Institut für Mathematik
Arbeitsgruppe Geometrie

Variational Problems on Supermanifolds

Dissertation
zur Erlangung des akademischen Grades
“doctor rerum naturalium” (Dr. rer. nat.)
in der Wissenschaftsdisziplin Mathematik/Differentialgeometrie

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät
der Universität Potsdam

von
Florian Hanisch

Potsdam, den 21.09.2011

Abstract

In this thesis, we discuss the formulation of variational problems on supermanifolds. Supermanifolds incorporate bosonic as well as fermionic degrees of freedom. Fermionic fields take values in the odd part of an appropriate Grassmann algebra and are thus showing an anticommutative behaviour. However, a systematic treatment of these Grassmann parameters requires a description of spaces as functors, e.g. from the category of Grassmann algberas into the category of sets (or topological spaces, manifolds,...). After an introduction to the general ideas of this approach, we use it to give a description of the resulting supermanifolds of fields/maps. We show that each map is uniquely characterized by a family of differential operators of appropriate order. Moreover, we demonstrate that each of this maps is uniquely characterized by its component fields, i.e. by the coefficients in a Taylor expansion w.r.t. the odd coordinates. In general, the component fields are only locally defined. We present a way how to circumvent this limitation. In fact, by enlarging the supermanifold in question, we show that it is possible to work with globally defined components. We eventually use this formalism to study variational problems. More precisely, we study a super version of the geodesic and a generalization of harmonic maps to supermanifolds. Equations of motion are derived from an energy functional and we show how to decompose them into components. Finally, in special cases, we can prove the existence of critical points by reducing the problem to equations from ordinary geometric analysis. After solving these component equations, it is possible to show that their solutions give rise to critical points in the functor spaces of fields.

In dieser Dissertation wird die Formulierung von Variationsproblemen auf Supermannigfaltigkeiten diskutiert. Supermannigfaltigkeiten enthalten sowohl bosonische als auch fermionische Freiheitsgrade. Fermionische Felder nehmen Werte im ungeraden Teil einer Grassmannalgebra an, sie antikommutieren deshalb untereinander. Eine systematische Behandlung dieser Grassmann-Parameter erfordert jedoch die Beschreibung von Räumen durch Funktoren, z.B. von der Kategorie der Grassmannalgebren in diejenige der Mengen (der topologischen Räume, Mannigfaltigkeiten, ...). Nach einer Einführung in das allgemeine Konzept dieses Zugangs verwenden wir es um eine Beschreibung der resultierenden Supermannigfaltigkeit der Felder bzw. Abbildungen anzugeben. Wir zeigen, dass jede Abbildung eindeutig durch eine Familie von Differentialoperatoren geeigneter Ordnung charakterisiert wird. Darüber hinaus beweisen wir, dass jede solche Abbildung eineindeutig durch ihre Komponentenfelder, d.h. durch die Koeffizienten einer Taylorentwickelung bzgl. ungerader Koordinaten bestimmt ist.

Im Allgemeinen sind Komponentenfelder nur lokal definiert. Wir stellen einen Weg vor, der diese Einschränkung umgeht: Durch das Vergrößern der betreffenden Supermannigfaltigkeit ist es immer möglich, mit globalen Koordinaten zu arbeiten. Schließlich wenden wir diesen Formalismus an, um Variationsprobleme zu untersuchen, genauer betrachten wir eine super-Version der Geodäte und eine Verallgemeinerung von harmonischen Abbildungen auf Supermannigfaltigkeiten. Bewegungsgleichungen werden von Energiefunktionalen abgeleitet und wir zeigen, wie sie sich in Komponenten zerlegen lassen. Schließlich kann in Spezialfällen die Existenz von kritischen Punkten gezeigt werden, indem das Problem auf Gleichungen der gewöhnlichen geometrischen Analysis reduziert wird. Es kann dann gezeigt werden, dass die Lösungen dieser Gleichungen sich zu kritischen Punkten im betreffenden Funktor-Raum der Felder zusammensetzen.

Acknowledgments

I would like to thank my advisor Christian Bär for suggesting this interesting topic to me, for his continued support and for his patience even though the work took far more time than we originally thought.
Special thanks also to the members of the Potsdam geometry group, especially to Frank Pfäffle, Christoph Stephan, Christian Becker, Volker Branding and my former roommate Dennis Koh for various enlightening discussions and feedback. It was (and still is) a great time.

I am grateful to Gregor Weingart for very helpful email conversations concerning the theory of jet bundles and differential operators, as well as to Christoph Sachse for explaining parts of the Molotkov-Sachse approach to supergeometry to me at points where I got stuck.

Moreover, I want to thank Vicente Cortés, Sylvie Paycha and Thomas Schick, who were referees or member of the board of examination, for making very useful suggestions for improvement of this text.

Finally, I would like to thank the "IMPRS for Geometric Analysis, Gravitation and String Theory" at the Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut) and the Sonderforschungsbereich Raum-Zeit-Materie (SFB 647) for financial support during the time I worked on this thesis.

Statement of Originality

This thesis contains no material which has been accepted for the award of any other degree or diploma at any other university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying.

Potsdam, September 21, 2011
Florian Hanisch

Contents

1 Introduction

The research in this thesis addresses the problem of the formulation of geometric variational problems on supermanifolds and the development of technics to find and describe its critical points. The problems in question generalize the notions of geodesics and harmonic maps to supergeometry.

The development of supergeometry took its origin from concepts of theoretical physics: In quantum physics, there are two different kinds of particles, bosons and fermions. Whereas bosons behave according to the Bose-Einstein statistic and are allowed to occupy the same quantum state, fermions are not able to do so, they behave according to the Fermi-Dirac statistic. As a consequence, bosonic fields are described by operators Φ_1, Φ_2 which commute among each other up to order $\hbar$, whereas fermionic fields Ψ_1, Ψ_2 lead to anticommutator relations:

$$[\Phi_1, \Phi_2] = \mathcal{O}(\hbar) \qquad\qquad \{\Psi_1, \Psi_2\} = \mathcal{O}(\hbar)$$

In a classical limit $\hbar \to 0$, this leads to a theory of commuting and anticommuting classical fields. Supergeometry is the attempt to incorporate commuting and anticommuting (i.e. bosonic and fermionic) objects in a unified framework in order to obtain a geometric approach to physics which treats both types of fields on the same footing. There are several reasons to look for such a framework. In quantum physics, the path integral formalism for fermions requires a classical field theory which includes anticommuting quantities. In general, it is then desirable to be able to work on an arbitrary curved space. Moreover, the concept of superspaces allows for the construction of theories which are manifestly supersymmetric in a nice geometric fashion (although not all supersymmetric theories arise in this way). Finally, the subject is also interesting from a mathematical point of view. It leads to interesting generalizations of concepts from classical geometry and geometric analysis such as geodesics or harmonic maps which get anticommuting contributions, even on ordinary smooth manifolds. These behave differently from their commuting partners. Insights from supermathematics also play an important role in index theory and there are ideas to give "supersymmetric proofs" of the index theorem which still have to be made rigorous.

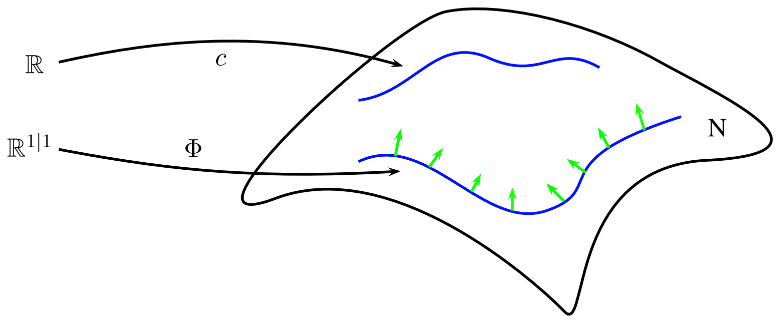
In this work, we will discuss variational problems on spaces containing commuting and anticommuting degrees of freedom rather than questions arising from supersymmetry itself. In particular, we will neither discuss the complex supersymmetric theories from modern physics nor the relation of spin and statistics - there will be no spinor fields. The focus is mainly on a formalism, which enables us to define variational problems, introduce a concept of critical points and to demonstrate the existence of solutions for certain geometrical functionals defined on supermanifolds. We will use the approach introduced by Berezin, Kostant and Leites which defines a supermanifold in terms of its sheaf of functions using the ringed space language from algebraic geometry. A superfunction which anticommutes with itself must be nilpotent and in fact, it is the existence of nontrivial nilpotent functions which is responsible for most of the new features of supergeometry. In particular, this geometry can not be described in terms of points of a topological space.

In fact, it turns out that it is necessary to define a "space of maps" between supermanifolds to give a precise meaning to functionals and their critical points. Following [61], this "space" has to be defined as a functor and we will adopt the categorical approach to supergeometry by Molotkov ([44]) and Sachse ([52]) to describe this construction. Although it would be possible to work exclusively inside the Molotkov-Sachse framework, we will choose a hybrid approach. We will avoid functorial language when dealing with finite dimensional supergeometry and use the construction by Berezin, Kostant and Leites instead. On the other hand, the notion of critical points relies on the functorial language and we will show how to merge both concepts.

We close the introduction with a naive discussion of a simple variational problem which nevertheless shows some interesting features. Looking at a map Φ from a space $\mathbb{R}^{1|1}$ with one commuting and one anticommuting dimension into a Riemannian manifold $(N, \langle , \rangle)$, we may do a naive Taylor expansion w.r.t. an odd coordinate θ:

$$\Phi(t,\theta) = c(t) + \psi(t)\theta \tag{1.1}$$

This has to stop because $\theta^2 = 0$. It turns out that c is just an ordinary (commuting) curve whereas ψ is an anticommuting vector field along c, so that we have the following picture:



It is possible to write down an energy functional for such a super curve and the resulting equations of motion read

$$\nabla_t \dot{c} = R(\psi,\psi)\dot{c} \qquad\qquad \nabla_t \psi = 0 \tag{1.2}$$

We see that one map Φ gives rise to 2 geometric objects (c, ψ) whose equations of motions are coupled in a natural way. What's more, we obtain completely new expressions since a term as $R(\psi,\psi)\dot{c}$ does not automatically vanish.

This work is organized as follows:
The first chapter contains basic material on super linear algebra. All notions and constructions will be introduced on the level of modules. In particular, we will discuss the Berezinian of a module and a map as well as the operation of changing the ring of a module, which is needed during the discussion of pullbacks and the development of component formalism in chapter 4.
The second chapter is an introduction in the theory of supermanifolds by Berezin, Kostant

and Leites. We will discuss their basic structure theory and that of morphisms among them. Furthermore, we will discuss super vector bundles and geometric structures on them which generalize the notion of Riemannian metrics and connections. The construction of pullback bundles and connections will be developed in detail since the approach here is different from that on ordinary smooth manifolds and not much material on that topic is available in the literature. Finally, we describe integration on Riemannian supermanifolds and discuss some variational calculus as well as the divergence theorem.
The third chapter discusses the categorical (or functorial) approach to supergeometry. We introduce the general idea of this concept which consists in replacing supermanifolds by functors (from the category of supermanifolds to the category of sets) and maps by natural transformation between such functors. As an example, we will discuss how to construct the total space of a vector bundle in terms of the functor representing it. The second part of chapter 3 is devoted to the Molotkov-Sachse approach to supergeometry which uses certain functors from the category of finite dimensional Grassmann algebras into a category of manifolds to describe a supermanifold. After discussing this approach in some detail, the space of all maps between two supermanifolds is introduced (however, without equipping it with a smooth structure). This notion is crucial to deal with variational problems.
In chapter four, we analyze the structure of the space of maps defined in the previous chapter. It turns out that it can be described in terms of differential operators on supermanifolds along maps. To make this statement precise, we give a short introduction to the theory of jet bundles and linear differential operators on smooth manifolds. We then include a detailed exposition of the algebraic definition of differential operators on modules and their characterization using product rules similar to the Leibniz rule. Based on such product rules, we use combinatorial arguments to prove that the space of maps can be described by differential operators. Finally, we introduce component fields of maps and vector fields and prove that the latter ones are uniquely determined by their components. Since the definition of these components is based on a choice of global odd coordinate fields on a given supermanifold, we introduce a method which will enable us to apply the component formalism on arbitrary supermanifolds.
The last chapter is devoted to special variational problems for maps from supermanifolds to ordinary Riemannian manifolds $(N, \langle , \rangle)$. We will look at supergeodesics as discussed e.g. in [61] in a first part and then consider a generalization of harmonic maps to supermanifolds. The chapter starts with a discussion of functionals and their critical points using the categorical approach from chapter 4. Then we give an overview on classical theorems on closed geodesics, harmonic maps and elliptic theory on vector bundles. Based on these well known results, we discuss the existence of critical points by reducing the super equations of motion to differential equations on smooth manifolds and vector bundles. This also reveals the geometric meaning of these equations. We find the following general pattern resulting from the reduction: There is one nonlinear equation and a family of linear equations, which depend on the solution of the nonlinear one and can be thought of as corrections from the super world. In the case of supergeodesics defined on $\mathbb{R}^{1|1}$, it is possible to give an explicit expression for the functor of critical points which is represented by the supermanifold $TN \oplus \Pi TN$. This is already mentioned in [10]. For closed supergeodesics and superharmonic maps, this explicit description in terms of a representing supermanifold is not available and we will describe the functor of critical points in terms of a "bundle structure".

2 Elements of Superalgebra

In this section, we will give a short introduction into concepts of superalgebra. The general concepts are well known in the literature (e.g. [10], chapter 1, [66], chapter 3 or [42], chapter 3). However, some notions occur in the literature in an ambiguous way so we will also use this section to settle them. Finally we collect some algebraic results that will be needed in later sections, which are, to the authors knowledge, not contained in the literature on supergeometry.

2.1 Super modules and algebras

The word "super" is always to mean $\mathbb{Z}_2$-graded, indices in $\mathbb{Z}_2$ referring to this grading will be denoted $\overline{0}, \overline{1}$ and $\overline{i}, \overline{j}$ respectively. They may occur as lower or upper indices. The parity of an element x of a $\mathbb{Z}_2$-graded object is denoted by $|x|$. If $G = G_{\overline{0}} \oplus G_{\overline{1}}$ is some $\mathbb{Z}_2$-graded Abelian group, elements of $G_{\overline{0}} \cup G_{\overline{1}} \setminus \{0\}$ are called *homogeneous*.

We introduce the basis algebraic concepts:

Definition 2.1

(a) *A superring R is a $\mathbb{Z}_2$-graded ring, i.e. it satisfies $R = \mathbb{R}_{\overline{0}} \oplus \mathbb{R}_{\overline{1}}$ as an Abelian group and $R_{\overline{i}} \cdot R_{\overline{j}} \subset R_{\overline{i}+\overline{j}}$ for the multiplication. R is called supercommutative if the supercommutator*

$$[r_1, r_2] := r_1 r_2 - (-1)^{|r_1||r_2|} r_2 r_1 \tag{2.1}$$

vanishes for all homogeneous elements of $r_1, r_2 \in R$.

