+\alpha}(S^1, N)$. This can then be used to improve the regularity of ψ_∞. Iterating this procedure, we conclude that the pair $(\phi_\infty, \psi_\infty)$ is smooth.
This completes the proof of Theorem 4.1.

Remark 4.21. Unfortunately, it is not possible to decide if the limiting map $(\phi_\infty, \psi_\infty)$ is a coupled or an uncoupled Dirac-harmonic map. Moreover, it could also happen that the spinor ψ becomes trivial as $\varepsilon \to 0$.

5. Dirac-harmonic Maps from Riemann Surfaces

5.1. Introduction and Results

Throughout this chapter we assume that (M, h) is a closed, oriented Riemannian surface with fixed spin structure. It is known that every orientable Riemannian surface admits a spin structure, the number of different spin structures can be counted by the genus of the surface, see for example [LM89], p. 88. Thus, it is a topological information.
Again, we study the evolution equations (with $(\phi_0(x), \psi_0(x)) = (\phi(x, 0), \psi(x, 0))$)

$$\frac{\partial \phi_t}{\partial t} = \tau(\phi_t) - \mathcal{R}(\phi_t, \psi_t) - \varepsilon \mathcal{R}_c(\phi_t, \psi_t), \tag{5.1}$$

$$\frac{\tilde{\nabla} \psi_t}{\partial t} = \varepsilon \tilde{\Delta} \psi_t - \not{D} \psi_t. \tag{5.2}$$

The aim of this chapter is to prove a result similar to Struwe's result [Str85] for the harmonic map heat flow from surfaces. Due to the coupling between the fields ϕ and ψ new analytical difficulties arise. Our final result is given by the following

Theorem 5.1. *Suppose (M, h) is a closed Riemannian surface with fixed spin structure and (N, g) is a closed Riemannian manifold. Then for any smooth initial data (ϕ_0, ψ_0) and ε sufficiently large, there exists a global distribution solution*

$$\phi\colon M \times [0, \infty) \to N, \qquad \psi\colon M \times [0, \infty) \to \Sigma M \otimes \phi^{-1} TN$$

of (5.1) and (5.2) on $M \times [0, \infty)$, which is smooth away from at most finitely many singular points $(x_k, t_k), 1 \leq k \leq K$ with $K = K(\varepsilon)$. The solution is unique in this class. The pair $(\phi(\cdot, t), \psi(\cdot, t))$ converges weakly in $H^{1,2}(M, N) \times H^{1,2}(M, \Sigma M \otimes \phi_t^{-1} TN)$ to a regularized Dirac-harmonic map $(\phi_\infty, \psi_\infty)$ as $t \to \infty$ suitably and smoothly away from finitely many points $(x_k, t_k = \infty)$. The pair $(\phi_\infty, \psi_\infty)$ is smooth on $M \setminus \{x_1, \ldots, x_K\}$.

In addition, we sketch how to perform a blowup analysis of the singular points. Namely, at each singular point (x_k, t_k) a non-constant, smooth harmonic map $\bar{\phi} : S^2 \to N$ separates such that for sequences $R_m \to 0, t_m \to t, x_m \to x$ as $m \to \infty$, we have

$$\phi_m(x) = \phi(exp_{x_m}(R_m x), t_m) \to \bar{\phi} \qquad \text{in } H^{2,2}_{loc}(\mathbb{R}^2, N).$$

The spinor ψ becomes trivial during the blowup process.

Remark 5.2. By studying the classical results for the harmonic map heat flow in dimension two we can get an idea what to expect when analyzing the evolution equations for regularized Dirac-harmonic maps. It would be quite a surprise if the more complicated regularized Dirac-harmonic map heat flow would behave better analytically than the harmonic map heat flow. We cannot hope to find a global smooth solution, as already the harmonic map heat flow develops singularities in finite time [CD90]. In addition, we cannot expect to find a unique solution in general since in [BDPvdH02] and [Top02], solutions that are different from Struwe's solution, were constructed.

Remark 5.3. Assume that M is a compact Riemann surface. Then the following terms are invariant under conformal transformations:

$$\int_M |d\phi|^2 dM, \qquad \int_M \langle\psi, \not{D}\psi\rangle dM, \qquad \int_M |\psi|^4 dM.$$

A proof can for example be found in [CJLW06], p. 416, Lemma 3.1. In particular, this means that the Dirac-harmonic map functional $E(\phi,\psi)$ is conformally invariant in dimension two. We will see later that the L^4 norm of the spinor ψ plays an important role in the context of a removable singularity theorem. On the other hand, we note that through the regularization the conformal invariance is broken.

5.2. Energy Estimates and Monotonicity Formulas

In the case that $M = S^1$ we could derive pointwise energy estimates by the maximum principle. These estimates cannot be carried over for M being a closed Riemannian surface due to several non-linearities. Consequently, we are forced to establish integral estimates, both locally on balls and globally on the whole surface M. Before we do so, we will shortly present the tools that we are using in the following.
The first one is a covering argument due to Struwe [Str85], p. 563, Lemma 3.3.

Lemma 5.4. *There exist constants $K, R_0 > 0$ depending only on the manifold M such that for any $R \in (0, R_0]$, there exists a cover on M by balls $B_{\frac{R}{2}}(x_i)$ with the property that at any point $x \in M$ at most K of the balls $B_R(x_i)$ meet.*

We will often make use of the following Sobolev type inequality:

Lemma 5.5. *Assume that $v \in H^{1,2}(M)$. Then the following inequality holds:*

$$\int_M |v|^4 dM \leq C \int_M |v|^2 dM \int_M |\nabla v|^2 dM. \tag{5.3}$$

Proof. The statement follows directly from the two-dimensional Sobolev embedding $|v|_{L^2} \leq C|\nabla v|_{L^1}$ and the Cauchy-Schwarz inequality. □

In addition, we need a local version of the Sobolev inequality from above. Therefore, we set $Q = M \times [0, T)$. By $B_R(x)$ we denote the geodesic ball of radius R around $x \in M$ and i_M denotes the injectivity radius of M. In terms of these quantities we can formulate the following:

Lemma 5.6 (Local Sobolev inequality)**.** *Assume that $v \in H^{1,2}(M)$. Then there exists a constant C such that for any $R \in (0, i_M)$ the following inequality holds:*

$$\int_Q |\nabla v|^4 dM dt \leq C \sup_{(x,t)} \int_{B_{R(x)}} |\nabla v|^2 dM \left(\int_Q |\nabla^2 v|^2 dM dt + \frac{1}{R^2} \int_Q |\nabla v|^2 dM dt \right). \quad (5.4)$$

Proof. A proof can for example be found in [Str08], p. 225, Lemma 6.7. □

Lemma 5.7. *If* $\dim(M) = 2$ *the space $C^\infty(M, N)$ is dense in $H^{1,2}(M, N)$.*

Due to this Lemma sequences in $H^{1,2}(M, N)$ can be approximated by smooth ones.
As a first step, we want to obtain a pointwise bound for the norm of the spinor ψ_t.

Lemma 5.8. *Let $\psi_t \in C^2(M \times [0, T), \Sigma M \otimes \phi_t^{-1} TN))$ be a solution of (5.2). We get a uniform bound on ψ_t*

$$|\psi_t|^2_{L^\infty(M \times [0,T))} \leq C e^{\frac{1}{\varepsilon}}, \quad (5.5)$$

if ε is large enough. The constant C depends on M and the L^2 norm of ψ_0.

Proof. First of all, we derive a pointwise estimate using (5.2)

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} |\psi_t|^2 &= \frac{\varepsilon}{2} \Delta |\psi_t|^2 - \langle \psi_t, \not{D} \psi_t \rangle - \varepsilon |\tilde{\nabla} \psi_t|^2 \\ &\leq \frac{\varepsilon}{2} \Delta |\psi_t|^2 + \sqrt{2} |\psi_t| |\tilde{\nabla} \psi_t| - \varepsilon |\tilde{\nabla} \psi_t|^2 \\ &\leq \frac{\varepsilon}{2} \Delta |\psi_t|^2 + \frac{1}{2\varepsilon} |\psi_t|^2. \end{aligned}$$

If in addition we can also bound the L^2 norm of the spinor ψ_t, we get a uniform pointwise bound on ψ_t by (B.13). Therefore, we compute

$$\begin{aligned} \frac{\partial}{\partial t} \frac{1}{2} \int_M |\psi_t|^2 dM &= -\int_M \langle \psi_t, \not{D} \psi_t \rangle dM - \varepsilon \int_M |\tilde{\nabla} \psi_t|^2 dM \\ &\leq \left(\int_M |\psi_t|^2 dM \right)^{\frac{1}{2}} \left(\int_M \sqrt{2} |\tilde{\nabla} \psi_t|^2 dM \right)^{\frac{1}{2}} - \varepsilon \int_M |\tilde{\nabla} \psi_t|^2 dM \\ &\leq (2^{\frac{1}{4}} C_S \sqrt{\mathrm{vol(M)}} - \varepsilon) \int_M |\tilde{\nabla} \psi|^2 dM, \end{aligned}$$

where we used the Sobolev embedding theorem in dimension two. Thus, if the regularizing parameter ε is large enough, we have obtained a uniform bound on the L^2 norm of ψ_t. □

Since our evolution equations are originating from a variational problem, we get bounds in terms of the initial data (ϕ_0, ψ_0).

Lemma 5.9. *Let $(\phi_t, \psi_t) \in C^2(M \times [0,T), N) \times C^2(M \times [0,T), \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (5.1) and (5.2). If $\int_M |\psi_t|^2 dM \leq C$ we have for all $t \in [0,T)$*

$$\int_M (|d\phi_t|^2 + \varepsilon|\tilde{\nabla}\psi_t|^2) dM + \int_Q \left(\left|\frac{\partial \phi_t}{\partial t}\right|^2 + \left|\frac{\tilde{\nabla}\psi_t}{\partial t}\right|^2 \right) dMdt \leq C.$$

The constant C depends on $M, \varepsilon, E_\varepsilon(\phi_0, \psi_0)$ and ψ_0.

Proof. A direct consequence of the gradient flow is the following equality

$$E_\varepsilon(\phi_t, \psi_t) + \int_Q \left(\left|\frac{\partial \phi_t}{\partial t}\right|^2 + \left|\frac{\tilde{\nabla}\psi_t}{\partial t}\right|^2 \right) dMdt = E_\varepsilon(\phi_0, \psi_0).$$

Subtracting $\frac{1}{2}\langle \psi_t, \not{D}\psi_t \rangle$ on both sides and using

$$-\frac{1}{2}\langle \psi_t, \not{D}\psi_t \rangle \leq \frac{\varepsilon}{8}|\tilde{\nabla}\psi_t|^2 + \frac{1}{2\varepsilon}|\psi_t|^2,$$

we get the desired result. $\square$

In the following we will often need the following combination of quantities

$$\begin{aligned}
E_\varepsilon(\phi_t, \psi_t, B_R) &:= \frac{1}{2}\int_{B_R} (|d\phi_t|^2 + \langle \psi_t, \not{D}\psi_t \rangle + \varepsilon|\tilde{\nabla}\psi_t|^2) dM, \\
F(\phi_t, \psi_t, B_R) &:= \frac{1}{2}\int_{B_R} (|d\phi_t|^2 + \varepsilon|\tilde{\nabla}\psi_t|^2) dM, \\
F(\phi_t, \psi_t) &:= \frac{1}{2}\int_M (|d\phi_t|^2 + \varepsilon|\tilde{\nabla}\psi_t|^2) dM.
\end{aligned}$$

For the further analysis it turns out to be useful to introduce the following function space:

$$V := \left\{ \sup_{0 \leq t \leq T} F(\phi_t, \psi_t) + \int_Q (|\nabla^2\phi|^2 + |\tilde{\nabla}^2\psi|^2) dQ + \int_Q \left(\left|\frac{\partial \phi_t}{\partial t}\right|^2 + \left|\frac{\tilde{\nabla}\psi_t}{\partial t}\right|^2 \right) dQ \right\}.$$

The next Lemma is the analogue of Lemma 3.6 from [Str85]. We want to get local bounds of the L^2 norms of $d\phi_t$ and $\tilde{\nabla}\psi_t$.

Lemma 5.10. *Let $(\phi_t, \psi_t) \in V$ be a solution of (5.1) and (5.2). There exists a constant C such that for $R \in (0, i_M)$ and any $(x,t) \in Q$ there holds the estimate*

$$E_\varepsilon(\phi_t, \psi_t, B_R) \leq \frac{C}{R^2}\int_Q (|d\phi_t|^2 + |\psi_t|^2 + \varepsilon^2|\tilde{\nabla}\psi_t|^2) dMdt + E_\varepsilon(\phi_0, \psi_0, B_{2R}), \tag{5.6}$$

where the constant C only depends on M.

Proof. First of all, we choose a smooth cut-off function η with the following properties

$$\eta \in C^\infty(M), \qquad \eta \geq 0, \qquad \eta = 1 \text{ on } B_R(x_0),$$
$$\eta = 0 \text{ on } M \setminus B_{2R}(x_0), \qquad |\nabla \eta|_{L^\infty} \leq \frac{C}{R},$$

where again $B_R(x_0)$ denotes the geodesic ball of radius R around $x_0 \in M$. In addition, we choose an orthonormal basis $\{e_\alpha, \alpha = 1, 2\}$ on M such that $\nabla_{e_\alpha} e_\beta = \nabla_{\partial_t} e_\alpha = 0$ at the considered point. By a direct calculation we find

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{2}|d\phi_t|^2 &= \partial_{e_\alpha}\langle \frac{\partial \phi_t}{\partial t}, d\phi_t(e_\alpha)\rangle - \langle \frac{\partial \phi_t}{\partial t}, \tau(\phi_t)\rangle, \\
\frac{\partial}{\partial t}\frac{1}{2}\langle \psi_t, \not{D}\psi_t\rangle &= -\partial_{e_\alpha}\frac{1}{2}\langle \frac{\tilde{\nabla}\psi_t}{\partial t}, e_\alpha \cdot \psi_t\rangle + \langle \frac{\tilde{\nabla}\psi_t}{\partial t}, \not{D}\psi_t\rangle + \langle \frac{\partial \phi_t}{\partial t}, \mathcal{R}(\phi_t, \psi_t)\rangle, \\
\frac{\partial}{\partial t}\frac{1}{2}|\tilde{\nabla}\psi_t|^2 &= \langle \mathcal{R}_c(\phi_t, \psi_t), \frac{\partial \phi_t}{\partial t}\rangle + \partial_{e_\alpha}\langle \frac{\tilde{\nabla}\psi_t}{\partial t}, \tilde{\nabla}_{e_\alpha}\psi_t\rangle - \langle \frac{\tilde{\nabla}\psi_t}{\partial t}, \tilde{\nabla}_{e_\alpha}\tilde{\nabla}_{e_\alpha}\psi_t\rangle.
\end{aligned}$$

Multiplying each of the terms with the cut-off function η^2, adding up the three terms and using the evolution equations (5.1) and (5.2), we find

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{1}{2}\int_M \eta^2(|d\phi_t|^2 + \langle \psi_t, \not{D}\psi_t\rangle + \varepsilon|\tilde{\nabla}\psi_t|^2)dM + \int_M \eta^2\left(\left|\frac{\tilde{\nabla}\psi_t}{\partial t}\right|^2 + \left|\frac{\partial \phi_t}{\partial t}\right|^2\right)dM \\
&= \int_M \eta^2 \partial_{e_\alpha}\left(\langle \frac{\partial \phi_t}{\partial t}, d\phi_t(e_\alpha)\rangle - \frac{1}{2}\langle \frac{\tilde{\nabla}\psi_t}{\partial t}, e_\alpha \cdot \psi_t\rangle - \varepsilon\langle \frac{\tilde{\nabla}\psi_t}{\partial t}, \tilde{\nabla}_{e_\alpha}\psi_t\rangle\right)dM.
\end{aligned}$$

Using integration by parts we derive

$$\begin{aligned}
\int_M \eta^2 \partial_{e_\alpha}\langle \frac{\partial \phi_t}{\partial t}, d\phi_t(e_\alpha)\rangle dM &\leq C\int_M |\eta||\nabla \eta||\frac{\partial \phi_t}{\partial t}||d\phi_t|dM, \\
\int_M \eta^2 \partial_{e_\alpha}\langle \frac{\tilde{\nabla}\psi_t}{\partial t}, e_\alpha \cdot \psi_t\rangle dM &\leq C\int_M |\eta||\nabla \eta||\frac{\tilde{\nabla}\psi_t}{\partial t}||\psi_t|dM, \\
\int_M \eta^2 \partial_{e_\alpha}\langle \frac{\tilde{\nabla}\psi_t}{\partial t}, \tilde{\nabla}_{e_\alpha}\psi_t\rangle dM &\leq C\int_M |\eta||\nabla \eta||\frac{\tilde{\nabla}\psi_t}{\partial t}||\tilde{\nabla}\psi_t|dM.
\end{aligned}$$

Applying Young's inequality and by the properties of the cut-off function η, we find

$$\frac{\partial}{\partial t}E_\varepsilon(\phi_t, \psi_t, B_R) \leq \frac{C}{R^2}\int_M (|d\phi_t|^2 + |\psi_t|^2 + \varepsilon^2|\tilde{\nabla}\psi_t|^2)dM.$$

