Increasing coupling of Probabilistic Cellular Automata

P.-Y. Louis

Institut für Mathematik, Potsdam Universität,
Am neuen Palais, Sans Souci,
Potsfach 60 15 53, D-14 415 Potsdam

Abstract

We give a necessary and sufficient condition for the existence of an increasing coupling of N ($N \geqslant 2$) synchronous dynamics on $S^{\mathbb{Z}^d}$(PCA). Increasing means the coupling preserves stochastic ordering. We first present our main construction theorem in the case where S is totally ordered; applications to attractive PCA's are given. When S is only partially ordered, we show on two examples that a coupling of more than two synchronous dynamics may not exist. We also prove an extension of our main result for a particular class of partially ordered spaces.

Key words:
Probabilistic Cellular Automata, Stochastic ordering, Monotone Coupling
2000 MSC: 60K35, 60E15, 60J10, 82C20, 37B15, 68W10

1 Introduction

Probabilistic Cellular Automata (abbreviated in PCA) are discrete-time Markov chains on a product space S^Λ (*configuration space*) whose transition probability is a *product measure*. S is assumed to be a finite set (*spin space*). We denote by Λ (set of *sites*) a subset, finite or infinite, of $\mathbb{Z}^d$. Since the transition probability kernel $P(d\sigma|\sigma')$ ($\sigma, \sigma' \in S^\Lambda$) is a product measure, all interacting elementary components (*spins*) $\{\sigma_k : k \in \Lambda\}$ are simultaneously and independently updated (*parallel updating*). This synchronous transition is the main feature of PCA and differs from the one in the most common Gibbs samplers, where only one site is updated at each time step (*sequential updating*). In opposition to these sequential updating dynamics, it is simple to define PCA's on the infinite set $S^{\mathbb{Z}^d}$ without passing to continuous time.

Probabilistic Cellular Automata were first studied as Markov chains in the 70's (see Toom et al. (1978)). We refer for instance to Louis (2002) for a recent historical overview and a list of applications of Cellular Automata dynamics, which are to be found in physics, biology, image restoration (see Younes (1998))... PCA dynamics may present a variety of behaviours. Let us only mention the

Email address: louis@math.uni-potsdam.de (P.-Y. Louis).
URL: http://www.math.uni-potsdam.de/∼louis (P.-Y. Louis).

following: contrarily to the usual discrete time sequential updating dynamics, for a given measure μ, there is no canonical way of constructing a PCA for which μ is stationary. Moreover, there exist Gibbs measures on $S^{\mathbb{Z}^2}$ such that no PCA admits them as stationary reversible measures (see Theorem 4.2 in Dawson (1974)).

Coupling refers to the construction of a product probability space on which several dynamics may evolve simultaneously, and having the property that the marginals coincide with each one of these dynamics. Coupling techniques for stochastic processes are now well established, powerful tools of investigation. We refer to Lindvall (1992) and Thorisson (2000) for a more extensive review and applications to a large scope of probabilistic objects. The first use of a coupling of Probabilistic Cellular Automata is to be found in Vasershtein (1969). It was also used in Maes (1993). Recently, the coupling constructed in this paper was used to state some necessary and sufficient condition for the exponential ergodicity of attractive PCA's (see Louis (2004)). This last result relies on the fact that our coupling preserves a stochastic order between the configurations (so called *increasing coupling*). In López and Sanz (2000), the authors gave necessary and sufficient condition for the existence of a coupling preserving the stochastic order between two PCA's on $S^{\mathbb{Z}^d}$, where S is a partially ordered set. In this paper, we give a necessary and sufficient condition for the existence of an increasing coupling of any finite number of possibly different PCA dynamics. As some counter examples will show, there is a gap between the construction of an increasing coupling of two PCA's and that of an increasing coupling of N PCA's with $N \geqslant 3$. Moreover, we give here an explicit algorithmic construction of this coupling, which is a kind of graphical construction. We also give several examples and general applications of the constructed coupling. Indeed, the motivation for coupling together three or more PCA's comes from the paper Louis (2004), where a comparison between four different PCA dynamics proved to be useful.

In section 2 we state our main result, namely the existence, under some necessary and sufficient condition of monotonicity (Definition 2.2), of an increasing coupling of several PCA dynamics (Theorem 2.4). Corollary 2.5 states the existence of some universal coupling of any attractive PCA and some significant examples are also presented. In section 3 we prove these results, and state some important property (Lemma 3.2) of coherence between the different couplings. In section 4, we then present some useful applications of the coupling just constructed. In section 5 we consider the case where S is a partially ordered set. Two counter-examples show that it may happen that an increasing coupling of N PCA dynamics does not exist when $N \geqslant 3$. A generalisation of Theorem 2.4 and Corollary 2.5 to the case where S is partially linearly ordered is presented.

Finally, let us point out that our motivation for considering partially ordered spin spaces comes from the study of 'block dynamics', where the siteds are not updated individually, but rather blockwise. This amounts to consider PCA's on $(S^r)^{\mathbb{Z}^d}$ where r is the number of sites in these blocks (even if S is totally ordered, S^r is not totally ordered in a natural way).

2 Definitions and main results

Let S be a finite set, with a partial order denoted by $\preccurlyeq$. The conjunction of $s \neq s'$ and $s \preccurlyeq s'$ will be denoted by $s \precnsim s'$. Let P denote a PCA dynamics on the product space $S^{\mathbb{Z}^d}$, which means a time-homogeneous Markov Chain on $S^{\mathbb{Z}^d}$ whose transition probability kernel P verifies, for all configurations $\eta \in S^{\mathbb{Z}^d}$, $\sigma = (\sigma_k)_{k\in\mathbb{Z}^d} \in S^{\mathbb{Z}^d}$, $P(\ d\sigma \mid \eta\ ) = \bigotimes_{k\in\mathbb{Z}^d} p_k(\ d\sigma_k \mid \eta\ )$, where for all site $k \in \mathbb{Z}^d$, $p_k(\ .\ |\eta)$ is a probability measure on S, called *updating rule*. In other words, *given the previous time step* $(n-1)$, all the spin values $(\omega_k(n))_{k\in\mathbb{Z}^d}$ at time n are *simultaneously and independently updated*, each one according to the probabilistic rule $p_k(\ .\ \mid (\omega_{k'}(n-1))_{k'\in\mathbb{Z}^d})$. We let $P = \bigotimes_{k\in\mathbb{Z}^d} p_k$. All PCA dynamics considered in this paper are local, which means $\forall k \in \mathbb{Z}^d, \exists\, V_k \Subset \mathbb{Z}^d, p_k(\ .\ |\eta) = p_k(\ .\ |\eta_{V_k})$. The notation $\Lambda \Subset \mathbb{Z}^d$ means Λ is a finite subset of $\mathbb{Z}^d$. For any subset Δ of $\mathbb{Z}^d$ and for all configurations σ and η of $S^{\mathbb{Z}^d}$, the configuration $\sigma_\Delta \eta_{\Delta^c}$ is defined by σ_k for $k \in \Delta$, η_k elsewhere. We also let $\sigma_\Delta := (\sigma_k)_{k\in\Delta}$ too.

