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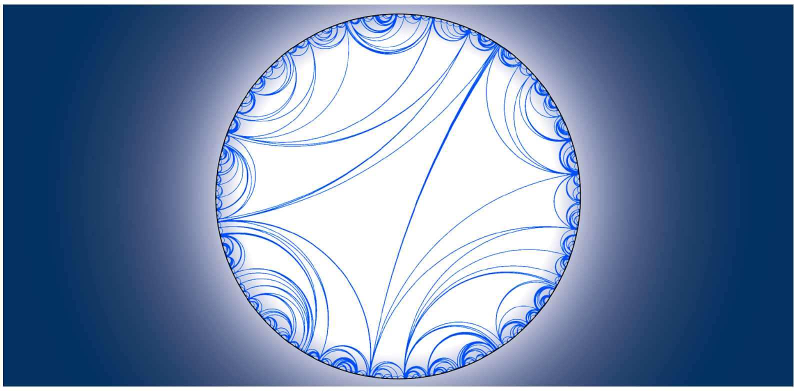

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TEMPORAL SYMMETRY OF SOME CLASSES OF STOCHASTIC PROCESSES

CHRISTIAN LÉONARD, SYLVIE RŒLLY, AND JEAN-CLAUDE ZAMBRINI

Abstract. In this article we analyse the structure of Markov processes and reciprocal processes to underline their time symmetrical properties, and to compare them. Our originality consists in adopting a unifying approach of reciprocal processes, independently of special frameworks in which the theory was developped till now (diffusions, or pure jump processes). This leads to some new results too.

Contents

KEY-WORDS: Markov processes, reciprocal processes, time symmetry.

Introduction

The Markov property, classic concept of the probabilistic landscape since more than one century, was presented by Doob in 1953 [Doo53] in a symmetrical way. This remarkable point of view, often replaced by an asymmetric notion of the directed time, was developped further by numerous authors, see references in the monograph from Chung and Walsh [CW05].
A few decades after Markov, Schrödinger in his paper *"Über die Umkehrung der Naturgesetze"* [Sch31] published in 1931, and then Bernstein [Ber32] one year later, introduced in the framework of diffusion processes the concept of reciprocal processes which, as one senses, is insensitive with respect to the direction of time.

We propose in this paper a new *structural* approach of the law of reciprocal processes, called reciprocal probabilities, which disregards the specific contexts already treated in the past - diffusions with continuous paths by Jamison [Jam74] (see also [Thi02], [TZ97b] or [TZ97a]) or pure jump processes by Murr [Mur12]. We then present a unifying vision of *reciprocal* notions and compare, in a general framework, different type of time symmetries satisfied either by Markov or by reciprocal processes.

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So, in Theorem 1.8, the multiplicative -characteristic- structure of the density of a Markov probability with respect to an other one is made clear. In Theorem 2.20, we describe more precisely this structure for elements of a reciprocal family.

Intentionally, we treat in this paper only the case of *probability* measures. We could generalise most of the results to σ-finite measures, as for example diffusions having as marginal at initial time a measure with infinite mass. More details on this generalised framework are given in [Léoa].

Some notations. We consider the set $\Omega = \mathbb{D}([0,1], \mathcal{X}) \subset \mathcal{X}^{[0,1]}$ of càdlàg paths defined on the finite time interval $[0,1]$ with state space $\mathcal{X}$, which is supposed to be polish, endowed with its Borel σ-algebra. As usual Ω is endowed with the canonical filtration $\mathcal{A}$, generated by the *canonical process* $X = (X_t)_{t\in[0,1]}$:

$$X_t(\omega) := \omega_t, \quad \omega = (\omega_s)_{s\in[0,1]} \in \Omega,\ t \in [0,1].$$

For any subset $\mathcal{S} \subset [0,1]$ and for any probability measure P on Ω one denotes

- $X_{\mathcal{S}} = (X_s)_{s\in\mathcal{S}}$ the canonical process restricted to $\mathcal{S}$,
- $\mathcal{A}_{\mathcal{S}} = \sigma(X_s; s \in \mathcal{S})$ the σ-algebra of the events observed during $\mathcal{S}$,
- $P_{\mathcal{S}} = (X_{\mathcal{S}})_\# P$ the restriction of P to $\Omega_{\mathcal{S}} := X_{\mathcal{S}}(\Omega)$.

For $\mathcal{S} = [s,u] \subset [0,1]$ we use the peculiar notations:

- $X_{[s,u]} := (X_t; s \le t \le u)$
- $\mathcal{A}_{[s,u]} := \sigma(X_{[s,u]})$, the σ-algebra generated by the events occured between time s and time u
- $P_s := (X_s)_\# P$ is the projection of P at time s
- $P_{su} := (X_s, X_u)_\# P$ is the marginal of P at times s and u simultaneously (P_{01} is therefore the joint law of the initial and final conditions)
- $P_{[s,u]} := (X_{[s,u]})_\# P$ is the projection of P on the time interval $[s,u]$
- $P^{sx} := P(\cdot \mid X_s = x)$ is the measure P conditioned to be equal at δ_x at time s
- $P^{sx,uy} := P(\cdot \mid X_s = x, X_u = y)$ is the measure P conditioned to be equal at δ_x at time s and δ_y at time u.

In particular, for $x, y \in \mathcal{X}$, we denote by

$$P^{xy} := P^{0x,1y} = P(\cdot \mid X_0 = x, X_1 = y)$$

the bridge of P on $[0,1]$, concentrated on x at initial time and on y at final time (a.s. $X_0 = x$ and $X_1 = y$).
The probability measure

$$P^* = (X^*)_\# P$$

is the law under P of the time reversed canonical process $X^* := (X_{1-t})_{0\le t\le 1}$.

1. Time symmetry of Markov probabilities

We present in this section structural properties of Markov probabilities and of their bridges. We especially underline their time symmetry, which has already been studied in specific frameworks (see for example [CW05]). To make this paper self-contained, we sketch results even if they are already known in some special case (as in the diffusion framework).

1.1. **Definition and essential properties.** Let us begin with the symmetrical definition of the Markov property.

Definition 1.1 (Markov probability). *A probability measure P on Ω is called* Markov *(or the law of a Markov process) if for any $t \in [0,1]$ and for any events $A \in \mathcal{A}_{[0,t]}, B \in \mathcal{A}_{[t,1]}$*

$$P(A \cap B \mid X_t) = P(A \mid X_t)P(B \mid X_t), \quad P\text{-}a.s. \tag{1.2}$$

The above property states that, under P, the future $\mathcal{A}_{[t,1]}$ and the past $\mathcal{A}_{[0,t]}$ are conditionally independent, given the present time t. It is invariant with respect to time-reversal.

In the Theorem below, we recall equivalent descriptions of the Markov property, especially the identity (2) which states that a Markov probability forgets its past history.

Theorem 1.3. *Let P be a probability measure on Ω. Then the following are equivalent:*

(1) *The probability P is Markov.*

(1*) *The time-reversed probability P^* is Markov.*

(2) *For all $0 \le t \le 1$, and all sets $B \in \mathcal{A}_{[t,1]}$,*
$$P(B \mid X_{[0,t]}) = P(B \mid X_t), \quad P\text{-}a.e.$$

(2*) *For all $0 \le t \le 1$, and all sets $A \in \mathcal{A}_{[0,t]}$,*
$$P(A \mid X_{[t,1]}) = P(A \mid X_t), \quad P\text{-}a.e.$$

(3) *For all $0 \le s \le u \le 1$, and all sets $A \in \mathcal{A}_{[0,s]}, C \in \mathcal{A}_{[u,1]}$*
$$P(A \cap C \mid X_{[s,u]}) = P(A \mid X_s)P(C \mid X_u), \quad P\text{-}a.e.$$

Proof. We will proof (3) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

• Proof of (3) $\Rightarrow$ (1). It is clear taking $s = u$.

• Proof of (2) $\Rightarrow$ (3). For all sets $A \in \mathcal{A}_{[0,s]}$ and $C \in \mathcal{A}_{[u,1]}$ and all sets $B \in \mathcal{A}_{[s,u]}$, the equality

$$P(A \cap B \cap C) = E[\mathbf{1}_B P(A \cap C \mid X_{[s,u]})]$$

holds, just as

$$\begin{aligned} P(A \cap B \cap C) &= E[P(A \cap B \cap C \mid X_{[0,u]})] \\ &= E[\mathbf{1}_A \mathbf{1}_B P(C \mid X_{[0,u]})] \\ &= E[\mathbf{1}_A \mathbf{1}_B P(C \mid X_{[s,u]})] \\ &= E[\mathbf{1}_B P(A \mid X_{[s,u]}) P(C \mid X_{[s,u]})] \end{aligned}$$

where property (2) is used in the second to last equality. Therefore

$$P(A \cap C \mid X_{[s,u]}) = P(A \mid X_{[s,u]})P(C \mid X_{[s,u]}).$$

• Proof of (1) $\Rightarrow$ (2). Let us show that if (1.2) is satisfied under P then P forgets its past history. Let $A \in \mathcal{A}_{[0,t]}$ and $B \in \mathcal{A}_{[t,1]}$ be some events. Let us compute $P(B \mid X_{[0,t]})$ by using (1.2):

$$E[\mathbf{1}_A P(B \mid X_{[0,t]})] = P(A \cap B) = E(P(A \cap B \mid X_t)) = E[P(A \mid X_t)P(B \mid X_t)].$$

On the other hand,

$$E[\mathbf{1}_A P(B \mid X_t)] = E[P(A \mid X_t)P(B \mid X_t)].$$

One obtains for any set $A \in \mathcal{A}_{[0,t]}$, $E[\mathbf{1}_A P(B \mid X_{[0,t]})] = E[\mathbf{1}_A P(B \mid X_t)]$, which implies $P(B \mid X_{[0,t]}) = P(B \mid X_t)$. This completes (1) $\Rightarrow$ (2) and finally the proof of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).
Eventually the symmetry of the formulation of (3) leads to the equivalence between (2) and (1*). Assertion (2*) corresponds to (2) applied to P^*. □

One finds a first proof of (1) $\Leftrightarrow$ (2) in the monograph by Doob [Doo53, Eq. (6.8) & (6.8')]. Dynkin [Dyn61] and then Chung [Chu68, Thm 9.2.4] took it over.

