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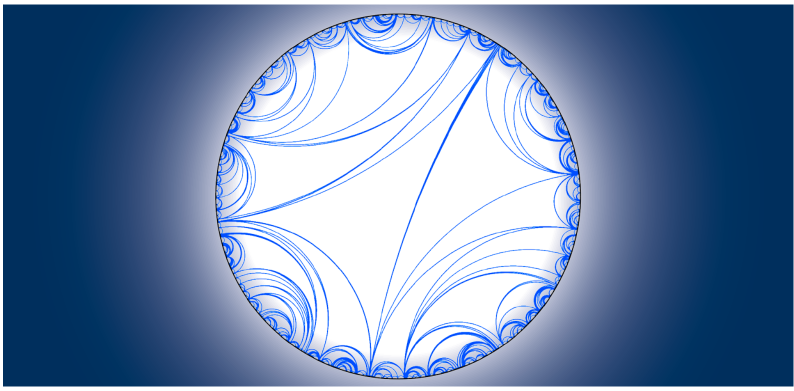

Yulia Bagderina | Nikolai Tarkhanov

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Differential invariants of a class of Lagrangian systems with two degrees of freedom

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Abstract

We consider systems of Euler-Lagrange equations with two degrees of freedom and with Lagrangian being quadratic in velocities. For this class of equations the generic case of the equivalence problem is solved with respect to point transformations. Using Lie's infinitesimal method we construct a basis of differential invariants and invariant differentiation operators for such systems. We describe certain types of Lagrangian systems in terms of their invariants. The results are illustrated by several examples.

Keywords:
Equivalence, Differential invariant, Euler-Lagrange equations

1. Introduction

Applying the variational principle in mechanics one reduces mechanical problems to systems of ordinary differential equations (ODEs) of the form

$$\frac{d}{dt}L_{\dot{x}_i} - L_{x_i} = 0, \quad i = 1, \dots, n, \quad \dot{x}_i = \frac{dx_i}{dt}. \tag{1}$$

known as Euler-Lagrange equations. While any scalar second-order ODE has a Lagrangian representation, for $n \geq 2$ there are systems which fail to admit Lagrangians $L(t, x, \dot{x})$, $x = (x_1, \dots, x_n)$, $\dot{x} = (\dot{x}_1, \dots, \dot{x}_n)$. Corresponding criteria for a system of two second-order ODEs are established in [1].

It is known [2] that the class of equations (1) is closed with respect to point transformations. That is, any nondegenerate change of variables

$$\tilde{t} = \theta(t, x), \quad \tilde{x}_i = \varphi_i(t, x), \quad i = 1, \dots, n \tag{2}$$

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transforms system (1) to a system of the same form

$$\frac{d}{d\tilde{t}}\tilde{L}_{\dot{\tilde{x}}_i} - \tilde{L}_{\tilde{x}_i} = 0, \quad i = 1, \ldots, n, \tag{3}$$

with possibly different Lagrangian $\tilde{L}(\tilde{t}, \tilde{x}, \dot{\tilde{x}})$. A point change of variables (2) is called canonical if the Lagrangian $\tilde{L}(\tilde{t}, \tilde{x}, \dot{\tilde{x}})$ of the transformed system (3) coincides with the Lagrangian $L(t, x, \dot{x})$ of system (1) written in variables (2). Noncanonical changes of variables are accepted as more interesting [3], since they can transform system (1) into a system (3) with more simple Lagrangian (e.g., several of its variables become cyclic or the resulting system is decoupled).

If system (1) admits a variational symmetry (or a constant of motion in the case of Hamiltonian representation of system (1)), one can reduce the order $2n$ of the system (1) to $2n - 2$. Solution of the so-called integrable systems which possess sufficient number of constants of motion can be reduced to quadratures. At the beginning of the 20 th century it became known from the works of H. Poincaré that the global existence of constants of motion is rather exceptional case. Many examples of integrable systems are considered in [4]. In particular, there are listed integrable natural systems with two degrees of freedom having the Lagrangian of the form $L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - U(x_1, x_2)$.

It should be noticed that such properties of system (1) as the natural form of its Lagrangian, representation in the form of Liouville system or in decoupled form are not invariant under arbitrary change of variables (2). As usual, these properties become implicit for the transformed system (3). A system of Euler-Lagrange equations one obtains in some applications may possess such hidden properties. The problem of finding the simplest equivalent form for a given system (1) can be solved with the use of invariants of equations (1).

Transformation (2) is an equivalence transformation of the class of equations (1), i.e. the most general transformation preserving the form of equations. Two systems (1) and (3) are said to be equivalent, if there is an invertible change of variables (2) which transforms the systems to each other. The equivalence problem can be solved using the invariants of the equivalence transformation group of a given class of equations. Indeed, if systems (1) and (3) are equivalent with respect to a point transformation (2) then their invariants coincide, i.e.

$$I_j(t, x, \dot{x}) = \tilde{I}_j(\tilde{t}, \tilde{x}, \dot{\tilde{x}}), \quad j = 1, 2, 3, \ldots. \tag{4}$$

By invariants of system (1) are meant the invariants of its group E of equivalence transformations. The invariants of some subgroup of E are called relative invariants of system (1).

The group of transformations may possess infinitely many differential invariants depending on arbitrary element of the given class of equations and its derivatives,

$$I_j = I_j(t, x, \dot{x}, L, L_t, L_x, L_{\dot{x}}, L_{tt}, L_{tx}, \ldots, L_{\dot{x}\cdots\dot{x}}). \tag{5}$$

The order of an invariant I_j is defined by the highest order of the derivatives of function L which are involved in I_j. The invariant (5) takes the form $I_j = I_j(t, x, \dot{x})$ (just as in equalities (4)) when we substitute the given function $L(t, x, \dot{x})$ into (5). As follows from [5, 6], the infinite set of differential invariants of the transformation group possesses a finite basis in the sense that an arbitrary invariant of the group can be obtained from basis invariants by algebraic operations and invariant differentiations. The operators $\mathcal{D}$ of invariant differentiation bear the property that if I is an invariant of system (1) then $\mathcal{D}I$ is its invariant, too. The number of such independent operators just amounts to the number of arguments in an arbitrary element (function L) of the given class of equations.

The equivalence problem in the case of one dependent variable has been solved for the first-order Lagrangians [7] and Lagrangians of higher order in [8, 9, 10]. In the present paper we solve the generic case of the equivalence problem for a system of Euler-Lagrange equations when $n = 2$. Furthermore, since most of the systems arising in applications are natural, we restrict our attention to constructing the invariants for equations with Lagrangians depending quadratically on velocities $\dot{x}$:

$$\frac{d}{dt}L_{\dot{x}_1} - L_{x_1} = 0, \quad \frac{d}{dt}L_{\dot{x}_2} - L_{x_2} = 0, \tag{6}$$

$$L_{\dot{x}_1\dot{x}_1\dot{x}_1} = 0, \quad L_{\dot{x}_1\dot{x}_1\dot{x}_2} = 0, \quad L_{\dot{x}_1\dot{x}_2\dot{x}_2} = 0, \quad L_{\dot{x}_2\dot{x}_2\dot{x}_2} = 0. \tag{7}$$

This problem has not been studied yet. Only for the case of two dependent and two independent variables it has been solved with respect to linear changes of variables for the quadratic Lagrangians with constant coefficients [11]. In [12] for a system of n second-order ODEs some fundamental relative invariants are introduced and the criteria of its equivalence to the simplest form $\ddot{\tilde{x}} = 0$ are obtained. Note that for systems (1) with two degrees of freedom some symmetry properties (which may be used for their integration) were previously studied in [13, 14, 15]. More precisely, in [13] one deals with Lie point symmetries of autonomous systems (6), (7). The recent papers [14, 15] are devoted to constructing the Noether type symmetries and first integrals for those particular systems (6) which are equivalent in a complex domain to a single equation $d^2u/dt^2 = f(t, u, du/dt)$ for a complex-valued function $u = x_1 + ix_2$, where $i^2 = -1$.

It can be easily shown that the class of equations (6), (7) is closed with respect to the point transformations of the form

$$\tilde{t} = \theta(t), \qquad \tilde{x}_1 = \varphi_1(t, x), \qquad \tilde{x}_2 = \varphi_2(t, x). \tag{8}$$

In Section 2 we apply Lie's infinitesimal method (see its description, e.g., in [6, 16, 17] and examples of application in [18, 19, 20, 21]) to construct a basis of invariants of the corresponding equivalence transformation group and to compute the operators of invariant differentiation. Note also that there exist other methods for finding the invariants, namely, Cartan's equivalence method [22, 23] and an approach based on using pseudovector fields [24]. In Section 3 we apply formulas of Section II to specify invariants for some classes of Lagrangian systems. Finally, in Section 4 we consider several examples showing how the invariants work in solving the equivalence problem.

2. Invariants of systems with quadratic Lagrangian

Suppose that the Hessian of the function $L(t, x, \dot{x})$ with respect to the velocities $\dot{x}$ does not vanish identically. Then the system (6), (7) is solved with respect to the second-order derivatives in the form

$$\ddot{x}_1 = f_1(t, x, \dot{x}), \quad \ddot{x}_2 = f_2(t, x, \dot{x}).$$

Here the solvability is rather formal, e.g., in the sense of formal power series. In constructing the invariants of system (6), (7) we use the operator of differentiation by virtue of system (6), (7)

$$D_0 = D_t + p_1 D_{x_1} + p_2 D_{x_2} + f_1 D_{p_1} + f_2 D_{p_2}$$

and the operators

$$D_i = D_{x_i} + \frac{1}{2}(f_{1p_i} D_{p_1} + f_{2p_i} D_{p_2}), \quad i = 1, 2,$$

where $f_{jp_i} = D_{p_i} f_j$ and D_t, D_{x_j}, D_{p_j} are the operators of total differentiation with respect to t, x_j, p_j, respectively (e.g., $D_t = \partial_t + L_t\partial_L + L_{tt}\partial_{L_t} + \sum_{i=1}^{2}(L_{tx_i}\partial_{L_{x_i}} + L_{tp_i}\partial_{L_{p_i}}) + \ldots$ and so on). To avoid confusion, in this section and in the Appendix we use the notation $p_i = \dot{x}_i$ for the first-order derivatives. The following theorem provides a solution of the equivalence problem for systems of Euler-Lagrange equations with two degrees of freedom.

Theorem 1. *For a class of systems (6), (7) with non-vanishing relative invariants j_0, J_0, I_0 the following nine fifth-order invariants*

$$I_1 = \frac{J_0 J_1}{j_0^{1/2} I_0}, \quad I_2 = \frac{J_0^{1/2} J_2}{I_0}, \quad I_3 = \frac{j_0^{1/2} J_0^{3/2} J_3}{I_0}, \quad I_4 = \frac{J_4}{j_0^{1/2} J_0^{5/4}}, \quad I_5 = \frac{J_0^{1/2} J_5}{j_0^{1/2} I_0},$$
$$I_6 = \frac{J_0^{5/4} J_6}{I_0}, \quad I_7 = \frac{j_0^{1/2} J_0^{3/4} J_7}{I_0}, \quad I_8 = \frac{J_0^{1/4} J_8}{j_0^{1/2} I_0}, \quad I_9 = \frac{J_0^{1/4} J_9}{j_0^{1/2} I_0} \tag{9}$$

form a basis of differential invariants with respect to point transformations (8). The invariant differentiations are defined by the operators

$$\begin{aligned}
\mathcal{D}_0 &= J_0^{-1/4} D_0,\\
\mathcal{D}_1 &= j_0^{1/2} J_0^{3/4} I_0^{-1}\Big((b_1 a_2 - b_2 a_1) D_1 + (b_1 a_1 - b_0 a_2) D_2\\
&\quad + (b_2 A_1 - b_1 A_2) D_{p_1} + (b_0 A_2 - b_1 A_1) D_{p_2}\Big),\\
\mathcal{D}_2 &= J_0^{5/4} I_0^{-1}\Big((L_{p_1p_2} a_2 - L_{p_2p_2} a_1) D_1 + (L_{p_1p_2} a_1 - L_{p_1p_1} a_2) D_2\\
&\quad + (L_{p_2p_2} A_1 - L_{p_1p_2} A_2) D_{p_1} + (L_{p_1p_1} A_2 - L_{p_1p_2} A_1) D_{p_2}\Big),\\
\mathcal{D}_3 &= j_0^{1/2} J_0 I_0^{-1}\Big((b_1 a_2 - b_2 a_1) D_{p_1} + (b_1 a_1 - b_0 a_2) D_{p_2}\Big),\\
\mathcal{D}_4 &= J_0^{3/2} I_0^{-1}\Big((L_{p_1p_2} a_2 - L_{p_2p_2} a_1) D_{p_1} + (L_{p_1p_2} a_1 - L_{p_1p_1} a_2) D_{p_2}\Big).
\end{aligned} \tag{10}$$

Any other differential invariant of system (6), (7) is a function of invariants (9) and their invariant derivatives.