All the measures considered in this paper are probability measures. For a probability measure ν on $S^{\mathbb{Z}^d}$ (equipped with the Borel σ-field associated to the product topology), νP refers to the law at time 1 of the PCA dynamics with law ν at time 0: $\nu P(d\sigma) = \int P(d\sigma|\eta)\nu(d\eta)$. Recursively, $\nu P^{(n)} = (\nu P^{(n-1)})P$ is the law at time n of the system evolving according to the PCA dynamics P and having initial law ν. For each measurable function $f : S^{\mathbb{Z}^d} \to \mathbb{R}_+$, $P(f)$ denotes the function on $S^{\mathbb{Z}^d}$ defined by $P(f)(\eta) = \int f(\sigma)P(d\sigma|\eta)$.

Let us now define basic notions of stochastic ordering $\preccurlyeq$. Two configurations σ and η of S^Λ (with $\Lambda \subset \mathbb{Z}^d$) satisfy $\sigma \preccurlyeq \eta$ if $\forall k \in \Lambda, \sigma_k \preccurlyeq \eta_k$. A real function f on S^Λ will be increasing if $\sigma \preccurlyeq \eta \Rightarrow f(\sigma) \leqslant f(\eta)$. Thus two probability measures ν_1 and ν_2 satisfy the stochastic ordering $\nu_1 \preccurlyeq \nu_2$ if, for all increasing functions f on S^Λ, $\nu_1(f) \leqslant \nu_2(f)$, with the notation $\nu(f) = \int f(\sigma)\nu(d\sigma)$. Considered as a Markov chain, a PCA dynamics P on S^Λ ($\Lambda \subset \mathbb{Z}^d$) is said to be *attractive* if for all increasing functions f, $P(f)$ is still increasing. This requirement is equivalent to ($\mu_1 \preccurlyeq \mu_2 \Rightarrow \mu_1 P \preccurlyeq \mu_2 P$), where μ_1, μ_2 are two probability measures on $S^{\mathbb{Z}^d}$.

Definition 2.1 (Synchronous coupling of PCA dynamics)

Let $P^1, P^2, \ldots, P^N$ be N probabilistic cellular automata dynamics, with $P^i = \underset{k\in\mathbb{Z}^d}{\otimes} p^i_k$. A synchronous coupling *of $(P^i)_{1\leqslant i\leqslant N}$ is a Markovian dynamics Q on $(S^{\mathbb{Z}^d})^N$, which is also a PCA dynamics whose marginals coinncide respectively with $P^1, P^2, \ldots, P^N$. Thus, Q is such that $Q = \underset{k\in\mathbb{Z}^d}{\otimes} q_k$ and*

$\forall i \in \{1, \ldots, N\}, \quad \forall s^i \in S,\ \forall \zeta^1, \ldots, \zeta^N \in S^{\mathbb{Z}^d}$,

$$p^i_k(s^i \mid \zeta^i) = \sum_{s^j\in S, j\neq i} q_k\big(\ (s^1, \ldots, s^N)\ \big|\ (\zeta^1, \ldots, \zeta^N)\big). \tag{1}$$

Definition 2.2 (Increasing N-tuple of PCA dynamics) *Let $(P^1, P^2, \ldots, P^N)$ be an N-tuple of PCA dynamics, where $N \geqslant 2$ and $P^i = \underset{k\in\mathbb{Z}^d}{\otimes} p^i_k$ $(1 \leqslant i \leqslant N)$. This N-tuple is said to be* increasing *if*

$$\zeta^1 \preccurlyeq \zeta^2 \preccurlyeq \ldots \preccurlyeq \zeta^N \quad \Rightarrow \quad P^1(\ .\mid \zeta^1) \preccurlyeq P^2(\ .\mid \zeta^2) \preccurlyeq \ldots \preccurlyeq P^N(\ .\mid \zeta^N). \tag{2}$$

Since $P(\ .\ |\sigma)$ is a product measure, according to Proposition 2.9 in Toom et al. (1978), condition (2) is equivalent to: $\forall k \in \mathbb{Z}^d$,

$$\zeta^1 \preccurlyeq \zeta^2 \preccurlyeq \ldots \preccurlyeq \zeta^N \quad \Rightarrow \quad p^1_k(\ .\mid \zeta^1) \preccurlyeq p^2_k(\ .\mid \zeta^2) \preccurlyeq \ldots \preccurlyeq p^N_k(\ .\mid \zeta^N). \tag{3}$$

Definition 2.3 (Increasing synchronous coupling) *A synchronous coupling* $\mathbf{Q}$ *of an N-tuple $(P^i)_{1\leqslant i\leqslant N}$ of PCA dynamics is said to be an* increasing coupling *of $(P^1, P^2, \ldots, P^N)$ if the following property holds: for any initial configurations $\sigma^1 \preccurlyeq \sigma^2 \preccurlyeq \ldots \preccurlyeq \sigma^N$, for any time $n \geqslant 1$,*

$$\mathbf{Q}\ \big(\ \omega^1(n) \preccurlyeq \ldots \preccurlyeq \omega^N(n)\ \big|\ (\omega^1, \ldots, \omega^N)(0) = (\sigma^1, \ldots, \sigma^N)\ \big) = 1. \tag{4}$$

We now state:

Theorem 2.4 *Let S be a totally ordered space. Let $(P^i)_{1\leqslant i\leqslant N}$ be an N-tuple of PCA dynamics on $S^{\mathbb{Z}^d}$. There exists a synchronous coupling $\mathbf{Q}$ of $(P^i)_{1\leqslant i\leqslant N}$ if and only if $(P^1,\ldots,P^N)$ is increasing.*

The increasing coupling we are constructing will be denoted by $P^1 \circledast P^2 \circledast \ldots \circledast P^N$. Note that the property of preserving the order implies that the coupling has the *coalescence property*. This means that if two components are taking the same value at some time, then they will remain equal from this time on (as well as all the components in between). In Lemma 3.3 we will see that if a PCA dynamics P is attractive, then for all $N \geqslant 2$, the N-tuple $(P,P,\ldots,P)$ is increasing. As an immediate consequence of Theorem 2.4 and Lemma 3.3 we then have the

Corollary 2.5 *Let S be a totally ordered space, P be a PCA dynamics on $S^{\mathbb{Z}^d}$ and $N \geqslant 2$. There exists an increasing coupling $P^{\circledast N}$ of $(P,\ldots,P)$ if and only if P is attractive.*

Lemma 3.1 from section 3 gives a practical constructive criterion for testing if an N-tuple of PCA dynamics is increasing or if a PCA is attractive. We use it in the following examples.

A family of PCA dynamics

Let $(P^{\beta_i,h_i})_{1\leqslant i\leqslant N}$ be a family of N PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$, defined by $\forall k \in \mathbb{Z}^d$, $\forall \eta \in \{-1,+1\}^{\mathbb{Z}^d}$, $\forall s \in S = \{-1,+1\}$

$$p_k^i(s \mid \eta) = \frac{1}{2}\Big(1 + s\tanh(\beta_i \sum_{k'\in V_0} \mathcal{K}(k'+k)\eta_{k'} + \beta_i h_i)\Big), \tag{5}$$

where $(\beta_i)_{1\leqslant i\leqslant N}$ are positive real numbers, $(h_i)_{1\leqslant i\leqslant N}$ real numbers, $V_0 \Subset \mathbb{Z}^d$ and $\mathcal{K} : V_0 \to \mathbb{R}$ is an interaction function between sites which is symmetric.