Integration with respect to t yields the result. □

We can use the previous Lemma to formulate monotonicity formulas for $F(\phi_t, \psi_t, B_R)$, namely by Young's inequality and the "monotonicity formula" for the local energy $E_\varepsilon(\phi_t, \psi_t, B_R)$, we get

$$F(\phi_t, \psi_t, B_R) \leq 2E_\varepsilon(\phi_0, \psi_0, B_{2R}) + C\frac{T}{R^2} + \frac{1}{2\varepsilon}\int_{B_R} |\psi_t|^2 dM. \tag{5.7}$$

Roughly speaking, we want to make the left hand side of this inequality as small as we have to. This can be achieved by choosing the initial data (ϕ_0, ψ_0), the radius R of the ball B_R and the time T appropriately. More precisely, we get the following

Corollary 5.11. *Let $(\phi_t, \psi_t) \in V$ be a solution of (5.1) and (5.2). Assume that $|\psi_t|_{L^\infty(M\times[0,T))} \leq C$. Then there exists a constant δ_1 such that for $R \in (0, i_M)$ and any $(x,t) \in Q$ there holds the estimate*

$$\sup_{(x,t)\in Q} F(\phi_t, \psi_t, B_R) \leq \delta_1. \tag{5.8}$$

Proof. From Lemma 5.10 and the bound on the norm of ψ_t, it follows that for any δ_1 and (ϕ_0, ψ_0) suitably, there exists a number $R > 0$ for which

$$\sup_{x\in M}\left(2E_\varepsilon(\phi_0, \psi_0, B_{2R}) + \frac{1}{2\varepsilon}\int_{B_R} |\psi_t|^2 dM\right) < \frac{\delta_1}{2}. \tag{5.9}$$

For $T_1 = \frac{\delta_1 R^2}{2C}$ we then get

$$\sup_{\substack{x\in M\\ 0\leq t\leq T_1}} F(\phi_t, \psi_t, B_R(x)) \leq \delta_1,$$

such that the desired estimate holds. □

In order to turn the Laplace type terms into full second derivatives, we will make use of the following Bochner type formulas:

Lemma 5.12 (Bochner type formulas)**.** *For a map $\phi : M \to N$ and a spinor along the map $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$ the following Bochner type formulas hold:*

$$\begin{aligned}\int_M |\tau(\phi)|^2 dM &= \int_M |\nabla d\phi|^2 dM - \int_M \langle R^N(d\phi(e_\alpha), d\phi(e_\beta))d\phi(e_\alpha), d\phi(e_\beta)\rangle dM \\ &\quad + \int_M \langle d\phi(Ric^M(e_\beta)), d\phi(e_\beta)\rangle dM. \qquad (5.10)\\ \int_M |\tilde\Delta\psi|^2 dM &= \int_M |\tilde\nabla^2\psi|^2 dM + \langle R^{E_1}(e_\alpha, e_\beta)\tilde\nabla_{e_\beta}\psi, \tilde\nabla_{e_\alpha}\psi\rangle_{E_1} dM \\ &\quad + \langle R^{E_2}(e_\alpha, e_\beta)\psi, \tilde\nabla_{e_\beta}\tilde\nabla_{e_\alpha}\psi\rangle_{E_2} dM \qquad (5.11)\end{aligned}$$

with the vector bundles $E_1 = T^*M \otimes \Sigma M \otimes \phi^{-1}TN$ and $E_2 = T^*M \otimes E_1$.

Proof. The first statement follows from

$$\nabla_{e_\beta}\nabla_{e_\alpha} d\phi(e_\alpha) = \nabla_{e_\alpha}\nabla_{e_\beta} d\phi(e_\alpha) + R^N(d\phi(e_\beta), d\phi(e_\alpha))d\phi(e_\alpha) - d\phi(Ric^M(e_\beta))$$

and integration by parts. For the second statement we compute

$$\begin{aligned}\int_M |\tilde\nabla^*_{e_\alpha}\tilde\nabla_{e_\alpha}\psi|^2 dM &= \int_M \langle \tilde\nabla_{e_\beta}\tilde\nabla^*_{e_\alpha}\tilde\nabla_{e_\alpha}\psi, \tilde\nabla_{e_\beta}\psi\rangle dM \\ &= \int_M \langle \tilde\nabla^*_{e_\alpha}\tilde\nabla_{e_\beta}\tilde\nabla_{e_\alpha}\psi, \tilde\nabla_{e_\beta}\psi\rangle dM + \langle R^{E_1}(e_\alpha, e_\beta)\tilde\nabla_{e_\beta}\psi, \tilde\nabla_{e_\alpha}\psi\rangle dM.\end{aligned}$$

Using integration by parts again and interchanging second derivatives the curvature term on the bundle E_2 pops up. $\square$

We are now able to bound the L^2 norm of the second derivatives of (ϕ_t, ψ_t) on $M \times [0, T)$.

Theorem 5.13. *Let $(\phi_t, \psi_t) \in V$ be a solution of (5.1) and (5.2) and moreover, assume that $|\psi_t|_{L^\infty(M\times[0,T))} \leq C$. For $R \in (0, i_M)$ there exists $\delta_1 > 0$ such that if $\sup_{(x,t)\in M\times[0,T)} F(\phi_t, \psi_t, B_R(x)) < \delta_1$, we have for all $t \in [0, T)$*

$$\int_Q \left(|\nabla d\phi_t|^2 + \varepsilon^2|\tilde{\nabla}^2\psi_t|^2\right) dMdt \leq C\left(1 + \frac{T}{R^2}\right), \tag{5.12}$$

where the constant C depends on $M, N, \varepsilon, \psi_0, d\phi_0$ and $\tilde{\nabla}\psi_0$.

Proof. Using the evolution equations (5.1) and (5.2), we compute

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{1}{2}\int_M (|d\phi_t|^2 + \varepsilon|\tilde{\nabla}\psi_t|^2)dM + \int_M (|\tau(\phi_t)|^2 + \varepsilon^2|\tilde{\Delta}\psi_t|^2)dM \\
&\quad = \int_M (\varepsilon\langle\mathcal{R}(\phi_t,\psi_t), \mathcal{R}_c(\phi_t,\psi_t)\rangle + \varepsilon\langle\tilde{\Delta}\psi_t, \not{D}\psi_t\rangle - \varepsilon^2|\mathcal{R}_c(\phi_t,\psi_t)|^2)dM \\
&\quad\quad + \int_M (\langle\tau(\phi_t), \mathcal{R}(\phi_t,\psi_t) + 2\varepsilon\mathcal{R}_c(\phi_t,\psi_t)\rangle)dM.
\end{aligned}$$

Applying Young's inequality and estimating the terms on the right hand side, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{1}{2}\int_M (|d\phi_t|^2 + \varepsilon|\tilde{\nabla}\psi_t|^2)dM + \frac{1}{2}\int_M |\tau(\phi_t)|^2 + \varepsilon^2|\tilde{\Delta}\psi_t|^2)dM \\
&\leq \int_M \big(\frac{1}{2}|\mathcal{R}(\phi_t,\psi_t)|^2 + \varepsilon\langle\mathcal{R}(\phi_t,\psi_t), \mathcal{R}_c(\phi_t,\psi_t)\rangle + \varepsilon^2|\mathcal{R}_c(\phi_t,\psi_t)|^2 + \frac{1}{2}|\not{D}\psi_t|^2\big)dM \\
&\leq C\int_M (|d\phi_t|^2|\psi_t|^4 + \varepsilon^2|d\phi_t|^2|\tilde{\nabla}\psi_t|^2|\psi_t|^2 + |\tilde{\nabla}\psi_t|^2)dM \\
&\leq C\int_M (|d\phi_t|^2 + \varepsilon^2|d\phi_t|^2|\tilde{\nabla}\psi_t|^2 + |\tilde{\nabla}\psi_t|^2)dM.
\end{aligned}$$

As a next step we transform the Laplace type terms into second derivatives, therefore we apply the Bochner type formulas (5.10), (5.11), from which we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{1}{2}\int_M (|d\phi_t|^2 + \varepsilon|\tilde{\nabla}\psi_t|^2)dM + C\int_M (|\nabla d\phi_t|^2 + \varepsilon^2|\tilde{\nabla}^2\psi_t|^2)dM \\
&\quad \leq C\left(\int_M (|d\phi_t|^2 + |\tilde{\nabla}\psi_t|^2)dM + \int_M (|d\phi_t|^4 + \varepsilon^2|\tilde{\nabla}\psi_t|^4)dM + C\right),
\end{aligned}$$

where we estimated all curvature contributions. Finally, we apply the local Sobolev inequality (5.4) to $\int_M |d\phi_t|^4 dM$ and $\int_M |\tilde{\nabla}\psi_t|^4 dM$, which leads to

$$\begin{aligned}&\frac{\partial}{\partial t}\frac{1}{2}\int_M (|d\phi_t|^2 + \varepsilon|\tilde{\nabla}\psi_t|^2)dM + C\int_M (|\nabla d\phi_t|^2 + \varepsilon^2|\tilde{\nabla}^2\psi_t|^2)dM \\ &\leq C\Big(\int_M (|d\phi_t|^2 + |\tilde{\nabla}\psi_t|^2)dM + \frac{\delta_1}{R^2}\int_M (|d\phi_t|^2 + \varepsilon|\tilde{\nabla}\psi_t|^2)dM \\ &+ \delta_1 \int_M (|\nabla d\phi_t|^2 + \varepsilon^2|\tilde{\nabla}^2\psi_t|^2)dM + C\Big).\end{aligned}$$

Choosing δ_1 small enough, the terms containing the second derivatives on the right hand side can be absorbed into the left hand side. Integrating with respect to t yields the result. □

Using the bounds on the second derivatives, we can apply the Sobolev embedding theorem to bound $\int_Q |d\phi_t|^4 dMdt$ and $\int_Q |\tilde{\nabla}\psi_t|^4 dMdt$.

Corollary 5.14. *Let $(\phi_t, \psi_t) \in V$ be a solution of (5.1) and (5.2). In addition, assume that $|\psi_t|_{L^\infty(M\times[0,T))} \leq C$. Then for $R \in (0, i_M)$ there exists $\delta_1 > 0$ such that if $\sup_{(x,t)\in M\times[0,T)} F(\phi_t, \psi_t, B_R(x)) < \delta_1$, we have for all $t \in [0,T)$*

$$\int_Q |d\phi_t|^4 dMdt \leq Cf_1(t), \tag{5.13}$$

$$\int_Q |\tilde{\nabla}\psi_t|^4 dMdt \leq Cf_2(t), \tag{5.14}$$

with $f_i(t)$ satisfying $f_i(t) \to 0$ as $t \to 0$ for $i = 1, 2$. In particular, we also get the bounds

$$\int_Q |du|^4 dMdt \leq Cf_1(t), \tag{5.15}$$

$$\int_Q (|\nabla\psi|^4 + |\mathrm{I\!I}(\psi, du(e_\alpha))|^4)dMdt \leq Cf_2(t) \tag{5.16}$$

using the isometric embedding ι. Here, $u = \iota\circ\phi : M \to \mathbb{R}^q$ and $\psi \in \Gamma(\Sigma M \otimes T\mathbb{R}^q)$. The constant C depends on $M, N, \varepsilon, \psi_0, d\phi_0$ and $\tilde{\nabla}\psi_0$.

Proof. The bounds follow from the Sobolev embedding in two dimensions and the previous estimates, namely

$$\begin{aligned}\int_Q |d\phi_t|^4 dMdt &\leq C\int_Q |d\phi_t|^2 dMdt \int_Q |\nabla d\phi_t|^2 dMdt \\ &\leq Cf_1(t).\end{aligned}$$

The estimate on $\int_Q |\tilde{\nabla}\psi_t|^4 dMdt$ can be derived by the same method. □

Corollary 5.15. *Choosing δ_1 small enough and integrating over a small time interval $|t-s| \leq \delta_2$, we can achieve*

$$\int_s^t \int_M |d\phi_t|^4 dM dt \leq C, \qquad \int_s^t \int_M |\tilde{\nabla}\psi_t|^4 dM dt \leq C \tag{5.17}$$

and the right hand side can be made as small as needed. The constant C depends on $M, N, R, \delta_1, \delta_2, \varepsilon, \psi_0, d\phi_0$ and $\tilde{\nabla}\psi_0$.

So far, we have derived integral estimates of the second derivatives on $Q = M \times [0,T)$. In order to turn these estimates into estimates on M, we have to gain control over the derivatives with respect to t of the pair (ϕ_t, ψ_t). For this purpose we make use of the following

Lemma 5.16. *Let $(\phi_t, \psi_t) \in V$ be a solution of (5.1) and (5.2). Then we have for all $t \in [0,T)$*

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{2}\int_M |\frac{\partial \phi_t}{\partial t}|^2 dM &= \int_M \Big(-|\nabla \frac{\partial \phi_t}{\partial t}|^2 + \langle R^N(d\phi_t(e_\alpha), \frac{\partial \phi_t}{\partial t}) d\phi_t(e_\alpha), \frac{\partial \phi_t}{\partial t}\rangle \qquad (5.18)\\
&\quad -\langle \frac{\nabla}{\partial t}\mathcal{R}(\psi_t,\phi_t), \frac{\partial \phi_t}{\partial t}\rangle - \varepsilon \langle \frac{\nabla}{\partial t}\mathcal{R}_c(\psi_t,\phi_t), \frac{\partial \phi_t}{\partial t}\rangle \Big) dM \\
\frac{\partial}{\partial t}\frac{1}{2}\int_M |\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 dM &= \int_M \Big(-\varepsilon |\tilde{\nabla}\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 - \langle \frac{\tilde{\nabla}\psi_t}{\partial t}, \frac{\tilde{\nabla}}{\partial t} \not{D}\psi_t\rangle \qquad (5.19)\\
&\quad +\varepsilon \langle \frac{\tilde{\nabla}\psi_t}{\partial t}, R^{E_1}(\partial_t, e_\alpha)\tilde{\nabla}_{e_\alpha}\psi_t\rangle - \varepsilon \langle \tilde{\nabla}_{e_\alpha}\frac{\tilde{\nabla}\psi_t}{\partial t}, R^{E_1}(\partial_t, e_\alpha)\psi_t\rangle \Big) dM
\end{aligned}$$

with the vector bundle $E_1 = T^*M \otimes \Sigma M \otimes \phi_t^{-1}TN$.

Proof. The first equation directly follows from integrating the pointwise equation (3.8) and the divergence theorem. For the second one consider

$$\frac{\partial}{\partial t}\frac{1}{2}|\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 = \langle \frac{\tilde{\nabla}}{\partial t}(-\not{D}\psi_t + \varepsilon\tilde{\Delta}\psi_t), \frac{\tilde{\nabla}\psi_t}{\partial t}\rangle.$$

Commuting space and time derivatives, we find

$$\frac{\tilde{\nabla}}{\partial t}\tilde{\nabla}^*_{e_\alpha}\tilde{\nabla}_{e_\alpha}\psi = R^{E_1}(\partial_t, e_\alpha)\tilde{\nabla}_{e_\alpha}\psi + \tilde{\nabla}_{e_\alpha}R^{E_1}(\partial_t, e_\alpha)\psi + \tilde{\nabla}^*_{e_\alpha}\tilde{\nabla}_{e_\alpha}\frac{\tilde{\nabla}\psi}{\partial t}$$

with the vector bundle $E_1 = T^*M \otimes \Sigma M \otimes \phi_t^{-1}TN$. Combining both equations and integrating by parts yields the result. □

Theorem 5.17. *Let $(\phi_t, \psi_t) \in V$ be a solution of (5.1) and (5.2). Furthermore, assume that $|\psi_t|_{L^\infty(M\times[0,T))} \leq C$. For $\tau > 0$, provided $\sup_{(x,t)\in M\times[0,T)} F(\phi_t, \psi_t, B_R(x))) < \delta_1$ is small enough, we get that*

$$\sup_{2\tau \leq t \leq T} \int_M \left(|\frac{\partial \phi(\cdot,t)}{\partial t}|^2 + |\frac{\tilde{\nabla}\psi(\cdot,t)}{\partial t}|^2 \right) dM \leq C(1+\tau^{-1}), \tag{5.20}$$

where the constant C depends on $M, N, R, \delta_1, \delta_2, \varepsilon, \tau, \psi_0, d\phi_0$ and $\tilde{\nabla}\psi_0$.

