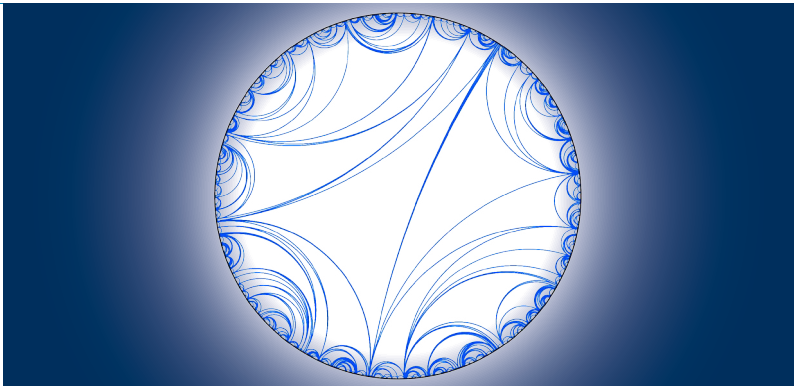




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THE METHOD OF FISCHER-RIESZ EQUATIONS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS

A. ALSAEDY AND N. TARKHANOV

ABSTRACT. We develop the method of Fischer-Riesz equations for general boundary value problems elliptic in the sense of Douglis-Nirenberg. To this end we reduce them to a boundary problem for a (possibly overdetermined) first order system whose classical symbol has a left inverse. For such a problem there is a uniquely determined boundary value problem which is adjoint to the given one with respect to the Green formula. On using a well elaborated theory of approximation by solutions of the adjoint problem, we find the Cauchy data of solutions of our problem.

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INTRODUCTION

The method of Fischer-Riesz equations can be specified within a larger approach which is usually referred to as the boundary element method. By this latter is meant a numerical method of solving boundary value problems which have been formulated as boundary integral equations. It can be applied in many areas of engineering and science including fluid mechanics, acoustics, electromagnetics, and fracture mechanics, see [WA02], [Kat02], [Gib08], [BSD08].

The boundary elements method attempts to use the given boundary conditions and other data of the problem to fit boundary values into the integral equation, rather than values throughout the space defined by a system of partial differential equations. Once this is done, the boundary integral equation can be used again

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to calculate numerically the solution directly at any desired point in the solution domain. More precisely, from the Cauchy data of the solution on the whole boundary one calculates readily the solution in the domain provided a left fundamental solution of the system is available in an explicit form, see for instance Lemma 10.2.3 in [Tar95].

The idea of the method of Fischer-Riesz equations goes back at least as far as [Pic40]. The paper [PF50] was given by Picone as an invited address before the Second Austrian Mathematical Congress in Innsbruck in 1949. He states in the introduction that he asked Fichera to write a certain part of the report. It is a crystallisation in the form of an abstract theory of some of the methods used by the authors and their associates at the National Institute for Applied Mathematics in Rome in the solution of problems involving differential and integro-differential equations. The method is based on some functional interpretation of the relations of mathematical physics analogous to Green's formula. The central point of the method is a construction of a suitable sequence of functions which are complete in a Lebesgue space L^2 on the boundary and satisfy the formal adjoint system in a neighbourhood of the closure of the domain. In [PF50] there were no indication to any solution of this problem. In [Kup67] a general process of constructing a necessary complete sequence of functions was elaborated assuming an explicit fundamental solution of the system. However, this paper fell short of providing a function-theoretic description of the method. In [KT93] the second author published a proof of a theorem of functional analysis that had already been obtained at the end of the 1980s. As it became clear later, this theorem was just an abstract exposition of the Fischer-Riesz equations method mentioned in [Kup67]. In [KT93] this method was developed for studying the ill-posed Cauchy problem with data on a piece of the boundary for solutions of overdetermined elliptic systems, see also Chapter 11 in [Tar95].

The purpose of the present paper is to develop the method of Fischer-Riesz equations for general boundary value problems for systems of partial differential equations elliptic in the sense of Douglis-Nirenberg [DN55]. To escape technicalities related to assigning weights we exploit the result of [Pro88] and reduce the system to a (possibly overdetermined) first order system whose classical symbol has a left inverse away from the zero section of the cotangent bundle. In this way we obtain what is often referred to as the first order system with injective symbol. The advantage of such systems lies in the fact that the Cauchy data of a solution just amount to the restriction of the solution to the boundary of the domain. Moreover, to any first order system there corresponds a unique Green operator which leads to a canonical Green formula for solutions. Any normal boundary conditions for solutions of the source system then reduce to an inhomogeneous linear system in the space of Cauchy data.

In contrast to [KT93] we elaborate the method of Fischer-Riesz equations for elliptic boundary value problems in Sobolev spaces, for these latter fit well the Fredholm property. If the boundary value problem is Fredholm then the conditions of solvability obtained by the Fischer-Riesz equations method come to those obtained from the Fredholm theory, i.e., the orthogonality to solutions of the homogeneous boundary value problem adjoint relative to the Green formula. The method of Fischer-Riesz equations may then be developed as a tool to get effective approximate solutions, cf. [Kup67].

1. REDUCTION TO A FIRST ORDER SYSTEM

One of the fundamental problems in the theory of partial differential equations is the problem of classifying equations and systems by type. A specific problem associated with the definition of ellipticity is that when a higher-order equation or system is reduced to a first order system, ellipticity may be destroyed. (We manipulate the concept “elliptic” freely. This concept can be given a strong sense only in an operator algebra with symbol map, where by the ellipticity of an operator is meant the invertibility of its symbol.) One approach to this problem was introduced in the paper [DN55] which gave a definition of ellipticity for systems which involved assigning weights to each of the equations and dependent variables and then defining the principal part of the system in terms of those weights. This concept can also be interpreted in terms of generalised homogeneity based on certain group actions in the spaces of preimages and images, see equality (1.4) in [Vol65]. The advantage of the definition of ellipticity given in [DN55] is that ellipticity can be preserved while a higher-order equation or system is reduced to an equivalent first order system. The disadvantage is that the definition is not invariant under nonsingular changes of variables. Therefore, the approach via weights fails to properly recognise elliptic systems. An alternative approach suggested in [Pro88] is to reduce the original equation or system to an overdetermined first order system and then use the classical symbol, which is natural and invariant way for such systems. In [Cos91] this result is strengthened by showing that any determined or overdetermined system with smooth coefficients and injective Douglis-Nirenberg symbol can be reduced to an overdetermined first order system with smooth coefficients and injective classical symbol. This reduction is accomplished by introducing as new dependent variables the derivatives of some of the original variables, and adjoining equations describing the relations between the new variables and the old or among the new variables. Moreover, any overdetermined first order system with smooth coefficients and injective classical symbol can be converted to a determined second order systems which is elliptic in the sense of Douglis-Nirenberg, or under any reasonable definition of ellipticity. The conversion is accomplished by operating on the original system with an appropriately chosen first order operator. The conversion to a second order system allows the application of the regularity results of [Mor54]. In fact, second order systems are treated in detail in [Mor54]. Note that the systems of partial differential equations usually still contain hidden integrability conditions. The process of their explicit construction is called completion (to involution). In [KST06] it is shown that the completion of any determined or overdetermined system with injective Douglis-Nirenberg symbol leads to an equivalent system whose classical symbol has a left inverse. To formulate the main result of [Cos91] more precisely, we extend the concept of ellipticity in the sense of Douglis-Nirenberg to overdetermined systems.