This example is important, since any reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$ can be presented in this form[1]. When β is fixed and $h_1 \leqslant \ldots \leqslant h_N$, the N-tuple $(P^{\beta,h_i})_{1\leqslant i\leqslant N}$ is increasing. On the other hand, note that in the case $h_i = 0$, the assumption $\beta_1 \leqslant \ldots \leqslant \beta_N$ does not imply that the N-tuple $(P^{\beta_i,0})_{1\leqslant i\leqslant N}$ is increasing. Consider for instance $\beta_1 = \frac{1}{2}$, $\beta_2 = 3$, $d = 2$, $\mathcal{K}$ such that $V_0 = \{-e_1, e_1, -e_2, e_2\}$ where (e_1,e_2) is a basis of $\mathbb{R}^2$. Condition (3) is false considering $k = 0$, $\zeta^1_{V_0}$ consisting of four -1, and $\zeta^2_{V_0}$ of three -1 and one $+1$.

Example of an attractive PCA dynamics

Let $P^{\beta,h}$ be some PCA dynamics defined by the updating rule (5) ($\beta \geqslant 0$, $h \in \mathbb{R}$). We know from Proposition 4.1.2 in Louis (2002) that this dynamics is attractive if and only if $\mathcal{K}(.) \geqslant 0$. For a more systematic study of this class, we refer to Dai Pra et al. (2002) and Louis (2004). From Corollary 2.5, we can then construct an increasing coupling of such PCA. We show in section 4 how this can be used.

Example of an attractive PCA dynamics with $\#\mathbf{S} = \mathbf{q}$, $\mathbf{q} \geqslant \mathbf{2}$

Let $S = \{1,\ldots,q\}$ ($q \geqslant 2$), and consider the updating rule
$\forall k \in \mathbb{Z}^d, \quad \forall s \in S, \quad \forall \sigma \in S^{\mathbb{Z}^d}$,

$$p_k(s|\sigma) = \frac{e^{\beta N_k(s,\sigma)}}{\sum_{s'\in S} e^{\beta N_k(s',\sigma)}}$$

[1] A PCA dynamics is said to be reversible if it admits at least one reversible probability measure (see subsection 4.1.1 in Louis (2002)).

where $\beta \geqslant 0$, V_k is a finite neighbourhood of k and $N_k(s,\sigma)$ is the number of $\sigma_{k'}$ ($k' \in V_k$) which are larger than s. This dynamics is attractive for any β non-negative.

3 Proof of the main results

Assume in this section that S is a totally ordered set. Let us then enumerate the spin set elements as $S = \{-, \ldots, s, s+1, \ldots, +\}$, where we denote with $+$ (resp. $-$) the (necessarily unique) maximum (resp. minimum) value of S and for $s \in S$, $(s+1)$ denotes the unique element in S such that there is no $s'' \in S$, $s \precneqq s'' \precneqq s+1$. A real valued function f on $S^{\mathbb{Z}^d}$ is said to be *local* if $\exists \Lambda_f \Subset \mathbb{Z}^d$, $\forall \sigma \in S^{\mathbb{Z}^d}$, $f(\sigma) = f(\sigma_{\Lambda_f})$.

Lemma 3.1 *When S is a totally ordered space, the condition (3) of monotonicity is equivalent to $\forall k \in \mathbb{Z}^d, \forall \zeta^1 \preccurlyeq \zeta^2 \preccurlyeq \ldots \preccurlyeq \zeta^N \in (S^{\mathbb{Z}^d})^N, \forall s \in S$*

$$F_k^1(s,\zeta^1) \geqslant F_k^2(s,\zeta^2) \geqslant \ldots \geqslant F_k^N(s,\zeta^N), \tag{6}$$

where $F_k^i(s,\sigma)$ is the distribution function of $p_k^i(.|\sigma)$: $F_k^i(s,\sigma) = \sum_{s' \leqslant s} p_k^i(s'|\sigma)$.

Proof. The implication (3) $\Rightarrow$ (6) is straightforward using the increasing function $f(s') = \mathbb{1}_{\{s'>s\}}$. To prove (6) $\Rightarrow$ (3), it is enough to remark that, for any function $f : S \to \mathbb{R}$,

$$p_k^i(f|\sigma) = f(+) + \sum_{s \precneqq +} (f(s) - f(s+1)) F_k^i(s,\sigma). \qquad \square \tag{7}$$

Proof of Theorem 2.4
Let us explain how to construct explicitly the increasing coupling $P^1 \circledast P^2 \circledast \ldots \circledast P^N$.
n being a fixed time index, we need to describe the stochastic transition from $(\omega^1, \ldots, \omega^N)(n)$ (element of S^N) to $(\omega^1, \ldots, \omega^N)(n+1)$. Let $(U_k)_{k \in \Lambda}$ be a family of independent random variables, distributed uniformaly on $]0,1[$. Since we are constructing a synchronous coupling, it is enough to define the rule for a fixed site $k \in \mathbb{Z}^d$. Let r denote a fixed realisation of the random variable U_k and use the following *algorithmic rule* to choose the value $\omega_k^i(n+1)$ for any i ($1 \leqslant i \leqslant N$):

$$\begin{cases} \text{if } F_k^i(s-1,\omega^i(n)) < r \leqslant F_k^i(s,\omega^i(n)), \quad - \precneqq s, \text{ assign } \omega_k^i(n+1) = s \\ \text{if } 0 \leqslant r \leqslant F_k^i(-,\omega^i(n)) \text{ assign } \omega_k^i(n+1) = - \ . \end{cases} \tag{8}$$

This rule corresponds to the definition of the coupling between times n and $n+1$ according to

$$\forall k \in \mathbb{Z}^d, \quad \Big(\omega_k^i(n+1)\Big)_{1 \leqslant i \leqslant N} = \Big(\big(F_k^i(\ .\ ,\omega^i(n))\big)^{-1}(U_k)\Big)_{1 \leqslant i \leqslant N} \tag{9}$$

where $(F_k^i)^{-1}$ denotes the Lévy probability transform (generalised inverse probability transform) of the F_k^i distribution function

$$(F_k^i)^{-1}(t) = \inf_{\preccurlyeq} \{s \in S : F_k^i(s) \geqslant t\}, \quad t \in]0,1[, \ i \in \{1, \ldots, N\}.$$

Finally, remark that the stochastic dependence between the components $1 \leqslant i \leqslant N$ comes from the fact that we are using the *same* realisation r of U_k for *all* components. It is easy to check that this coupling preserves stochastic ordering assuming that $(P^1, \ldots, P^N)$ is increasing, since it is equivalent to check (6) (Lemma 3.1).