Proof. First of all, we choose an orthonormal basis $\{e_\alpha, \alpha = 1,2\}$ on M such that $\nabla_{\partial_t} e_\alpha = 0$ at a considered point. Combining both equations from Lemma 5.16 we get

$$
\begin{aligned}
&\frac{\partial}{\partial t}\frac{1}{2}\int_M \Big(|\frac{\partial\phi_t}{\partial t}|^2 + |\frac{\tilde\nabla\psi_t}{\partial t}|^2\Big)dM + \int_M \Big(|\nabla\frac{\partial\phi_t}{\partial t}|^2 + \varepsilon|\tilde\nabla\frac{\tilde\nabla\psi_t}{\partial t}|^2\Big)dM\\
&= \int_M \Big(\langle R^N(d\phi_t(e_\alpha),\frac{\partial\phi_t}{\partial t})d\phi_t(e_\alpha),\frac{\partial\phi_t}{\partial t}\rangle - \langle\frac{\nabla}{\partial t}\mathcal{R}(\psi_t,\phi_t),\frac{\partial\phi_t}{\partial t}\rangle - \varepsilon\langle\frac{\nabla}{\partial t}\mathcal{R}_c(\psi_t,\phi_t),\frac{\partial\phi_t}{\partial t}\rangle\\
&\quad - \langle\frac{\tilde\nabla\psi_t}{\partial t},\frac{\tilde\nabla}{\partial t}\not{D}\psi_t\rangle + \varepsilon\langle\frac{\tilde\nabla\psi_t}{\partial t},R^{E_1}(\partial_t,e_\alpha)\tilde\nabla_{e_\alpha}\psi_t\rangle - \varepsilon\langle\tilde\nabla_{e_\alpha}\frac{\tilde\nabla\psi_t}{\partial t},R^{E_1}(\partial_t,e_\alpha)\psi_t\rangle\Big)dM\\
&= A_1 + A_2 + A_3 + A_4 + A_5 + A_6.
\end{aligned}
$$

We have to estimate all terms on the right hand side, starting with the A_1 term

$$
\int_M \langle R^N(d\phi_t(e_\alpha),\frac{\partial\phi_t}{\partial t})d\phi_t(e_\alpha),\frac{\partial\phi_t}{\partial t}\rangle dM \leq C\int_M |d\phi_t|^2|\frac{\partial\phi_t}{\partial t}|^2 dM := I_1.
$$

Using the pointwise estimate

$$
\begin{aligned}
|\langle\frac{\nabla}{\partial t}\mathcal{R}(\phi_t,\psi_t),\frac{\partial\phi_t}{\partial t}\rangle| \quad&\leq\quad C\big(|\frac{\partial\phi_t}{\partial t}|^2|d\phi_t||\psi_t|^2 + |d\phi_t||\frac{\partial\phi_t}{\partial t}||\frac{\tilde\nabla\psi_t}{\partial t}||\psi_t|\\
&\qquad + |\frac{\nabla}{\partial t}d\phi_t||\frac{\partial\phi_t}{\partial t}||\psi_t|^2\big)
\end{aligned}
$$

and the fact that ψ_t is bounded uniformly, we estimate the A_2 term as

$$
\begin{aligned}
\int_M |\langle\frac{\nabla}{\partial t}\mathcal{R}(\phi_t,\psi_t),\frac{\partial\phi_t}{\partial t}\rangle| dM \quad&\leq\quad C\Big(\int_M |\frac{\partial\phi_t}{\partial t}|^2|d\phi_t|dM + \int_M |d\phi_t||\frac{\partial\phi_t}{\partial t}||\frac{\tilde\nabla\psi_t}{\partial t}|dM\\
&\qquad + \int_M |\frac{\nabla}{\partial t}d\phi_t||\frac{\partial\phi_t}{\partial t}|dM\Big) := I_2 + I_3 + I_4.
\end{aligned}
$$

To proceed we again consider the pointwise estimate

$$
\begin{aligned}
|\langle\frac{\nabla}{\partial t}\mathcal{R}_c(\phi_t,\psi_t),\frac{\partial\phi_t}{\partial t}\rangle| \quad&\leq\quad C\big(|\frac{\partial\phi_t}{\partial t}|^2|d\phi_t||\tilde\nabla\psi_t||\psi_t| + |\tilde\nabla\psi_t||d\phi_t||\frac{\partial\phi_t}{\partial t}||\frac{\tilde\nabla\psi_t}{\partial t}||\psi_t|\\
&\qquad + |\frac{\tilde\nabla}{\partial t}\tilde\nabla\psi_t||d\phi_t||\frac{\partial\phi_t}{\partial t}||\psi_t| + |\tilde\nabla\psi_t||\frac{\nabla}{\partial t}d\phi_t||\frac{\partial\phi_t}{\partial t}||\psi_t|\big)
\end{aligned}
$$

and together with the bound on ψ_t, we estimate the A_3 term as

$$
\begin{aligned}
&\int_M |\langle\frac{\nabla}{\partial t}\mathcal{R}_c(\phi_t,\psi_t),\frac{\partial\phi_t}{\partial t}\rangle| dM\\
&\qquad\leq C\Big(\int_M |\frac{\partial\phi_t}{\partial t}|^2|d\phi_t||\tilde\nabla\psi_t|dM + \int_M |\tilde\nabla\psi_t||d\phi_t||\frac{\partial\phi_t}{\partial t}||\frac{\tilde\nabla\psi_t}{\partial t}|dM\\
&\qquad\quad + \int_M |\frac{\tilde\nabla}{\partial t}\tilde\nabla\psi_t||d\phi_t||\frac{\partial\phi_t}{\partial t}|dM + \int_M |\tilde\nabla\psi_t||\frac{\nabla}{\partial t}d\phi_t||\frac{\partial\phi_t}{\partial t}|dM\Big)\\
&\qquad := I_5 + I_6 + I_7 + I_8.
\end{aligned}
$$

As a next step, we want to control the terms arising from interchanging covariant spinorial derivatives, namely A_4, A_5 and A_6. Starting with A_4, we first of all calculate

$$\frac{\tilde{\nabla}}{\partial t}\tilde{\nabla}_{e_\alpha}\psi_t = R^N(d\phi_t(\partial_t), d\phi_t(e_\alpha))\psi_t + \tilde{\nabla}_{e_\alpha}\frac{\tilde{\nabla}\psi_t}{\partial t},$$

and by Young's inequality we get the following estimate

$$\begin{aligned}
&\int_M |\langle\frac{\tilde{\nabla}\psi_t}{\partial t}, \frac{\tilde{\nabla}}{\partial t}\not{D}\psi_t\rangle| dM \\
&\quad\leq C\Big(\int_M |\frac{\partial\phi_t}{\partial t}||d\phi_t||\psi_t||\frac{\tilde{\nabla}\psi_t}{\partial t}|dM + \int_M |\frac{\tilde{\nabla}\psi_t}{\partial t}||\tilde{\nabla}\frac{\tilde{\nabla}\psi_t}{\partial t}|dM\Big) \\
&\quad\leq C\Big(\int_M |\frac{\partial\phi_t}{\partial t}|^2|d\phi_t|^2 dM + \int_M |\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 dM\Big) + \frac{\varepsilon}{8}\int_M |\tilde{\nabla}\frac{\tilde{\nabla}\psi_t}{\partial t}|dM,
\end{aligned}$$

where we again used the bound on ψ_t. We continue with the A_5 and A_6 term

$$\begin{aligned}
A_5 \leq \int_M |\langle\frac{\tilde{\nabla}\psi_t}{\partial t}, R^{E_1}(\partial_t, e_\alpha)\tilde{\nabla}_{e_\alpha}\psi_t\rangle|dM &\leq C\int_M |d\phi_t||\frac{\partial\phi_t}{\partial t}||\frac{\tilde{\nabla}\psi_t}{\partial t}||\tilde{\nabla}\psi_t|dM := I_9, \\
A_6 \leq \int_M |\langle\tilde{\nabla}_{e_\alpha}\frac{\tilde{\nabla}\psi_t}{\partial t}, R^{E_1}(\partial_t, e_\alpha)\psi_t\rangle|dM &\leq C\int_M |d\phi_t||\frac{\partial\phi_t}{\partial t}||\tilde{\nabla}\frac{\tilde{\nabla}\psi_t}{\partial t}|dM := I_{10},
\end{aligned}$$

and move on by rearranging the terms I_1 up to I_{10}. The I_1 term is already in the shape that we need it. Concerning the other contributions, we apply Young's inequality several times:

$$\begin{aligned}
I_2 &= \int_M |\frac{\partial\phi_t}{\partial t}|^2|d\phi_t|dM \leq \frac{1}{2}\int_M |\frac{\partial\phi_t}{\partial t}|^2|d\phi_t|^2 dM + \frac{1}{2}\int_M |\frac{\partial\phi_t}{\partial t}|^2 dM, \\
I_3 &= \int_M |d\phi_t||\frac{\partial\phi_t}{\partial t}||\frac{\tilde{\nabla}\psi_t}{\partial t}|dM \leq \frac{1}{2}\int_M |d\phi_t|^2|\frac{\partial\phi_t}{\partial t}|^2 dM + \frac{1}{2}\int_M |\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 dM, \\
I_4 &= C\int_M |\frac{\nabla}{\partial t}d\phi_t||\frac{\partial\phi_t}{\partial t}|dM \leq \frac{1}{4}\int_M |\frac{\nabla}{\partial t}d\phi_t|^2 dM + C^2\int_M |\frac{\partial\phi_t}{\partial t}|^2 dM, \\
I_5 &= \int_M |\frac{\partial\phi_t}{\partial t}|^2|d\phi_t||\tilde{\nabla}\psi_t|dM \leq \frac{1}{2}\int_M |\frac{\partial\phi_t}{\partial t}|^2|d\phi_t|^2 dM + \frac{1}{2}\int_M |\frac{\partial\phi_t}{\partial t}|^2|\tilde{\nabla}\psi_t|^2 dM, \\
I_6 &= \int_M |\tilde{\nabla}\psi_t||d\phi_t||\frac{\partial\phi_t}{\partial t}||\frac{\tilde{\nabla}\psi_t}{\partial t}|dM \leq \frac{1}{2}\int_M |d\phi_t|^2|\frac{\partial\phi_t}{\partial t}|^2 dM + \frac{1}{2}\int_M |\tilde{\nabla}\psi_t|^2|\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 dM.
\end{aligned}$$

We note that the I_{10} contribution can be absorbed into the I_7 term, which may be estimated as

$$I_7 = C\int_M |d\phi_t||\frac{\partial\phi_t}{\partial t}||\frac{\tilde{\nabla}}{\partial t}\tilde{\nabla}\psi_t|dM \leq C\int_M |d\phi|^2|\frac{\partial\phi_t}{\partial t}|^2 dM + \frac{\varepsilon}{8}\int_M |\tilde{\nabla}\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 dM.$$

We interchanged covariant derivatives and estimated the curvature contributions. To estimate the I_8 term, we apply Young's inequality once more

$$I_8 = C\int_M |\tilde{\nabla}\psi_t||\frac{\nabla}{\partial t}d\phi_t||\frac{\partial\phi_t}{\partial t}|dM \quad\leq\quad \frac{1}{4}\int_M |\frac{\nabla}{\partial t}d\phi_t|^2 dM + C^2\int_M |\frac{\partial\phi_t}{\partial t}|^2|\tilde{\nabla}\psi_t|^2 dM.$$

Finally, the I_9 term can be absorbed into the I_6 term. Note that $\nabla \frac{\partial \phi}{\partial t} = \frac{\nabla}{\partial t} d\phi$, which is due to the torsion freeness of the connection. We sum up the different contributions and find the following inequality

$$\begin{aligned} &\frac{\partial}{\partial t} \frac{1}{2} \int_M \left(|\frac{\partial \phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla} \psi_t}{\partial t}|^2 \right) dM + \frac{1}{2} \int_M \left(|\nabla \frac{\partial \phi_t}{\partial t}|^2 + \varepsilon |\tilde{\nabla} \frac{\tilde{\nabla} \psi_t}{\partial t}|^2 \right) dM \\ \leq & C \Big(\int_M |d\phi_t|^2 |\frac{\partial \phi_t}{\partial t}|^2 dM + \int_M |\frac{\partial \phi_t}{\partial t}|^2 |\tilde{\nabla} \psi_t|^2 dM \\ &+ \int_M |\frac{\tilde{\nabla} \psi_t}{\partial t}|^2 |\tilde{\nabla} \psi_t|^2 dM + \int_M \left(|\frac{\partial \phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla} \psi_t}{\partial t}|^2 \right) dM \Big). \end{aligned}$$

We used part of the second order terms on the left hand side to absorb the second order terms from the right hand side.
Integrating with respect to t over the domain $\tau \leq s < t \leq T$ we get

$$\begin{aligned} &\int_s^t dt \frac{\partial}{\partial t} \frac{1}{2} \int_M \left(|\frac{\partial \phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla} \psi_t}{\partial t}|^2 \right) dM + \frac{1}{2} \int_s^t \int_M \left(|\nabla \frac{\partial \phi_t}{\partial t}|^2 + \varepsilon |\tilde{\nabla} \frac{\tilde{\nabla} \psi_t}{\partial t}|^2 \right) dM dt \\ \leq & C \Big(\int_s^t \int_M |d\phi_t|^2 |\frac{\partial \phi_t}{\partial t}|^2 dM dt + \int_s^t \int_M |\frac{\partial \phi_t}{\partial t}|^2 |\tilde{\nabla} \psi_t|^2 dM dt \\ &+ \int_s^t \int_M |\frac{\tilde{\nabla} \psi_t}{\partial t}|^2 |\tilde{\nabla} \psi_t|^2 dM dt + \int_s^t \int_M \left(|\frac{\partial \phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla} \psi_t}{\partial t}|^2 \right) dM dt \Big). \end{aligned}$$

The last term can be bounded in terms of the initial data and the L^2-norm of ψ_t by Lemma 5.9. We use another type of Sobolev inequality(similar to (5.4) for $|t-s| \leq 1$) to bound the mixed terms like $\int_s^t \int_M |\frac{\partial \phi_t}{\partial t}|^2 |\tilde{\nabla} \psi_t|^2 dM dt$, more precisely

$$\begin{aligned} &\int_s^t \int_M |d\phi_t|^2 |\frac{\partial \phi_t}{\partial t}|^2 dM dt \\ &\leq \left(\int_s^t \int_M |d\phi_t|^4 dM dt \right)^{\frac{1}{2}} \left(\sup_{s \leq \theta \leq t} \int_M |\frac{\partial \phi}{\partial t}(\cdot, \theta)|^2 dM + \int_s^t \int_M |\nabla \frac{\partial \phi_t}{\partial t}|^2 dM dt \right) \end{aligned}$$

and similarly for both of the other two terms.
Choosing $t - s < \delta_2$ sufficiently small, applying the Sobolev inequality and the estimates from Corollary 5.15, we can absorb part of the right hand side in the left and obtain

$$\begin{aligned} &\int_M \left(|\frac{\partial \phi_t(\cdot, t)}{\partial t}|^2 + |\frac{\tilde{\nabla} \psi_t(\cdot, t)}{\partial t}|^2 \right) dM \\ &\qquad \leq \inf_{t-\delta_2 \leq s \leq t} C \int_M \left(|\frac{\partial \phi_t(\cdot, s)}{\partial t}|^2 + |\frac{\tilde{\nabla} \psi_t(\cdot, s)}{\partial t}|^2 \right) dM + C. \end{aligned}$$

Finally, we estimate the infimum by the mean value, more precisely

$$\begin{aligned}\sup_{2\tau\le t\le T}\int_M\Big(|\frac{\partial\phi(\cdot,t)}{\partial t}|^2+|\frac{\tilde\nabla\psi(\cdot,t)}{\partial t}|^2\Big)dM&\\ &\le C(1+\tau^{-1})\int_s^t\int_M\left(|\frac{\partial\phi_t}{\partial t}|^2+|\frac{\tilde\nabla\psi_t}{\partial t}|^2\right)dMdt+C\\ &\le C(1+\tau^{-1})+C.\end{aligned}$$

Hence, we get the desired bound. ☐

Theorem 5.18 (Bounding second derivatives)**.** *Let $(\phi_t,\psi_t)\in V$ be a solution of (5.1) and (5.2). Assume that $|\psi_t|_{L^\infty(M\times[0,T))}\le C$ and $\sup_{(x,t)\in M\times[0,T)}F(\phi_t,\psi_t,B_R(x)))<\delta_1$, then we have*

$$\int_M(|\nabla^2\phi(\cdot,t)|^2+\varepsilon^2|\tilde\nabla^2\psi(\cdot,t)|^2)dM\le C, \tag{5.21}$$

where the constant C depends on $M,N,R,\delta_1,\delta_2,\varepsilon,\tau,\psi_0,d\phi_0$ and $\tilde\nabla\psi_0$.

