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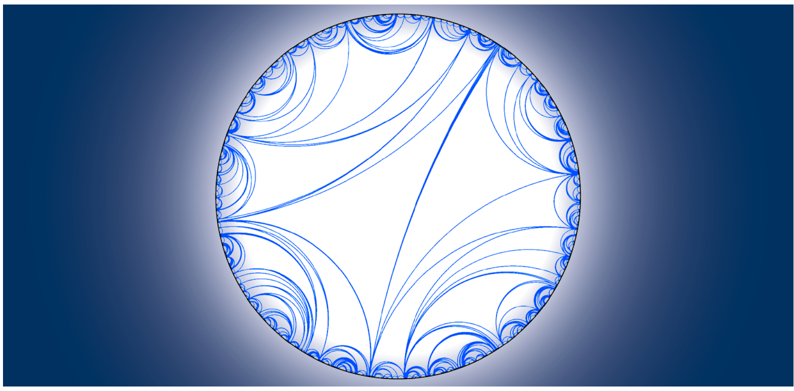

Olimdjan Makhmudov | Nikolai Tarkhanov

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An Extremal Problem Related to Analytic Continuation

O. Makhmudov and N. Tarkhanov

Abstract. We show that the usual variational formulation of the problem of analytic continuation from an arc on the boundary of a plane domain does not lead to a relaxation of this overdetermined problem. To attain such a relaxation, we bound the domain of the functional, thus changing the Euler equations.

Contents

Introduction

Let $\mathcal{X}$ be a bounded domain with smooth boundary in the complex plane $\mathbb{C}$ and $\mathcal{S}$ a nonempty open arc on the boundary of $\mathcal{X}$. The problem of analytic continuation of functions given on $\mathcal{S}$ to $\mathcal{X}$ is of great importance in analysis, see [**Aiz93**]. It reads as follows: Given any function u_0 on $\mathcal{S}$, find an analytic function u in $\mathcal{X}$ whose limit values exist in a reasonable sense and coincide with u_0 at $\mathcal{S}$. This problem is not normally solvable unless $\mathcal{S} = \partial\mathcal{X}$, for no nonzero smooth function u_0 of compact support in $\mathcal{S}$ extends analytically to $\mathcal{X}$.

Analytic functions in $\mathcal{X}$ are solutions of the Cauchy-Riemann system $\bar{\partial}u = 0$ in the domain. To get an approximate solution of the problem of analytic continuation, one can relax the system $\bar{\partial}u = 0$ and require that $\bar{\partial}u$ be "small" in some sense in $\mathcal{X}$. In other words, the problem is replaced by a variational problem which has the advantage of being constructively solvable. When looking for a solution u of Sobolev

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space $W^{1,p}(\mathcal{X})$ with $p > 1$, one considers the variational problem of minimizing the functional

$$I(u) := \int_{\mathcal{X}} \frac{1}{p} \, |\bar{\partial} u|^p \, dx \mapsto \min \qquad (0.1)$$

over the set $\mathcal{A}$ of all $u \in W^{1,p}(\mathcal{X})$ satisfying $u = u_0$ on $\mathcal{S}$. By the Sobolev trace theorem, the condition $p > 1$ implies that u has boundary values belonging to $W^{1-1/p,p}(\partial\mathcal{X})$, hence the equality $u = u_0$ is well defined almost everywhere on $\mathcal{S}$ for all $u_0 \in W^{1-1/p,p}(\mathcal{S})$.

In the language of partial differential equations the problem of analytic continuation from $\mathcal{S}$ is called the Cauchy problem with data at $\mathcal{S}$ for solutions of the elliptic system $\bar{\partial} u = 0$ in $\mathcal{X}$. The variational approach to such problems was first elaborated in [**LT09**].

If $u \in W^{1,p}(\mathcal{X})$ is an analytic continuation of u_0, then $u \in \mathcal{A}$ and $I(u) = 0$. Hence, u is a solution of the variational problem $I(u) \mapsto \min$ over the set $u \in \mathcal{A}$. Conversely, if the functional I attains a minimum at a function $u \in \mathcal{A}$ and this minimum just amounts to zero, then u is actually an analytic continuation of u_0. However, the minimum need not vanish. The infima of I over $\mathcal{A}$ belong to the set of all critical points of I in $\mathcal{A}$. A function $u \in \mathcal{A}$ is proved to be a critical point of the functional I in $\mathcal{A}$ if and only if it satisfies the so-called Euler equations for I. They look like

$$\left\{ \begin{array}{rcll} \bar{\partial}^* \left(|\bar{\partial} u|^{p-2} \bar{\partial} u \right) & = & 0 & \text{in} \quad \mathcal{X}, \\ u & = & u_0 & \text{at} \quad \mathcal{S}, \\ n(|\bar{\partial} u|^{p-2} \bar{\partial} u) & = & 0 & \text{at} \quad \partial\mathcal{X} \setminus \mathcal{S}, \end{array} \right. \qquad (0.2)$$

where $\bar{\partial}^*$ is the formal adjoint of $\bar{\partial}$ and $n(|\bar{\partial} u|^{p-2} \bar{\partial} u) = \sigma(\bar{\partial}^*)(-\imath\nu)(|\bar{\partial} u|^{p-2} \bar{\partial} u)$ the Cauchy data of $|\bar{\partial} u|^{p-2} \bar{\partial} u$ on $\partial\mathcal{X}$ with respect to $\bar{\partial}^*$. Here, $\sigma(\bar{\partial}^*)(-\imath\nu)$ stands for the principal symbol of $\bar{\partial}^*$ evaluated at the cotangent vector $-\imath\nu$, $\nu = (\nu_1, \nu_2)$ being the unit outward normal vector of $\partial\mathcal{X}$.

The case $p = 2$ is of particular interest, since the Euler equations for functional (0.1) are linear in this case. If moreover $\mathcal{S}$ is the whole boundary, the extremal problem goes back at least as far as the theory of harmonic integrals on complex manifolds, see [**Koh63**].

The nonlinear second order differential operator $L_p(u) = \bar{\partial}^* \left(|\bar{\partial} u|^{p-2} \bar{\partial} u \right)$ is called the complex p-Laplace operator. This is an analogue of the p-Laplace operator $\Delta_p(u) = \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right)$ in $\mathbb{R}^n$ which plays an important role in nonlinear potential theory and appears often in physics and engineering. A mixed boundary value problem for Δ_p similar to (0.2) was recently studied in [**She13**]. For $p = 2$, this is precisely the well known mixed problem for the Laplace equation first studied by Zaremba [**Zar10**].