Identity (2) is often used as the definition of the Markov property, while $\mathcal{A}_{[0,t]}$ and $\mathcal{A}_{[t,1]}$ are interpreted as past and future of the present time t. It is usually called *one-sided property*, and creates the illusion that the Markov property is time asymmetric , which is inaccurate.
Meyer already remarked in [Mey67] that *the Markov property is invariant under time reversal*, unlike other interesting properties of processes.

Since each Markov process can be defined via its forward and backward transition probability kernels, we recall how to construct it in a symmetrical way.

Definitions 1.4. *Let P be a Markov probability on Ω.*

1. *The* forward transition probability kernel *associated with P is the family of conditional probabilities $\big(p(s,x;t,\cdot); 0 \le s \le t \le 1, x \in \mathcal{X}\big)$ defined for any $0 \le s \le t \le 1$, and P_s-almost all x, by*
$$p(s,x;t,dy) = P(X_t \in dy \mid X_s = x).$$
2. *The* backward transition probability kernel *associated with P is the family of conditional probabilities $\big(p^*(s,\cdot;t,y); 0 \le s \le t \le 1, y \in \mathcal{X}\big)$ defined for any $0 \le s \le t \le 1$, and P_t-almost all y, by*
$$p^*(s,dx;t,y) := P(X_s \in dx \mid X_t = y).$$

Since these kernels satisfy the celebrated Chapman-Kolmogorov relations

$$\forall 0 \le s \le t \le u \le 1,$$
$$p(s,x;u,\cdot) = \int_{\mathcal{X}} p(s,x;t,dy)p(t,y;u,\cdot) \text{ for } P_s\text{-a.a. } x \tag{1.5}$$
$$p^*(s,\cdot;u,z) = \int_{\mathcal{X}} p^*(s,\cdot;t,y)p^*(t,dy;u,z) \text{ for } P_u\text{-a.a. } z, \tag{1.6}$$

one can construct the probability measure P in the following way.

Proposition 1.7. *The Markov probability measure P is uniquely determined by one time marginal P_u at some time $u \in [0,1]$, its forward transition probability kernels starting from time u, $\big(p(s,x;t,\cdot); u \le s \le t \le 1, x \in \mathcal{X}\big)$ and the backward transition probability kernels until time u, $\big(p^*(s,\cdot;t,y); 0 \le s \le t \le u, y \in \mathcal{X}\big)$.*

Indeed, for any $0 \le s_1 \le \dots s_k \le u \le t_1 \le \dots t_l \le 1$*, and* $k, l \ge 1$*, the finite dimensional projection of* P *are given by*

$$P_{s_1,\dots,s_k,u,t_1,\dots,t_l} = p^*_{s_1;s_2} \otimes \dots \otimes p^*_{s_k;u} \otimes P_u \otimes p_{u;t_1} \otimes \dots \otimes p_{t_{l-1};t_l}.$$

where we used the following intuitive notation

$$P_u \otimes p_{u;t}(dx, dy) := P_u(dx)p(u, x; t, dy).$$

1.2. **Probability measure dominated by a Markov probability measure.** We now identify when a probability measure on Ω, which is dominated by a given Markov one, inherits its Markovianity.
In the following result, we present a criterium concerning the multiplicative structure of the Radon-Nikodym density, which should be time symmetrical.

Theorem 1.8. *Let* R *be a reference probability measure on* Ω *and let* P *be a probability measure dominated by* R*. Then the following are equivalent:*

(1) The probability measure P *is Markov*
(2) For any time $t \in [0, 1]$*, the Radon-Nikodym density of* P *with respect to* R *factorizes in the following way:*

$$\frac{dP}{dR} = \alpha_t\, \beta_t \quad R\text{-a.e.} \tag{1.9}$$

where α_t *resp.* β_t *is a non negative* $\mathcal{A}_{[0,t]}$*- resp.* $\mathcal{A}_{[t,1]}$*-measurable functional.*

Proof. • Proof of (2) ⇒ (1). Take two events, $A \in \mathcal{A}_{[0,t]}$ and $B \in \mathcal{A}_{[t,1]}$. In terms of Definition 1.1, we have to show that

$$P(A \cap B \mid X_t) = P(A \mid X_t)P(B \mid X_t), \quad P\text{-a.e.}. \tag{1.10}$$

To this aim, note that though the product $\alpha_t\beta_t$ is R-integrable, it is not clear why α_t or β_t are separately integrable. Since we need this property, we may use the following fine Lemma of integration theory, which assures the $R(. \mid X_t)$-integrability of the functionals α_t and β_t P-a.e..

Lemma 1.11. *Under hypothesis (2) of the above Theorem, the functionals* α_t *and* β_t *are* $R(. \mid X_t)$*-integrable* P*-a.e., and*

$$\begin{cases} 0 < E_R(\alpha_t\beta_t \mid X_t) = E_R(\alpha_t \mid X_t)E_R(\beta_t \mid X_t), & P\text{-a.e.} \\ 0 \le E_R(\alpha_t\beta_t \mid X_t) = \mathbf{1}_{\{E_R(\alpha_t|X_t)E_R(\beta_t|X_t)<+\infty\}}E_R(\alpha_t \mid X_t)E_R(\beta_t \mid X_t), & R\text{-a.e.} \end{cases}$$

Proof. See [Léoa, § 3]. □

Now Lemma 1.11 leads to

$$P(A \cap B \mid X_t) = \frac{E_R(\alpha_t\beta_t\, \mathbf{1}_A\mathbf{1}_B \mid X_t)}{E_R(\alpha_t\beta_t \mid X_t)} = \frac{E_R(\alpha_t\, \mathbf{1}_A \mid X_t)}{E_R(\alpha_t \mid X_t)}\frac{E_R(\beta_t\, \mathbf{1}_B \mid X_t)}{E_R(\beta_t \mid X_t)}, \ P\text{-a.e.}.$$

Choosing $A = \Omega$ or $B = \Omega$ in this formula, we obtain

$$P(B \mid X_t) = E_R(\beta_t\, \mathbf{1}_B \mid X_t)/E_R(\beta_t \mid X_t) \text{ and } P(A \mid X_t) = E_R(\alpha_t\, \mathbf{1}_A \mid X_t)/E_R(\alpha_t \mid X_t).$$

This completes the proof of (1.10).

• Proof of (1) ⇒ (2). Take a Markov probability measure P with density Z with respect to $R : dP = Z\,dR$. We denote by

$$Z_t := E_R(Z \mid X_{[0,t]}),\ Z^*_t := E_R(Z \mid X_{[t,1]}) \text{ and } \zeta_t(z) := E_R(Z \mid X_t = z) = \frac{dP_t}{dR_t}(z).$$

Remark that the last equality implies that $\zeta_t(X_t) > 0$, P-a.e.,

$$\zeta_t(X_t) = E_R(Z_t \mid X_t) = E_R(Z^*_t \mid X_t), \quad R\text{-a.e.} \tag{1.12}$$

and that $\zeta_t(X_t)$ is R-integrable.
Fix three bounded non negative functions f, g, h respectively $\mathcal{A}_{[0,t]}$-, $\mathcal{A}_t$- and $\mathcal{A}_{[t,1]}$-measurable. One gets

$$\begin{aligned} E_P(fgh) &\overset{\text{(i)}}{=} E_P\left[E_P(f \mid X_t)\ g\ E_P(h \mid X_t)\right] \\ &\overset{\text{(ii)}}{=} E_P\left[\frac{E_R(fZ_t \mid X_t)}{E_R(Z_t \mid X_t)}\ g\ \frac{E_R(hZ^*_t \mid X_t)}{E_R(Z^*_t \mid X_t)}\right] \\ &\overset{\text{(iii)}}{=} E_P\left[g\frac{E_R(fhZ_tZ^*_t \mid X_t)}{\zeta_t(X_t)^2}\right] \\ &\overset{\text{(iv)}}{=} E_P[gE_{\widetilde{P}}(fh \mid X_t)] \end{aligned}$$

where we successively used in (i): Markovianity of P, in (iii) identity (1.12) and Markovianity of R and in (iv), we introduce the probability measure

$$\widetilde{P} := \mathbf{1}_{\{\zeta_t(X_t)>0\}} \frac{Z_tZ^*_t}{\zeta_t(X_t)}\, R. \tag{1.13}$$

From all these identities one deduces that

$$P(\cdot \mid X_t) = \widetilde{P}(\cdot \mid X_t), \quad P\text{-a.e..} \tag{1.14}$$

Define

$$\begin{cases} \alpha_t &= \mathbf{1}_{\{\zeta_t(X_t)>0\}}\, Z_t/\zeta_t(X_t) \\ \beta_t &= Z^*_t. \end{cases}$$

Therefore (1.13) becomes

$$\widetilde{P} = \alpha_t\beta_t\, R \tag{1.15}$$

and

$$E_R(\alpha_t \mid X_t) = \mathbf{1}_{\{\zeta_t(X_t)>0\}} \text{ and } E_R(\beta_t \mid X_t) = \zeta_t(X_t).$$

In order to identify P with $\widetilde{P}$, since (1.14) is satisfied, it is enough to show that their marginals at time t are the same, which we now prove.

$$\begin{aligned} \widetilde{P}_t(dz) &= E_R\left(\alpha_t\beta_t \mid X_t = z\right)\ R_t(dz) \\ &\overset{(i)}{=} E_R\left(\alpha_t \mid X_t = z\right) E_R\left(\beta_t \mid X_t = z\right)\ R_t(dz) \\ &= \zeta_t(z)\, R_t(dz) = P_t(dz) \end{aligned}$$

where the Markovianity of R is used in (i). This fact, together with (1.14), implies the equality $P = \widetilde{P}$. Eventually, since Z_t is $\mathcal{A}_{[0,t]}$-measurable and Z^*_t is $\mathcal{A}_{[t,1]}$-measurable, α_t and β_t are $\mathcal{A}_{[0,t]}$- resp. $\mathcal{A}_{[t,1]}$-measurable functionals. □

Example 1.16. In the extremal case in which $\alpha_t = f(X_0)$ is $\mathcal{A}_0$-measurable and $\beta_t = g(X_1)$ is $\mathcal{A}_1$-measurable, one obtains from the above Theorem that any probability P of the form

$$P = f(X_0)g(X_1)\, R \tag{1.17}$$

is Markov.