The proof of Theorem 1 is given in Appendix. Invariants (9) are functions of variables

$$\begin{aligned}
&j_0 = L_{p_1p_1} L_{p_2p_2} - L_{p_1p_2}^2, \quad I_0 = b_0(\Gamma_2^2 - \Gamma_1\Gamma_3) + b_1(\Gamma_0\Gamma_3 - \Gamma_1\Gamma_2) + b_2(\Gamma_1^2 - \Gamma_0\Gamma_2),\\
&J_0 = b_1^2 - b_0 b_2, \quad J_1 = L_{p_2p_2} a_1^2 - 2L_{p_1p_2} a_1 a_2 + L_{p_1p_1} a_2^2, \quad J_2 = b_2 a_1^2 - 2b_1 a_1 a_2 + b_0 a_2^2,\\
&J_3 = (D_{p_1} D_2^2 - D_{p_2} D_2 D_1) L_{p_1} + (D_{p_2} D_1^2 - D_{p_1} D_1 D_2) L_{p_2},\\
&J_4 = L_{p_1p_1}(b_2 B_1 - b_1 B_2) + L_{p_1p_2}(b_0 B_2 - b_2 B_0) + L_{p_2p_2}(b_1 B_0 - b_0 B_1),\\
&J_5 = L_{p_1p_1}(\Gamma_2^2 - \Gamma_1\Gamma_3) + L_{p_1p_2}(\Gamma_0\Gamma_3 - \Gamma_1\Gamma_2) + L_{p_2p_2}(\Gamma_1^2 - \Gamma_0\Gamma_2),\\
&J_6 = L_{p_2p_2} E_0 - 2L_{p_1p_2} E_1 + L_{p_1p_1} E_2, \quad J_7 = b_2 E_0 - 2b_1 E_1 + b_0 E_2,\\
&J_8 = (L_{p_2p_2} a_1 - L_{p_1p_2} a_2)(b_2\Gamma_0 - 2b_1\Gamma_1 + b_0\Gamma_2) + (L_{p_1p_1} a_2 - L_{p_1p_2} a_1)(b_2\Gamma_1 - 2b_1\Gamma_2 + b_0\Gamma_3),\\
&J_9 = (b_2 a_1 - b_1 a_2)(L_{p_2p_2}\Gamma_0 - 2L_{p_1p_2}\Gamma_1 + L_{p_1p_1}\Gamma_2)\\
&\qquad + (b_0 a_2 - b_1 a_1)(L_{p_2p_2}\Gamma_1 - 2L_{p_1p_2}\Gamma_2 + L_{p_1p_1}\Gamma_3)
\end{aligned} \tag{11}$$

depending on relative invariants of the fourth order

$$a_i = 2(D_2D_1 - D_1D_2)L_{p_i}, \quad b_1 = \frac{1}{4}D_0(f_{1p_1} - f_{2p_2}) + \frac{1}{8}(f_{2p_2}^2 - f_{1p_1}^2) + \frac{1}{2}(f_{2x_2} - f_{1x_1}),$$
$$b_0 = -\beta_1, \quad b_2 = \beta_2, \quad \beta_i = \frac{1}{2}D_0 f_{kp_i} - \frac{1}{4}f_{kp_i}(f_{1p_1} + f_{2p_2}) - f_{kx_i}, \tag{12}$$

where $i = 1, 2$, $k = 3 - i$, and relative invariants of the fifth order

$$A_i = \frac{1}{3}\Big(D_0a_i + a_i(f_{ip_i} + \frac{1}{2}f_{kp_k}) + \frac{1}{2}a_kf_{kp_i}\Big), \quad B_0 = D_0b_0 + \frac{1}{2}b_0(f_{1p_1} - f_{2p_2}) + b_1f_{2p_1},$$
$$B_1 = D_0b_1 + \frac{1}{2}(b_0f_{1p_2} + b_2f_{2p_1}), \quad B_2 = D_0b_2 + b_1f_{1p_2} + \frac{1}{2}b_2(f_{2p_2} - f_{1p_1}),$$
$$\Gamma_0 = \gamma_0, \quad \Gamma_1 = \gamma_1 + 2A_1, \quad \Gamma_2 = \gamma_2 - 2A_2, \quad \Gamma_3 = \gamma_3,$$
$$\gamma_{2i+j-3} = (4D_0D_iD_j - 4D_iD_0D_j - 2D_jD_0D_i)L_{p_i} + 2D_jL_{x_ix_i} + 4D_iL_{x_ix_j}$$
$$-(-1)^ja_if_{lp_i} - 4L_{x_ix_ix_j} + \frac{1}{2}(L_{x_jp_1p_1}f_{1p_i}^2 + 2L_{x_jp_1p_2}f_{1p_i}f_{2p_i} + L_{x_jp_2p_2}f_{2p_i}^2)$$
$$+L_{x_ip_1p_1}f_{1p_i}f_{1p_j} + L_{x_ip_1p_2}(f_{1p_i}f_{2p_j} + f_{1p_j}f_{2p_i}) + L_{x_ip_2p_2}f_{2p_i}f_{2p_j},$$
$$E_0 = \varepsilon_{11}, \quad E_1 = \frac{1}{2}(\varepsilon_{12} + \varepsilon_{21}), \quad E_2 = \varepsilon_{22},$$
$$\varepsilon_{ij} = D_ja_i + a_iD_{p_j}(f_{ip_i} + \frac{1}{2}f_{kp_k}) + \frac{1}{2}a_kf_{kp_ip_j}, \quad i, j = 1, 2, \quad k = 3 - i, \quad l = 3 - j. \tag{13}$$

Theorem 1 solves the equivalence problem for non-degenerate systems (6), (7). The first assumption $j_0 \neq 0$ means that the Hessian of function L is nonzero. The restrictions $J_0 \neq 0$ and $I_0 \neq 0$ are not so clear. Note only that the invariants b_0, b_1, b_2 coincide with the invariants $\tilde{P}^i_j$ introduced in [12] for systems of second-order ODEs. As it follows from [12], any system with quadratic dependence of the right-hand side on the first-order derivatives and vanishing invariants $\tilde{P}^i_j$ is reducible by a local transformation (2) to the form $\ddot{\bar{x}} = 0$. The condition $b_1^2 - b_0b_2 = 0$ seems to characterize systems (6), (7) of sufficiently degenerate form. For example, in the case of natural Lagrangian $L = \frac{1}{2}(p_1^2 + p_2^2) - F$ with real potential function $F = F(t, x_1, x_2)$ the condition $J_0 = 0$ implies that the corresponding system of Euler-Lagrange equations is linear and decoupled (see Section 3.3).

3. Invariants of some classes of Lagrangian systems

Here we describe the invariants for some classes of easily integrable systems of Euler-Lagrange equations or systems having standard form.

3.1. Decoupled systems

A system of Euler-Lagrange equations breaks up into two decoupled equations

$$\ddot{x}_i = -g_i^{-1}\left(\frac{1}{2}\dot{x}_i^2g_{ix_i} + \dot{x}_ig_{i\,t}\right) + e_i(t, x_i), \quad i = 1, 2,$$

if the Lagrangian is equal to

$$L = \frac{1}{2}\left(\dot{x}_1^2g_1(t, x_1) + \dot{x}_2^2g_2(t, x_2)\right) + \dot{x}_1c_1(t, x_1, x_2) + \dot{x}_2c_2(t, x_1, x_2) + c_0(t, x_1, x_2), \tag{14}$$

the functions c_0, c_1, c_2 satisfying

$$c_{2x_1} - c_{1x_2} = 0, \quad c_{0x_i} = c_{i\,t} + g_i e_i, \quad i = 1, 2.$$

For a system with Lagrangian (14) the only nonzero relative invariants (11) are

$$j_0 = g_1 g_2, \quad J_0 = b_1^2, \quad I_0 = -16 g_1 g_2 b_1 b_{1x_1} b_{1x_2},$$

where

$$b_1 = \frac{1}{4}\sum_{i=1}^{2}(-1)^i g_i^{-1/2}\Big(2(e_i\sqrt{g_i})_{x_i} + (g_{i\,t}/\sqrt{g_i})_t\Big).$$

Thus, for this system all invariants (9) vanish:

$$I_1 = 0, \qquad \dots, \qquad I_9 = 0. \tag{15}$$

3.2. Systems with two cyclic coordinates

If the Lagrangian does not depend explicitly on x_i, for some i, then this coordinate is called cyclic. A system with Lagrangian $L = L(t, \dot{x}_1, \dot{x}_2)$ has all invariants (11) equal to zero, except for the invariants j_0, J_0, J_4 which depend on the variable t only.

3.3. Standard form of a natural system

Most of the systems arising in mechanics have the form of a system with Lagrangian

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - F(t, x_1, x_2) \tag{16}$$

in some coordinates. Such a system has the invariants

$$I_1 = 0, \quad I_2 = 0, \quad I_3 = 0, \quad I_4 = J_0^{-5/4}\Big(F_{x_1x_2}D_0(F_{x_1x_1} - F_{x_2x_2}) + (F_{x_2x_2} - F_{x_1x_1})D_0F_{x_1x_2}\Big),$$
$$I_5 = 4J_0^{1/2}I_0^{-1}\Big(F_{x_1x_1x_2}^2 - F_{x_1x_1x_1}F_{x_1x_2x_2} + F_{x_1x_2x_2}^2 - F_{x_1x_1x_2}F_{x_2x_2x_2}\Big),$$
$$I_6 = 0, \quad I_7 = 0, \quad I_8 = 0, \quad I_9 = 0,$$

where

$$J_0 = F_{x_1x_2}^2 + (F_{x_1x_1} - F_{x_2x_2})^2/4, \quad D_0 = \partial_t + \dot{x}_1\partial_{x_1} + \dot{x}_2\partial_{x_2},$$
$$I_0 = 4F_{x_1x_2}(F_{x_1x_1x_2}^2 - F_{x_1x_1x_1}F_{x_1x_2x_2} - F_{x_1x_2x_2}^2 + F_{x_1x_1x_2}F_{x_2x_2x_2})$$
$$+2(F_{x_1x_1} - F_{x_2x_2})(F_{x_1x_1x_1}F_{x_2x_2x_2} - F_{x_1x_1x_2}F_{x_1x_2x_2}).$$

Therefore, any system (6), (7) reducible by a transformation (8) to the standard form with Lagrangian (16) should have zero invariants I_1, I_2, I_3, I_6, I_7, I_8, I_9, invariant I_5 depending on t, x_1, x_2 only, and invariant I_4 depending linearly on $\dot{x}_1$, $\dot{x}_2$.

Remark. Suppose $I_4 = 0$ for the system with Lagrangian (16). If we split this equality by powers of $\dot{x}_1$, $\dot{x}_2$, we obtain three relations which imply that either $F_{x_1x_2} = 0$, and then this system is decoupled, or $F_{x_1x_1} - F_{x_2x_2} = 2cF_{x_1x_2}$ (if $F_{x_1x_2} \neq 0$), where c is a constant. From this equality it follows that $I_5 = 0$, too. Therefore, the function $F(t, x_1, x_2)$ has the form

$$F = F_1(t, x_2 + (c + \sqrt{c^2+1})x_1) + F_2(t, x_2 + (c - \sqrt{c^2+1})x_1)$$

and the corresponding system of Euler-Lagrange equations becomes decoupled in the variables $\tilde{x}_i = x_2 + x_1(c + (-1)^i\sqrt{c^2+1})$.