Proof. With the help of the previous estimates we can now bound the full second derivatives of (ϕ_t,ψ_t) in L^2. By the evolution equations (5.1), (5.2) and Young's inequality, we find

$$\begin{aligned}\int_M|\tau(\phi_t)|^2dM&\le C\int_M(|\mathcal{R}(\phi_t,\psi_t)|^2+\varepsilon^2|\mathcal{R}_c(\phi_t,\psi_t)|^2+|\frac{\partial\phi_t}{\partial t}|^2)dM\\ &\le C\int_M(|\psi_t|^4|d\phi_t|^2+\varepsilon^2|\psi_t|^2|\tilde\nabla\psi_t|^2|d\phi_t|^2+|\frac{\partial\phi_t}{\partial t}|^2)dM,\\ \int_M\varepsilon^2|\tilde\nabla^*_{e_\alpha}\tilde\nabla_{e_\alpha}\psi_t|^2dM&\le C\int_M(|\not{D}\psi_t|^2+|\frac{\tilde\nabla\psi_t}{\partial t}|^2)dM.\end{aligned}$$

Applying the pointwise bound on ψ_t, the Bochner formulas (5.10), (5.11) and using

$$\nabla d\phi=\nabla^2\phi+\Gamma(\phi)(d\phi,d\phi),$$

we turn the Laplace type terms into full second derivatives and obtain

$$\begin{aligned}&\int_M|\nabla^2\phi_t|^2dM\le C\Big(\int_M(|d\phi_t|^4+\varepsilon^4|\tilde\nabla\psi_t|^4)dM+\int_M|d\phi_t|^2dM+\int_M|\frac{\partial\phi_t}{\partial t}|^2dM\Big),\\ &\varepsilon^2\int_M|\tilde\nabla^2\psi_t|^2dM\le C\Big(\int_M|\tilde\nabla\psi_t|^2dM+\int_M(|d\phi_t|^4+\varepsilon^4|\tilde\nabla\psi_t|^4)dM\\ &\qquad+\int_M|\frac{\tilde\nabla\psi_t}{\partial t}|^2dM+C\Big).\end{aligned}$$

Applying the previous estimates and Young's inequality, we get

$$\begin{aligned}
\int_M &(|\nabla^2\phi|^2+\varepsilon^2|\tilde{\nabla}^2\psi|^2)dM \\
&\leq C\int_M(|d\phi_t|^2+\varepsilon|\tilde{\nabla}\psi_t|^2)dM+C\int_M(|d\phi_t|^4+\varepsilon^2|\tilde{\nabla}\psi_t|^4)dM+C \\
&\quad +C\int_M\left(|\frac{\partial\phi_t}{\partial t}|^2+|\frac{\tilde{\nabla}\psi_t}{\partial t}|^2\right)dM \\
&\leq C+C\int_M\left(|d\phi_t|^4+\varepsilon^2|\tilde{\nabla}\psi_t|^4\right)dM \\
&\leq C+C\delta_1\int_M(|\nabla^2\phi_t|^2+\varepsilon^2|\tilde{\nabla}^2\psi_t|^2)dM,
\end{aligned}$$

where we again applied the local Sobolev inequality (5.4) in the last step. Choosing δ_1 small enough, such that the right hand side can be absorbed into the left hand side, the result follows.

$\square$

Corollary 5.19 (Higher regularity)**.** *Suppose that M is a closed Riemann surface and assume that the pair $(\phi_t,\psi_t)\colon (M\times[0,T)\to N)\times(M\times[0,T)\to \Sigma M\otimes\phi_t^{-1}TN)$ is a regular solution of (5.1) and (5.2). The pair (ϕ_t,ψ_t) is smooth as long as δ_1,δ_2 are small enough.*

Proof. Since we have a bound on the L^2 norm of the second derivatives of ϕ_t and ψ_t by (5.21), we can apply the Sobolev embedding theorem and get that both $|d\phi_t|\in L^p$ and $|\tilde{\nabla}\psi_t|\in L^p$ for $p<\infty$. From the evolution equations (5.1) and (5.2) we may conclude that $\left|\frac{\partial\phi_t}{\partial t}\right|,|\nabla^2\phi_t|\in L^p$ and also $\left|\frac{\tilde{\nabla}\psi_t}{\partial t}\right|,|\tilde{\nabla}^2\psi_t|\in L^p$. By the embedding $H^{2,2}\hookrightarrow C^\alpha$ for some $\alpha<1$ we get that $|d\phi_t|$ and $|\tilde{\nabla}\psi_t|$ are Hölder continuous. At this point the same reasoning as in Theorem 3.24 yields that the pair (ϕ_t,ψ_t) is smooth. $\square$

5.3. Long-time Existence and Singularities

As in the one-dimensional case, we first of all derive a uniqueness statement before turning to the proof of the long-time existence of the evolution equations, see Theorem 4.11. To avoid the problem of identifying sections in different vector bundles, we will make use of the Nash embedding theorem.

Proposition 5.20 (Stability and uniqueness)**.** *Assume that M is a closed Riemann surface with fixed spin structure, N compact and let $(\phi,\psi)\in V$ and $(\xi,\chi)\in V$ be solutions of (5.1) and (5.2). The spinor $\psi\in\Gamma(\Sigma M\otimes\phi^{-1}TN)$ is defined along the map ϕ and the spinor $\chi\in\Gamma(\Sigma M\otimes\xi^{-1}TN)$ along the map ξ. Suppose that $|\chi|_{L^\infty(M\times[0,T))}\leq C$ and $|\chi|_{L^\infty(M\times[0,T))}\leq C$. If the initial data coincide, $(\phi_0,\psi_0)=(\xi_0,\chi_0)$, then we have $(\phi,\psi)=(\xi,\chi)$ throughout $M\times[0,T)$.*

Proof. We follow [Str08], p. 235. We apply the embedding theorem of Nash and regard u, v as vector valued functions in $\mathbb{R}^q$, more precisely $u, v\colon M \times [0,T) \to \iota(N) \subset \mathbb{R}^q$ with $u = \iota \circ \phi, v = \iota \circ \xi$. The spinors ψ and χ are defined along the maps u and v. We set

$$h_1 = u(x,t) - v(x,t), \qquad h_2 = \psi(x,t) - \chi(x,t)$$

with $\chi, \psi \in \Gamma(\Sigma M \otimes T\mathbb{R}^q)$ and combine h_1 and h_2 into a function $h(x,t)$ by

$$h = (h_1, h_2) = (u(x,t) - v(x,t), \psi(x,t) - \chi(x,t)).$$

First, we study the evolution of h_1 and h_2 separately and add up both contributions in the end. We compute using the evolution equation for u (3.24)

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{2}\int_M |h_1|^2 dM = &-\int_M |dh_1|^2 dM + \int_M \langle \mathrm{I\!I}_u(du,du) - \mathrm{I\!I}_v(dv,dv), h_1\rangle dM \\
&+ \int_M \langle h_1, P(\mathrm{I\!I}_u(du(e_\alpha), e_\alpha \cdot \psi), \psi) - P(\mathrm{I\!I}_v(dv(e_\alpha), e_\alpha \cdot \chi), \chi)\rangle dM \\
&+ \varepsilon \int_M \langle h_1, P(\mathrm{I\!I}_u(du(e_\alpha), \nabla_{e_\alpha}\psi), \psi) - P(\mathrm{I\!I}_v(dv(e_\alpha), \nabla_{e_\alpha}\chi), \chi)\rangle dM \\
&+ \varepsilon \int_M \langle h_1, P(\mathrm{I\!I}_u(du(e_\alpha), \psi), \nabla_{e_\alpha}\psi) - P(\mathrm{I\!I}_v(dv(e_\alpha), \chi), \nabla_{e_\alpha}\chi)\rangle dM \\
&+ \varepsilon \int_M \langle h_1, B_u(du, \psi, du, \psi) - B_v(dv, \chi, dv, \chi)\rangle dM.
\end{aligned}$$

After integrating with respect to t, we estimate the right hand side in terms of h_1 and h_2, where we apply the estimates derived previously. We set $Q = M \times [0,T)$ and $dQ = dMdt$. Applying the mean value theorem, we find

$$\begin{aligned}
&\int_Q |\langle \mathrm{I\!I}_u(du,du) - \mathrm{I\!I}_v(dv,dv), h_1\rangle| dQ \\
&\qquad \leq C\int_Q |h_1|^2(|du|^2 + |dv|^2) dQ + C\int_Q |h_1||dh_1|(|du| + |dv|) dQ.
\end{aligned}$$

Both of these terms can be further manipulated, the first one as

$$\begin{aligned}
\int_Q |h_1|^2(|du|^2 + |dv|^2) dQ &\leq C\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{2}}\left(\int_Q (|du|^4 + |dv|^4) dQ\right)^{\frac{1}{2}} \\
&\leq C\sqrt{f_1(t)}\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{2}}
\end{aligned}$$

and the second one as

$$\begin{aligned}
&\int_Q |h_1||dh_1|(|du| + |dv|) dQ \\
&\leq C\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{4}}\left(\int_Q |dh_1|^2 dQ\right)^{\frac{1}{2}}\left(\int_Q (|du|^4 + |dv|^4) dQ\right)^{\frac{1}{4}} \\
&\leq C\sqrt{f_1(t)}\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{2}} + Cf_1(t)\int_Q |dh_1|^2 dQ.
\end{aligned}$$

As in the one-dimensional case, we rewrite and estimate

$$\begin{aligned}&|\langle h_1, P(\mathrm{I\!I}_u(du(e_\alpha), e_\alpha\cdot\psi),\psi) - P(\mathrm{I\!I}_v(dv(e_\alpha), e_\alpha\cdot\chi),\chi)\rangle|\\ &\qquad\qquad\leq C(|du||h_1|^2|\psi|^2 + |dh_1||\psi|^2|h_1| + |dv||\psi||h_1||h_2|).\end{aligned}$$

Integrating over the manifold M and with respect to t for $0\leq t\leq T$, we estimate

$$\begin{aligned}\int_Q |du||h_1|^2|\psi|^2 dQ &\leq C\left(\int_Q |du|^2 dQ\right)^{\frac{1}{2}}\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{2}}\\ &\leq C\sqrt{t}\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{2}}.\end{aligned}$$

The next term can easily be estimated as

$$\begin{aligned}C\int_Q |\psi|^2|dh_1||h_1| dQ &\leq C\int_Q |h_1|^2 dQ + \frac{1}{8}\int_Q |dh_1|^2 dQ\\ &\leq C\sqrt{t}\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{2}} + \frac{1}{8}\int_Q |dh_1|^2 dQ,\end{aligned}$$

where we used Young's and Hölder's inequality. Applying Young's inequality to

$$|dv||\psi||h_1||h_2| \leq C(|dv||h_1|^2 + |dv||h_2|^2),$$

we realize that the first term has already been estimated, whereas the second one can be rearranged as

$$\int_Q |dv||h_2|^2 dQ \leq \left(\int_Q |dv|^2 dQ\right)^{\frac{1}{2}}\left(\int_Q |h_2|^4 dQ\right)^{\frac{1}{2}} \leq C\sqrt{t}\left(\int_Q |h_2|^4 dQ\right)^{\frac{1}{2}}.$$

Again, as in the one-dimensional case we rewrite

$$\begin{aligned}&|\langle h_1, P(\mathrm{I\!I}_u(du(e_\alpha),\psi),\nabla_{e_\alpha}\psi) - P(\mathrm{I\!I}_v(dv(e_\alpha),\chi),\nabla_{e_\alpha}\chi)\rangle|\\ &\quad\leq C(|du||h_1|^2|\psi||\nabla\psi| + |dh_1||\psi||\nabla\psi||h_1| + |dv||\nabla\psi||h_1||h_2| + |dv||\psi||\nabla h_2||h_1|),\end{aligned}$$

such that we can estimate again

$$\begin{aligned}\int_Q |du|h_1|^2|\psi||\nabla\psi| dQ &\leq C\left(\int_Q |du|^4 dQ\right)^{\frac{1}{4}}\left(\int_Q |\nabla\psi|^4 dQ\right)^{\frac{1}{4}}\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{2}}\\ &\leq C(\sqrt{f_1(t)} + \sqrt{f_2(t)})\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{2}}.\end{aligned}$$

For the next term we perform the following manipulation

$$\begin{aligned}C\int_Q |dh_1||\psi||\nabla\psi||h_1| dQ &\leq C\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{4}}\left(\int_Q |dh_1|^2 dQ\right)^{\frac{1}{2}}\left(\int_Q |\nabla\psi|^4 dQ\right)^{\frac{1}{4}}\\ &\leq C\sqrt{f_2(t)}\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{2}} + \frac{1}{8}\int_Q |dh_1|^2 dQ.\end{aligned}$$

Again, by Young's inequality we have

$$|dv||\nabla\psi||h_1||h_2| \leq C(|\nabla\psi|^2|h_1|^2 + |dv|^2|h_2|^2).$$

We already took care of the first term, the second term can be estimated by the same method. Finally, we manipulate

$$\begin{aligned}\int_Q |dv||\psi||h_1||\nabla h_2|dQ &\leq C\left(\int_Q |dv|^4 dQ\right)^{\frac{1}{4}}\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{4}}\left(\int_Q |\nabla h_2|^2 dQ\right)^{\frac{1}{2}} \\ &\leq C\sqrt{f_1(t)}\left(\int_Q |h_1|^4 dQ\right)^{\frac{1}{2}} + \frac{\varepsilon}{8}\int_Q |\nabla h_2|^2 dQ.\end{aligned}$$

Note that the contribution

$$\varepsilon\langle h_1, P(\mathrm{I\!I}_u(du(e_\alpha), \nabla_{e_\alpha}\psi), \psi) - P(\mathrm{I\!I}_v(dv(e_\alpha), \nabla_{e_\alpha}\chi), \chi)\rangle$$

can be estimated by the same methods. As in the one-dimensional case we have

$$\begin{aligned}&|\langle h_1, B_u(du, \psi, du, \psi) - B_v(dv, \chi, dv, \chi)\rangle| \\ &\quad\leq C(|h_1|^2|\psi|^2|du|^2 + |h_1||dh_1||\psi|^2|du| + |du||\psi||dv||h_2||h_1| \\ &\qquad + |h_1||dv||\chi||\psi||dh_1| + |h_1||dv|^2|\chi||h_2|).\end{aligned}$$

By application of Young's inequality we find

$$|\langle h_1, B_u(du, \psi, du, \psi) - B_v(dv, \chi, dv, \chi)\rangle| \leq C(|h_1|^2(|du|^2 + |dv|^2) + |h_2|^2|dv|^2) + \frac{1}{8}|dh_1|^2.$$

The first terms on the right hand side have already been estimated, the last one can later be absorbed into the left hand side.
As a second step, we turn to the function h_2. With the the help of (3.25) we find

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{2}\int_M |h_2|^2 dM &= -\int_M \langle \not{\partial} h_2, h_2\rangle dM - \varepsilon\int_M |\nabla h_2|^2 dM \\ &\quad -\varepsilon\int_M \langle h_2, (\nabla_{e_\alpha}\mathrm{I\!I}_u)(du(e_\alpha), \psi) - (\nabla_{e_\alpha}\mathrm{I\!I}_v)(dv(e_\alpha), \chi)\rangle dM.\end{aligned}$$

The other terms involving the second fundamental form vanish since $\mathrm{I\!I} \perp \psi$. Again, we first of all integrate with respect to t and estimate

$$\int_Q \langle \not{\partial} h_2, h_2\rangle \leq C\sqrt{t}\left(\int_Q |h_2|^4 dQ\right)^{\frac{1}{2}} + \frac{\varepsilon}{8}\int_Q |\nabla h_2|^2 dQ.$$

To estimate the last term, we rearrange

$$\begin{aligned}&|\langle \psi - \chi, (\nabla_{e_\alpha}\mathrm{I\!I}_u)(du(e_\alpha), \psi) - (\nabla_{e_\alpha}\mathrm{I\!I}_v)(dv(e_\alpha), \chi)\rangle| \\ &\qquad\leq C(|h_1||h_2||du||dv| + |dv||dh_1||\psi||h_2| + |dv|^2|h_2|^2).\end{aligned}$$

Again, by Young's inequality we may estimate

$$|\langle \psi - \chi, (\nabla_{e_\alpha} \mathrm{II}_u)(du(e_\alpha), \psi) - (\nabla_{e_\alpha} \mathrm{II}_v)(dv(e_\alpha), \chi)\rangle| \leq C(|h_1||du|^2 + |dv|^2|h_2|^2) + \frac{1}{4}|dh_1|^2$$

and all of the terms on the right hand side have already been considered. Adding up the inequalities for $|h_1|^2$ and $|h_2|^2$ and applying the Sobolev embedding theorem, we find

$$\begin{aligned} &\frac{1}{2}\int_M (|h_1|^2 + |h_2|^2) dM + \frac{1}{2}\int_0^T \int_M (|dh_1|^2 + \varepsilon|\nabla h_2|^2) dQ \\ &\quad \leq C f(t) \left(\sup_{[0,T)} \int_M (|h_1|^2 + |h_2|^2) dM + \int_0^T \int_M (|dh_1|^2 + \varepsilon|\nabla h_2|^2) dQ \right) \\ &\quad\quad + \frac{1}{2}\int_M (|h_1(0)|^2 + |h_2(0)|^2) dM, \end{aligned}$$

where $f(t)$ denotes the supremum of $\sqrt{f_1(t)}, \sqrt{f_2(t)}$ and $\sqrt{t}$. We know that $f(t) \to 0$ as $t \to 0$ and by assumption $h_1(0) = h_2(0) = 0$. Choosing t small enough, we have $u = v$ and $\psi = \chi$ throughout Q. □

Proposition 5.21 (Long-time existence)**.** *Let $(\phi_t, \psi_t) \in V$ be a solution of (5.1) and (5.2). Assume that $|\psi_t|_{L^\infty(M\times[0,T))} \leq C$. Then the evolution equations admit a solution for $0 \leq t < \infty$.*

Proof. The first singular time T_0 is characterized by the condition

$$\limsup_{t\to T_0} F(\phi_t, \psi_t, B_R(x)) \geq \delta_1.$$

Since we have $\partial_t \phi, \tilde{\nabla}_t \psi \in L^2(M \times [0,T))$ and also $F(\phi_t, \psi_t) \leq CF(\phi_0, \psi_0) + C$ for $0 < t < T$, there exists $(\phi(\cdot, T), \psi(\cdot, T)) \in H^{1,2}(M, N) \times H^{1,2}(M, \Sigma M \otimes \phi_t^{-1}TN)$ such that $(\phi(\cdot, t), \psi(\cdot, t)) \to (\phi(\cdot, T), \psi(\cdot, T))$ weakly in $H^{1,2}(M, N) \times H^{1,2}(M, \Sigma M \otimes \phi_t^{-1}TN)$ as t approaches T. In particular, we have

$$F(\phi_T, \psi_T) \leq \liminf_{s\to t} CF(\phi_s, \psi_s) + C \leq CF(\phi_t, \psi_t) + C, \qquad 0 \leq t \leq T.$$