The systems we consider are of the form

$$A = (A_{i,j})_{\substack{i=1,\dots,l \\ j=1,\dots,k}}, \quad (1.1)$$

where $A_{i,j}$ are scalar partial differential operators of order $m_{i,j}$ on an open set \mathcal{X} in \mathbb{R}^n . (We will generally use the convention that our source system has k dependent variables, l equations and n independent variables.)

Definition 1.1. Suppose there are weights s_1, \dots, s_l and t_1, \dots, t_k in \mathbb{Z} , such that $m_{i,j} \leq s_i + t_j$. With this structure, the principal symbol of (1.1) is the matrix

$$\sigma_{\text{DN}}(A)(x, \xi) = (\sigma^{s_i+t_j}(A_{i,j})(x, \xi))_{\substack{i=1, \dots, l \\ j=1, \dots, k}}$$

for $(x, \xi) \in T^*\mathcal{X}$, where $\sigma^{s_i+t_j}(A_{i,j})$ is the homogeneous component of degree $s_i + t_j$ of the full symbol of $A_{i,j}$.

System (1.1) is said to have injective symbol in the sense of Douglis-Nirenberg at $x_0 \in \mathcal{X}$ if $\sigma_{\text{DN}}(A)(x, \xi)$ has maximal rank (that is, rank k) for $x = x_0$ and all $\xi \in \mathbb{R}^n \setminus \{0\}$.

The numbers s_1, \dots, s_l and t_1, \dots, t_k are determined uniquely up to an additive constant. Hence, the weights can be normalised by the condition $s_1, \dots, s_l \leq 0$ and $\max s_i = 0$. Then t_j satisfy automatically $t_1, \dots, t_k \geq 0$, for if $t_j < 0$, then $s_i + t_j < 0$ and so all the operators $A_{1,j}, \dots, A_{l,j}$ vanish identically. This amounts to saying that the j th dependent variable does not enter into the system, which is impossible.

Let $(\kappa_\lambda)_{\lambda>0}$ and $(\tilde{\kappa}_\lambda)_{\lambda>0}$ be the group actions in \mathbb{C}^k and \mathbb{C}^l , respectively, given by

$$\begin{aligned} \kappa_\lambda &= \lambda^{|t|} \text{diag}(\lambda^{-t_1}, \dots, \lambda^{-t_k}), \\ \tilde{\kappa}_\lambda &= \lambda^{-|s|} \text{diag}(\lambda^{s_1}, \dots, \lambda^{s_l}), \end{aligned}$$

where $|s| = s_1 + \dots + s_l$ and $|t| = t_1 + \dots + t_k$. Then the principal symbol $\sigma_{\text{DN}}(A)$ is homogeneous of degree $|s| + |t|$ in the sense that

$$\sigma_{\text{DN}}(A)(x, \lambda\xi) = \lambda^{|s|+|t|} \tilde{\kappa}_\lambda \sigma_{\text{DN}}(A)(x, \xi) \kappa_\lambda^{-1} \quad (1.2)$$

for all $\lambda > 0$. For $s_1 = \dots = s_l = 0$, one recovers the principal part of A considered by I. G. Petrovskii. For $s_1 = \dots = s_l = 0$ and $t_1 = \dots = t_k = m$, one obtains the classical principal symbol. The so-called twisted homogeneity of type (1.2) is of great importance in the calculus of pseudodifferential operators with operator-valued symbols.

The following result is due to [Cos91].

Theorem 1.2. Any system (1.1) with coefficients of class $C^{s,h}$ and injective symbol in the sense of Douglis-Nirenberg in \mathcal{X} can be converted to an equivalent overdetermined first order system whose coefficients are of class $C^{s-1,h}$ and whose classical symbol is injective.

Proof. As is mentioned in [Cos91], the reduction procedure used here is related to that attributed to Atiyah and Singer. It is probably not optimal in the sense that it may lead to a first order system which is not the smallest possible representation of the original system. \square

2. GREEN FORMULA

From what has been proved in Section 1 it follows that there is no restriction of generality in assuming that A is a (possibly overdetermined) first order partial differential operator with injective symbol on an open set $\mathcal{X} \subset \mathbb{R}^n$. Thus, A is of the form

$$A(x, D) = \sum_{j=1}^n A_j(x) D_j + A_0(x),$$

where $A_1(x), \dots, A_n(x)$ and $A_0(x)$ are $(l \times k)$ -matrices of smooth functions on \mathcal{X} and $D_j = -\imath \partial_{x_j}$ with $\imath = \sqrt{-1}$. We require

$$\sigma^1(A)(x, \xi) := \sum_{j=1}^n A_j(x) \xi_j$$

to have maximal rank (that is rank k) for all (x, ξ) away from the zero section of $T^*\mathcal{X}$.

In order to get asymptotic results, it is necessary to put some restrictions on A . Our basic assumption is that A satisfies the uniqueness condition of the local Cauchy problem in \mathcal{X} (condition $(U)_s$, cf. [Tar95, p. 185]). I.e., if u is a solution of $Au = 0$ on a connected open set $U \subset \mathcal{X}$ and u vanishes on a nonempty open subset of U then $u \equiv 0$ in U .

Lemma 2.1. *If A satisfies the condition $(U)_s$ in \mathcal{X} , then it has a pseudodifferential left fundamental solution, i.e., there is an $(k \times l)$ -matrix Φ of classical pseudodifferential operators of order -1 on \mathcal{X} , such that $\Phi A = I$ on compactly supported distributions in \mathcal{X} with values in \mathbb{C}^k .*

Proof. See Theorem 4.4.3 of [Tar95]. By the very construction, Φ has rational symbol, i.e., it satisfies the transmission condition with respect to each hypersurface in \mathcal{X} . \square

Let \mathcal{D} be a relatively compact domain with smooth boundary in \mathcal{X} and B any $(k' \times k)$ -matrix of smooth functions on the boundary $\partial\mathcal{D}$ of \mathcal{D} , such that the rank of $B(x)$ is equal to k' for all $x \in \partial\mathcal{X}$. We are interested in the boundary value problem

$$\begin{cases} Au = f & \text{in } \mathcal{D}, \\ Bu = u_0 & \text{at } \partial\mathcal{D} \end{cases} \quad (2.1)$$

with data u_0 on $\partial\mathcal{D}$. The most conventional Hilbert space setting of this problem is $H_1 := W^{1,2}$, hence we choose u_0 in $H^{1/2}(\partial\mathcal{D}, \mathbb{C}^{k'})$ and look for a $u \in H^1(\mathcal{D}, \mathbb{C}^k)$ satisfying (2.1).

Lemma 2.2. *Let C be a $((k - k') \times k)$ -matrix of smooth functions on $\partial\mathcal{D}$, such that*

$$\text{rank} \begin{pmatrix} B(x) \\ C(x) \end{pmatrix} = k$$

for all $x \in \partial\mathcal{D}$. Then there are unique matrices B^ and C^* of continuous functions on $\partial\mathcal{D}$ with the property that*

$$\int_{\partial\mathcal{D}} ((Bu, C^*g)_x - (Cu, B^*g)_x) ds = \int_{\mathcal{D}} ((Au, g)_x - (u, A^*g)_x) dx \quad (2.2)$$

for all $u \in H^1(\mathcal{D}, \mathbb{C}^k)$ and $g \in H^1(\mathcal{D}, \mathbb{C}^l)$, where ds is the surface measure on the boundary.