4. Examples of equivalent systems

We now consider a few examples, which illustrate application of the invariants of Euler-Lagrange equations to solving the equivalence problem.

Example 1. The Hénon-Heiles system [26]

$$\ddot{q}_1 + \omega_1 q_1 = bq_1^2 - aq_2^2, \quad \ddot{q}_2 + \omega_2 q_2 = -2aq_1q_2, \quad a, b, \omega_1, \omega_2 = \text{const}, \tag{17}$$

has the Langrangian

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 - \omega_1 q_1^2 - \omega_2 q_2^2\right) - aq_1q_2^2 + \frac{1}{3}bq_1^3$$

and, when $b \neq a$, the invariants

$$I_1 = 0, \quad I_2 = 0, \quad I_3 = 0, \quad I_4 = a\Big(4(a+b)(q_1\dot{q}_2 - q_2\dot{q}_1) + 2(\omega_2 - \omega_1)\dot{q}_2\Big)J_0^{-5/4},$$
$$I_5 = \frac{(a+b)\sqrt{J_0}}{2a(b-a)q_2}, \quad I_6 = 0, \qquad I_7 = 0, \quad I_8 = 0, \quad I_9 = 0,$$

where $J_0 = (a+b)^2q_1^2 + 4a^2q_2^2 + (a+b)(\omega_2 - \omega_1)q_1 + (\omega_2 - \omega_1)^2/4$. System (17) may be reduced to a decoupled form if it has invariants of the form (15). It is readily seen that system (17) has vanishing invariants I_4, I_5 when $b/a = -1$, $\omega_2 = \omega_1$, in which case system (17) breaks up in the variables $q_1 \pm q_2$, as it is noticed in [26].

The latter relation on the parameters of the system specifies one of the three integrable cases where q_1 satisfies a fourth-order ODE corresponding to the stationary solution of the Sawada-Kotera equation [27, 28]. Here we have established that this is the only case when the system reduces to a decoupled form after a transformation of the form (8).

Example 2. The paper [29] studies a three-dimensional system whose maximal superintegrability depends on the existence of an extra constant of motion for the two-dimensional system with Hamiltonian

$$H = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\psi^2\right) + \frac{F(\psi)}{r^2}.$$

For the potential

$$F(\psi) = \frac{k}{\sin^2 \lambda\psi}, \qquad k = \text{const}, \qquad k \neq 0, \qquad \lambda \in N, \tag{18}$$

[29] gives an explicit expression of this extra first integral.

Here we consider the corresponding system of Euler-Lagrange equations

$$\ddot{r} = r\dot{\psi}^2 + 2\frac{F(\psi)}{r^3}, \qquad \ddot{\psi} = -\frac{2}{r}\dot{r}\dot{\psi} - \frac{F'(\psi)}{r^4} \tag{19}$$

with the Lagrangian $L = \frac{1}{2}(\dot{r} + r^2\dot{\psi}^2) - r^{-2}F(\psi)$. It has the invariants

$$I_1 = 0, \qquad I_2 = 0, \qquad I_3 = 0, \qquad I_4 = \frac{8r^2\dot{\psi}\Phi_1}{\sqrt{2F}\Phi_0^{5/4}}, \qquad I_5 = -\frac{\sqrt{\Phi_0}\Phi_1}{2\Phi_2},$$
$$I_6 = 0, \qquad I_7 = 0, \qquad I_8 = 0, \qquad I_9 = 0, \tag{20}$$

where

$$\Phi_0 = (x^{4/3}y + x^2 - 8)^2 + 36x^2, \quad \Phi_1 = 3x^{11/3}y\frac{dy}{dx} + x^{4/3}(x^2+40)y - (x^4 + 20x^2 + 64),$$
$$\Phi_2 = \frac{1}{x}(x^{4/3}y - 2x^2 - 8)\Phi_1 + \frac{2}{x}(2x^{4/3}y - x^2 - 4)\Phi_0,$$
$$x = \frac{F'}{F}, \quad y = F''\left(\frac{F}{F'^4}\right)^{1/3} - \left(\frac{F'}{F}\right)^{2/3}.$$

Let us find the conditions under which the invariants I_4, I_5 of system (19) vanish and, therefore, necessary condition of its reducibility to a decoupled system holds. These conditions become sufficient if we find a suitable change of variables (8) which transforms the system to a decoupled form.

The condition $\Phi_1 = 0$ represents an Abel equation of the second kind with general solution given by

$$C_1x^4(2x^{4/3}y - x^2 - 4) + C_2(2x^{4/3}y - x^2 - 16)^2\Phi_0 = 0, \qquad C_1, C_2 = \text{const}.$$

In the variables ψ, F this equality takes the form

$$C_1F'^4(2FF'' - 3F'^2 - 4F^2) + C_2(2FF'' - 3F'^2 - 16F^2)^2((F'' - 8F)^2 + 36F'^2) = 0. \quad (21)$$

Equation (21) can be easily integrated for certain values of parameters C_1, C_2.

If $C_2 = 0$ then the resulting equation $2FF'' - 3F'^2 - 4F^2 = 0$ has the solution

$$F(\psi) = \frac{1}{(c_1\sin\psi + c_2\cos\psi)^2}, \qquad c_1, c_2 = \text{const}. \quad (22)$$

It is not difficult to see that in the variables

$$x_1 = r(c_1\sin\psi + c_2\cos\psi), \qquad x_2 = r(c_1\cos\psi - c_2\sin\psi)$$

system (19), (22) breaks up into two decoupled equations

$$x_1'' = \frac{2(c_1^2 + c_2^2)}{x_1^3}, \qquad x_2'' = 0.$$

If $C_1 = 0$, $C_2 = 1$ then (21) reduces to the equation $2FF'' - 3F'^2 - 16F^2 = 0$, which has the solution

$$F(\psi) = \frac{1}{(c_1\sin 2\psi + c_2\cos 2\psi)^2}, \qquad c_1, c_2 = \text{const}. \quad (23)$$

In the variables

$$y_j = r(K_j\cos\psi - c_2\sin\psi), \qquad K_j = c_1 + (-1)^j\sqrt{c_1^2 + c_2^2}, \qquad j = 1, 2, \qquad c_2 \neq 0$$

system (19), (23) takes the form

$$y_1'' = \frac{2K_1^2}{y_1^3}, \qquad y_2'' = \frac{2K_2^2}{y_2^3}.$$

When $c_2 = 0$, system (19), (23) breaks up into

$$z_1'' = \frac{1}{2c_1^2 z_1^3}, \qquad z_2'' = \frac{1}{2c_1^2 z_2^3}$$

in the polar coordinates $z_1 = r\cos\psi$, $z_2 = r\sin\psi$.

In the general case a change of variables which reduces system (19) to a decoupled form can be found in two steps. The system (19) is first reduced to the system with the Lagrangian $L = \frac{1}{2}(\dot z_1^2 + \dot z_2^2) - (z_1^2 + z_2^2)^{-1}F(\arctan z_2/z_1)$ in the polar coordinates. And then one can use the remark of Section 3.3.

Note that potential (18) satisfies equality (21) only for the parameter values $\lambda = 1$ and $\lambda = 2$. This example and the previous one demonstrate that the integrability of a Hamiltonian system is not related directly to the separability of the corresponding system of Euler-Lagrange equations.

Example 3. The paper [30] deals with the generalized nonlinear Schrödinger equation

$$u_t - ia_0u_{xx} + a_3u_{xxx} - iN|u|^2u + a_1|u|^2u_x + a_2u(|u|^2)_x = 0, \quad N, a_j = \text{const}, \ \ a_3 \neq 0. \tag{24}$$

As usual, its particular solution is sought in the form

$$u = r(\tau)\exp(\varphi(\tau) - kt), \quad \tau = x - vt, \quad k, v = \text{const}. \tag{25}$$

Substituting (25) into (24) leads to the system of two third-order ODEs for $r(\tau)$, $\varphi(\tau)$. Once integrated, for the functions $r(\tau)$, $\phi(\tau) = \varphi'(\tau) - a_0/3a_3$ it takes the form of two second-order ODEs

$$\begin{aligned}
&rr'' - \frac{1}{2}(r'^2 + 3r^2\phi^2) + (\beta_0 + \beta_1)r^4 + \beta_2r^2 + c = 0, \quad c = \text{const},\\
&r\phi'' + 3r'\phi' + 3r''\phi - r\phi^3 + (4\beta_1r^3 + 2\beta_2r)\phi - \beta_3r^3 - \beta_4r = 0,
\end{aligned} \tag{26}$$

where

$$\beta_0 = \frac{a_2}{2a_3}, \ \beta_1 = \frac{a_1}{4a_3}, \ \beta_2 = \frac{a_0^2}{6a_3^2} - \frac{v}{2a_3}, \ \beta_3 = \frac{N}{a_3} - \frac{a_0a_1}{3a_3^2}, \ \beta_4 = \frac{k}{a_3} + \frac{va_0}{3a_3^2} - \frac{2a_0^3}{27a_3^3}.$$

When $\beta_0 = 0$, this is a system of Euler-Lagrange equations with Lagrangian

$$L = \frac{3}{2}\phi r'^2 + rr'\phi' + \frac{1}{2}r^2\phi^3 - (\beta_1r^4 + \beta_2r^2 + c)\phi + \frac{1}{4}\beta_3r^4 + \frac{1}{2}\beta_4r^2$$

having the invariants

$$I_1 = 0, \quad I_2 = 0, \quad I_3 = 0, \quad I_4 = \frac{3\Big(r\phi'F_0 + \phi r'(F_2 - F_0)\Big)}{2\sqrt{-r^2}J_0^{5/4}},$$

$$I_5 = -\frac{4\phi\sqrt{-r^2}F_2\sqrt{J_0}}{r\Big(15\phi^3(5F_1 - 2F_2) - F_0F_1 - F_1^2\Big)}, \quad I_6 = 0, \quad I_7 = 0, \quad I_8 = 0, \quad I_9 = 0,$$

where

$$\begin{aligned}
&J_0 = \frac{3}{4}\phi(F_1 - F_0), \quad F_1 = 30\phi^3 - 20\beta_1r^2\phi + 7\beta_3r^2 + 3\beta_4 - 12c\phi r^{-2},\\
&F_0 = 15\phi^3 - 7\beta_3r^2 - 3\beta_4, \quad F_2 = 30\phi^3 - 40\beta_1r^2\phi + 21\beta_3r^2 + 3\beta_4.
\end{aligned}$$

The invariants of system (26) with $\beta_0 = 0$ satisfy the conditions listed in Section 3.3. It is readily verified that in the variables

$$x_1 = \sqrt{r}(1 - r\phi), x_2 = i\sqrt{r}(1 + r\phi),$$

$i^2 = -1$, equations (25) assemble into a system with Lagrangian of the form (16). But note that in the variables $y_1 = \sqrt{r}$, $y_2 = r^{3/2}\phi$ the system has more simple (real) Lagrangian

$$L = y_1' y_2' + \frac{1}{4} y_2^3 y_1^{-5} - \frac{1}{2} y_2 (\beta_1 y_1^5 + \beta_2 y_1 + c y_1^{-3}) + \frac{1}{8}\beta_3 y_1^8 + \frac{1}{4}\beta_4 y_1^4.$$

The invariants I_4, I_5 of system (26) cannot be equal to zero. Hence, the system does not reduce to a decoupled system.

