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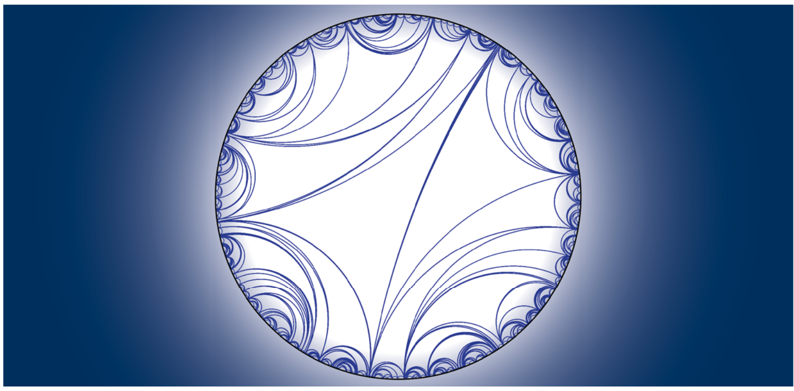

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ASYMPTOTIC SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE HEAT EQUATION AT A CHARACTERISTIC POINT

A. ANTONIOUK, O. KISELEV, V. A. STEPANENKO, AND N. TARKHANOV

ABSTRACT. The Dirichlet problem for the heat equation in a bounded domain $\mathcal{G} \subset \mathbb{R}^{n+1}$ is characteristic, for there are boundary points at which the boundary touches a characteristic hyperplane $t = c$, c being a constant. It was I.G. Petrovskii (1934) who first found necessary and sufficient conditions on the boundary which guarantee that the solution is continuous up to the characteristic point, provided that the Dirichlet data are continuous. This paper initiated standing interest in studying general boundary value problems for parabolic equations in bounded domains. We contribute to the study by constructing a formal solution of the Dirichlet problem for the heat equation in a neighbourhood of a characteristic boundary point and showing its asymptotic character.

CONTENTS

INTRODUCTION

The problem we consider in this paper goes back at least as far as [Gev13] who proved the existence of a classical solution to the first boundary value problem for the heat equation in a non-cylindrical plane domain. By classical is meant "continuous up to the boundary," and a boundary point is called regular if any

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weak solution of the problem is continuous up to the point, provided the boundary data are continuous. The domain is assumed to be bounded by an interval $[a, b]$ of the x-axis and two curves $x = X_1(t)$ and $x = X_2(t)$ in the upper half-plane through $(a, 0)$ and $(b, 0)$, respectively. The Dirichlet data are posed on the interval and both lateral curves. All points of the interval $[a, b]$ are characteristic. The interval $[a, b]$ may shrink up to a point (say $(0, 0)$) in which case the origin is the only characteristic point.

The theory of [Gev13] applies in particular to the plane domains $\mathcal{G}$ consisting of all $(x, t) \in \mathbb{R}^2$, such that $|x| < 1$ and $\mathfrak{f}(|x|) < t < \mathfrak{f}(1)$, where $\mathfrak{f}(r)$ is a C^1 function on $(0, 1]$ satisfying $\mathfrak{f}(r) > 0$, $\mathfrak{f}'(r) \neq 0$ for all $r \in (0, 1]$ and $\mathfrak{f}(0+) = 0$. The boundary point $(0, 0)$ proves to be regular if $\mathfrak{f}^{-1}(t)$ satisfies the Hölder condition of exponent larger than $1/2$. When applied to the function $\mathfrak{f}(r) = r^p$, this obviously implies $0 < p < 2$. Note that for $1 < p < 2$ the origin is a true (i.e., smooth) characteristic point at the boundary while for $0 < p < 1$ this is a cuspidal (i.e., singular) boundary point.

The paper [Gev13] exploited the fundamental solution of the heat equation and integral equations of potential theory. A more careful analysis led Petrovskii in [Pet34] to an explicit necessary and sufficient condition for a boundary point to be regular. This latter paper initiated an extensive literature devoted to general boundary value problems for parabolic equations, see [Mik63], [Kon66], etc. Mention that the classical paper [Slo58] was actually motivated by the first boundary problem for the heat equation in a bounded domain $\mathcal{G} \subset \mathbb{R}^n$. On the other hand, [Kon66] made essential use of function spaces of Slobodetskii [Slo58]. Unfortunately, [Kon66] suffers several drawbacks which, however, do not affect the main result of this seminal paper.

The most cited paper of Kondrat'ev is [Kon67] studying boundary value problems for elliptic equations in domains with conical points on the boundary. Asymptotics of solutions of general boundary value problems for elliptic equations in domains with cusps remains still a challenge for mathematicians, see [KS10].

According to the MathSciNet of the AMS there has been merely 8 citations to the paper [Kon66] while this latter already contains all of the techniques of [Kon67], especially the asymptotics of solutions at conical points. At the end of the '90s Kondrat'ev called the last author's attention to the paper [Kon66] saying "Here are cusps." In spite of the fact that [Kon66] deals with C^∞ boundaries the analysis near characteristic boundary points reveals Fuchs-type operators typical for conical singularities, provided that the contact degree of the boundary and characteristic plane is at least the anisotropy quotient (2 for the heat equation). If the contact degree is less than the anisotropy quotient, the analysis close to the characteristic point requires pseudodifferential operators typical for cuspidal points on the boundary, cf. [Gev13] discussed above.

The structure of asymptotics at a conical point is completely determined by the spectrum of the problem frozen at the singular point. To an eigenvalue λ_n of multiplicity μ_n there correspond eigenfunctions $|x|^{-\imath\lambda_n}(\log|x|)^j$ with $j = 0, 1, \ldots, \mu_n - 1$. Each horizontal strip of finite width in the complex plane contains finitely many values λ_n, and the set of all λ_n is infinite. The expansions of solutions over these basic functions fail usually to converge, and so the series should be thought of as asymptotic.

Moreover, in the absence of embedding theorems the concept of asymptotic in the sense of Poincaré does not apply. We are thus led to asymptotic expansions related to certain filtrations on function spaces, a purely algebraic concept, which is a true substitution for Poincaré's asymptotics, see for instance [Len83] and elsewhere.

In mathematics, a (descending) filtration is an indexed set $\mathcal{F}_n$ of subspaces of a given vector space $\mathcal{F}$, with the index n running over entire numbers, subject to the condition that $S_{n+1} \subset S_n$ for all n. Let $\mathcal{F}_{-\infty}$ be the union of the $\mathcal{F}_n$. Given any $f \in \mathcal{F}_{-\infty}$, by

$$f \sim \sum_{n=n_f}^{\infty} f_n \tag{0.1}$$

with $f_n \in \mathcal{F}_n$ is meant that

$$f - \sum_{n=n_f}^{N} f_n \in \mathcal{F}_{N+1}$$

holds for every $N \geq n_f$. We intend to develop this generalisation of Poincaré's asymptotics in a forthcoming publication.

As filtration Kondrat'ev used in [Kon66] weighted Slobodetskii spaces, where the weight functions are powers of the distance to the characteristic point. Analysis on manifolds with point and more general singularities has since exploited weighted function spaces.

In [AB96], [AB98] the first boundary problem is studied for the heat equation in a bounded plane domain with cuspidal points at the boundary at which the tangent coincides with a characteristic $t = c$, where c is a constant. The paper [AT12] contributed to the study of the first boundary problem for the **1D** heat equation in a bounded plane domain by evaluating the first term of the asymptotic of a solution at the characteristic point. The goal of the present paper is to explicitly compute full asymptotic expansions and extend these results to higher order equations in many variables.

Our scheme of construction of asymptotic series for a solution near a characteristic point consists in the following. In Section 1 we resolve singularities at the characteristic point by blowing-up this point to a segment of the x-axis containing the origin. The domain $\mathcal{G}$ close to the origin blows up to a half-strip. In Section 2 we construct a formal solution of the transformed problem in the half-strip. This is actually a formal Puiseux series in fractional powers of t unless $p = 2$. In Section 3.1 we construct a formal solution in the case $p = 2$, which reveals immediately asymptotic expansions on manifolds with conical points. In Section 4 we show how these expansions are generalised to higher dimensions. To prove the asymptotic character of formal solution we need an existence theorem which is a part of Fredholm theory for the first boundary problem for the heat equation. To this end we describe in Section 5 a change of variables which transforms the characteristic point to the point at infinity along the t-axis. In Section 6 we discuss the Fredholm property of the first boundary problem for the heat equation. When the Fredholm property has been proved one obtains real solutions of the problem which expand as formal series. In this case one introduces the difference between the real solution and a partial sum of the formal series and substitutes this remainder to the equations. This yields a nonhomogeneous problem for the remainder, and the formal solvability might testify to the possibility of estimating the remainder. We follow this way to show in Section 7 the asymptotic character of formal solution in the sense (0.1). In the last Section 8 we present some explicit computations for an inverse parabolic equation.

Needless to say that our results go far beyond the first boundary value problem for the heat equation and extend to general boundary value problems for parabolic equations in bounded domains.

Part 1. Formal solution of the heat equation in the plane

1. Blow-up techniques

Consider the first boundary value problem for the heat equation in a bounded domain $\mathcal{G} \subset \mathbb{R}^2$. The boundary of $\mathcal{G}$ is assumed to be C^∞ except for a finite number of singular points. A boundary point is called characteristic if the boundary is smooth at this point and the tangent is orthogonal to the t-axis. Since $\mathcal{G}$ is bounded, there are at least two characteristic points on the boundary unless it has a singularity at a characteristic point. In this paper we restrict our discussion to characteristic points which may moreover bear boundary singularities. By the local principle of [Sim65] it is sufficient to study the problem only in a small neighbourhood of any characteristic (singular) point. Thus, the domain $\mathcal{G}$ looks like that of Figure 1 with $n = 1$, i.e. it is bounded by a curve $t = |x|^p$, with $p > 0$ an arbitrary real number, from below and by a horizontal segment from above. This is a typical domain for problems of such a type. As usual, no conditions are posed on the upper segment, see [TS72].

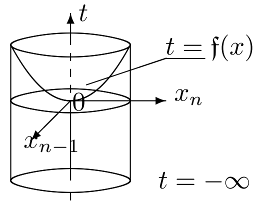

Figure 1. *Resolution of singularities at a characteristic point*

If $p > 1$, then the origin is a characteristic point of the boundary. If $0 < p \leq 1$, then the boundary has a singularity at the origin, which is a conical point for $p = 1$ and a cusp for $p < 1$. As mentioned, the case $p \geq 2$ was treated in [Kon66] in the framework of analysis of Fuchs-type operators. The paper [AT12] demonstrates rather strikingly that, for $0 < p < 2$, the problem to be considered is specified in analysis on manifolds with cusps. A modern approach to studying boundary value problems in domains with cuspidal boundary points is based on the so-called blow-up techniques, cf. [RST00]. While giving a complete characterisation of Fredholm problems, the approach falls short of providing asymptotics of solutions at singular points.