Conversely, the condition (6) is necessary. Assume the existence of a synchronous coupling $(q_k)_{k\in\mathbb{Z}^d}$ of N PCA dynamics on $S^{\mathbb{Z}^d}$ which preserves stochastic ordering. This means that for $\zeta^1 \preccurlyeq \ldots \preccurlyeq \zeta^N$, $q_k(\,.\, |(\zeta^1, \ldots, \zeta^N)) > 0$ only on $(S^N)^+$, where $(S^N)^+$ is the subset $\{(s^1, \ldots, s^N) : s^1 \preccurlyeq \ldots \preccurlyeq s^N\}$ of S^N. Let $s \in S$, $1 \leqslant i < N$, and $\zeta^1 \preccurlyeq \ldots \preccurlyeq \zeta^N$ be fixed. Using the condition (1) on the i-th marginal of a coupling, we have

$$F_k^i(s, \zeta^i) = \sum_{(s^1,\ldots,s^N)\in A_s^i} q_k((s^1, \ldots, s^N)|(\zeta^1, \ldots, \zeta^N)),$$

where $A_s^i = \{(s^1, \ldots, s^N) \in (S^N)^+ : s^i \preccurlyeq s\}$. Decompose $A_s^i = A_s^{i+1} \sqcup \Delta_s^i$ with $\Delta_s^i = \{(s^1, \ldots, s^N) \in (S^N)^+ : s^i \precneqq s \preccurlyeq s^{i+1}\}$ ($\sqcup$ denotes the disjoint union). Finally note that

$$F_k^i(s, \zeta^i) = F_k^{i+1}(s, \zeta^{i+1}) + \sum_{(s^1,\ldots,s^N)\in \Delta_s^i} q_k((s^1, \ldots, s^N)|(\zeta^1, \ldots, \zeta^N))$$

where the last term is non-negative. □

One may readily notice the *compatibility property* satisfied by this coupling, and whose proof is straightforward according to its construction.

Lemma 3.2 *Let N and N' be two integers such that $1 \leqslant N < N'$. Let $(P^1, \ldots, P^{N'})$ be N' PCA dynamics. The projection of the coupling $P^1 \circledast P^2 \ldots \circledast P^{N'}$ on any N components $(i_1, \ldots, i_N)$ coincides with the coupling $P^{i_1} \circledast \ldots \circledast P^{i_N}$.*

Lemma 3.3 *Let P be a PCA dynamics on $S^{\mathbb{Z}^d}$. It is an attractive dynamics if and only if, for all $N \geqslant 2$, the N-tuple $(P, P, \ldots, P)$ is increasing.*

Proof. Assume P is attractive. Let $k \in \mathbb{Z}^d$ be fixed, and let f_0 be an increasing function on S. We consider the function f on $S^{\mathbb{Z}^d}$ defined by $f(\sigma) = f_0(\sigma_k), \forall \sigma \in S^{\mathbb{Z}^d}$. Since $P(f) = p_k(f_0)$ is an increasing function, relation (3) holds with $p_k^i = p_k, \forall i$. The equivalence (3) $\Longleftrightarrow$ (2) gives $(P, \ldots, P)$ increasing for any $N \geqslant 2$.

Conversely, assume (P, P) is increasing. Then relation (6) holds with the same dynamics on the two components. Let f be an increasing function on $S^{\mathbb{Z}^d}$ such that $\exists k \in \mathbb{Z}^d$, $\forall \sigma \in S^{\mathbb{Z}^d}$, $f(\sigma) = f(\sigma_k)$. According to formula (7), we conclude that $P(f)$ is increasing. Recursively, we can state the same result for all local functions, because of the product form of the kernels. Since S is finite, $S^{\mathbb{Z}^d}$ is compact, and a density argument gives the conclusion. □

4 Applications

Using the increasing coupling, we develop a precise analysis of the structure of the set of PCA dynamics' stationary measures. Moreover, the time-asymptotical behaviour is investigated.

Let us first prove a property (Proposition 4.1) for stationary measures associated to PCA restricted on S^Λ ($\Lambda \Subset \mathbb{Z}^d$) (see formula (10)). The relation between these measures and the stationary measures for the PCA dynamics on $S^{\mathbb{Z}^d}$ is then established (Proposition 4.2). In particular, we formulate an identity relating spatial limits and temporal limits (see equations (13) and (14)). Proposition 4.4 establishes a comparison between infinite volume PCA's (*e.g.* $\Lambda = \mathbb{Z}^d$) and PCA's in a large but finite volume. See also Louis (2004) for applications.

In the following P is an attractive PCA dynamics on $S^{\mathbb{Z}^d}$, where S is a totally ordered space.

4.1 Finite volume PCA dynamics

Let $\Lambda \Subset \mathbb{Z}^d$ be a finite subset of $\mathbb{Z}^d$, called finite volume. We call *finite volume PCA dynamics with boundary condition* τ ($\tau \in S^{\mathbb{Z}^d}$ or $\tau \in S^{\Lambda^c}$), the Markov Chain on S^Λ whose transition probability P^τ_Λ is defined by:

$$P^\tau_\Lambda(d\sigma_\Lambda \mid \eta_\Lambda) = \underset{k\in\Lambda}{\otimes} p_k(\, d\sigma_k \mid \eta_\Lambda \tau_{\Lambda^c}\,). \tag{10}$$

It may be identified with the following infinite volume PCA dynamics on $S^{\mathbb{Z}^d}$:

$$P^\tau_\Lambda(d\sigma \mid \eta_\Lambda) = \underset{k\in\Lambda}{\otimes} p_k(\, d\sigma_k \mid \eta_\Lambda \tau_{\Lambda^c}\,) \otimes \delta_{\tau_{\Lambda^c}}(d\sigma_{\Lambda^c}) \tag{11}$$

where the spins of Λ evolve according to P^τ_Λ, and those of Λ^c are almost surely 'frozen' at the value τ. We assume that the finite volume PCA dynamics P^τ_Λ are irreducible and aperiodic Markov Chains. They then admit one (and only one) stationary probability measure, called ν^τ_Λ (*i.e.* $\nu^\tau_\Lambda P^\tau_\Lambda = \nu^\tau_\Lambda$); furthermore P^τ_Λ is ergodic, which means $\lim_{n\to\infty} \rho_\Lambda (P^\tau_\Lambda)^{(n)} = \nu^\tau_\Lambda$ in the weak sense, for any initial condition ρ_Λ.

A sufficient condition for the irreducibility and aperiodicity of P^τ_Λ is for instance to assume that the PCA dynamics under study are *non degenerate*. This means: $\forall k \in \mathbb{Z}^d$, $\forall \eta \in S^{\mathbb{Z}^d}$, $\forall s \in S$, $p_k(\, s \mid \eta\,) > 0$. The following Proposition states that the finite volume stationary measures associated with extremal boundary conditions satisfy some sub/super-DLR relation (which means these measures are 'sub/super-Gibbs measures'). In the very special case where $S = \{-1, +1\}$ and for P reversible, this result was shown in Dai Pra et al. (2002).