The complex p-Laplace operator has been investigated also within complex analysis in the study of extremal problems in Bergman spaces of analytic function, see [**KS97**]. In [**KS97**], the regularity problem for solutions of the Dirichlet problem for the homogeneous equation $L_p u = 0$ is discussed, i.e. problem (0.2) in case $\mathcal{S}$ is all of $\partial\mathcal{X}$.

The differential equation (0.2) in the domain $\mathcal{X}$ is the so-called degenerate elliptic equation. It can also be viewed as an elliptic system of two real such equations. Ellipticity fails exactly at the points where $\bar{\partial} u = 0$. In the case $\mathcal{S} = \partial\mathcal{X}$ the problems similar to (0.2) or its variational source $I(u) \mapsto \min$ have been studied by Morrey in [**Mor38**], [**Mor59**], [**Mor66**] and many others, cf., e.g., [**LU68**], [**Lieb88**] and the references given there. The corollary from those investigations is that, if u_0 is a polynomial, the Dirichlet problem for L_p possesses a unique solution of Hölder class $C^{1,\lambda}(\overline{\mathcal{X}})$ with exponent $\lambda > 0$ depending on p (see [**KS97**], Theorem D).

In this paper we are interested in the variational problem $I(u) \mapsto \min$ over $\mathcal{A}$ in the case where $\mathcal{S}$ is nonempty and different from the whole boundary $\partial\mathcal{X}$. For the study of mixed boundary value problem (0.2) we invoke the theory of weak boundary values of solutions to elliptic systems developed in [**Tar95**]. We prove that a function $u \in W^{1,p}(\mathcal{X})$ satisfies (0.2) if and only if it is an analytic extension of u_0 in $\mathcal{X}$. Hence it follows that problem (0.1) has actually no solutions u different from the analytic continuation of u_0 in $\mathcal{X}$, if there is any. Thus, (0.1) is not suited to be a good relaxation of the Cauchy problem for the Cauchy-Riemann system in $\mathcal{X}$ with data on $\mathcal{S}$. This result differs considerably from that of the paper [**She13**] which asserts that the analogous mixed problem for the p-Laplace operator in $\mathbb{R}^n$ is uniquely solvable for all data $u_0 \in W^{1,p}(\mathcal{S})$. (In fact [**She13**] allows also nonzero Neumann data $|\nabla u|^{p-2} n(\nabla u) = u_1$ at $\partial\mathcal{X} \setminus \mathcal{S}$, where u_1 is a continuous linear functional on $W^{1-1/p,p}(\partial\mathcal{X} \setminus \mathcal{S})$.)

In order to achieve a true relaxation of the problem of analytic continuation one has to look for local infima of the functional $I(u)$ on bounded closed subsets of $\mathcal{A}$. To this end, given any $R > 0$, we denote by $\mathcal{A}_R$ the set of all functions $u \in W^{1,p}(\mathcal{X})$, such that $u = u_0$ at $\mathcal{S}$ and $\|u\|_{W^{1,p}(\mathcal{X})} \leq R$. Obviously, $\mathcal{A}_R$ is a convex bounded closed subset of $\mathcal{A}$ and the family $\mathcal{A}_R$ increases and exhausts $\mathcal{A}$, when $R \to \infty$. The main result of this paper is the following theorem which goes back at least as far as [**Ros65**].

THEOREM 0.1. *Suppose $\mathcal{S} \subset \partial\mathcal{X}$ is a nonempty arc different from the whole boundary and $u_0 \in W^{1-1/p,p}(\mathcal{S})$, where $1 < p < \infty$. For each $R > 0$, there is a unique function $u_R \in W^{1,p}(\mathcal{X})$ in $\mathcal{A}_R$ minimising functional (0.1) over $\mathcal{A}_R$.*

It is clear that the Euler equations for the critical points of I on the set $\mathcal{A}_R$ are different from mixed problem (0.2). However, if the family $\{u_R\}_{R>0}$ is bounded, then it stabilises for R large enough. The limit function u is the minimum of I over all of $\mathcal{A}$, and so u is an analytic extension of u_0. As but one consequence we deduce that for a function $u_0 \in W^{1-1/p,p}(\mathcal{S})$ to admit an analytic continuation in $\mathcal{X}$ it is necessary and sufficient that the family $\{u_R\}_{R>0}$ of Theorem 0.1 be bounded in $W^{1,p}(\mathcal{X})$.

Note that the "approximate" solutions u_R can be constructed, e.g., by the classical Ritz method, see [**Rit09**].

The above results extend to the Cauchy problem for solutions of first order elliptic systems $Au = 0$ in a smoothly bounded domain $\mathcal{X} \subset \mathbb{R}^n$ with data on an open part $\mathcal{S}$ of the boundary. More precisely, A is assumed to be a square matrix of first order scalar partial differential operators in a neighbourhood U of the closure of $\mathcal{X}$ and the principal symbol $\sigma(A)(x, \xi)$ of A to be invertible for all nonzero $\xi \in \mathbb{R}^n$. Then the formal adjoint A^* of A is also elliptic and we require A^* to satisfy the so-called uniqueness condition for the local Cauchy problem in U, see [**Tar95**, p. 185]. In particular, one can choose a square Dirac operator as A, e.g., the Cauchy-Riemann operator in Clifford analysis, etc. The corresponding p-Laplace operator is $u \mapsto A^*(|Au|^{p-2} Au$. The p-Laplace operator Δ_p in $\mathbb{R}^n$ does not belong to this class of operators, for the gradient operator fails to be elliptic unless $n = 1$.

Let us dwell on the contents of the paper. Section 1 presents some preliminaries on the Cauchy-Riemann system from the viewpoint of partial differential equations. In Section 2 we give a variational formulation of the problem of analytic continuation from a part of boundary and derive the corresponding Euler equations which form a mixed boundary value problem in $\mathcal{X}$. In Section 3 we show that the mixed problem actually reduces to the original problem of analytic continuation. Section 4 is devoted to further development of the variational approach to the problem of analytic continuation. Finally, in Section 5 we touch a few aspects of the theory of p-Laplace operators.