In Theorem 2.20 we will see that, under some restrictions for R, the probabilities P of this form are the unique ones which are Markov in the class of the probabilities of the form $P = h(X_0, X_1)\, R$.

1.3. **A fundamental example: bridges.** Since our aim is to carefully analyse the time symmetry of probabilities on path spaces, it is reasonable to desintegrate them along their initial and final values. One then describes a probability P on Ω as a mixture of pinned probabilities at time $t = 0$ and at time $t = 1$, that is as a mixture of its own bridges:

$$P = \int_{\mathcal{X}\times\mathcal{X}} P^{xy}\, P_{01}(dxdy). \tag{1.18}$$

Since $\mathcal{X}$ is polnish, the product space $\mathcal{X}^2$ is polnish too and this desintagration is meaningful.

It is known but nevertheless remarkable that, to pin a Markov probability measure P at initial and final times, does not perturb its Markovianity. We recall this important result below. Note that the bridges of P are only P-a.s. defined (in the article [FPY92] one finds a precise construction of bridges in the general framework of right processes.).

Proposition 1.19. *(Almost-) all bridges of a Markov probability measure on Ω are Markov.*

Proof. Let P be a Markov probability, t be a time in $[0,1]$, $A \in \mathcal{A}_{[0,t]}$ and $B \in \mathcal{A}_{[t,1]}$ be two events. We first show the following equality:

$$P(A \cap B \mid X_0, X_t, X_1) = P(A \mid X_0, X_t) P(B \mid X_0, X_t, X_1), P\text{-a.s. .} \tag{1.20}$$

Indeed,

$$\begin{aligned} P(A \cap B \mid X_0, X_t, X_1) &= E[P(A \cap B \mid X_0, X_{[t,1]}) \mid X_0, X_t, X_1] \\ &= E[\mathbf{1}_B P(A \mid X_0, X_{[t,1]}) \mid X_0, X_t, X_1] \\ &= E[\mathbf{1}_B P(A \mid X_0, X_t) \mid X_0, X_t, X_1] \\ &= P(A \mid X_0, X_t) P(B \mid X_0, X_t, X_1). \end{aligned}$$

Moreover, by Theoreme 1.3 (2*), $P(A \mid X_0, X_t) = P(A \mid X_0, X_t, X_1)$. Therefore

$$P^{X_0,X_1}(A \cap B \mid X_t) = P^{X_0,X_1}(A \mid X_t) P^{X_0,X_1}(B \mid X_t)\, P\text{-a.s. ,}$$

which characterizes the Markovianity of (P-almost) all bridges P^{X_0,X_1} via (1.2). □

In the sequel, we treat in particular the case where the Markov transition probability kernels are sufficiently regular, that is they admit densities. We describe this situation as follows.

Hypothesis (H). There exists a σ-finite positive measure m on $\mathcal{X}$ such that the transition probability kernels of the Markov probability measure P satisfy

$$\forall 0 \le s < t \le 1,\ p(s,x;t,\cdot) \ll m \text{ pour } P_s\text{-p.t. } x \text{ and } p^*(s,\cdot;t,y) \ll m \text{ pour } P_t\text{-p.t. } y.$$

By sake of simplicity, one also writes p, p^* for the density functions with respect to m: for all $0 \le s < t \le 1$,

$$\begin{aligned} p(s,x;t,y) &:= \frac{dp(s,x;t,\cdot)}{dm}(y) \text{ for } R_s \otimes m\text{-a.a. } (x,y) \\ \text{et} \quad p^*(s,x;t,y) &:= \frac{dp^*(s,\cdot;t,y)}{dm}(x) \text{ for } m \otimes R_t\text{-a.a. } (x,y). \end{aligned}$$

Remark that Hypothesis (H) is not always satisfied. If P is a Poisson process with random initial condition having a density on $\mathbb{R}$, at any time s, P_s admits a density too. But the support of the measure $p(s,x;t,dy)$ is discrete and equal to $x+\mathbb{N}$. Therefore there does not exist any measure m such that for a.a. x, $p(s,x;t,dy) \ll m(dy)$. We will see in Example 1.30 (ii) how get round this difficulty.

In the rest of this section, we assume that the reference Markov Probability measure R which we consider satisfies Hypothesis (H) with transition probability density denoted by r. Therefore

$$R_0(dx) = \int r^*(0,x;1,y) m(dx) R_1(dy) = \int r^*(0,x;1,y) R_1(dy)\, m(dx) =: r_0(x)\, m(dx)$$

and symetrically,

$$R_1(dy) = \int r(0,x;1,y) m(dy)\, R_0(dx) =: r_1(y)\, m(dy).$$

This leads to

$$R_{01}(dxdy) = r_0(x)\, m(dx) r(0,x;1,y) m(dy) = r_1(y) m(dy) r^*(0,x;1,y) m(dx),$$

in such a way that the function defined by

$$c(x,y) := r_0(x) r(0,x;1,y) = r_1(y) r^*(0,x;1,y),$$

is the density of the joint marginal $R_{01}(dxdy)$ with respect to $m \otimes m$.

We now recall the general structural relation between the probability R and its bridges. These latters are not globally absolutely continuous with respect to R (as the probabilities considered in the last section), but they are locally absolutely continuous with respect to R, on each time interval $[s,t]$ strictly included in $[0,1]$. The density is time symmetrical and we do exhibit it in a simple way.

Theorem 1.21. *Consider a Markov probability R on Ω satisfying* (H). *Denote $r_0 := dR_0/dm$, $r_1 := dR_1/dm$, and*

$$c(x,y) := r_0(x) r(0,x;1,y) = r^*(0,x;1,y) r_1(y) \quad m \otimes m\text{-a.e.}.$$

Then for all $0 < s \le t < 1$ and for R_{01}-a.a.(x, y), the bridge R^{xy} of R restricted to $\mathcal{A}_{[s,t]}$ is dominated by $R_{[s,t]}$ with density given by

$$(R^{xy})_{[s,t]} = \frac{r^*(0, x; s, X_s)\, r(t, X_t; 1, y)}{c(x, y)}\, R_{[s,t]}. \tag{1.22}$$

Proof. We first show the following property:

$$c(x, y) = 0 \Rightarrow r^*(0, x; s, z) r(t, z'; 1, y) = 0, \quad \forall (z, z'),\ R_{st}\text{-a.e.} \tag{1.23}$$

On one side,

$$R_{01}(dxdy) = c(x, y) m(dx) m(dy)$$

and on the other side, following Proposition 1.7,

$$\begin{aligned} R_{01}(dxdy) &= \int_{\mathcal{X}\times\mathcal{X}} R_{0,s,t,1}(dx, dz, dz', dy) \\ &= \int_{\mathcal{X}\times\mathcal{X}} r^*(0, dx; s, z) R_s(dz) r(s, z; t, dz') r(t, z'; 1, dy) \\ &= \int_{\mathcal{X}\times\mathcal{X}} r^*(0, x; s, z) r(s, z; t, z') r(t, z'; 1, y) R_s(dz) m(dz')\, m(dx) m(dy). \end{aligned}$$

Then

$$c(x, y) = \int_{\mathcal{X}\times\mathcal{X}} r^*(0, x; s, z) r(s, z; t, z') r(t, z'; 1, y) R_s(dz) m(dz')$$

and (1.23) holds. Moreover, for R_{st}-a.a. (z, z'), the probability measure $r^*(0, dx; s, z) r(t, z'; 1, dy)$ is dominated by $R_{01}(dxdy)$ and satisfies

$$r^*(0, dx; s, z) r(t, z'; 1, dy) = \frac{r^*(0, x; s, z) r(t, z'; 1, y)}{c(x, y)}\, R_{01}(dxdy). \tag{1.24}$$

Take two bounded measurable functions f, g and an event $B \in \mathcal{A}_{[s,t]}$. Thus,

$$\begin{aligned} & E_R[f(X_0)\ \mathbf{1}_B\ g(X_1)] \\ =\ & E_R\left[\mathbf{1}_B\ E_R(f(X_0) \mid X_{[s,t]})\ E_R(g(X_1) \mid X_{[s,t]})\right] \\ =\ & E_R\left[\mathbf{1}_B\ E_R(f(X_0) \mid X_s)\ E_R(g(X_1) \mid X_t)\right] \\ =\ & E_R\left[\mathbf{1}_B \int_{\mathcal{X}} f(x)\, r^*(0, dx; s, X_s) \int_{\mathcal{X}} g(y)\, r(t, X_t; 1, dy)\right] \\ =\ & E_R\left[\mathbf{1}_B \int_{\mathcal{X}\times\mathcal{X}} f(x) g(y) r^*(0, dx; s, X_s) r(t, X_t; 1, dy)\right] \\ \overset{\checkmark}{=}\ & E_R\left[\mathbf{1}_B \int_{\mathcal{X}\times\mathcal{X}} f(x) \frac{r^*(0, x; s, X_s) r(t, X_t; 1, y)}{c(x, y)} g(y)\, R_{01}(dxdy)\right] \\ =\ & \int_{\mathcal{X}\times\mathcal{X}} f(x) E_R\left[\mathbf{1}_B \frac{r^*(0, x; s, X_s)\, r(t, X_t; 1, y)}{c(x, y)}\right] g(y)\, R_{01}(dxdy), \end{aligned}$$

where we used (1.24) at the marked equality. This proves (1.22). □

Corollary 1.25 (Decomposition of a bridge)**.** *By introducing* $f_s(z) := r^*(0,x;s,z)$ *and* $g_t(z') =: c(x,y)^{-1} r(t,z';1,y)$, (1.22) *becomes*

$$(R^{xy})_{[s,t]} = f_s(X_s)\, g_t(X_t)\, R_{[s,t]}. \tag{1.26}$$

In particular, at each time $t \in]0,1[$*, the one dimensional marginal of the bridge* R^{xy} *is dominated by the marginal* R_t *of the Markov probability* R*. Its satisfies*

$$R_t^{xy} = f_t(X_t)\, g_t(X_t)\, R_t.$$

One interprets (1.26) as a modulation of (1.17) on the time interval $[s,t]$: the density of the bridge decomposes into a product of functions of the process at boundary times s and t. This assures its Markovianity.