The first boundary value problem for the heat equation in the domain $\mathcal{G}$ is formulated as follows: Write Σ for the set of all characteristic points $0, \ldots$ on the boundary of $\mathcal{G}$. Given functions f in $\mathcal{G}$ and u_0 at $\partial\mathcal{G} \setminus \Sigma$, find a function u on $\overline{\mathcal{G}} \setminus \Sigma$ which satisfies

$$\begin{array}{rcll} u'_t - u''_{x,x} & = & f & \text{in } \mathcal{G}, \\ u & = & u_0 & \text{at } \partial\mathcal{G} \setminus \Sigma. \end{array} \tag{1.1}$$

By the local principle of Simonenko [Sim65], the Fredholm property of problem (1.1) in suitable function spaces is equivalent to the local invertibility of this problem at each point of the closure of $\mathcal{G}$. Here we focus upon the characteristic points like the origin 0.

Suppose the domain $\mathcal{G}$ is described in a neighbourhood of the origin by the inequality

$$t > |x|^p, \tag{1.2}$$

where p is a positive real number. There is no loss of generality in assuming that $|x| \le 1$.

We now blow up the domain $\mathcal{G}$ at P_3 by introducing new coordinates (ω, r) with the aid of

$$\begin{array}{rcl} x & = & t^{1/p}\,\omega, \\ t & = & r, \end{array} \tag{1.3}$$

where $|\omega| < 1$ and $r \in (0,1)$. It is clear that the new coordinates are singular at $r = 0$, for the entire segment $[-1,1]$ on the ω-axis is blown down into the origin by (1.3). The rectangle $(-1,1) \times (0,1)$ transforms under the change of coordinates (1.3) into the part of the domain $\mathcal{G}$ nearby 0 lying below the line $t = 1$.

In the domain of coordinates (ω, r) problem (1.1) reduces to an ordinary differential equation with respect to the variable r with operator-valued coefficients. More precisely, under transformation (1.3) the derivatives in t and x change by the formulas

$$\begin{array}{rcl} \dfrac{\partial u}{\partial t} & = & \dfrac{\partial u}{\partial r} - \dfrac{1}{r}\dfrac{\omega}{p}\dfrac{\partial u}{\partial \omega}, \\ \dfrac{\partial u}{\partial x} & = & \dfrac{1}{r^{1/p}}\dfrac{\partial u}{\partial \omega}, \end{array}$$

and so (1.1) transforms into

$$\begin{array}{rclll} r^Q\, U'_r - U''_{\omega,\omega} - r^{Q-1}\dfrac{\omega}{p}\, U'_\omega & = & r^Q F & \text{in} & (-1,1)\times(0,1), \\ U & = & U_0 & \text{at} & \{\pm 1\}\times(0,1), \end{array} \tag{1.4}$$

where $U(\omega, r)$ and $F(\omega, r)$ are pullbacks of $u(x,t)$ and $f(x,t)$ under transformation (1.3), respectively, and

$$Q = \frac{2}{p}.$$

We are now interested in the local solvability of problem (1.4) near the edge $r = 0$ in the rectangle $(-1,1)\times(0,1)$. Note that the equation degenerates at $r = 0$, since the coefficient $r^{2/p}$ of the higher order derivative in r vanishes at $r = 0$. The exponent Q is of crucial importance for specifying the ordinary differential equation. If $p = 2$ then it is a Fuchs-type equation, these are also called regular singular equations. The Fuchs-type equations fit well into an algebra of pseudodifferential operators based on the Mellin transform. If $p > 2$, then the singularity of the equation at $r = 0$ is weak and so regular theory of finite smoothness applies. In the case $p < 2$ the degeneracy at $r = 0$ is strong and the equation can not be treated except by the theory of slowly varying coefficients [RST00].

2. Formal asymptotic solution

To determine appropriate function spaces in which a solution of problem (1.4) is sought, one constructs formal asymptotic solutions of the corresponding homogeneous problem. That is

$$\begin{array}{rcll} r^Q\, U'_r - U''_{\omega,\omega} - r^{Q-1}\dfrac{\omega}{p}\, U'_\omega & = & 0 & \text{in} \quad (-1,1)\times(0,\infty), \\ U(\pm 1, r) & = & 0 & \text{on} \quad (0,\infty). \end{array} \tag{2.1}$$

We first consider the case $p \neq 2$. We look for a formal solution to (2.1) of the form

$$U(\omega,r) = e^{S(r)}\, V(\omega,r), \tag{2.2}$$

where S is a differentiable function of $r > 0$ and V expands as a formal Puiseux series with nontrivial principal part

$$V(\omega,r) = \frac{1}{r^{\mathfrak{e}N}} \sum_{j=0}^{\infty} V_{j-N}(\omega)\, r^{\mathfrak{e}j},$$

the (possibly) complex exponent N and real exponent $\mathfrak{e}$ have to be determined. Perhaps the factor $r^{-\mathfrak{e}N}$ might be included into the definition of $\exp S$ as $\exp(-\mathfrak{e}N \ln r)$, however, we prefer to highlight the key role of Puiseux series.

Substituting (2.2) into (2.1) yields

$$\begin{array}{rcll} r^Q\, (S'V + V'_r) - V''_{\omega,\omega} - r^{Q-1}\dfrac{\omega}{p}\, V'_\omega & = & 0 & \text{in} \quad (-1,1)\times(0,\infty), \\ V(\pm 1, r) & = & 0 & \text{on} \quad (0,\infty). \end{array}$$

In order to reduce this boundary value problem to an eigenvalue problem we require the function S to satisfy the eikonal equation $r^Q S' = \lambda$ with a complex constant λ. This implies

$$S(r) = \lambda\, \frac{r^{1-Q}}{1-Q}$$

up to an inessential constant to be included into a factor of $\exp S$. In this manner the problem reduces to

$$\begin{array}{rcll} r^Q\, V'_r - V''_{\omega,\omega} - r^{Q-1}\dfrac{\omega}{p}\, V'_\omega & = & -\lambda V & \text{in} \quad (-1,1)\times(0,\infty), \\ V(\pm 1, r) & = & 0 & \text{on} \quad (0,\infty). \end{array} \tag{2.3}$$

If $\mathfrak{e} = \dfrac{Q-1}{k}$ for some natural number k, then

$$\begin{array}{rcl} r^Q\, V'_r & = & \displaystyle\sum_{j=k}^{\infty} \mathfrak{e}(j-N-k)\, V_{j-N-k} r^{\mathfrak{e}(j-N)}, \\ V''_{\omega,\omega} & = & \displaystyle\sum_{j=0}^{\infty} V''_{j-N} r^{\mathfrak{e}(j-N)}, \\ r^{Q-1}\, V'_\omega & = & \displaystyle\sum_{j=k}^{\infty} V'_{j-N-k} r^{\mathfrak{e}(j-N)}, \end{array}$$

as is easy to check. On substituting these equalities into (2.3) and equating the coefficients of the same powers of r we get two collections of Sturm-Liouville problems

$$\begin{array}{rcll} -V''_{j-N} + \lambda\, V_{j-N} & = & 0 & \text{in} \quad (-1,1), \\ V_{j-N} & = & 0 & \text{at} \quad \mp 1, \end{array} \tag{2.4}$$

for $j = 0, 1, \ldots, k-1$, and

$$\begin{array}{rcll} -V''_{j-N} + \lambda\, V_{j-N} & = & \dfrac{\omega}{p}\, V'_{j-N-k} - \mathfrak{e}(j-N-k)\, V_{j-N-k} & \text{in} \quad (-1,1), \\ V_{j-N} & = & 0 & \text{at} \quad \mp 1, \end{array} \tag{2.5}$$

for $j = mk, mk+1, \ldots, mk+(k-1)$, where m takes on all natural values.

Given any $j = 0, 1, \ldots, k-1$, the Sturm-Liouville problem (2.4) has obviously simple eigenvalues

$$\lambda_n = -\Big(\frac{\pi}{2} n\Big)^2$$

for $n = 1, 2, \ldots$, a nonzero eigenfunction corresponding to λ_n being $\sin \dfrac{\pi}{2} n(\omega + 1)$. It follows that

$$V_{j-N}(\omega) = c_{j-N}\, \sin \frac{\pi}{2} n(\omega+1), \tag{2.6}$$

for $j = 0, 1, \ldots, k-1$, where c_{j-N} are constant. Without restriction of generality we can assume that the first coefficient V_{-N} in the Puiseux expansion of V is different from zero. Hence, $V_{j-N} = c_{j-N} V_{-N}$ for $j = 1, \ldots, k-1$. For simplicity of notation, we drop the index n.

On having determined the functions $V_{-N}, \ldots, V_{k-1-N}$, we turn our attention to problems (2.5) with $j = k, \ldots, 2k-1$. Set

$$f_{j-N} = \frac{\omega}{p}\, V'_{j-N-k} - \mathfrak{e}(j-N-k)\, V_{j-N-k},$$

then for the inhomogeneous problem (2.5) to possess a nonzero solution V_{j-N} it is necessary and sufficient that the right-hand side f_{j-N} be orthogonal to all solutions of the corresponding homogeneous problem, to wit V_{-N}. The orthogonality refers to the scalar product in $L^2(-1,1)$. Let us evaluate the scalar product (f_{j-N}, V_{-N}). We get

$$(f_{j-N}, V_{-N}) = c_{j-N-k} \Big(\frac{1}{p}\, (\omega V'_{-N}, V_{-N}) - \mathfrak{e}(j-N-k)\, (V_{-N}, V_{-N}) \Big)$$

and

$$\begin{array}{rcl} (\omega V'_{-N}, V_{-N}) & = & \omega\, |V_{-N}|^2 \Big|_{-1}^{1} - (V_{-N}, V_{-N}) - (V_{-N}, \omega V'_{-N}) \\ & = & -(V_{-N}, V_{-N}) - (\omega V'_{-N}, V_{-N}), \end{array}$$

the latter equality being due to the fact that V_{-N} is real-valued and vanishes at ± 1. Hence,

$$(\omega V'_{-N}, V_{-N}) = -\frac{1}{2}\, (V_{-N}, V_{-N})$$

and

$$(f_{j-N}, V_{-N}) = -c_{j-N-k} \Big(\frac{1}{2p} + \mathfrak{e}(j-N-k) \Big) (V_{-N}, V_{-N}) \tag{2.7}$$

for $j = k, \ldots, 2k-1$.

Since $V_{-N} \neq 0$, the condition $(f_{j-N}, V_{-N}) = 0$ fulfills for $j = k$ if and only if

$$\mathfrak{e} N = \frac{1}{2p}. \tag{2.8}$$

Under this condition, problem (2.5) with $j = k$ is solvable and its general solution has the form

$$V_{k-N} = V_{k-N,0} + c_{k-N} V_{-N},$$

where $V_{k-N,0}$ is a particular solution of (2.5) and c_{k-N} an arbitrary constant. Moreover, for $(f_{j-N}, V_{-N}) = 0$ to fulfill for $j = k+1, \ldots, 2k-1$ it is necessary and sufficient that $c_{1-N} = \ldots = c_{k-1-N} = 0$, i.e., all of $V_{1-N}, \ldots, V_{k-1-N}$ vanish. This in turn implies that $f_{k+1-N} = \ldots = f_{2k-1-N} = 0$, whence $V_{j-N} = c_{j-N} V_{-N}$ for all $j = k+1, \ldots, 2k-1$, where c_{j-N} are arbitrary constants. We choose the constants $c_{k-N}, \ldots, c_{2k-1}$ in such a way that the solvability conditions of the next k problems are fulfilled.