1. The Cauchy-Riemann system

The Cauchy-Riemann operator in the complex plane of variable $z = x_1 + \imath x_2$ is defined by

$$\bar{\partial} u := \frac{1}{2} (\partial_1 + \imath \partial_2) \, u,$$

where $\partial_j = \dfrac{\partial}{\partial x_j}$ for $j = 1, 2$.

When identifying a complex-valued function $u = u_1 + \imath u_2$ with the two-column of real-valued functions $u_1 =: \Re u$ and $u_2 =: \Im u$, one specifies the operator $\bar{\partial}$ within (2×2)-matrices of first order partial differential operators with real coefficients. More precisely,

$$\bar{\partial} u = \frac{1}{2} \begin{pmatrix} \partial_1 & -\partial_2 \\ \partial_2 & \partial_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{1.1}$$

Endowing the complex plane $\mathbb{C}$ with the usual Hermitean structure we introduce the Hilbert space $L^2(\mathbb{C})$. The formal adjoint $\bar{\partial}^*$ of $\bar{\partial}$ is defined by requiring $(\bar{\partial} u, g)_{L^2(\mathbb{C})} = (u, \bar{\partial}^* g)_{L^2(\mathbb{C})}$ for all smooth functions u and g of compact support. When identifying complex-valued functions with two-columns of real-valued ones and using the Hermitean structure in $\mathbb{R}^2$, we get precisely the same formal adjoint operator. That is

$$\begin{aligned} \bar{\partial}^* g &= -\partial g \\ &= \frac{1}{2} \begin{pmatrix} -\partial_1 & -\partial_2 \\ \partial_2 & -\partial_1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \end{aligned}$$

for $g = g_1 + \imath g_2$.

The classical principal symbol of the Cauchy-Riemann operator is the family of (2×2)-matrices

$$\sigma(\bar{\partial})(\xi) = \frac{1}{2} \begin{pmatrix} \imath \xi_1 & -\imath \xi_2 \\ \imath \xi_2 & \imath \xi_1 \end{pmatrix} \tag{1.2}$$

parametrised by $\xi \in \mathbb{R}^2$. The operator $\bar{\partial}$ is elliptic in the sense that the family (1.2) is invertible for all $\xi \in \mathbb{R}^2 \setminus \{0\}$.

By operation with symbols, we get $\sigma(\bar{\partial}^*) = \sigma(\bar{\partial})^*$, the asterisk on the right-hand side indicating the adjoint matrix. Hence it follows that the formal adjoint is also elliptic.

Moreover, it is easy to verify that

$$\bar{\partial}^* \bar{\partial} = -\frac{1}{4} E_2 \, \Delta,$$

where E_2 is the unit matrix of size 2×2 and Δ the Laplace operator in $\mathbb{R}^2$. This can be equivalently reformulated by saying that (2 times) $\bar{\partial}$ is a Dirac operator in the plane.

Solutions of the system $\bar{\partial} u = 0$ in a domain $\mathcal{X} \subset \mathbb{R}^2$ are known to be analytic (or holomorphic) functions in $\mathcal{X}$. In this paper we restrict ourselves to functions u of Sobolev space $W^{1,p}(\mathcal{X})$ with $1 < p < \infty$. If $\mathcal{X}$ is bounded by a smooth curve, then each function $u \in W^{1,p}(\mathcal{X})$ possesses a trace on $\partial\mathcal{X}$ in the sense of Sobolev spaces which is an element of $W^{1/p',p}(\partial\mathcal{X})$, where $1/p + 1/p' = 1$, see for instance [**AH96**].

The following formula is known as the Green formula in complex analysis. It is a very particular case of Green formulas for general partial differential operators, see [**Tar95**, p. 300].

LEMMA 1.1. *Suppose $\mathcal{X}$ is a bounded domain with piecewise smooth boundary in $\mathbb{R}^2$. Then*

$$\int_{\mathcal{X}} \left((\bar{\partial} u, g)_x - (u, \bar{\partial}^* g)_x \right) dx = - \int_{\partial\mathcal{X}} (u, \sigma(\bar{\partial}^*)(-\imath\nu) g)_x ds,$$

for all $u \in W^{1,p}(\mathcal{X})$ and $g \in W^{1,p'}(\mathcal{X})$, where $\nu = (\nu_1, \nu_2)$ is the unit outward normal vector of the boundary.

The restriction of $\sigma(\bar{\partial}^*)(-\imath\nu)g$ to the boundary is called the Cauchy data of g at $\partial\mathcal{X}$ with respect to the operator $\bar{\partial}^*$, see [**Tar95**, p. 301]. It is usually denoted by $n(g)$.

2. An extremal problem

Let $\mathcal{X} \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $\mathcal{S}$ a nonempty open arc on $\partial\mathcal{X}$. If $u \in W^{1,p}(\mathcal{X})$ is an analytic function in $\mathcal{X}$ vanishing at $\mathcal{S}$, then u is identically zero in all of $\mathcal{X}$, see, e.g., Theorem 10.3.5 of [**Tar95**]. This raises the following problem of analytic continuation going beyond function theory: Given a function $u_0 \in W^{1/p',p}(\mathcal{S})$, find $u \in W^{1,p}(\mathcal{X})$ which is analytic in $\mathcal{X}$ and satisfies $u = u_0$ at $\mathcal{S}$. Throughout the paper we tacitly assume that $u_0 \neq 0$, since otherwise the problem is trivial.

If S' is a nonempty open arc on $\partial\mathcal{X}$ whose closure belongs to $\mathcal{S}$, then the analytic continuation u is uniquely defined by the values of u_0 at S'. Hence, the problem of analytic continuation from $\mathcal{S}$ is overdetermined. In particular, if u_0 is a smooth function with compact support in $\mathcal{S}$, then u_0 extends to an analytic function $u \in W^{1,p}(\mathcal{X})$ if and only if $u_0 \equiv 0$.

To construct a variational relaxation of the problem of analytic continuation, we introduce the functional

$$I(u) := \int_{\mathcal{X}} \frac{1}{p} |\bar{\partial} u|^p \, dx \tag{2.1}$$

and give I the domain $\mathcal{A}$ consisting of all $u \in W^{1,p}(\mathcal{X})$, such that $u = u_0$ at $\mathcal{S}$. For the calculus of variations it is important that $\mathcal{A}$ is a convex closed subset of $W^{1,p}(\mathcal{X})$.