(b) *A left R-supermodule is a $\mathbb{Z}_2$-graded left R-module $M = M_{\overline{0}} \oplus M_{\overline{1}}$ such that $R_{\overline{i}} \cdot M_{\overline{j}} \subset M_{\overline{i}+\overline{j}}$. Right supermodules are defined in a similar way.*

(c) *A morphism $f : M \longrightarrow N$ of left R-supermodules is a morphism of the underlying R- modules which preserves the $\mathbb{Z}_2$-grading. The set of morphisms is denoted by $\mathrm{Hom}_{_RSMod}(M, N)$, the corresponding category is denoted by $_RSMod$ (and $SMod_R$ for right modules).*

In case there is no danger of confusion, we will simply write $\mathrm{Hom}_R(M, N)$ instead of $\mathrm{Hom}_{_RSMod}(M, N)$ etc. Superalgebras are modules with an additional multiplication:

Definition 2.2 *An left R-supermodule A is called an R-superalgebra, if A is an ordinary R-algebra and its multiplication is compatible with the grading in the sense $A_{\overline{i}} A_{\overline{j}} \subset A_{\overline{i}+\overline{j}}$. Together with the morphisms*

$$\mathrm{Hom}_{_RSAlg}(A, B) = \{f \in \mathrm{Hom}_R(A, B) | f(ab) = f(a)f(b) \text{ for all } a, b \in A\}$$

they form the category $_RSAlg$. Again, the case of right R-superalgebras is treated in an analogous way.

Defining submodules and ideals, it is necessary to make sure that the corresponding subspaces are compatible with the grading:

Definition 2.3 *Let M be a left R-supermodule and $N \subset M$.*

(a) N is called a R-super submodule of M if it is an ordinary submodule and in addition, the $\mathbb{Z}_2$-gradings are compatible in the sense $N_{\overline{i}} = N \cap M_{\overline{i}}$ for $\overline{i} = \overline{0}, \overline{1}$. In other words, N must not lie "transversal" to the grading of M.

(b) If M is a R-superalgebra, then N is called a superideal (or homogeneous ideal) of M if it is an ordinary ideal and a super submodule.

Remark 2.4

(a) $\mathbb{R}$ becomes a superring by setting $\mathbb{R}_{\overline{0}} := \mathbb{R}$ and $\mathbb{R}_{\overline{1}} := \{0\}$. The resulting category of left (and right) supermodules is then denoted $SVec$, its objects are called $\mathbb{R}$-super vector spaces.

(b) In many cases, the superring in question will already be a unital, supercommutative $\mathbb{R}$- superalgebra A. The corresponding A-supermodules then of course always have an underlying $\mathbb{R}$-super vector space.

(c) Equations like (2.1) make sense only for homogenous elements. In the rest of this work, we will use similar expressions for general elements of R interpreting them in the following way: The statement is true if all elements involved are homogenous and it is extended by multilinearity to the general case.

(d) In commutative algebra, left and right R-module structures correspond bijectively to each other. In the same manner, a left R-supermodule M over a supercommutative superring R gives rise to a right R-supermodule by defining

$$m \cdot r = (-1)^{|m||r|} r \cdot m \qquad \text{for } m \in M, r \in R \tag{2.2}$$

and vice versa.

(e) In this work, we will assume that the superrings in question have a **unit** 1 (which has to be an element of $R_{\overline{0}}$) and are **supercommutative** as well as **associative**. Thus, by (d), there is no need to distinguish between left and right supermodules. We will often indicate which structure is used just by putting the ring element r to the left or to the right of the module element m.

The concept of morphisms of supermodules from definition 2.1 can be extended as follows:

Definition 2.5 *Let M, N be R-supermodules, then the set of* inner morphisms *is defined by*

$$\underline{\mathrm{Hom}}_{RSMod}(M, N) := \{f : M \longrightarrow N | f \text{ is } R-linear \}$$

Again, we will simply write $\underline{\mathrm{Hom}}_R(M, N)$.

At the moment, this is an ad hoc definition. We will see in example 4.3 using the categorical point of view that this notion occurs very naturally in many situations (e.g. if the modules are free). It is important to point out that this is not the set of homomorphisms in the category of supermodules. *Morphisms in a category of superobjects will always preserve the $\mathbb{Z}_2$-grading.* Inner morphism have the following properties (see [10], §1.6):

Lemma 2.6 *Let R be a superring and M, N be R-supermodules. Then $\underline{\mathrm{Hom}}(M,N)$ becomes an left R-supermodule by the following specifications for $r \in R$:*

$$\underline{\mathrm{Hom}}(M,N)_{\bar{i}} := \{f : M \longrightarrow N R-linear\ |f(M_{\bar{j}}) \subset M_{\bar{j}+\bar{i}}\}$$
$$r \cdot f := (m \mapsto r \cdot (f(m)))$$

Moreover, we have $\underline{\mathrm{Hom}}(M,N)_{\bar{0}} = \mathrm{Hom}(M,N)$ and for all $r \in R, m \in M, f \in \underline{\mathrm{Hom}}(M,N)$:

$$f(rm) = (-1)^{|r||f|} r f(m) \qquad\qquad f(mr) = f(m)r \tag{2.3}$$

so that f is R-linear w.r.t. to right module structure from equation (2.2)*.*

Remark 2.7 The left side of equation (2.3) is an example for the following important rule of thumb: *If a calculation involves super objects x and y and these two are interchanged, then a sign $(-1)^{|x||y|}$ occurs.* Thus, objects of parity $\bar{1}$ behave like fermions.
From a more abstract point of view, the rule of thumb is a consequence of the fact that the categories of interest are tensor categories in a super sense (see the next example for the definition of the tensor product). This means that there are, as in usual tensor categories, commutativity-isomorphisms satisfying the usual identities except for signs that have to be added, according to the parity of the interchanged objects (see [10] §1.2 or [66] 3.7 for a discussion).

Given R-supermodules M and N, we can perform the obvious algebraic constructions:

Example 2.8

(a) **The module with exchanged parity:** As additive group, we set $(\Pi M)_{\bar{i}} := M_{\bar{i}+\bar{1}}$. Denoting by $\pi : M \longrightarrow \Pi M$ the canonical map which is the identity on the level of sets, the multiplication $\cdot^\pi$ on ΠM depends on whether M is a left or right module. It is defined by

$$r \cdot^\pi \pi(m) := (-1)^{|r|}\pi(r \cdot m) \qquad\qquad \pi(m) \cdot^\pi r := \pi(m \cdot r)$$

These equations reflect the fact, that π, as an inner morphism $M \longrightarrow \Pi M$, is odd.
In case that M is a supermodule over a superalgebra A, ΠM can be identified with $\Pi\mathbb{R} \otimes_{\mathbb{R}} M = \mathbb{R}^{0|1} \otimes_{\mathbb{R}} M$ for the tensor product defined below.

(b) **The sum of modules and algebras:** The direct sum $M \oplus N$ is defined as the direct sum of ordinary modules, equipped with the following $\mathbb{Z}_2$-grading:

$$(M \oplus N)_{\bar{i}} := M_{\bar{i}} \oplus N_{\bar{i}}$$

In case that M and N are superalgebras, the sum $M \oplus N$ inherits this structure by

$$(m,n) \cdot (m',n') = (-1)^{|n||m'|}(mm', nn')$$

(c) **The dual of a module:** The dual left R-supermodule M^* of M is defined to be $\underline{\mathrm{Hom}}_R(M, R)$, where R is considered as a left supermodule over itself. This is a left R-supermodule by 2.6. Note that we took the inner morphisms because these carry a $\mathbb{Z}_2$-grading, whereas the set $\mathrm{Hom}_R(M, R)$ does not have a useful superstructure.

(d) **The tensor product of modules and algebras:** By (2.2), we can regard M as a right module since we assume that R is supercommutative. The tensor product of ordinary modules $M \otimes_R N$, together with the following grading and left action by R,

$$(M \otimes_R N)_{\overline{k}} = \bigoplus_{\overline{i}+\overline{j}=\overline{k}} M_{\overline{i}} \otimes N_{\overline{j}}$$

$$r \cdot (m \otimes n) := (rm \otimes n)$$

then gives the super tensor product of M and N. The tensor product, defined in this way, has the usual universal property of tensor products, see [2] p.6 ff. In case that M, N are in fact R-superalgebras, the tensor product becomes a superalgebra again by defining the multiplication according to the rule of thumb:

$$(m \otimes n) \cdot (m' \otimes n') = (-1)^{|n||m'|}(mm') \otimes (nn')$$

Remark 2.9 Using the tensor product of supermodules, it is possible to rewrite the definition of a superalgebra in a useful way: It is an R-supermodule A, together with an element

$$\mu : A \otimes_R A \longrightarrow A$$

of $\mathrm{Hom}_R(A \otimes A, A)$. In fact, since μ preserves the grading, the rule $a \cdot b := \mu(a \otimes b)$ defines the multiplication and vice versa.

Following [20], appendix A2.3, super algebras may be used to give a unified definition of symmetric and exterior algebra. Let V be a super vector space and provide $T(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}$ with the following super structure:

$$(T(V))_{\overline{i}} := \bigoplus_{\overline{j}_1+\cdots+\overline{j}_k \equiv \overline{i}} V_{\overline{j}_1} \otimes \ldots \otimes V_{\overline{j}_k}$$

This is just a generalization of the tensor product from example 2.8. Thus, for $V = V_{\overline{0}}$, this is the trivial graduation on $T(V)$ whereas for $V = V_{\overline{1}}$, we have the natural $\mathbb{Z}_2$-graduation on $T(V)$ which is given by tensor powers. In general, we could have started with a $\mathbb{Z}$-graded vector space or even module but we are not going to deal with that.

Definition 2.10 *The symmetric algebra $\mathcal{S}(V)$ is defined to be $T(V)/I$, where I is the two-sided ideal generated by the elements*

$$I := \{[a, b] | a, b \in T(V)\} = \{ab - (-1)^{p(a)p(b)} ba | a, b, \in T(V)\}$$

The d-th symmetric power $\mathcal{S}^d(V)$ is the image of $V^{\otimes d} \subset T(V)$ under the quotient map.

By definition of the "ordinary" symmetric and exterior algebra, we get

Proposition 2.11 ([20] A.2.2) *We have the following isomorphisms :*

(a) If $V = V_{\overline{0}}$, then $\mathcal{S}(V) \cong \mathrm{Sym}^\bullet(V)$ and $\mathcal{S}^d(V) \cong \mathrm{Sym}^d(V)$. Furthermore, $\mathcal{S}$ carries the trivial graduation.

(b) If $V = V_{\overline{1}}$, then $\mathcal{S}(V) \cong \bigwedge^\bullet(V)$ and $\mathcal{S}^d(V) \cong \bigwedge^d(V)$. Furthermore, $\mathcal{S}$ carries the natural super structure: $\mathcal{S}(V)_{\overline{0}} = \bigwedge^{ev}(V)$, $\mathcal{S}(V)_{\overline{1}} = \bigwedge^{od}(V)$

This proposition will provide a nice justification for the definition of supermanifolds in 3.1. Symmetric algebras over a super vector space V form one of the most important classes of supercommutative superalgebras over $\mathbb{R}$. It of course contains, as a special case, the exterior algebra of a finite dimensional vector space.

2.2 Free modules and linear algebra

A particular important class of supermodules is given by those that admit a basis. They are for instance used to define the notion of super vector bundles. Many familiar theorems from the linear algebra of vector spaces carry over to this class of modules.

Definition 2.12 *An R-super module M is called free of rank (or dimension) $p|q$ if it admits a homogeneous basis $\{m^1_{\overline{0}}, \dots, m^p_{\overline{0}}, m^1_{\overline{1}}, \dots, m^q_{\overline{1}}\}$ such that $m^i_{\overline{j}} \in M_{\overline{j}}$.*

Under the assumption, that R is unital, the rank is indeed well defined:

Proposition 2.13 *The rank $p|q$ of a free super module M over a supercommutative unital superring is uniquely defined.*

We will not give a detailed proof (see [66] p. 114 for details) but just sketch the method. If $J \subset R$ denotes the ideal of nilpotent elements, R/J is a commutative ring with 1. Thus, there is a unital homomorphism $\varphi : R \longrightarrow K$ into some field K (in later applications usually $K = \mathbb{R}$, in general we can take the quotient of R/J by some maximal ideal) which becomes an R-module by $r \cdot k := \varphi(r)k$. Forming the tensor product, we obtain $M \otimes_R K \cong K^{p|q}$. But the right hand side is a super vector space whose even and odd dimension is well defined.

Remark 2.14 The technique, that was applied in the preceding proof can be roughly described by "setting the odd variables to zero". This is due to the fact, that $R_{\overline{1}} \subset J$ so that we in particular take a quotient by $R_{\overline{1}}$ when we take the product $M \otimes_R K$. Taking an appropriate tensor product to remove odd and nilpotent components will prove crucial for many constructions and arguments in subsequent parts of this work.

Having fixed a basis, we clearly have the following isomorphism:

Lemma 2.15 (and Definition) *If M is a free R-super module of rank $p|q$, then we have $M \cong R^{m|n}$ where $R^{m|n}$ is the (left) R-supermodule, given by the free R-module R^{m+n}, provided with the grading*

$$(R^{m|n})_{\overline{0}} = R^m_{\overline{0}} \oplus R^n_{\overline{1}} \qquad\qquad (R^{m|n})_{\overline{1}} = R^n_{\overline{0}} \oplus R^m_{\overline{1}}$$

Hence, it is possible to do linear algebra of free modules using matrices with entries in the superring R. The set of these $(m+n)\times(s+t)$ matrices will be denoted by $\underline{Mat}(m|n, s|t, R)$, invertible matrices by $\underline{GL}(m|n, R)$. We will not go into details but only note that we can assign a parity to matrices P which describe elements of $\underline{\mathrm{Hom}}_R(R^{m|n}, R^{s|t})$ according to the parity of the corresponding inner homomorphism of modules. Writing P in the obvious block decomposition

$$P := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.4}$$

we have $|P| = \overline{0}$ if all entries of A, D are even and those of B, C are odd, whereas $|P| = \overline{1}$ if the entries of A, D are odd and those of B, C are even.

For free modules, we have the usual duality relations:

Proposition 2.16 *If M is a free R-supermodule with homogeneous basis $\{m_1, \ldots, m_{p+q}\}$, then the set $\{\varphi^1, \ldots, \varphi^{m+n}\} \subset \underline{\mathrm{Hom}}(M, R)$, uniquely defined by $\varphi^i(e_j) = \delta^i_j$, provides a homogeneous basis for E^*, where the parities are given by $|\varphi^i| = |m_i|$. In particular, we have $M \cong M^* \cong R^{p|q}$.*

Bearing in mind the rule of thumb (remark 2.7), it is necessary to fix an order for the natural pairing of M^* and M. We will always use

$$\langle \cdot, \cdot \rangle : M^* \otimes_R M \longrightarrow R \qquad\qquad \langle \varphi, m \rangle := \varphi(m) \tag{2.5}$$

Using this contraction, the super trace is defined as follows on a free module M:

$$\mathrm{str} : \underline{\mathrm{Hom}}_R(M, M) \longrightarrow R \qquad F \mapsto \textstyle\sum_i (-1)^{|e_i|} e^i(F(e_i)) = \mathrm{tr}(A) - \mathrm{tr}(D) \tag{2.6}$$

where on the right hand side, a homogeneous basis was chosen and F is represented by a matrix of the form (2.4).