As usual, we write A^* for the formal adjoint of the differential operator A on the open set \mathcal{X} .

Proof. By assumption, the $(k \times k)$ -matrix

$$T(x) = \begin{pmatrix} B(x) \\ C(x) \end{pmatrix}$$

is invertible for all $x \in \partial\mathcal{D}$. Write $(T(x))^{-1} = (T_1(x), T_2(x))$ where T_1 and T_2 are $(k \times k')$ - and $(k \times (k - k'))$ -matrices of smooth functions on $\partial\mathcal{D}$, respectively. The equalities $T^{-1}T = E_k$ and $TT^{-1} = E_k$ amount to $T_1B + T_2C = E_k$ and

$$\begin{aligned} BT_1 &= E_{k'}, & BT_2 &= 0, \\ CT_1 &= 0, & CT_2 &= E_{k-k'}, \end{aligned} \quad (2.3)$$

where E_k stands for the unity $(k \times k)$ -matrix.

Given any $u \in H^1(\mathcal{D}, \mathbb{C}^k)$ and $g \in H^1(\mathcal{D}, \mathbb{C}^l)$, the Green formula of [Tar95, 9.2.2] shows that

$$\int_{\partial\mathcal{D}} (\sigma(x)u, g)_x ds = \int_{\mathcal{D}} ((Au, g)_x - (u, A^*g)_x) dx$$

where $\sigma(x)$ is the principal symbol of A evaluated at the point $(x, -\nu(x)\xi)$ of the complexified cotangent bundle of \mathcal{X} , $\nu(x)$ being the outward normal unit vector of the boundary at $x \in \partial\mathcal{D}$. Substituting $u = (T_1B + T_2C)u$ into this formula yields (2.2) with

$$\begin{aligned} C^* &= (\sigma T_1)^*, \\ B^* &= -(\sigma T_2)^*, \end{aligned} \quad (2.4)$$

as desired. \square

From (2.4) it follows immediately that the rank of C^* is equal to k' and the rank of B^* is $k - k'$.

Elliptic boundary value problems (2.1) require $k = l$ to be even and $k' = k/2$, in which case also the problem

$$\begin{cases} A^*g &= v & \text{in } \mathcal{D}, \\ B^*g &= g_0 & \text{at } \partial\mathcal{D} \end{cases} \quad (2.5)$$

called adjoint to (2.1) with respect to the Green formula is actually elliptic, cf. [Agr69].

Given any first order partial differential operators A with injective symbol on \mathcal{X} , the composition $\Delta = A^*A$ is a second order elliptic operator in the classical sense. This operator is usually referred to as the Laplacian of A . An easy manipulation of Green formula (2.2) leads to a fairly structural Green formula for the Laplacian Δ .

Theorem 2.3. *Under the above notation, any functions $u, v \in H^2(\mathcal{D}, \mathbb{C}^k)$ satisfy the integral equality*

$$\begin{aligned} &\int_{\partial\mathcal{D}} ((Bu, C^*Av)_x - (C^*Au, Bv)_x - (Cu, B^*Av)_x + (B^*Au, Cv)_x) ds \\ &= \int_{\mathcal{D}} ((\Delta u, v)_x - (u, \Delta v)_x) dx. \end{aligned}$$

Proof. It suffices to apply (2.2) twice to the left-hand side of this equality. Cf. Corollary 9.2.12 of [Tar95]. \square

3. FUNCTION SPACES

Denote by $H^{1/2}(\mathcal{D}, \mathbb{C}^k)$ the Slobodetskii space of functions of fractional smoothness $1/2$ in \mathcal{D} with values in \mathbb{C}^k , i.e. the completion of $C^\infty(\overline{\mathcal{D}}, \mathbb{C}^k)$ with respect to

the norm

$$\|u\|_{H^{1/2}(\mathcal{D}, \mathbb{C}^k)} = \left(\|u\|_{L^2(\mathcal{D}, \mathbb{C}^k)}^2 + \iint_{\mathcal{D} \times \mathcal{D}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+1}} dx dy \right)^{1/2}.$$

Obviously, $H^{1/2}(\mathcal{D}, \mathbb{C}^k)$ is a Hilbert space. We use the continuous embedding $H^{1/2}(\mathcal{D}, \mathbb{C}^l) \hookrightarrow L^2(\mathcal{D}, \mathbb{C}^l)$ to specify the dual space of $H^{1/2}(\mathcal{D}, \mathbb{C}^l)$ via the pairing in $L^2(\mathcal{D}, \mathbb{C}^l)$. Namely, let $H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$ be the completion of $C^\infty(\overline{\mathcal{D}}, \mathbb{C}^l)$ with respect to the norm

$$\|f\|_{H^{-1/2}(\mathcal{D}, \mathbb{C}^l)} = \sup_{\substack{g \in C^\infty(\overline{\mathcal{D}}, \mathbb{C}^l) \\ g \neq 0}} \frac{|(f, g)_{L^2(\mathcal{D}, \mathbb{C}^l)}|}{\|g\|_{H^{1/2}(\mathcal{D}, \mathbb{C}^l)}}.$$

Using these spaces, we are in a position to enlarge the domain of problem (2.1). To this end, we write $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ for the completion of $C^\infty(\overline{\mathcal{D}}, \mathbb{C}^k)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)} = \left(\|u\|_{H^{1/2}(\mathcal{D}, \mathbb{C}^k)}^2 + \|u\|_{L^2(\partial\mathcal{D}, \mathbb{C}^k)}^2 \right)^{1/2}. \quad (3.1)$$

By the trace theorem, the space $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ contains any space $H^s(\mathcal{D}, \mathbb{C}^k)$ with $s > 1/2$. However, the norm of $L^2(\partial\mathcal{D}, \mathbb{C}^k)$ is not majorised by the norm of $H^{1/2}(\mathcal{D}, \mathbb{C}^k)$.

Suppose $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ and $\{u_j\}$ is a sequence in $C^\infty(\overline{\mathcal{D}}, \mathbb{C}^k)$ converging to u in the norm (3.1). Then $\{u_j\}$ is a Cauchy sequence in $H^{1/2}(\mathcal{D}, \mathbb{C}^k)$, and so it converges to an element $u_i \in H^{1/2}(\mathcal{D}, \mathbb{C}^k)$. Moreover, the restrictions of u_j to the boundary form a Cauchy sequence in $L^2(\partial\mathcal{D}, \mathbb{C}^k)$. Hence, the sequence $\{u_j|_{\partial\mathcal{D}}\}$ converges in the space $L^2(\partial\mathcal{D}, \mathbb{C}^k)$ to an element u_b . It follows immediately that the closure of the mapping $u \mapsto (u|_{\mathcal{D}}, u|_{\partial\mathcal{D}})$ is an isometry of $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ onto a subspace of the Cartesian product $H^{1/2}(\mathcal{D}, \mathbb{C}^k) \times L^2(\partial\mathcal{D}, \mathbb{C}^k)$. For this reason, each element $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ can be identified with its image (u_i, u_b) in the Cartesian product. We call $u_b \in L^2(\partial\mathcal{D}, \mathbb{C}^k)$ the (generalised) trace of $u_i \in H^{1/2}(\mathcal{D}, \mathbb{C}^k)$ on the boundary of \mathcal{D} in spite of the fact that the trace u_b does not depend continuously on u_i .