LEMMA 2.1. *The functional I is strongly convex on $\mathcal{A}$.*

PROOF. We have to show that if $u, v \in \mathcal{A}$ then

$$I(tu + (1-t)v) < t\, I(u) + (1-t)\, I(v)$$

for all $t \in (0,1)$. Note that if u and v are two different elements of $\mathcal{A}$, then $\bar{\partial} u$ and $\bar{\partial} v$ are different functions on $\mathcal{X}$. Indeed, if $\bar{\partial} u = \bar{\partial} v$ almost everywhere in $\mathcal{X}$, then the difference $u - v \in W^{1,p}(\mathcal{X})$ is holomorphic in $\mathcal{X}$ and vanishes at $\mathcal{S}$. By uniqueness, we get $u - v \equiv 0$ in $\mathcal{X}$, a contradiction. Now the strong convexity of the function $|y|^p$, $p > 1$, implies

$$\begin{array}{rcl} I(tu + (1-t)v) & = & \displaystyle\int_{\mathcal{X}} \frac{1}{p} \, |\bar{\partial}\, (tu + (1-t)v)\, |^p \, dx \\ & < & t \displaystyle\int_{\mathcal{X}} \frac{1}{p} \, |\bar{\partial} u|^p \, dx + (1-t) \displaystyle\int_{\mathcal{X}} \frac{1}{p} \, |\bar{\partial} v|^p \, dx \\ & = & t\, I(u) + (1-t)\, I(v) \end{array}$$

for all $u, v \in \mathcal{A}$ with $u \neq v$ and all $t \in (0,1)$, as desired. □

It follows from the lemma that there is at most one function $u \in \mathcal{A}$ at which I attains its infimum over $\mathcal{A}$.

Consider the extremal problem of finding the minimum of the functional I on the set $\mathcal{A}$, if there is any. Clearly, if $u \in W^{1,p}(\mathcal{X})$ is an analytic extension of u_0 into $\mathcal{X}$, then u is a solution of the extremal problem, the minimal value of I on $\mathcal{A}$ being $I(u) = 0$.

The following lemma describes all critical points of the functional I on $\mathcal{A}$. The corresponding equations are known as Euler equations for the extremal problem $I(u) \mapsto \min$ over $u \in \mathcal{A}$, see [**Mor66**].

Lemma 2.2. *Assume that the functional $I : \mathcal{A} \to \mathbb{R}$ attains a local minimum at a function $u \in \mathcal{A}$. Then u satisfies*

$$\left\{ \begin{array}{rcll} \bar{\partial}^* \left(|\bar{\partial} u|^{p-2} \bar{\partial} u \right) & = & 0 & \mathit{in} \ \ \mathcal{X}, \\ u & = & u_0 & \mathit{at} \ \ \mathcal{S}, \\ n(|\bar{\partial} u|^{p-2} \bar{\partial} u) & = & 0 & \mathit{at} \ \ \partial\mathcal{X} \setminus \mathcal{S}, \end{array} \right.$$

Proof. Let $v \in C^\infty(\overline{\mathcal{X}})$ be an arbitrary complex-valued function vanishing on $\mathcal{S}$. Write $u = u_1 + \imath u_2$ and $v = v_1 + \imath v_2$. For each $\varepsilon = (\varepsilon_1, \varepsilon_2)$ in $\mathbb{R}^2$, the variation $(u_1 + \varepsilon_1 v_1, u_2 + \varepsilon_2 v_2)$ is left in $\mathcal{A}$. Therefore, if I attains a local minimum at u, then the function $F(\varepsilon) = I(u_1 + \varepsilon_1 v_1, u_2 + \varepsilon_2 v_2)$ takes on a local minimum at $\varepsilon = 0$. It follows that $\varepsilon = 0$ is a critical point of F, i.e. both derivatives F'_{ε_1} and F'_{ε_2} vanish at the origin.

An easy computation shows that

$$\begin{array}{rcl} F'_{\varepsilon_1}(0) & = & \displaystyle\int_{\mathcal{X}} \Big(|\bar{\partial} u|^{p-2} \bar{\partial} u, \frac{1}{2} \Big(\begin{array}{c} \partial_1 v_1 \\ \partial_2 v_1 \end{array} \Big) \Big)_x dx, \\ F'_{\varepsilon_2}(0) & = & \displaystyle\int_{\mathcal{X}} \Big(|\bar{\partial} u|^{p-2} \bar{\partial} u, \frac{1}{2} \Big(\begin{array}{c} -\partial_2 v_2 \\ \partial_1 v_2 \end{array} \Big) \Big)_x dx \end{array}$$

whence

$$\begin{array}{rcl} F'_{\varepsilon_1}(0) + F'_{\varepsilon_2}(0) & = & \displaystyle\int_{\mathcal{X}} \left(|\bar{\partial} u|^{p-2} \bar{\partial} u, \bar{\partial} v \right)_x dx \\ & = & 0 \end{array} \qquad (2.2)$$

for all smooth functions v on the closure of $\mathcal{X}$ which vanish at $\mathcal{S}$. In particular, equality (2.2) holds for all smooth functions with compact support in $\mathcal{X}$, thus implying

$$\bar{\partial}^* \left(|\bar{\partial} u|^{p-2} \bar{\partial} u \right) = 0 \qquad (2.3)$$

in the sense of distributions in $\mathcal{X}$.

The function $|\bar{\partial}u|^{p-2}\bar{\partial}u$ is easily verified to belong to $L^{p'}(\mathcal{X})$, where p' is the real number with $1/p+1/p' = 1$. Since $\bar{\partial}$ is a Dirac operator, $|\bar{\partial}u|^{p-2}\bar{\partial}u$ is harmonic in the sense of distributions in $\mathcal{X}$. By Weyl's lemma, this function is harmonic in $\mathcal{X}$. We can now invoke the theory of [**Str84**], [**Tar95**, 9.4] to conclude that $|\bar{\partial}u|^{p-2}\bar{\partial}u$ admits a distribution boundary value in $W^{-1/p',p'}(\partial\mathcal{X})$.