To be able to do integration on supermanifold, the notion of a determinant is needed. The ordinary Leibniz formula clearly can be generalized to matrices containing only elements of $R_{\overline{0}}$. In general, the concept of the Berezinian is introduced:

Proposition 2.17 (and definition [42] 3.3.4, 3.3.5) *Let $P \in GL(m|n, R)$ be written in block form (2.4). Its Berezinian is defined by*

$$Ber(P) := det(A - BD^{-1}C) det D^{-1}$$

$Ber : GL(m|n, R) \longrightarrow GL(1|0, R_{\overline{0}})$ is a well defined group homomorphism agreeing with the determinant in case $n = 0$.

The notion of a Berezinian can then be generalized to automorphisms of free supermodules. The following concept will play a role similar to that of the top degree exterior power of a vector space:

Proposition 2.18 ([10] §1.11, [42], III.3.7) *Let M be a free R-supermodule of rank $p|q$. The Berezinian $Ber(M)$ of M is then the R-supermodule, generated by elements $[m_{\overline{0}}^1 \cdots m_{\overline{1}}^q]$ for any homogeneous basis $\{m_{\overline{0}}^1, \ldots, m_{\overline{0}}^p, m_{\overline{1}}^1, \ldots, m_{\overline{1}}^q\}$ subject to the following relations:*

$$[T(m_{\overline{0}}^1) \cdots T(m_{\overline{1}}^q)] = Ber(T)[m_{\overline{0}}^1 \cdots m_{\overline{1}}^q] \qquad \forall T \in \mathrm{Aut}_R(M)$$

$Ber(M)$ is a free R-supermodule of rank $1|0$ for $q \in 2\mathbb{N}$ and $0|1$ for $q \in 2\mathbb{N}_0 + 1$.
If N is another free R-supermodule of rank $p|q$ and $f \in \mathrm{Hom}_R(M, N)$ is invertible, then there is an induced morphism

$$Ber(f) : Ber(M) \longrightarrow Ber(N) \qquad [m_{\overline{0}}^1 \cdots m_{\overline{1}}^q] \mapsto [f(m_{\overline{0}}^1) \cdots f(m_{\overline{1}}^q)]$$

If $M = N$, then $Ber(f)$ is given by multiplication by $Ber(F)$, where F is any matrix representing f.

Note that similar to definition 2.10, it is possible to give the definition of the exterior algebra of a free supermodule. However, there is in general no top degree component because the exterior algebra of $R^{0|s}$, considered as an ungraded algebra, is isomorphic to the symmetric algebra $\mathrm{Sym}^\bullet(R^s)$.

Since the Berezinian is only defined for invertible matrices, it can not be used to test whether an element of $\underline{\mathrm{Mat}}(m|n, R)$ is invertible. However, the problem can be reduced to the computation of a determinant. First recall that a matrix with entries in an ordinary unital commutative Ring S is invertible if and only if its determinant is an element of the group of units $R^\times$. This can be applied to the construction discussed on top of remark 2.14. The projection $\pi : R \longrightarrow R/J$ induces a projection

$$\pi : \underline{\mathrm{Mat}}(m|n, R) \longrightarrow \underline{Mat}(m + n, R/J)$$

which can be used to characterize the invertibility of a matrix:

Lemma 2.19 ([8], 2.22) *For $A \in \underline{\mathrm{Mat}}(m|n, R)$, the following statements are equivalent:*

(a) A is invertible.

(b) $\pi(A) \in Mat(m + n, R/J)$ is invertible

(c) $\det(\pi(A))$ is in the group of units $(R/J)^\times$

The proof $(a) \Leftrightarrow (b)$ uses the fact that if $\pi(A)$ is invertible, A is invertible up to a nilpotent element N. But then, it is easy to write down an inverse for A using a Neumann's series which converges since N is nilpotent (see the quoted reference for details). A typical example for this situation in supergeometry is the following:

Example 2.20 Let $U \subset \mathbb{R}^p$ be an open set and $q \in \mathbb{N}$. Let $R := C^\infty(U) \otimes_{\mathbb{R}} \bigwedge^\bullet \mathbb{R}^q$ with the grading induced from $\bigwedge^\bullet \mathbb{R}^q$. We then have

$$J = C^\infty(U) \otimes_{\mathbb{R}} \textstyle\bigwedge^{\geq 1} \mathbb{R}^q \qquad\qquad R/J \cong C^\infty(U) \otimes_{\mathbb{R}} \mathbb{R} \cong C^\infty(U)$$

Thus, a matrix in $\underline{\mathrm{Mat}}(s, R)$ is invertible if and only if the the projected matrix $\pi(A) \in \mathrm{Mat}(s, C^\infty(U))$ is invertible, i.e. if and only if we have $det(\pi(A)) \in C^\infty(U, \mathbb{R}^\times)$. We will also write $ev(A)$ instead of $\pi(A)$.

Finally, we can localize this statement at a point $x \in U$. The localization R_x of R at the maximal ideal $\mathfrak{m}_x := \mathfrak{m}'_x \otimes_{\mathbb{R}} \bigwedge^\bullet \mathbb{R}^q \oplus \bigwedge^{\geq 1} \mathbb{R}^q$ (where $\mathfrak{m}'_x = \{[f] \in C^\infty_{\mathbb{R}^p,x} | f(x) = 0\}$) is given by $R_x = C^\infty_x \otimes \bigwedge^\bullet \mathbb{R}^q$. Here, C^∞_x denotes the ring of germs of smooth functions on U near x and R_x ist just the stalk of germs of sections of R near x (where R is considered as a sheaf of rings on U). Since the ideal of nilpotent elements in R_x is given by $J_x = C^\infty_x \otimes \bigwedge^{\geq 1} \mathbb{R}^q$, a matrix $M \in \underline{\mathrm{Mat}}(m|n, R_x)$ “at x” is thus invertible if and only if its image $\pi_x(M)$ under the projection

$$\pi_x : \underline{\mathrm{Mat}}(m|n, R_x) \longrightarrow \mathrm{Mat}(m + n, R_x/J_x) = \mathrm{Mat}(m + n, C^\infty_x)$$

is an invertible matrix with entries in C^∞_x. Furthermore, this is equivalent to the statement that the matrix $ev_x(M) := \pi_x(M)(x) \in \mathrm{Mat}(m + n, \mathbb{R} = R_x/\mathfrak{m}_x)$, obtained by evaluating all germs at x, is invertible.

2.3 Extension and restriction of scalars

In the following, we collect some results concerning the change of the ring of a given (super)-module. The reference is [6], chapter II.1.13 and II.5. All superrings are again assumed to be unital and supercommutative.

Definition 2.21 *Let R and S be two superrings and $\rho : S \longrightarrow R$ a homomorphism of superrings (which preserves the grading).*

(a) Let M be an R-supermodule. The S-supermodule $\rho_* M$ (also denoted by M^ρ) is given by M as an Abelian group, equipped with the action of S defined by $s \cdot m := \rho(s)m$. It is called the module obtained from M by restricting the ring of scalars.

(b) For each $f \in \underline{\mathrm{Hom}}_R(M, N)$ let $\rho_* f \in \underline{\mathrm{Hom}}_S(\rho_* M, \rho_* N)$ be the canonically induced morphism.

This construction has the following properties:

Proposition 2.22 ([6],II.1.13)

(a) The assignment $f \mapsto \rho_* f$ is injective, it is bijective in case ρ is surjective.

(b) If ρ is surjective (injective), every generating (free) family in M is generating (free) in $\rho_* M$.

(c) For $\sigma : T \longrightarrow S$ is another homomorphism of superrings, then $(\rho \circ \sigma)_* M = \sigma_* \rho_* M$.

To describe the operation adjoint to the restriction of scalars, let again $\rho : R \longrightarrow S$ be a homomorphism of superrings. By definition 2.21 and remark 2.7, S becomes a right R-supermodule $\rho_*(R)$. Thus, it is possible to change the scalars in the following way:

Definition 2.23 *Let M be a left R-supermodule.*

(a) The S-supermodule $\rho^* M$ is given by $\rho_* S \otimes_R M$ as an Abelian group, equipped with the action of S defined by $s \cdot (s' \otimes m) := (ss') \otimes M$. It is the supermodule derived from M by extending the ring of scalars. We will also use the notation $S \otimes_\rho M$ in subsequent chapters.

(b) For each $f \in \underline{\mathrm{Hom}}_R(M, N)$, let $f_\rho := id_S \otimes f \in \underline{\mathrm{Hom}}_S(\rho^* M, \rho^* N)$ be the induced morphism.

By the following result, restriction and extension of scalars are adjoint operations:

Proposition 2.24 ([6],II.5.1.1) *Let $\rho : R \longrightarrow S$ be a homomorphism of superrings as before. Let M be a R- and N be an S-supermodule. Then there is a canonical bijection*

$$\underline{\mathrm{Hom}}_R(M, \rho_* N) \cong \underline{\mathrm{Hom}}_S(\rho^* M, N) \qquad\qquad f \mapsto \overline{f}$$

where $\overline{f}$ is uniquely characterized by the condition $\overline{f}(1 \otimes m) = f(m)$ for all $m \in M$.

In the following, we again collect some elementary properties of this construction. Let N be another R-supermodule:

Proposition 2.25 ([6], II.5)

(a) Let $\sigma : S \longrightarrow T$ be another homomorphism of superrings. Then there is an isomorphism $(\sigma \circ \rho)^* M \cong \sigma^* \rho^* M$ of T-supermodules mapping $1 \otimes m$ to $1 \otimes (1 \otimes m)$ for all $m \in M$.

(b) There is a unique isomorphism $\rho^* M \otimes_R \rho^* N \cong \rho^*(M \otimes_R N)$ of S-supermodules mapping $(1 \otimes m) \otimes (1 \otimes n)$ to $1 \otimes (m \otimes n)$.

(c) Let P be a right S-module, then there is an isomorphism of R-modules $\rho_* P \otimes_R M \longrightarrow P \otimes_S \rho^* M$ mapping $p \otimes m$ to $p \otimes (1 \otimes m)$.

(d) If M is free with basis $\{m_i\}_{i \in I}$, then $\rho^* M$ is free with basis $\{1 \otimes m_i\}_{i \in I}$. If moreover ρ is injective, then $m \mapsto m \otimes 1$ is injective too.

(e) If M is free and of finite rank, there is an isomorphism $S \otimes_R \underline{\mathrm{Hom}}_{\mathbb{R}}(M, N) \longrightarrow \underline{\mathrm{Hom}}_S(\rho^* E, \rho^* F)$ given by $b \otimes f \mapsto ((b' \otimes m) \mapsto (-1)^{|b'||f|} bb' \otimes f(m))$

(f) If M is a free module of finite rank, then there is an isomorphism $v : \rho^*(E^*) \cong (\rho^* E)^*$ of S- supermodules given by $\langle v(s' \otimes \varphi), s \otimes m \rangle = (-1)^{|s||\varphi|} s' s \rho(\varphi(m))$

3 Supermanifolds

In this chapter, we will give an introduction into the general theory of supermanifolds. There are two different approaches to this subject. The first is inspired by the way, spaces (e.g. schemes, varieties) are defined in algebraic geometry and goes back to Berezin (see [4]), Kostant (see [35]), Leites (see [39]) and others and describes a space by specifying the algebra of functions defined on this space. The other approach by Rogers (see [51] for an overview) and DeWitt (see [9]) is closer to the concepts of differential geometry and uses charts to a model space. These approaches are not equivalent and their relation was clarified by Molotkov and Sachse (see [44] section 4.7 and [53], chapter 5) using a categorical approach to supermanifolds which will be discussed in chapter 4 . In this work, we will use this categorical point of view and the Berezin-Kostant-Leites (BKL) approach. They are equivalent in finite dimensions. This chapter serves as an introduction to the BKL construction and the corresponding notion of morphisms (or "smooth maps") between these supermanifolds. However, concepts like the "component formalism", i.e. the expansion of a morphisms w.r.t. odd coordinates, will only become clear using the categorical point of view discussed in the next chapters. As in the previous chapter, most of the general theory is well known. Only at the end of the chapter, we will provide some additional results in supergeometry related to the construction of connections and their pullbacks.

3.1 Elementary structure of supermanifolds

Following the general principles for the definition and construction of spaces in algebraic geometry (which includes smooth and complex manifolds and virtually all other types of spaces considered in "geometry"), the general concept of a superspace is defined as follows (see [66], 4.1)

Definition 3.1 *A superspace is a super ringed space $(X, \mathcal{O})$ (i.e. a topological space X equipped with a sheaf of super algebras $\mathcal{O}$) such that the stalks are local supercommutative rings.*

Here, a local supercommutative ring is a supercommutative superring containing a unique maximal homogeneous ideal (see definition 2.3 (b)). There is a natural concept of morphisms between superspaces, the concept of local morphisms among ringed spaces :

Definition 3.2 *A morphism of superspaces $(\varphi, \Phi^*) : (X, \mathcal{O}) \longrightarrow (Y, \mathcal{R})$ is a local morphism of the ringed spaces which preserves parity. More precisely, it is given by a continuous map $\varphi : X \longrightarrow Y$ and a morphism of sheaves of superalgebras $\Phi : \mathcal{R} \longrightarrow \varphi_*\mathcal{O}$ (where $\varphi_*\mathcal{O}$ is the direct image of $\mathcal{O}$ under φ, see definition 3.13) such that*

(a) The homomorphisms $\Phi^*_V : \mathcal{R}(V) \longrightarrow \mathcal{O}(\varphi^{-1}(V))$ of supercommutative algebras preserve the $\mathbb{Z}_2$-grading.

(b) The induced homomorphisms $\Phi^*_y : \mathcal{R}_y \longrightarrow \varphi_*\mathcal{O}_y$ on stalks are local, i.e. $\Phi_y(\mathfrak{m}'_y) \subset \mathfrak{m}_y$ where $\mathfrak{m}'_y \subset \mathcal{R}_y$ and $\mathfrak{m}_y \subset \varphi_*\mathcal{O}_y$ are the unique maximal ideals of the stalks.