Lemma 3.1. *There is a constant $c > 0$ such that*

$$\|Au\|_{H^{-1/2}(\mathcal{D}, \mathbb{C}^l)} \leq c \|u\|_{\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)}$$

for all $u \in C^\infty(\overline{\mathcal{D}}, \mathbb{C}^k)$.

Proof. The proof is based on manipulations of Green formula (2.2). See Lemma 2.3.1 in [Roi96]. \square

It follows from Lemma 3.1 that the closure A of the mapping $u \mapsto Au$ for $u \in C^\infty(\overline{\mathcal{D}}, \mathbb{C}^k)$, acts continuously from $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ into $H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$. Indeed, if $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ and $\{u_j\}$ is a sequence in $C^\infty(\overline{\mathcal{D}}, \mathbb{C}^k)$ converging to u in $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$, then $\{Au_j\}$ is, by Lemma 3.1, a Cauchy sequence in $H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$. Let f be the limit of $\{Au_j\}$ in $H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$. It is immaterial which sequence $\{u_j\}$ we choose to define f , and so we may set $Au = f$. Substituting u_j into the estimate of Lemma 3.1 and letting $j \rightarrow \infty$, we deduce that this estimate actually holds for all $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$. Thus, for each $u = (u_i, u_b)$ in $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$, the element Au is defined in $H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$, and the mapping $u \mapsto Au$ is continuous in the

corresponding norms. A passage to the limit similar to the above implies that $Au = f$ holds for $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ and $f \in H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$ if and only if the couple (u_i, u_b) satisfy the equation

$$\int_{\mathcal{D}} (u_i, A^* g)_x dx + \int_{\partial \mathcal{D}} ((Bu_b, C^* g)_x - (Cu_b, B^* g)_x) ds = \int_{\mathcal{D}} (f, g)_x dx$$

for all $g \in C^\infty(\overline{\mathcal{D}}, \mathbb{C}^l)$. In other words, Green's formula (2.2) is still valid for functions $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$.

If $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ then $Au \in H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$, and so Au can be approximated by functions of $C_{\text{comp}}^\infty(\mathcal{D}, \mathbb{C}^l)$ in the $H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$ -norm. On multiplying Au by the characteristic function $\chi_{\mathcal{D}}$ of \mathcal{D} we get an element of $H^{-1/2}(\mathcal{X}, \mathbb{C}^l)$ with support in $\overline{\mathcal{D}}$. It follows that $\Phi(\chi_{\mathcal{D}} Au)$ is well defined and belongs to the local space $H_{\text{loc}}^{1/2}(\mathcal{X}, \mathbb{C}^k)$. We now show that any $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ can be restored through the data Au and $u_b = u|_{\partial \mathcal{D}}$.

To shorten notation we use the same letter $\Phi(x, y)$ for the Schwartz kernel of the pseudodifferential operator Φ .

Lemma 3.2. *For each $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$, it follows that*

$$\begin{aligned} & - \int_{\partial \mathcal{D}} ((Bu, C^* \Phi(x, \cdot)^*)_y - (Cu, B^* \Phi(x, \cdot)^*)_y) ds + \int_{\mathcal{D}} (Au, \Phi(x, \cdot)^*)_y dy \\ & = \begin{cases} u(x), & \text{if } x \in \mathcal{D}, \\ 0, & \text{if } x \in \mathcal{X} \setminus \overline{\mathcal{D}}. \end{cases} \end{aligned} \quad (3.2)$$

Proof. The proof of Lemma 2.3 shows that formula (3.2) is actually equivalent to the equality

$$-\Phi([\partial \mathcal{D}] \sigma u) + \Phi(\chi_{\mathcal{D}} Au) = \chi_{\mathcal{D}} u$$

in the sense of distributions on \mathcal{X} , where $[\partial \mathcal{D}]$ is the surface layer on $\partial \mathcal{D}$. This follows in turn from the Green formula and the fact that Φ is a left fundamental solution of A in \mathcal{X} , for

$$\begin{aligned} \Phi([\partial \mathcal{D}] \sigma u) &= \Phi(\chi_{\mathcal{D}} Au - A(\chi_{\mathcal{D}} u)) \\ &= \Phi(\chi_{\mathcal{D}} Au) - \chi_{\mathcal{D}} u, \end{aligned}$$

as desired. \square

4. OPERATOR-THEORETIC FOUNDATIONS

The operator-theoretic foundations of the method of Fischer-Riesz equations are elaborated in [Tar95, 11.1]. It goes back at least as far as [PF50]. In this section we adapt this method for study of boundary value problem (2.1) in the Hilbert space $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$.

Set

$$\begin{aligned} H_1 &= \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k), \\ H &= H_2 \oplus L^2(\partial \mathcal{D}, \mathbb{C}^{k-k'}), \end{aligned}$$

where $H_2 = H^{-1/2}(\mathcal{D}, \mathbb{C}^l) \oplus L^2(\partial \mathcal{D}, \mathbb{C}^{k'})$.

Consider the mapping $M : H_1 \rightarrow H$ given by $Mu = (Au, Bu, Cu)$, which corresponds to the Cauchy problem for solutions of $Au = f$ in \mathcal{D} with Cauchy data $Bu = u_0$ and $Cu = u_1$ on $\partial \mathcal{D}$. By the above, M is continuous. In Section 5 we will prove that M has closed range.

Denote by $M^* : H \rightarrow H_1$ the operator that is adjoint to $M : H_1 \rightarrow H$ in the sense of Hilbert spaces.

Lemma 4.1. *The null-space $\ker M^*$ of the operator M^* is separable in the topology induced from H .*

Proof. This is true by the school fact that any subspace of a separable metric space is separable. \square

By $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$ we denote the space of all infinitely differentiable solutions of the formal adjoint system $A^*g = 0$ in a neighbourhood of the closure of \mathcal{D} .

Lemma 4.2. *Assume that $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$. Then the couple $(g \ominus C^*g, B^*g)$ belongs to $\ker M^*$.*

Proof. One has to show that $(Mu, (g \ominus C^*g, B^*g))_H = 0$ for all $u \in H_1$. By the Green formula, we get

$$\begin{aligned} (Mu, (g \ominus C^*g, B^*g))_H &= \int_{\mathcal{D}} (Au, g)_x dx - \int_{\partial\mathcal{D}} ((Bu, C^*g)_x - (Cu, B^*g)_x) ds \\ &= \int_{\mathcal{D}} (u, A^*g)_x dx \\ &= 0, \end{aligned}$$

as desired. \square

The subspace of $\ker M^*$ consisting of all elements of the form $(g \ominus C^*g, B^*g)$, where $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$, is separable. Hence, there are many ways to choose a sequence $\{g_i\}_{i=1,2,\dots}$ in $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$, such that the system $\{(g_i \ominus C^*g_i, B^*g_i)\}$ is complete in this subspace.

In Example 4.5 we will show some explicit sequences $\{g_i\}$ with this property. For the moment we fix one of such sequences.