Example 4. The paper [31] studies the following two families of Hamiltonians H_1, H_2 and K_1, K_2 which define two-dimensional generalisations of the second Painléve transcendent (κ = const):

$$\begin{aligned}
H_1 &= P_1^2(Q_2 - Q_1 - t_1) + 2Q_2P_1P_2 + P_2^2 + 2P_1(Q_1^2 - t_1^2 + t_2Q_2) + 2P_2(Q_1Q_2 + t_1Q_2 + t_2) \\
&\quad + 2\kappa Q_1, \\
H_2 &= Q_2P_1^2 + 2P_1P_2 + 2P_1(Q_1Q_2 + t_1Q_2 + t_2) + 2P_2(Q_2^2 - Q_1 + t_1) + 2\kappa Q_2, \\
K_1 &= \frac{1}{2}(q_1 - q_2)^{-1}\left(p_1^2 - p_2^2 - p_1(2q_1^3 + 2\tau_2 q_1 + \tau_1) + p_2(2q_2^3 + 2\tau_2 q_2 + \tau_1)\right) - \kappa(q_1 + q_2), \\
K_2 &= \frac{1}{2}(q_1 - q_2)^{-1}\left(q_1p_2^2 - q_2p_1^2 - p_1 + p_2 + q_2p_1(2q_1^3 + 2\tau_2 q_1 + \tau_1)\right. \\
&\quad \left. - q_1p_2(2q_2^3 + 2\tau_2 q_2 + \tau_1)\right) + \kappa q_1 q_2.
\end{aligned}$$

Here, for H_1, t_1 is an independent variable and t_2 is a parameter. For H_2, t_1 is a parameter and t_2 is an independent variable. In a similar manner τ_1, τ_2 are thought of with respect to Hamiltonians K_1, K_2.

The Lagrangian

$$\begin{aligned}
L_2 &= \frac{1}{2}\dot{Q}_1\dot{Q}_2 - \frac{1}{4}Q_2\dot{Q}_2^2 + (\dot{Q}_1 + Q_2^3 - 3Q_1Q_2 - t_1Q_2 - 2t)(Q_1 - Q_2^2 - t_1) \\
&\quad + \dot{Q}_2(Q_2^3 - 2Q_1Q_2 - t) - 2\kappa Q_2, \quad t = t_2,
\end{aligned}$$

corresponds to the Hamiltonian H_2. In this case the system of Euler-Lagrange equations

$$\begin{aligned}
\ddot{Q}_1 &= \frac{1}{2}\dot{Q}_2^2 + 2(1 - 3Q_1^2 - Q_2^4 + t_1^2) + 4(3Q_1Q_2^2 + t_1(Q_1 + Q_2^2) + tQ_2 - \kappa), \\
\ddot{Q}_2 &= 4(2Q_2^3 - 3Q_1Q_2 + t_1Q_2 - t) \qquad (27)
\end{aligned}$$

has the invariants

$$\begin{aligned}
&I_1 = 0, \quad I_2 = 0, \quad I_3 = 0, \quad I_4 = \frac{2(5Q_2^2\dot{Q}_1 - (5Q_2^3 + t)\dot{Q}_2 + Q_2)}{\sqrt{-6}(Q_2\Phi)^{5/4}}, \\
&I_5 = -\frac{5Q_2\sqrt{-Q_2}\Phi^{1/2}}{10Q_2^3 + 20Q_1Q_2 + 4t_1Q_2 + 2t}, \quad I_6 = 0, \quad I_7 = 0, \quad I_8 = 0, \quad I_9 = 0, \qquad (28)
\end{aligned}$$

where $\Phi = 5Q_2^3 - 2(5Q_1 + t_1)Q_2 - 2t$.

It is readily seen that the necessary condition of reducibility to the standard form (16) is fulfilled. Indeed, in the variables $y_1 = Q_1 - Q_2^2/4$, $y_2 = Q_2$ system (27) reduces to a system with Lagrangian

$$\tilde{L}_2 = \dot{y}_1\dot{y}_2 + t(3y_2^2 - 4y_1) - \frac{3}{8}y_2^5 + (5y_1 + t_1)y_2^3 + 2(2t_1y_1 - 3y_1^2 + 1 + t_1^2 - 2\kappa)y_2,$$

and in the variables $x_1 = Q_1 - Q_2^2/4 + Q_2$, $x_2 = i(Q_1 - Q_2^2/4 - Q_2)$, where $i^2 = -1$, it becomes a system with Lagrangian (16).

On using the invariants one can easily show that system (27) is equivalent to the system of Euler-Lagrange equations, which corresponds to the Hamiltonian K_1. This is the system with Lagrangian

$$\Lambda_1 = \frac{1}{2}(q_1 - q_2)(\dot{q}_1^2 - \dot{q}_2^2) + \frac{1}{2}(2\dot{q}_1 + q_1^2 + q_1q_2 + q_2^2 + \tau_2)(q_1^3 + \tau_2 q_1 + \tau/2)$$
$$+\frac{1}{2}(2\dot{q}_2 + q_1^2 + q_1q_2 + q_2^2 + \tau_2)(q_2^3 + \tau_2 q_2 + \tau/2) + \kappa(q_1 + q_2), \quad \tau = \tau_1,$$

having the invariants

$$I_1 = 0, \quad I_2 = 0, \quad I_3 = 0, \quad I_4 = \frac{2(10(q_1 + q_2)^2(q_1\dot{q}_1 + q_2\dot{q}_2) - \tau(\dot{q}_1 + \dot{q}_2) + q_1 + q_2)}{\sqrt{-6}(q_1 + q_2)^{5/4}\phi^{5/4}},$$
$$I_5 = -\frac{5(q_1 + q_2)\sqrt{-(q_1 + q_2)}\phi^{1/2}}{10(q_1^3 + 5q_1^2q_2 + 5q_1q_2^2 + q_2^3) - 8\tau_2(q_1 + q_2) - \tau},$$
$$I_6 = 0, \quad I_7 = 0, \quad I_8 = 0, \quad I_9 = 0, \quad \phi = 5(q_1^3 + q_1^2q_2 + q_1q_2^2 + q_2^3) + 4\tau_2(q_1 + q_2) + \tau.$$

It is not difficult to see that they coincide with invariants (28) of system (27) if

$$t = -\tau/2, \quad Q_1 = q_1q_2 + c, \quad Q_2 = q_1 + q_2, \quad c = \text{const},$$
$$\dot{Q}_1 = -2(q_2\dot{q}_1 + q_1\dot{q}_2), \quad \dot{Q}_2 = -2(\dot{q}_1 + \dot{q}_2), \quad t_1 = -2\tau_2 - 5c. \tag{29}$$

Substituting (29) into (27) shows that this transformation relates the corresponding systems of Euler-Lagrange equations with each other if and only if $c = -\tau_2/2$.

Similarly one can consider the Lagrangian

$$L_1 = \frac{\dot{Q}_2^2}{4} - \frac{(Q_2\dot{Q}_2 - \dot{Q}_1)^2}{4(Q_1 + t)} + (\dot{Q}_1 - Q_1^2 - t_2Q_2 + t^2)(Q_1 - Q_2^2 - t)$$
$$+(\dot{Q}_2 - Q_1Q_2 - tQ_2 - t_2)(Q_2^3 - 2Q_1Q_2 - t_2) - 2\kappa Q_1, \quad t = t_1,$$

corresponding to the Hamiltonian H_1, and the Lagrangian

$$\Lambda_2 = \frac{1}{2}(q_1 - q_2)\left(\frac{\dot{q}_2^2}{q_1} - \frac{\dot{q}_1^2}{q_2}\right) - \kappa q_1 q_2$$
$$+\left(\dot{q}_1 - \frac{1}{2}q_1q_2(q_1 + q_2) - \frac{1}{4}q_2(q_2^2 + \tau) + \frac{1}{8}(T_2 + q_1^{-1})\right)\left(q_1^3 + \tau q_1 + \frac{1}{2}(\tau_1 - q_2^{-1})\right)$$
$$+\left(\dot{q}_2 - \frac{1}{2}q_1q_2(q_1 + q_2) - \frac{1}{4}q_1(q_1^2 + \tau) + \frac{1}{8}(\tau_1 + q_2^{-1})\right)\left(q_2^3 + \tau q_2 + \frac{1}{2}(\tau_1 - q_1^{-1})\right),$$

where $\tau = \tau_2$, corresponding to the Hamiltonian K_2. Their invariants are too cumbersome and so are not given here. But similarly to the case of Lagrangians L_2 and Λ_1, the systems of Euler-Lagrange equations with Lagrangians L_1 and Λ_2 are related to each other by transformation

$$t = \tau/2, \quad Q_1 = q_1 q_2 - \tau/2, \quad Q_2 = q_1 + q_2, \quad t_2 = -2\tau_1. \tag{30}$$

Comparing (29), (30) one readily sees that the relations

$$t_1 = \frac{\tau_2}{2}, \quad t_2 = -\frac{\tau_1}{2}, \quad Q_1 = q_1 q_2 - \frac{\tau_2}{2}, \quad Q_2 = q_1 + q_2$$

define transformation which relate with each other the Euler-Lagrange equations corresponding to the Hamiltonians H_1, H_2 and K_2, K_1.

5. Conclusion

Integration of nonlinear equations proves to be a complicated problem. Applying invariants of a given class of equations allows one to reduce it to finding an equivalent equation with known solution or an equation being more simple for integration. The invariants may be effective also when we need to prove nonequivalence of two given equations or their irreducibility to a special form.

In the present paper a basis of invariants of Euler-Lagrange equations (1) is constructed when $n = 2$ and the Lagrangian has quadratic dependence on velocities. With a number of examples it is shown how the invariants can either facilitate the integration of a given system or prove the inefficiency of some known method for constructing an analytical solution of the system. Note that the equivalence problem for the more general class of Euler-Lagrange equations with $n > 2$ degrees of freedom remains open.

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Appendix A. Proof of Theorem 1

To prove the statement of Theorem 1 we use Lie's infinitesimal method [6, Chapter 7]. In the calculations presented here, we used the symbolic package Maple for some tedious computations and checking the results obtained. The invariants of system (6), (7) are found from the condition of their invariance under the infinitesimal operator

$$X = \xi_0(t,x)\partial_t + \xi_1(t,x)\partial_{x_1} + \xi_2(t,x)\partial_{x_2} + \eta(t,x,p,L,L_t,L_x,L_p,L_{tt},\ \dots\ ,L_{pp})\partial_L \tag{A.1}$$

corresponding to the group E of equivalence transformations of system (6), (7). When extended to the derivatives $p_j = \dot{x}_j$, $\ddot{x}_j$ and to the derivatives of L with respect to t, $x = (x_1, x_2)$, $p = (p_1, p_2)$, operator (A.1) should leave invariant the system (6), (7). If we set $(z_0, z_1, z_2, z_3, z_4) = (t, x_1, x_2, p_1, p_2)$, then the coordinates $\xi_3, \xi_4, \xi_5, \xi_6$ of extended operator (A.1)

$$X = \eta\partial_L + \sum_{j=0}^{4}\left(\xi_j\partial_{z_j} + \eta_j\partial_{L_{z_j}} + \sum_{k=0}^{j}\left(\eta_{jk}\partial_{L_{z_j z_k}} + \sum_{l=0}^{k}\eta_{jkl}\partial_{L_{z_j z_k z_l}}\right)\right) + \xi_5\partial_{\ddot{x}_1} + \xi_6\partial_{\ddot{x}_2}$$

are calculated by the standard prolongation formulas [6]

$$\xi_3 = \frac{d}{dt}\xi_1 - p_1\frac{d}{dt}\xi_0, \quad \xi_4 = \frac{d}{dt}\xi_2 - p_2\frac{d}{dt}\xi_0, \quad \xi_5 = \frac{d}{dt}\xi_3 - \ddot{x}_1\frac{d}{dt}\xi_0, \quad \xi_6 = \frac{d}{dt}\xi_4 - \ddot{x}_2\frac{d}{dt}\xi_0.$$

In order to calculate the coordinates η_j, η_{jk}, η_{jkl} we regard L as dependent variable and $z_0, \ldots, z_4$ as independent ones, thus obtaining

$$\eta_j = D_{z_j}\eta - \sum_{i=0}^{4} L_{z_i} D_{z_j}\xi_i, \quad \eta_{jk} = D_{z_k}\eta_j - \sum_{i=0}^{4} L_{z_j z_i} D_{z_k}\xi_i,$$
$$\eta_{jkl} = D_{z_l}\eta_{jk} - \sum_{i=0}^{4} L_{z_j z_k z_i} D_{z_l}\xi_i, \quad j,k,l = 0, \ldots, 4. \tag{A.2}$$