Proposition 4.1 *Let ν^+_Λ (resp. ν^-_Λ) be the unique stationary probability measure associated with the finite volume PCA dynamics P^+_Λ (resp. P^-_Λ) with $+$ (resp. $-$) extremal boundary condition. Let $\Lambda \subset \Lambda' \Subset \mathbb{Z}^d$. One has, for any configuration σ,*

$$\nu^-_{\Lambda'}(.|\sigma_{\Lambda'\setminus\Lambda}) \succcurlyeq \nu^-_\Lambda(.) \quad \textit{and} \quad \nu^+_{\Lambda'}(.|\sigma_{\Lambda'\setminus\Lambda}) \preccurlyeq \nu^+_\Lambda(.). \tag{12}$$

Proof. First, using (3) shows that the pair of PCA's $(P^+_{\Lambda'}, P^+_\Lambda \otimes \delta_{+_{\Lambda'\setminus\Lambda}})$ (resp. $(P^-_\Lambda \otimes \delta_{-_{\Lambda'\setminus\Lambda}}, P^-_{\Lambda'})$ on $S^{\Lambda'}$) is increasing. Using the increasing coupling defined in Theorem 2.4, we may then state: for any initial condition σ and for $n \geqslant 1$ that

$$P^+_{\Lambda'} \circledast \left(P^+_\Lambda \otimes \delta_{+_{\Lambda'\setminus\Lambda}}\right)\left(f(\omega^2(n)) - f(\omega^1(n))|\ (\omega^1,\omega^2)(0) = (\sigma,\sigma)\ \right) \geqslant 0,$$

where f is any increasing function on $S^{\mathbb{Z}^d}$. Thus

$$P^+_{\Lambda'}(f(\omega(n)) \mid \omega(0) = \sigma) \leqslant P^+_\Lambda \otimes \delta_{+_{\Lambda'\setminus\Lambda}}(f(\omega(n)) \mid \omega(0) = \sigma).$$

Letting $n \to \infty$ and using finite volume ergodicity yields $\nu^+_{\Lambda'} \preccurlyeq \nu^+_\Lambda \otimes \delta_{+_{\Lambda'\setminus\Lambda}}$. Similarly, $\nu^-_\Lambda \otimes \delta_{-_{\Lambda'\setminus\Lambda}} \preccurlyeq \nu^-_{\Lambda'}$.

Let $\sigma_{\Lambda'\setminus\Lambda} \in S^{\Lambda'\setminus\Lambda}$. Let B be the event $B = \{\omega \in S^{\Lambda'} : \omega_{\Lambda'\setminus\Lambda} = \sigma_{\Lambda'\setminus\Lambda}\}$. Consider a sequence of independent, identically distributed random variables $(Z_n)_{n\geqslant 1}$, with distribution $\nu^+_{\Lambda'}$. Let Y be a random variable with distribution $\nu^+_\Lambda \otimes \delta_{+_{\Lambda'\setminus\Lambda}}$. Let T be the stopping time $\inf\{n \geqslant 1 : Z_n \in B\}$. Checking that $\forall n \geqslant 1$, $Z_n \preccurlyeq Y$ almost surely, one then has $Z_T \preccurlyeq Y$. This in turn means that $\nu^+_{\Lambda'}(.|\sigma_{\Lambda'\setminus\Lambda}) \preccurlyeq \nu^+_\Lambda(.)$ and the other inequality is proved in the same way. □

Proposition 4.2 *Let $\Lambda \Subset \mathbb{Z}^d$. The measure ν^+_Λ (resp. ν^-_Λ) is the maximal (resp. minimal) measure of the set $\{\nu^\tau_\Lambda : \tau \in S^{\Lambda^c}\}$. Let ν^+ and ν^- denote the maximal and the minimal elements of the set $\mathcal{S}$ of stationary measures on $S^{\mathbb{Z}^d}$ associated to the PCA dynamics P.*

The following relations hold:

$$\nu^+ = \lim_{L\to\infty} \nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}} = \lim_{n\to\infty} \delta_+ P^{(n)} \tag{13}$$

$$\nu^- = \lim_{L\to\infty} \nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}} = \lim_{n\to\infty} \delta_- P^{(n)}, \tag{14}$$

where for L integer, $\mathcal{B}(L)$ is the l^1-ball $\{k \in \mathbb{Z}^d : \|k\|_1 = \sum_{j=1}^d |k_j| \leqslant 1\}$. In particular, P admits a unique stationary measure ν if and only if $\nu^- = \nu^+$.

Proof. Let us first prove that: $\tau \preccurlyeq \tau' \Rightarrow \nu^\tau_\Lambda \preccurlyeq \nu^{\tau'}_\Lambda$. Let f be an increasing function on $S^{\mathbb{Z}^d}$. It is easy to check that $(P^\tau_\Lambda, P^{\tau'}_\Lambda)$ is a increasing pair, thus $P^\tau_\Lambda \circledast P^{\tau'}_\Lambda$ preserves stochastic order. Let $\sigma \in S^{\mathbb{Z}^d}$ be an initial condition. Since $\sigma_\Lambda \tau_{\Lambda^c} \preccurlyeq \sigma_\Lambda \tau'_{\Lambda^c}$, such an inequality is at time n still valid. Using the monotonicity of f, we have:

$$P^\tau_\Lambda \circledast P^{\tau'}_\Lambda \big(f(\omega^2(n)) - f(\omega^1(n))|\ (\omega^1,\omega^2)(0) = (\sigma,\sigma)\ \big) \geqslant 0\ .$$

Thus $P^\tau_\Lambda(f(\omega(n)) \mid \omega(0) = \sigma) \leqslant P^{\tau'}_\Lambda(f(\omega(n)) \mid \omega(0) = \sigma)$. The first result thus follows by letting $n \to \infty$ and using finite volume ergodicity; the extremality of ν^+_Λ and ν^-_Λ follows.

Then, note that $\lim_{L\to\infty}(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}})$ and $\lim_{L\to\infty}(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}})$ exist due to monotonicity of the following sequences: $(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}})_L$ and $(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}})_L$. This comes from the fact that $\wp_\Lambda\ \nu^+_{\Lambda'} \preccurlyeq \nu^+_\Lambda$ (where $\Lambda \Subset \Lambda' \Subset \mathbb{Z}^d$ and $\wp_\Lambda$ denotes the projection on Λ) which is easily checked using the increasing coupling $(P^+_{\Lambda'}, P^+_\Lambda)$. Since $\nu^+_{\mathcal{B}(L)}$ is P^+_Λ-stationary (resp. $\nu^-_{\mathcal{B}(L)}$ is P^-_Λ-stationary), the limits $\lim_{L\to\infty}(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}})$ and $\lim_{L\to\infty}(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}})$ are P-stationary.

Let ν be a P-stationary measure, and L any positive integer. Since the coupling $P^-_{\mathcal{B}(L)} \circledast P \circledast P^+_{\mathcal{B}(L)}$ preserves stochastic order, using finite volume ergodicity, one can state:
$\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}} \preccurlyeq \nu \preccurlyeq \nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}}$. We then have:

$$\lim_{L\to\infty} \nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}} \preccurlyeq \nu \preccurlyeq \lim_{L\to\infty} \nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}}. \tag{15}$$

On the other hand, it is easy to check $\delta_+ P \preccurlyeq \delta_+$, so that, using P's attractivity, $(\delta_+ P^{(n)})_{n\in\mathbb{N}}$ is decreasing. Analogously, $(\delta_- P^{(n)})_{n\in\mathbb{N}}$ is increasing. Thus, the limits $\lim_{n\to\infty} \delta_- P^{(n)}$ and $\lim_{n\to\infty} \delta_+ P^{(n)}$ exist and are obviously P-stationary measures.