Lemma 4.3. *As defined above, the system $\{(g_i \ominus C^*g_i, B^*g_i)\}_{i=1,2,\dots}$ is complete in $\ker M^*$.*

Proof. Let \mathcal{F} be a continuous linear functional on $\ker M^*$ vanishing on each element of the system $\{(g_i \ominus C^*g_i, B^*g_i)\}$. Since $\ker M^*$ is a closed subspace of H , the Riesz representation theorem implies the existence of an element $(f, u_0, u_1) \in \ker M^*$, such that the action of \mathcal{F} on $\ker M^*$ consists in scalar multiplication with the element (f, u_0, u_1) . In particular,

$$\begin{aligned} \mathcal{F}(g_i \ominus C^*g_i, B^*g_i) &= \int_{\mathcal{D}} (g_i, f)_x dx - \int_{\partial\mathcal{D}} ((C^*g_i, u_0)_x - (B^*g_i, u_1)_x) ds \\ &= 0 \end{aligned}$$

for all $i = 1, 2, \dots$. Since the system $\{(g_i \ominus C^*g_i, B^*g_i)\}_{i=1,2,\dots}$ is dense in the subspace of $\ker M^*$ consisting of all elements of the form $(g \ominus C^*g, B^*g)$, where $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$, we get

$$\int_{\partial\mathcal{D}} ((u_0, C^*g)_x - (u_1, B^*g)_x) ds = \int_{\mathcal{D}} (f, g)_x dx$$

for all $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$. We now use Theorem 5.1 which says that there exists a function $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ such that $Au = f$ in \mathcal{D} and $Bu = u_0$, $Cu = u_1$ at the boundary of \mathcal{D} . In other words, $(f, u_0, u_1) = Mu$. Hence it follows that $\mathcal{F}(h) = (h, Mu)_H = 0$ for all $h \in \ker M^*$. Thus, $\mathcal{F} \equiv 0$ and the standard application of the Hahn-Banach theorem completes the proof. \square

Write P for the orthogonal projection of H onto its direct summand H_2 . The composition $PM = (A, B)$ acting from H_1 to H_2 just amounts to the operator of boundary value problem (2.1) in the updated setting. More precisely, given any $f \in H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$ and $u_0 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k'})$, find $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ satisfying $Au = f$ in \mathcal{D} and $Bu = u_0$ at $\partial\mathcal{D}$. The following lemma expresses the most important property of system $\{g_i\}$.

Lemma 4.4. *The system $\{B^*g_i\}_{i=1,2,\dots}$ is complete in $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$ if and only if PM is injective.*

Proof. By the Hahn-Banach theorem, $\{B^*g_i\}$ is complete in $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$ if and only if any continuous linear functional \mathcal{F} on $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$ vanishing on each element of the system, is zero. Pick such a functional \mathcal{F} . By the Riesz representation theorem there is a function $u_1 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$ such that $\mathcal{F}(h) = (h, u_1)$ for all $h \in L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$. So we get

$$\begin{aligned} ((0, 0, u_1), (g_i \ominus C^*g_i, B^*g_i))_H &= \overline{(B^*g_i, u_1)_{L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})}} \\ &= \overline{\mathcal{F}(B^*g_i)} \\ &= 0 \end{aligned}$$

for all $i = 1, 2, \dots$. Applying Lemma 4.3 we deduce that the element $(0, 0, u_1)$ belongs to the orthogonal complement of the subspace $\ker M^*$ in H . Since the operator M has closed range, the orthogonal complement of $\ker M^*$ coincides with the range of M . Hence, there is a function $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ satisfying $Au = 0$ in \mathcal{D} and $Bu = 0$, $Cu = u_1$ at $\partial\mathcal{D}$. If the operator PM is injective, then $u = 0$ whence $u_1 = 0$ and $\mathcal{F} = 0$. Conversely, if the functional $\mathcal{F} = 0$ is different from zero, then u_1 is not zero and so PM fails to be injective, which is precisely the desired conclusion. \square

After removing the elements which are linear combinations of the previous ones from the system $\{B^*g_i\}_{i=1,2,\dots}$, we get a sequence $\{g_{i_n}\}$ in $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$, such that the system $\{B^*g_{i_n}\}$ is linearly independent. Applying then the Gram-Schmidt orthogonalisation to the system $\{B^*g_{i_n}\}$ in $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$, we obtain a new system $\{e_n\}_{n=1,2,\dots}$ in $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$, such that $\{B^*e_n\}$ is an orthonormal system in the space $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$. Moreover, $\{B^*e_n\}$ is an orthonormal basis in $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$, provided that PM is injective. Note that the elements e_n of the new system have explicit expressions through the elements $\{g_{i_1}, \dots, g_{i_n}\}$ of the old system in the form of Gram's determinants.

Example 4.5. Assume that A has real analytic coefficients outside the closure of \mathcal{D} in \mathcal{X} . Then a familiar trick with the Laplacian A^*A shows that A has a left fundamental solution Φ whose Schwartz kernel is real analytic away from the diagonal of $(\mathcal{X} \setminus \overline{\mathcal{D}}) \times (\mathcal{X} \setminus \overline{\mathcal{D}})$. Let $\{x_i\}$ be a finite or countable set of points in $\mathcal{X} \setminus \overline{\mathcal{D}}$,

such that each connected component of $\mathcal{X} \setminus \overline{\mathcal{D}}$ contains at least one point x_i . Then the columns of $D_x^\alpha \Phi(x_i, \cdot)^*$ belong to $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$ and the system $\{B^* D_x^\alpha \Phi(x_i, \cdot)^*\}$ is complete in the subspace of $L^2(\mathcal{D}, \mathbb{C}^{k-k'})$ formed by elements of the type $\{B^* g\}$ with $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$.

The proof of this fact actually repeats the reasoning of Example 11.4.14 in [Tar95]. Apparently the system of Example 4.5 is most convenient for numerical simulations.

5. THE CAUCHY PROBLEM

The Green formula (2.2) displays the Cauchy data of $u \in H^1(\mathcal{D}, \mathbb{C}^k)$ at the boundary of \mathcal{D} with respect to the operator A . These are Bu and Cu at $\partial\mathcal{D}$. Hence we formulate the Cauchy problem as follows: Given any $f \in H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$ and $u_0 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k'})$, $u_1 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$, find a function $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$, satisfying $Au = f$ in \mathcal{D} and

$$\begin{cases} Bu &= u_0, \\ Cu &= u_1 \end{cases} \quad (5.1)$$

at $\partial\mathcal{D}$.

The Cauchy problem for solutions of systems with injective symbol and data on the whole boundary was intensively studied in the 1960s. This study was motivated to a certain extent by the paper [Cal63]. For a recent account of the theory we refer to [VS00], [SS11].

Theorem 5.1. *Let $f \in H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$ and $u_0 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k'})$, $u_1 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$ be given functions. In order that there might exist a solution $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ to the system $Au = f$ in \mathcal{D} subject to boundary conditions (5.1), it is necessary and sufficient that*

$$\int_{\partial\mathcal{D}} ((u_0, C^*g)_x - (u_1, B^*g)_x) ds = \int_{\mathcal{D}} (f, g)_x dx \quad (5.2)$$

for all $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$.

Proof. Necessity. If $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ is a solution of the Cauchy problem with data f and u_0, u_1 , then by the Green formula

$$\begin{aligned} \int_{\partial\mathcal{D}} ((u_0, C^*g)_x - (u_1, B^*g)_x) ds &= \int_{\partial\mathcal{D}} ((Bu, C^*g)_x - (Cu, B^*g)_x) ds \\ &= \int_{\mathcal{D}} ((Au, g)_x - (u, A^*g)_x) dx \\ &= \int_{\mathcal{D}} (f, g)_x dx \end{aligned}$$

for all $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$, as required.