Action of X on system (6), (7) and substitution of $\ddot{x}_1$, $\ddot{x}_2$, $L_{p_1p_1p_1}, \ldots, L_{p_2p_2p_2}$ by virtue of this system provide six determining equations. On equating the coefficients of the same powers of the third and fourth-order derivatives of L in these equations one obtains the conditions

$$\eta_{L_{z_i}} = 0, \quad \eta_{L_{z_i z_j}} = 0, \quad i, j = 0, \ldots, 4,$$

i.e. $\eta = \eta(t, x, p, L)$. Then equating the coefficients of the same powers of the first- and second-order derivatives of L yields the conditions

$$\xi_{0x_i} = 0, \quad \eta_{LL} = 0, \quad \eta_{p_i L} = 0, \quad \eta_{p_i p_j} = 0, \quad i, j = 1, 2.$$

Substituting $\xi_0 = \tau(t)$, $\eta = F_0(t,x) + p_1F_1(t,x) + p_2F_2(t,x) + LF_3(t,x)$ into the determining equations and equating the coefficients of p_j, L, L_{p_j}, $j = 1, 2$, one immediately obtains

$$F_{0x_j} - F_{jt} = 0, \quad F_{1x_2} - F_{2x_1} = 0, \quad F_{3x_j} = 0, \quad F_{3t} + \tau_{tt} = 0, \quad j = 1, 2.$$

Therefore, with a function $\zeta = \zeta(t, x_1, x_2)$ and a constant c, the operator of the equivalence transformation group of system (6), (7) is given by

$$X = \tau(t)\partial_t + \sum_{j=1}^{2}\Big(\xi_j(t, x_1, x_2)\partial_{x_j} + (\xi_{jt} + p_j(\xi_{jx_j} - \tau_t) + p_{3-j}\xi_{jx_{3-j}})\partial_{p_j}\Big)$$
$$+(\zeta_t + p_1\zeta_{x_1} + p_2\zeta_{x_2} - L(\tau_t + c))\partial_L. \tag{A.3}$$

An arbitrary element L of system (6), (7) and its invariant (5) do not depend on variables $\ddot{x}_1$, $\ddot{x}_2$. So, here one needs merely an extension of operator X to the velocities $p_j = \dot{x}_j$. The function ζ and constant c in operator (A.3) result from the known fact that the multiplication of the Lagrangian by a non-zero constant and the addition of the total derivative of a function of t, x to L do not alter the corresponding system of Euler-Lagrange equations.

The fifth-order invariants of system (6), (7), which depend on 191 variables

$$t,\ x,\ p,\ L,\ L_t,\ L_x,\ L_p,\ L_{tt},\ L_{tx},\ L_{xx},\ L_{tp},\ L_{xp},\ L_{pp},$$
$$L_{ttt}, \ldots, L_{xpp}, L_{tttt}, \ldots, L_{xxpp}, L_{ttttt}, \ldots, L_{xxxpp}, \tag{A.4}$$

are found from the invariance condition $\tilde{X}I = 0$. We assume that all derivatives L_{ppp} are equal to zero, so the collection (A.4) does not include the derivatives L_{ppp}, L_{tppp}, L_{xppp}, L_{pppp}, $L_{ttppp}, \ldots, L_{ppppp}$. (Hereinafter we use the notation L_{ppp} for all third-order derivatives of L with respect to p_1, p_2, L_{pppp} for those of the fourth-order, and so on). Write $\tilde{X}$ for a fifth-order extension of operator (A.3)

$$\tilde{X} = X + \sum_{j=0}^{4}\Big(\eta_j\partial_{L_{z_j}} + \sum_{k=0}^{j}\Big(\eta_{jk}\partial_{L_{z_jz_k}} + \sum_{l=0}^{k}\Big(\eta_{jkl}\partial_{L_{z_jz_kz_l}} + \sum_{m=0}^{l}\Big(\eta_{jklm}\partial_{L_{z_jz_kz_lz_m}} + \sum_{n=0}^{m}\eta_{jklmn}\partial_{L_{z_jz_kz_lz_mz_n}}\Big)\Big)\Big)\Big),$$

with coordinates calculated by formulas (A.2) and

$$\eta_{jklm} = D_{z_m}\eta_{jkl} - \sum_{i=0}^{4} L_{z_jz_kz_lz_i}D_{z_m}\xi_i, \quad \eta_{jklmn} = D_{z_n}\eta_{jklm} - \sum_{i=0}^{4} L_{z_jz_kz_lz_mz_i}D_{z_n}\xi_i, \tag{A.5}$$

$j, k, l, m, n = 0, \dots, 4$. From (A.2), (A.3), (A.5) it is not difficult to see that the operator $\tilde{X}$ depends linearly on arbitrary functions ζ, ξ_1, ξ_2, τ and their derivatives up to the sixth order. On the other hand, an invariant I depends neither on these functions nor on their derivatives. Hence, according to the theory of invariants of infinite transformation groups [6], the relation $\tilde{X}I = 0$ should be split by these functions and their derivatives. This gives rise to a homogeneous system of linear first-order partial differential equations

$$X_0(c)I = 0, \quad X_1(\xi_1)I = 0, \quad \dots, \quad X_{258}(\partial^6\tau/\partial t^6)I = 0, \tag{A.6}$$

where every operator X_i is the coefficient of some derivative in $\tilde{X}$ (which is displayed in the parentheses). The functionally independent solutions of system (A.6) provide all independent differential invariants of system (6), (7) up to the fifth order.

The solution of system (A.6) is found in several steps. First we consider the subsystem of equations (A.6) with three operators $X_1(\xi_1) = \partial_{x_1}$, $X_2(\xi_2) = \partial_{x_2}$, $X_3(\tau) = \partial_t$ and 85 operators

$$\begin{aligned}
&X_{174}(\partial^6\zeta/\partial t^6) = \partial_{L_{ttttt}}, \quad X_{174+j}(\partial^6\zeta/\partial t^5\partial x_j) = \partial_{L_{ttttx_j}} + p_j\partial_{L_{ttttt}}, \ \dots,\\
&X_{195}(\partial^6\zeta/\partial x_1^6) = p_1\partial_{L_{x_1x_1x_1x_1x_1}}, \ \dots, \ X_{201}(\partial^6\zeta/\partial x_2^6) = p_2\partial_{L_{x_2x_2x_2x_2x_2}}, \ \dots,\\
&X_{201+j}(\partial^6\xi_j/\partial t^6) = -L_{p_j}\partial_{L_{ttttt}}, \ \dots, \ X_{258}(\partial^6\tau/\partial t^6) = (p_1L_{p_1} + p_2L_{p_2} - L)\partial_{L_{ttttt}},
\end{aligned} \tag{A.7}$$

$j = 1, 2$, being the coefficients of the sixth-order derivatives of ζ, ξ_1, ξ_2, τ in $\tilde{X}$ and acting on 21 variables L_{ttttt}, L_{ttttx}, $\dots$, L_{xxxxx}. Only 21 of these operators are independent and the remaining 64 operators are represented as their linear combinations. In the space of variables (A.4) the subsystem of equations (A.6) with operators X_1, X_2, X_3 and (A.7) has 167 functionally independent solutions

$$\begin{aligned}
&p,\ L,\ L_t,\ L_x,\ L_p,\ L_{tt},\ \dots,\ L_{pp},\ L_{ttt},\ \dots,\ L_{xpp},\\
&L_{tttt},\ \dots,\ L_{xxpp},\ L_{ttttp},\ \dots,\ L_{xxxxp},\ L_{tttpp},\ \dots,\ L_{xxxpp}.
\end{aligned}$$

In these variables the next 64 operators in system (A.6) (the coefficients of the fifth-order derivatives of ζ, ξ_1, ξ_2, τ in $\tilde{X}$) become

$$\begin{aligned}
&X_{110}(\partial^5\zeta/\partial t^5) = \partial_{L_{tttt}}, \quad X_{110+j}(\partial^5\zeta/\partial t^4\partial x_j) = \partial_{L_{tttx_j}} + p_j\partial_{L_{tttt}} + \partial_{L_{ttttp_j}}, \ \dots,\\
&X_{130+j}(\partial^5\xi_j/\partial t^5) = -L_{p_j}\partial_{L_{tttt}} - L_{p_1p_j}\partial_{L_{ttttp_1}} - L_{p_jp_2}\partial_{L_{ttttp_2}}, \ \dots,\\
&X_{173}(\partial^5\tau/\partial t^5) = -L\partial_{L_{tttt}} + \sum_{i=1}^{2} p_i(L_{p_i}\partial_{L_{tttt}} + L_{p_1p_i}\partial_{L_{ttttp_1}} + L_{p_ip_2}\partial_{L_{ttttp_2}}), \quad j = 1, 2,
\end{aligned}$$

acting on 45 variables L_{tttt}, L_{tttx}, ..., L_{xxxx}, L_{ttttp}, ..., L_{xxxxp}. Note that 19 of these operators are linear functions of the remaining 45 operators and the subsystem of the corresponding 64 equations (A.6) has 122 independent solutions

$$p,\ L,\ L_t,\ L_x,\ L_p,\ L_{tt},\ \dots,\ L_{pp},\ L_{ttt},\ \dots,\ L_{xpp},$$
$$L_{tttp},\ \dots,\ L_{xxxp},\ L_{ttpp},\ L_{txpp},\ L_{xxpp},\ L_{tttpp},\ \dots,\ L_{xxxpp}.$$

Further we consider the subsystem of 46 equations (A.6) with operators

$$X_{64}(\partial^4\zeta/\partial t^4) = \partial_{L_{ttt}}, \quad X_{64+j}(\partial^4\zeta/\partial t^3\partial x_j) = \partial_{L_{ttx_j}} + p_j\partial_{L_{ttt}} + \partial_{L_{tttp_j}}, \quad \dots,$$
$$X_{79+j}(\partial^4\xi_j/\partial t^4) = -L_{p_j}\partial_{L_{ttt}} - L_{p_1p_j}\partial_{L_{tttp_1}} - L_{p_jp_2}\partial_{L_{tttp_2}}, \quad \dots, \quad j = 1,2,$$
$$X_{109}(\partial^4\tau/\partial t^4) = -L\partial_{L_{ttt}} - \sum_{i=1}^{2} p_iX_{79+i} + L_{p_1p_1}\partial_{L_{tttp_1p_1}} + L_{p_1p_2}\partial_{L_{tttp_1p_2}} + L_{p_2p_2}\partial_{L_{tttp_2p_2}},$$

acting on 60 variables L_{ttt}, ..., L_{xxx}, L_{tttp}, ..., L_{xxxp}, L_{tttpp}, ..., L_{xxxpp}. As 76 independent solutions of this subsystem we can take

$$p,\ L,\ L_t,\ L_x,\ L_p,\ L_{tt},\ \dots, L_{pp},\ L_{ttp},\ L_{txp},\ L_{xxp},\ L_{tpp},\ L_{xpp},\ L_{ttpp},\ L_{txpp},\ L_{xxpp},$$
$$B_i,\ A_i,\ \Gamma_0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \varepsilon_{ij},\ \epsilon_i = D_iJ_3 - J_3j_0^{-1}D_ij_0, \quad i,j = 1,2,$$

where the variables B_i, A_i, Γ_0, Γ_1, Γ_2, Γ_3, ε_{ij} are defined by formulas (13). For the sake of symmetry one can add here a variable B_0 which is related to B_1, B_2 by the equality $L_{p_2p_2}B_0 - 2L_{p_1p_2}B_1 + L_{p_1p_1}B_2 = 0$.