The distribution boundary value of $|\bar{\partial}u|^{p-2}\bar{\partial}u$ is referred to as weak limit value in [**Tar95**]. In Section 9.4.3 *ibid.* the weak limit values are proved to be equivalent to the so-called strong limit values. These latter are precisely those values on the boundary which take part in the Green formula, see Section 9.4.1 in [**Tar95**]. By Lemma 1.1, we get

$$\int_{\mathcal{X}} \left(|\bar{\partial}u|^{p-2}\bar{\partial}u, \bar{\partial}v\right)_x dx = -\int_{\partial\mathcal{X}} \left(n(|\bar{\partial}u|^{p-2}\bar{\partial}u), v\right)_x ds,$$

and so by (2.3) the right-hand side is equal to zero for all $v \in C^\infty(\overline{\mathcal{X}})$ vanishing at $\mathcal{S}$. Hence it follows that the distribution $n(|\bar{\partial}u|^{p-2}\bar{\partial}u) \in W^{-1/p',p'}(\partial\mathcal{X})$ has support in $\partial\mathcal{X} \setminus \mathcal{S}$. □

It is worth emphasizing that both $\bar{\partial}^*\left(|\bar{\partial}u|^{p-2}\bar{\partial}u\right)$ in $\mathcal{X}$ and $n(|\bar{\partial}u|^{p-2}\bar{\partial}u)$ at $\partial\mathcal{X}$ are understood in the sense of distributions.

3. Complex p-Laplace operator

The Euler equations for the variational problem $I(u) \mapsto \min$ over $u \in \mathcal{A}$ form a mixed problem for solutions of the nonlinear differential equation $L_p u = 0$ in $\mathcal{X}$, where

$$L_p u := \bar{\partial}^*\left(|\bar{\partial}u|^{p-2}\bar{\partial}u\right)$$

for $u \in W^{1,p}(\mathcal{X})$. This operator is called the complex p-Laplace operator by analogy with the p-Laplace operator $\Delta_p(u) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ in $\mathbb{R}^n$.

An easy computation shows that L_p is a degenerate elliptic operator. Ellipticity is violated exactly at the points where $\bar{\partial}u = 0$. We endow this operator with the domain $W^{1,p}(\mathcal{X})$ and interpret it in the weak sense, see (2.2). Moreover, given any $u \in W^{1,p}(\mathcal{X})$, the distribution $L_p(u)$ in $\mathcal{X}$ extends (non-uniquely) to a continuous linear functional on $W^{1,p}(\mathcal{X})$ by

$$(L_p(u), v) := \left(|\bar{\partial}u|^{p-2}\bar{\partial}u, \bar{\partial}v\right)_{L^2(\mathcal{X})} \tag{3.1}$$

for all $v \in W^{1,p}(\mathcal{X})$.

If $u \in W^{1,p}(\mathcal{X})$ satisfies $L_p u = 0$ in $\mathcal{X}$, then $f = |\bar{\partial}u|^{p-2}\bar{\partial}u$ is a weak solution of the equation $\bar{\partial}^* f = 0$ in $\mathcal{X}$. Since $\bar{\partial}\bar{\partial}^* = -1/4\,\Delta$, it follows by Weyl's lemma that f is a harmonic function in the domain $\mathcal{X}$. Moreover, from $f \in L^{p'}(\mathcal{X})$ we deduce that f is of finite order of growth near the boundary. Therefore, the function f admits boundary values at $\partial\mathcal{X}$ in the sense of distributions, see [**Str84**] and Section 9.4 of [**Tar95**]. In the sequel by $n(f) = \sigma(\bar{\partial}^*)(-\imath\nu)f$ we mean the Cauchy data of f on $\partial\mathcal{X}$ with respect to $\bar{\partial}^*$.

LEMMA 3.1. *Assume that $\mathcal{S} \neq \partial\mathcal{X}$. If $u \in W^{1,p}(\mathcal{X})$ is a solution of the mixed boundary value problem*

$$\left\{\begin{array}{rcll} \bar{\partial}^*\left(|\bar{\partial}u|^{p-2}\bar{\partial}u\right) & = & 0 & \text{in} \quad \mathcal{X}, \\ u & = & u_0 & \text{at} \quad \mathcal{S}, \\ n(|\bar{\partial}u|^{p-2}\bar{\partial}u) & = & 0 & \text{at} \quad \partial\mathcal{X}\setminus\mathcal{S}, \end{array}\right.$$

then $\bar{\partial} u = 0$ *in* $\mathcal{X}$.

PROOF. This lemma is actually a very particular case of Theorem 10.3.5 of [**Tar95**] applied to the elliptic differential operator $\bar{\partial}^* = -\partial$. More precisely, set $f = |\bar{\partial} u|^{p-2} \bar{\partial} u$. By the above, f is an antiholomorphic function in the domain $\mathcal{X}$ of finite order of growth at the boundary. Moreover, the Cauchy data of f with respect to $\bar{\partial}^*$ vanish in the complement of $\bar{\mathcal{S}}$. Since $\mathcal{S}$ is different from the whole boundary, it follows that the complement of $\bar{\mathcal{S}}$ contains at least one interior point at the boundary. Applying Theorem 10.3.5 of [**Tar95**] yields $f \equiv 0$ in $\mathcal{X}$. Since $\bar{\partial} u \in L^p(\mathcal{X})$, we get $|f| = |\bar{\partial} u|^{p-1}$ whence $\bar{\partial} u = 0$ almost everywhere in $\mathcal{X}$, as desired. □

If $u \in W^{1,p}(\mathcal{X})$ is an analytic extension of u_0 into $\mathcal{X}$, then u is obviously a solution of the mixed problem. Hence, the mixed problem of Lemma 3.1 is actually an equivalent reformulation of the problem of analytic extension from $\mathcal{S}$ into $\mathcal{X}$, provided $\mathcal{S} \neq \partial \mathcal{X}$. The problem of analytic continuation from the whole boundary is stable. In the case $\mathcal{S} = \partial \mathcal{X}$ the mixed problem becomes the Dirichlet problem for the complex p-Laplace operator. Its solutions need not be analytic functions in $\mathcal{X}$. We conclude that the Euler equations of the extremal problem $I \mapsto \min$ over $\mathcal{A}$ can be thought of as relaxation of the problem of analytic continuation only in the case $\mathcal{S} = \partial \mathcal{X}$.

4. Relaxation of the problem of analytic continuation

We now restrict ourselves to the case where $\mathcal{S}$ is different from the whole boundary.