Sufficiency. We introduce a function U in $\mathcal{X} \setminus \partial\mathcal{D}$ with values in \mathbb{C}^k by the Green-type integral

$$U(x) = - \int_{\partial\mathcal{D}} ((u_0, C^* \Phi(x, \cdot)^*)_y - (u_1, B^* \Phi(x, \cdot)^*)_y) ds + \int_{\mathcal{D}} (f, \Phi(x, \cdot)^*)_y dy, \quad (5.3)$$

where $x \in \mathcal{X} \setminus \partial\mathcal{D}$. An easy calculation using (2.4) shows that

$$(u_0, C^* \Phi(x, \cdot)^*)_y - (u_1, B^* \Phi(x, \cdot)^*)_y = \Phi(x, \cdot) \sigma u_b$$

on $\partial\mathcal{D}$, where

$$u_b = T^{-1} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

It is clear that u_b is of class $L^2(\partial\mathcal{D}, \mathbb{C}^k)$ if and only if $u_0 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k'})$ and $u_1 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$. Thus, formula (5.3) reduces to

$$U = -[\partial\mathcal{D}] \sigma u_b + \Phi(\chi_{\mathcal{D}} f)$$

in $\mathcal{X} \setminus \partial\mathcal{D}$.

For each fixed $x \in \mathcal{X} \setminus \overline{\mathcal{D}}$, the columns of the matrix $\Phi(x, \cdot)^*$ belong to $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$. Hence, (5.2) implies that U vanishes in the complement of $\overline{\mathcal{D}}$.

Set $u = U \upharpoonright_{\mathcal{D}}$. We next prove that u is the desired solution of the Cauchy problem. This is equivalent to saying that $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ and $Au = f$ in \mathcal{D} , $u \upharpoonright_{\partial\mathcal{D}} = u_b$ at $\partial\mathcal{D}$.

By Lemma 3.2 of [SS11], the double layer potential $\Phi([\partial\mathcal{D}] \sigma u_b)$ in \mathcal{D} belongs to $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$. Moreover, the volume potential $\Phi(\chi_{\mathcal{D}} f)$ in \mathcal{D} is of class $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$, which is due to Lemma 3.1 *ibid*. Hence it follows that $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$. Be the function f zero, we would be able to deduce the rest of the proof from Theorem 10.3.4 of [Tar95].

In the general case we complete Φ to a fundamental solution at step 0 of a compatibility complex of A , cf. Theorem 4.4.3. An easy computation using solvability condition (5.2) yields

$$AU = -[\partial\mathcal{D}] \sigma u_b + \chi_{\mathcal{D}} f$$

in the sense of distributions in \mathcal{X} . In particular, $Au = f$ in \mathcal{D} .

Since $u_b \in L^2(\partial\mathcal{D}, \mathbb{C}^k)$, the jump of the double layer potential $\Phi([\partial\mathcal{D}] \sigma u_b)$ under crossing the surface $\partial\mathcal{D}$ from $\mathcal{X} \setminus \overline{\mathcal{D}}$ to \mathcal{D} just amounts to u_b . This is true even for all distributions u_b on $\partial\mathcal{D}$ taking their values in \mathbb{C}^k , see Theorem 10.1.5 in [Tar95]. For the square integrable densities u_b the jump is understood in an appropriate sense including the $L^2(\partial\mathcal{D}, \mathbb{C}^k)$ -norm.

On the other hand, the volume potential $\Phi(\chi_{\mathcal{D}} f)$ has no jump at the boundary of \mathcal{D} , for $\chi_{\mathcal{D}} f \in H_{\overline{\mathcal{D}}}^{-1/2}(\mathcal{X}, \mathbb{C}^k)$. Summarising we conclude that $u \upharpoonright_{\partial\mathcal{D}} = u_b$, for U vanishes in $\mathcal{X} \setminus \overline{\mathcal{D}}$. For a thorough treatment of this equality we refer the reader to Theorem 4.3 of [SS11]. \square

6. THE FISCHER-RIESZ EQUATIONS

Let $\{g_i\}_{i=1,2,\dots}$ be an arbitrary sequence in $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$ with the property that the system $\{(g_i \ominus C^* g_i, B^* g_i)\}$ is complete in $\ker M^*$. Applying the Gram-Schmidt orthogonalisation to the system $\{B^* g_i\}$ in $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$, we obtain a new system $\{e_n\}_{n=1,2,\dots}$ in $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$, such that the system $\{B^* e_n\}$ is orthonormal in the space $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$.

Given any $u_1 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$, we denote by $k_n(u_1)$ the Fourier coefficients of u_1 with respect to the system $\{B^* e_n\}$, i.e.,

$$k_n(u_1) = \int_{\partial\mathcal{D}} (u_1, B^* e_n)_y ds$$

for $n = 1, 2, \dots$

Lemma 6.1. *If $u \in \mathcal{H}^{1/2}(\partial\mathcal{D}, \mathbb{C}^k)$, then*

$$k_n(Cu) = \int_{\partial\mathcal{D}} (Bu, C^* e_n)_y ds - \int_{\mathcal{D}} (Au, e_n)_y dy,$$

where $n = 1, 2, \dots$

Proof. Using Lemma 4.2 we obtain

$$\begin{aligned} k_n(Cu) &= \int_{\partial\mathcal{D}} (Cu, B^* e_n)_y ds - (Mu, (e_n \ominus C^* e_n, B^* e_n))_H \\ &= \int_{\partial\mathcal{D}} (Bu, C^* e_n)_y ds - \int_{\mathcal{D}} (Au, e_n)_y dy, \end{aligned}$$

as desired. \square

Thus, in order to find the Fourier coefficients of the data Cu on the boundary with respect to the system $\{B^* e_n\}$ in $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$, it suffices to know only the data Au and Bu of problem (2.1).

Theorem 6.2. *Let $f \in H^{-1/2}(\mathcal{D}, \mathbb{C}^l)$ and $u_0 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k'})$. In order that there be a $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ such that $Au = f$ in \mathcal{D} and $Bu = u_0$ at $\partial\mathcal{D}$, it is necessary and sufficient that*

- 1) $\sum_{n=1}^{\infty} |c_n|^2 < \infty$, where $c_n = \int_{\partial\mathcal{D}} (u_0, C^* e_n)_y ds - \int_{\mathcal{D}} (f, e_n)_y dy$, and
- 2) $\int_{\partial\mathcal{D}} (u_0, C^* g)_y ds - \int_{\mathcal{D}} (f, g)_y dy = 0$ for all $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$ satisfying $B^* g = 0$ at the boundary.

Proof. Necessity. Suppose there is a function $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ satisfying $Au = f$ in \mathcal{D} and $Bu = u_0$ at $\partial\mathcal{D}$. Then $c_n = k_n(Cu)$ for all $n = 1, 2, \dots$, which is due to Lemma 6.1. Applying the Bessel inequality yields

$$\sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} |k_n(Cu)|^2 \leq \int_{\partial\mathcal{D}} |Cu|^2 ds < \infty,$$

and 1) is proved. On the other hand, 2) follows immediately from the Green formula, as desired.