Now let $(\tilde{\phi}_t, \tilde{\psi}_t)\colon (M \times [T, T+T_1) \to N) \times (M \times [T, T+T_1) \to \Sigma M \otimes \phi_t^{-1}TN)$ be a solution of (5.1) and (5.2). Assume that $(\tilde{\phi}, \tilde{\psi})(x,t) = (\phi, \psi)(x,t)$. We define

$$(\hat{\phi}_t, \hat{\psi}_t) = \begin{cases} (\phi_t, \psi_t), & 0 \leq t \leq T, \\ (\tilde{\phi}_t, \tilde{\psi}_t), & T \leq t \leq T + T_1. \end{cases}$$

One can now verify that $(\hat{\phi}_t, \hat{\psi}_t)\colon (M \times [0, T_1) \to N) \times (M \times [0, T_1) \to \Sigma M \times \hat{\phi}_t^{-1}TN)$ is a weak solution of (5.1) and (5.2). By iteration, we obtain a weak solution (ϕ_t, ψ_t) on a maximal time interval $T + \delta$ for some $\delta > 0$. If $T + \delta < \infty$ then by the above argument the solution (ϕ_t, ψ_t) may be extended to infinity, hence $T = \infty$. The uniqueness follows from Proposition (5.20). □

Theorem 5.22 (Finitely many singularities). *Assume that M is a closed Riemann surface and suppose the pair (ϕ_t, ψ_t) is a solution of (5.1) and (5.2). Assume that $|\psi_t|_{L^\infty(M\times[0,\infty))} \leq C$. There are only finitely many singular points $(x_k, t_k), 1 \leq k \leq K$. The number K depends on $M, \varepsilon, \psi_0, d\phi_0$ and $\tilde{\nabla}\psi_0$.*

Proof. We follow the presentation in [LW08], p. 138, for the harmonic map heat flow. To prove that there exist only finitely many singular points for the regularized Dirac-harmonic map heat flow, we assume that $T_0 > 0$ is the first singular time and define the singular set as

$$S(\phi, \psi, T_0) = \bigcap_{R>0} \left\{ x \in M \mid \limsup_{t\to T_0} F(\phi_t, \psi_t, B_R(x)) \geq \delta_1 \right\}.$$

Now, let $\{x_j\}_{j=1}^K$ be any finite subset of $S(\phi, \psi, T_0)$. Then we have for $R > 0$

$$\limsup_{t\to T_0} \int_{B_R(x_j)} (|d\phi|^2 + \varepsilon|\tilde{\nabla}\psi|^2) dM \geq \delta_1, \qquad 1 \leq j \leq K.$$

By (5.7) we have the following local inequality for the quantity $F(\phi_t, \psi_t, B_R)$

$$F(\phi_t, \psi_t, B_R(x)) \leq 2E_\varepsilon(\phi_0, \psi_0, B_{2R}(x)) + \delta_3 \frac{T}{R^2} + \frac{1}{\varepsilon} \int_{B_R} |\psi_t|^2 dM \tag{5.22}$$

with $\delta_3 = C \int_M (|d\phi_t|^2 + \varepsilon^2 |\tilde{\nabla}\psi_t|^2 + |\psi_t|^2) dM$. Moreover, we have the global estimate

$$F(\phi_t, \psi_t) \leq \delta_4 F(\phi_0, \psi_0) + \delta_5$$

with $\delta_4 = 2$ and $\delta_5 = \frac{4}{\varepsilon} \int_M |\psi_0|^2$. We choose $R > 0$ such that all $B_{2R}(x_j), 1 \leq j \leq K$ are mutually disjoint and small enough to have

$$\frac{1}{\varepsilon} \int_{B_R} |\psi_t|^2 dM \leq \frac{\delta_1}{4}.$$

Then, we have by (5.22)

$$\begin{aligned}
K\delta_1 &\leq \sum_{j=1}^K \limsup_{t\to T_0} F(\phi_t, \psi_t, B_R(x_j)) \\
&\leq \sum_{j=1}^K \left(\limsup_{t\to T_0} 2E_\varepsilon(\phi_\tau, \psi_\tau, B_{2R}(x_j)) + \frac{\delta_1}{2} \right) \\
&\leq 2E_\varepsilon(\phi_\tau, \psi_\tau) + \frac{K\delta_1}{2} \\
&\leq 2E_\varepsilon(\phi_0, \psi_0) + \frac{K\delta_1}{2}
\end{aligned}$$

for any $\tau \in [T_0 - \frac{\delta_1 R^2}{4\delta_3}, T_0]$. We conclude that

$$K \leq 4\frac{E_\varepsilon(\phi_0, \psi_0)}{\delta_1},$$

which implies the finiteness of the singular set $S(\phi, \psi, T_0)$. Our next aim is to show that there are only finitely many singular spatial points. Therefore we set

$$\tilde{M} = M \setminus \bigcup_{1\leq j\leq K} B_{2R}(x_j)$$

and in addition, we calculate

$$\begin{aligned} F(\phi_{T_0}, \psi_{T_0}) &= \lim_{R\to 0} F(\phi_{T_0}, \psi_{T_0}, \tilde{M}) \\ &\leq \lim_{R\to 0} \limsup_{t\to T_0} F(\phi_t, \psi_t, \tilde{M}) \\ &\leq F(\phi_t, \psi_t) - \lim_{R\to 0} \sum_{j=1}^{K} \liminf_{t\to T_0} F(\phi_t, \psi_t, B_{2R}(x_j)) \\ &\leq \delta_4 F(\phi_0, \psi_0) + \delta_5 - \lim_{R\to 0} \sum_{j=1}^{K} \limsup_{t\to T_0} F(\phi_t, \psi_t, B_R(x_j)) \\ &\leq \delta_4 F(\phi_0, \psi_0) + \delta_5 - K\delta_1. \end{aligned} \tag{5.23}$$

Now suppose $T_0 < \ldots < T_j$ are j singular times and by $K_0, \ldots, K_j$ we denote the number of singular points at each singular time. Set

$$(\phi_i, \psi_i) = \lim_{t\to T_i} (\phi_t, \psi_t), \qquad 0 \leq i \leq j.$$

By iterating (5.23) we get

$$\begin{aligned} F(\phi_j, \psi_j) &\leq \delta_4 F(\phi_{j-1}, \psi_{j-1}) + \delta_5 - \delta_1 K_{j-1} \\ &\leq \delta_4^2 F(\phi_{j-2}, \psi_{j-2}) + \delta_5(1 + \delta_4) - \delta_1(K_{j-1} + \delta_4 K_{j-2}) \\ &\leq \ldots \\ &\leq \delta_4^j F(\phi_0, \psi_0) + \delta_5 \sum_{i=0}^{j-1} \delta_4^i - \delta_1 \sum_{i=0}^{j-1} K_i \delta_4^i, \end{aligned}$$

which can be rearranged as

$$\sum_{i=0}^{j-1} K_i \delta_4^i \leq \frac{\delta_4^j F(\phi_0, \psi_0) + \delta_5 \sum_{i=0}^{j-1} \delta_4^i}{\delta_1}. \tag{5.24}$$

We conclude that there are only finitely many singularities. □

Remark 5.23. If we compare the bound on the number of singularities for the regularized Dirac-harmonic map heat flow with the bound on the number of singularities in the harmonic map heat flow, then we realize that the former one will encounter more singularities. In the case of the harmonic map heat flow we would have $\delta_4 = 1$, $\delta_5 = 1$ and $F(\phi_0, \psi_0) = \frac{1}{2} \int_M |d\phi_0|^2$, which lowers the upper bound in (5.24).

5.4. Convergence and Blowup Analysis

In this section we discuss the convergence of the evolution equations (5.1) and (5.2). In addition, we sketch how to perform a blowup analysis of the singular points.

Theorem 5.24. *Let $(\phi_t, \psi_t) \in V$ be a solution of (5.1) and (5.2). Moreover, assume that $|\psi_t|_{L^\infty(M\times[0,\infty))} \leq C$. Then the pair (ϕ_t, ψ_t) converges strongly in L^2 to a regularized Dirac-harmonic map on the set $M \setminus \{x_1, \ldots, x_k\}$. The limiting map $(\phi_\infty, \psi_\infty)$ is smooth on $M \setminus \{x_1, \ldots, x_k\}$.*

Proof. Since we have a uniform bound on the L^2 norm of the t derivatives of (ϕ_t, ψ_t) by Lemma 5.9, we can achieve for $t_m \to \infty$ suitably

$$\int_M \left(|\frac{\partial \phi_t}{\partial t}|^2 + |\frac{\tilde{\nabla}\psi_t}{\partial t}|^2 \right) dM\big|_{t=t_m} \to 0$$

and in addition, we suppose that $T = \infty$ is non-singular

$$\limsup_{t\to\infty}(\sup_{x\in M} F(\phi_t, \psi_t, B_R(x))) \leq \delta_1$$

for some $R > 0$. By (5.21) we have a bound on the second derivatives

$$\int_M \Big(|\nabla^2\phi|^2(\cdot, t_m) + \varepsilon^2|\tilde{\nabla}^2\psi|^2(\cdot, t_m) \Big) dM \leq C,$$

such that due to the Rellich-Kondrachov embedding theorem we may assume that

$$\begin{aligned} \phi(\cdot, t_m) \to \phi_\infty \qquad & \text{strongly in } H^{1,p}(M, N), \\ \psi(\cdot, t_m) \to \psi_\infty \qquad & \text{strongly in } H^{1,p}(M, \Sigma M \otimes \phi_{t_m}^{-1}TN) \end{aligned}$$

for any $p < \infty$. But then by (5.1) and (5.2) we get convergence of the evolution equations

$$\tau(\phi_\infty) = \mathcal{R}(\phi_\infty, \psi_\infty) + \varepsilon \mathcal{R}_c(\phi_\infty, \psi_\infty), \tag{5.25}$$

$$\varepsilon\tilde{\Delta}\psi_\infty = \not{D}\psi_\infty \tag{5.26}$$

in L^2 and hence the pair $(\phi_\infty, \psi_\infty)$ is a regularized Dirac-harmonic map, which satisfies $(\phi_\infty, \psi_\infty) \in H^{2,2}(M, N) \times H^{2,2}(M, \Sigma M \otimes \phi_\infty^{-1}TN)$.
If $T = \infty$ is singular, meaning that at the points $\{x_1, \ldots, x_k\}$

$$\limsup_{t\to\infty} F(\phi_t, \psi_t, B_R(x_j)) \geq \delta_1, \qquad 1 \leq j \leq k$$

for all $R > 0$, then for suitable numbers $t_m \to \infty$ the family (ϕ_{t_m}, ψ_{t_m}) will be bounded in $H^{2,2}(M, N) \times H^{2,2}(M, \Sigma M \otimes \phi_{t_m}^{-1}TN)$ on the set $M \setminus \{x_1, \ldots, x_k\}$. Consequently, the family (ϕ_{t_m}, ψ_{t_m}) will accumulate as follows

$$\begin{aligned} &\phi_\infty \colon M \setminus \{x_1, \ldots, x_k\} \to N, \\ &\psi_\infty \colon M \setminus \{x_1, \ldots, x_k\} \to \Sigma(M \setminus \{x_1, \ldots, x_k\}) \otimes \phi_\infty^{-1}TN. \end{aligned}$$

We set $\tilde{M} := M \setminus \{x_1, \ldots, x_k\}$. Concerning the regularity of $(\phi_\infty, \psi_\infty)$ on $\tilde{M}$, we have $\phi_\infty \in H^{1,p}(\tilde{M}, N)$ for any $0 < p < \infty$, since $\phi_\infty \in H^{2,2}(\tilde{M}, N)$. In addition, we have $\psi_\infty \in H^{2,2}(\tilde{M}, \Sigma\tilde{M} \otimes \phi_\infty^{-1}TN)$ and consequently also $\psi_\infty \in H^{1,p}(\tilde{M}, \Sigma\tilde{M} \otimes \phi_\infty^{-1}TN)$ for any $0 < p < \infty$. Hence, the right hand sides of both (5.25) and (5.26) are in L^p for $2 < p < \infty$. Writing $\tau(\phi) = \Delta\phi + \Gamma(\phi)(d\phi, d\phi)$ and by elliptic estimates for second order operators we then get $\phi_\infty \in H^{2,p}$ for any $0 < p < \infty$ and by the Sobolev embedding theorem it follows that $\phi_\infty \in C^{1,\alpha}(\tilde{M}, N)$. By the same argumentation we find that also $\psi_\infty \in C^{1,\alpha}(\tilde{M}, \Sigma\tilde{M} \otimes \phi_\infty^{-1}TN)$. The smoothness of $(\phi_\infty, \psi_\infty)$ then follows from a standard bootstrap argument. □

Of course, one would like to apply a removable singularity theorem to get a smooth solution not only on $\tilde{M}$, but on the whole manifold M. This issue will be addressed in the next section.

This completes the proof of Theorem 5.1.

Our next aim is to get a better understanding of the singular points (x_k, t_k). In the case of the harmonic map heat flow one can perform a blowup analysis, which finally leads to the "bubbling off of harmonic spheres", see for example [Str85]. An important ingredient in that calculation is the scaling behaviour of the evolution equation for harmonic maps. Therefore, let us analyze the scaling of the regularized Dirac-harmonic heat flow.

Remark 5.25 (Scaling of the evolution equations)**.** By regularizing the functional $E(\phi, \psi)$, we haven broken the conformal invariance and consequently the evolution equations for (ϕ_t, ψ_t) do not scale in a "nice" way. Nevertheless, it is possible to do a rescaling if one allows to rescale ε as well. It is easy to see that the evolution equations

$$\begin{aligned}
\frac{\partial \phi_t}{\partial t} &= \tau(\phi_t) - \frac{1}{2} R^N(e_\alpha \cdot \psi_t, \psi_t) d\phi_t(e_\alpha) - \varepsilon R^N(\tilde{\nabla}_{e_\alpha} \psi_t, \psi_t) d\phi_t(e_\alpha), \\
\frac{\tilde{\nabla} \psi_t}{\partial t} &= \varepsilon \tilde{\Delta} \psi_t - \not{D} \psi_t
\end{aligned}$$

are invariant under the following rescaling

$$\begin{aligned}
\phi(x,t) &\to \phi(x_0 + Rx, t_0 + R^2 t), \\
\psi(x,t) &\to \sqrt{R}\psi(x_0 + Rx, t_0 + Rt), \\
\varepsilon &\to \frac{\varepsilon}{R}
\end{aligned} \tag{5.27}$$

for $R > 0$. We would like to mention that also a dimensional analysis of the evolution equation for ψ motivates to also rescale ε. Unfortunately, the two evolution equations scale differently. The evolution equation for ϕ scales like a heat type equation, whereas the evolution equation for ψ scales like a first order evolution equation. We should also mention that it may be problematic to rescale the regularizing parameter ε. First of all, all the estimates we derived so far depend on ε in a non-trivial way. Even worse, remember that the maximal existence interval T_{max} of the short-time solution (3.22) in general also depends on ε. Another problem we would like to address, is that up to now we have deformed the pair (ϕ, ψ) simultaneously. But the rescaling presented above requires to deform ϕ and ψ independently of each other.

If we ignore the problems just mentioned, we can analyze the singular points as in the harmonic map heat flow. To perform a blowup analysis of the singular points, we recall Moser's Harnack inequality for subsolutions of the heat equation [Mos64]. To this end, we define the parabolic cylinder $P_R(z_0)$ for $z_0 = (x_0, t_0) \in M \times (0, \infty)$ and $0 < R < \min\{i_M, \sqrt{t_0}\}$ by

$$P_R(z_0) = \{z = (x,t) \in M \times [0,t) \big| |x - x_0| \leq R,\ t_0 - R^2 \leq t \leq t_0\}. \tag{5.28}$$

With these preparations we can state

Lemma 5.26 (Moser's parabolic Harnack inequality). *Suppose $v \in C^\infty(P_R(z_0))$ is non-negative and satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta\right) v \leq Cv$$

with $C > 0$. Then there exists a constant $C_1 > 0$ such that

$$v(z_0) \leq \frac{C_1}{R^{m+2}} \int_{P_R(z_0)} v\, dxdt, \tag{5.29}$$

where m is the dimension of the manifold M.

With the help of Moser's Harnack inequality we can sketch how to perform a blowup analysis.