Write m for the infimum of $I(u)$ over $u \in \mathcal{A}$. We are aimed at finding those function $u \in \mathcal{A}$ at which the functional $I(u)$ takes on the value m. The integrand in (2.1) is

$$L(v) = \frac{2^{-p}}{p} \left((v_1^1 - v_2^2)^2 + (v_2^1 + v_1^2)^2 \right)^{p/2}, \tag{4.1}$$

where $v = (v_1, v_2)$ is a (2×2)-matrix whose columns are substitutions for $\partial_1 u$ and $\partial_1 u$, respectively.

LEMMA 4.1. *For each fixed* $p \geq 1$, *the function* $L(v)$ *is convex in the entries of* $v \in \mathbb{R}^{2 \times 2}$.

PROOF. We neglect the factor $(p\, 2^p)^{-1}$ in (4.1), which is inessential for the proof. Write

$$v_j = \begin{pmatrix} v_j^1 \\ v_j^2 \end{pmatrix}$$

for $j = 1, 2$. A cumbersome but quite elementary computation shows that

$$\begin{aligned} & \sum_{\substack{i,k=1,2 \\ j,l=1,2}} L''_{v_j^i, v_l^k}(v)\, w_j^i w_l^k \\ = \ & p\, L(v)^{\frac{p-4}{p}} \left(L(v)^{\frac{2}{p}} L(w)^{\frac{2}{p}} + (p-2) \left((v_1^1 - v_2^2)(w_1^1 - w_2^2) + (v_2^1 + v_1^2)(w_2^1 + w_1^2) \right)^2 \right) \end{aligned} \tag{4.2}$$

for all $w_j = \left(\begin{array}{c} w_j^1 \\ w_j^2 \end{array} \right)$, $j = 1, 2$, in $\mathbb{R}^2$. Since $p \geq 1$, the right-hand side of (4.2) is greater than or equal to

$$p\, L(v)^{\frac{p-4}{p}} \left(L(v)^{\frac{2}{p}} L(w)^{\frac{2}{p}} - \left((v_1^1 - v_2^2)(w_1^1 - w_2^2) + (v_2^1 + v_1^2)(w_2^1 + w_1^2) \right)^2 \right) \geq 0,$$

the last inequality being due to the Cauchy-Schwarz inequality for the scalar product in $\mathbb{R}^2$. The nonnegative definiteness of the quadratic form in (4.2) is equivalent to the convexity of L as a function of the entries of v, see Lemma 1.8.1 of [**Mor66**] and elsewhere. □

Thus, the general hypothesis of [**Mor66**, p. 91] concerning the integrand function $L(v)$ are satisfied. It should be noted that $L(v)$ fails obviously to be strongly convex.

If $u \in \mathcal{A}$ furnishes a local minimum to (2.1), then necessarily

$$\left(\begin{array}{c} y^1 \\ y^2 \end{array} \right)^* \left(\begin{array}{cc} \sum\limits_{j,l=1}^{2} L''_{v_j^1, v_l^1} \xi_j \xi_l & \sum\limits_{k,l=1}^{2} L''_{v_j^1, v_l^2} \xi_j \xi_l \\ \sum\limits_{j,l=1}^{2} L''_{v_j^2, v_l^1} \xi_j \xi_l & \sum\limits_{j,l=1}^{2} L''_{v_j^2, v_l^2} \xi_j \xi_l \end{array} \right) \left(\begin{array}{c} y^1 \\ y^2 \end{array} \right) \geq 0 \tag{4.3}$$

is fulfilled for all $\xi \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$, the derivatives of L being evaluated at u'. This classical necessary condition is known as the Legendre-Hadamard condition, see [**Mor66**, 1.5]. In this case, one says that the integrand function L is regular if the inequality holds in (4.3) for all $\xi \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$ which are different from zero.

LEMMA 4.2. *For $p = 2$, the integrand function $L(v)$ is regular.*

PROOF. Using equality (4.2) we deduce that, for $p = 2$, the left-hand side in (4.3) is equal to

$$\sum_{\substack{i,k=1,\ldots,\ell \\ j,l=1,\ldots,n}} L''_{v_j^i, v_l^k}(v)\, (y^i \xi_j)(y^k \xi_l) = L(w)$$

where w is the (2×2)-matrix with entries $w_j^i = y^i \xi_j$. Moreover,

$$L(w) = \sum_{\substack{i=1,2 \\ j=1,2}} (y^i \xi_j)^2,$$

showing that $L(w)$ is positive for all $\xi \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$ different from zero, as desired. □

If $p \neq 2$, the lemma seems to fail, for the complex p-Laplace operator degenerates at those $x \in \mathcal{X}$ where $\bar{\partial} u(x)$ vanishes.

Having disposed of this preliminary step, we proceed with searching for local minima of (2.1). By definition, there is a sequence $\{u_\nu\}$ in $\mathcal{A}$, such that $I(u_\nu) \searrow m$. It is called minimising. Any subsequence of a minimising sequence is also minimising. Were it possible to extract a subsequence $\{u_{\nu_\iota}\}$ converging to an element $u \in \mathcal{A}$ in the $W^{1,p}(\mathcal{X})$ norm, then $I(u_{\nu_\iota})$ would converge to $I(u) = m$, and so u would be a desired solution of our extremal problem. It is possible to require the convergence of a minimising sequence in a weaker topology than that of $W^{1,p}(\mathcal{X})$. However, the functional I should be lower semicontinuous with respect to correspondingly more general types of convergence. In order to find a convergent subsequence of a minimising sequence, one uses a compactness argument. The space $W^{1,p}(\mathcal{X})$ is reflexive, for we assume $1 < p < \infty$. Hence, each bounded sequence in $W^{1,p}(\mathcal{X})$ has a weakly convergent subsequence. Thus, any bounded minimising sequence $\{u_\nu\}$ has a subsequence $\{u_{\nu_\iota}\}$ which converges weakly in $W^{1,p}(\mathcal{X})$ to some function u. By a theorem of Mazur, see [**Yos65**] and elsewhere, any convex closed subset of a reflexive Banach space is weakly closed. It follows that the limit function u satisfies $u = u_0$ on $\mathcal{S}$, i.e., it belongs to $\mathcal{A}$. Moreover, Theorem 3.4.4 of [**Mor66**] says that the subsequence $\{u_{\nu_\iota}\}$ converges also strongly in $L^1(\mathcal{X})$ to u. Although $L(v)$ is neither normal nor strictly convex in v, Theorem 4.1.1 of [**Mor66**] applies to the functional I.