There is a second, equivalent description of morphisms of ringed spaces, which will be discussed in proposition 3.14. Supermanifolds can now defined in the same way, manifolds are defined in ordinary “non super” geometry: They are superspaces which are locally isomorphic to a model space. The latter is defined as follows:

Definition 3.3 *The superspace $\mathbb{R}^{p|q}$ is defined as the super ringed space $(R^p, \mathcal{O}^{p|q})$ given by the underlying space $\mathbb{R}^p$ and the sheaf $\mathcal{O}^{p|q}$ where $\mathcal{O}^{p|q}(U) := C^\infty(U, \mathbb{R}) \otimes_{\mathbb{R}} \bigwedge^\bullet \mathbb{R}^q$ for open sets $U \subset \mathbb{R}^p$. Open subspaces of $\mathbb{R}^{p|q}$ are superspaces of the form $(W, \mathcal{O}^{p|q}|_W)$ for $W \subset \mathbb{R}^p$.*

As already mentioned in example 2.20, the stalk at $x \in \mathbb{R}^p$ is given by $\mathcal{O}_x^{p|q} = C^\infty_{\mathbb{R}^p,x} \otimes \bigwedge^\bullet \mathbb{R}^q$ and the ideal $\mathfrak{m}_x \otimes_{\mathbb{R}} \bigwedge^\bullet \mathbb{R}^q \oplus \bigwedge^{\geq 1} \mathbb{R}^q$ is its unique maximal graded ideal where $\mathfrak{m}_x = \{[f] \in C^\infty_{\mathbb{R}^p,x} | f(x) = 0\}$.

Definition 3.4 *A supermanifold is a superspace $X = (\tilde{X}, \mathcal{O})$ such that the following properties are satisfied:*

(a) The topology of $\tilde{X}$ is Hausdorff and second countable

(b) The superspace $(\tilde{X}, \mathcal{O})$ is locally isomorphic to $\mathbb{R}^{p|q}$ ($p, q \in \mathbb{N}_0$ fixed)

The pair $p|q$ is called the dimension of $(X, \mathcal{O})$ with even part p and odd part q. Together with the morphisms defined in 3.2, supermanifolds form a category denoted by BKL[1]. *If $U \subset \tilde{X}$ is an open set, then $X|_U := (U, \mathcal{O}|_U)$ is called an open subsupermanifold of X.*

For $q = 0$, the resulting supermanifold ist just an ordinary smooth manifold of dimension p (cf. proposition 3.6). If $(\tilde{X}, \mathcal{O})$ is a supermanifold, (local) sections of $\mathcal{O}$ are called (local) superfunctions. It is possible to assign to superfunctions $s \in \mathcal{O}(U)$ a ”value” for each $x \in U$, more precisely:

Lemma 3.5 ([66], p.133) *Let $s \in \mathcal{O}(U)$ be a superfunction and $x \in U$. Then there is a unique real number $ev_x(s)$ such that $s - ev_x(s)$ is not invertible in any neighborhood of x, called the value of s in x. The real valued function $x \ni U \mapsto ev_x(s)$ will be denoted by $ev(s)$.*

The assignment $s \mapsto ev(s)$ is clearly a homomorphism of $\mathbb{R}$-algebras. Since there are no nontrivial nilpotent real-valued functions, a superfunction s can not be determined by $ev_x(s)$, in fact, the values of a nilpotent superfunction are always 0. There is no other invariant concept of “value at x” that resolves this problem. Nevertheless, the concept is still useful: Defining $\widetilde{\mathcal{O}}$ to be the presheaf given by real functions of the form $ev(s)$, it may be shown, that in fact, $\widetilde{\mathcal{O}}$ is a sheaf over X and locally isomorph to $C^\infty_{\mathbb{R}^p}$. Thus, we have:

Proposition 3.6 ([66], p.133 and [8] 4.5) *Let X be a supermanifold. Then, $\tilde{X}$ carries a canonical structure of an ordinary smooth manifold and there is an inclusion of supermanifolds $\iota : (\tilde{X}, C^\infty_{\tilde{X}}) \longrightarrow X$, where ι is given by $id_{\tilde{X}}$ and the surjective sheaf morphism*

$$\iota_U^* : \mathcal{O}(U) \longrightarrow C^\infty_{\tilde{X}}(U), \; s \mapsto ev(s)$$

ι^* is uniquely determined by X and $\iota^*_{\varphi^{-1}(V)} \Phi^*_V = \varphi^*_V \iota^*_V$ for all morphisms $(\varphi, \Phi^*) : X \longrightarrow Y$.

[1] We avoid using “SMan” at this point, since this will be used for a different category later.

In particular, φ is already determined by Φ^* and we will simply write $\Phi : X \longrightarrow Y$ for morphisms in the rest of this work. We will not proof this theorem (see [8], Satz 4.5) but sketch one instructive way to construct ι following [66]. Let

$$\mathcal{J}(U) := \{s \in \mathcal{O}(U) \mid s \text{ is nilpotent}\}$$

It may be shown that $\mathcal{J}$ is a sheaf of ideals of $\mathcal{O}$ over $\tilde{X}$, the subsheaf of nilpotent elements of $\mathcal{O}$ and that $\mathcal{J}(U) = \{s \in \mathcal{O}(U) | ev(s) = 0\}$. The sheaf $\mathcal{J}$ fits into the following exact sequence:

$$0 \longrightarrow \mathcal{J} \xrightarrow{\subset} \mathcal{O} \xrightarrow{\iota^*} C^\infty_{\tilde{X}} \longrightarrow 0 \tag{3.1}$$

Thus, ι^* naturally occurs as the quotient map $\mathcal{O} \twoheadrightarrow \mathcal{O}/\mathcal{J}$. This quotient is a sheaf of $\mathbb{R}$-algebras on $\tilde{X}$ locally isomorphic to $C^\infty(\mathbb{R}^p)$ and we just write $C^\infty_{\tilde{X}}$ for it. This exact sequence always splits but not canonically (see [8] p.126 and [35] 2.4). In fact, any splitting leads, by definition, to a decomposition

$$\mathcal{O} = C \oplus \mathcal{J} \cong C^\infty_{\tilde{X}} \oplus \mathcal{J}$$

Thus, it determines the way, smooth real-valued functions are embedded in the larger sheaf of superfunctions:

Definition 3.7 ([35] (2.2.3)) *A subalgebra $C(U) \subset \mathcal{O}(U)$ for $U \subset \tilde{X}$ is called a (local) function factor, if*

$$\iota^*_U|_{C(U)} : C(U) \longrightarrow C^\infty(U)$$

is an isomorphism of $\mathbb{R}$-algebras. The same terminology is used for the corresponding sheaves of algebras.

It can be shown that function factors exist on all open sets (see [8] 4.21) but they are not unique. In fact, a choice for a function factor corresponds to a choice of the even part of a coordinate system. We will not define coordinates at this point but refer to [8] 4.22 and 4.26. The relation between coordinates and function factors is discussed in 4.23 of this reference.

While the quotient $\mathcal{O}/J$ only determines the structure of the smooth manifold $\tilde{X}$, the sheaf J can be used to characterize a superfunction completely as well as to specify the global structure of $(\tilde{X}, \mathcal{O})$ up to (non-canonical) isomorphism. For $x \in U \subset \tilde{X}$ we define the ideal of superfunctions vanishing in p by

$$\mathcal{J}_x(U) := \ker(ev_x) = \{s \in \mathcal{O}(U) \mid \iota^*_U(s)(x) = 0\}$$

and we denote by $\mathcal{J}^k_x(U)$ its k-th power. A superfunction f is then characterized by its images $f + \mathcal{J}^{q+1}_x(U)$ in the (q+1)-th infinitesimal neighborhood $\mathcal{O}(U)/\mathcal{J}^{q+1}_x(U)$ as follows:

Lemma 3.8 ([8] 4.13 - 4.16) *Let $(\tilde{X}, \mathcal{O})$ be a supermanifold of dimension $p|q$. Then, $\mathcal{O}$ is q-separated, i.e. for each $U \subset \tilde{X}$, we have $\bigcap_{x \in U} \mathcal{J}^{q+1}_x(U) = \{0\}$.*

To obtain a global description of X based on $\mathcal{J}$, let $\mathcal{E} := \mathcal{J}/\mathcal{J}^2$ be the quotient sheaf, which is a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G} := \mathrm{Sym}^\bullet(\mathcal{E})$ be the sheaf of symmetric algebras, as defined analogously to 2.10.

Theorem 3.9 (Batchelor, see [3] 2.2 and [42], 4 §2.2) *Let X be a supermanifold of dimension $p|q$. Then, the sheaves $\mathcal{G}$ and $\mathcal{O}$ are isomorphic, i.e. we have an isomorphism of supermanifolds $X \cong (\tilde{X}, \mathcal{G})$. The isomorphism is not canonical.*
In fact, there exist a vector bundle $E \longrightarrow \tilde{X}$ such that $\mathcal{E} \cong \Gamma(-, E)$ and consequently $\mathcal{O} \cong \Gamma(-, \bigwedge^\bullet E)$. The bundle E is called a "Batchelor bundle".

The preceding theorem reduces the classification of supermanifolds to the classification of vector bundles over $\tilde{X}$, see also [42], 4 §2.7. This description of supermanifolds in terms of vector bundles is very useful and will be referred to as "Batchelor picture". One application is the construction of products. Let $E_1 \longrightarrow \tilde{X}_1$ and $E_2 \longrightarrow \tilde{X}_2$ describe two supermanifolds given by a Batchelor bundle and let $pr_i : \tilde{X}_1 \times \tilde{X}_2 \twoheadrightarrow \tilde{X}_i$ $(i = 1, 2)$ be the canonical projection. It is then easy to see that $(\tilde{X}_1 \times \tilde{X}_2, \Gamma(-, \bigwedge^\bullet(pr_1^* E_1 \oplus pr_2^* E_2)))$ satisfies the universal property of a product in the category BKL and hence, we have

Proposition 3.10 ([8] 5.21) *The category BKL of supermanifolds admits finite products. The product of $X = (\tilde{X}, \mathcal{O})$ and $Y = (\tilde{Y}, \mathcal{R})$ will be denoted by $X \times Y = (\tilde{X} \times \tilde{Y}, \mathcal{O}\hat{\otimes}\mathcal{R})$.*

Since $\tilde{X}$ carries the structure of a smooth manifold, some technical results from ordinary differential geometry are still valid in supergeometry. Recall that the support of a local section s of a presheaf of Abelian groups is defined by

$$\mathrm{supp}(s) := U \setminus \bigcup \{V \mid V \subset U \text{ is open and } \rho_V^U(s) = 0\}$$

We then have the existence of partitions of unity:

Proposition 3.11 ([39], 3.1.7) *Every open covering of a supermanifold $(\tilde{X}, \mathcal{O})$ admits a local finite refinement $\{U_i\}$ and a family $\{f_i\}$ of superfunctions $f_i \in \mathcal{O}(X)_{\overline{0}}$ such that:*

$$\mathrm{supp}(f_i) \in U_i \qquad\qquad \sum_i f_i = 1 \qquad\qquad \tilde{f}_i \geq 0$$

The sum in the last line of the proposition is locally finite and hence well defined.

As a corollary, the following localization principle can be obtained:

Lemma 3.12 ([39], 3.1.8) *Let $(\tilde{X}, \mathcal{O})$ be a supermanifold, $C \subset \tilde{X}$ be a closed and $U \subset \tilde{X}$ an open subset such that $C \subset U \subset \tilde{X}$. Then, for each $f \in \mathcal{O}(U)$ there is an open set V satisfying $C \subset V \subset U$ and a superfunction $g \in \mathcal{O}(\tilde{X})$ such that*

$$\rho_V^U(f) = \rho_V^{\tilde{X}}(g) \qquad \text{and} \qquad \mathrm{supp}(g) \subset \mathrm{supp}(f)$$

3.2 Morphisms between supermanifolds

Since supermanifolds are not described by the points of a topological space, their morphisms are not simply maps between sets but are defined as morphisms of ringed spaces. To discuss their structure in more detail, recall that every sheaf $\mathcal{R}$ on $\tilde{Y}$ can be described or even defined by using its sheaf space $\pi : L\mathcal{R} \twoheadrightarrow \tilde{Y}$, i.e. the bundle of stalks over Y (see [62] chapter 2.3). Sections of $\mathcal{R}$ then correspond to continuous sections of π. The change of the base space of a sheaf can be described in two different ways

Definition 3.13 (and Proposition, see [62] 3.7.3, 3.7.11) *Let $(\tilde{X}, \mathcal{O})$, $(\tilde{Y}, \mathcal{R})$ be ringed spaces and $\varphi : \tilde{X} \longrightarrow \tilde{Y}$ a continuous map:*

(a) The direct image $\varphi_*\mathcal{O}$ of $\mathcal{O}$ is defined by $\varphi_*\mathcal{O}(V) := \mathcal{O}(\varphi^{-1}(V))$ for open sets $V \subset \tilde{Y}$, together with the obvious restriction map. This construction yields a sheaf of rings over $\tilde{Y}$ called the direct image of $\mathcal{O}$ under φ.

(b) Consider the pullback of the sheaf space $L\mathcal{R}$ along φ,

$$\varphi^* L\mathcal{R} = \{(x, r) \in X \times L\mathcal{R} | \varphi(x) = p(r)\},$$

provided with the topology induced from $X \times L\mathcal{R}$ and the projection $\pi' : \varphi^* L\mathcal{R} \longrightarrow X$ on the first component. This defines a sheaf space over $\tilde{X}$ and its sheaf of sections is called the inverse image of $\mathcal{R}$ along φ. It is sometimes also denoted by $\varphi^{-1}L\mathcal{R}$.

Both approaches can be used to describe morphisms of ringed spaces and it is useful to switch between them since both have advantages and disadvantages. This is justified by the following theorem:

Theorem 3.14 ([62], 7.13) *Let $(\tilde{X}, \mathcal{O})$ and $(\tilde{Y}, \mathcal{R})$ be ringed spaces and $\varphi : \tilde{X} \longrightarrow \tilde{Y}$ a continuous map. Then, there is a natural bijection*

$$\mathrm{Hom}_{RS-\tilde{X}}(\varphi^*\mathcal{R}, \mathcal{O}) \xrightarrow{\cong} \mathrm{Hom}_{RS-\tilde{Y}}(\mathcal{R}, \varphi_*\mathcal{O})$$
$$\psi \mapsto (\sigma \mapsto \psi(\varphi^*\sigma))$$

where $RS-\tilde{X}$ and $RS-\tilde{Y}$ denote the categories of ringed spaces over X and Y respectively. In other words, the functor $\varphi_* : RS-\tilde{X} \longrightarrow RS-\tilde{Y}$ is right adjoint to $\varphi^* : RS-\tilde{Y} \longrightarrow RS-\tilde{X}$ and vice versa.

Remark 3.15

(a) It should be noted that, even though it is possible to change the base from $\tilde{X}$ to $\tilde{Y}$, a morphism of ringed spaces is described in both cases by a morphism which generalizes the pullback of functions. It maps "functions" on the target space (i.e. elements of $\mathcal{R}$ or $\varphi^*\mathcal{R}$) to those of the domain space (i.e. elements of $\mathcal{O}$ or $\varphi_*\mathcal{O}$).

(b) We will use both equivalent interpretations without documenting the change if there is no danger of confusion.