More precisely, we consider the problem (2.5) for $j = 2k$, the right-hand side being

$$\begin{aligned} f_{2k-N} &= \Big(\frac{\omega}{p} V'_{k-N,0} - \mathfrak{e}(k-N)\, V_{k-N,0} \Big) + c_{k-N} \Big(\frac{\omega}{p} V'_{-N} - \mathfrak{e}(k-N)\, V_{-N} \Big) \\ &= \Big(\frac{\omega}{p} V'_{k-N,0} - \mathfrak{e}(k-N)\, V_{k-N,0} \Big) + c_{k-N} \Big(f_{k-N} - \mathfrak{e} k\, V_{-N} \Big). \end{aligned}$$

Combining (2.7) and (2.8) we conclude that

$$\begin{aligned} (f_{k-N} - \mathfrak{e} k\, V_{-N}, V_{-N}) &= -\mathfrak{e} k\, (V_{-N}, V_{-N}) \\ &= (1-Q)\, (V_{-N}, V_{-N}) \end{aligned}$$

is different from zero. Hence, the constant c_{k-N} can be uniquely defined in such a way that $(f_{2k-N}, V_{-N}) = 0$. Moreover, the functions $f_{2k+1-N}, \ldots, f_{3k-1-N}$ are orthogonal to V_{-N} if and only if $c_{k+1-N} = \ldots = c_{2k-1-N} = 0$. It follows that V_{j-N} vanishes for each $j = k+1, \ldots, 2k-1$.

Continuing in this fashion we construct a sequence of functions $V_{j-N}(\omega)$, for $j = 0, 1, \ldots$, satisfying equations (2.4) and (2.5). The functions $V_{j-N}(\omega)$ are defined uniquely up to a common constant factor c_{-N}. Moreover, V_{j-N} vanishes identically unless $j = mk$ with $m = 0, 1, \ldots$. Therefore,

$$\begin{aligned} V(\omega, r) &= \frac{1}{r^{\mathfrak{e} N}} \sum_{m=0}^{\infty} V_{mk-N}(\omega)\, r^{\mathfrak{e} mk} \\ &= \frac{1}{r^{Q/4}} \sum_{m=0}^{\infty} \tilde{V}_m(\omega)\, r^{(Q-1)m} \end{aligned}$$

is a unique (up to a constant factor) formal asymptotic solution of problem (2.3) corresponding to $\lambda = \lambda_n$.

Theorem 2.1. *Let $p \neq 2$. Then an arbitrary formal asymptotic solution of homogeneous problem (2.1) has the form*

$$U(\omega, r) = \frac{c}{r^{Q/4}} \exp\Big(\lambda \frac{r^{1-Q}}{1-Q} \Big) \sum_{m=0}^{\infty} \frac{\tilde{V}_m(\omega)}{r^{(1-Q)m}},$$

where λ is one of eigenvalues $\lambda_n = -\Big(\dfrac{\pi}{2} n\Big)^2$.

Proof. The theorem follows readily from (2.2). □

In the original coordinates (x,t) close to the point 0 in $\mathcal{G}$ the formal asymptotic solution looks like

$$u(x,t) = \frac{c}{t^{Q/4}} \exp\Big(\lambda \frac{t^{1-Q}}{1-Q}\Big) \sum_{m=0}^{\infty} \tilde{V}_m \Big(\frac{x}{t^{1/p}}\Big)\Big(\frac{1}{t}\Big)^{(1-Q)m}. \tag{2.9}$$

If $1-Q>0$, i.e., $p>2$, expansion (2.9) behaves in much the same way as boundary layer expansion in singular perturbation problems, since the eigenvalues are all negative. The threshold value $p=2$ is a turning contact order under which the boundary layer degenerates.

3. The exceptional case $p = 2$

In this section we consider the case $p=2$ in detail. For $p=2$, problem (2.1) takes the form

$$\begin{array}{rcll} r\,U'_r - U''_{\omega,\omega} - \dfrac{\omega}{2}\,U'_\omega &=& 0 & \text{in} \quad (-1,1)\times(0,\infty), \\ U(\pm 1, r) &=& 0 & \text{on} \quad (0,\infty). \end{array} \tag{3.1}$$

The problem is specified as Fuchs-type equation on the half-axis with coefficients in boundary value problems on the interval $[-1,1]$. Such equations have been well understood, see [Esk80] and elsewhere.

If one searches for a formal solution to (3.1) of the form $U(\omega,r) = e^{S(r)}\,V(\omega,r)$, then the eikonal equation $rS' = \lambda$ gives $S(r) = \lambda \ln r$, and so $e^{S(r)} = r^\lambda$, where λ is a complex number. It makes therefore no sense to looking for $V(\omega,r)$ being a formal Puiseux series in fractional powers of r. The choice $\mathfrak{e} = (Q-1)/k$ no longer works, and so a good substitute for a fractional power of r is the function $1/\ln r$. Thus,

$$V(\omega,r) = \sum_{j=0}^{\infty} V_{j-N}(\omega)\,\Big(\frac{1}{\ln r}\Big)^{j-N}$$

has to be a formal asymptotic solution of

$$\begin{array}{rcll} r\,V'_r - V''_{\omega,\omega} - \dfrac{\omega}{2}\,V'_\omega &=& -\lambda V & \text{in} \quad (-1,1)\times(0,\infty), \\ V(\pm 1, r) &=& 0 & \text{on} \quad (0,\infty), \end{array}$$

N being a nonnegative integer. Substituting the series for $V(\omega,r)$ into these equations and equating the coefficients of the same powers of $\ln r$ yields two collections of Sturm-Liouville problems

$$\begin{array}{rcll} -V''_{-N} - \dfrac{\omega}{2}\,V'_{-N} + \lambda\,V_{-N} &=& 0 & \text{in} \quad (-1,1), \\ V_{-N} &=& 0 & \text{at} \quad \mp 1, \end{array} \tag{3.2}$$

for $j=0$, and

$$\begin{array}{rcll} -V''_{j-N} - \dfrac{\omega}{2}\,V'_{j-N} + \lambda\,V_{j-N} &=& (j-N-1)V_{j-N-1} & \text{in} \quad (-1,1), \\ V_{j-N} &=& 0 & \text{at} \quad \mp 1, \end{array} \tag{3.3}$$

for $j \geq 1$.

Problem (3.2) has a nonzero solution V_{-N} if and only if λ is an eigenvalue of the operator

$$v \mapsto v'' + \frac{1}{2}\omega\, v'$$

whose domain consists of all functions $v \in H^2(-1,1)$ vanishing at ∓ 1. Then, equalities (3.3) for $j = 1, \ldots, N$ mean that $V_{-N+1}, \ldots, V_0$ are actually root functions of the operator corresponding to the eigenvalue λ. In other words, $V_{-N}, \ldots, V_0$ is a Jordan chain of length $N+1$ corresponding to the eigenvalue λ. Note that for $j = N+1$ the right-hand side of (3.3) vanishes, and so $V_1, V_2, \ldots$ is also a Jordan chain corresponding to the eigenvalue λ. This suggests that the series breaks beginning at $j = N+1$. Furthermore, it follows from the Sturm-Liouville theory that problem (3.2) has a discrete sequence $\{\lambda_n\}_{n=1,2,\ldots}$ of real eigenvalues. If

$$-v'' - \frac{1}{2}\omega\, v' + \lambda v = 0$$

on $(-1,1)$ for some function $v \in H^2(-1,1)$ vanishing at ∓ 1, then

$$\|v'\|^2 + \lambda \|v\|^2 = \frac{1}{2}\,(\omega v', v), \tag{3.4}$$

where the scalar product and norm are those of $L^2(-1,1)$. By the Schwarz inequality, we get $|(\omega v', v)| \le \|v'\|\,\|v\|$. Since

$$\begin{aligned} \|v'\|^2 + \lambda\|v\|^2 &= \frac{1}{2}\,\|v'\|\,\|v\| + \Big(\|v'\| - \frac{1}{4}\,\|v\|\Big)^2 + \Big(\lambda - \frac{1}{16}\Big)\|v\|^2 \\ &\ge \frac{1}{2}\,\|v'\|\,\|v\| + \Big(\lambda - \frac{1}{16}\Big)\|v\|^2, \end{aligned}$$

we conclude that inequality (3.4) fulfills only for the function $v = 0$ unless $\lambda \le 1/16$. Hence,

$$\lambda_n \le \frac{1}{16}$$

for all $n = 1, 2, \ldots$. Each eigenvalue λ_n is simple whence $N = 0$.

Theorem 3.1. *Suppose $p = 2$. Then an arbitrary formal asymptotic solution of homogeneous problem (2.1) has the form $U(\omega, r) = r^\lambda\, V_0(\omega)$, where λ is one of the eigenvalues λ_n.*

Proof. The theorem follows immediately from the above discussion. □

In the original coordinates (x,t) near the point P_3 in $\mathcal{G}$ the formal asymptotic solution proves to be

$$u(x,t) = c\, t^\lambda\, V_0\Big(\frac{x}{t^{1/2}}\Big).$$

Of course, Theorem 3.1 can be proved immediately, for the homogeneous problem (2.1) admits a separation of variables. Namely, set $U(\omega, r) = R(r)\varOmega(\omega)$. Substituting this into equation (3.1) yields

$$rR'\varOmega - \varOmega'' - \frac{\omega}{2}\varOmega' R = 0,$$

which is equivalent to

$$\begin{aligned} rR' &= \lambda R, \\ \varOmega'' - \frac{\omega}{2}\varOmega' &= \lambda \varOmega. \end{aligned}$$

Then $R(r) = r^\lambda$, where the parameter λ is determined from the boundary value problem for $\varOmega$. The function $\varOmega$ can be described in terms of parabolic cylinder functions, see [AS64]. To transform the equation for Ω to the equation of parabolic cylinder, set

$$\Omega(\omega) = \exp\Big(\frac{\omega^2}{8}\Big) y(\omega).$$

Then y satisfies

$$y'' + \Big(\Big(\frac{\omega}{4}\Big)^2 + \lambda - \frac{1}{4}\Big) y = 0.$$

Two linearly independent solutions of this equation are called functions of parabolic cylinder.

4. Generalisation to higher dimensions

The explicit formulas obtained above generalise easily to the evolution equation related to the b th power of the Laplace operator in $\mathbb{R}^n$, where b is a natural number. Consider the first boundary value problem for the operator $\partial_t + (-\Delta)^b$ in a bounded domain $\mathcal{G} \subset \mathbb{R}^{n+1}$. Note that the choice of the sign $(-1)^b$ is explained exceptionally by our wish to deal with parabolic (not backward parabolic) equation.

The boundary of $\mathcal{G}$ is assumed to be C^∞ except for a finite number of characteristic points. These are those points of $\partial\mathcal{G}$ at which the boundary touches with a hyperplane in $\mathbb{R}^{n+1}$ orthogonal to the t-axis. As above, we restrict our attention to analysis of the first boundary problem near a characteristic point like 0 in Figure 1.