Theorem 5.27 (Blowup analysis). *Suppose that M is a closed Riemann surface and assume that the pair (ϕ_t, ψ_t) is a solution of (5.1) and (5.2) and $|\psi_t|_{L^\infty(M\times[0,\infty))} \leq C$. At the singular points $(\bar{x}, \bar{t})$ harmonic spheres $\phi : S^2 \to N$ separate and the spinor ψ becomes trivial during the blowup process.*

Proof. We follow the presentation in [Str08], p. 233, for the blowup analysis of the harmonic map heat flow. Suppose $(\bar{x}, \bar{t})$ is a singular point in the sense that for any $R \in (0, \frac{1}{2} i_M)$ we have

$$\limsup_{t \to \bar{t}} F(\phi_t, \psi_t, B_R(\bar{x})) \geq \delta_1.$$

By finiteness of the singular set we know that $\bar{x}$ is isolated among concentration points. Consequently, for $R_m \to 0$ we may choose $\bar{x}_m \to \bar{x}$ and $\bar{t}_m \to \bar{t}$ such that for $R_0 > 0$ we have

$$F(\phi_{\bar{t}_m}, \psi_{\bar{t}_m}, B_{R_m}(\bar{x}_m)) = \sup_{\substack{x \in B_{2R_0(\bar{x})} \\ \bar{t}_m - \tau_m \leq t \leq \bar{t}_m}} F(\phi_t, \psi_t, B_{R_m}(x)) = \frac{\delta_1}{4}.$$

where $\tau_m = \frac{\delta_1 R_m^2}{8\delta_3}$. We assume that $\bar{x}_m \in B_{R_0}(\bar{x})$ and perform the rescaling

$$\begin{aligned}
\phi_m(x,t) &:= \phi(\bar{x}_m + R_m x, \bar{t}_m + R_m^2 t), \\
\psi_m(x,t) &:= \sqrt{R_m}\psi(\bar{x}_m + R_m x, \bar{t}_m + R_m t), \\
\varepsilon_m &:= \frac{\varepsilon}{R_m}.
\end{aligned}$$

We have to assume that the rescaling of ε does not get in conflict with the existence of the short-time solution (3.22). Since the norm of the rescaled spinor $\psi_m(x,t)$ satisfies

$$|\psi_m(x,t)|^2 = R_m|\psi(\bar{x}_m + R_m x, \bar{t}_m + R_m t)|^2,$$

we get a first hint that ψ will become trivial as $R_m \to 0$. Note that

$$\begin{aligned}
\phi_m &: B_{\frac{R_0}{R_m}} \times [t_0, 0] \to N \\
\psi_m &: B_{\frac{R_0}{R_m}} \times [t_0, 0] \to \Sigma M \otimes \phi_m^{-1}TN
\end{aligned}$$

solve (5.1) and (5.2) classically with $t_0 = -\frac{\delta_1}{8\delta_3}$. Moreover, we have

$$\sup_{\substack{R_m|x|<R_0 \\ t_0<t<0}} F(\phi_m(\cdot,t), \psi_m(\cdot,t), B_1(x)) \leq F(\phi_m(\cdot,0), \psi_m(\cdot,0), B_1(0)) = \frac{\delta_1}{4}.$$

In addition, we find

$$\tau_m^\phi = -R_m^2 t_0, \qquad \tau_m^\psi = -R_m t_0,$$

which reflects the first and second order character of the evolution equations. We deduce

$$\begin{aligned}
\int_{t_0}^0 \int_{B_{\frac{R_0}{R_m}}} |\frac{\partial \phi_m}{\partial t}|^2 dxdt &\leq \int_{\bar{t}_m - \tau_m^\phi}^{\bar{t}_m} \int_M |\frac{\partial \phi_t}{\partial t}|^2 dMdt \to 0, \\
\int_{t_0}^0 \int_{B_{\frac{R_0}{R_m}}} |\frac{\tilde{\nabla} \psi_m}{\partial t}|^2 dxdt &\leq \int_{\bar{t}_m - \tau_m^\psi}^{\bar{t}_m} \int_M |\frac{\tilde{\nabla} \psi_t}{\partial t}|^2 dMdt \to 0,
\end{aligned}$$

as $m \to \infty$. To get a better understanding how the spinor ψ_t behaves during the blowup process, we apply Moser's Harnack inequality to

$$\frac{\partial}{\partial t}|\psi_t|^4 \leq \varepsilon \Delta |\psi_t|^4 + \frac{2}{\varepsilon}|\psi_t|^4$$

and find

$$\begin{aligned}
|\psi|^4(x_0, t_0) &\leq \frac{C}{(\frac{R_0}{R_m})^4} \int_{t_0}^0 \int_{B_{\frac{R_0}{R_m}}} |\psi_t|^4 dxdt \\
&\leq \frac{C}{(\frac{R_0}{R_m})^4} \int_{t_0}^0 \int_M |\psi_t|^4 dMdt \\
&\leq -C\frac{t_0 R_m^4}{R_0^4}
\end{aligned}$$

with the parabolic cylinder $P_{\frac{R_0}{R_m}}$ around (x_0, t_0) defined by

$$P_{\frac{R_0}{R_m}} = \{z = (x,t) \in M \times [t_0, 0) | |x - x_0| \leq \frac{R_0}{R_m}, t_0 \leq t < 0\}.$$

Note that $P_{\frac{R_0}{R_m}} \to \mathbb{R}^2 \times \mathbb{R}_-$ as $m \to \infty$. Consequently, $\psi \to 0$ in the limit $R_m \to 0$. Of course, one would like to localize the bound on the second derivative of ϕ_t (5.21) to $B_{\frac{R_0}{2R_m}}$. Therefore, we need to analyze the behaviour of the curvature terms during the blowup process

$$\begin{aligned}
\int_{t_0}^0 \int_{B_{\frac{R_0}{2R_m}}} |\mathcal{R}(\phi_m, \psi_m)|^2 dx dt &\leq \int_0^t \int_M |\mathcal{R}(\phi_t, \psi_t)|^2 dM dt \\
&\leq C \int_0^t \int_M |d\phi_t|^2 |\psi_t|^4 dM dt \leq Ct|\psi_t|^4_{L^\infty(M\times[0,T))} \\
\int_{t_0}^0 \int_{B_{\frac{R_0}{2R_m}}} \varepsilon_m^2 |\mathcal{R}(\phi_m, \psi_m)|^2 dx dt &\leq \int_0^t \int_M \varepsilon^2 |\mathcal{R}_c(\phi_t, \psi_t)|^2 dM dt \\
&\leq C\varepsilon^2 |\psi_t|^2 \int_0^t \int_M |d\phi_t|^2 |\tilde{\nabla}\psi_t|^2 dM dt \\
&\leq C\varepsilon^2 |\psi_t|^2_{L^\infty(M\times[0,T))} (f_1(t))^{\frac{1}{2}} (f_2(t))^{\frac{1}{2}}.
\end{aligned}$$

By application of Moser's Harnack inequality to the norm of ψ_t both of these integrals tend to zero as $m \to \infty$. Consequently, we can apply the estimates for the harmonic heat map flow from the original proof of Struwe, see for example [Str08], p. 234, and find

$$\int_{t_0}^0 \int_{B_{\frac{R_0}{2R_m}}} |\nabla^2 \phi_m|^2 dx dt \leq C.$$

Hence, ϕ_m converges strongly locally in $H^{2,2}(\mathbb{R}^2, N)$ against a limiting map ϕ_∞. Since

$$\frac{1}{2} \int_M |d\phi_\infty|^2 dM \leq E_\varepsilon(\phi_0, \psi_0)$$

we can apply a classical theorem from Sacks-Uhlenbeck ([SU81], Theorem 3.6) and extend ϕ_∞ to a non-constant harmonic map from $\phi : S^2 \cong \mathbb{R}^2 \to N$. □

When analyzing the bubbling phenomena of Dirac-harmonic maps, it is important to have control over the energy of the bubbles, such that now concentration phenomena can happen. This control is usually given by what is called *energy identity*. For Dirac-harmonic maps the energy identity was first proven for the case $M = S^2, N = S^n \subset \mathbb{R}^{n+1}$ in [CJLW05], which was later generalized to the case of M being an arbitrary Riemannian spin surface and N a compact Riemannian manifold in [Zha07b], p. 131.

Definition 5.28 (Blowup Set)**.** Let $(\phi_k, \psi_k) : M \to N$ be a sequence of smooth Dirac-harmonic maps with uniformly bounded energy

$$\int_M (|d\phi_k|^2 + |\psi_k|^4) dM \leq C$$

and furthermore assume that (ϕ_k, ψ_k) converges weakly to a Dirac-harmonic map (ϕ, ψ) in $H^{1,2}(M,N) \times L^4(\Sigma M \otimes T\mathbb{R}^q)$, then we call

$$S := \bigcap_{R>0} \{x \in M \mid \liminf_{k\to\infty} \int_{B_R(x)} (|d\phi_k|^2 + |\psi_k|^4) dM > \delta\}$$

the blow-up set of $\{\phi_k, \psi_k\}$.

Theorem 5.29 (Energy identity for Dirac-harmonic maps)**.** *Consider a sequence of smooth Dirac-harmonic maps (ϕ_k, ψ_k) with uniformly bounded energy*

$$\int_M |(d\phi_k|^2 + |\psi_k|^4) dM \leq C$$

and assume that (ϕ_k, ψ_k) weakly converges to a Dirac-harmonic map (ϕ, ψ) in the space $H^{1,2}(M,N) \times L^4(\Sigma M \otimes \mathbb{R}^q)$ with a finite set of blow-up points denoted by $\{p_1, \cdots, p_m\}$. Then after passing to a subsequence, still denoted by (ϕ_k, ψ_k), we can find a finite set of Dirac-harmonic spheres

$$(\sigma_i^l, \xi_i^l) : S^2 \to N, \qquad i = 1, \cdots, I,\ l = 1, \cdots, L_i,$$

such that the following energy identities hold

$$\lim_{k\to\infty} E(\phi_k) = E(\phi) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} E(\sigma_i^L), \tag{5.30}$$

$$\lim_{k\to\infty} E(\psi_k) = E(\psi) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} E(\xi_i^L). \tag{5.31}$$

Note that the quantity we need to control to apply the energy identity is different from what we needed to control the evolution equations. In addition, the energy identity deals with a sequence of Dirac-harmonic maps, whereas we have a sequence of regularized Dirac-harmonic maps.

Remark 5.30. In the context of the regularity of harmonic maps, one important ingredient is the so called *ε-regularity theorem*. This was first proven in the stationary case by Schoen [Sch84] and then extended to the parabolic case by Struwe [Str88b] and Chen-Struwe [CS89]. Once again the important point is the scaling behaviour of the corresponding equations. To establish the regularity for Dirac-harmonic maps an *ε-regularity theorem* was established in [WX09]. It was used that

$$\int_M (|d\phi|^2 + |\psi|^4) dM$$

is scale invariant in dimension two. If one tries to extend these ideas to the evolution equations for regularized Dirac-harmonic maps, one encounters a scaling problem. The scaling

$$\phi(x,t) \to \phi(Rx, R^2 t), \qquad \psi(x,t) = \sqrt{R}\psi(Rx, Rt), \qquad \varepsilon \to \frac{\varepsilon}{R}$$

leaves the evolution equations (5.1) and (5.2) invariant. On the other hand, the linear combination

$$F(x,t,\varepsilon) := \frac{1}{2}(|d\phi|^2 + \varepsilon|\tilde{\nabla}\psi|^2)$$

satisfies

$$F(Rx, t, \frac{\varepsilon}{R}) = R^2 F(x,t,\varepsilon),$$

but to set up a parabolic ε-regularity theorem, we would also need a nice scaling behaviour in the t variable.

Before discussing the limiting process $\varepsilon \to 0$, we analyze the structure of Dirac-harmonic maps for the case that M and N are surfaces of lower genus.

5.5. Dirac-harmonic Maps between Surfaces

In this section we discuss Dirac-harmonic maps between surfaces. We will assume that M is compact and oriented. A criterion if the Euler-Lagrange equations for Dirac-harmonic maps decouple, is given by the following theorem from [Yan09], p. 410:

Theorem 5.31. *Let M and N both be compact oriented Riemann surfaces and suppose that (ϕ, ψ) is a Dirac-harmonic map from M to N. If*

$$g_M = 0 \qquad or \qquad |g_M - 1| < |deg(\phi)||2g_N - 2|, \tag{5.32}$$

then ϕ has to be a harmonic map.

We now analyze Dirac-harmonic maps between some explicit surfaces. To this end, we make the following definition

Definition 5.32 (Twistor spinor)**.** A spinor $\chi \in \Gamma(\Sigma M)$ is called twistor spinor if it satisfies

$$P\chi := \nabla^{\Sigma M}_{e_\alpha}\chi + \frac{1}{2}e_\alpha \cdot \partial\!\!\!/\chi = 0. \tag{5.33}$$

Note that this equation is conformally invariant in dimension two. Dirac-harmonic maps from $S^2 \to S^2$ are characterized by the following theorem, also from [Yan09], p. 410:

Theorem 5.33. *Let $M = N = S^2$ with arbitrary metric and suppose that (ϕ, ψ) is a non-trivial Dirac-harmonic map from M to N. Then ϕ has to be holomorphic or antiholomorphic and the spinor ψ can be written in the form*

$$\psi = e_\alpha \cdot \chi \otimes d\phi(e_\alpha), \tag{5.34}$$

where χ is a twistor spinor.

For the sake of completeness we want to mention that several generalizations of Theorem 5.33 are given in [Mo10]. Twistor spinors on S^2 can be further characterized by [BFGK91]:

Lemma 5.34. *Suppose that $\chi \in \Gamma(\Sigma M)$. If M is closed, then $\ker(P)$ is finite dimensional. In the case $\chi \neq 0$ and if furthermore R is constant, then either $R = 0$ and χ is parallel or $R > 0$ and χ is the sum of two real non-parallel Killing spinors. Here, R denotes the scalar curvature on M.*

Hence, the spinor χ in (5.34) is actually the sum of two Killing spinors.
Furthermore, it is known that the the only compact orientable surfaces admitting twistor-spinors are S^2 and T^2 carrying any conformal class, the latter one being endowed with its trivial spin structure. The space of twistor spinors on S^2 is four-dimensional, whereas on T^2 it has only dimension two [Gin09], p. 126.
Consequently, one has to be careful when generalizing the construction (5.34) to arbitrary surfaces and arbitrary target manifolds N as was done in [JMZ09], since as we have just seen the only compact, orientable surfaces admitting twistor spinors are S^2 and T^2. This fact has already been noted in [Gin11].

Corollary 5.35. *There is no Dirac-harmonic map from $T^2 \to S^2$ with $\deg \phi = \pm 1$.*

Proof. The proof is by contradiction. Assume that (ϕ, ψ) is a Dirac-harmonic map from $T^2 \to S^2$ with $\deg(\phi) \pm 1$. By Theorem 5.31 the map ϕ has to be harmonic in this case. But on the other hand, Eells and Wood showed in [EW76] that there is no harmonic map from $T^2 \to S^2$ of degree ± 1 independently of the metrics chosen on M and N. $\square$

Remark 5.36. Since the degree of a map is homotopy-invariant, we cannot find a Dirac-harmonic map from $T^2 \to S^2$ in the homotopy class of ϕ with $\deg \phi = \pm 1$. This example motivates the occurrence of singularities in the heat flow for Dirac-harmonic maps.

Since the twisted Dirac operator $\not{D}$ is elliptic and self-adjoint, one can analyze its spectrum. A fundamental inequality regarding the spectrum of the usual Dirac operator $\not{\partial}$ is Friedrich's inequality [Fri80]. This inequality can easily be generalized to the Dirac-harmonic map case.

Lemma 5.37 (Friedrich inequality for Dirac-harmonic maps)**.** *All Eigenvalues of the Dirac operator $\not{D}$ on $\Sigma M \otimes \phi^{-1}TN$ satisfy the inequality*

$$\lambda^2 \geq \frac{1}{4}\frac{m}{m-1}\inf_M R - \frac{1}{\int_M |\psi|^2 dM}\frac{m}{m-1}\int_M R_{ijkl}\langle \nabla\phi^l \cdot \psi^i, \nabla\phi^k \cdot \psi^j\rangle dM, \tag{5.35}$$

where m is the dimension of the manifold M.

Proof. By the Weitzenboeck formula (2.5) we have

$$\int_M |\not{D}\psi|^2 dM = \int_M |\tilde{\nabla}\psi|^2 dM + \int_M \big(\frac{R}{4}|\psi|^2 + \frac{1}{2}\langle e_\alpha \cdot e_\beta \cdot R^N(d\phi(e_\alpha), d\phi(e_\beta))\psi, \psi\rangle\big) dM$$

and by the Cauchy Schwarz inequality on the other hand

$$|\not{D}\psi|^2 \leq m|\tilde{\nabla}\psi|^2.$$

Combining both equations yields the result. □

Corollary 5.38. *In particular, this means that there are no non-trivial Dirac-harmonic maps from S^2 to N with $N = \mathbb{R}^n, T^n$.*

Proof. For $M = S^2$, the first term on the right hand side in (5.35) is always positive. On the other hand, for $N = T^n$ or $\mathbb{R}^n$, the second term on the right hand side in (5.35) vanishes. □

We want to summarize our considerations in the following tabular:

M	N	Existence of Dirac-harmonic maps?
S^2	S^2	ϕ conformal, ψ is of the form (5.34)
S^2	T^2	no non-trivial Dirac-harmonic map
T^2	S^2	no Dirac-harmonic maps if $\deg(\phi) = \pm 1$
T^2	T^2	ϕ harmonic map, ψ harmonic spinor

If we further increase the genus of the surface M, for example $g_M = 2$, it is not that easy to state some general results. The existence of harmonic spinors on surfaces of genus $g_M \geq 2$ and their dependence on the spin structure and the Riemannian metric was investigated in [BS92].

5.6. Removing the Regularization

In this section we analyze the limit $\varepsilon \to 0$. We saw in Theorem 5.24 that the regularized Dirac-harmonic map heat flow converges to a smooth regularized Dirac-harmonic map $(\phi_\infty, \psi_\infty)$ on M away from finitely many singular points. The smoothness of the limiting map depends on the estimates that were derived before. Therefore the question is, which of these estimates we still need to control after taking the limit $\varepsilon \to 0$.
In particular, we would like to

1. Keep the number of singularities bounded,
2. Remove the singularities of the solution $(\phi_\infty, \psi_\infty)$,
3. Control the regularity of the solution $(\phi_\infty, \psi_\infty)$.