In analogy to ordinary differential geometry, morphisms can be locally described using coordinates, we follow [8] section 4.3. Let $(U, \mathcal{O}^{p|q}|_U)$ be a super domain with (standard) coordinate system $(x^1, \ldots, x^p, \xi^1, \ldots, \xi^q)$. Now, let $\Phi : X \longrightarrow (U, \mathcal{O}^{p|q}|_U)$ be a morphism of supermanifolds with underlying smooth map $\varphi : U \longrightarrow \tilde{X}$. Then, we can define the following functions on X:

$$f^i := \Phi^*(x^i) \in \mathcal{O}_X(\varphi^{-1}(U))_{\overline{0}} \tag{3.2}$$
$$h^j := \Phi^*(\xi^j) \in \mathcal{O}_X(\varphi^{-1}(U))_{\overline{1}}$$

These superfunctions on X represent the coordinates of Φ. By proposition (3.6), we have $(\iota^*(f^1), \ldots, \iota^*(f^p)) = \varphi \in C^\infty(U, \mathbb{R}^p)$. In particular, the tuple satisfies the following property:

Definition 3.16 *Let $g^1, \ldots, g^p \in \mathcal{O}(X)_{\overline{0}}$ and $U \subset \mathbb{R}^p$ an open set. Then $(g^1, \ldots, g^p)$ satisfy the* mapping condition *with respect to U, if for all $x \in \tilde{X}$,*

$$(\iota^*_{\tilde{X}}(g^1)(x), \ldots, \iota^*_{\tilde{X}}(g^p)(x)) \in U$$

The next theorem states that it is possible to reverse this construction, i.e. to construct a unique morphism such that its coordinates are given by (3.2). Hence, coordinates are still sufficient to describe morphisms and consequently all other structures in supergeometry in the sense of BKL supermanifolds. This implies in particular that tensors, connections, etc. can still be decomposed into components in the way familiar from ordinary differential geometry.

Theorem 3.17 ([8] 4.18) *Let $U^{p|q}$ be an open sub-supermanifold of $\mathbb{R}^{p|q}$ with standard coordinates $x^i \in \mathcal{O}^{p|q}(U)_{\overline{0}}$, $\theta^\alpha \in \mathcal{O}^{p|q}(U)_{\overline{1}}$ and X an arbitrary supermanifold. Then we have:*

(a) For any morphism $\Phi : X \longrightarrow U^{p|q}$, $f^i := \Phi^*(x^i) \in \mathcal{O}_{\overline{0}}$ and $g^\alpha := \Phi^*(\theta^\alpha) \in \mathcal{O}_{\overline{1}}$.

(b) For given $f^i \in \mathcal{O}_{\overline{0}}$, $g^\alpha \in \mathcal{O}_{\overline{1}}$ satisfying the mapping condition 3.16, there is a unique morphism $\Phi : X \longrightarrow U^{p|q}$ of supermanifolds such that $f^i = \Phi^*(x^i) \in (\mathcal{O}_{\overline{0}})$ and $g^\alpha = \Phi^*(\theta^\alpha) \in \mathcal{O}_{\overline{1}}$.

Another important theorem which is useful for simplifying arguments roughly says, that every morphism can be defined by prescribing the homomorphism between the global algebras of superfunctions. The proof relies on the existence of partitions of unity and thus, it is only valid in the category of smooth supermanifolds but not in that of analytic or holomorphic ones.

Theorem 3.18 ([8] Satz 4.8, compare also [35] section 2.15) *Let $X = (\tilde{X}, \mathcal{O})$ and $Y = (\tilde{Y}, \mathcal{R})$ be supermanifolds. Then, the following map is a bijection:*

$$\mathrm{Hom}_{\mathrm{BKL}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathrm{SAlg}}(\mathcal{R}(\tilde{Y}), \mathcal{O}(\tilde{X}))$$
$$\Phi \mapsto \Phi^*_{\tilde{Y}}$$

Motivated by this result, we will no longer keep track of the open sets $U \subset \tilde{X}$ unless there is danger of confusion and just write $\mathcal{O}$ for the global algebra of superfunctions.

3.3 Super vector bundles

This section discusses vector bundles on supermanifolds. On a smooth manifold, a vector bundle can be equivalently described by its total space or by its sheaf of sections. In fact, there is a one-to-one correspondence between the isomorphism classes of vector bundles over M and the isomorphism classes of locally free C^∞_M-modules, see [67] II.1.13.

Definition 3.19 ([10], §3.2) *A super vector bundle over a super manifold $X = (\tilde{X}, \mathcal{O})$ is a locally free $\mathcal{O}$-module $\mathcal{E}$, i.e. $\mathcal{E}$ is a sheaf of $\mathcal{O}$-supermodules on X, which is locally isomorphic to a sheaf of the form (cf. example 2.8 (a) for the definition of $\Pi\mathcal{O}$)*

$$\underbrace{\mathcal{O} \oplus \cdots \oplus \mathcal{O}}_{s\text{-times}} \oplus \underbrace{\Pi\mathcal{O} \oplus \cdots \oplus \Pi\mathcal{O}}_{t\text{-times}}$$

The pair $s|t$ is called the rank of the super vector bundle.

Remark 3.20

(a) A super vector bundle can also defined as a fibre bundle with fibres isomorphic to $\mathbb{R}^{s|t}$ and structure group $Gl(s|t)$. We are not going to discuss this equivalent approach at this point. Since $\mathcal{E}$ only describes sections of the bundle, it is necessary to construct the sheaf of "all functions on the total space". It is much easier (and much more useful) to do define it as a functor. This will be done in example 4.7

(b) A super vector bundle of rank $s|0$ has odd sections. It is locally isomorphic to $\mathcal{O}^s$ and this module has a non trivial odd part, given by $(\mathcal{O}_{\bar{1}})^s$.

(c) The algebraic constructions of taking sums, tensor products and dualization generalize to super vector bundles in a straightforward way: They are described by the corresponding super operations of the $\mathcal{O}$-modules defined in example 2.8. Recall that the dual sheaf is defined by the internal Hom functor, $\mathcal{E}^* = \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{O})$ (see 2.8 (c)).

(d) The parity reversed bundle $\Pi\mathcal{E}$ is the locally free sheaf of $\mathcal{O}$-supermodules obtained from applying Π (see example 2.8 (a)) to $\mathcal{E}$.

Is is clear that the direct sums, the dual and the parity reversed sheaves of $\mathcal{O}$-supermodules are again sheaves. This is nontrivial for the tensor product. In general, the assignment $U \mapsto E_1(U) \otimes_{\mathcal{O}(U)} E_2(U)$ for two sheaves of $\mathcal{O}$-supermodules is not a sheaf but has to be sheafified. The following result simplifies the situation for supermanifolds:

Lemma 3.21 ([57] 7.13, p.133 or [50] proposition 4.6) *Let X be supermanifold, E, F sheaves of $\mathcal{O}$-supermodules and assume that E is locally free. Then $U \mapsto E(U) \otimes_{\mathcal{O}(U)} F(U)$ defines a sheaf.*

We will need the pullback of bundles and of connections along a given morphism of supermanifolds. On smooth manifolds, the pullback of a bundle $E \longrightarrow N$ along a (smooth) map $\varphi : M \longrightarrow N$ is, roughly speaking, the bundle over M, construct by attaching the fibre $E_{\varphi(m)}$ of the bundle E at the point $m \in M$. This construction can not be generalized directly because a supermanifold is not completely described by the points $x \in \tilde{X}$. Following the philosophy of the ringed-space formalism, we have to construct its sheaf of sections which is - by definition - a locally free $\mathcal{O}$-module. The following construction follows the general ideas described in [62], 4.4.13 and is also sketched in textbooks on algebraic geometry.

Let $(\varphi, \Phi^*) : X = (\tilde{X}, \mathcal{O}) \longrightarrow Y = (\tilde{Y}, \mathcal{R})$ be a morphism of super manifolds. The general pullback of a sheaf along a continuous map $\varphi : \tilde{X} \longrightarrow \tilde{Y}$ was introduced at the beginning of section 3.2. In particular, to any sheaf $\mathcal{E}$ of $\mathcal{R}$-supermodules over $\tilde{Y}$, we get a sheaf $\varphi^*\mathcal{E}$ of $\varphi^*\mathcal{R}$-supermodules over $\tilde{X}$. Note that the latter is not the sheaf of superfunctions $\mathcal{O}$ - it is obtained by putting at each $x \in \tilde{X}$ the stalk $\mathcal{R}_{\varphi(x)}$ of superfunctions from the target of Φ. By theorem 3.14, Φ can be considered as a morphism of sheaves $\Phi^* : \varphi^*\mathcal{R} \longrightarrow \mathcal{O}$ over $\tilde{X}$. Using restriction of scalars from definition 2.21, we can turn $\mathcal{O}$ in a $\varphi^*\mathcal{R}$-module sheaf by

$$\eta \cdot f := \Phi^*_U(\eta) f \qquad \text{for } \eta \in (\varphi^*\mathcal{R})(U), f \in \mathcal{O}_X(U) \tag{3.3}$$

However, taking into account proposition 2.22 and the fact, that Φ^* will be in general neither surjective nor injective, we can not expect that this sheaf has nice properties:

Remark 3.22 The $\varphi^*\mathcal{R}$-module $\mathcal{O}$ is not free of finite rank. This is not a problem that is special to supergeometry, it already occurs if all spaces are smooth manifolds. As an example, consider the (super-)manifolds $(\mathbb{R}, C^\infty_{\mathbb{R}^2})$ and $(\mathbb{R}, C^\infty_{\mathbb{R}})$ and the morphism given by $\varphi(x, y) := x$. Its action on functions (more precisely, germs of functions on $\mathbb{R}$ pulled back to $\mathbb{R}^2$) in $\varphi^* C^\infty_{\mathbb{R}^1}$ is then given by $\Phi(g) = ((x, y) \mapsto g(\varphi(x, y))) \cong \varphi^* g$, i.e. by pullback. Obviously, all functions of the form $\varphi^* g$ are constant in y-direction. Hence, it is clearly impossible that every smooth function on $\mathbb{R}^2$ can be written as a linear combination of some fixed functions $f_1 \dots, f_N \in C^\infty_{\mathbb{R}^2}$ with coefficients of the form $\varphi^* g$. This proves that, even without odd contributions, $\mathcal{O}$ is not a free $\varphi^*\mathcal{R}$-module.
On a super manifold with (local) odd coordinate functions $\{\theta^\alpha\}$, we can in particular not expect that $\{\theta^I\}_{\|I\| \leq q}$ is a $\varphi^*\mathcal{R}$-basis for $\mathcal{O}$, although it is a basis w.r.t the action of $C^\infty_{\tilde{X}}$. Here $I = (i_1, \dots, i_k)$ is a multi-index[2], $\|I\| := \sum_{s=1}^k i_k$ and $\theta^I := (\theta^1)^{i_1} \cdots (\theta^k)^{i_k}$.

By using extension of scalars 2.23, we obtain a presheaf $\mathcal{O} \otimes_\Phi \varphi^*\mathcal{E}$ of $\mathcal{O}$-supermodules. The module structure is given by 2.23:

$$f'(f \otimes e) := (f'f) \otimes e \qquad \text{for } e \in (\varphi^*\mathcal{E})(U),\ f, f' \in \mathcal{O}(U)$$

The general definition (see [62], 4.4.13) then reads:

Definition 3.23 *The pullback or inverse image $\Phi^*\mathcal{E}$ of an $\mathcal{R}$-module $\mathcal{E}$ along a morphism $\Phi : X \longrightarrow Y$ is the presheaf of $\mathcal{O}$-supermodules given by*

$$\Phi^*\mathcal{E}(U) := \mathcal{O}(U) \otimes_\Phi \varphi^*\mathcal{E}(U) \qquad \textit{for } U \subset \tilde{X}$$

[2]Multi-indices are discussed in section 5.2 and 5.3, in particular, we use $\|I\|$ to denote their absolute values and not $|I|$, because this notation will be used for a $\mathbb{Z}_2$-parity, see (5.2).

Since $\Phi^*\mathcal{E}$ was defined as a tensor product of sheaves, the resulting presheaf is a priori not complete and $\Phi^*\mathcal{E}$ should be defined as its sheafification. To show that this is not necessary in our situation, we collect some properties of the pullback presheaves:

Lemma 3.24 *Let $\Phi : X \longrightarrow Y$ a morphism of supermanifolds and $\mathcal{E}$ a locally free $\mathcal{R}$-module of rank $s|t$ over Y, then we have:*

(a) $\varphi^*\mathcal{E}$ is a locally free $\varphi^*\mathcal{R}$-module of rank $s|t$ over $\tilde{X}$.

(b) $\Phi^*\mathcal{E}$ is a locally free presheaf of $\mathcal{O}$-modules of rank $s|t$ over $\tilde{X}$.

Proof To see the first part, let $x \in X$ and choose an open neighborhood V of $\varphi(x)$, s.t. $\mathcal{E}|_V$ admits a homogeneous basis $\{\sigma_i\}_{i=1\ldots,s+t}$. Let $U := \varphi^{-1}(V)$. Then, the sections $\varphi^*\sigma_i := (x \in U \mapsto (x, \sigma_i(\varphi(x))))$ form a basis of $\varphi^*\mathcal{E}(U)$. To see this, let ω be another section in $\varphi^*\mathcal{E}_U$. Locally on small open sets $W \subset V$, we may write $\omega|_W = \varphi^*\sigma^W$ for some local section σ^W in $\mathcal{E}|_V$. We then have a unique decomposition $\sigma^W = \sum_i \lambda_i^W \sigma_i$, where λ_i^W are local sections in $\mathcal{R}$. By construction and uniqueness of the coefficients, the pulled back sections $\varphi^*\lambda_i^W$ coincide for fixed i on intersections. By the sheaf property of $\varphi^*\mathcal{R}$, they glue to sections $\lambda_i \in \varphi^*\mathcal{R}(U)$, which are uniquely determined. This proves that $\{\varphi^*\sigma_i\}$ is a basis for $\varphi^*\mathcal{E}(U)$.
By proposition 2.25 d), the sections $\{1 \otimes_\Phi \varphi^*\sigma_i\}$ then form a local basis for $\Phi^*\mathcal{E} = \mathcal{O} \otimes_\Phi \varphi^*\mathcal{E}$ because $\{\varphi^*\sigma_i\}$ is a local basis for $\varphi^*\mathcal{E}$. This proves the second assertion.

□

Using lemma 3.21 together with part a) of the preceding lemma, we finally obtain:

Corollary 3.25 *$\Phi^*\mathcal{E}$ is a locally free sheaf of $\mathcal{O}$-modules and we have* $\mathrm{rk}(\Phi^*\mathcal{E}) = \mathrm{rk}(\mathcal{E})$.

If $F : \mathcal{E} \longrightarrow \mathcal{E}'$ is a morphism of $\mathcal{R}$-modules $\mathcal{E}$ and $\mathcal{E}'$, we have an induced morphism $\varphi^*F : \varphi^*\mathcal{E} \longrightarrow \varphi^*\mathcal{E}'$ of $\varphi^*\mathcal{R}$-supermodule sheaves (see [62], remark below 7.12, p.58). Extending $\mathcal{O}$-linear as in definition 2.23, we obtain a morphism $\Phi^*\mathcal{E} \longrightarrow \Phi^*\mathcal{E}'$, given by

$$\Phi^*F(f \otimes \varphi^*e) := f \otimes \varphi^*(F(e)) \qquad \text{for } e \in \varphi^*\mathcal{E}, f \in \mathcal{O}_X$$

Together this yields a functor

$$\Phi^* : \{\text{locally free } \mathcal{R}\text{-modules over } \tilde{Y}\} \longrightarrow \{\text{locally free } \mathcal{O}\text{-modules over } \tilde{X}\} \tag{3.4}$$

The following proposition indicates, that this construction indeed yields the correct notion of “pullback bundle”:

Proposition 3.26 *Let $\varphi : M \longrightarrow N$ be a (smooth) map and $E \longrightarrow N$ a (smooth) vector bundle with sheaf of sections $\mathcal{E}$. Then, we have*

$$\Phi^*\mathcal{E} \cong \Gamma(-, \varphi^*E) \tag{3.5}$$

where Φ denotes the morphism, canonically associated to φ by taking pullbacks.