The first boundary value problem for the evolution equation in $\mathcal{G}$ is formulated as follows: Let Σ be the set of all characteristic points of the boundary of $\mathcal{G}$. Given any functions f in $\mathcal{G} \to \mathbb{R}$ $u_0, u_1, \ldots, u_{b-1}$ on $\partial\mathcal{G} \setminus \Sigma$, find a function u on $\overline{\mathcal{G}} \setminus \Sigma$ satisfying

$$\begin{array}{rcll} u'_t + (-\Delta)^b u & = & f & \text{in} \quad \mathcal{G}, \\ \partial_\nu^j u & = & u_j & \text{at} \quad \partial\mathcal{G} \setminus \Sigma, \end{array} \qquad (4.1)$$

for $j = 0, 1, \ldots, b-1$, where ∂_ν is the derivative along the outward unit normal vector of the boundary. We focus upon a characteristic point 0 of the boundary which is assumed to be the origin in $\mathbb{R}^{n+1}$.

Suppose the domain $\mathcal{G}$ is described in a neighbourhood of the origin by the inequality

$$t > \mathfrak{f}(x), \qquad (4.2)$$

where $\mathfrak{f}$ is a smooth function of $x \in \mathbb{R}^n \setminus 0$ homogeneous of degree $p > 0$. We blow up the domain $\mathcal{G}$ at 0 by introducing new coordinates $(\omega, r) \in D \times (0, 1)$ with the aid of

$$\begin{array}{rcl} x & = & t^{1/p}\,\omega, \\ t & = & r, \end{array} \qquad (4.3)$$

where D is the domain in $\mathbb{R}^n$ consisting of those points $\omega \in \mathbb{R}^n$ which satisfy $f(\omega) < 1$. Under this change of variables the domain $\mathcal{G}$ nearby 0 transforms into the half-cylinder $D \times (0, \infty)$, the cross-section $D \times \{0\}$ blowing down into the origin by (4.3).

In the domain of coordinates (ω, r) problem (4.1) reduces to an ordinary differential equation with respect to the variable r with operator-valued coefficients. It is easy to see that under transformation (4.3) the derivatives in t and x change by the formulas

$$\begin{array}{rcl} u'_t & = & u'_r - \dfrac{1}{p}\dfrac{1}{r}\,(\omega, u'_\omega), \\ u'_{x_k} & = & \dfrac{1}{r^{1/p}}\,u'_{\omega_k} \end{array}$$

for $k = 1, \ldots, n$, where $(\omega, u'_\omega) = \displaystyle\sum_{k=1}^{n} \omega_k \frac{\partial u}{\partial \omega_k}$ stands for the Euler derivative. Thus, (4.1) transforms into

$$\begin{array}{rcll} r^Q\,U'_r + (-\Delta_\omega)^b U - \dfrac{1}{p} r^{Q-1}\,(\omega, U'_\omega) & = & r^Q F & \text{in} \quad D \times (0,1), \\ \partial_\nu^j\,U & = & U_j & \text{at} \quad \partial D \times (0,1) \end{array} \tag{4.4}$$

for $j = 0, 1, \ldots, b-1$, where $U(\omega, r)$ and $F(\omega, r)$ are pullbacks of $u(x,t)$ and $f(x,t)$ under transformation (4.3), respectively, and

$$Q = \frac{2b}{p}.$$

We are interested in the local solvability of problem (4.4) near the base $r = 0$ in the cylinder $D \times (0,1)$. Note that the equation degenerates at $r = 0$, since the coefficient r^Q of the higher order derivative in r vanishes at $r = 0$. The theory of [RST00] still applies to characterise those problems (4.4) which are locally invertible.

To describe function spaces which give the best fit for solutions of problem (4.4), one constructs formal asymptotic solutions of the corresponding homogeneous problem. That is

$$\begin{array}{rcll} r^Q\,U'_r + (-\Delta_\omega)^b U - \dfrac{1}{p} r^{Q-1}\,(\omega, U'_\omega) & = & 0 & \text{in} \quad D \times (0,\infty), \\ \partial_\omega^\alpha\,U & = & 0 & \text{on} \quad \partial D \times (0,\infty) \end{array} \tag{4.5}$$

for all $|\alpha| \le b-1$.

We assume that $p \neq 2b$. Similar arguments apply to the case $p = 2b$, the only difference being in the choice of the Ansatz, see Section 3. We look for a formal solution to (4.5) of the form $U(\omega, r) = e^{S(r)}\,V(\omega, r)$, where S is a differentiable function of $r > 0$ and V expands as a formal Puiseux series with nontrivial principal part

$$V(\omega, r) = \frac{1}{r^{\mathfrak{e}N}} \sum_{j=0}^{\infty} V_{j-N}(\omega)\,r^{\mathfrak{e}j},$$

where N is a complex number and $\mathfrak{e}$ a real exponent to be determined.

On substituting $U(\omega, r)$ into (2.1) we extract the eikonal equation $r^Q S' = \lambda$ for the function $S(r)$, where λ is a (possibly complex) constant to be defined. For $Q \neq 1$ this implies

$$S(r) = \lambda\,\frac{r^{1-Q}}{1-Q}$$

up to an inessential constant factor. In this way the problem reduces to

$$\begin{array}{rcll} r^Q\,V'_r + (-\Delta_\omega)^b V - \dfrac{1}{p} r^{Q-1}\,(\omega, V'_\omega) & = & -\lambda V & \text{in} \quad D \times (0,\infty), \\ \partial_\omega^\alpha V & = & 0 & \text{on} \quad \partial D \times (0,\infty) \end{array} \tag{4.6}$$

for all $|\alpha| \le b-1$.

Analysis similar to that in Section 2 shows that a right choice of $\mathfrak{e}$ is $\mathfrak{e} = (Q-1)/k$ for some natural number k. On substituting the formal series for $V(\omega, r)$ into (4.6) and equating the coefficients of the same powers of r we get two collections of problems

$$\begin{array}{rcll} (-\Delta)^b V_{j-N} + \lambda\, V_{j-N} & = & 0 & \text{in} \quad D, \\ \partial^\alpha\, V_{j-N} & = & 0 & \text{at} \quad \partial D \end{array} \tag{4.7}$$

for all $|\alpha| \leq b-1$, where $j = 0, 1, \ldots, k-1$, and

$$\begin{array}{rcll} (-\Delta)^b V_{j-N} + \lambda\, V_{j-N} & = & \frac{1}{p}\, (\omega, V'_{j-N-k}) - \mathfrak{e}(j-N-k)\, V_{j-N-k} & \text{in} \quad D, \\ \partial^\alpha\, V_{j-N} & = & 0 & \text{at} \quad \partial D \end{array} \tag{4.8}$$

for all $|\alpha| \leq b-1$, where $j = k, k+1, \ldots, 2k-1$, and so on.

Given any $j = 0, 1, \ldots, k-1$, problem (4.7) is essentially an eigenvalue problem for the strongly nonnegative operator $(-\Delta)^b$ in $L^2(D)$ whose domain consists of all functions of $H^{2b}(D)$ vanishing up to order $b-1$ at ∂D. The eigenvalues of the latter operator are known to be all positive and form a nondecreasing sequence $\lambda'_1, \lambda'_2, \ldots$ which converges to ∞. Hence, (4.7) admits nonzero solutions only for a discrete sequence

$$\lambda_n = -\lambda'_n < 0$$

where $n = 1, 2, \ldots$.

In general, the eigenvalues $\{\lambda_n\}$ fail to be simple. The generic simplicity of the eigenvalues of the Dirichlet problem for self-adjoint elliptic operators with respect to variations of the boundary have been investigated by several authors, see [PP08] and the references given there. We focus on an eigenvalue λ_n of multiplicity 1, in which case the formal asymptotic solution is especially simple. By the above, this condition is not particularly restrictive.

If $\lambda = \lambda_n$, there is a nonzero solution $e_n(\omega)$ of this problem which is determined uniquely up to a constant factor. This yields

$$V_{j-N}(\omega) = c_{j-N}\, e_n(\omega), \tag{4.9}$$

for $j = 0, 1, \ldots, k-1$, where c_{j-N} are constant. Without restriction of generality we can assume that the first coefficient V_{-N} in the Puiseux expansion of V is different from zero. Hence, $V_{j-N} = c_{j-N} V_{-N}$ for $j = 1, \ldots, k-1$. For simplicity of notation, we drop the index n.

On taking the functions $V_{-N}, \ldots, V_{k-1-N}$ for granted, we now turn to problems (2.5) with $j = k, \ldots, 2k-1$. Set

$$f_{j-N} = \frac{1}{p}\, (\omega, V'_{j-N-k}) - \mathfrak{e}(j-N-k)\, V_{j-N-k},$$

then for the inhomogeneous problem (4.8) to admit a nonzero solution V_{j-N} it is necessary and sufficient that the right-hand side f_{j-N} be orthogonal to all solutions of the corresponding homogeneous problem, to wit V_{-N}. The orthogonality refers to the scalar product in $L^2(D)$. Let us evaluate the scalar product (f_{j-N}, V_{-N}). We get

$$(f_{j-N}, V_{-N}) = c_{j-N-k} \Big(\frac{1}{p}\, ((\omega, V'_{-N}), V_{-N}) - \mathfrak{e}(j-N-k)\, (V_{-N}, V_{-N}) \Big)$$

and, by Stokes' formula,

$$\begin{aligned}((\omega, V'_{-N}), V_{-N}) &= \int_{\partial D} |V_{-N}|^2 (\omega, \nu)\, ds - \sum_{k=1}^{n} \int_D V_{-N} \frac{\partial}{\partial \omega_k} (\omega_k \overline{V_{-N}})\, d\omega \\ &= -n \|V_{-N}\|^2 - ((\omega, V'_{-N}), V_{-N}),\end{aligned}$$

the latter equality being due to the fact that V_{-N} is real-valued and vanishes at ∂D. Hence,

$$((\omega, V'_{-N}), V_{-N}) = -\frac{n}{2}\, \|V_{-N}\|^2$$

and

$$(f_{j-N}, V_{-N}) = -c_{j-N-k} \left(\frac{n}{2p} + \mathfrak{e}(j - N - k) \right) \|V_{-N}\|^2 \tag{4.10}$$

for $j = k, \ldots, 2k - 1$.

Since $V_{-N} \neq 0$, the condition $(f_{j-N}, V_{-N}) = 0$ fulfills for $j = k$ if and only if

$$\mathfrak{e}N = \frac{n}{2p}. \tag{4.11}$$

Under this condition, problem (4.8) with $j = k$ is solvable and its general solution has the form

$$V_{k-N} = V_{k-N,0} + c_{k-N} V_{-N},$$

where $V_{k-N,0}$ is a particular solution of (4.8) and c_{k-N} an arbitrary constant. Moreover, for $(f_{j-N}, V_{-N}) = 0$ to fulfill for $j = k + 1, \ldots, 2k - 1$ it is necessary and sufficient that $c_{1-N} = \ldots = c_{k-1-N} = 0$, i.e., all of $V_{1-N}, \ldots, V_{k-1-N}$ vanish. This in turn implies that $f_{k+1-N} = \ldots = f_{2k-1-N} = 0$, whence $V_{j-N} = c_{j-N} V_{-N}$ for all $j = k + 1, \ldots, 2k - 1$, where c_{j-N} are arbitrary constants. We choose the constants $c_{k-N}, \ldots, c_{2k-1}$ in such a way that the solvability conditions of the next k problems are fulfilled.