In these variables the subsystem of 31 equations (A.6) with operators

$$X_{33}(\partial^3\zeta/\partial t^3) = \partial_{L_{tt}}, \quad X_{33+j}(\partial^3\zeta/\partial t^2\partial x_j) = \partial_{L_{tx_j}} + p_j\partial_{L_{tt}} + \partial_{L_{ttp_j}}, \quad \dots,$$
$$X_{42+j}(\partial^3\xi_j/\partial t^3) = -L_{p_j}\partial_{L_{tt}} - L_{p_1p_j}\partial_{L_{ttp_1}} - L_{p_jp_2}\partial_{L_{ttp_2}}, \quad \dots, \quad j = 1,2,$$
$$X_{63}(\partial^3\tau/\partial t^3) = -L\partial_{L_{tt}} - \sum_{i=1}^{2} p_iX_{42+i} + L_{p_1p_1}\partial_{L_{ttp_1p_1}} + L_{p_1p_2}\partial_{L_{ttp_1p_2}} + L_{p_2p_2}\partial_{L_{ttp_2p_2}},$$

has 45 functionally independent solutions

$$p,\ L,\ L_t,\ L_x,\ L_p,\ L_{tp},\ L_{xp},\ L_{pp},\ L_{tpp},\ L_{xpp},$$
$$b_i,\ a_i,\ J_3,\ B_i,\ A_i,\ \Gamma_0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \varepsilon_{ij},\ \epsilon_i, \quad i,j = 1,2, \tag{A.8}$$

where b_i, a_i, J_3 are defined by (11), (12). It is also convenient to add a variable b_0 satisfying $L_{p_2p_2}b_0 - 2L_{p_1p_2}b_1 + L_{p_1p_1}b_2 = 0$.

The next 18 operators of system (A.6) act on first 26 variables (A.8) only. These are

$$X_{14}(\zeta_{tt}) = \partial_{L_t}, \quad X_{14+j}(\zeta_{tx_j}) = \partial_{L_{x_j}} + p_j\partial_{L_t} + \partial_{L_{tp_j}}, \quad X_{17}(\zeta_{x_1x_1}) = p_1\partial_{L_{x_1}} + \partial_{L_{x_1p_1}},$$
$$X_{18}(\zeta_{x_1x_2}) = p_1\partial_{L_{x_2}} + p_2\partial_{L_{x_1}} + \partial_{L_{x_2p_1}} + \partial_{L_{x_1p_2}}, \quad X_{19}(\zeta_{x_2x_2}) = p_2\partial_{L_{x_2}} + \partial_{L_{x_2p_2}},$$
$$X_{19+j}(\xi_{j\,tt}) = -L_{p_j}\partial_{L_t} - L_{p_1p_j}\partial_{L_{tp_1}} - L_{p_jp_2}\partial_{L_{tp_2}},$$
$$X_{21+j}(\xi_{j\,tx_1}) = p_1X_{20} + L_{p_j}(p_1X_{14} - X_{15}) - L_{p_1p_j}(\partial_{L_{x_1p_1}} + 2\partial_{L_{tp_1p_1}}) - L_{p_jp_2}(\partial_{L_{x_1p_2}} + \partial_{L_{tp_1p_2}}),$$
$$X_{23+j}(\xi_{j\,tx_2}) = p_2X_{21} + L_{p_j}(p_2X_{14} - X_{16}) - L_{p_1p_j}(\partial_{L_{x_2p_1}} + \partial_{L_{tp_1p_2}}) - L_{p_jp_2}(\partial_{L_{x_2p_2}} + 2\partial_{L_{tp_2p_2}}),$$
$$X_{25+j}(\xi_{j\,x_1x_1}) = -L_{p_j}X_{17} - L_{p_1p_j}(p_1\partial_{L_{x_1p_1}} + 2\partial_{L_{x_1p_1p_1}}) - L_{p_jp_2}(p_1\partial_{L_{x_1p_2}} + \partial_{L_{x_1p_1p_2}}),$$
$$X_{27+j}(\xi_{j\,x_1x_2}) = -L_{p_j}X_{18} - \sum_{i=1}^{2} L_{p_ip_j}(p_2\partial_{L_{x_1p_i}} + p_1\partial_{L_{x_2p_i}} + \partial_{L_{x_ip_1p_2}} + 2\partial_{L_{x_{3-i}p_ip_i}}),$$
$$X_{29+j}(\xi_{j\,x_2x_2}) = -L_{p_j}X_{19} - L_{p_1p_j}(p_2\partial_{L_{x_2p_1}} + \partial_{L_{x_2p_1p_2}}) - L_{p_jp_2}(p_2\partial_{L_{x_2p_2}} + 2\partial_{L_{x_2p_2p_2}}), \tag{A.9}$$

$j = 1, 2$, having 27 independent invariants

$$p,\ L,\ L_p,\ L_{pp},\ b_i,\ a_i,\ J_3,\ B_i,\ A_i,\ \Gamma_0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \varepsilon_{ij},\ \epsilon_i,\quad i, j = 1, 2. \tag{A.10}$$

In these variables (where we add B_0 and b_0 for symmetry), the remaining 12 operators of system (A.6) look like

$$\begin{aligned}
X_{32}(\tau_{tt}) &= -2(b_0\partial_{B_0} + b_1\partial_{B_1} + b_2\partial_{B_2}) - \frac{1}{2}(a_1\partial_{A_1} + a_2\partial_{A_2}) + \frac{1}{2}J_3(\partial_{\varepsilon_{12}} - \partial_{\varepsilon_{21}}),\\
X_4(\zeta_t) &= \partial_L,\quad X_5(\zeta_{x_1}) = \partial_{L_{p_1}} + p_1\partial_L,\quad X_6(\zeta_{x_2}) = \partial_{L_{p_2}} + p_2\partial_L,\\
X_7(\xi_{1t}) &= \partial_{p_1},\quad X_8(\xi_{2t}) = \partial_{p_2},\\
X_9(\xi_{1x_1}) &= p_1\partial_{p_1} - L_{p_1}\partial_{L_{p_1}} - 2L_{p_1p_1}\partial_{L_{p_1p_1}} - L_{p_1p_2}\partial_{L_{p_1p_2}} - b_0\partial_{b_0} + b_2\partial_{b_2} - 2a_1\partial_{a_1}\\
&\quad -a_2\partial_{a_2} - 2J_3\partial_{J_3} - B_0\partial_{B_0} + B_2\partial_{B_2} - 2A_1\partial_{A_1} - A_2\partial_{A_2} - 3\Gamma_0\partial_{\Gamma_0} - 2\Gamma_1\partial_{\Gamma_1}\\
&\quad -\Gamma_2\partial_{\Gamma_2} - 3\varepsilon_{11}\partial_{\varepsilon_{11}} - 2\varepsilon_{12}\partial_{\varepsilon_{12}} - 2\varepsilon_{21}\partial_{\varepsilon_{21}} - \varepsilon_{22}\partial_{\varepsilon_{22}} - 3\epsilon_1\partial_{\epsilon_1} - 2\epsilon_2\partial_{\epsilon_2},\\
X_{10}(\xi_{2x_1}) &= p_1\partial_{p_2} - L_{p_2}\partial_{L_{p_1}} - 2L_{p_1p_2}\partial_{L_{p_1p_1}} - L_{p_2p_2}\partial_{L_{p_1p_2}} - 2b_1\partial_{b_0} - b_2\partial_{b_1}\\
&\quad -a_2\partial_{a_1} - 2B_1\partial_{B_0} - B_2\partial_{B_1} - A_2\partial_{A_1} - 3\Gamma_1\partial_{\Gamma_0} - 2\Gamma_2\partial_{\Gamma_1} - \Gamma_3\partial_{\Gamma_2}\\
&\quad -(\varepsilon_{12} + \varepsilon_{21})\partial_{\varepsilon_{11}} - \varepsilon_{22}(\partial_{\varepsilon_{12}} + \partial_{\varepsilon_{21}}) - \epsilon_2\partial_{\epsilon_1},\\
X_{11}(\xi_{1x_2}) &= p_2\partial_{p_1} - L_{p_1}\partial_{L_{p_2}} - L_{p_1p_1}\partial_{L_{p_1p_2}} - 2L_{p_1p_2}\partial_{L_{p_2p_2}} - b_0\partial_{b_1} - 2b_1\partial_{b_2}\\
&\quad -a_1\partial_{a_2} - B_0\partial_{B_1} - 2B_1\partial_{B_2} - A_1\partial_{A_2} - \Gamma_0\partial_{\Gamma_1} - 2\Gamma_1\partial_{\Gamma_2} - 3\Gamma_2\partial_{\Gamma_3}\\
&\quad -\varepsilon_{11}(\partial_{\varepsilon_{12}} + \partial_{\varepsilon_{21}}) - (\varepsilon_{12} + \varepsilon_{21})\partial_{\varepsilon_{22}} - \epsilon_1\partial_{\epsilon_2},\\
X_{12}(\xi_{2x_2}) &= p_2\partial_{p_2} - L_{p_2}\partial_{L_{p_2}} - L_{p_1p_2}\partial_{L_{p_1p_2}} - 2L_{p_2p_2}\partial_{L_{p_2p_2}} + b_0\partial_{b_0} - b_2\partial_{b_2} - a_1\partial_{a_1}\\
&\quad -2a_2\partial_{a_2} - 2J_3\partial_{J_3} + B_0\partial_{B_0} - B_2\partial_{B_2} - A_1\partial_{A_1} - 2A_2\partial_{A_2} - \Gamma_1\partial_{\Gamma_1} - 2\Gamma_2\partial_{\Gamma_2}\\
&\quad -3\Gamma_3\partial_{\Gamma_3} - \varepsilon_{11}\partial_{\varepsilon_{11}} - 2\varepsilon_{12}\partial_{\varepsilon_{12}} - 2\varepsilon_{21}\partial_{\varepsilon_{21}} - 3\varepsilon_{22}\partial_{\varepsilon_{22}} - 2\epsilon_1\partial_{\epsilon_1} - 3\epsilon_2\partial_{\epsilon_2},\\
X_{13}(\tau_t) &= -L\partial_L + L_{p_1p_1}\partial_{L_{p_1p_1}} + L_{p_1p_2}\partial_{L_{p_1p_2}} + L_{p_2p_2}\partial_{L_{p_2p_2}} + J_3\partial_{J_3}\\
&\quad + \sum_{i=1}^{2}\Big(- p_i\partial_{p_i} - 2b_i\partial_{b_i} - 3B_i\partial_{B_i} - A_i\partial_{A_i} + \epsilon_i\partial_{\epsilon_i} + \sum_{j=1}^{2}\Gamma_{2i+j-3}\partial_{\Gamma_{2i+j-3}}\Big),\\
X_0(c) &= -L\partial_L - L_{p_1p_1}\partial_{L_{p_1p_1}} - L_{p_1p_2}\partial_{L_{p_1p_2}} - L_{p_2p_2}\partial_{L_{p_2p_2}} - J_3\partial_{J_3}\\
&\quad - \sum_{i=1}^{2}\Big(L_{p_i}\partial_{L_{p_i}} + a_i\partial_{a_i} + A_i\partial_{A_i} + \epsilon_i\partial_{\epsilon_i} + \sum_{j=1}^{2}(\Gamma_{2i+j-3}\partial_{\Gamma_{2i+j-3}} + \varepsilon_{ij}\partial_{\varepsilon_{ij}})\Big).
\end{aligned} \tag{A.11}$$

The subsystem of nine equations $X_iI = 0$, $(X_9 - X_{12})I = 0$, $i = 4, \ldots, 8, 10, 11, 32$, has 18 functionally independent solutions (11) and

$$\begin{aligned}
J_{10} &= a_2A_1 - a_1A_2,\quad J_{11} = a_2^2E_0 - 2a_1a_2E_1 + a_1^2E_2,\\
J_{12} &= a_2\epsilon_1 - a_1\epsilon_2,\quad J_{13} = (L_{p_2p_2}a_1 - L_{p_1p_2}a_2)\epsilon_1 + (L_{p_1p_1}a_2 - L_{p_1p_2}a_1)\epsilon_2,\\
J_{14} &= (L_{p_2p_2}a_1 - L_{p_1p_2}a_2)A_1 + (L_{p_1p_1}a_2 - L_{p_1p_2}a_1)A_2 + \frac{J_1}{8J_0}(b_2B_0 - 2b_1B_1 + b_0B_2),\\
J_{15} &= \varepsilon_{21} - \varepsilon_{12} + \frac{J_3}{4J_0}(b_2B_0 - 2b_1B_1 + b_0B_2).
\end{aligned} \tag{A.12}$$