Naturally, forward and backward dynamics of the bridge are directly related to dynamics of the original process.

Proposition 1.27. *Let* R *be a Markov probability measure on* Ω *which satisfies hypothesis* (H).

1. *For any time* $t < 1$ *and for* R_{01}*-a.a.* (x,y)*, the bridge* R^{xy} *of* R*, restricted to* $\mathcal{A}_{[0,t]}$ *is given by*

$$(R^{xy})_{[0,t]} = \frac{r(t,X_t;1,y)}{r(0,x;1,y)}\, R_{[0,t]}(\cdot \mid X_0 = x). \tag{1.28}$$

2. *Analogously, for any time* $s > 0$ *and for* R_{01}*-a.a.* (x,y)*, the bridge* R^{xy} *of* R *restricted to* $\mathcal{A}_{[s,1]}$ *is given by*

$$(R^{xy})_{[s,1]} = \frac{r^*(0,x;s,X_s)}{r^*(0,x;1,y)}\, R_{[s,1]}(\cdot \mid X_1 = y). \tag{1.29}$$

3. *The forward and backward transition probability kernels of* R^{xy} *satisfy for all* $0 \le s < t \le 1$ *and* R_{st}*-a.a* (z,z'),

$$\begin{aligned} r^{xy}(s,z;t,dz') &= \mathbf{1}_{\{r(s,z;1,y)>0\}} \frac{r(s,z;t,z') r(t,z';1,y)}{r(s,z;1,y)}\, m(dz') \\ r_*^{xy}(s,dz;t,z') &= \mathbf{1}_{\{r^*(0,x;t,z')>0\}} \frac{r^*(0,x;s,z) r^*(s,z;t,z')}{r^*(0,x;t,z')}\, m(dz) \end{aligned}$$

with the conventions $r(1,z;1,y) = \mathbf{1}_{\{z=y\}}$ *and* $r^*(0,x;0,z) = \mathbf{1}_{\{z=x\}}$.

Proof. • Proof of (1). Define $P^{\widetilde{xy}} := \dfrac{r(t,X_t;1,y)}{r(0,x;1,y)} R_{[0,t]}(\cdot \mid X_0 = x)$ and take a bounded non negative map f and an event $B \in \mathcal{A}_{[0,t]}$. Then,

$$\begin{aligned}
E_R\left(P^{\widetilde{xX_1}}(B)f(X_1) \mid X_0 = x\right) &= \int_{\mathcal{X}} r(0,x;1,y)P^{\widetilde{xy}}(B)f(y)\,m(dy) \\
&= \int_{\mathcal{X}} E_R[\mathbf{1}_B\, r(t,X_t;1,y)f(y) \mid X_0 = x]\,m(dy) \\
&= E_R[\mathbf{1}_B \int_{\mathcal{X}} r(t,X_t;1,dy)f(y)m(dy) \mid X_0 = x] \\
&= E_R[\mathbf{1}_B\, E_R(f(X_1) \mid X_t) \mid X_0 = x] \\
&= E_R[\mathbf{1}_B\, E_R(f(X_1) \mid X_{[0,t]}) \mid X_0 = x] \\
&= E_R[\mathbf{1}_B\, f(X_1) \mid X_0 = x] \\
&= E_R[R^{xX_1}(B)\, f(X_1) \mid X_0 = x]
\end{aligned}$$

which proves (1.28).
• Proof of (2). It is analogous with (1).
• Proof of (3). It is a direct corollary of (1) and (2). □

Examples 1.30. Several kind of bridges.

(i) The first example is classic. Let $\mathbf{W}$ be a *Wiener measure* on the set of real-valued continuous paths on $[0,1]$, with initial marginal law $\mathbf{W}_0$, admiting a density function r_0 with respect to Lebesgue measure $m(dx) \equiv dx$. Hypothesis (H) is then satisfied and the forward and backward transition probability densities are given, for $x, y \in \mathbb{R}$, by:

$$r(s,x;t,y) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}}, r^*(s,x;t,y) = \frac{\int r_0(z) r(0,z;s,x) r(s,x;t,y)\,dz}{\int r_0(z) r(0,z;t,y)\,dz}.$$

Therefore, due to (1.28), the Brownian bridge restricted to $\mathcal{A}_{[0,t]}$ satisfies

$$(\mathbf{W}^{x,y})_{[0,t]} = \frac{1}{\sqrt{1-t}} e^{-\left(\frac{(y-X_t)^2}{2(1-t)} - \frac{(y-x)^2}{2}\right)} \mathbf{W}_{[0,t]}(\cdot \mid X_0 = x).$$

Similarly

$$(\mathbf{W}^{x,y})_{[s,1]} = F^{xy}(s,X_s)\,\mathbf{W}_{[s,1]}(\cdot \mid X_1 = y)$$

where

$$\begin{aligned}
F^{xy}(s,z) &= \frac{r^*(0,x;s,z)}{r^*(0,x;1,y)} = \frac{\int r_0(x') r(0,x';1,y)\,dx'}{\int r_0(x') r(0,x';s,z)\,dx'} \frac{r(0,x;s,z)}{r(0,x;1,y)} \\
&= \frac{\int r_0(x') e^{-\frac{(y-x')^2}{2}}\,dx'}{\int r_0(x') e^{-\frac{(z-x')^2}{2s}}\,dx'} e^{-\left(\frac{(z-x)^2}{2s} - \frac{(y-x)^2}{2}\right)}.
\end{aligned}$$

Moreover, the density of the marginal at time $t > 0$ of the Brownian bridge (Brownian motion pinned in x and y) with respect to the marginal of the "free"

Brownian motion is given by

$$\sqrt{\frac{1}{1-t}}\,\frac{r_0(x)}{\int r_0(x')e^{-\frac{(X_t-x')^2}{2t}}dx'}\,e^{-\left(\frac{(X_t-x)^2}{2t}+\frac{(y-X_t)^2}{2(1-t)}\right)}.$$

(ii) Let $\mathbf{P}$ be the law of a *Poisson process* on the set of counting processes with general initial condition, that is the càdlàg step functions with positive unit jumps. One assumes that its marginal law at time 0 is a probability measure $\mathbf{P}_0(dx) = r_0(x)dx$ on $\mathbb{R}$. As already remarked, such a process does not satisfy hypothesis (H). However its dynamics is space- (and time-) homogeneous:

$$r(s,x;t,dy) = \delta_x * r(0,0;t-s,dy)$$

and the transition kernel $r(0,0;u,dy)$ admits a poissonian density r with respect to the counting measure m on $\mathbb{N}$:

$$r(0,0;u,dy) = r(u,y)\,m(dy) \quad \text{where} \quad r(u,n) = e^{-u}\frac{u^n}{n!}.$$

Therefore the proof of (1.28) can be generalised to this case, since it is enough to exhibit a density of the bridge of the Poisson process between 0 and n on the time interval $[0,t]$ with respect to the standart Poisson process starting in 0.
Now, the density on the time interval $[0,t]$ of the Poisson process pinned in x et y with respect to the Poisson process starting in x satisfies for $\mathbf{P}_0$-a.a. x and $y \in x + \mathbb{N}$,

$$\begin{aligned}(\mathbf{P}^{xy})_{[0,t]} &= \frac{r(1-t,y-X_t)}{r(1,y-x)}\,\mathbf{P}_{[0,t]}(\cdot \mid X_0 = x)\\ &= e^t(1-t)^{y-X_t}\frac{(y-x)!}{(y-X_t)!}\,\mathbf{P}_{[0,t]}(\cdot \mid X_0 = x).\end{aligned}$$

(iii) Let $\mathbf{C}$ be the law of a *Cauchy process* on Ω. We denote by $\mathbf{C}_0$ its marginal law at time 0. The forward transition density $r(s,x;t,y)$ is given, for each $x, y \in \mathbb{R}$, by the Cauchy law with parameter $t-s$:

$$r(s,x;t,y) = \frac{t-s}{\pi((t-s)^2 + (y-x)^2)}$$

and for $\mathbf{C}_0$-almost all x,

$$(\mathbf{C}^{xy})_{[0,t]} = (1-t)\,\frac{1+(y-x)^2}{(1-t)^2+(y-X_t)^2}\quad \mathbf{C}_{[0,t]}(\cdot \mid X_0 = x).$$

The computation of the density of the bridge on the time interval $[s,1]$ follows the same shema, using the backward transition density and the initial value $\mathbf{C}_0$. We also could consider the reversible situation, corresponding to $\mathbf{C}_0(dx) = dx$. This reversible measure can not be normalised but these techniques remain valuable also for σ-finite measures, see [Léoa].

2. Reciprocal probabilities and temporal symmetry

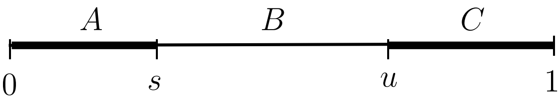
We now enlarge our framework to the class of probability measures called reciprocal, which are not necessarily Markov but show a sort of natural time symmetry. Bernstein introduced them in the particular framework of diffusion processes in his talk [Ber32] at the International Congress in Zürich in 1932, characterizing their associated transition as *stochastiquement parfaites*. Their symmetry property justifies their use in particular in the study of Quantum mechanical (random) systems, as in [Nel67], [Nag93] or in [CZ03].

2.1. Definition and essential properties.

Let us begin with the definition.