Sufficiency. We now assume that 1) and 2) are satisfied. Condition 1) implies, by the Fischer-Riesz theorem, that there exists a function $u_1 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$, such that $\{c_n\}_{n=1,2,\dots}$ are the Fourier coefficients of u_1 with respect to the orthonormal system $\{B^* e_n\}$ in $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$. In other words, we get $c_n = k_n(u_1)$ for all $n = 1, 2, \dots$. On substituting formulas for c_n from 1) to these equalities we arrive at the orthogonality relations

$$\int_{\mathcal{D}} (f, e_n)_y dy - \int_{\partial\mathcal{D}} ((u_0, C^* e_n)_y - (u_1, B^* e_n)_y) ds = 0 \quad (6.1)$$

for $n = 1, 2, \dots$, cf. (5.2).

Our next goal is to prove that the element $(f, u_0, u_1) \in H$ is actually orthogonal to all elements of the system $\{(g_i \ominus C^* g_i, B^* g_i)\}_{i=1,2,\dots}$ in H , this latter being complete in $\ker M^*$. To do this, let us recall how the system $\{e_n\}$ has been obtained from the system $\{g_i\}$.

Even if the system $\{(g_i \ominus C^* g_i, B^* g_i)\}$ is linearly independent in H , the system $\{B^* g_i\}$ may have elements which are linear combinations of the previous ones in the space $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$. Such elements should be eliminated from the system before applying the Gram-Schmidt orthogonalisation.

For example, suppose that, for some i , the equality

$$B^* g_i = \sum_{j=1}^{i-1} c_{i,j} B^* g_j$$

is fulfilled with suitable complex numbers $c_{i,j}$. Consider the function

$$g'_i = g_i - \sum_{j=1}^{i-1} c_{i,j} g_j$$

which belongs to $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$. Obviously, $(g'_i \ominus C^* g'_i, B^* g'_i)$ lies in $\ker M^*$ and satisfies $B^* g'_i = 0$. It follows that

$$g_i = \sum_{j=1}^{i-1} c_{i,j} g_j + g'_i.$$

All the other elements $(g_i \ominus C^* g_i, B^* g_i)$, except for the eliminated ones, are expressed, by the contents of Gram-Schmidt orthogonalisation, as linear combinations of the elements $\{(e_n \ominus C^* e_n, B^* e_n)\}_{n=1, \dots, i}$. Thus, any element of the system $\{(g_i \ominus C^* g_i, B^* g_i)\}$ has a unique expression through the elements of the system $\{(e_n \ominus C^* e_n, B^* e_n)\}_{n=1, 2, \dots}$ in the form

$$g_i = \sum_{n=1}^i c_{i,n} e_n + g'_i, \quad (6.2)$$

where $g'_i \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$ satisfies $B^* g'_i = 0$ at the boundary $\partial\mathcal{D}$.

From equalities (6.1) and (6.2) and condition 2) of the theorem it follows immediately that

$$\begin{aligned} & ((f, u_0, u_1), (g_i \ominus C^* g_i, B^* g_i))_H \\ &= \sum_{n=1}^i c_{i,n} ((f, u_0, u_1), (e_n \ominus C^* e_n, B^* e_n))_H + ((f, u_0, u_1), (g'_i \ominus C^* g'_i, B^* g'_i))_H \\ &= 0 \end{aligned}$$

for all $i = 1, 2, \dots$. Since the system $\{(g_i \ominus C^* g_i, B^* g_i)\}_{i=1, 2, \dots}$ is complete in $\ker M^*$, the element (f, u_0, u_1) belongs to the orthogonal complement of this subspace in H . Using the lemma of operator kernel annihilator, we deduce that there exists a function $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ satisfying $Mu = (f, u_0, u_1)$. In particular, $Au = f$ in \mathcal{D} and $Bu = u_0$ at $\partial\mathcal{D}$, i.e., u is the desired solution of boundary value problem (2.1). \square

The convergence of the series in 1) guarantees the stability of boundary value problem (2.1). Under this condition, the range of the mapping PM is described in terms of continuous linear functionals on the space H , cf. 2), which is impossible in the general case.

Corollary 6.3. *Under the hypotheses of Theorem 6.2, if moreover the homogeneous adjoint boundary value problem (2.5) has no smooth solutions in $\overline{\mathcal{D}}$ different from*

zero, then for problem (2.1) to have a solution $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ it is necessary and sufficient that

$$\sum_{n=1}^{\infty} |c_n|^2 < \infty.$$

Proof. This follows immediately from Theorem 6.2 since condition 2) is automatically fulfilled. \square

7. REGULARISATION OF SOLUTIONS

Note that the proof of Theorem 6.2 works without the assumption that the operator PM in H is injective. Our next objective will be to construct an approximate solution to boundary value problem (2.1). To this end it is natural to assume that the homogeneous boundary value problem corresponding to (2.1) has only zero solution in the space $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$, i.e., the mapping PM is injective. In this case the orthonormal system $\{B^*e_n\}$ is actually complete in the space $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$. The orthonormal bases in $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$ of this form are said to be special, cf. [KT93], [Tar95, 11.3].

For $x \in \mathcal{X} \setminus \partial\mathcal{D}$, we denote by $k_n(B^*\Phi(x, \cdot)^*)$ the k -row whose entries are the Fourier coefficients of the columns of the $((k - k') \times k)$ -matrix $B^*\Phi(x, \cdot)^*$ with respect to the orthonormal basis $\{B^*e_n\}_{n=1,2,\dots}$ in $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$. More precisely, we set

$$k_n(B^*\Phi(x, \cdot)^*) = \int_{\partial\mathcal{D}} (B^*\Phi(x, \cdot)^*, B^*e_n)_y ds$$

for $n = 1, 2, \dots$

Lemma 7.1. *For $n = 1, 2, \dots$, the coefficients $k_n(B^*\Phi(x, \cdot)^*)$ are infinitely differentiable functions in $\mathcal{X} \setminus \partial\mathcal{D}$ with values in $(\mathbb{C}^k)^*$.*

Proof. The assertion is obvious, for the fundamental solution $\Phi(x, y)$ is C^∞ away from the diagonal of $\mathcal{X} \times \mathcal{X}$. \square

Consider the following (Schwartz) kernels R_N defined for $x \in \mathcal{X} \setminus \partial\mathcal{D}$ and y in a neighbourhood of $\overline{\mathcal{D}}$:

$$R_N(x, y) = \Phi(x, y) - \sum_{n=1}^N k_n(B^*\Phi(x, \cdot)^*)^* e_n(y)^*,$$

where $N = 1, 2, \dots$

Lemma 7.2. *As defined above, the kernels R_N are C^∞ in $x \in \mathcal{X} \setminus \partial\mathcal{D}$ and y in a neighbourhood of $\overline{\mathcal{D}}$ except for the diagonal $\{x = y\}$, and $A^*(y, D)R_N(\cdot, y)^* = 0$ on this set.*

Proof. This follows immediately from Lemma 7.1 and the fact that $e_n \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$, as desired. \square

The sequence $\{R_N\}$ provides a very special approximation of the fundamental solution Φ .