In general, we cannot expect that the limit $\varepsilon \to 0$ will exist. First of all, we study two simple examples.

Example 5.39. 1. Assume that $M = S^2$ and $N = T^2$. We have seen that in this case there exist no non-trivial Dirac-harmonic maps. Consequently, the limit $\varepsilon \to 0$ cannot exist and this fact should be reflected by the calculation.

2. If both $M = N = T^2$, we have seen that the system decouples and we have to look for harmonic spinors ψ^i on T^2. The two-dimensional torus has four spin structures and not all of them admit harmonic spinors. Hence, the limit $\varepsilon \to 0$ cannot be trivial in this case, too.

Number of singularities after $\varepsilon \to 0$

To study the dependence of the bound on the number of singularities on ε, we perform the following analysis. First of all, we supposed that $t = \infty$ is non-singular, in particular

$$\limsup_{t\to\infty}(\sup_{x\in M} F(\phi_t, \psi_t, B_R(x))) \leq \delta_1.$$

In addition, we have to ensure that there are only finitely many spatial singularities at $t = \infty$. Therefore, let us analyze how the bound (5.24) depends on ε. Rearranging (5.24) yields

$$\sum_{i=0}^{j-1} K_i \leq C\frac{F(\phi_0, \psi_0)(\varepsilon) + \delta_5(\varepsilon)}{\delta_1(\varepsilon)}. \tag{5.36}$$

It is easy to see that

$$\lim_{\varepsilon\to 0} F(\phi_0, \psi_0) = E(\phi_0) \leq C,$$

but on the other hand the limits

$$\lim_{\varepsilon\to 0} \delta_1(\varepsilon), \qquad \lim_{\varepsilon\to 0} \delta_5(\varepsilon)$$

do not exist in general as can easily be seen from the definitions of δ_1 and δ_5. Moreover, there is no cancellation of the different ε's on the right hand side of (5.36).

Removal of singularities after $\varepsilon \to 0$

To remove the singularities of the solution $(\phi_\infty, \psi_\infty)$ we would like to apply the following (Theorem 4.6 in [CJLW06], p. 426):

Theorem 5.40 (Removable singularity theorem)**.** *For $U \subset M$ let (ϕ, ψ) be a Dirac-harmonic map which is C^∞ on $U \setminus \{p\}$ for some $p \in U$. If*

$$\int_U \left(|d\phi|^2 + |\psi|^4\right) dM \leq C$$

then (ϕ, ψ) extends to a C^∞ solution on U.

In our case, the L^2 norm of $d\phi_\infty$ can be controlled, but in general we cannot bound the L^4 norm of ψ_∞ after $\varepsilon \to 0$. By the Sobolev embedding theorem it would be enough to control the L^2 norm of $\tilde{\nabla}\psi$. If $\psi^i_\infty \in \Gamma(\Sigma M)$ would be a Killing spinor, then $\psi_\infty \in L^4$.

Regularity of $(\phi_\infty, \psi_\infty)$ after $\varepsilon \to 0$

The question of the regularity of Dirac-harmonic maps has been studied in ([WX09]).

Definition 5.41 (Weakly Dirac-harmonic map)**.** A weakly Dirac-harmonic map is a pair $(\phi, \psi) \in H^{1,2}(M, N) \times S^{1,\frac{4}{3}}(M, \Sigma M \otimes \phi^{-1}TN)$, which is a critical point of $E(\phi, \psi)$ over the Sobolev space $H^{1,2}(M, N) \times S^{1,\frac{4}{3}}(M, \Sigma M \otimes \phi^{-1}TN)$. The spinor ψ is in the space $S^{1,\frac{4}{3}}(M, \Sigma M \otimes \phi^{-1}TN)$ if $\psi^i \in L^4(\Sigma M)$ and $\nabla^{\Sigma M}\psi^i \in L^{\frac{4}{3}}(\Sigma M)$.

The relation between weak and smooth Dirac-harmonic maps in dimension two is given by the following ([WX09], Theorem 1.5, p. 3764)

Theorem 5.42. *Assume that M is a compact Riemannian spin surface and that the pair $(\phi, \psi) \in H^{1,2}(M, N) \times S^{1,\frac{4}{3}}(M, \Sigma M \otimes \phi^{-1}TN)$ is a weakly Dirac-harmonic map. Then $(\phi, \psi) \in C^\infty(M, N) \times C^\infty(M, \Sigma M \otimes \phi^{-1}TN)$.*

Hence, we have to ensure that the estimates necessary for the existence of a weakly Dirac-harmonic map can be carried over to the limit $\varepsilon \to 0$. By the Sobolev embedding theorem in two dimensions $\psi^i \in L^4(\Sigma M)$ if $\nabla^{\Sigma M}\psi^i \in L^{\frac{4}{3}}(\Sigma M)$. The regularity of the map ϕ can easily be assured by plugging the spinor ψ_∞ into the inequality for the energy functional $E_\varepsilon(\phi, \psi)$

$$\int_M |d\phi_\infty|^2 dM \leq E_\varepsilon(\phi_0, \psi_0).$$

The difficult question is, if we can also achieve that $\psi \in S^{1,\frac{4}{3}}(M, \Sigma M \otimes \phi^{-1}TN)$ after taking the limit $\varepsilon \to 0$. When analyzing the regularity of solutions of

$$\varepsilon\tilde{\Delta}\psi = \not{D}\psi$$

as a function of ε, we first of all note that the operator $L := \varepsilon\tilde{\Delta} - \not{D}$ is uniformly elliptic as long as $\varepsilon \neq 0$. In the limit $\varepsilon \to 0$ the operator L becomes weakly elliptic and we cannot apply estimates for uniformly elliptic operators any longer. Hence, we should think of

$$\not{D}\psi = \varepsilon\tilde{\Delta}\psi \tag{5.37}$$

as a Dirac equation with right hand side. It is well known that the right hand side determines the regularity of the solution. Consequently, we should utilize elliptic estimates for first order operators in combination with the regularity theory for Dirac-harmonic maps developed so far.

A rough attempt to ensure $\nabla\psi^i \in L^{\frac{4}{3}}(\Sigma M)$ is to try to bound the H^1 norm. To control the H^1 norm of the spinor ψ^i we make use of elliptic estimates for first order equations. Expanding (5.37) we find

$$\begin{aligned} |\psi^i|_{H^1} &\leq C(|\psi^i|_{L^2} + |\not{\partial}\psi^i|_{L^2}) \\ &\leq C(|\psi^i|_{L^2} + ||\psi^i||d\phi||_{L^2} \\ &\quad +\varepsilon^2(||\nabla^{\Sigma M}\psi^i||d\phi||_{L^2} + ||\psi^i||d\phi|^2|_{L^2} + ||\psi^i||\Delta\phi||_{L^2} + ||\Delta^{\Sigma M}\psi^i|||_{L^2})). \end{aligned}$$

We would get a smooth Dirac-harmonic map from $M \to N$ if the right hand side survives the limit $\varepsilon \to 0$.

1. *Decoupling*
   If the limiting map $(\phi_\infty, \psi_\infty)$ decouples, then ϕ_∞ becomes a smooth harmonic map from $M \to N$. Nevertheless, we still have to ensure the regularity of the spinors $\psi^i \in \Gamma(\Sigma M)$ after $\varepsilon \to 0$.

2. *Killing spinor*
   If the limiting spinor $\psi^i_\infty \in \Gamma(\Sigma M)$ would be a Killing spinor, then we would have $|\psi^i_\infty|^2 \leq C$, independently of ε. Unfortunately, the existence of Killing spinors is heavily restricted.

We may summarize that in general we cannot expect that the limit $\varepsilon \to 0$ exists in a reasonable sense.
We want to finish this section by stating some general properties of the spinor ψ_∞.
We can analyze the local behaviour of the spinor ψ_∞ by

Remark 5.43 (Morrey type estimate)**.** If we have a smooth spinor ψ satisfying

$$\varepsilon\tilde{\Delta}\psi = \not{D}\psi,$$

then we find the following Bochner-type formula

$$\begin{aligned}
\Delta|\psi|^2 &= 2|\tilde{\nabla}\psi|^2 + 2\langle\psi, \tilde{\Delta}\psi\rangle \\
&= 2|\tilde{\nabla}\psi|^2 + \frac{2}{\varepsilon}\langle\psi, \not{D}\psi\rangle \\
&\geq -\frac{C}{\varepsilon^2}|\psi|^2.
\end{aligned}$$

Applying a Morrey type estimate [Mor08], Theorem 5.3.1, we find for any $x_0 \in M$ and $\rho > 0$, that

$$\sup_{B_{x_0}(\rho)} |\psi|^2 \leq \frac{C}{\varepsilon^2 R^2}\int_{B_{x_0}(R+\rho)} |\psi|^2 dM. \tag{5.38}$$

If the manifold M would be non-compact and we would have a bound on the L^2 norm of ψ, then we can conclude that ψ is trivial by letting $\rho \to \infty$.

Lemma 5.44 (Rayleigh quotient)**.** *Assume that ψ is a smooth solution of $\varepsilon\tilde{\Delta}\psi = \not{D}\psi$. Then the Rayleigh quotient of $\not{D}^2$ evaluated at ψ satisfies the inequality*

$$c_1\varepsilon^2 \leq \frac{\int_M |\not{D}\psi|^2 dM}{\int_M |\psi|^2 dM} \leq \frac{c_2}{\varepsilon^2} \tag{5.39}$$

with the constants $c_1 = \frac{1}{C_S^4 Vol(M)^2}$ and $c_2 = 4$. A consequence of this inequality is that in the limit $\varepsilon \to 0$ the spectrum of the operator $\not{D}$ becomes unbounded again.

Proof. We compute

$$\begin{aligned}
\varepsilon\int_M |\tilde{\nabla}\psi|^2 dM &= -\int_M \langle\psi, \not{D}\psi\rangle dM \leq \int_M |\psi|\sqrt{2}|\tilde{\nabla}\psi| dM \\
&\leq \frac{1}{\varepsilon}\int_M |\psi|^2 dM + \frac{\varepsilon}{2}\int_M |\tilde{\nabla}\psi|^2 dM
\end{aligned}$$

and hence we find

$$\varepsilon\int_M|\tilde{\nabla}\psi|^2dM\leq\frac{2}{\varepsilon}\int_M|\psi|^2dM.$$

Together with

$$\int_M|\not{D}\psi|^2dM\leq 2\int_M|\tilde{\nabla}\psi|^2dM$$

the first inequality follows. For the other direction we calculate

$$\left(\int_M|\psi|^2dM\right)^{\frac{1}{2}}\leq C_S\int_M|\tilde{\nabla}\psi|dM\leq C_S\sqrt{Vol(M)}\left(\int_M|\tilde{\nabla}\psi|^2dM\right)^{\frac{1}{2}},$$

where we used the Sobolev embedding theorem. Of course, one could also apply the Poincaré inequality here. With the help of the equation for ψ again, we find

$$\begin{aligned}\int_M|\psi|^2dM\leq C_S^2Vol(M)\int_M|\tilde{\nabla}\psi|^2dM &= C_S^2\frac{Vol(M)}{\varepsilon}\int_M\langle\psi,\not{D}\psi\rangle dM\\ &\leq C_S^4\frac{Vol(M)^2}{2\varepsilon^2}\int_M|\not{D}\psi|^2dM+\frac{1}{2}\int_M|\psi|^2dM\end{aligned}$$

and hence the result follows. The constants c_1 and c_2 may not be optimal. The statement concerning the spectrum of $\not{D}$ relies on the min-max principle for self-adjoint elliptic operators, see [Cha84], pp. 16-17. □

Remark 5.45. If we assume that the spinor $\psi\in\Gamma(\Sigma M\otimes\phi^{-1}TN)$ is of the form $\psi=e_\alpha\cdot\chi\otimes d\phi(e_\alpha)$, then a direct calculation yields ([AG11], p. 4)

$$\not{D}\psi=-2P_{e_\alpha}\chi\otimes d\phi(e_\alpha)+\frac{2-m}{m}e_\alpha\cdot\not{\partial}\chi\otimes d\phi(e_\alpha)-\chi\otimes\tau(\phi),\tag{5.40}$$

where P denotes the twistor operator and m denotes the dimension of M. In addition, one can check that $\mathcal{R}(\phi,\psi)=0$, when inserting ψ from above. We realize that for $m=2$, ϕ harmonic and $P_{e_\alpha}\chi=0$ we get a Dirac-harmonic map (ϕ,ψ). The last condition $P_{e_\alpha}\chi=0$ is satisfied if the spinor $\chi\in\Gamma(\Sigma M)$ is a twistor spinor, or a constant. It may be possible that our solution $(\phi_\infty,\psi_\infty)$ is of the form ϕ_∞ harmonic, χ is constant and $\psi=e_\alpha\cdot\chi\otimes d\phi(e_\alpha)$.

Appendices

A. Spin Geometry

We briefly recall the basic notions of spin geometry, see for example [Hij01].

Definition A.1 (Spin structure)**.** Let (M, h) be an m-dimensional oriented Riemannian manifold. A spin structure on M is a pair $(\mathrm{Spin}(M), \eta)$, where $\mathrm{Spin}(M)$ is a Spin_m-principal fibre bundle over M and η a 2-fold covering such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Spin}(M) \times \mathrm{Spin}_m \arrow[r] \arrow[dd, "\eta\times\mathrm{Ad}"] & \mathrm{Spin}(M) \arrow[dd, "\eta"] \arrow[dr, "\pi"] & \\
 & & M \\
\mathrm{SO}(M) \times \mathrm{SO}_m \arrow[r] & \mathrm{SO}(M) \arrow[ur, "\pi"] &
\end{tikzcd}$$

The maps in the rows are the actions of Spin_m and SO_m on the principal fibre bundles $\mathrm{Spin}(M)$ and $\mathrm{SO}(M)$.

The existence of a spin structure on M is equivalent to the second Stiefel-Whitney class $\omega_2(M)$ being zero, which is a topological restriction.

Definition A.2 (Spinor bundle)**.** The (complex) spinor bundle associated to a spin structure $\mathrm{Spin}(M)$ of M is the (complex) vector bundle

$$\Sigma M := \mathrm{Spin}(M) \times_\rho \Sigma_m,$$

where $\rho\colon \mathrm{Spin}_n \to \mathrm{Aut}(\Sigma_m)$ is the (complex) Spin_n representation.

On the spinor bundle ΣM we have the Clifford relations

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2h(X, Y)\psi$$

for all $X, Y \in \Gamma(TM), \psi \in \Gamma(\Sigma M)$ and the metric h on M. We choose a hermitian metric on the spinor bundle ΣM. The Clifford multiplication is skew-symmetric, namely

$$\langle X \cdot \psi, \chi\rangle_{\Sigma M} = -\langle \psi, X \cdot \chi\rangle_{\Sigma M}$$

for all $X \in \Gamma(TM)$ and $\psi, \chi \in \Gamma(\Sigma M)$. On the spinor bundle ΣM we have a connection, denoted by $\nabla^{\Sigma M}$, which is induced from the connection on the manifold M.

Proposition A.3 (Local description of $\nabla^{\Sigma M}$ and $R^{\Sigma M}$)**.** *The covariant derivative on ΣM is locally given by*

$$\nabla^{\Sigma M}\psi = \frac{1}{4}h(\nabla e_\alpha, e_\beta)e_\alpha \cdot e_\beta \cdot \psi.$$

If $R^M(X,Y)$ denotes the curvature tensor on the tangent bundle of M, then it can be related to the curvature on ΣM by

$$R^{\Sigma M}(X,Y)\psi = \frac{1}{4}h(R^M(X,Y)e_\alpha, e_\beta)e_\alpha \cdot e_\beta \cdot \psi$$

for all $X, Y \in \Gamma(TM)$ and $\{e_\alpha\}$ denoting a local orthonormal basis of TM.

Proposition A.4. *The connection on the spinor bundle $\nabla^{\Sigma M}$ is compatible with both Clifford multiplication and the bundle metric, namely*

$$\begin{aligned} \partial_X\langle\psi,\chi\rangle_{\Sigma M} &= \langle\nabla^{\Sigma M}_X\psi,\chi\rangle_{\Sigma M} + \langle\psi,\nabla^{\Sigma M}_X\chi\rangle_{\Sigma M}, \\ \nabla^{\Sigma M}_X(Y\cdot\psi) &= (\nabla_X Y)\cdot\psi + Y\cdot\nabla^{\Sigma M}_X\psi \end{aligned}$$

for all $X, Y \in \Gamma(TM)$ and $\psi, \chi \in \Gamma(\Sigma M)$.

Definition A.5 (Dirac operator)**.** The Dirac operator $\not{\partial}$ is the composition of the covariant derivative acting on sections of ΣM with Clifford multiplication, which is locally given by

$$\not{\partial}\psi := e_\alpha \cdot \nabla^{\Sigma M}_{e_\alpha}\psi,$$

where $\{e_\alpha\}$ is a local orthonormal basis of TM.