In these variables, the remaining three operators (A.11) take the form

$$X_9 + X_{12} = -4j_0\partial_{j_0} - 6I_0\partial_{I_0} - 8J_1\partial_{J_1} - 6J_2\partial_{J_2} - 4J_3\partial_{J_3} - 2J_4\partial_{J_4} - 8J_5\partial_{J_5}$$

$$-6J_6\partial_{J_6} - 4J_7\partial_{J_7} - 8J_8\partial_{J_8} - 8J_9\partial_{J_9} - 6J_{10}\partial_{J_{10}} - 10J_{11}\partial_{J_{11}} - 8J_{12}\partial_{J_{12}}$$
$$-10J_{13}\partial_{J_{13}} - 8J_{14}\partial_{J_{14}} - 4J_{15}\partial_{J_{15}},$$
$$X_{13} = 2j_0\partial_{j_0} - 4J_0\partial_{J_0} - 4I_0\partial_{I_0} + J_1\partial_{J_1} - 2J_2\partial_{J_2} + J_3\partial_{J_3} - 4J_4\partial_{J_4} - J_5\partial_{J_5} + J_6\partial_{J_6}$$
$$-2J_7\partial_{J_7} - 2J_8\partial_{J_8} - 2J_9\partial_{J_9} - J_{10}\partial_{J_{10}} + J_{12}\partial_{J_{12}} + 2J_{13}\partial_{J_{13}},$$
$$X_0 = -2j_0\partial_{j_0} - 2I_0\partial_{I_0} - 3J_1\partial_{J_1} - 2J_2\partial_{J_2} - J_3\partial_{J_3} - J_4\partial_{J_4} - 3J_5\partial_{J_5} - 2J_6\partial_{J_6} - J_7\partial_{J_7}$$
$$-3J_8\partial_{J_8} - 3J_9\partial_{J_9} - 2J_{10}\partial_{J_{10}} - 3J_{11}\partial_{J_{11}} - 2J_{12}\partial_{J_{12}} - 3J_{13}\partial_{J_{13}} - 3J_{14}\partial_{J_{14}} - J_{15}\partial_{J_{15}}.$$

Operators $X_9 + X_{12}$ and X_{13} have 16 independent invariants

$$i_0 = \frac{I_0}{j_0^{3/2}J_0^{7/4}}, \quad i_1 = \frac{J_1}{j_0^2 J_0^{3/4}}, \quad i_2 = \frac{J_2}{j_0^{3/2}J_0^{5/4}}, \quad i_3 = \frac{J_3}{j_0 J_0^{1/4}}, \quad i_4 = \frac{J_4}{j_0^{1/2}J_0^{5/4}},$$
$$i_5 = \frac{J_5}{j_0^2 J_0^{5/4}}, \quad i_6 = \frac{J_6}{j_0^{3/2}J_0^{1/2}}, \quad i_7 = \frac{J_7}{j_0 J_0}, \quad i_8 = \frac{J_8}{j_0^2 J_0^{3/2}}, \quad i_9 = \frac{J_9}{j_0^2 J_0^{3/2}}, \quad i_{10} = \frac{J_{10}}{j_0^{3/2}J_0},$$
$$i_{11} = \frac{J_{11}}{j_0^{5/2}J_0^{5/4}}, \quad i_{12} = \frac{J_{12}}{j_0^2 J_0^{3/4}}, \quad i_{13} = \frac{J_{13}}{j_0^{5/2}J_0^{3/4}}, \quad i_{14} = \frac{J_{14}}{j_0^2 J_0}, \quad i_{15} = \frac{J_{15}}{j_0 J_0^{1/2}}. \tag{A.13}$$

The remaining operator X_0 has 15 invariants

$$I_k = \frac{i_k}{i_0},\ k = 1,2,3,5,\ldots,10,14,15, \quad I_4 = i_4, \quad I_l = \frac{i_l}{i_0^2},\ l = 11,12,13. \tag{A.14}$$

In order to obtain an arbitrary invariant of system (6), (7), we need to find the operators of invariant differentiation. According to the theory of [6, Chapter 7], the coefficients ψ_j of an invariant differentiation operator $\mathcal{D} = \sum_{j=0}^{4} \psi_j D_{z_j}$ satisfy

$$\tilde{X}\psi_j = \sum_{i=0}^{4} \psi_i D_{z_i}\xi_j, \quad j = 0,\ldots,4, \tag{A.15}$$

where we continue to designate $(z_0,\ldots,z_4) = (t, x_1, x_2, p_1, p_2)$ and $\xi_0,\ldots,\xi_4$ are the coefficients of operator (A.3) at the partial derivatives $\partial_t,\ldots,\partial_{p_2}$, respectively. We assume that the functions ψ_j depend on variables (A.4) and, similarly to the invariance criterion $\tilde{X}I = 0$, the equalities (A.15) should be split by functions ζ, ξ_1, ξ_2, τ and their derivatives. This yields a system of linear first-order partial differential equations which contains the nonhomogeneous equations

$$X_7\psi_1 = \psi_0, \quad X_9\psi_1 = \psi_1, \quad X_{11}\psi_1 = \psi_2, \quad X_8\psi_2 = \psi_0, \quad X_{10}\psi_2 = \psi_1, \quad X_{12}\psi_2 = \psi_2,$$
$$X_{8+j}\psi_k = \psi_3, \quad X_{10+j}\psi_k = \psi_4, \quad X_{13}\psi_k = -\psi_k, \quad X_{19+j}\psi_k = \psi_0,$$
$$X_{21+j}\psi_k = \psi_1 + p_1\psi_0, \quad X_{23+j}\psi_k = \psi_2 + p_2\psi_0, \quad X_{25+j}\psi_k = p_1\psi_1,$$
$$X_{27+j}\psi_k = p_2\psi_1 + p_1\psi_2, \quad X_{29+j}\psi_k = p_2\psi_2, \quad X_{32}\psi_k = -p_j\psi_0,$$
$$X_{13}\psi_0 = \psi_0, \quad j = 1,2, \quad k = 2+j, \tag{A.16}$$

the remaining equations of the system being homogeneous. Hence it follows that ψ_0 is a function of variables (11), (A.12), the functions ψ_1, ψ_2 depend on variables (A.10) and ψ_3, ψ_4 depend on variables (A.8). Note that operators (A.9), (A.11) act on the right-hand sides f_1, f_2 of system (6), (7), solved with respect to $\ddot{x}_1$, $\ddot{x}_2$, and their derivatives f_{jp_i} as follows:

$$
\begin{aligned}
&X_{19+j} = \partial_{f_j}, \quad X_{21+j} = 2\partial_{f_{jp_1}} + 2p_1\partial_{f_j}, \quad X_{23+j} = 2\partial_{f_{jp_2}} + 2p_2\partial_{f_j},\\
&X_{25+j} = 2p_1\partial_{f_{jp_1}} + p_1^2\partial_{f_j}, \quad X_{27+j} = 2p_2\partial_{f_{jp_1}} + 2p_1\partial_{f_{jp_2}} + 2p_1p_2\partial_{f_j},\\
&X_{29+j} = 2p_2\partial_{f_{jp_2}} + p_2^2\partial_{f_j}, \quad j = 1,2, \quad X_{32} = -p_1\partial_{f_1} - p_2\partial_{f_2} - \partial_{f_{1p_1}} - \partial_{f_{2p_2}},\\
&X_9 = f_1\partial_{f_1} + f_{1p_2}\partial_{f_{1p_2}} - f_{2p_1}\partial_{f_{2p_1}}, \quad X_{12} = f_2\partial_{f_2} - f_{1p_2}\partial_{f_{1p_2}} + f_{2p_1}\partial_{f_{2p_1}},\\
&X_{10} = f_1\partial_{f_2} + f_{1p_2}(\partial_{f_{2p_2}} - \partial_{f_{1p_1}}) + (f_{1p_1} - f_{2p_2})\partial_{f_{2p_1}},\\
&X_{11} = f_2\partial_{f_1} + f_{2p_1}(\partial_{f_{1p_1}} - \partial_{f_{2p_2}}) + (f_{2p_2} - f_{1p_1})\partial_{f_{1p_2}},\\
&X_{13} = -2f_1\partial_{f_1} - 2f_2\partial_{f_2} - f_{1p_1}\partial_{f_{1p_1}} - f_{1p_2}\partial_{f_{1p_2}} - f_{2p_1}\partial_{f_{2p_1}} - f_{2p_2}\partial_{f_{2p_2}}. \qquad \text{(A.17)}
\end{aligned}
$$

Taking into account (A.17), we see readily that the system for $\psi_0, \ldots, \psi_4$ has five independent solutions corresponding to operators (10). These are

1) one solution of the form $\psi_0 \neq 0$, $\psi_j = p_j\psi_0$, $\psi_{2+j} = f_j\psi_0$, $j = 1,2$, where $\psi_0 = J_0^{-1/4}$;

2) two solutions of the form $\psi_0 = 0$, $\psi_1 = 0$, $\psi_2 = 0$, $\psi_3 \neq 0$, $\psi_4 \neq 0$, where for (ψ_3, ψ_4) one can take the functions $j_0^{1/2}J_0I_0^{-1}\,(b_1a_2 - b_2a_1, b_1a_1 - b_0a_2)$ and $J_0^{3/2}I_0^{-1}\,(L_{p_1p_2}a_2 - L_{p_2p_2}a_1, L_{p_1p_2}a_1 - L_{p_1p_1}a_2)$;

3) two solutions of the form $\psi_0 = 0$, $\psi_1 \neq 0$, $\psi_2 \neq 0$, $\psi_{2+j} = \chi_j + \dfrac{1}{2}(f_{jp_1}\psi_1 + f_{jp_2}\psi_2)$, where $X_9\chi_1 = \chi_1$, $X_{11}\chi_1 = \chi_2$, $X_{10}\chi_2 = \chi_1$, $X_{12}\chi_2 = \chi_2$, $X_{13}\chi_j = -\chi_j$, $X_{32}\chi_j = \psi_j/2$, $j = 1,2$, and other operators (A.9), (A.11) act on χ_1, χ_2 homogeneously. The functions $(\psi_1, \psi_2, \chi_1, \chi_2)$ satisfy these conditions, if they are equal to $j_0^{1/2}J_0^{3/4}I_0^{-1}\,(b_1a_2 - b_2a_1, b_1a_1 - b_0a_2, b_2A_1 - b_1A_2, b_0A_2 - b_1A_1)$ or $J_0^{5/4}I_0^{-1}\,(L_{p_1p_2}a_2 - L_{p_2p_2}a_1, L_{p_1p_2}a_1 - L_{p_1p_1}a_2, L_{p_2p_2}A_1 - L_{p_1p_2}A_2, L_{p_1p_1}A_2 - L_{p_1p_2}A_1)$.