More precisely, we consider the problem (4.8) for $j = 2k$, the right-hand side being

$$\begin{aligned}f_{2k-N} &= \Big(\frac{1}{p} (\omega, V'_{k-N,0}) - \mathfrak{e}(k-N) V_{k-N,0} \Big) + c_{k-N} \Big(\frac{1}{p} (\omega, V'_{-N}) - \mathfrak{e}(k-N) V_{-N} \Big) \\ &= \Big(\frac{1}{p} (\omega, V'_{k-N,0}) - \mathfrak{e}(k-N) V_{k-N,0} \Big) + c_{k-N} \Big(f_{k-N} - \mathfrak{e}k V_{-N} \Big).\end{aligned}$$

Combining (4.10) and (4.11) we conclude that

$$\begin{aligned}(f_{k-N} - \mathfrak{e}k\, V_{-N}, V_{-N}) &= -\mathfrak{e}k\, (V_{-N}, V_{-N}) \\ &= (1 - Q)\, (V_{-N}, V_{-N})\end{aligned}$$

is different from zero. Hence, the constant c_{k-N} can be uniquely defined in such a way that $(f_{2k-N}, V_{-N}) = 0$. Moreover, the functions $f_{2k+1-N}, \ldots, f_{3k-1-N}$ are orthogonal to V_{-N} if and only if $c_{k+1-N} = \ldots = c_{2k-1-N} = 0$. It follows that V_{j-N} vanishes for each $j = k + 1, \ldots, 2k - 1$.

Continuing in this manner we construct a sequence of functions $V_{j-N}(\omega)$, for $j = 0, 1, \ldots$, satisfying equations (4.7) and (4.8). The functions $V_{j-N}(\omega)$ are defined uniquely up to a common constant factor c_{-N}. Moreover, V_{j-N} vanishes identically unless j is an integer multiple of k, i.e., $j = mk$ with $m = 0, 1, \ldots$. Hence it follows that

$$\begin{aligned} V(\omega, r) &= \frac{1}{r^{\mathfrak{c}N}} \sum_{m=0}^{\infty} V_{mk-N}(\omega)\, r^{\mathfrak{c}mk} \\ &= \frac{1}{r^{n/2p}} \sum_{m=0}^{\infty} \tilde{V}_m(\omega)\, r^{(Q-1)m} \end{aligned}$$

is a unique (up to a constant factor) formal asymptotic solution of problem (4.6) corresponding to $\lambda = \lambda_n$. Summarising, we arrive at the following generalisation of Theorem 2.1.

Theorem 4.1. *Let $p \neq 2b$. Then an arbitrary formal asymptotic solution of homogeneous problem (4.5) has the form*

$$U(\omega, r) = \frac{c}{r^{n/2p}} \exp\Big(\lambda \frac{r^{1-Q}}{1-Q}\Big) \sum_{m=0}^{\infty} \frac{\tilde{V}_m(\omega)}{r^{(1-Q)m}},$$

where λ is one of eigenvalues $\lambda_n = -\lambda_n'$.

Thus, the construction of formal asymptotic solution U of general problem (4.1) follows by the same method as in Section 2.

In the original coordinates (x,t) close to the point 0 in $\mathcal{G}$ the formal asymptotic solution looks like

$$u(x,t) = \frac{c}{t^{n/2p}} \exp\Big(\lambda \frac{t^{1-Q}}{1-Q}\Big) \sum_{m=0}^{\infty} \tilde{V}_m\Big(\frac{x}{t^{1/p}}\Big) \Big(\frac{1}{t}\Big)^{(1-Q)m}. \tag{4.12}$$

The computations of this section extend obviously both to eigenvalues λ_n of higher multiplicity and arbitrary self-adjoint elliptic operators $A(x,D)$ in place of $(-\Delta)^b$. Mention that under small perturbations of $A(x,D)$ the eigenvalues of multiplicity μ break down into μ simple eigenvalues. When solving nonhomogeneous equations (4.8), one chooses the only solution which is orthogonal to all solutions of the corresponding homogeneous problem (4.7). This special solution actually determines what is known as Green operator. However, formula (4.12) becomes less transparent. And so we omit the details.

Part 2. Proof of asymptotic character

5. Resolution of singularities at infinity

Throughout this part we will assume that $0 < p < 2$, i.e., $Q = 2/p$ is greater than 1. As mentioned in the Introduction, this case is not included in the treatise [Kon66] and it was first studied in [AT12]. For $1 < p < 2$, the origin is a characteristic boundary point of the domain $\mathcal{G}$. For $0 < p < 1$, the origin is a cuspidal point at the boundary.

We are actually interested in the local solvability of problem (1.4) near the edge $r = 0$ in the rectangle $(-1,1) \times (0,1)$. Note that the equation degenerates at $r = 0$, since the coefficient r^Q of the higher order derivative in r vanishes at $r = 0$. If $Q = 1$, the equation is of Fuchs type and is studied within the framework of Mellin calculus. In order to handle this degeneration in an orderly fashion for $Q > 1$, we find a change of coordinates $s = \delta(r)$ in the interval $(0, 1)$, such that

$$r^Q \frac{d}{dr} = \frac{d}{ds}.$$

Such a function δ is determined uniquely up to an inessential constant from the equation $\delta'(r) = r^{-Q}$ and is given by

$$\delta(r) = \frac{r^{1-Q}}{1-Q} \tag{5.1}$$

for $r > 0$. Note that $\delta(0+) = -\infty$. Problem (1.4) becomes

$$\begin{array}{rclll} U'_s - U''_{\omega,\omega} + \dfrac{1}{2-p}\dfrac{1}{s}\,\omega U'_\omega & = & \Big(\dfrac{\delta(1)}{s}\Big)^{\frac{2}{2-p}} F & \text{in} & (-1,1)\times(-\infty,\delta(1)), \\ U & = & U_0 & \text{at} & \{\pm 1\}\times(-\infty,\delta(1)), \end{array} \tag{5.2}$$

where we use the same letter to designate U and the push-forward of U under the transformation $s = \delta(r)$, and similarly for F.

The advantage of reducing the problem in the half-cylinder $[-1, 1] \times [0, \infty)$ to that in the infinite cylinder $[-1, 1] \times (-\infty, \infty)$ lies in the fact that it allows one to exploit the Fredholm theory of pseudodifferential operators with slowly varying symbols, see [Gru70]. More precisely, the operator of boundary value problem (5.2) can be written as pseudodifferential operator

$$AU\,(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\imath s\sigma} a(s,\sigma)\hat{U}(\sigma)d\sigma$$

for $s < 0$, where

$$a(s,\sigma) = \begin{pmatrix} \imath\sigma - \Big(\dfrac{\partial}{\partial\omega}\Big)^2 + \dfrac{1}{2-p}\dfrac{1}{s}\,\omega\dfrac{\partial}{\partial\omega} \\ r' \end{pmatrix} \tag{5.3}$$

is a symbol with values in the Sturm-Liouville boundary value problems on the interval $[-1, 1]$ (r' standing for the restriction to the boundary), and $\hat{U}$ the Fourier transformation of U in the variable s. An important property of $a(s, \sigma)$ is that it is slowly varying at $s \to -\infty$ which means that, for each $\alpha \geq 0$ and $\beta \geq 1$, the derivative $D_s^\beta D_\sigma^\alpha a(s, \sigma)$ tends to zero in a suitable norm as $s \to -\infty$, see [Gru70], [RST00].

It is now straightforward that the inverse change of variables $r = \delta^{-1}(s)$ pushes the Fourier operators near $s = -\infty$ forward to operators near $r = 0$ on the half-axis $r \geq 0$. These latter are based on an abstract Fourier transform corresponding to a group structure on $\mathbb{R}_{>0}$.

We now rewrite the series for formal solutions of homogeneous problem (2.1) in the new coordinates (ω, s). On substituting (5.1) into Theorem 2.1 we get immediately

$$U(\omega,s) = c\,((1-Q)s)^{\frac{1}{4}\frac{Q}{Q-1}} \exp(\lambda s) \sum_{m=0}^{\infty} \frac{\tilde{V}_m(\omega)}{((1-Q)s)^m} \tag{5.4}$$

for s in a neighbourhood of $-\infty$, where λ is one of eigenvalues $\lambda_n = -\Big(\dfrac{\pi}{2}n\Big)^2$.

Formula (5.4) demonstrates rather strikingly that the solutions of homogeneous problem (5.2) are relevant perturbations of solutions of ordinary differential equations with constant coefficients without source term. Therefore, the natural setting of problem (5.2) is constituted by functions of exponential type in s. More precisely, the local ring of functions corresponding to problem (5.2) at the singular point $s = -\infty$ consists of those functions which admit asymptotic expansions of the form

$$U(\omega, s) \sim s^N \exp(\lambda s) \sum_{m=0}^{\infty} \frac{V_m(\omega)}{s^m}$$

near $s = -\infty$, where $N \geq 0$ and λ are real numbers and $V_m(\omega)$ differentiable functions on $[-1, 1]$.

6. Fredholm property of the first boundary problem

In this section we present the results of [AT12] in the particular case $\mathfrak{f}(r) = r^p$, where $0 < p < 2$.

For those pseudodifferential operators whose symbols are slowly varying at the point $s = -\infty$, the paper [RST00] gives a criterion of local solvability at $-\infty$. However, this criterion does not apply immediately to problem (5.2), for [RST00] deals with classical polyhomogeneous symbols while our problem requires quasihomogeneous symbols. However, the approach of [RST00] still works in the anisotropic case if the derivatives in s are counted with weight factor 2. The details are left to an interested reader.

Using transformations rather standard in the Sturm-Liouville theory we reduce problem (5.2) to a simpler form. Introduce

$$q(\omega, s) = \exp\Big(-\frac{1}{2-p} \frac{1}{s} \frac{\omega^2}{2} \Big)$$

which is a bounded differentiable function with positive values in the half-strip $[-1, 1] \times (-\infty, \delta(1))$. A trivial verification shows that (5.2) transforms to the problem

$$\begin{array}{rcll} U'_s - \dfrac{1}{q} (qU'_\omega)'_\omega & = & \tilde{F} & \text{in} \quad (-1,1) \times (-\infty, \delta(1)), \\ U & = & U_0 & \text{at} \quad \{\pm 1\} \times (-\infty, \delta(1)). \end{array}$$

On replacing the unknown function by $U = \dfrac{1}{\sqrt{q}} v$ we finally arrive at the boundary value problem

$$\begin{array}{rcll} v'_s - v''_{\omega,\omega} + cv & = & \sqrt{q}\, \tilde{F} & \text{in} \quad (-1,1) \times (-\infty, \delta(1)), \\ v & = & \sqrt{q} U_0 & \text{at} \quad \{\pm 1\} \times (-\infty, \delta(1)), \end{array} \tag{6.1}$$

where

$$c(\omega, s) = -\frac{1}{2} \frac{1}{2-p} \frac{1}{s} + \frac{1}{4} \frac{p-1}{(p-2)^2} \frac{\omega^2}{s^2},$$

cf. [CH68, v. I, p. 250].