Lemma 4.3. *If u_ν and u lie in $W^{1,p}(\mathcal{X})$ and $u_\nu \to u$ in $L^1(K)$ for each compact set K interior to $\mathcal{X}$, then*

$$I(u) \leq \liminf I(u_\nu).$$

Proof. See Theorem 4.1.1 of [**Mor66**]. □

We have thus proved that if there is a bounded minimising sequence $\{u_\nu\}$ and u is a weak limit point of this sequence in $W^{1,p}(\mathcal{X})$, then $u \in \mathcal{A}$ and $I(u) = m$, i.e., u is a minimiser. It is clear that any minimising sequence is bounded in $W^{1,p}(\mathcal{X})$ if the functional I majorises the norm of u in $\mathcal{A}$ in the sense that

$$\int_{\mathcal{X}} |\bar{\partial} u|^p \, dx \geq c \, \|u\|^p_{W^{1,p}(\mathcal{X})} - Q \tag{4.4}$$

for all $u \in \mathcal{A}$, with c and Q constants independent of u. This is obviously a far reaching generalisation of A. Korn's (1908) inequality for the case of nonlinear problems.

Since the boundary value problem for $\bar{\partial}$ with boundary data $\Re u = u_0$ is elliptic, estimate (4.4) holds provided that $\mathcal{S} = \partial\mathcal{X}$. If however $\mathcal{S} \neq \partial\mathcal{X}$, no a priori estimate (4.4) is possible, see [**ST10**].

Hence, we are not able to prove the boundedness of any minimising sequence $\{u_\nu\}$, so the above arguments do not apply. To get over this difficulty a general idea is to confine the set $\mathcal{A}$ of competing functions so that $\mathcal{A}$ be itself bounded. In linear analysis this idea goes back at least as far as [**Ros65**]. While additional assumptions on $\mathcal{A}$ may depend on the concrete problem there is also an abstract prescription.

Given any $R > 0$, we denote by $\mathcal{A}_R$ the subset of $W^{1,p}(\mathcal{X})$ consisting of all $u \in W^{1,p}(\mathcal{X})$, such that $u = u_0$ at $\mathcal{S}$ and, moreover,

$$\|\Re u\|_{W^{1/p',p}(\partial\mathcal{X}\setminus\mathcal{S})} \leq R. \tag{4.5}$$

Note that $\mathcal{A}_R \subset \mathcal{A}$ and each $u \in \mathcal{A}$ lies in some $\mathcal{A}_R$, i.e. the family $\mathcal{A}_R$ actually exhausts $\mathcal{A}$. It is clear that the local minima of the functional I over $\mathcal{A}_R$ need not satisfy Euler equations (0.2).

Theorem 4.4. *Let $R > 0$. Then, for each data $u_0 \in W^{1/p',p}(\mathcal{S})$, there is a unique function $u \in \mathcal{A}_R$ minimising the functional I over $\mathcal{A}_R$.*

Proof. One can assume without loss of generality that R is large enough, since otherwise the family $\mathcal{A}_R$ is empty and the assertion makes no sense. Since the boundary value problem

$$\left\{ \begin{array}{rcll} \bar{\partial} u & = & f & \text{in} \ \ \mathcal{X}, \\ \Re u & = & u_0 & \text{at} \ \ \partial \mathcal{X} \end{array} \right.$$

is elliptic, there is a constant C depending only on $\mathcal{X}$ and p, such that

$$\|u\|_{W^{1,p}(\mathcal{X})} \leq C \left(\|\bar{\partial} u\|_{L^p(\mathcal{X})} + \|\Re u\|_{W^{1/p',p}(\partial \mathcal{X})} \right) \tag{4.6}$$

for all $u \in W^{1,p}(\mathcal{X})$. Pick a minimising sequence $\{u_\nu\}$ in $\mathcal{A}_R$. Then the sequence $\{\bar{\partial} u_\nu\}$ is bounded in $L^p(\mathcal{X})$. Combining (4.6) and (4.5) we readily deduce that the sequence $\{u_\nu\}$ is actually bounded in the $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$-norm. We now argue as above. The bounded sets in $W^{1,p}(\mathcal{X})$ are relatively compact with respect to weak convergence in $W^{1,p}(\mathcal{X})$. Hence, there is a subsequence $\{u_{\nu_\iota}\}$ of $\{u_\nu\}$ which tends weakly in $W^{1,p}(\mathcal{X})$ to some function $u \in W^{1,p}(\mathcal{X})$. Since $\mathcal{A}_R$ is a convex closed set in the reflexive Banach space $W^{1,p}(\mathcal{X})$, it is, by a theorem of Mazur cited above, weakly closed. Hence we conclude that the limit function u belongs to the set $\mathcal{A}_R$. Moreover, Theorem 3.4.4 of [**Mor66**] says that $\{u_{\nu_\iota}\}$ converges also strongly in $L^1(\mathcal{X})$ to u. On applying the lower semicontinuity of I stated by Lemma 4.3 we obtain

$$I(u) \leq \liminf I(u_{\nu_\iota}) = m_R,$$

where m_R is the infimum of I over $\mathcal{A}_R$. Since $u \in \mathcal{A}$, it follows that $I(u) = m_R$, as desired. What is left is to show that the minimiser is unique. To this end, we assume that u and v are two different minimisers in $\mathcal{A}_R$. Then $(1/2)(u+v) \in \mathcal{A}$ and by Lemma 2.1

$$\begin{array}{rcl} I\Big(\dfrac{u+v}{2}\Big) & < & \dfrac{1}{2} I(u) + \dfrac{1}{2} I(v) \\ & = & m_R, \end{array}$$

a contradiction. □

The same proof still goes when we replace condition (4.5) in the definition of $\mathcal{A}_R$ by $\|u\|_{W^{1,p}(\mathcal{X})} \leq R$. This implies Theorem 0.1.