Definition 2.1 (Reciprocal probability)**.** *A probability measure P on Ω is called* reciprocal *(or the law of a reciprocal process) if for any times $s \le u$ in $[0,1]$ and for any events $A \in \mathcal{A}_{[0,s]}, B \in \mathcal{A}_{[s,u]}, C \in \mathcal{A}_{[u,1]}$*

$$P(A \cap B \cap C \mid X_s, X_u) = P(A \cap C \mid X_s, X_u) P(B \mid X_s, X_u), \quad P\text{-a.s.}. \tag{2.2}$$



The above property, formalised by [Jam74], states that under P the future of the time u and the past of the time s are conditionally independent, given the knowledge of the process at both times s and u. It is clearly time symmetrical.

In parallel to Theorem 1.3, we now present several characterisations of the reciprocal property.

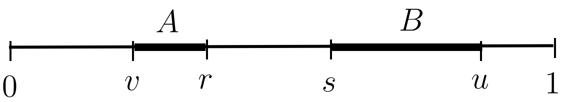
Theorem 2.3. *Let P be a probability measure on Ω. Then the following are equivalent:*

(1) *The probability P is reciprocal.*
(1*) *The time-reversed probability P^* is reciprocal.*
(2) *For all $0 \le s \le u \le 1$, and all sets $B \in \mathcal{A}_{[s,u]}$,*

$$P(B \mid X_{[0,s]}, X_{[u,1]}) = P(B \mid X_s, X_u). \tag{2.4}$$

(3) *For all $0 \le v \le r \le s \le u \le 1$, and all sets $A \in \mathcal{A}_{[v,r]}$, $B \in \mathcal{A}_{[s,u]}$,*

$$P(A \cap B \mid X_{[0,v]}, X_{[r,s]}, X_{[u,1]}) = P(A \mid X_v, X_r) P(B \mid X_s, X_u).$$



Proof. • Proof of (1) ⇔ (1*). Straightforward.
• Proof of (1) ⇒ (2).
Let take $B \in \mathcal{A}_{[s,u]}$. $P(B \mid X_{[0,s]}, X_{[u,1]})$ is the unique random variable $\mathcal{A}_{[0,s]} \vee \mathcal{A}_{[u,1]}$-measurable such that, for all $A \in \mathcal{A}_{[0,s]}$ and $C \in \mathcal{A}_{[u,1]}$,

$$P(A \cap B \cap C) = E[\mathbf{1}_A \mathbf{1}_C P(B \mid X_{[0,s]}, X_{[u,1]})].$$

But, due to (2.2), one has

$$\begin{aligned}
P(A\cap B\cap C) = &= E(P(A\cap B\cap C \mid X_s,X_u))\\
&= E[P(A\cap C\mid X_s,X_u)P(B\mid X_s,X_u)]\\
&= E[E(\mathbf{1}_A\mathbf{1}_C P(B\mid X_s,X_u)\mid X_s,X_u)]\\
&= E[\mathbf{1}_A\mathbf{1}_C P(B\mid X_s,X_u)].
\end{aligned}$$

This implies (2).
• Proof of (2) $\Rightarrow$ (1).
Let take $0\le s\le u\le 1$, $A\in\mathcal{A}_{[0,s]}, B\in\mathcal{A}_{[s,u]}$, $C\in\mathcal{A}_{[u,1]}$ and f,g some measurable non negative functions. By definition,

$$E[\mathbf{1}_A\mathbf{1}_B\mathbf{1}_C f(X_s)g(X_u)] = E[P(A\cap B\cap C\mid X_s,X_u)f(X_s)g(X_u)]$$

holds. On another side,

$$\begin{aligned}
E[\mathbf{1}_A\mathbf{1}_B\mathbf{1}_C f(X_s)g(X_u)] &= E[E(\mathbf{1}_A\mathbf{1}_B\mathbf{1}_C f(X_s)g(X_u)\mid X_{[0,s]},X_{[u,1]})]\\
&= E[\mathbf{1}_A\mathbf{1}_C P(B\mid X_s,X_u)f(X_s)g(X_u)]\\
&= E[P(A\cap C\mid X_s,X_u)P(B\mid X_s,X_u)f(X_s)g(X_u)].
\end{aligned}$$

Therefore

$$P(A\cap B\cap C\mid X_s,X_u) = P(A\cap C\mid X_s,X_u)P(B\mid X_s,X_u).$$

• Proof of (2) $\Rightarrow$ (3).
Take $A\in\mathcal{A}_{[v,r]}$ and $B\in\mathcal{A}_{[s,u]}$. Then

$$\begin{aligned}
&P(A\cap B\mid X_{[0,v]},X_{[r,s]},X_{[u,1]})\\
=\ & E\big[P(A\cap B\mid X_{[0,v]},X_{[r,1]})\mid X_{[0,v]},X_{[r,s]},X_{[u,1]}\big]\\
\overset{\checkmark}{=}\ & E\big[P(A\mid X_v,X_r)\,\mathbf{1}_B\ \mid X_{[0,v]},X_{[r,s]},X_{[u,1]}\big]\\
=\ & E\Big[E\big(P(A\mid X_v,X_r)\,\mathbf{1}_B\ \mid X_{[0,s]},X_{[u,1]}\big)\mid X_{[0,v]},X_{[r,s]},X_{[u,1]}\Big]\\
\overset{\checkmark}{=}\ & E\big[P(A\mid X_v,X_r)P(B\mid X_s,X_u)\mid X_{[0,v]},X_{[r,s]},X_{[u,1]}\big]\\
=\ & P(A\mid X_v,X_r)P(B\mid X_s,X_u)
\end{aligned}$$

where we used assumption (2) at the ✓-marked equalities.
• Proof of (3) $\Rightarrow$ (2).
It is enough to take $A=\Omega$ and $v=t=s$. □

Identity (2.4) states that a reciprocal probability is indeed a Markov field indexed by the time, as one-dimensional continuous parameter: if one conditions the probability evolving during the time interval $[s,u]$ by the knowledge of the past of s and of the future of u, this is equivalent to to condition it by the only knowledge at both boundary times s and u. This property is sometimes called *two-side Markov property*, which is inadequate, because one could get mixed up with (1.2) or 1.3-(3).

2.2. **Fundamental examples.** For a probability, to be Markov is stronger than to be reciprocal:

Proposition 2.5. *Any Markov probability measure on Ω is reciprocal, but the contrary is false.*

Proof. Take P a Markov probability measure, $0 \leq s \leq u \leq 1$ and $A \in \mathcal{A}_{[0,s]}, B \in \mathcal{A}_{[s,u]}$ et $C \in \mathcal{A}_{[u,1]}$. The following holds:

$$\begin{aligned}
P(A \cap B \cap C) &= E[P(A \cap B \cap C \mid X_{[s,u]})] \\
&\overset{(i)}{=} E[P(A \mid X_s)\mathbf{1}_B P(C \mid X_u)] \\
&= E[P(A \mid X_s)P(B \mid X_s, X_u)P(C \mid X_u)] \\
&\overset{(ii)}{=} E[P(A \mid X_s)P(B \mid X_s, X_u)P(C \mid X_{[0,u]})] \\
&= E[P(A \mid X_s)P(B \mid X_s, X_u)\mathbf{1}_C] \\
&\overset{(iii)}{=} E[P(A \mid X_{[s,1]})P(B \mid X_s, X_u)\mathbf{1}_C] \\
&= E[\mathbf{1}_A P(B \mid X_s, X_u)\mathbf{1}_C]
\end{aligned}$$

Equality *(i)* is due to Theorem 1.3 (3). To prove *(ii)* et *(iii)* we use the Markov property. Therefore (2.4) holds.
In Examples 2.7 *(ii)* we obtain a counter-example which shows that the set of Markov probability measures is strictly included in the set of reciprocal probabilities. □

The first proof of this assertion was done in [Jam70] in a Gaussian framework.

Let us mention the following class of reciprocal - but not Markov - probabilities. Take a Markov probability R on Ω and m a probability measure on $\mathcal{X}$. Suppose that m-a.e., the bridges R^{xx} are well defined. Then

$$R_{per} := \int_{\mathcal{X}} R^{xx}\, m(dx)$$

is a probability concentrated on periodical paths, with initial marginal law m. Due to Proposition 2.8, R_{per} is reciprocal, associated with the mixture probability $\pi(dxdy) = m(dx)\delta_x(dy)$. Nevertheless R_{per} is not Markov since condition (2^*) of Theorem 1.3 is denied: Let $A \in \mathcal{A}_0$ and t be a positive time. Following (2^*), $P(A|\mathcal{A}_{[t,1]})$ depends only on X_t. Otherwise, $A \in \mathcal{A}_0 = \mathcal{A}_1 \subset \mathcal{A}_{[t,1]}$. Thus $P(A|\mathcal{A}_{[t,1]}) = \mathbf{1}_A$, which only holds in the degenerate case of Dirac probabilities.

We will discuss in a short while the typical structure of reciprocal probabilities.

2.3. **To pin returns Markovianity.** In Proposition 1.19 we recalled the stability of the Markov property by pinning. Indeed, it is remarkable that pinning a reciprocal probability not only respects its reciprocality but also transforms it in a Markov one.

Theorem 2.6. *Let P be a reciprocal probability on Ω. Then, for P_{01}-almost all $(x, y) \in \mathcal{X}^2$, the bridge P^{xy} is a Markov probability.*

Proof. Take two events $A \in \mathcal{A}_{[0,t]}$ and $B \in \mathcal{A}_{[t,1]}$. Equation (1.20) holds under P: it is enough to do similar computations as in the proof of Theorem 1.19 and to apply property (2.4) to A with $s = 0$ and $t = u$. By symmetry one then obtains

$$\begin{aligned} P(A \cap B \mid X_0, X_t, X_1) &= P(A \mid X_0, X_t) P(B \mid X_0, X_t, X_1) \\ &= P(A \mid X_0, X_t, X_1) P(B \mid X_0, X_t, X_1), \end{aligned}$$

which characterises the Markovianity of any bridge P^{X_0,X_1} via (1.2). □

In the next subsection we study how to mix pinned probabilities without perturbing their nice properties.