Let ν be a P-stationary measure. Since P is attractive and $\delta_- \preccurlyeq \nu \preccurlyeq \delta_+$, we have:

$$\lim_{n\to\infty} \delta_- P^{(n)} \preccurlyeq \nu \preccurlyeq \lim_{n\to\infty} \delta_+ P^{(n)}. \tag{16}$$

Using the fact that the measures $\lim_{L\to\infty}(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}})$, $\lim_{L\to\infty}(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\Lambda^c}})$, $\lim_{n\to\infty} \delta_- P^{(n)}$ and $\lim_{n\to\infty} \delta_+ P^{(n)}$ are P-stationary, we apply inequalities (15) and (16), and the conclusion follows. □

Thanks to the above constructed coupling, we investigate the time-asymptotical behaviour. The PCA dynamics P on the infinite volume space $S^{\mathbb{Z}^d}$ considered in this subsection is assumed to be translation invariant (or *space homogeneous*): $\forall k \in \mathbb{Z}^d$, $\forall s \in S$, $\forall \eta \in S^{\mathbb{Z}^d}$, $p_k(\ s \mid \eta\ ) = p_0(\ s \mid \theta_{-k}\eta\ )$, where $\theta_{k_0}(\sigma) = (\sigma_{k-k_0})_{k\in\mathbb{Z}^d}$. Remark that if the PCA dynamics P^i are translation invariant, so is the coupled dynamics $P^1 \circledast \ldots \circledast P^N$.

We will use the notation $\mathbf{P}$ to denote the coupling $P \circledast P \circledast \ldots \circledast P$ of N times the same attractive PCA dynamics P. Using the compatibility property of the Lemma 3.2, the marginal of $P^{\circledast N'}$ on N components chosen in $\{1, .., N'\}$ is the same as the coupling $P^{\circledast N}$. Here, it is enough to choose $N = 4$.

As Proposition 4.2 shows, in order to study the behaviour of a PCA dynamics P on $S^{\mathbb{Z}^d}$, one may turn its attention to the finite volume associated dynamics P_Λ^τ on S^Λ, where $\Lambda \Subset \mathbb{Z}^d$. Note that their time asymptotics are known. Using the increasing coupling, Proposition 4.4 below shows how the time-asymptotical behaviour of our PCA is controlled by the sequence $(\rho(n))_{n\geqslant 1}$, where

$$\rho(n) = \mathbf{P}\Big(\omega_0^1(n) \neq \omega_0^2(n) \Big| (\omega^1, \omega^2)(0) = (-,+)\Big), \tag{17}$$

where $\mathbf{P}$ is the coupling introduced in Corollary 2.5. In the paper Louis (2004) we gave conditions to ensure the convergence of $(\rho(n))_{n\geqslant 1}$ and stated conditions for the ergodicity with exponential speed of the dynamics P.

Let $\Lambda \Subset \mathbb{Z}^d$. Let P_Λ^+ (resp. P_Λ^-) be the dynamics on S^Λ defined in (11) with the maximal (resp. minimal) boundary condition $+$ (resp. $-$). First note the easily checked fact:

Lemma 4.3 *If the PCA dynamics P is attractive then $(P_\Lambda^-, P, \ldots, P, P_\Lambda^+)$ is increasing, and thus the increasing coupling $P_\Lambda^- \circledast P \circledast \ldots \circledast P \circledast P_\Lambda^+$ can be defined.*

Proposition 4.4 *Let $\sigma \preccurlyeq \eta \in S^{\mathbb{Z}^d}$ and P be an attractive PCA dynamics. The following inequality holds:*

$$\mathbf{P}\Big(\omega_0^1(n) \neq \omega_0^2(n) \Big| (\omega^1, \omega^2)(0) = (\sigma, \eta)\Big) \leqslant \rho(n) \leqslant P_\Lambda^- \circledast P_\Lambda^+ (\omega_0^1(n) \neq \omega_0^2(n)\ |(\omega^1, \omega^2)(0) = (-,+)) \tag{18}$$

where $(\rho(n))_{n\in\mathbb{N}^*}$ is defined by (17). For each initial condition ξ on $S^{\mathbb{Z}^d}$ and for any time n, it holds:

$$P_\Lambda^- \big(\omega(n) \in .\big|\omega(0) = \xi_\Lambda(-)_{\Lambda^c}\big) \preccurlyeq P\big(\omega(n) \in .\big|\omega(0) = \xi\big) \preccurlyeq P_\Lambda^+ \big(\omega(n) \in .\big|\omega(0) = \xi_\Lambda(+)_{\Lambda^c}\big). \tag{19}$$

The sequence $(\rho(n))_{n\in\mathbb{N}^*}$ is decreasing, and P is ergodic if and only if $\lim_{n\to\infty} \rho(n) = 0$. Moreover, in this case,

$$\sup_\sigma \Big| P\Big(f(\omega(n))|\omega(0) = \sigma\Big) - \nu(f)\Big| \leqslant 2\ |||\ f\ |||\ \ \rho(n) \tag{20}$$

where ν denotes the unique stationary measure and where, for each f continuous function on the compact $S^{\mathbb{Z}^d}$ and for all k in $\mathbb{Z}^d$, $\Delta_f(k) = \sup\Big\{\Big|f(\sigma) - f(\eta)\Big| \ :\ (\sigma, \eta) \in (S^{\mathbb{Z}^d})^2, \sigma_{\{k\}^c} \equiv \eta_{\{k\}^c}\Big\}$, whereas $|||\ f\ ||| = \sum_{k\in\mathbb{Z}^d} \Delta_f(k)$.

Proof. The proof of the left inequality in (18) is straightforward using the compatibility property from Lemma 3.2. The right inequality comes from the confunction of two properties: preservation of the stochastic order as well as the compatibility property of the coupling $P_\Lambda^- \circledast P \circledast P \circledast P_\Lambda^+$.

Since the coupling $P_\Lambda^- \circledast P \circledast P_\Lambda^+$ is increasing, (19) is a consequence of the fact that any initial condition ξ in $S^{\mathbb{Z}^d}$ satisfies $\xi_\Lambda(-)_{\Lambda^c} \preccurlyeq \xi \preccurlyeq \xi_\Lambda(+)_{\Lambda^c}$.

The monotonicity of the sequence $(\rho(n))_{n\in\mathbb{N}^*}$ comes from the coalescence property of the increasing coupling $\mathbf{P}$.

If P is ergodic, there can be only one stationary measure on $S^{\mathbb{Z}^d}$ and so $\lim_{n\to\infty}\rho(n)=0$.

Conversely, let f be a local function. For any σ,η configurations in $S^{\mathbb{Z}^d}$, let us write:

$$\begin{aligned}&\Big|P(f(\omega(n))|\omega(0)=\sigma)-P(f(\omega(n))|\omega(0)=\eta)\Big|\\ &\leqslant\Big|\mathbf{P}\Big(f(\omega^1(n))-f(\omega^2(n))\ \Big|(\omega^1,\omega^2)(0)=(-,\sigma)\Big)\Big|+\Big|\mathbf{P}\Big(f(\omega^1(n))-f(\omega^2(n))\ \Big|(\omega^1,\omega^2)(0)=(-,\eta)\Big)\Big|.\end{aligned}$$

Since f is local, for all ξ^1,ξ^2, $\Big|f(\xi^1)-f(\xi^2)\Big|$ depends only on $\xi^1_{\Lambda_f}$ and $\xi^2_{\Lambda_f}$, which differ only in a finite number of sites. Using interpolating configurations between $\xi^1_{\Lambda_f}$ and $\xi^2_{\Lambda_f}$, we write:
$|f(\xi^1)-f(\xi^2)|\leqslant\sum_{k\in\Lambda_f}\Delta_f(k)\mathbb{1}_{\{\sigma_k\neq\eta_k\}}$, so that the translation invariance assumption and the left part of (18) then yield: $\Big|P(f(\omega(n))|\omega(0)=\sigma)-P(f(\omega(n))|\omega(0)=\eta)\Big|\leqslant 2\ |||\ f\ |||\ \rho(n)$, which is enough to conclude and state (20). □