Proof [sketch following the proof of lemma 3.24] It is well known that the sections $\{e_i \circ \varphi\}$ define a local frame for $\varphi^* E$ if $\{e_i\}$ is a local frame for E. These sections can be mapped to the sections $1 \otimes_{\varphi^*} e_i \circ \varphi$ in $\Phi^*\mathcal{E}$ and it can be verified, that this prescription extends to an isomorphism of C^∞_M-module sheaves.

□

Example 3.27 (and Warning) If U is a coordinate neighborhood in X with odd coordinates $\{\theta^i\}$, it is tempting to decompose a section $\xi \in \Phi^*\mathcal{F}(U)$ in the form

$$\xi = \sum_I \theta_I \otimes_\Phi \xi^I \tag{3.6}$$

where ξ_I are suitable sections of $\varphi^*\mathcal{F}$. It should be emphasized that due to the fact that $\mathcal{O}$ is not a free $\varphi^*\mathcal{R}$-module (see remark 3.22) this decomposition in general does not exist. Hence, this is not the right way to define the "component fields" of ξ. We will give a different approach in chapter 5.
In general, vector fields along Φ given by $d\Phi(X)$ do not admit such a decomposition, even not locally. Again, this is nothing special to supergeometry, we may consider the smooth curve given by

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2 \qquad\qquad t \mapsto \begin{cases} e^{-\frac{1}{t^2}}(sin(\frac{1}{t}), sin(\frac{2}{t})) & t \neq 0 \\ 0 & t = 0 \end{cases}$$

Obviously, there is a cluster point at 0, the image points converge to $(0,0)$ and the curve enters $(0,0)$ from different directions. Thus, it is impossible to write $d\gamma(\frac{d}{dt})(0) = \dot{\gamma}(0) = \zeta \otimes_{\varphi^*} f \cong (\zeta \circ \varphi) f$ for some local section $\zeta \in \varphi^*\mathcal{T}_{\mathbb{R}^2}$. Such a section would have to be multivalued.

3.4 Tangent sheaf and tangent spaces

The tangent space of a smooth finite dimensional manifold M at p can be defined by

$$\begin{aligned} T_pM &:= Der(C^\infty_p, \mathbb{R}) \\ &:= \{v : \mathbb{C}^\infty_p \longrightarrow \mathbb{R} | v \text{ is } \mathbb{R}-\text{linear}, v(fg) = v(f)ev_p(g) + ev_p(f)v(g)\} \end{aligned}$$

where C^∞_p denotes the germs of smooth functions at p and $ev_p : C^\infty_p \longrightarrow \mathbb{R}$ the evaluation map. A vector field on an open set $U \subset M$ is correspondingly given by elements of

$$Der(C^\infty(U), C^\infty(U)) = \{V \in End_{\mathbb{R}}(C^\infty(U)) | V(fg) = V(f)g + fV(g)\}$$

We may evaluate a vector field V at $p \in M$ obtaining an element of T_pM by the following formula

$$(ev_p(V))(f) := ev_p(V(f))$$

Here f is a function defined locally near p. Note that a locally defined vector field V is uniquely specified by all the derivations $ev_p(V)$ for $p \in U$.

These constructions can be generalized to a supermanifold $X = (\tilde{X}, \mathcal{O})$. Let $p|q$ be its dimension.

Definition 3.28 *Let $U \subset M$ be an open set. The superderivations of $\mathcal{O}(U)$ are defined by*

$$\begin{aligned}\mathcal{T}_X(U) &:= Der(\mathcal{O}(U), \mathcal{O}(U)) \\ &:= \{V : \mathcal{O}(U) \longrightarrow \mathcal{O}(U) \mid V \textit{is linear}, V(fg) = V(f)g + (-1)^{|V||f|} fV(g)\}\end{aligned}$$

$|V| \in \mathbb{Z}_2$ is called the parity of V and a general element of $\mathcal{T}_X(U)$ can be uniquely decomposed as $V = V_{\bar{0}} + V_{\bar{1}}$.

Fixing local coordinates $(x^1, \dots, x^p, \theta^1, \dots, \theta^q)$ on X, each superfunction can locally be written as $f = \sum_I f_I(x)\theta^I$ and the following super derivations can be introduced:

$$\frac{\partial}{\partial x^\mu} f = \sum_I \frac{\partial f_I}{\partial x^\mu}\theta^I \qquad \frac{\partial}{\partial \theta^\alpha}\left(\sum_{I\alpha \notin I} f_I\theta^I + f_{\alpha,I}\theta^\alpha\theta^I\right) = \sum_{\alpha\notin I} f_{\alpha,I}\theta^I \tag{3.7}$$

Super derivations are local operators (see e.g. [35] 3.1.9, we will prove a more general result in proposition 5.14) . Thus it is possible to introduce a restriction map $\rho^U_V : \mathcal{T}_X(U) \longrightarrow \mathcal{T}_X(V)$ and $\mathcal{T}_X$ becomes a presheaf. In fact, it is a super vector bundle over X:

Theorem 3.29 ([10] 3.3.1) *The assignment $U \mapsto \mathcal{T}_X(U)$ defines a sheaf of $\mathcal{O}$-supermodules over $\tilde{X}$. It is locally free of rank $p|q$, a local basis is given by the superderivations in* (3.7)*.*

PropSmfSmooth This allows to introduce the common tangent structures:

Definition 3.30 *Let X be a supermanifold.*

(a) $\mathcal{T}_X$ is called its tangent sheaf, $\Omega^1_X := \mathcal{T}^*_X$ the cotangent sheaf of X. Their sections are called super vector fields and super 1-forms respectively, the dual pairing between them is defined in accordance with (2.5).

(b) For 2 super vector fields V, W, the super Lie bracket is defined by

$$[V, W](f) := V(W(f)) - (-1)^{|V||W|}W(V(f))$$

(c) Let $\Phi : X \longrightarrow Y$ be a morphism of supermanifolds. Its differential is defined as the morphism of sheaves of Abelian groups given by

$$d\Phi : \mathcal{T}_X \longrightarrow \mathcal{T}_Y \qquad V \mapsto V \circ \Phi^*$$

It can be shown that $d\Phi$ is a section in $\mathcal{T}^*_X \otimes_{\mathcal{O}} \Phi^*\mathcal{T}_Y$ with the pullback introduced in definition 3.23.

Using the evaluation $ev_p : \mathcal{O}(U) \longrightarrow \mathbb{R}$ at $p \in U$, each super vector field V induces a super derivation of the stalk $\mathcal{O}_p$ by

$$X(p) : \mathcal{O}_p \longrightarrow \mathbb{R} \qquad\qquad X(p)([f]_p) := ev_p(X(f))$$

As in the smooth setting, these objects form a vector space (see [35] 2.10, 2.12):

Definition 3.31 (and Lemma) *For $p \in \tilde{X}$, the tangent space at p to X is defined to be*

$$T_pX := \{v \in \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{O}_p, \mathbb{R}) \mid v(fg) = v(f)ev_p(g) + (-1)^{|v||f|}ev_p(f)v(g)\}$$

It is a real super vector space having the same dimension as X. All tangent spaces form a bundle of super vector spaces over $\tilde{X}$ denoted by TX.

Remark 3.32

(a) A super vector field X is not determined by its values $X(p)$. In fact, using coordinate vector fields,

$$ev_p(\textstyle\sum_i f^i \frac{\partial}{\partial x^i} + \sum_\alpha g^\alpha \frac{\partial}{\partial \theta^\alpha}) = \sum_i f^i_\varnothing(p) \frac{\partial}{\partial x^i}|_p + \sum_\alpha g^\alpha_\varnothing(p) \frac{\partial}{\partial \theta^\alpha}|_p$$

where $f_\varnothing$ and $g_\varnothing$ are the real components of the superfunctions f, g. Thus, $ev_p(\theta\frac{\partial}{\partial x}) = 0$ whereas the odd derivation $\theta\frac{\partial}{\partial x}$ is nonzero. The same holds for the even derivation $\theta\frac{\partial}{\partial \theta}$. Nevertheless, the concept of super tangent space will prove useful when discussing metrics.

(b) We already observed that it is not sufficient to look at points in $\tilde{X}$ to describe a supermanifold $(\tilde{X}, \mathcal{O})$. Similarly, the points TX of $\mathcal{T}_X$ do not specify the complete tangent sheaf. Example 4.7 will show how to generalize the notion of point in order to obtain a proper description of the total space of $\mathcal{T}_X$.

If $\mathcal{E} = \mathcal{T}_Y$ is the tangent sheaf of $(\tilde{Y}, \mathcal{R})$, its pullback $\Phi^*\mathcal{T}_Y$ along a morphism $\Phi : X \longrightarrow Y$ also acts on a suitable sheaf of functions. As $(\Phi^*\mathcal{T}_Y)_x \cong (\mathcal{O})_x \otimes_{\mathcal{R}_{\varphi(x)}} (\mathcal{T}_Y)_{\varphi(x)}$, the stalks of this sheaf should consist of germs of functions in $\mathcal{R}$ defined along φ, in other words, $\Phi^*\mathcal{T}_Y$ should act on sections of $\varphi^*\mathcal{R}$. In particular, each function $g \in \mathcal{R}(V)$ gives rise to such a section $x \mapsto g_{\varphi(x)}$, but not each section is globally of this form. Using the elementary identification

$$\mathcal{O} \cong \mathcal{O} \otimes_\Phi \varphi^*\mathcal{R} = \Phi^*\mathcal{R} \qquad \text{where} \qquad f \cong f \otimes 1 \qquad 1 \otimes \eta \cong \Phi^*(\eta)$$

we obtain:

Proposition 3.33 *The following action of $f \otimes \xi \in \Phi^*\mathcal{T}_Y$ on sections λ of $\varphi^*\mathcal{R}$ defines a derivation along the morphism Φ:*

$$(f \otimes_\Phi \xi)(\lambda) := f \otimes_\Phi \xi(\lambda) \in \Phi^*\mathcal{R} \cong \mathcal{O}$$

PROOF It is clear that the construction is $\mathcal{O}$-super linear in f and hence can be extended by super linearity. To check the derivation property along Φ, let $\lambda, \mu \in \varphi^*\mathcal{R}$:

$$\begin{aligned}(f\otimes\xi)(\lambda\mu) &= f\otimes\xi(\lambda\mu)\\ &= f\otimes\xi(\lambda)\mu + (-1)^{|\xi||\lambda|} f\otimes\lambda\xi(mu)\\ &= (f\otimes\xi(\lambda))\Phi^*(\mu) + (-1)^{(|\xi|+|f|)|\lambda|}\Phi^*(\lambda)(f\otimes\xi(\mu))\end{aligned}$$

This proves that $f\otimes\xi$ is a superderivation of parity $|f\otimes\xi| = |\xi| + |f|$.

□

3.5 Metrics, frames and integration

The discussion in the previous section showed that a super vector field $X \in Der(\mathcal{O}_U)$ is not determined by the super tangent vectors $X(p)$ for $p \in U$. Consequently, the notion of a metric has to be defined on the level of sheaves:

Definition 3.34 ([12], definition 23, [26], section 4.1, [51] 12.3.1) *A super pseudo-Riemannian metric is a morphism of sheaves of $\mathcal{O}_X$-super modules*

$$\langle,\rangle : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \longrightarrow \mathcal{O}_M$$

such that the following conditions are satisfied:

(a) $\langle,\rangle$ is supersymmetric, that is, we have for all $X, Y \in Der(\mathcal{O}_M)$:

$$\langle X, Y\rangle = (-1)^{|X||Y|}\langle Y, X\rangle$$

(b) $\langle,\rangle$ is non-degenerated, that is, it induces an isomorphism

$$Der(\mathcal{O}_M) \longrightarrow Der(\mathcal{O}_M)^* = \underline{Hom}(Der(\mathcal{O}_M), \mathcal{O}_M) \qquad X \mapsto \langle X, \cdot\rangle$$

Remark 3.35

(a) The fact that the metric is a morphism of sheaves of supermodules implies that it is even.

(b) Up to this point, there is no notion of positive definiteness. This has two reasons: The product of a purely odd vector field with itself has to vanish by supersymmetry and, even if we restricted to even vector fields, the product would take values in $\mathcal{O}_{\overline{0}}$, so that there is no notion of positivity.

(c) In principle, there is a second choice for the isomorphism in part (b): $X \mapsto \langle\cdot, X\rangle$. We stick to the other convention, because it (clearly) satisfies $p(\langle X,\cdot\rangle) = p(X)$ in the sense that $\langle X, aY\rangle = (-1)^{|X||a|}a\langle X, Y\rangle$. Nevertheless, the other choice leads to an equivalent concept of non-degeneracy.

Given a super pseudo-Riemannian metric, we can evaluate it at p an obtain a well-defined bilinear form on the super tangent space T_pX:

$$\langle , \rangle_p : T_pX \otimes_{\mathbb{R}} T_pX \longrightarrow \mathbb{R} \quad : \quad \langle U(p), V(p) \rangle_p := ev_p(\langle X, Y \rangle)$$

where $U, V \in Der(O_p)$ are super vector fields such that evaluation at p yields $U(p)$ and $V(p)$. It is clear that $\langle U, V \rangle \in \mathcal{O}_{\overline{1}}$ if the vector fields U and V have different parities. Thus, $ev_p \langle U, V \rangle = 0$ and we have:

Lemma 3.36 *Let $\langle , \rangle$ be a super pseudo-Riemannian metric on a supermanifold X:*

(a) $\langle , \rangle_p$ is a well defined, supersymmetric, non-degenerated $\mathbb{R}$-bilinear form on the super vector space T_pX, i.e. its restrictions to $(T_pX)_{\overline{0}}$ and $(T_pX)_{\overline{1}}$ are symmetric and anti-symmetric respectively.

(b) The splitting $T_pX = (T_pX)_0 \oplus (T_pX)_1$ is orthogonal with respect to $\langle , \rangle$.