2.4. **To mix in a right way preserves reciprocality.** To complete the previous subsection we analyse in which way mixing probabilities perturbs their reciprocality and/or Markovianity.

To mix Markov probabilities sometimes preserves the Markovianity - but not always. Similarly, by mixing reciprocal probabilities the result is sometimes reciprocal - but not always. The following examples illustrate these sentences. Moreover, we construct in (ii) an example of a reciprocal probability which is not Markov.

Examples 2.7 (Various mixtures of determinist paths.). Let $\mathcal{X} = \{\mathsf{a},\mathsf{b},\mathsf{c}\}$ be a finite state space with three elements. We denote by δ_w, $w \in \Omega$, the Dirac measure at the path w. Any δ_w is Markov since the path w is determinist.

(i) One denotes by $\mathsf{acb} \in \Omega$ the following path:

$$\mathsf{acb}(t) := \mathbf{1}_{[0,1/3)}(t)\ \mathsf{a} + \mathbf{1}_{[1/3,2/3)}(t)\ \mathsf{c} + \mathbf{1}_{[2/3,1]}(t)\ \mathsf{b}.$$

Similar notations are used for paths which jump only at time 0, 1/3 or 2/3.
The probability measure P on Ω defined by

$$P = \frac{1}{4}(\delta_{\mathsf{abc}} + \delta_{\mathsf{aba}} + \delta_{\mathsf{cba}} + \delta_{\mathsf{cbc}})$$

is a uniform mixture of determinist Markov paths, and therefore is Markov too. Indeed $P_0 = \frac{1}{2}(\delta_{\mathsf{a}} + \delta_{\mathsf{c}})$ and the - non trivial - transition probabilities are given by

$$P(X_{1/3} = \mathsf{b} \mid X_0 = \mathsf{a}) = P(X_{1/3} = \mathsf{b} \mid X_0 = \mathsf{c}) = 1$$

and

$$P(X_{2/3} = \mathsf{a} \mid X_{1/3} = \mathsf{b}) = P(X_{1/3} = \mathsf{c} \mid X_{1/3} = \mathsf{b}) = 1/2.$$

Moreover, since $P^{\mathsf{a},\mathsf{c}} = \delta_{\mathsf{abc}}, P^{\mathsf{a},\mathsf{a}} = \delta_{\mathsf{aba}}, P^{\mathsf{c},\mathsf{a}} = \delta_{\mathsf{cba}}, P^{\mathsf{c},\mathsf{c}} = \delta_{\mathsf{cbc}}$, we observe that P is the uniform mixture of its four bridges.

(ii) The probability on Ω

$$P = \frac{1}{2}(\delta_{\mathsf{abc}} + \delta_{\mathsf{cba}}),$$

is reciprocal but not Markov. Indeed, it is clearly reciprocal since each boundary condition determines the path, as in (i) : $P^{\mathsf{a},\mathsf{c}} = \delta_{\mathsf{abc}}, P^{\mathsf{c},\mathsf{a}} = \delta_{\mathsf{cba}}$. Nevertheless we observe that P is not Markov since

$$P(X_1 = \mathsf{a} \mid X_0 = \mathsf{a}, X_{1/2} = \mathsf{b}) = 0$$

while

$$P(X_1 = \mathsf{a} \mid X_{1/2} = \mathsf{b}) = 1/2.$$

(iii) Let us now define paths with four states and three jumps at fixed times $1/4, 1/2$ et $3/4$, like

$$\mathsf{abab}(t) := \mathbf{1}_{[0,1/4)}(t)\ \mathsf{a} + \mathbf{1}_{[1/4,1/2)}(t)\ \mathsf{b} + \mathbf{1}_{[1/2,3/4)}(t)\ \mathsf{a} + \mathbf{1}_{[3/4,1]}(t)\ \mathsf{b}.$$

The probability measure $P := \frac{1}{2}(\delta_{\mathsf{abab}} + \delta_{\mathsf{cbcb}})$ on Ω, mixture of reciprocal determinist paths (which are its own bridges) is no more reciprocal. Indeed

$$P(X_{2/3} = \mathsf{a} \mid X_{[0,1/3]}, X_{[4/5,1]}) = \mathbf{1}_{\{X_0=\mathsf{a}\}}$$

while

$$P(X_{2/3} = \mathsf{a} \mid X_{1/3}, X_{4/5}) = P(X_{2/3} = \mathsf{a}) = 1/2.$$

Let R be a reciprocal probability measure on Ω. We would like to test the reciprocal character of any mixture of its bridges. If one wants to work in a general framework, the first difficulty comes from the fact that the bridges R^{xy} are only R_{01}-a.s. well defined. Therefore there are two possibilities: either one works with probabilities R whose bridges are defined for all x, y and smooth enough as function of the boundary conditions (x, y). In that case, any mixture is allowed. Or one considers only special mixture measures, whose support is included in the support of R_{01}.
Let us first assume that there exists a regular version of the bridges of R.

Proposition 2.8. *Let R be a reciprocal probability on Ω, such that the map $(x, y) \in \mathcal{X}\times\mathcal{X} \mapsto R^{xy}$ is well defined and continuous. Then, for any probability measure π on $\mathcal{X}\times\mathcal{X}$, the probability measure*

$$P(\cdot) = \int_{\mathcal{X}\times\mathcal{X}} R^{xy}(\cdot)\,\pi(dxdy)$$

is reciprocal. Moreover, the bridges of P coincide with those of R P-a.s..

Proof. Let us show (2.4) under P. Let $0 \le s \le t \le 1$, $A \in \mathcal{A}_{[0,s]}$, $B \in \mathcal{A}_{[s,u]}$ and $C \in \mathcal{A}_{[u,1]}$. Then

$$\begin{aligned} E_P[\mathbf{1}_A P(B \mid X_{[0,s]}, X_{[u,1]})\mathbf{1}_C] &= P(A\cap B\cap C) \\ &= \int_{\mathcal{X}\times\mathcal{X}} R^{xy}(A\cap B\cap C)\,\pi(dxdy) \\ &\overset{\checkmark}{=} \int_{\mathcal{X}\times\mathcal{X}} E_{R^{xy}}[\mathbf{1}_A R(B \mid X_s, X_t)\mathbf{1}_C]\,\pi(dxdy) \\ &= E_P[\mathbf{1}_A R(B \mid X_s, X_t)\mathbf{1}_C] \end{aligned}$$

where reciprocality was used at the marked equality. Thus $P(B \mid X_{[0,s]}, X_{[t,1]})$ only depends on (X_s, X_t) and

$$P(B \mid X_{[0,s]}, X_{[t,1]}) = R(B \mid X_s, X_t),\ P\text{-a.e..}$$

which completes the proof. □

Remark that this results does not contradict Example 2.7-(iii): There, P was expressed as a mixture of its own bridges, but not as a mixture of bridges of a reciprocal probability. There does not exist indeed any reciprocal probability R such that $\delta_{abab} = R^{ab}$ and

$\delta_{cbcb} = R^{cb}$.

The previous Proposition allows to construct classes of reciprocal probabilities based on some reference one, letting vary the way to mix the bridges. Therefore, we now recall the definition of an important concept.

Definition 2.9 (Reciprocal class associated with R)**.** *Suppose that R is a reciprocal probability on Ω such that $(x,y) \in \mathcal{X}\times\mathcal{X} \mapsto R^{xy}$ is well defined und continuous. The set of probabilities on Ω defined by*

$$\mathfrak{R}_c(R) := \{P : P(\cdot) = \int_{\mathcal{X}\times\mathcal{X}} R^{xy}(\cdot)\,\pi(dxdy),\ \textit{with } \pi \textit{ any probability measure on } \mathcal{X}\times\mathcal{X}\} \tag{2.10}$$

is called the reciprocal class associated with R.

The index c in $\mathfrak{R}_c(R)$ recalls the first letter of the word c̲lass. Later we will introduce another set of probabilities called reciprocal f̲amily, which will be denoted by $\mathfrak{R}_f(R)$.

In the case of a *discrete* state space $\mathcal{X}$, the continuity hypothesis of $(x,y) \mapsto R^{xy}$ is useless. One only should make sure that the support of the mixture measure π is included in the support of R_{01}, in such a way that (2.10) makes sense.

Remarks 2.11 (about this definition). The concept of reciprocal class is due to Jamison, [Jam74] Section 3, for a Markov reference probability whose transition kernels satisfy hypothesis (H).
In the particular case where R is a Brownian diffusion defined on the space of continuous paths, the class $\mathfrak{R}_c(R)$ can be characterized by two functionals of the drift of R, called reciprocal invariants. This was conjectured by Krener in [Kre88] and proved by Clark in [Cla91, Thm 1]. Thereafter, Thieullen and the second author gave rise to an integration by parts on the path space, in which appear the reciprocal invariants of the Brownian diffusion R and which characterises completely the associated reciprocal class. See [RT04] for one-dimensional diffusions and [RT05] for the more-dimensional case.
When R is a counting process ($\mathcal{X}$ is one-to-one with $\mathbb{N}$), one finds in [Mur12] a description of reciprocal invariant associated with $\mathfrak{R}_c(R)$, as well as a characterisation of the reciprocal class through a duality formula. An extension of this work for more general jump processes is in preparation.

2.5. **Time reversal and reciprocality.** We already have seen in Theorem 2.3 that a probability is reciprocal if and only if its time-reversed probability measure on Ω is reciprocal too. We can now precise this assertion.

Proposition 2.12. *Let R be a reciprocal probability on Ω as in Definition 2.9. Then*

$$P \in \mathfrak{R}_c(R) \iff P^* \in \mathfrak{R}_c(R^*).$$

Proof of Proposition 2.12. We first prove following auxiliary lemma.

Lemma 2.13.