5. Generalisation to Dirac operators

Let A be an $(l \times k)$-matrix of first order scalar partial differential operators with constant coefficients in $\mathbb{R}^n$. It acts locally on k-columns of smooth functions or distributions in $\mathbb{R}^n$ by multiplying these with A from the left. As usual, the formal adjoint A^* is defined for A by requiring $(Au, v)_{L^2(\mathbb{R}^n, \mathbb{C}^l)} = (u, A^* v)_{L^2(\mathbb{R}^n, \mathbb{C}^k)}$ for all smooth functions u, v of compact support with values in $\mathbb{C}^k$ and $\mathbb{C}^l$, respectively. The operator A^* is specified within $(l \times k)$-matrices of first order scalar partial differential operators with constant coefficients in $\mathbb{R}^n$. We say A is a Dirac operator if $A^* A = -E_k \,\Delta$, where E_k is the unit $(k \times k)$-matrix. If A is a Dirac operator, then A^* is a Dirac operator, too, if $l = k$, and is not, if $l \neq k$. Each Dirac operator A is overdetermined elliptic, i.e. its principal symbol $\sigma(A)(\xi)$ has a left inverse for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

As developed above for the Cauchy-Riemann operator in one complex variable, the theory generalises to those Dirac operators A which are elliptic in the classical sense, i.e. with $k = l$.

Assume that $\mathcal{X}$ is a bounded domain with smooth boundary in $\mathbb{R}^n$ and $\mathcal{S}$ a nonempty open set in $\partial \mathcal{X}$. If $u \in W^{1,p}(\mathcal{X}, \mathbb{C}^k)$ satisfies the system $Au = 0$ in $\mathcal{X}$

and the trace of u on $\mathcal{S}$ is zero, then u vanishes identically in $\mathcal{X}$, see Theorem 10.3.5 of [**Tar95**]. Hence, the following Cauchy problem has at most one solution: Given $u_0 \in W^{1/p',p}(\mathcal{S}, \mathbb{C}^k)$, find a function $u \in W^{1,p}(\mathcal{X}, \mathbb{C}^k)$ satisfying $Au = 0$ in $\mathcal{X}$ and $u = u_0$ at $\mathcal{S}$.

For the study of the Cauchy problem for solutions of elliptic equations along more classical lines we refer the reader to [**Tar95**] and more recent papers [**ST05**], [**MNT08**], [**MMT11**].

In order to construct a relaxation of the Cauchy problem we introduce the set $\mathcal{A}$ which consists of all functions $u \in W^{1,p}(\mathcal{X}, \mathbb{C}^k)$, such that $u = u_0$ at $\mathcal{S}$. One sees readily that $\mathcal{A}$ is a convex closed subset of $W^{1,p}(\mathcal{X}, \mathbb{C}^k)$. Consider a functional I on $\mathcal{A}$ given by

$$I(u) = \int_{\mathcal{X}} |Au|^p \, dx$$

for $u \in \mathcal{A}$. Arguing as in the proof of Lemma 2.1 we deduce that the functional I is strongly convex on $\mathcal{A}$.

The Euler equations for the extremal problem $I(u) \mapsto \min$ over $u \in \mathcal{A}$ constitute the mixed boundary value problem

$$\left\{ \begin{array}{rcll} A^* \left(|Au|^{p-2} Au \right) & = & 0 & \text{in } \ \mathcal{X}, \\ u & = & u_0 & \text{at } \ \mathcal{S}, \\ n(|Au|^{p-2} Au) & = & 0 & \text{at } \ \partial\mathcal{X} \setminus \mathcal{S}, \end{array} \right. \tag{5.1}$$

where $n(|Au|^{p-2}Au) = \sigma(A)^*(-\imath\nu)(|Au|^{p-2}Au)$ stands for the Cauchy data of $|Au|^{p-2}Au$ on $\partial\mathcal{X}$ with respect to A^*. The second order partial differential operator $u \mapsto A^*(|Au|^{p-2}Au)$ is called the p-Laplace operator associated with A. Since A^* is elliptic, analysis similar to that in the proof of Lemma 3.1 shows that the function $f := |Au|^{p-2}Au$ has weak limit values on the boundary belonging to $W^{-1/p',p'}(\partial\mathcal{X}, \mathbb{C}^l)$. If $\mathcal{S}$ is different from the entire boundary, then $f \equiv 0$ in $\mathcal{X}$, implying $Au = 0$ in $\mathcal{X}$. Therefore, the extremal problem $I(u) \mapsto \min$ over $u \in \mathcal{A}$ is equivalent to the Cauchy problem.

Let $R > 0$. Denote by $\mathcal{A}_R$ the subset of $W^{1,p}(\mathcal{X}, \mathbb{C}^k)$ consisting of all $u \in W^{1,p}(\mathcal{X}, \mathbb{C}^k)$, satisfying $u = u_0$ at $\mathcal{S}$ and, moreover,

$$\|u\|_{W^{1/p',p}(\partial\mathcal{X}\setminus\mathcal{S}, \mathbb{C}^k)} \leq R,$$

cf. (4.5). By the Sobolev trace theorem, $\mathcal{A}_R$ is a convex bounded closed subset of $\mathcal{A}$ and the family $\mathcal{A}_R$ exhausts $\mathcal{A}$.

THEOREM 5.1. *Suppose $R > 0$. Then, for each data $u_0 \in W^{1/p',p,\mathbb{C}^k}(\mathcal{S}, \mathbb{C}^k)$ with $1 < p < \infty$, the functional I takes on its minimum over $\mathcal{A}_R$ precisely at one function $u_R \in \mathcal{A}_R$.*

PROOF. This follows by the same method as in Theorem 4.4. The details are left to the reader. □

The behaviour of u_R for large R proves thus to be crucial for the solvability of the Cauchy problem. If the family $\{u_R\}_{R>0}$ is bounded in $W^{1,p}(\mathcal{X}, \mathbb{C}^k)$, then it stabilises for sufficiently large R. The limit function u is the minimum of I over all of $\mathcal{A}$, and so u is a solution of mixed problem (5.1), provided that $\mathcal{S}$ is different from $\partial\mathcal{X}$.

COROLLARY 5.2. *Let* $u_0 \in W^{1/p',p}(\mathcal{S}, \mathbb{C}^k)$*, where* $\mathcal{S} \neq \partial \mathcal{X}$*. For the existence of a function* $u \in W^{1,p}(\mathcal{X}, \mathbb{C}^k)$ *satisfying* $Au = 0$ *in* $\mathcal{X}$ *and* $u = u_0$ *at* $\mathcal{S}$ *it is necessary and sufficient that the family* $\{u_R\}_{R>0}$ *of Theorem 5.2 be bounded in* $W^{1,p}(\mathcal{X}, \mathbb{C}^k)$*.*

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