Our approach to solving problem (6.1) is fairly standard in the theory of linear equations. On choosing a proper scale of weighted Sobolev spaces in the strip $\mathcal{C} = [-1, 1] \times \mathbb{R}$ and taking the data $v_0 = \sqrt{q} U_0$ in the corresponding trace spaces on the boundary $\omega = \pm 1$ of $\mathcal{C}$ we can assume without loss of generality that $v_0 \equiv 0$. We think of (6.1) as a perturbation of the problem with homogeneous boundary conditions

$$\begin{array}{rcll} v'_s - v''_{\omega,\omega} & = & \sqrt{q}\, \tilde{F} & \text{in} \quad \mathcal{C}, \\ v & = & 0 & \text{at} \quad \partial\mathcal{C}. \end{array} \tag{6.2}$$

This is exactly the first boundary value problem for the heat equation in the cylinder $\mathcal{C}$ which is nowadays well understood, cf. for instance Chapter 3 in [TS72]. If $g = \sqrt{q}\,\tilde{F}$ vanishes, problem (6.2) possesses infinitely many linearly independent solutions of the form

$$v_n(\omega, s) = c_n \exp\Big(- \Big(\frac{\pi}{2}\,n\Big)^2 s\Big) \sin \frac{\pi}{2}\,n(\omega+1) \tag{6.3}$$

with n a natural number. In order to eliminate the solutions with n large enough it is necessary to pose growth restrictions on $v(\omega, s)$ for $s \to -\infty$. As but one possibility to do that we mention Sobolev spaces with exponential and powerlike weight factors, see [RST00]. Since the coefficients of the operator are stationary, the Fourier transform in s applies to reduce the problem to a Sturm-Liouville eigenvalue problem on the interval $(-1, 1)$, see Chapter 5 in [CH68, v. 1]. Instead of the Fourier transform one can use orthogonal decompositions over the eigenfunctions, which leads immediately to formal solutions of the unperturbed problem at the point $s = -\infty$.

On returning to problem (6.1) we observe that it differs from the unperturbed problem by the multiplication operator $v \mapsto cv$. If the unperturbed problem is Fredholm and the perturbation $v \mapsto cv$ is compact, then the perturbed problem is Fredholm as well. The local version of this assertion states that if the unperturbed problem is invertible and the perturbation $v \mapsto cv$ is small, then the perturbed problem is also invertible. Since $c(\omega, s) \to 0$ uniformly in $\omega \in [-1, 1]$, as $s \to -\infty$, the operator $v \mapsto cv$ is compact in natural scales of weighted Sobolev spaces, to be introduced.

We write the ordinary differential equation with operator-valued coefficients of (6.1) in the form

$$v'_s + C(s)v = g \tag{6.4}$$

where

$$C(s) = -\Big(\frac{d}{d\omega}\Big)^2 + c(\omega, s)$$

is a continuous function on $(-\infty, \delta(1))$ with values in second order ordinary differential operators on $(-1, 1)$. We think of $C(s)$ as unbounded operator in $L^2(-1, 1)$ whose domain consists of all $v \in H^2(-1, 1)$ satisfying $v(-1) = v(1) = 0$. As but one result of the theory of Sturm-Liouville boundary value problems we mention that $C(s)$ is closed.

As usual in the theory of ordinary differential equations with operator-valued coefficients, we associate the operator pencil $a(s, \sigma) = \imath\sigma + C(s)$ with (6.4). It depends on parameters $s \in (-\infty, \delta(1))$ and $\sigma \in \mathbb{C}$. By the above, $a(s, \sigma)$ stabilises to an operator pencil $a(-\infty, \sigma)$ independent of s, as $s \to -\infty$. More precisely, we get

$$a(-\infty, \sigma) = \imath\sigma - \Big(\frac{d}{d\omega}\Big)^2.$$

For each integer $k \geq 1$, the symbol $a(-\infty, \sigma)$ acting from $H^{2k}(-1, 1) \cap \overset{\circ}{H}(-1, 1)$ to $H^{2(k-1)}(-1, 1)$ has a bounded inverse everywhere in the entire complex plane $\mathbb{C}$ except for the discrete set

$$\sigma_n = -\imath\lambda_n = \imath\Big(\frac{\pi}{2}n\Big)^2$$

with $n \in \mathbb{N}$. It is worth pointing out that $a(-\infty, \sigma)^{-1} = \mathcal{R}_{C(-\infty)}(-\imath\sigma)$, the resolvent of $C(-\infty)$ at $-\imath\sigma$.

Lemma 6.1. *There exists a constant c with the property that, for all complex σ lying away from any angular sector containing the positive imaginary axis, the inequality*

$$\begin{aligned} & \|v\|^2_{H^{2k}(-1,1)} + |\sigma|^{2k} \|v\|^2_{L^2(-1,1)} \\ \leq \; & c \Big(\|a(-\infty,\sigma)v\|^2_{H^{2(k-1)}(-1,1)} + |\sigma|^{2(k-1)} \|a(-\infty,\sigma)v\|^2_{L^2(-1,1)} \Big) \end{aligned}$$

is fulfilled whenever $v \in H^{2k}(-1,1) \cap \overset{\circ}{H}{}^1(-1,1)$ with $k \geq 1$.

Proof. The operator pencils $\mathfrak{s}(-\infty,\sigma)$ with this property are said to be anisotropic elliptic. See [AV64] for a more general estimate. □

By a solution of (6.4) is meant any function $v(s)$ with values in $H^2(-1,1)$ satisfying $v(-1) = v(1) = 0$, which has a strong derivative in $L^2(-1,1)$ for almost all $s < \delta(1)$, and which fulfills (6.4).

Formula (5.4) suggests readily a scale of Hilbert spaces to control the solutions of problem (6.1). For any $k = 0, 1, \ldots$ and $\gamma \in \mathbb{R}$, we introduce $H^{k,\gamma}(-\infty,\delta(1))$ to be the space of all functions on $(-\infty,\delta(1))$ with values in $H^{2k}(-1,1)$, such that the norm

$$\|v\|_{H^{k,\gamma}(-\infty,\delta(1))} := \Big(\int_{-\infty}^{\delta(1)} e^{-2\gamma s} \sum_{j=0}^{k} \|v^{(j)}(s)\|^2_{H^{2(k-j)}(-1,1)} ds \Big)^{1/2}$$

is finite, cf. Slobodetskii [Slo58]. In particular, $H^{0,\gamma}(-\infty,\delta(1))$ consists of all square integrable functions on $(-\infty,\delta(1))$ with values in $L^2(-1,1)$ with respect to the measure $e^{-2\gamma s} ds$.

Recall that the numbers σ_n are usually referred to as eigenvalues of the operator pencil $a(-\infty,\sigma)$, for there are nonzero functions v_n in $H^2(-1,1)$ vanishing at ∓ 1 and satisfying $a(-\infty,\sigma_n)v_n = 0$. The functions v_n are called eigenfunctions of $a(-\infty,\sigma)$ at σ_n. The following theorem fits well the abstract theory of [MP72]. However, [MP72] is a straightforward generalisation of the asymptotic formula of Evgrafov [Evg61] for solutions of first order equations to equations of an arbitrary order. Our results go thus back at least as far as [Evg61] while we refer to the more available paper [MP72].

Theorem 6.2. *Suppose in the strip $-\mu < \Im\sigma < -\gamma$ there lie exactly N of the eigenvalues σ_n, and that there are no eigenvalues σ_n on the lines $\Im\sigma = -\mu$ and $\Im\sigma = -\gamma$. Then the solution $v \in H^{1,\gamma}(-\infty,\delta(1))$ of problem (6.4) with $g \in H^{0,\mu}(-\infty,\delta(1))$ has the form*

$$v(s) = c_1\, s_1(s) + \ldots + c_N\, s_N(s) + R(s)$$

where $s_1, \ldots, s_N$ are solutions of the homogeneous problem in $H^{1,\gamma}(-\infty,\delta(1))$ which do not depend on v, $c_1, \ldots, c_N$ constants, and $R \in H^{1,\mu}(-\infty,\delta(1))$.

Proof. Obviously, $c(\omega,s) \to c(\omega,-\infty)$ in the $L^2(-1,1)$-norm when $s \to -\infty$. Since the embedding

$$H^1(-1,1) \hookrightarrow C[-1,1]$$

is continuous, we see that $C(s) \to C(-\infty)$ in the norm of $\mathcal{L}(H^2(-1,1), L^2(-1,1))$, as $s \to -\infty$. Hence the desired conclusion is a direct consequence of Theorem 3 in

[MP72] with

$$\begin{array}{rcl} H_0 &=& L^2(-1,1), \\ H_1 &=& H^2(-1,1) \cap \overset{\circ}{H}{}^1(-1,1). \end{array}$$

□

Thus, any solution $v \in H^{1,\gamma}(-\infty, \delta(1))$ of (6.4) with a "good" right-hand side g can be written as the sum of several singular functions and a "remainder" which behaves better at infinity. The singular functions $s_1, \ldots, s_N$ are linearly independent and do not depend on the particular solution v. What is still lacking is that they are not explicit.

By the local solvability of problem (6.1) at $s = -\infty$ is meant that there is a number $S \ll \delta(1)$, such that for each function $\tilde{F}$ in $\mathcal{C}$ with $\sqrt{q}\,\tilde{F} \in H^{0,\gamma}(-\infty, \delta(1))$ there is a function $v \in H^{1,\gamma}(-\infty, \delta(1))$ satisfying (6.1) for all $s < S$. Yet another designation for the local solvability is the local invertibility from the right at the point $s = -\infty$. For a deeper discussion of local invertibility we refer the reader to [RST00].

Theorem 6.3. *Suppose that $\gamma \in \mathbb{R}$ is different from λ_n for all $n = 1, 2, \ldots$. Then problem (6.1) is locally solvable at $s = -\infty$.*

Proof. Our problems fits into the framework of analysis of pseudodifferential operators with slowly varying symbols. Hence, the desired assertion follows in much the same way as Corollary 23.2 of [RST00]. On the other hand, the idea of the proof is as classical as the construction of Neumann series and can be easily explained. We write

$$Av\,(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\imath s\sigma} a(-\infty, \sigma) \hat{v}(\sigma) d\sigma + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\imath s\sigma} (a(s, \sigma) - a(-\infty, \sigma)) \hat{v}(\sigma) d\sigma$$

for $s < \delta(1)$. The resolvent $a(-\infty, \sigma)^{-1}$ exists for all σ on the horizontal line $\Im\sigma = -\gamma$. Furthermore, the Fourier transform of each function $v \in H^{1,\gamma}(-\infty, \delta(1))$ vanishing close to $\delta(1)$ extends to an analytic function in the half-plane $\Im\sigma \geq -\gamma$. Hence, the first summand on the right-hand side is invertible on such functions v. On the other hand, the second summand is small for s which are sufficiently close to $-\infty$. Hence, the operator A is invertible on functions $v \in H^{1,\gamma}(-\infty, \delta(1))$ with the property that Av vanishes away from a small neighbourhood of $-\infty$, as desired. □

On changing the coordinates by

$$\begin{array}{rcl} \omega &=& \dfrac{x}{t^{1/p}}, \\ s &=& \dfrac{t^{1-Q}}{1-Q}, \end{array}$$

we pull back the function spaces $H^{k,\gamma}(-\infty, \delta(1))$ to the original domain $\mathcal{G}$. Theorem 6.3 then yields a condition of local solvability of problem (1.1) at the characteristic point, see [AT12].