(a) *Consider the diagram* $\Omega \xrightarrow{\Phi} \Phi(\Omega) \xrightarrow{\theta} \mathcal{Y}$ *where mentioned sets and maps are measurable. Then, for any bounded measurable function* $f : \Phi(\Omega) \to \mathbb{R}$*,*

$$E_{\Phi_\# P}(f|\theta) = \alpha(\theta) \text{ with } \alpha(y) := E_P(f(\Phi)|\theta(\Phi) = y).$$

(b) *Consider the diagram* $\mathcal{Y} \xleftarrow{\theta} \Omega \xrightarrow{\Phi} \Omega$ *where mentioned sets and maps are measurable. Suppose that* Φ *is one-to-one with measurable inverse* Φ^{-1}*. Then,*

$$\Phi_\# \Big[P(\cdot \mid \theta = y)\Big] = \big[\Phi_\# P\big](\cdot \mid \theta \circ \Phi^{-1} = y), \quad y \in \mathcal{Y}.$$

• Proof of (a). For any bounded measurable function $u : \mathcal{Y} \to \mathbb{R}$,

$$\begin{aligned} E_{\Phi_\# P}\Big[E_{\Phi_\# P}(f|\theta)u(\theta)\Big] &= E_{\Phi_\# P}(fu(\theta)) = E_P\Big[f(\Phi)u(\theta(\Phi))\Big] \\ &= E_P\Big[E_P(f(\Phi)|\theta(\Phi))\, u(\theta(\Phi))\Big] = E_{\Phi_\# P}(\alpha(\theta)\, u(\theta)) \end{aligned}$$

• Proof of (b). We add a bounded measurable function u to the diagram: $\mathcal{Y} \xleftarrow{\theta} \Omega \xrightarrow{\Phi} \Omega \xrightarrow{u} \mathbb{R}$ and compute, for $y \in \mathcal{Y}$,

$$E_{\Phi_\# P(\cdot|\theta=y)}(u) = E_P\big[u(\Phi)|\theta = y\big] = E_P\big[u(\Phi)|\theta \circ \Phi^{-1} \circ \Phi = y\big] \overset{(i)}{=} E_{\Phi_\# P}(u|\theta \circ \Phi^{-1} = y)$$

where equality (i) is a consequence of the above result (a).

In particular (b) implies that

$$(R^{xy})^* = (R^*)^{yx}, \quad \text{pour } R\text{-a.e. } x, y \in \mathcal{X}. \tag{2.14}$$

Let $P \in \mathfrak{R}_c(R)$, then $P(\cdot) = \int_{\mathcal{X}\times\mathcal{X}} R^{xy}(\cdot)\, P_{01}(dxdy)$. We now compute the integral of a function u under P^* :

$$\begin{aligned} E_{P^*}[u(X)] &= E_P[u(X^*)] = \int_{\mathcal{X}\times\mathcal{X}} E_{(R^{xy})^*}(u)\, P_{01}(dxdy) \\ &\overset{(2.14)}{=} \int_{\mathcal{X}\times\mathcal{X}} E_{(R^*)^{yx}}(u)\, P_{01}(dxdy) = \int_{\mathcal{X}\times\mathcal{X}} E_{(R^*)^{xy}}(u)\, (P^*)_{01}(dxdy). \end{aligned}$$

This means that $P^*(\cdot) = \int_{\mathcal{X}\times\mathcal{X}} (R^*)^{xy}(\cdot)\, (P^*)_{01}(dxdy)$, and the proof of Proposition 2.12 is completed. □

2.6. **Reciprocal families.** To precise our structural analysis of reciprocal probabilities staying in a general framework, we prefer to introduce a slightly more restrictive concept than the one of reciprocal class, which we call *reciprocal family* associated with R. This set only contains probabilities which are dominated by the reference probability R.

Definition 2.15 (Reciprocal family associated with R)**.** *Suppose that R is a reciprocal probability on Ω. The set of probabilities on Ω defined by*

$$\mathfrak{R}_f(R) := \{P \in \mathfrak{R}_c(R) \text{ with mixture measures } \pi \text{ on } \mathcal{X}\times\mathcal{X} \text{ satisfying } \pi \ll R_{01}\} \tag{2.16}$$

is called the reciprocal family associated with R.

Remarks 2.17 (about this definition). (a) Due to the domination from π by R_{01}, it is no more necessary to suppose any regularity of R^{xy} as function of x, y.

(b) Due to Proposition 2.8, we notice that a reciprocal family, as subset of a reciprocal class, contains probabilities which are reciprocal.

(c) We denote by $P \prec R$ if P desintegrates as in (2.16). Note that the relation $\prec$ is transitive but not a priori symmetrical, if the marginal laws at time 0 and 1 are not equivalent. Therefore a reciprocal family is not an equivalence class. If one wants to define a "true" equivalence relation $\sim$ between probabilities on Ω one should assume that marginal laws at time 0 and 1 are equivalent. Then $P \sim R$ if and only if $P \prec R$ and $R \prec P$.

Now, the structure of probabilities in a reciprocal family is remarkably simple.

Theorem 2.18. *Each probability measure P in the reciprocal family $\mathfrak{R}_f(R)$, defined by* (2.16)*, is dominated by R and satisfies*

$$P = \frac{d\pi}{dR_{01}}(X_0, X_1)\, R.$$

Reciprocally, if P is defined by

$$P = h(X_0, X_1)\, R \tag{2.19}$$

where h is a non negative measurable map, then $P \in \mathfrak{R}_f(R)$ and more precisely, it is a π-mixture of bridges of R where

$$\pi(dxdy) := h(x, y)\, R_{01}(dxdy).$$

Proof. Let $P \in \mathfrak{R}_f(R)$ and f any non negative bounded map. Due to Definition (2.16), since $\pi \ll R_{01}$,

$$\begin{aligned}
E_P(f) &= \int_{\mathcal{X}\times\mathcal{X}} E_R(f \mid X_0 = x, X_1 = y)\, \pi(dxdy) \\
&= \int_{\mathcal{X}\times\mathcal{X}} E_R(f \mid X_0 = x, X_1 = y) \frac{d\pi}{dR_{01}}(x, y)\, R_{01}(dxdy) \\
&= E_R\left(E_R(f \mid X_0, X_1) \frac{d\pi}{dR_{01}}(X_0, X_1) \right) \\
&= E_R\left(\frac{d\pi}{dR_{01}}(X_0, X_1) f \right),
\end{aligned}$$

which proves the first assertion.
For the second assertion, note that

$$P(\cdot) = \int_{\mathcal{X}\times\mathcal{X}} P^{xy}(\cdot)\, \pi(dxdy) = \int_{\mathcal{X}\times\mathcal{X}} h(x, y) R^{xy}(\cdot)\, R_{01}(dxdy).$$

□

The specific structure of P which appears in (2.19) reminds the h-transform of Doob ([Doo57]), symmetrised on the time interval $[0, 1]$.

2.7. **Markov probabilities of a reciprocal family.** Since Markovianity is more restrictive than reciprocality, we naturally would like to describe the subset of a reciprocal family containing probabilities which are Markov. With others words, we are looking at the specific mixture probabilities which preserve Markovianity.

In the rest of the subsection, R is a reference Markov probability on Ω. If a probability of $\mathfrak{R}_f(R)$ admits a density which is decomposable as a product map as in Example 1.16, we already know that it is Markov. This property is (almost) characteristic, as we now prove.

Theorem 2.20. *Let $P \in \mathfrak{R}_f(R)$ where R a reference Markov probability. One considers following assertions:*

(1) The probability P is Markov.
(2) There exists two measurable non negative maps f and g such that

$$\frac{dP}{dR} = f(X_0)g(X_1), \quad R\text{-a.e.}. \tag{2.21}$$

Then assertion (2) implies assertion (1).
If we suppose, moreover, that there exists $0 < t_o < 1$ and a measurable subset $\mathcal{X}_o \subset \mathcal{X}$ such that $R_{t_o}(\mathcal{X}_o) > 0$ and, for all $z \in \mathcal{X}_o$,

$$R_{01} \ll R_{01}^{t_0 z} := R((X_0, X_1) \in \cdot | X_{t_o} = z), \tag{2.22}$$

then assertions (1) and (2) are equivalent.

Proof. • Proof of (2) ⇒ (1). It is contained in Example 1.16; indeed Hypothesis (2.22) is not necessary.
• Proof of (1) ⇒ (2). Since P is Markov, Theoreme 1.8 applied with $t = t_o$ leads to

$$\frac{dP}{dR} = \alpha(X_{[0,t_o]})\beta(X_{[t_o,1]}) \quad R\text{-a.e.} \tag{2.23}$$

with α and β two measurable non negative functionals. On an other side, since P belongs to the reciprocal family of R, following Proposition 2.18, its Radon-Nikodym derivative looks like

$$\frac{dP}{dR} = h(X_0, X_1)$$

with h a measurable non negative map on $\mathcal{X}^2$. This implies

$$\alpha(X_{[0,t_o]})\beta(X_{[t_o,1]}) = h(X_0, X_1) \quad R\text{-a.e.}$$

which can hold only if the functionals α and β have the form

$$\alpha(X_{[0,t_o]}) = a(X_0, X_{t_o}) \text{ and } \beta(X_{[t_o,1]}) = b(X_{t_o}, X_1),\ R\text{-a.e.}$$

with a and b two measurable non negative functions on $\mathcal{X}^2$. They satisfy

$$a(x,z)b(z,y) = h(x,y) \quad \forall (x,z,y) \in \mathcal{N}^c \subset \mathcal{X}^3,$$

where the set $\mathcal{N} \subset \mathcal{X}^3$ is $R_{0,t_o,1}$-negligible. Now, with the notation

$$\mathcal{N}_z := \{(x,y); (x,z,y) \in \mathcal{N}\} \subset \mathcal{X}^2,$$

we get

$$0 = R_{0,t_o,1}(\mathcal{N}) = \int_{\mathcal{X}} R_{01}^{t_0 1}(\mathcal{N}_z)\, R_{t_o}(dz)$$

which implies that $R_{01}^{t_0 1}(\mathcal{N}_z) = 0$ for R_{t_o}-a.a. $z \in \mathcal{X}_0$. Due to assumption (2.22), one deduces that there exists $z_o \in \mathcal{X}_o$ such that $R_{01}(\mathcal{N}_{z_o}) = 0$. Taking $f = a(\cdot, z_o)$ et $g = b(z_o, \cdot)$, we obtain

$$h(x,y) = f(x)g(y), \quad R_{01}(dxdy)\text{-a.e.},$$

which proves that dP/dR has the form expressed in (2.21). □

Remarks 2.24. (a) This result belongs to "folk" facts in the framework of reciprocal processes, and is often used without correct detailed proof. In particular hypothesis (2.22) does rarely appear. A partial version of Theorem 2.20 could be found in [Jam74, Thm. 3.1]. Jamison proved, under the assumption that the Markov probability R admits smooth transition densities, that $P \in \mathfrak{R}_c(R)$ if and only if there exists two probabilities on $\mathcal{X}$, ν_0 and ν_1, such that

$$P_{01}(dxdy) = r(0,x;1,y)\nu_0(dx)\nu_1(dy).$$

Unlike Jamison, we underline here the importance of the multiplicative structure of the density between P and R.