Lemma 7.3. *The sequence $\{B^* R_N(x, \cdot)^*\}_{N=1,2,\dots}$ converges to zero in the norm of $L^2(\partial\mathcal{D}, \mathbb{C}^{(k-k') \times k})$ uniformly in x on compact subsets of $\mathcal{X} \setminus \partial\mathcal{D}$.*

Proof. In fact, we get

$$\begin{aligned} B^* R_N(x, \cdot)^* &= B^* \Phi(x, \cdot)^* - \sum_{n=1}^N B^* e_n k_n(B^* \Phi(x, \cdot)^*) \\ &= \sum_{n=N+1}^{\infty} B^* e_n k_n(B^* \Phi(x, \cdot)^*) \end{aligned}$$

for each fixed $x \in \mathcal{X} \setminus \partial\mathcal{D}$. The right-hand side of this equality is a remainder of the Fourier series of the element $B^* R_N(x, \cdot)^*$ with respect to the orthonormal basis $\{B^* e_n\}$ in $L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$. Hence, it tends to zero in the $L^2(\partial\mathcal{D}, \mathbb{C}^{(k-k') \times k})$ -norm, as $N \rightarrow \infty$. This proves the first part of the lemma. The second part follows from a general remark on Fourier series, for the mapping of $\mathcal{X} \setminus \partial\mathcal{D}$ to $L^2(\partial\mathcal{D}, \mathbb{C}^{(k-k') \times k})$ given by $x \mapsto B^* \Phi(x, \cdot)^*$ is continuous. \square

The convergence of the approximations allows one to reconstruct solutions u of the class $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ through their data Au and Bu .

Theorem 7.4. *Every function $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ can be represented by the integral formula*

$$u(x) = \lim_{N \rightarrow \infty} \left(- \int_{\partial\mathcal{D}} (Bu, C^* R_N(x, \cdot)^*)_y ds + \int_{\mathcal{D}} (Au, R_N(x, \cdot)^*)_y dy \right)$$

for all $x \in \mathcal{D}$.

Proof. Fix a point $x \in \mathcal{D}$. Since $R_N(x, \cdot)^*$ and $\Phi(x, \cdot)^*$ differ by a k -row of smooth solutions of the system $A^*g = 0$ in a neighbourhood of $\overline{\mathcal{D}}$, one can write by the Green formula

$$u(x) = - \int_{\partial\mathcal{D}} ((Bu, C^* R_N(x, \cdot)^*)_y - (Cu, B^* R_N(x, \cdot)^*)_y) ds + \int_{\mathcal{D}} (Au, R_N(x, \cdot)^*)_y dy \quad (7.1)$$

for any $N = 1, 2, \dots$. From $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ we deduce that $Cu \in L^2(\partial\mathcal{D}, \mathbb{C}^{k-k'})$. Hence it follows by Lemma 7.3 that

$$\lim_{N \rightarrow \infty} \int_{\partial\mathcal{D}} (Cu, B^* R_N(x, \cdot)^*)_y ds = 0.$$

Thus, letting $N \rightarrow \infty$ in (7.1) establishes the formula. \square

As mentioned, for many problems of mathematical physics formulas for approximate solution like that of Theorem 7.4 were earlier obtained by Kupradze and his colleagues, see [Kup67].

8. SOLVABILITY OF ELLIPTIC BOUNDARY VALUE PROBLEMS

We can now return to the classical setting of boundary value problem (2.1) which is $H_1 = H^1(\mathcal{D}, \mathbb{C}^k)$. Given any $u \in H^1(\mathcal{D}, \mathbb{C}^k)$, both Au and Bu are well defined in $L^2(\mathcal{D}, \mathbb{C}^l)$ and $H^{1/2}(\partial\mathcal{D}, \mathbb{C}^{k'})$, respectively. Hence, the analysis does not require any function spaces of negative smoothness but distributions. More generally, let s

be a natural number. Given any $f \in H^{s-1}(\mathcal{D}, \mathbb{C}^l)$ and u_0 in $H^{s-1/2}(\partial\mathcal{D}, \mathbb{C}^{k'})$, we look for a $u \in H^s(\mathcal{D}, \mathbb{C}^k)$ satisfying (2.1). Theorem 6.2 still applies to establish the existence of a weak solution $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$, provided that the conditions 1) and 2) are fulfilled. To infer the existence of a classical solution, one needs a regularity theorem for weak solutions in $\mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$ saying that any weak solution is actually a classical one. This is the case if (2.1) is an elliptic boundary value problem, i.e., A is elliptic ($l = k$) and the pair (A, B) satisfies the Shapiro-Lopatinskii condition at the boundary of \mathcal{D} , see Section 10.5 in [Roi96]. For general operators A with injective symbol the regularity problem may be reduced to a regularity theorem for weak solutions of $A^*Au = A^*f$ in \mathcal{D} with boundary data $Bu = u_0$ and $B^*Au = B^*f$ at $\partial\mathcal{D}$, see Lemma 2.3.

Corollary 8.1. *Suppose a regularity theorem holds for (2.1). Let $f \in H^{s-1}(\mathcal{D}, \mathbb{C}^l)$ and $u_0 \in H^{s-1/2}(\partial\mathcal{D}, \mathbb{C}^{k'})$, where $s = 1, 2, \dots$. Then, in order that there be a $u \in H^s(\mathcal{D}, \mathbb{C}^k)$ such that $Au = f$ in \mathcal{D} and $Bu = u_0$ at $\partial\mathcal{D}$, it is necessary and sufficient that*

- 1) $\sum_{n=1}^{\infty} |c_n|^2 < \infty$, where $c_n = \int_{\partial\mathcal{D}} (u_0, C^*e_n)_y ds - \int_{\mathcal{D}} (f, e_n)_y dy$, and
- 2) $\int_{\partial\mathcal{D}} (u_0, C^*g)_y ds - \int_{\mathcal{D}} (f, g)_y dy = 0$ for all $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$ satisfying $B^*g = 0$ at the boundary.

Proof. It is sufficient to prove the sufficiency of conditions 1) and 2). If the conditions 1) and 2) are satisfied, then there exists a function $u \in \mathcal{H}^{1/2}(\mathcal{D}, \mathbb{C}^k)$, such that $Au = f$ in \mathcal{D} and $Bu = u_0$ at $\partial\mathcal{D}$. Since $Au \in H^{s-1}(\mathcal{D}, \mathbb{C}^l)$ and $Bu \in H^{s-1/2}(\partial\mathcal{D}, \mathbb{C}^{k'})$, the regularity theorem implies that $u \in H^s(\mathcal{D}, \mathbb{C}^k)$, as desired. \square

If (2.1) is elliptic then so is the problem (A^*, B^*) adjoint to (2.1) with respect to the Green formula. By the Fredholm property, the space of all $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$ satisfying $B^*g = 0$ at $\partial\mathcal{D}$, is finite dimensional. Moreover, the condition 2) alone is sufficient for the existence of a solution $u \in H^s(\mathcal{D}, \mathbb{C}^k)$ to problem (2.1). Hence it follows that for elliptic boundary value problems the condition 2) is automatically fulfilled.

Thus, the regularity problem for weak solutions of (2.1) is still of primary character in the study of boundary value problems. On the other hand, our approach demonstrates rather strikingly that Theorem 7.4 is of great importance for numerical simulation.

Corollary 8.1 applies in particular to boundary value problems for generalised Cauchy-Riemann systems in the space [Ste91, Ste93a, Ste93b], see also [FL65], [Agr69].

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