(c) If $p|q$ is the dimension of X then $q \in 2\mathbb{N}_0$

The last point of the lemma uses the fact, that there are non non-degenerate antisymmetric forms on vector spaces of odd dimension. It is also clear that there is no reasonable notion of positivity for $\langle , \rangle_p$ restricted to $(T_pX)_{\overline{1}}$. Thus, we use the following concept of Riemannian metric

Definition 3.37 *A super pseudo-Riemannian metric $\langle , \rangle$ is called Riemannian if the restriction of $\langle , \rangle_p$ to $(T_pX)_{\overline{0}}$ is positive definite for all $p \in \tilde{X}$.*

After choosing a homogeneous basis $\{b_1^0, \ldots, b_{p+q}^1\}$ of $Der(\mathcal{O}_U)$, $\langle , \rangle$ can be represented by the matrix

$$G_{ij} := \langle b_i, b_j \rangle \in \underline{\mathrm{Mat}}(p|q, \mathcal{O})_{\overline{0}}$$

The non-degeneracy of the bilinear forms on $\mathcal{T}_X$ and T_pX is clearly equivalent to the invertibility of the corresponding matrix G. By lemma 2.19, we obtain

Lemma 3.38 *$\langle , \rangle$ is non-degenerate on $\mathcal{T}_X$ if and only if $\langle , \rangle_p$ is non-degenerate for all $p \in \tilde{X}$.*

For calculations, it is convenient to have orthogonal frames. The following result is mentioned in [51] 12.3 and proven for similar $\mathbb{C}$-valued products in [9] (2.8.16) and [63] IV.7.8. :

Proposition 3.39 *Let $\langle , \rangle$ be a super Riemannian metric. Then, there are local frames $\{e_1, \ldots, e_p\}$ of $\mathcal{T}_{\overline{0}}$ and $\{e_{p+1}, \ldots, e_{p+q}\}$ of $\mathcal{T}_{\overline{1}}$ s.t. the matrix $\langle e_i, e_j \rangle$ takes the form*

$$N := \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ \hline & & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & \\ & 0 & & & \ddots & \\ & & & & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \end{array} \right) \tag{3.8}$$

We will only sketch a proof. Since $G = \begin{pmatrix} G_{\overline{00}} & G_{\overline{01}} \\ G_{\overline{10}} & G_{\overline{11}} \end{pmatrix} \in Mat(p|q, \mathcal{O})_0$ is invertible, this is also true for the skew symmetric matrix $ev(G_{\overline{11}}) \in Mat(q|C^\infty_{\tilde{X}})$. Thus we can adapt the standard proof for the existence of symplectic bases on symplectic vector spaces and obtain a symplectic basis $\{e_{p+1}, \ldots, e_{p+q}\}$ for $\mathcal{T}_{\overline{1}}$. For $i = 1 \ldots p$, we form $b'_i := b_i - \sum_{k=p+1}^{q} \lambda_i^k$ where $\lambda_i^k = \langle b_i, e_{k-1} \rangle$ for k even and $\lambda_i^k = -\langle b_i, e_{k+1} \rangle$ for k odd. Since the representing matrix $G'_{\overline{00}}$ of $\langle , \rangle$ w.r.t $\{b'_i\}$ is invertible and $ev(G'_{\overline{00}})$ is positive definite at each point $x \in \tilde{X}$, we can finally adapt the Gram-Schmidt procedure[3] to $\{b'_i\}$ and obtain the desired basis.
Since $\langle e_i, e_i \rangle = 0$ for $|e_i| = 1$, we introduce the following (even) map defined locally w.r.t an orthonormal frame:

$$Je_k := \begin{cases} e_k & \text{if } k \leq p \\ -e_{k+1} & \text{if } k = p + 2l - 1 \text{ for } l \in \{1, \ldots, q'\} \\ e_{k-1} & \text{if } k = p + 2l \text{ for } l \in \{1, \ldots, q'\} \end{cases} \tag{3.9}$$

It clearly satisfies the following identities:

$$\langle e_k, Je_l \rangle = \delta_{kl} \qquad \langle JX, JY \rangle = \langle X, Y \rangle \qquad J^2 = \mathrm{pr}_{(\mathcal{T}_X)_{\overline{0}}} - \mathrm{pr}_{(\mathcal{T}_X)_{\overline{1}}}$$

The decomposition of a vector field into its component reads

$$X = \sum_i g(X, Je_i) e_i = \sum_i (-1)^{|e_i|} g(X, e_i) Je_i = \sum_i Je_i g(e_i, X) = \sum_i (-1)^{|e_i|} e_i g(Je_i, X)$$

Finally, denoting by $\{e^i\}$ the frame dual to $\{e_i\}$ given by $e^i(e_j) = \delta_{ij}$, then we have

$$e^i = (-1)^{|e_i|} \langle Je_i, \cdot \rangle$$

Choosing a Batchelor bundle E, we have[4] $TX_{\overline{1}} \cong E^*$, so that the Riemannian metric, restricted to the odd tangent spaces, defines a symplectic form on E^*. The existence of a symplectic form is related to the structure of the bundle E in the following way (see [43], proposition 2.61 and theorem 2.60):

Proposition 3.40 *Let $E \longrightarrow M$ be a $2n$-dimensional vector bundle, then we have*

(a) For each symplectic form ω on E, there is a compatible complex structure J on E, i.e. a complex structure s.t. $\omega(\cdot, J\cdot)$ is symmetric and positive definite. In particular, E is a complex vector bundle.

(b) For each complex structure J on E, there is a symplectic form ω compatible with J.

(c) Two symplectic bundles (E_i, ω_i) $(i = 1, 2)$ are isomorphic if and only if the underlying complex vector bundles are isomorphic.

[3] As usual, the square root of an even superfunction f such that $ev(f) > 0$ is defined by means of a Taylor expansion in the nilpotent part.

[4] An element $V \in \Gamma(E^*)$ acts on $\Gamma(\Lambda^\bullet E)$ as odd derivation. Since superfunctions are identified with elements of $\Gamma(\bigwedge^\bullet E)$, sections of $TX_{\overline{1}}$ are identified with elements of the form $ev \circ V$ where $ev : \Gamma(\bigwedge^\bullet E) \to C^\infty(\tilde{X})$ assigns to each superfunction its value.

We thus obtain the following corollary which will prove useful later:

Corollary 3.41 *The supermanifold X admits a super Riemannian metric in the sense of definition 3.37 if and only if one (and hence every) Batchelor bundle is a complex bundle.*

PROOF We only have to show that a symplectic form ω on E^* defines a super Riemannian metric. Sections of E^* can be interpreted as odd vector fields on X by extending their action from $\Gamma(E)$ to $\Gamma(\bigwedge^\bullet E)$ as an odd derivation. Thus, extending by $\Gamma(\bigwedge^\bullet E)$-linearity (i.e. super linearity), we get a well defined metric on $(\mathcal{T}_X)_{\bar{1}}$. Similarly, a Riemannian metric g on $\tilde{X}$ can be extended by super linearity, the orthogonal sum $g+\omega$ then gives a super Riemannian metric.

□

Remark 3.42 If we start with a super Riemannian metric, restrict it to TX and then extend it again by super linearity, we do not recover the original metric. In particular, the metric obtained on $(\mathcal{T}_X)_{\bar{1}}$ by super linear extension is always flat in the sense that there are local coordinate frames which are orthonormal. Every local symplectic basis of E^*, extended to $\Gamma(\bigwedge^\bullet E)$ has this property.
This also shows that on $(\mathcal{T}_X)_{\bar{1}}$, the situation is quite different from the geometry on smooth manifolds: Metrics do not always exist, but if they exist, there is even a flat one.

Integration theory is based upon the notion of the Berezinian of Ω^1_X rather than on its exterior power. The latter is a well defined object, but, as mentioned in chapter 2, there are no top degree forms if the supermanifold X has odd dimension $q > 0$. Moreover, as discussed in [10] §3.3, k-forms can only be integrated on $k|0$-dimensional submanifolds of X which in particular excludes integrating objects over all of X in case $q > 0$. Sections of $Ber(\Omega^1_X)$ can be integrated over X. We will discuss integration under the assumption that $\tilde{X}$ is compact and orientable (see [10] 3.10 for the general concept):

Theorem 3.43 *There is a unique linear functional*

$$\int_X : \Gamma(Ber(\Omega^1_X)) \longrightarrow \mathbb{R}$$

such that for each section of $[dx^1 \cdots dx^p d\theta^1 \cdots d\theta^q]g \in Ber(\Omega^1_X)$ with support contained in a coordinate neighborhood U, the integral takes the value

$$\int_X [dx^1 \cdots dx^p d\theta^1 \cdots d\theta^q]g = \int_{\tilde{M}} dx^1 \cdots dx^n g_{(q\cdots 1)} \tag{3.10}$$

where $g \in \mathcal{O}(U)$ is expanded $g_{(q\cdots 1)}\theta^q \cdots \theta^1 + \sum_{\|I\|<q} g_I \theta^I$ and $\int_{\tilde{X}}$ is the ordinary integral of the p-form $dx^1 \cdots dx^n g_{(q\cdots 1)}$.

To be able to do calculus of variations, we need some properties of the integral. By the definition of products in 3.10, there is a canonical projection $\mathrm{pr}_X : \mathbb{R} \times X \twoheadrightarrow X$ and functions f_t depending smoothly on some parameter can be identified with sections of $\mathrm{pr}_X^*\mathcal{O} = C^\infty_{\mathbb{R}} \hat{\otimes} \mathcal{O}$.

Lemma 3.44 *Let X be a compact orientable supermanifold and $\omega \in Ber(\Omega^1_X)$.*

(a) Let f be a function depending on t. Then, we have

$$\tfrac{d}{dt}\big|_0 \int_X \omega f = \int_X \omega \, \tfrac{\partial f}{\partial t}\Big|_0$$

(b) Let $f \in \mathcal{O}$ and assume that ω locally generates $Ber(\Omega^1_X)$. If $\int_X \omega f g = 0$ for all $g \in \mathcal{O}$, then $f = 0$.

Proof For the first statement, we can choose a finite covering of X with coordinate neighborhoods and a subordinated partition of unity (see proposition 3.11). Then, the problem is reduced to expressions of the form (3.10). All integrals on $\tilde{X}$ have compactly supported integrands and we can exchange integration and differentiation. This proves the first part. For the second assertion, assume that $\mathrm{supp}(g)$ is contained in a coordinate neighborhood. Writing $\omega|_U = [dx^1 \cdots d\theta^q]h$, $h \in \mathcal{O}(U)$ is invertible by assumption. Let $f' := hf$. Expanding the functions with respect to the coordinates, we obtain

$$0 = \int_X \omega f g = \int_X [dx^1 \cdots d\theta^q] f' g = \sum_{I \subset \underline{q}} \int_U dx^1 \cdots dx^p f'_I g_{I^c}$$

Since all components g_{I^c} can be chosen independently, we obtain $f'_I = 0$ from the ordinary theory of integration. Thus, $f' = 0$ on U and since h is invertible, we have $f = 0$ on all coordinate neighborhoods.

□

We will eventually need a version of Stokes' theorem. It is formulated using integral forms, which are used instead of differential forms of lower degree. The complex of integral forms is defined as a free $\mathcal{O}$-supermodule $I^\bullet$, which is also $\mathbb{Z}$-graded but bounded from above with respect to this grading. The highest nontrivial component is at degree p and given by $I^p := Ber(\Omega^1_X)$. Lower degree components are obtained by contractions $i_X : I^\bullet \longrightarrow I^{\bullet-1}$ for $X \in \mathcal{T}_X$. In analogy to differential forms, there are compatible operations $\alpha\wedge : I^\bullet \longrightarrow I^{\bullet+1}$ and $d : I^\bullet \longrightarrow I^{\bullet+1}$, see [10], 3.12 for the details. Here, we will only need integral forms of degree $p-1$, which satisfy $I^{p-1} \cong \mathcal{T}_X \otimes_{\mathcal{O}} Ber(\Omega^1_X)$ ([10], (3.12.2)) and the differential is given by ([10], 3.11)

$$d : \mathcal{T}_X \otimes_{\mathcal{O}} Ber(\Omega^1_X) \longrightarrow Ber(\Omega^1_X) \qquad\qquad d(V \otimes \omega) = \mathcal{L}_V \omega$$

where $\mathcal{L}$ is the Lie derivative.

Theorem 3.45 ([10], 3.12.3) *Let X be a compact supermanifold of dimension $p|q$ and α an integral form of degree $p-1$. Then $\int_X d\alpha = 0$.*

We now discuss integration with respect to a Riemannian volume element. Let (X, G) be a Riemannian supermanifold, such that $\tilde{X}$ is orientable and compact. Moreover, since there exists a super Riemannian metric, the bundle $TX_{\bar{1}} \longrightarrow \tilde{X}$ is complex and carries a canonical orientation. We propose the following definition of a volume "form" (see [9] p. 112 for a similar construction in the context of the definition of metric used in this reference)

Definition 3.46 (and Lemma) *Let $\{e_1, \dots, e_p\, e_{p+1}, \dots, e_{p+q}\}$ be an local orthonormal frame for $\mathcal{T}_X$, which is assumed to be oriented in the sense that $ev(e_1), \dots, ev(e_p)$ is an oriented basis of $TX_{\overline{0}}$ and $\{ev(e_{p+1}), \dots, ev(e_{p+q})\}$ is an oriented basis of $TX_{\overline{1}}$. Then, the volume element of G is locally defined by the section $[e_1 \dots e_{p+q}]$ of $Ber(\Omega^1_X)$. These local sections yield the globally defined volume element $vol_G \in Ber(\Omega^1_X)$. The integral of a function $f \in \mathcal{O}$ is finally defined as the integral of $vol_G f$.*

To see that vol_G is well-defined, we note, that two different orthonormal bases are related by an orthosymplectic matrix S, i.e. $S^{sT} G_0 S = G_0$ where G_0 is the matrix from (3.8). Taking the Berezinian, we find $Ber(S)^2 = 1$. Since orientation is preserved by construction, we have $Ber(S) = 1$ and vol_G is well-defined.