It is not difficult to see that the relative invariants i_1, i_2, i_3 in (A.13) are of the fourth order. The operators of invariant differentiation (10) act on them as follows:

$$
\begin{aligned}
&\mathcal{D}_0 i_1 = 6i_{14}, \quad \mathcal{D}_0 i_2 = 6i_1^{-1}(i_2i_{14} - \bar{I}_1i_{10}) + \frac{1}{2}\bar{I}_1i_4, \quad \mathcal{D}_0 i_3 = i_{15},\\
&i_0\mathcal{D}_1 i_1 = \bar{I}_1i_{15} - i_1i_7 - i_2i_6 + \frac{3}{8}(\bar{I}_1i_8 + 3i_1\bar{I}_5 + i_2\bar{I}_0) - 2i_3i_1^{-1}(\bar{I}_1^{-1}i_2i_{14} + i_2i_{10}),\\
&i_0\mathcal{D}_1 i_2 = -2i_{11} - \frac{1}{4}i_1\bar{I}_0 + i_2(\bar{I}_5 - 2i_7) + (\bar{I}_1 - 4i_3)i_{10} + i_2^2i_1^{-1}(i_6 - i_{14})\\
&\qquad + \frac{5}{8}i_2i_1^{-1}(\bar{I}_1i_8 + i_1\bar{I}_5 + i_2\bar{I}_0) + 2i_3i_1^{-1}(2\bar{I}_1i_{10} - i_2i_{14}) + 2i_2i_3i_1^{-2}(\bar{I}_1i_{14} + i_2i_{10}),\\
&i_0\mathcal{D}_1 i_3 = i_1^{-1}(\bar{I}_1i_{12} - i_2i_{13}) + \frac{1}{8}i_3i_1^{-1}(\bar{I}_1i_8 + 3i_1\bar{I}_5 + i_2\bar{I}_0),\\
&i_0\mathcal{D}_2 i_1 = 2(i_{11} + i_3i_{10} - i_1i_6) + \frac{3}{8}i_1i_2^{-1}(\bar{I}_1i_9 + 3i_1\bar{I}_5 + i_2\bar{I}_0),\\
&i_0\mathcal{D}_2 i_2 = -\bar{I}_1i_{15} - i_1i_7 - i_2i_{14} + \frac{1}{8}(\bar{I}_1(2i_8 + 3i_9) + 13i_1\bar{I}_5 + 3i_2\bar{I}_0) + 2i_3i_1^{-1}(\bar{I}_1i_{14} + i_2i_{10}),\\
&i_0\mathcal{D}_2 i_3 = -i_{13} + \frac{1}{8}i_3i_2^{-1}(\bar{I}_1i_9 + 3i_1\bar{I}_5 + i_2\bar{I}_0), \quad i_0\mathcal{D}_3 i_1 = 2\bar{I}_1i_3 - \frac{3}{4}i_1^2, \quad i_0\mathcal{D}_3 i_2 = -\frac{3}{4}i_1i_2,\\
&i_0\mathcal{D}_3 i_3 = -\frac{1}{4}i_1i_3, \quad i_0\mathcal{D}_4 i_1 = -\frac{3}{4}i_1i_2, \quad i_0\mathcal{D}_4 i_2 = \frac{1}{2}i_1^2 - \frac{5}{4}i_2^2 - 2\bar{I}_1i_3, \quad i_0\mathcal{D}_4 i_3 = -\frac{1}{4}i_2i_3,
\end{aligned}
$$

where $\bar{I}_0$, $\bar{I}_1$, $\bar{I}_5$ are related to the invariants i_0, i_1, i_2, i_5 by the equalities

$$\bar{I}_1^2 + i_2^2 = i_1^2, \quad 2i_1i_2i_5 = i_2(\bar{I}_0^2 + i_8^2 - i_8i_9) + \bar{I}_0(\bar{I}_1i_9 + i_1\bar{I}_5), \quad 2i_1i_2i_0 = i_1(\bar{I}_5^2 + i_8i_9 - i_9^2) + \bar{I}_5(\bar{I}_1i_8 + i_2\bar{I}_0).$$

From the above relations we deduce that the action of operators $\mathcal{D}_3$, $\mathcal{D}_4$ on i_1, i_2, i_3 is expressible in terms of i_1, i_2, i_3. In much the same way one obtains similar relations for the fifth-order invariants (A.13). Therefore, the action of operators $\mathcal{D}_3$, $\mathcal{D}_4$ on invariants (A.14) does not yield any other invariants. The relations with $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$ mean that from the nine relations

$$\mathcal{D}_l I_k = i_0^{-1}\mathcal{D}_l i_k - i_k i_0^{-2}\mathcal{D}_l i_0, \quad k = 1, 2, 3, \quad l = 0, 1, 2,$$

one can find the values of $\mathcal{D}_0 i_0$, $\mathcal{D}_1 i_0$, $\mathcal{D}_2 i_0$ and six invariants $I_{10}, \dots, I_{15}$ in terms of other invariants $I_1, \dots, I_9$. Hence, the invariants $I_{10}, \dots, I_{15}$ are not basis ones, for they can be obtained by algebraic operations and invariant differentiations from invariants (9). It remains to show that all invariants of the sixth and higher order can also be obtained by these operations from invariants (9), which, therefore, form a basis of invariants of system (6), (7).

Note that 15 invariants (A.14) depend on the fifth-order derivatives of L via 14 variables B_i, A_i, γ_0, γ_1, γ_2, γ_3, ε_{ij}, ϵ_i, $i, j = 1, 2$. These 14 variables involve the derivatives in the following way

$$\begin{aligned}
&B_1 \sim \frac{1}{4} j_0^{-1}(L_{p_1p_1}L_{tttp_2p_2} - L_{p_2p_2}L_{tttp_1p_1}), \quad B_2 \sim \frac{1}{2} j_0^{-1}(L_{p_1p_2}L_{tttp_2p_2} - L_{p_2p_2}L_{tttp_1p_2}),\\
&A_i \sim \frac{1}{3}(L_{ttx_1p_ip_2} - L_{ttx_2p_1p_i}), \quad \gamma_{2i+j-3} \sim L_{ttx_jp_ip_i}, \quad \varepsilon_{ij} \sim L_{tx_1x_jp_ip_2} - L_{tx_jx_2p_1p_i},\\
&\epsilon_i \sim L_{x_1x_1x_ip_2p_2} - 2L_{x_1x_ix_2p_1p_2} + L_{x_ix_2x_2p_1p_1}, \quad i, j = 1, 2,
\end{aligned} \tag{A.18}$$

and, together with 13 variables

$$p,\ L,\ L_p,\ L_{pp},\ b_i,\ a_i,\ J_3, \quad i = 1, 2, \tag{A.19}$$

they form a set of independent invariants of the operators X_0, X_1, X_2, X_3, $X_{14}, \dots, X_{258}$. In the 27-dimensional space of variables (A.18), (A.19) 12 operators (A.11) have 15 invariants (A.14). Their invariant differentiations yield invariants, which depend on the sixth-order derivatives of L via 26 variables

$$\begin{aligned}
&D_0B_i, \quad D_0A_i, \quad D_0\gamma_{2i+j-3}, \quad D_iA_i, \quad D_2(\gamma_1 + 3A_1) \sim D_1(\gamma_2 - 3A_2), \quad D_j\gamma_{2i+j-3},\\
&D_2\gamma_{2i-2} \sim D_1\gamma_{2i-1}, \quad D_j\varepsilon_{ij}, \quad D_2\varepsilon_{i1} \sim D_1\varepsilon_{i2}, \quad D_i\epsilon_i, \quad D_1\epsilon_2 \sim D_2\epsilon_1,
\end{aligned} \tag{A.20}$$

$i, j = 1, 2$. It is not difficult to determine the number of these invariants, namely in the 53-dimensional space of variables (A.18)–(A.20) 12 operators (A.11) have 41 independent invariants and 26 of them are of the sixth order.

On the other hand, one can obtain the sixth-order invariants from the invariance condition $\tilde{X}I = 0$. On extending operator (A.3) to the sixth-order derivatives of L and splitting the equality $\tilde{X}I = 0$ by functions $\zeta, \xi_1, \xi_2, \tau$ and their derivatives up to the seventh order we arrive at a system

$$X_0(c)I = 0, \quad X_1(\xi_1)I = 0, \quad \dots, \quad X_{367}(\partial^7\tau/\partial t^7)I = 0,$$

where 103 operators (81+22) are represented as linear functions of the remaining 265 operators. Hence, in the 306-dimensional space of variables

$$t,\ x,\ p,\ L,\ L_t,\ L_x,\ L_p,\ L_{tt},\ \dots, L_{xxxpp},\ L_{tttttt},\ \dots, L_{xxxxpp}$$

this system has 41 functionally independent solutions. Fifteen of them are invariants of the fifth order and 26 are of the sixth order. This coincides with the number of independent invariants of the sixth order obtained by invariant differentiations of the fifth-order invariants. Similar reasoning extended to higher orders shows that all independent invariants of the n-th order can be obtained by invariant differentiations from invariants of the $(n-1)$-th order, $n \geq 6$. Therefore, invariants (9) form a basis. This completes the proof.

References

[1] J. Douglas, Solution of inverse problem of the calculus of variations, Trans. Amer. Math. Soc. 50 (1941) 71-128.

[2] C. Lanczos, The Variational Principles of Mechanics, University Press, Toronto, 1962.

[3] F. Gantmacher, Lectures in Analytical Mechanics, Mir Publications, Moscow, 1975.

[4] A.M. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras, Birkhauser Verlag, Basel, 1990.

[5] A. Tresse, Sur les invariants différentiels des groupes continus de transformations, Acta Math. 18 (1894) 1-88.

[6] L.V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York, 1982.

[7] N. Kamran, P.J. Olver, Equivalence problems for first order Lagrangians on the line, J. Diff. Equations 80 (1989) 32-78.

[8] W.F. Shadwick, The Hamiltonian formulation of regular r^{th} order Lagrangian field theories, Lett. Math. Phys. 6 (1982) 409-416.

[9] L. Hsu, N. Kamran, P.J. Olver, Equivalence of higher order Lagrangians II. The Cartan form for particle Lagrangians, J. Math. Phys. 30 (1989) 902-906.

[10] N. Kamran, P.J. Olver, Equivalence of higher order Lagrangians: III. New invariant differential equations, Nonlinearity 5 (1992) 601-621.

[11] P.J. Olver, The equivalence problem and canonical forms for quadratic Lagrangians, Adv. Appl. Math. 9 (1988) 226-257.

[12] M.E. Fels, The equivalence problem for systems of second-order ordinary differential equations, Proc. London Math. Soc. 71 (1995) 221-240.

[13] T. Sen, Lie symmetries and integrability, Phys. Lett. A 122 (1987) 327-330.

[14] M.U. Farooq, S. Ali, F.M. Mahomed, Two-dimensional systems that arise from the Noether classification of Lagrangians on the line, Appl. Math. Comput. 217 (2011) 6959-6973.

[15] M.U. Farooq, S. Ali, Asghar Qadir, Invariants of two-dimensional systems via complex Lagrangians with applications, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 1804-1810.

[16] S. Lie, Über Differentialinvarianten, Math. Ann. 24 (1884) 52-89.

[17] M. Ackerman, R. Hermann, Sophus Lie's 1884 Differential Invariant Paper, Math. Sci. Press, Brookline, Mass., 1976.

[18] C. Tsaousi, C. Sophocleous, Differential invariants for systems of linear hyperbolic equations, J. Math. Anal. Appl. 363 (2010) 238-248.

[19] F.M. Mahomed, M. Safdar, J. Zama, Ibragimov-type invariants for a system of two linear parabolic equations, Commun. Nonlinear Sci. Numer. Simul. 17 (2011) 3140-3147.

[20] N.H. Ibragimov, S.V. Meleshko, A solution to the problem of invariants for parabolic equations, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 2551-2558.

[21] Yu.Yu. Bagderina, Equivalence of third-order ordinary differential equations to Chazy equations I-XIII, Stud. Appl. Math. 120 (2008) 293-332.

[22] R.B. Gardner, The Method of Equivalence and Its Applications, SIAM, Philadelphia, 1989.

[23] P.J. Olver, Equivalence, Invariants and Symmetry. Graduate Texts in Mathematics, Cambridge University Press, Cambridge, 1995.

[24] R.A. Sharipov, Effective procedure of point-classification for the equations $y'' = P + 3Qy' + 3Ry'^2 + Sy'^3$, e-print arXiv: 9802027 [math DG].

[25] L.A. Pars, A Treatise on Analytical Dynamics, Heinemann, London, 1965.

[26] A.P. Fordy, The Hénon-Heiles system revisited, Physica D 52 (1991) 204-210.

[27] K. Sawada, T. Kotera, A method for finding N-soliton solutions of K.d.V. equation and K.d.V.-like equation, Progr. Theor. Phys. 51 (1974) 1355-1367.

[28] P.J. Caudrey, R.K. Dodd, J.D. Gibbon, A new hierarchy of Korteweg-de Vries equations, Proc. R. Soc. Lond. A 351 (1976) 407-422.

[29] C. Chanu, L. Degiovanni, G. Rastelli, Polynomial constants of motion for Calogero-type systems in three dimensions, J. Math. Phys. 52 (2011) 032903.

[30] M.J. Potasek, M. Tabor, Exact solutions for an extended nonlinear Schrödinger equation, Phys. Lett. A 154 (1991) 449-452.

[31] K. Okamoto, The Hamiltonians associated to the Painlevé equations, in The Painlevé Property: One Century Later (Ed. R. Conte), Springer, N.Y., 1999, pp. 735-787.