5 Partially ordered spin space case

In all this section, S is a partially ordered space. When S is totally ordered, a necessary and sufficient condition for the existence of an increasing coupling of PCA dynamics is given in section 3 by the inequality (6). It is done in term of the distribution function $F_k(.,\sigma)$ (σ given) of the probability $p_k(\ .\ |\sigma)$. We recall previous results, who gave a necessary and sufficient condition for the existence of an increasing coupling of two PCA's. In particular, P is attractive if and only if

$$\forall k\in\mathbb{Z}^d,\forall\sigma\preccurlyeq\eta,\forall\ \Gamma\text{ up-set in }S\Rightarrow\sum_{s\in\Gamma}p_k(s|\sigma)\leqslant\sum_{s\in\Gamma}p_k(s|\eta).\tag{21}$$

The quantity which now makes sense is the generalised function $F_k(\Gamma,\sigma):=\sum_{s'\in\Gamma}p_k(s'|\sigma)$, where Γ is an up-set of S (see Definition 5.1).

Nevertheless, there is a gap between coupling two PCA's or more than three PCA's. The counter-examples A and B presented here show that a satisfactory coupling of three PCA's may not exist and condition (21) of López and Sanz (2000) is not sufficient for the existence of an increasing 3-coupling when S is any partially ordered space.

These counter-examples rely on examples 1.1 and 5.7 in Fill and Machida (2001) of stochastically monotone families of distributions, indexed by a partially ordered set, which are not realisable monotone in the following sense. Let $(Q_\alpha)_{\alpha\in\mathcal{A}}$ be a family of probability distributions on a finite set $\mathcal{S}$ indexed by a partially ordered set $\mathcal{A}$. Fill and Machida (2001) define the system $(Q_\alpha)_{\alpha\in\mathcal{A}}$ as stochastically monotone if $\alpha_1\preccurlyeq_{\mathcal{A}}\alpha_2$ implies $Q_{\alpha_1}\preccurlyeq_{\mathcal{S}}Q_{\alpha_2}$. It is said to be realisable monotone if there exists a system of S-valued random variables $(X_\alpha)_{\alpha\in\mathcal{A}}$, defined on the same probability space, such that the distribution of X_α is Q_α and $\alpha_1\preccurlyeq_{\mathcal{A}}\alpha_2$ implies $X_{\alpha_1}\preccurlyeq_{\mathcal{S}}X_{\alpha_2}$ a.s.

In our case, the existence of a coupling of the N PCA dynamics $(P^1,\dots,P^N)$ implies that, for any $k\in\mathbb{Z}^d$ fixed, the system $\{p_k(.|\sigma_{V_k}):\sigma_{V_k}\in S^{V_k}\}$ of probability distributions on S, which is indexed by the partially ordered set S^{V_k}, is realisable monotone. In the counter-examples presented here, the distributions are stochastically monotone but not realisable monotone.

Definition 5.1 *A subset* Γ *of* S *is said to be an* up-set *(or increasing set) (resp. down-set or decreasing set) if:* $x\in\Gamma,y\in S,x\preccurlyeq y\Rightarrow y\in\Gamma$ *(resp.* $x\in\Gamma,y\in S,x\succcurlyeq y\Rightarrow y\in\Gamma$*).*

Note that the indicator function of an up-set (resp. down-set) is an increasing (resp. decreasing) function. Moreover, Theorem 1 in Kamae et al. (1977) states that for two measures μ_1, μ_2 on S, $\mu_1 \preccurlyeq \mu_2$ if and only if $\mu_1(\Gamma) \leqslant \mu_2(\Gamma)$ for all up-sets Γ of S, which is equivalent to $\mu_1(\Gamma) \geqslant \mu_2(\Gamma)$ for all down-sets Γ of S.

Counter-Example A

Let $S = S_A = \{0,1\}^2$ be equipped with the natural partial order $(0,0) \preccurlyeq (0,1)$, $(0,0) \preccurlyeq (1,0)$, $(0,1) \preccurlyeq (1,1)$, $(1,0) \preccurlyeq (1,1)$ (where $(0,1)$ and $(1,0)$ are not comparable). Let $P = \underset{k\in\mathbb{Z}^d}{\otimes} p_k$, where $p_k(\ .\ |\sigma) = p_k(\ .\ |\sigma_k)$ is defined as follows:

$$\begin{cases} p_k(\ .\ |(0,0)) = \frac{1}{2}(\delta_{(0,0)} + \delta_{(1,0)}) \\ p_k(\ .\ |(1,0)) = \frac{1}{2}(\delta_{(0,0)} + \delta_{(1,1)}) \\ p_k(\ .\ |(0,1)) = \frac{1}{2}(\delta_{(0,1)} + \delta_{(1,0)}) \\ p_k(\ .\ |(1,1)) = \frac{1}{2}(\delta_{(1,0)} + \delta_{(1,1)}). \end{cases} \tag{22}$$

It is simple to check that this PCA dynamics is attractive. Nevertheless, $P^{\circledast 4}$ can not exist since $(p_k(\ .\ |(0,0)), p_k(\ .\ |(1,0)), p_k(\ .\ |(0,1)), p_k(\ .\ |(1,1)))$ is a stochastically monotone family which is not realisable monotone (see example 1.1 in Fill and Machida (2001)). This PCA is in fact a collection of independent, S-valued Markov Chains, whose transition probability is $p_0(.|.)$. This example states the non-existence of a coupling of four particular Markov Chains.

Counter-Example B

Let $S = S_B = \{x,y,z,w\}$, considered with the following partial order $x \preccurlyeq z, y \preccurlyeq z, z \preccurlyeq w$ and x and y are not comparable. Consider the dimension $d = 1$, and the PCA $P = \underset{k\in\mathbb{Z}}{\otimes} p_k$ where $p_k(\ .\ |\sigma) = p_k(\ .\ |\sigma_{\{k,k+1\}})$ is defined as follows:

$$\begin{cases} p_k(\ .\ |(x,y)) = \frac{1}{2}(\delta_x + \delta_y) & p_k(\ .\ |(y,z)) = \delta_z \\ p_k(\ .\ |(x,z)) = \frac{1}{2}(\delta_x + \delta_w) & p_k(\ .\ |(z,x)) = \delta_z \\ p_k(\ .\ |(z,y)) = \frac{1}{2}(\delta_y + \delta_w) & p_k(\ .\ |(x,x)) = \delta_x \\ p_k(\ .\ |(z,z)) = \frac{1}{2}(\delta_z + \delta_w) & p_k(\ .\ |(y,x)) = \delta_z \\ p_k(\ .\ |(y,y)) = \delta_y & p_k(\ .\ |\text{otherwise}) = \delta_w \end{cases} \tag{23}$$

This is an attractive PCA, nevertheless a synchronous coupling $P^{\circledast 4}$ can not exist since $(p_k(\ .\ |(x,y)), p_k(\ .\ |(x,z)), p_k(\ .\ |(z,y)), p_k(\ .\ |(z,z)))$ is a stochastically monotone family which is also non realisable monotone (see example 5.7 in Fill and Machida (2001)).