(b) Since R is Markov, assumption (2.22) is equivalent to

$$\forall z \in \mathcal{X}_o, \quad R_{01} \ll R_0^{t_0 z} \otimes R_1^{t_0 z}.$$

(c) Without any additional condition on R, both assertions of the above Theorem are not equivalent. We furnish a counter-example, constructing a probability R which does not satisfy hypothesis (2.22), and a Markov probability P whose density with respect to R does not have the mentioned structure.
Let R be the Markov probability on Ω with state space $\mathcal{X} = \{\mathsf{a}, \mathsf{b}\}$, with initial law $R_0 = (\delta_{\mathsf{a}} + \delta_{\mathsf{b}})/2$ and infinitesimal generator $\begin{pmatrix} 0 & 0 \\ \lambda & -\lambda \end{pmatrix}$ for some $\lambda > 0$. The support of R is concentrated on two types of paths: those identically equal to a or à b, or those starting in b with one jump to a after an exponential waiting time with law $\mathcal{E}(\lambda)$ and realisation in $(0,1)$. One verifies that R does not satisfy (2.22). Indeed, for all $t \in]0,1[$,

$$R_{01}^{t\mathsf{a}}(\{(\mathsf{b},\mathsf{b})\}) = 0 \text{ but } R_{01}(\{(\mathsf{b},\mathsf{b})\}) = \frac{e^{-\lambda}}{2} > 0 \quad \text{thus } R_{01} \not\ll R_{01}^{t\mathsf{a}}$$

$$\text{and } R_{01}^{t\mathsf{b}}(\{(\mathsf{a},\mathsf{a})\}) = 0 \text{ but } R_{01}(\{(\mathsf{a},\mathsf{a})\}) = \frac{1}{2} > 0 \quad \text{thus } R_{01} \not\ll R_{01}^{t\mathsf{b}}.$$

One now consider the Markov probability P, charging uniformly two determinist constant paths equal to a or to b. It is dominated by R with density:

$$\frac{dP}{dR}(X) = \begin{cases} 1, & \text{if } X \equiv \mathsf{a} \\ e^{\lambda}, & \text{si } X \equiv \mathsf{b} \\ 0, & \text{si} X_0 \neq X_1 \end{cases}.$$

This density dP/dR has not the product form (2.21), since the system $\begin{cases} f(\mathsf{a})g(\mathsf{a}) &= 1 \\ f(\mathsf{b})g(\mathsf{b}) &= e^{\lambda} \\ f(\mathsf{b})g(\mathsf{a}) &= 0 \end{cases}$ does not have any solution.
Remark that the functionals α and β defined in (2.23) could be chosen as follows:

$\alpha(X) = \beta(X) = 1$ if $X \equiv \mathsf{a}$, $\alpha(X) = 1$ if $X \equiv \mathsf{b}$, $\beta(X) = e^{\lambda}$ if $X \equiv \mathsf{b}$ and $\alpha(X) = \beta(X) = 0$ otherwise.

3. Reciprocal probabilities as solution of the Schrödinger problem

We conclude this note by going back to the historical problem of Schrödinger introduced in [Sch31] and developped in [Sch32]. This problem comes from statistical physics and is the starting point of the theory of time-reversed Markov diffusions and reciprocal diffusions. We now present results which are detailed in the review paper [Léob] (see the references therein too).

The mail tool is the concept of relative entropy, defined as usual. The relative entropy of a probability p with respect to a probability r on a measurable space $\mathcal{Y}$ is given by

$$H(p|r) := \int_{\mathcal{Y}} \log\left(\frac{dp}{dr}\right) dp \in [0, +\infty]$$

when p is dominated by r, and $+\infty$ otherwise.

The *dynamical formulation* $\mathcal{S}_{\text{dyn}}$ *of the Schrödinger problem* means the following. As reference probability measure R one takes the Wiener measure on the space Ω_c of continuous paths with values in $\mathcal{X} = \mathbb{R}$ and μ_0, μ_1 are given probability measures on $\mathbb{R}$ (called the constraints). The aim is now to minimize the map $P \mapsto H(P|R)$ where P varies in the set of probability measures on Ω_c such that $P_0 = \mu_0, P_1 = \mu_1$.

By projection of this variational problem on the set $\mathbb{R}^2$ of initial and final positions, one obtains the following associated *static formulation* $\mathcal{S}$:to minimize the map $\pi \mapsto H(\pi|R_{01})$ where π varies in the set of probability measures on $\mathbb{R}^2$ such that $\pi_0 = \mu_0, \pi_1 = \mu_1$ ($\pi_0(dx) := \pi(dx \times \mathbb{R})$ and $\pi(dy) := \pi(\mathbb{R} \times dy)$ are the marginal laws of π).

Recall the uniqueness result obtained by Föllmer [Föl88] in the slightly more general case where R is a Brownian diffusion with drift.

Proposition 3.1 (Föllmer)**.** *Both Schrödinger problems - dynamic and static - admit at most one solution* $\widehat{P}$ *resp.* $\widehat{\pi}$. *If* $\mathcal{S}_{\text{dyn}}$ *has* $\widehat{P}$ *as solution, then* $\widehat{\pi} = \widehat{P}_{01}$ *is the solution of* $\mathcal{S}$. *Reciprocally, if* $\widehat{\pi}$ *solves* $\mathcal{S}$, *then* $\mathcal{S}_{\text{dyn}}$ *admits as solution*

$$\widehat{P}(\cdot) = \int_{\mathbb{R}\times\mathbb{R}} R^{xy}(\cdot)\, \widehat{\pi}(dxdy) \in \mathfrak{R}_f(R). \tag{3.2}$$

Sketch of the proof. As *strictly* convex minimisation problem, $\mathcal{S}_{\text{dyn}}$ and $\mathcal{S}$ admits at most one solution.

Using the desintegration formula

$$H(P|R) = H(P_{01}|R_{01}) + \int_{\mathbb{R}\times\mathbb{R}} H(P^{xy}|R^{xy})\, P_{01}(dxdy),$$

one obtains $H(P_{01}|R_{01}) \leq H(P|R)$ with equality (for $H(P|R) < +\infty$) if and only if $P^{xy} = R^{xy}$ for P_{01}-almost all $(x, y) \in \mathbb{R}^2$, which corresponds to $P \in \mathfrak{R}_f(R)$. Thus, $\widehat{P}$ is the solution of $\mathcal{S}_{\text{dyn}}$ if and only if its desintegrates as (3.2). □

We finally present an existence and uniqueness result for $\mathcal{S}_{\text{dyn}}$ and $\mathcal{S}$, obtained by the first author in a general framework where R is any Markov probability on the space Ω of càdlàg paths with values in a general set $\mathcal{X}$. The reader can find the proof in [Léob].

Theorem 3.3. *Let R be a reference Markov probability on Ω_c, with identical marginal laws at time 0 and 1, denoted by m. Suppose that R satisfies the following assumptions:*

(i) there exists $0 < t_o < 1$ and a measurable set $\mathcal{X}_o \subset \mathcal{X}$ such that $R_{t_o}(\mathcal{X}_o) > 0$ and

$$R_{01} \ll R\big((X_0, X_1) \in \cdot | X_{t_o} = z\big), \quad \forall z \in \mathcal{X}_o.$$

(ii) there exists a non negative measurable map A on $\mathcal{X}$ such that

$$R_{01}(dxdy) \geq e^{-A(x)-A(y)}\, m(dx)m(dy).$$

Suppose also that the constraints μ_0 and μ_1 satisfy

$$H(\mu_0|m) + H(\mu_1|m) < +\infty \text{ and } \int_{\mathcal{X}} A\, d\mu_0 + \int_{\mathcal{X}} A\, d\mu_1 < +\infty.$$

Then $\mathcal{S}$ admits a unique solution $\widehat{\pi}$ of the form

$$\widehat{\pi}(dxdy) = f_0(x) g_1(y)\, R_{01}(dxdy)$$

where the maps $f_0, g_1 : \mathcal{X} \to [0, \infty)$ are m-measurable, non negative and solutions of the so-called Schrödinger system*:*

$$\begin{cases} f_0(x)\, E_R[g_1(X_1) \mid X_0 = x] &= d\mu_0/dm(x), \quad \text{for } m\text{-a.a. } x \\ g_1(y)\, E_R[f_0(X_0) \mid X_1 = y] &= d\mu_1/dm(y), \quad \text{for } m\text{-a.a. } y \end{cases}$$

Moreover $\mathcal{S}_{\text{dyn}}$ admits as unique solution

$$\widehat{P} = f_0(X_0) g_1(X_1)\, R \tag{3.4}$$

which is Markov too.

Remark 3.5. In the Schrödinger system, $E_R[f_0(X_0) \mid X_1]$ and $E_R[g_1(X_1) \mid X_0]$ are well defined even if $f_0(X_0)$ and $g_1(X_1)$ are not R-integrable. In fact, f_0 and g_1 are measurable and non negative; therefore, only positive integration is needed, see [Léoa].

Generalising Föllmer's result, we obtain without additional effort.

Corollary 3.6. *The solution $\widehat{P}$ of the variable problem $\mathcal{S}_{\text{dyn}}$, if it exists, belongs to the reciprocal family $\mathfrak{R}_f(R)$.*

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