7. Asymptotic property of formal solution

We now turn to the proof of asymptotic property of the formal solution of the first boundary value problem for the heat equation at a characteristic point. We restrict ourselves to $n = 2$ and to homogeneous problem (2.1) in a $\mathcal{G}$. By (5.4), the formal solution has the form

$$U(\omega, s) = \sum_{n=1}^{\infty} \exp(\lambda_n s)\, V_n(\omega, s), \tag{7.1}$$

where

$$V_n(\omega, s) = c_n\, ((1-Q)s)^{\frac{1}{4}\frac{Q}{Q-1}} \sum_{m=0}^{\infty} \frac{V_{n,m}(\omega)}{((1-Q)s)^m}$$

and $\lambda_n = -\Big(n\dfrac{\pi}{2}\Big)$.

Obviously, $\exp(\lambda_n s) \in H^{1,\gamma}(-\infty, S)$ for some $S \ll \delta(1)$ if and only if $\gamma < \lambda_n$. Pick any γ with $\lambda_{n+1} < \gamma < \lambda_n$. By Theorem 6.3, the solutions $U \in H^{1,\gamma}(-\infty, S)$ of homogeneous problem (5.2) in the half-strip $(-1,1) \times (-\infty, S)$ form an one-dimensional space. Any solution U of this space is expected to possess a formal series expansion

$$U(\omega, s) \sim c\, ((1-Q)s)^{\frac{1}{4}\frac{Q}{Q-1}} \exp(\lambda_n s) \sum_{m=0}^{\infty} \frac{\tilde{V}_m(\omega)}{((1-Q)s)^m}$$

at $s = -\infty$, where c is a suitable constant.

For each $N = 0, 1, \ldots$, we introduce the function

$$U_N(\omega, s) = c\, ((1-Q)s)^{\frac{1}{4}\frac{Q}{Q-1}} \exp(\lambda_n s) \sum_{m=0}^{N} \frac{\tilde{V}_m(\omega)}{((1-Q)s)^m}$$

on $(-1,1) \times (-\infty, S)$. These functions belong to $H^{k,\gamma}(-\infty, S)$ for any natural number k. We get

$$\begin{aligned} AU_N &= c\, ((1-Q)s)^{\frac{1}{4}\frac{Q}{Q-1}} \exp(\lambda_n s) \\ &\times \Big(\sum_{m=0}^{N} \frac{-\tilde{V}_m'' + \lambda_n \tilde{V}_m}{((1-Q)s)^m} + \sum_{m=1}^{N+1} \frac{\frac{1}{2-p}\omega \tilde{V}_{m-1}' - \Big(\frac{Q}{4} + (m-1)(1-Q)\Big)\tilde{V}_{m-1}}{((1-Q)s)^m} \Big) \\ &= c\, ((1-Q)s)^{\frac{1}{4}\frac{Q}{Q-1}} \exp(\lambda_n s)\, \frac{\frac{1}{2-p}\omega \tilde{V}_N' - \Big(\frac{Q}{4} + N(1-Q)\Big)\tilde{V}_N}{((1-Q)s)^{N+1}}, \end{aligned}$$

the last equality being a consequence of the recurrent equations for V_m which look like

$$-\tilde{V}_m'' + \lambda_n \tilde{V}_m = -\frac{1}{2-p}\omega \tilde{V}_{m-1}' + \Big(\frac{Q}{4} + (m-1)(1-Q)\Big)\tilde{V}_{m-1} \tag{7.2}$$

for all $m = 0, 1, \ldots$. We thus conclude that the discrepancy is "small" with number N.

Since each function $\tilde{V}_m$ vanishes at $\omega = \mp 1$, we may apply a priori estimates for solutions of Sturm-Liouville problems (7.2) to evaluate the $H^2(-1,1)$-norms of $\tilde{V}_m$

through the $L^2(-1,1)$-norm of the right-hand sides. On repeating the estimate we arrive at inequalities

$$\|\tilde{V}_m\|_{H^2(-1,1)} \leq c^m\, m!\, \|\tilde{V}_0\|_{L^2(-1,1)}$$

with some constant c, valid for all $m = 1, 2, \ldots$. However, these inequalities are obviously insufficient to establish the convergence of the formal series expansion for U.

The main result of Part 2 reads as follows.

Theorem 7.1. *Suppose that $\lambda_{n+1} < \gamma < \lambda_n$. Then the formal series expansion (5.4) corresponding to a real solution $U \in H^{1,\gamma}(-\infty, S)$ is actually asymptotic in the sense (0.1).*

Proof. Given any nonnegative integer N, set

$$R_{N+1}(\omega, s) = U(\omega, s) - c\,((1-Q)s)^{\frac{1}{4}\frac{Q}{Q-1}} \exp(\lambda_n s) \sum_{m=0}^{N} \frac{\tilde{V}_m(\omega)}{((1-Q)s)^m}$$

for $(\omega, s) \in [-1,1] \times (-\infty, S)$. Moreover, define a function $X_{N+1}(\omega, s)$ from the equality

$$R_{N+1}(\omega, s) = c\,((1-Q)s)^{\frac{1}{4}\frac{Q}{Q-1}} \exp(\lambda_n s)\, \frac{X_{N+1}(\omega, s)}{((1-Q)s)^{N+1}},$$

which is possible since the coefficient of $X_{N+1}(\omega, s)$ does not vanish. Hence it follows that

$$U(\omega, s) = c\,((1-Q)s)^{\frac{1}{4}\frac{Q}{Q-1}} \exp(\lambda_n s) \Big(\sum_{m=0}^{N} \frac{\tilde{V}_m(\omega)}{((1-Q)s)^m} + \frac{X_{N+1}(\omega, s)}{((1-Q)s)^{N+1}} \Big),$$

and so the theorem is proved if we have established that $X_{N+1}(\omega, s)$ is bounded in a suitable sense in $[-1,1] \times (-\infty, S)$.

For this purpose we substitute $U(\omega, s)$ into equations (5.2). On taking into account recurrent equations (7.2) for $\tilde{V}_m(\omega)$ we obtain

$$\begin{array}{rcl} (X_{N+1})'_s - (X_{N+1})''_{\omega,\omega} + \dfrac{1}{2-p}\dfrac{\omega}{(1-Q)s}(X_{N+1})'_\omega + \Big(\lambda_n - \Big(\dfrac{1}{4}\dfrac{Q}{1-Q} + (N+1)\Big)\dfrac{1}{s}\Big) X_{N+1} & & \\ = & - \dfrac{1}{2-p}\omega \tilde{V}'_N + \Big(\dfrac{Q}{4} + N(1-Q)\Big)\tilde{V}_N & \\ = & -\tilde{V}''_{N+1} + \lambda_n \tilde{V}_{N+1} & \end{array} \qquad (7.3)$$

for $(\omega, s) \in (-1,1) \times (-\infty, S)$, the boundary conditions $X(\mp 1, s) = 0$ being obviously fulfilled.

Obviously, the remainder R_{N+1} belongs to $H^{1,\gamma}(-\infty, S)$ for each N. Hence it follows that

$$X_{N+1} = \frac{1}{c}\,((1-Q)s)^{-\frac{1}{4}\frac{Q}{Q-1} + (N+1)} \exp(-\lambda_n S) R_{N+1},$$

and so $X_{N+1} \in H^{1,\gamma-\lambda_n}(-\infty, S)$, which is due to the fact that $\gamma - \lambda_n < 0$. On the other hand,

$$(X_{N+1})'_s - (X_{N+1})''_{\omega,\omega} + \frac{1}{2-p}\frac{\omega}{(1-Q)s}(X_{N+1})'_\omega + \Big(\lambda_n - \Big(\frac{1}{4}\frac{Q}{1-Q} + (N+1)\Big)\frac{1}{s}\Big) X_{N+1} \in H^{0,0}(-\infty, S),$$

which is a consequence of (7.3).

The principal symbol of operator (7.3) at the singular point $-\infty$ just amounts to

$$\tilde{a}(-\infty, \sigma) = (\imath\sigma + \lambda_n) - \Big(\frac{\partial}{\partial\omega}\Big)^2,$$

where $\sigma \in \mathbb{C}$. The eigenvalues of this operator pencil are $\sigma_m = \imath(\lambda_n - \lambda_m)$ for $m = 1, 2, \ldots$. We now apply Theorem 6.2 with $\mu = 0$ and with γ replaced by $\gamma - \lambda_n$. Since $\gamma > \lambda_{n+1}$, there exist no eigenvalues σ_m which lie in the strip $0 < \Im\sigma < -\gamma + \lambda_n$. Hence it follows that X_{N+1} actually belongs to the space $H^{1,0}(-\infty, -0)$, as desired. □

8. Local spectrum of a backward parabolic equation

Here we evaluate explicitly the eigenvalues and eigenfunctions of the first boundary value problem for the backward parabolic equation $u'_t - u''''_{x,x,x,x} = 0$ in a bounded domain $\mathcal{G} \subset \mathbb{R}^2$ with a characteristic point at the origin. The authors gratefully acknowledge the help of Vitaly Stepanenko with the cumbersome computations used in this section.

The first boundary value problem for the backward parabolic equation in $\mathcal{G}$ is formulated as follows: Let Σ be the set of all characteristic points of the boundary of $\mathcal{G}$. Given any functions f in $\mathcal{G}$ and u_0, u_1 on $\partial\mathcal{G} \setminus \Sigma$, find a function u on $\overline{\mathcal{G}} \setminus \Sigma$ satisfying

$$\begin{array}{rcll} u'_t - u''''_{x,x,x,x} & = & f & \text{in} \quad \mathcal{G}, \\ \partial^j_\nu u & = & u_j & \text{at} \quad \partial\mathcal{G} \setminus \Sigma, \end{array} \tag{8.1}$$

for $j = 0, 1$, where ∂_ν is the derivative along the outward unit normal vector of the boundary. We focus upon a characteristic point 0 of the boundary which is assumed to be the origin in $\mathbb{R}^2$.

By the above, the domain $\mathcal{G}$ is described in a neighbourhood of the origin by the inequality $t > |x|^p$, where $p > 0$. We blow up the domain $\mathcal{G}$ at 0 by introducing new coordinates $(\omega, r) \in (-1, 1) \times (0, 1)$ with the aid of

$$\begin{array}{rcl} x & = & t^{1/p}\,\omega, \\ t & = & r. \end{array}$$

Under this change of variables the domain $\mathcal{G}$ nearby 0 transforms into the half-cylinder $(-1, 1) \times (0, \infty)$.

In the domain of coordinates (ω, r) problem (8.1) reduces to an ordinary differential equation with respect to the variable r with operator-valued coefficients. More precisely,

$$\begin{array}{rcll} r^Q\, U'_r - U''''_{\omega,\omega,\omega,\omega} - \dfrac{1}{p} r^{Q-1}\,\omega\, U'_\omega & = & r^Q F & \text{in} \quad (-1, 1) \times (0, 1), \\ \partial^j_\nu U & = & U_j & \text{at} \quad \{\mp 1\} \times (0, 1) \end{array} \tag{8.2}$$

for $j = 0, 1$, where $U(\omega, r)$ and $F(\omega, r)$ are pullbacks of $u(x, t)$ and $f(x, t)$ under the transformation, respectively, and $Q = 4/p$. We assume that $p \neq 4$, i.e., $Q \neq 1$.

We look for a formal solution to the homogeneous equation (8.2) (i.e., both F and U_j vanish) which has the form $U(\omega, r) = e^{S(r)}\, V(\omega, r)$, where S is a differentiable function of $r > 0$ and V expands as a formal Puiseux series with nontrivial principal part

$$V(\omega, r) = \frac{1}{r^{\mathfrak{e}N}} \sum_{j=0}^{\infty} V_{j-N}(\omega)\, r^{\mathfrak{e}j},$$

where N is a complex number and $\mathfrak{e}$ a real exponent to be determined.