Let us now present some generalisation of our main results, Theorem 2.4 and Corollary 2.5, when the spin space S belongs to a special class $\mathcal{Z}$ of partially ordered sets introduced in Fill and Machida (2001) and called *linearly ordered spaces.*

We call *predecessor* (resp. *successor*) of s ($s \in S$) any element s' such that $s \preccurlyeq s'$ (resp. $s \succcurlyeq s'$) and $s \preccurlyeq s'' \preccurlyeq s' \Rightarrow s'' \in \{s, s'\}$ (resp. $s \succcurlyeq s'' \succcurlyeq s' \Rightarrow s'' \in \{s, s'\}$). S belongs to the class $\mathcal{Z}$ if, for any $s \in S$, only one of the following situations occurs: s admits exactly one successor and one predecessor; s admits no predecessor and at most two successors; s admits no sucessor and at most two predecessors. One can then define on S a linear order $\leqslant_n$ by numbering the elements of S: $\{s_1, \ldots, s_n\}$ (where $n = \#S$) in such a way that s_{i+1} be a sucessor or a predecessor of S_i (for $i = 1, \ldots, n$) and

declaring that $s_i \leqslant_n s_j$ if $i \leqslant j$. Of course such linear order might be incompatible with the partial order $\preccurlyeq$ originally defined on S.
The set $S_C = \{s_i, 1 \leqslant i \leqslant 10\}$ with order relations: $s_1 \preccurlyeq s_2 \preccurlyeq s_3 \preccurlyeq s_4$, $s_6 \preccurlyeq s_5 \preccurlyeq s_4$, $s_6 \preccurlyeq s_7 \preccurlyeq s_8$, and $s_{10} \preccurlyeq s_9 \preccurlyeq s_8$ is an example of such a space. On the other hand, the spin space $S_D = \{x, y, z, u, v, w\}$ with order relations $y \preccurlyeq z$, $x \preccurlyeq z$, $w \preccurlyeq z$, $w \preccurlyeq u$, $w \preccurlyeq v$ does not belong to this class.

Define, for $s_i \in S$ $(1 \leqslant i \leqslant n)$, the subset $(\leftarrow, s_i]$ of S with $(\leftarrow, s_i] = \{s_j \in S \ : s_j \leqslant_n s_i\}$. The sets $(\leftarrow, s]$ (with $s \in S$) are either up-sets or down-sets of S (respective to the original order $\preccurlyeq$ on S). For instance, when $S = S_C$, $(\leftarrow, s_5]$ is an up-set and $(\leftarrow, s_6]$ is a down-set. It suffices to consider the generalised function $F_k(\Gamma, \sigma)$ for sets of the form $\Gamma = (\leftarrow, s]$ for which we have:

$$F_k(s,\sigma) = p_k(\ (\leftarrow, s] \mid \sigma) = \sum_{s' \in (\leftarrow, s]} p_k(s'|\sigma) \qquad (s \in S,\ \sigma \in S^{\mathbb{Z}^d}). \tag{24}$$

When S is a linearly ordered set, the monotonicity condition (3) is equivalent to the following conditions for the generalised associated distribution functions (Lemma 5.5 in Fill and Machida (2001)): $\forall k \in \mathbb{Z}^d, \forall (\zeta^1, \zeta^2, \ldots, \zeta^N) \in (S^{\mathbb{Z}^d})^N$ such that $\zeta^1 \preccurlyeq \zeta^2 \preccurlyeq \ldots \preccurlyeq \zeta^N$,

$$\forall s \in S \text{ with } (\leftarrow, s] \text{ down-set}, F_k^1(s \mid \zeta^1) \geqslant F_k^2(s \mid \zeta^2) \geqslant \ldots \geqslant F_k^N(s \mid \zeta^N). \tag{25}$$

$$\forall s \in S \text{ with } (\leftarrow, s] \text{ up-set}, F_k^1(s \mid \zeta^1) \leqslant F_k^2(s \mid \zeta^2) \leqslant \ldots \leqslant F_k^N(s \mid \zeta^N) \tag{26}$$

Proposition 5.2 *When S is a linearly ordered spin space, Theorem* 2.4 *and Corollary* 2.5 *still hold.*

Proof. The proof of such results relies on the following construction. Let us define the generalised probability transform, for $\sigma \in S^{\mathbb{Z}^d}$ and $k \in \mathbb{Z}^d$ fixed: $(F_k(\ .\ , \sigma))^{-1}(t) = \inf_{\leqslant_n} \{s_k : t < F_k(s_k, \sigma)\}$, $(t \in]0,1[)$, where the infimum is given in term of the linear order $\leqslant_n$. The construction of the increasing coupling holds as before thanks to the following evolution rule (9) between times n and $n+1$, where $(F_k^i)^{-1}$ denotes the generalised distribution function, as introduced before. The coherence of this coupling with the partial order $\preccurlyeq$ is proved in Lemma 6.2 in Fill and Machida (2001). □

Acknowledgements

The author thanks P. Dai Pra and G. Posta for helpful comments, O. Häggström for mentioning the reference Fill and Machida (2001), and S. Rœlly for a careful reading and comments to the preliminary version of this paper. He thanks as well M. Sortais for helping to improve the english exposition.

References

Dai Pra, P., Louis, P.-Y., Rœlly, S., 2002. Stationary measures and phase transition for a class of probabilistic cellular automata. ESAIM : Probability and Statistics 6, 89–104.

Dawson, D. A., 1974. Synchronous and asynchronous reversible Markov systems. Canad. Math. Bull. 17 (5), 633–649.

Fill, J. A., Machida, M., 2001. Stochastic monotonicity and realizable monotonicity. Ann. Probab. 29 (2), 938–978.

Kamae, T., Krengel, U., O'Brien, G., 1977. Stochastic inequalities on partially ordered spaces. Ann. Probab. 5 (6), 899–912.

Lindvall, T., 1992. Lectures on the coupling method. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, a Wiley-Interscience Publication.

López, F. J., Sanz, G., 2000. Stochastic comparisons for general probabilistic cellular automata. Statist. Probab. Lett. 46 (4), 401–410.

Louis, P.-Y., september 2002. Automates cellulaires probabilistes : mesures stationnaires, mesures de gibbs associées et ergodicité. Ph.D. thesis, Université de Lille 1 and Politecnico di Milano, available at URL: `http://tel.ccsd.cnrs.fr/documents/archives0/00/00/22/45/index_fr.html`.

Louis, P.-Y., 2004. Ergodicity of PCA: equivalence between spatial and temporal mixing conditions. Elec. Comm. in Prob. 9, 119–131.

Maes, C., 1993. Coupling interacting particle systems. Rev. in Math. Phys. 5 (3), 457–475.

Thorisson, H., 2000. Coupling, stationarity, and regeneration. Springer-Verlag, New York.

Toom, A., Vasilyev, N., Stavskaya, O., Mityushin, L., Kurdyumov, G., Pirogov, S., 1978. Locally interacting systems and their application in biology. Springer-Verlag, Berlin, lecture Notes in Mathematics, Vol. 653. See also *Stochastic Cellular Systems: ergodicity, memory, morphogenesis*, 1990, Manchester University Press.

Vasershtein, L. N., 1969. Markov processes over denumerable products of spaces describing large system of automata. Problemy Peredači Informacii 5 (3), 64–72.

Younes, L., 1998. Synchronous random fields and image restoration. IEEE TPAMI 20 (4).