Lemma A.6 (Properties of the Dirac operator)**.** *The Dirac operator is a first order partial differential operator, which is*

1. *(weakly) elliptic,*
2. *self-adjoint in L^2 for compact M*
3. *and the square of the Dirac operator satisfies the Schrödinger-Lichnerowicz formula*

$$\not{\partial}^2 = \nabla^*\nabla + \frac{1}{4}R,$$

where R denotes the scalar curvature of the manifold M.

B. Analytic Aspects

B.1. Differential Operators on Manifolds

Most of the definitions made here follow [Aub98]. Assume that (M,h) is a compact Riemannian manifold and let $E \to M$ be a Riemannian or hermitian vector bundle over M.

Definition B.1 (Principal symbol)**.** Let $P : \Gamma(E) \to \Gamma(E)$ be a differential operator of order k. For a point $x \in M$ and a covector $\xi \in T^*M$ we want to define the principal symbol $\sigma_k(P,\xi)\colon E_x \to E_x$. Assume that the operator P can be expressed as

$$P = \sum_{|\alpha|\leq k} A_\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \qquad \xi = \sum_l \xi_l dx_l.$$

Then the principal symbol is given by

$$\sigma_k(P,\xi) = \sum_{|\alpha|=k} A_\alpha \xi^\alpha. \tag{B.1}$$

Roughly spoken, the principal symbol picks the coefficient of the highest order term of a given differential operator P. The principal symbol encodes most of the important properties of a differential operator.

Definition B.2. A linear differential operator P of order k is (weakly) elliptic if the principal symbol $\sigma_k(P,\xi)$ is an isomorphism for every $\xi \neq 0$.

Definition B.3. A linear differential operator P is uniformly elliptic if the principal symbol $\sigma_k(P,\xi)$ satisfies the following inequality

$$\langle \sigma_k(P,\xi)\eta, \eta\rangle \geq C|\eta|^2 \tag{B.2}$$

for all $\eta \in \Gamma(E)$. The scalar product is taken with respect to the bundle metric on E.

Definition B.4. A differential operator $P\colon \Gamma(E) \to \Gamma(E)$

$$P\eta = F(x, \eta, \nabla\eta, \dots, \nabla^k\eta),$$

where F is assumed to be a differentiable map of its arguments, will be elliptic (uniformly elliptic) with respect to η if the linearized operator is elliptic (uniformly elliptic).

Definition B.5. A strictly parabolic equation is an equation of the type

$$\frac{\partial \eta_t}{\partial t} = P_t\eta_t, \tag{B.3}$$

where η_t is a t-dependent section of E and $P_t\colon \Gamma(E) \to \Gamma(E)$ is a smooth family of uniformly elliptic operators.

B.2. Hölder Spaces

We define Hölder spaces adapted to second order parabolic equations, see [LSU67] and [Kry96]. For $T > 0$ we set $Q = M \times [0, T)$ and $0 < \alpha < 1$. We first of all define Hölder spaces for functions taking values in $\mathbb{R}^q$ and then generalize them to functions on Riemannian manifolds and sections in vector bundles. For a function $u\colon M \to \mathbb{R}^q$ we define

$$|u|_Q = \sup_{(x,t)\in Q} |u(x,t)|,$$

$$\langle u \rangle_x^{(\alpha)} = \sup_{\substack{(x,t)(x',t)\in Q \\ x\neq x'}} \frac{|u(x,t) - u(x',t)|}{d^M(x,x')^\alpha}, \qquad \langle u \rangle_t^{(\alpha)} = \sup_{\substack{(x,t)(x,t')\in Q \\ t\neq t'}} \frac{|u(x,t) - u(x,t')|}{|t-t'|^\alpha},$$

and the norms $|u|_Q^{(\alpha,\alpha/2)}$, $|u|_Q^{(2+\alpha,1+\alpha/2)}$ by

$$\begin{aligned}
|u|_Q^{(\alpha,\alpha/2)} &= |u|_Q + \langle u \rangle_x^{(\alpha)} + \langle u \rangle_t^{(\alpha/2)}, \\
|u|_Q^{(2+\alpha,1+\alpha/2)} &= |u|_Q + |\partial_t u|_Q + |D_x u|_Q + |D_x^2 u|_Q \\
&\quad + \langle \partial_t u \rangle_t^{(\alpha/2)} + \langle D_x u \rangle_t^{(1/2+\alpha/2)} + \langle D_x^2 u \rangle_t^{(\alpha/2)} \\
&\quad + \langle \partial_t u \rangle_x^{(\alpha)} + \langle D_x^2 u \rangle_x^{(\alpha)}.
\end{aligned}$$

Here $d^M(x,x')$ denotes the Riemannian distance between x and x' on M. With the help of these norms we can define the following function spaces

$$\begin{aligned}
C^{\alpha,\alpha/2}(Q,\mathbb{R}^q) &= \{u \in C^0(Q) \mid |u|_Q^{(\alpha,\alpha/2)} < \infty\}, \\
C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q) &= \{u \in C^{2,1}(Q) \mid |u|_Q^{(2+\alpha,1+\alpha/2)} < \infty\}.
\end{aligned}$$

We want to sketch how these definitions can be extended to functions taking values in Riemannian manifolds and sections in vector bundles. Differences of vectors are replaced by differences of parallel transports of vectors along the shortest geodesic. This involves the Riemannian distance function on M.

B.3. Differentiability of Solutions and Schauder Estimates

The estimates presented in this section can for example be found in [WYW06], chapters 6 and 7. Let $U \subset \mathbb{R}^n$ be a bounded, connected and open set and let P be a second order linear elliptic differential operator of the form

$$P = a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(x)\frac{\partial}{\partial x^i} + d(x). \tag{B.4}$$

Theorem B.6. *For $0 < \alpha < 1$, assume that $a^{ij}, b^i, d, f \in C^\alpha(U)$. If $u \in C^2(U)$ satisfies the linear parabolic equation $Pu(x) = f(x)$, then $u \in C^{2+\alpha}(U)$. Additionally, if $a^{ij}, b^i, d, f \in C^{k+\alpha}(U)$ for a given $k \geq 1$, then a solution of $Pu(x) = f(x)$ is $C^{k+2+\alpha}(U)$. In particular, if $a^{ij}, b^i, d, f \in C^\infty(U)$, then $u \in C^\infty(U)$.*

A similar result holds for parabolic differential operators. Set $Q = U \times [0, T)$.

Theorem B.7. *1. For $0 < \alpha < 1$, assume that $a^{ij}, b^i, d \in C^\alpha(U)$ and $f \in C^{\alpha,\alpha/2}(Q)$. If $u \in C^{2,1}(Q)$ satisfies the following linear parabolic equation*

$$\left(P - \frac{\partial}{\partial t}\right) u(x,t) = f(x,t), \tag{B.5}$$

then $u \in C^{2+\alpha,1+\alpha/2}(Q)$.

2. *Let p, q be nonnegative integers. For given β, δ with $|\beta| \le p, |\beta| + 2\delta \le p, \delta \le q$, assume that $D_x^\beta a^{ij}, D_x^\beta b^i, D_x^\beta d \in C^\alpha(U)$ and $D_x^\beta D_t^\delta f \in C^{\alpha,\alpha/2}(Q)$. Then a solution u of (B.5) satisfies $D_x^\beta D_t^\delta f \in C^{\alpha,\alpha/2}(Q)$ for any β, δ with $|\beta| + 2\delta \le p+2, \delta \le q+1$. In particular, $a^{ij}, b^i, d, f \in C^\infty(U)$ and $f \in C^\infty(Q)$ imply that $u \in C^\infty(Q)$.*

In the following, we state the classical Schauder estimates for elliptic and parabolic partial differential equations. For $0 < \alpha < 1$, assume that

$$a^{ij}, b^i, d \in C^\alpha(B_r(0)), \qquad 1 \le i, j \le n,$$

and additionally that the operator P is uniformly elliptic

$$\lambda |\xi|^2 \le a^{ij}(x)\xi^i \xi^j \le \Lambda |\xi|^2$$

for some constants $0 < \lambda \le \Lambda < \infty$, for any $x \in B_r(0)$ and $\xi \in \mathbb{R}^n$. For the linear elliptic differential operator P defined above and the linear parabolic differential operator

$$L = P - \frac{\partial}{\partial t},$$

we then have the following

Theorem B.8 (Schauder estimates)**.** *1. If $f \in C^\alpha(B_r(0))$ and $u \in C^2(B_r(0))$ satisfy*

$$Pu(x) = f(x),$$

then $u \in C^{2+\alpha}(B_r(0))$ and the following inequalities hold

$$|u|_{C^{1+\alpha}(B_{\frac{r}{2}}(0))} \le C(|f|_{L^\infty(B_r(0))} + |u|_{L^\infty(B_r(0))}), \tag{B.6}$$

$$|u|_{C^{2+\alpha}(B_{\frac{r}{2}}(0))} \le C(|f|_{C^\alpha(B_r(0))} + |u|_{L^\infty(B_r(0))}). \tag{B.7}$$

The constant C depends on $n, \alpha, \lambda, \Lambda, |a^{ij}|_{C^\alpha(B(0,r))}, |b^i|_{C^\alpha(B(0,r))}, |d|_{C^\alpha(B(0,r))}$.

2. *Let $0 \le t \le T$. If $f(\cdot, t) \in C^\alpha(B_r(0))$ and $u(\cdot, t) \in C^2(B_r(0)$ satisfy*

$$Lu(x,t) = f(x,t),$$

then $u(\cdot,t) \in C^{2+\alpha}(B_r(0))$ and we have the estimates

$$|u(\cdot,t)|_{C^{\alpha}(B_{\frac{r}{2}}(0))} \leq C\left(\sup_{t\in[0,T)} |f(\cdot,t)|_{L^{\infty}(B_r(0))} + \sup_{t\in[0,T)} |u(\cdot,t)|_{L^{\infty}(B_r(0))}\right) \tag{B.8}$$

and

$$\begin{aligned} &|u(\cdot,t)|_{C^{2+\alpha}(B_{\frac{r}{2}}(0))} + |\frac{\partial u(\cdot,t)}{\partial t}|_{C^{\alpha}(B_r(0))} \\ &\qquad \leq C\left(\sup_{t\in[0,T)} |f(\cdot,t)|_{C^{\alpha}(B_r(0))} + \sup_{t\in[0,T)} |u(\cdot,t)|_{L^{\infty}(B_r(0))}\right). \end{aligned} \tag{B.9}$$

The constant C depends on $n, \alpha, \lambda, \Lambda, |a^{ij}|_{C^{\alpha}(B(0,r))}, |b^i|_{C^{\alpha}(B(0,r))}, |d|_{C^{\alpha}(B(0,r))}$.

All of these estimates can be carried over to the case that instead of u, we consider sections in a vector bundle over M.

B.4. Embedding Theorems

Theorem B.9 (Sobolev embedding theorem)**.** *Let (M,h) be a compact Riemannian manifold of dimension m. Let $k,l \in \mathbb{N}$, $p,q, \in [1,\infty)$ and $\alpha \in (0,1)$. Then the following statements hold:*

1. *If $\frac{1}{p} \leq \frac{1}{q} + \frac{k-l}{m}$, then $W^{k,p}(M)$ embeds continuously into $W^{l,q}(M)$ by inclusion.*
2. *If $k - \frac{m}{p} \geq l + \alpha$, then $W^{k,p}(M)$ embeds continuously into $C^{k,\alpha}(M)$ by inclusion.*

For $M = \mathbb{R}^m$ a proof can be found in [GT01], Thm. 7.10. In the case of M being a compact Riemannian manifold, see [Aub98], Thm. 2.20.

Theorem B.10 (Rellich-Kondrachov embedding theorem)**.** *Let (M,h) be a compact Riemannian manifold. Let $k,l \in \mathbb{N}$, $p,q \in [1,\infty)$ and $\alpha \in (0,1)$. Then the following statements hold:*

1. *If $\frac{1}{p} < \frac{1}{q} + \frac{k-l}{m}$, then the inclusion of $W^{k,p}$ into $W^{l,q}(M)$ is compact.*
2. *If $k - \frac{m}{p} > l + \alpha$, then the inclusion of $W^{k,p}(M)$ into $C^{l,\alpha}(M)$ is compact.*

For $M = \mathbb{R}^m$ a proof can be found in [GT01], Thm. 7.22. In the case of M being a compact Riemannian manifold, see [Aub98], Thm. 2.34.

B.5. Linear parabolic Equations

Theorem B.11. *Let (M,h) be a compact Riemannian manifold without boundary. For a vector valued function $u\colon Q \to \mathbb{R}^q$ we consider the linear parabolic differential operator*

$$Lu = \frac{\partial u}{\partial t} - \Delta u + b^i(x,t)D^i u + c(x,t)u \tag{B.10}$$

and the initial value problem

$$\begin{cases} Lu(x,t) = F(x,t), & (x,t) \in Q, \\ u(x,0) = f(x). \end{cases} \tag{B.11}$$

If $b^i, c \in C^{\alpha,\alpha/2}(Q, \mathbb{R}^q)$ for $0 < \alpha < 1$, then for any

$$F \in C^{\alpha,\alpha/2}(Q, \mathbb{R}^q), \qquad f \in C^{2+\alpha}(M, \mathbb{R}^q),$$

there exists a unique solution $u \in C^{2+\alpha,1+\alpha/2}(Q, \mathbb{R}^q)$ of (B.11) satisfying

$$|u|_Q^{(2+\alpha,1+\alpha/2)} \leq C(|F|_Q^{(\alpha,\alpha/2)} + |f|_M^{(2+\alpha)}). \tag{B.12}$$

The constant C depends on M, L, q, T, α.

A proof can be found in [WYW06], p. 251. The above theorem can be generalized for functions between Riemannian manifolds and sections in vector bundles.

B.6. Classical Tools

The most prominent tool to obtain pointwise estimates is the maximum principle. We state the following simple version:

Lemma B.12 (Maximum principle)**.** *Let (M, h) be a compact Riemannian manifold and let $L = \Delta - \frac{\partial}{\partial t}$ be the heat operator on M. If $u \in C^0(M \times [0,T)) \cap C^{2,1}(M \times [0,T))$ satisfies $Lu \geq 0$ in $M \times (0,T)$ then the following estimate holds*

$$\max_{M \times [0,T)} u = \max_{M \times \{0\}} u. \tag{B.13}$$

A proof can for example be found in [Lie96], p. 7. The following extension combines the pointwise maximum principle with an integral norm.

Lemma B.13. *Assume that (M, h) is a compact Riemannian manifold. If a function $u(x,t) \geq 0$ satisfies*

$$\frac{\partial u}{\partial t} \leq \Delta u + Cu,$$

and if in addition we have the bound

$$U(t) = \int_M u(x,t) dM \leq U_0,$$

then there exists a uniform bound on

$$u(x,t) \leq e^C K U_0$$

with the constant K depending on M.

Proof. A proof can for example be found in [Tay11], p. 284. □

It is also possible to express the constant K in terms of geometric quantities, therefore see [Jos88], p. 86, Lemma 2.3.1.

Lemma B.14 (Mean Value Theorem). *Let (M,h) be a Riemannian manifold and furthermore $u \in C^1(M)$. By $d^M(x,y)$ we denote the Riemannian distance function on M. Then for any compact convex subset $K \subset M$ the following inequality holds*

$$|u(x) - u(y)| \leq \sup_K |du| d^M(x,y)$$

for all $x, y \in K$.

B.7. Elliptic Operators and Spectral Theory

Assume that (M,h) is a compact Riemannian manifold and $E \to M$ is a Riemannian or hermitian vector bundle over M. All of the following statements can be found in [LM89], p. 192 ff., for more details the reader may take a look a the survey article [AB02]. Note that all of the statements here are true for weakly elliptic operators.

Theorem B.15 (Elliptic estimates). *Let $P\colon \Gamma(E) \to \Gamma(E)$ be an elliptic operator of order m. Then the following assertions hold:*

1. *For any open set $U \subset M$ and $u \in H^k(E)$ we have*

$$Pu\big|_U \in C^\infty \Rightarrow u\big|_U \in C^\infty.$$

2. *For each s there is a constant C such that the following inequalities hold:*

$$|u|_{H^s} \leq C(|u|_{H^{s-m}} + |Pu|_{H^{s-m}}) \tag{B.14}$$

and

$$|u|_{H^{ms}} \leq C(|u|_{L^2} + |P^s u|_{L^2}). \tag{B.15}$$

Theorem B.16. *Let $P\colon \Gamma(E) \to \Gamma(E)$ be a self-adjoint elliptic differential operator of order $m > 0$ over a compact Riemannian manifold. Then each eigenspace of P is finite-dimensional and consists of smooth sections. The eigenvalues of P are real, discrete and tend rapidly to infinity. Furthermore, the eigenspaces of P furnish complete orthonormal systems for $L^2(E)$.*

As a direct consequence we get

Corollary B.17. *Let $P\colon \Gamma(E) \to \Gamma(E)$ be a self-adjoint elliptic differential operator of order $m > 0$ over a compact Riemannian manifold. Then there exists a Hilbert space orthonormal basis $\psi_1, \psi_2, \ldots$ of $L^2(E)$ and real numbers $\lambda_1, \lambda_2, \ldots$ such that*

$$P\psi_k = \lambda_k \psi_k$$

with $\lambda_1 \leq \lambda_2 \leq \lambda_3, \ldots, \infty$. Each of the λ_k is repeated only finitely many times and all ψ_k are smooth, $\psi_k \in C^\infty(E)$.

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