On substituting $U(\omega, r)$ into (8.2) we extract the eikonal equation $r^Q S' = \lambda$ for the function $S(r)$, where λ is a (possibly complex) constant to be defined. For $Q \neq 1$ this implies

$$S(r) = \lambda\, \frac{r^{1-Q}}{1-Q}$$

up to an inessential constant factor. In this way the problem reduces to

$$\begin{array}{rcll} r^Q\, V'_r - V''_{\omega,\omega,\omega,\omega} - \dfrac{1}{p} r^{Q-1}\, \omega V'_\omega & = & -\lambda V & \text{in} \quad (-1,1) \times (0,\infty), \\ \partial^\alpha_\omega V & = & 0 & \text{at} \quad \{\mp 1\} \times (0,\infty) \end{array}$$

for all $|\alpha| \le 1$.

Analysis similar to that in Section 2 shows that a right choice of $\mathfrak{e}$ is $\mathfrak{e} = (Q-1)/k$ for some natural number k. On substituting the formal series for $V(\omega, r)$ into (4.6) and equating the coefficients of the same powers of r we get two collections of problems

$$\begin{array}{rcll} -V^{(4)}_{j-N} + \lambda\, V_{j-N} & = & 0 & \text{in} \quad (-1,1), \\ \partial^\alpha\, V_{j-N} & = & 0 & \text{at} \quad \{\mp 1\} \end{array} \tag{8.3}$$

for all $|\alpha| \le 1$, where $j = 0, 1, \ldots, k-1$, and

$$\begin{array}{rcll} -V^{(4)}_{j-N} + \lambda\, V_{j-N} & = & \dfrac{1}{p}\, (\omega, V'_{j-N-k}) - \mathfrak{e}(j-N-k)\, V_{j-N-k} & \text{in} \quad (-1,1), \\ \partial^\alpha\, V_{j-N} & = & 0 & \text{at} \quad \{\mp 1\} \end{array}$$

for all $|\alpha| \le 1$, where $j = k, k+1, \ldots, 2k-1$, and so on.

We restrict our discussion to Sturm-Liouville problem (8.3) for $j = 0$ merely. The remaining terms are determined in the same way as in Section 2. Set $v = V_{-N}$ and solve

$$\begin{array}{rcll} -v^{(4)} + \lambda\, v & = & 0 & \text{in} \quad (-1,1), \\ v = v' & = & 0 & \text{at} \quad \{\mp 1\}, \end{array} \tag{8.4}$$

Note that $\lambda > 0$, for $(v^{(4)}, v) = \lambda(v, v)$ and so $\|v''\|^2 = \lambda \|v\|^2$ by the Stokes formula, the scalar product and norm being those of $L^2(-1,1)$. On using the Ansatz $v(\omega) = \exp(z\omega)$ we get the characteristic equation $z^4 = \lambda$, whose roots are $z_0 = \sqrt[4]{\lambda}$, $z_1 = \imath z_0$, $z_2 = -z_0$ and $z_3 = -\imath z_0$. The general solution of the ordinary differential equation is

$$v = c_0 e^{z_0 \omega} + c_1 e^{\imath z_0 \omega} + c_2 e^{-z_0 \omega} + c_3 e^{-\imath z_0 \omega}.$$

Substituting this formula into the boundary conditions $v(\mp 1) = 0$ and $v'(\mp 1) = 0$ gives a system of four equations for four undetermined coefficients c_0, c_1, c_2, c_3. This system has a nonzero solution if and only is its determinant D vanishes. A direct computation shows that

$$D = \imath\, 8 z_0^2\, (\cosh(2z_0)\, \cos(2z_0) - 1) = 0.$$

If $z_0 = 0$ then $\lambda = 0$, which fails to be an eigenvalue. Hence the equation for eigenvalues reads

$$\cosh(2z_0)\cos(2z_0) = 1, \tag{8.5}$$

which has infinitely many solutions. The first solution is $2z_0 \approx 1,6\,\pi$, i.e., we get $\lambda_1 \approx (0,8\,\pi)^4$. For each z_0 satisfying (8.5) the rank of the matrix of the homogeneous linear system for the coefficients c_0, c_1, c_2, c_3 is equal to three. Hence, c_1, c_2, c_3 are multiples of c_0, i.e.,

$$c_1 = \frac{(\sinh 2z_0 \cos z_0 - (\cosh 2z_0 + 1)\sin z_0) + \imath(\sinh 2z_0 \sin z_0 + (\cosh 2z_0 - 1)\cos z_0)}{e^{z_0}\sin 2z_0 - e^{z_0}\cos 2z_0 + e^{-z_0}},$$

$$c_2 = \frac{e^{-z_0}\cos 2z_0 + e^{-z_0}\sin 2z_0 - e^{z_0}}{e^{z_0}\sin 2z_0 - e^{z_0}\cos 2z_0 + e^{-z_0}},$$

$$c_3 = \frac{(\sinh 2z_0 \cos z_0 - (\cosh 2z_0 + 1)\sin z_0) - \imath(\sinh 2z_0 \sin z_0 + (\cosh 2z_0 - 1)\cos z_0)}{e^{z_0}\sin 2z_0 - e^{z_0}\cos 2z_0 + e^{-z_0}}$$

up to a multiplicative real factor c_0. In particular, $\bar{c}_1 = c_3$ showing that v is real-valued. Thus, each eigenvalue λ is simple.

Mention that that the conditions $v(\mp 1) = 0$ and $v'(-1) = 0$ are for all z_0, not only for those satisfying (8.5). However, the evaluation of $v'(+1)$ gives

$$\begin{aligned} v'(+1) &= \frac{4c_0 z_0}{e^{z_0}\sin 2z_0 - e^{z_0}\cos 2z_0 + e^{-z_0}}\,(1 - \cosh(2z_0)\cos(2z_0)) \\ &= 0. \end{aligned}$$

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References

[AS64] Abramowitz, M., and Stegun, Irene A., eds., *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover Publications, New York, 1964.

[AN63] Agmon, S., and Nirenberg, L., *Properties of solutions of ordinary differential equations in Banach spaces*, Communs. Pure Appl. Math. **16** (1963), 121–239.

[AV64] Agranovich, M. S., and Vishik, M. I., *Elliptic problems with a parameter and parabolic problems of general form*, Russ. Math. Surv. **19** (1964), No. 3, 53–157.

[AT12] Antoniouk, A., and Tarkhanov, N., *The Dirichlet problem for the heat equation in domains with cuspidal points on the boundary*, In: Operator Theory: Advances and Applications, Birkhäuser, Basel et al., 2012, 20 pp.

[AB96] Aref'ev, V. N., and Bagirov, L. A., *Asymptotic behaviour of solutions of the Dirichlet problem for a parabolic equation in domains with singularities*, Mat. Zam. **59** (1996), No. 1, 12–23.

[AB98] Aref'ev, V. N., and Bagirov, L. A., *On solutions of the heat equation in domains with singularities*, Mat. Zam. **64** (1998), No. 2, 163–179.

[CH68] Courant, R., and Hilbert, D., *Methoden der Mathematischen Physik*, B. I u. II, 3. Auflage, Springer-Verlag, Berlin et al., 1968.

[Esk80] Eskin, G. I., *Boundary Value Problems for Elliptic Pseudodifferential Equations*, American Mathematical Society, Providence, RI, 1980.

[Evg61] Evgrafov, M. A., *Structure of solutions of exponential growth for some operator equations*, Trudy Mat. Inst. im. V. A. Steklova **60** (1961), 145–180.

[Fic56] Fichera, G., *Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine*, Atti Acc. Naz. Lincei Mem., Ser. 8 **5** (1956), 1–30.

[Gal09] Galaktionov, V. A., *On regularity of a boundary point for higher-order parabolic equations: Towards Petrovskii-type criterion by blow-up approach*, arXiv: 0901.3986vI [math.AP] 26 Jan 2009, 50 pp.

[GM11] Galaktionov, V. A., and Maz'ya, V. G., *Boundary characteristic point regularity for semilinear reaction-diffesion equations: Towards an ODE criteriom*, Journal of Mathematical Sciences **175** (2011), No. 3, 249–283.

[Gev13] Gevrey, M., *Sur les équations partielles du type parabolique*, J. math. pure appl. Paris **9** (1913), 305–471; **10** (1914), 105–148.

[Gru70] Grushin, V. V., *Pseudodifferential operators on* $\mathbb{R}^n$ *with bounded symbols*, Funkts. Analiz **4** (1970), 202–212.

[KS10] Kiselev, O., and Shestakov, I., *Asymptotics of solutions to the Laplace–Beltrami equation on a rotation surface with a cusp*, J. Math. Anal. Appl. **362** (2010), Issue 2, 393–400.

[Il'92] Il'in, A. M., *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems*, Translations of Mathematical Monographs, V. 102, AMS, Providence, RI, 1992.

[Kon66] Kondrat'ev, V. A., *Boundary problems for parabolic equations in closed domains*, Trans. Moscow Math. Soc. **15** (1966), 400–451.

[Kon67] Kondrat'ev, V. A., *Boundary value problems for elliptic equations in domains with conical points*, Trudy Mosk. Mat. Obshch. **16** (1967), 209–292.

[Len83] Leng, S., *Algebra*, Addison-Wesley, Reading, MA, 1983.

[Lev07] Levi, E. E., *Sull' equazione del calore*, Annali di Mat. **14** (1907), 187–264.

[MP72] Maz'ya, V. G., and Plamenevskii, B. A., *On the asymptotic behaviour of solutions of differential equations in Hilbert space*, Math. USSR Izvestiya **6** (1972), no. 5, 1067–1116.

[Mik63] Mikhaylov, V. P., *On the Dirichlet problem for a parabolic equation. I, II*, Mat. Sb. **61 (103)** (1963), 40–64; **62 (104)** (1963), 140–159.

[PP08] Pereira, A. L., and Pereira, M. C., *An eigenvalue problem for the biharmonic operator on* $\mathbb{Z}_2$ *-symmetric regions*, J. Lond. Math. Soc. **77** (2008), No. 2, 424–442.

[Pet34] Petrovskii, I. G., *Zur ersten Randwertaufgabe der Wärmeleitungsgleichung*, Compositio Math. **1** (1935), 389–419.

[RST00] Rabinovich, V., Schulze, B.-W., and Tarkhanov, N., *A calculus of boundary value problems in domains with non-Lipschitz singular points.* Math. Nachr. **215** (2000), 115–160.

[Sim65] Simonenko, I. B., *A new general method for investigating linear operator equations of the singular integral operator type. I, II*, Izv. Akad. Nauk SSSR, Ser. Mat. **29** (1965), 567–586; 757–782.

[Slo58] Slobodetskii, L. N., *The generalised spaces of S.L. Sobolev and their application to boundary value problems for partial differential equations*, Uch. Zapiski Leningr. Ped. Inst. im. A.I. Gertsena **197** (1958), 54–112.

[TS72] Tikhonov, A. N., and Samarskii, A. A., *Equations of Mathematical Physics*, 4 th ed., Nauka, Moscow, 1972.

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