To be able to use Stokes' theorem, it is necessary to compute $\mathcal{L}_V vol_G$. First note that for any smooth family $B(t)$ in $GL(p|q, R)_{\overline{0}}$ such that $B(0) = 1$, we obtain

$$\tfrac{d}{dt}|_0 Ber(B(t)) = \mathrm{tr}(\dot{B}_{\overline{00}}(0)) - \mathrm{tr}(\dot{B}_{\overline{11}}(0)) = \mathrm{str}(\dot{B}(0)) \tag{3.11}$$

from definition 2.17 using the usual rule for differentiation of the determinant. Let $|V| = \overline{0}$, $exp(tV)$ be the flow of V and $exp(tV)^*$ the induced actions on the different super vector bundles, i.e. on the corresponding free locally free modules. Strictly speaking, $exp(tV)^*$ is not an automorphism of these modules but rather satisfies $exp(tV)^*(fm) = exp(tV)^*(f)exp(tV)^*(m)$ where $f \in \mathcal{O}$ and m is a section of the module. Nevertheless, the action of $exp(tV)^*$ on $Ber(\Omega^1_X)$ is still given by multiplication by $Ber(exp(tV)^*)$ as in definition 2.18. Using (3.11) yields

$$\mathcal{L}_V vol_G = \tfrac{d}{dt}|_0 \big(exp(tV)^* vol_G\big) = \tfrac{d}{dt}|_0 Ber(exp(tV)^*) vol_G = \tfrac{d}{dt}|_0 \mathrm{str}(exp(tV)^*) vol_G$$

Expressing the super trace in the local frame, its derivative is given by $\sum_i (-1)^{|e_i|} G(Je_i, -[V, e_i])$. Since the Levi-Civita connection induced by G is free of torsion, it can be used to replace the Lie bracket and we find:

$$\mathcal{L}_V vol_G = \textstyle\sum_i G(\nabla_{e_i} V, Je_i) vol_G = \mathrm{str}(\nabla V) vol_G =: sdiv(V) vol_G$$

Thus, we obtain the following corollary to theorem 3.45

Corollary 3.47 *Let X be a compact Riemannian supermanifold of dimension $p|2q$ and V an even vector field on X. Then $\int_X sdiv(V) vol_G = 0$.*

3.6 Connections

The definition of a connection generalizes in a straightforward way to the superworld:

Definition 3.48 ([12], section 4.5, [42] 4.4.5) *Let $X = (\tilde{X}, \mathcal{O})$ be a supermanifold and $\mathcal{E}$ a locally free $\mathcal{O}$-module. A connection ∇ on $\mathcal{E}$ is a morphism of sheaves of $\mathbb{R}$-supermodules*

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E}^* \otimes_{\mathcal{O}} \mathcal{E}$$

satisfying the super Leibniz-rule for $e \in \mathcal{E}$ and $f \in \mathcal{O}$:

$$\nabla(f \cdot\ e) = df \otimes e + f \nabla e \tag{3.12}$$

By definition ∇ preserves the grading. It should be noted that ∇ itself is not $\mathcal{O}$-linear but only $\mathbb{R}$-linear and satisfies (3.12). The following properties are defined in analogy to the situation on a smooth vector bundle, see also [46], chapter 4 and [26], chapter 4:

Definition 3.49 ([12], section 4.5, [26] 4.2)
(a) If $(\cdot,\cdot) : \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{O}_X$ is a super scalar product on $\mathcal{E}$, then ∇ is called metric if

$$d\left(e, e'\right) = \left(\nabla e, e'\right) + \left(e, \nabla e'\right)$$

(b) If $\mathcal{E} = \mathcal{T}$ is the tangent sheaf, then the supertorsion T^{∇} of ∇ is defined by

$$T^{\nabla} : \mathcal{E} \otimes \mathcal{E} \longrightarrow \mathcal{E} : T^{\nabla}(V, W) := \nabla_V W - (-1)^{|V||W|}\nabla_W V - [V, W]$$

As in smooth Riemannian geometry, it is possible to prove the existence of a unique Levi-Cevita connection:

Proposition 3.50 ([46] 4.2 or [12] (22)) *Given a (pseudo) Riemannian supermanifold, there exists a unique metric and torsion free connection on $\mathcal{T}$. It is given by the super-analogue of the Koszul-formula for $X, Y, Z \in \mathcal{T}$:*

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y, Z) + (-1)^{|X||Y|}Yg(X, Z) - (-1)^{|Z|(|X|+|Y|)}Zg(X, Y)\\ &\quad + g([X, Y], Z) - (-1)^{|Y||Z|}g([X, Z], Y) - (-1)^{|X|(|Y|+|Z|)}g([Y, Z], X)\end{aligned}$$

It is interesting that the super Levi-Civita connection can be easily characterized as a derivation, i.e. by specifying its action on superfunctions. To the author's knowledge, this characterization is not included in the literature so far although it can be more natural in situations, when the action as a derivation is explicitly needed.

Definition 3.51 *Let $(X, \mathcal{O}_X, G)$ be a (pseudo) super Riemannian manifold and $f \in \mathcal{O}_X$. Its gradient $grad(f) \in \mathcal{T}$ is defined by $g(X, grad(f)) = X(f) = (-1)^{|X||f|}\langle df, X\rangle \qquad \forall X \in \mathcal{T}$*

Proposition 3.52 *Let $X, Y \in \mathcal{T}$ and $f \in \mathcal{O}_X$. Then the (unique) Levi-Civita connection is given by*

$$\begin{aligned}(\nabla_X Y)(f) &= X(Y(f)) - \tfrac{1}{2}\langle d(g(X, Y)), grad(f)\rangle\\ &\quad - \tfrac{1}{2}g(X, [Y, grad(f)]) - \tfrac{1}{2}(-1)^{p(X)p(Y)}g(Y, [X, grad(f)])\end{aligned}$$

The order of the symbols has been chosen in order to avoid signs arising from the parity of f.

We will omit the proof because the result is not used in the subsequent part of the thesis. The component formalism in chapter 5 relies on the notion of a pullback connection. We will generalize the approach from smooth differential geometry and give a step by step construction of the pullback ∇^{Φ} of ∇ on some super vector bundle $\mathcal{E}$. A similar construction has been recently published in [24], 2.15 for tangent bundles.

Let $\Phi : X \longrightarrow Y$ a morphism of supermanifolds. Let $\mathcal{E}$ a super vector bundle over Y of rank $m|n$ and $\nabla : \mathcal{E} \longrightarrow \mathcal{T}_Y^* \otimes_{\mathcal{R}} \mathcal{E}$ a connection. We give a construction in two steps of the connection on $\Phi^*\mathcal{E} = \mathcal{O} \otimes_{\Phi} \varphi^*\mathcal{E}$:

(a) **Definition on $\varphi^*\mathcal{E}$** :
On sections of the form $\varphi^* e$, we define $(\varphi^*\nabla)\varphi^* e \in \varphi^*\mathcal{T}_Y^* \otimes_{\varphi^*} \varphi^*\mathcal{E}$ by

$$\langle \varphi^* W, \varphi^*\nabla\varphi^* e\rangle := \varphi^*(\nabla_W e)$$

Each section of $\varphi^*\mathcal{E}$ is locally of the form $\varphi^* e$. Since ∇ satisfies the Leibniz rule, we can use a locality argument (see [8], 4.34, 4.35 and 4.36 for an analogous argument) to see that local representations fit together and form a well defined connection on $\varphi^*\nabla$.

(b) **Extension of scalars to $\mathcal{O}$**:
Using 2.25 b), we can naturally identify

$$\mathcal{O} \otimes_{\varphi^*} \left(\varphi^*\mathcal{T}_Y^* \otimes_{\varphi^*} \varphi^*\mathcal{E}\right) \cong (\mathcal{O} \otimes_{\varphi^*} \varphi^*\mathcal{T}_Y^*) \otimes_{\mathcal{O}} (\mathcal{O} \otimes_{\varphi^*} \varphi^*\mathcal{E}) = \Phi^*\mathcal{T}_Y^* \otimes_{\mathcal{O}} \Phi^*\mathcal{E} \tag{3.13}$$

Let ψ a local section of $\Phi^*\mathcal{E}$. Using a local basis $\{\sigma_i\}$ of $\varphi^*\mathcal{E}$, we can write $\psi = \sum_i f^i \otimes \sigma_i$ where the coefficients $f^i \in \mathcal{O}$ are uniquely determined by ψ. Choosing coordinates on Y, we can decompose $d\Phi(V) = \sum_k V(\Phi^*(\varphi^* y^k)) \otimes \varphi^* \frac{\partial}{\partial y^k}$ for each super vector field on V. Using the identification (3.13), we define the covariant derivative of ψ by

$$\begin{aligned} \nabla_V^\Phi \psi &= \sum_i (-1)^{|f_i||V|} f^i \nabla_{d\Phi(V)}(1 \otimes \sigma_i) + V(f^i) \otimes \sigma_i \\ &:= \sum_i (-1)^{|f_i||V|} f^i \sum_k V(\Phi^*(\varphi^* y^k)) \otimes_{\mathcal{O}} \varphi^*(\nabla_{\partial_k}\sigma_i) + V(f^i) \otimes \sigma_i \end{aligned} \tag{3.14}$$

Here, the second line serves as a definition for the $\nabla_{d\Phi(V)}$-expression, the first results from formally applying the Leibniz rule. It is clear that this construction does not depend on the choice of coordinates and the next proposition states that it is also independent of the choice of the frame $\{\sigma_i\}$.

Proposition 3.53 *There exist a connection along (φ, Φ^*), that is, a morphism of sheaves of super algebras*

$$\nabla^\Phi : \Phi^*\mathcal{E} \longrightarrow \mathcal{T}_X^* \otimes_{\mathcal{O}_X} \Phi^*\mathcal{E}$$

satisfying the Leibniz rule along (φ, Φ). It is uniquely determined by these properties and the values on sections of the form $1 \otimes_\Phi \varphi^* e$.

Proof First, we show that the definition is independent of the choice of the local basis $\{e_i\}$. We will omit the symbols φ^* and use Einstein's convention for sums. If $\{e'_j\}$ is another basis, we have $e'_j = \alpha_j^k e_k$ for a matrix $\alpha \in GL(m|n, \mathcal{R})$ and thus

$$f'_j \otimes e'_j = f'^j \otimes \alpha_j^k e_k = f'^j \Phi^*(\alpha_j^k) \otimes e_k \qquad \Longrightarrow \qquad f^k = f'^j \Phi^*(\alpha_j^k)$$

The super product rule yields for coordinates $\{y^\mu\}$ on Y:

$$V(f^k) = V(f'^j)\Phi^*(\alpha_j^k) + (-1)^{|V||f'^j|} f'^j V(\Phi^*(y^\mu))\Phi^*(\partial_\mu \alpha_i^k) \tag{3.15}$$

Inserting this in (3.14) and using Leibniz-rule, we obtain

$$
\begin{aligned}
&(-1)^{|f'^i||V|} f'^i V(\Phi^*(y^\mu)) \otimes \nabla_\mu e'_i + V(f'^i) \otimes e'_i \\
&\qquad = (-1)^{|f'^i||V|} f'^i V(\Phi^*(y^\mu)) \Phi^*(\partial_\mu \alpha_i^k) \otimes e_k + X(f'^i)\Phi(\alpha_i^k) \otimes e_k \\
&\qquad\quad + (-1)^{|f'^i||V| + |y^\mu||\alpha_i^k|} f'^i V(\Phi^*(y^\mu)) \Phi(\alpha_i^k) \otimes \nabla_\mu e_k
\end{aligned}
$$

Rearranging the third summand yields $(-1)^{|f^k||V|} f^k V(\Phi^*(y^\mu)) \otimes \nabla_\mu e_k$, whereas the first and the second add up to $V(f^k) \otimes e_k$ by (3.15). A locality argument then shows, that ∇^Φ defines a morphism $\Phi^*\mathcal{E} \longrightarrow \Phi^*\mathcal{T}_N^* \otimes_{\mathcal{O}_X} \Phi^*\mathcal{E}_X$ of sheaves.
By construction in (3.14), ∇^Φ is $\mathcal{O}_X$-linear in the first slot satisfies the Leibniz rule along Φ. The statement about uniqueness is straightforward because using $\mathcal{O}_X$-linearity and the Leibniz rule, every expression $\nabla^\Phi \psi$ for $\psi \in \Phi^*\mathcal{E}$ may be reduced to an expression involving only terms of the form $\nabla^\Phi(1 \otimes_\Phi \varphi^* e)$.

$\square$

We end this chapter with some comments that will be used in subsequent calculations:

Remark 3.54 If $\mathcal{E}$ is the tangent bundle of Y and ∇ is the Levi-Civita connection then ∇^Φ inherits the properties of being metric and free of torsion. The latter property deserves some comment, its general form is

$$\nabla_U^\Phi d\Phi(V) = (-1)^{|U||V|} \nabla_V^\Phi d\Phi(U)$$

provided $[U, V] = 0$. In case $U = V = \frac{\partial}{\partial\theta}$, this in particular implies $\nabla_\theta^\Phi d\Phi(\partial_\theta) = 0$.

Furthermore, the curvature of ∇^Φ is in this case given by

$$R^\Phi(X, Y)\xi = R^Y(d\Phi(X), d\Phi(Y))\xi$$

where $X, Y \in \mathcal{T}_X$ and $\xi \in \Phi^*\mathcal{E}$. In this expression, $d\Phi(X), d\Phi(Y)$ and ξ are vector fields along Φ. We have the following symmetry properties:

$$
\begin{aligned}
&R(\xi, \eta)\zeta = -(-1)^{|\xi||\eta|} R(\eta, \xi)\zeta \\
&R(\xi, \eta)\zeta + (-1)^{|\xi|(|\eta|+|\zeta|)} R(\eta, \zeta)\xi + (-1)^{|\zeta|(|\eta|+|\xi|)} R(\zeta, \xi)\eta = 0
\end{aligned}
$$

The second Bianchi identity can be adapted in a similar way.

4 Categorical aspects of supergeometry

In chapter 3, we discussed the ringed space approach to supergeometry. Instead of using "points" or values of a map at a point, supergeometric concepts were formulated in terms of rings of superfunctions and modules over them. However, there are certain problems which show that there is the need to introduce a layer of abstraction by describing these structures using the language of category theory.

As a starting point, we will consider one of these problems especially relevant for this work: Consider example (1.1) given as a motivation in the introduction, i.e. we consider morphisms $\Phi : \mathbb{R}^{1|1} \longrightarrow N$ where N is an ordinary smooth manifold. On the level of functions, Φ is described by

$$\Phi : C^\infty_N \longrightarrow \varphi_* \mathcal{O}^{1|1} \qquad\qquad \Phi(g) = A(g)1 + \psi(g)\theta$$

where $A, \psi : C^\infty_N \longrightarrow C^\infty_{\mathbb{R}}$. Since Φ is required to be even by definition, we have $\psi = 0$. A is a homomorphism of C^∞_N and hence corresponds to the underlying smooth map φ. Thus, the problem is reduced to a problem of ordinary differential geometry and in particular, there seems to be no vector field (be it odd or even) encoded in Φ. If we allowed Φ to break the parity, it would be possible to have a nontrivial contribution ψ. It is easy to see that it satisfies the Leibniz-rule of a φ-derivation. However, ψ would just be an ordinary vector field along φ and not "odd", because the tangent sheaf of N only contains even derivations. An expression $R^N(\psi, \psi)$ then clearly vanishes. The same problem is observed when looking at an even morphism $\Phi : \mathbb{R}^{1|2} \longrightarrow N$. Writing $\Phi(g) = A(g) + \psi(g)\theta^1\theta^2$, we now indeed find a vector field given by ψ in the case where Φ preserves parity. But again, this is just an ordinary vector field and we see that in general, it is not possible to obtain an anticommuting coefficient ψ simply because it takes values in the real numbers and not in a Grassmann algebra.
This problem can be resolved by introducing a suitable "space of all morphisms from $\mathbb{R}^{1|1}$ to N" and looking at its "points" rather than only at morphisms $\mathbb{R}^{1|1} \longrightarrow N$ as defined in 3.2. It will be shown that these points indeed contain odd vector fields. A similar approach to obtain odd spinor fields is described in [13], section 3.4. To define this space and to give meaning to the concept of a "point of this space", we have to introduce the categorical approach to supergeometry.

There are other reasons to introduce this more abstract concept which is based on ideas from algebraic geometry: In contrast to the theory of smooth manifolds where it is possible to define infinite dimensional Banach- or Fréchet manifolds, this is impossible in supergeometry as long as one keeps the ringed space approach. It was observed in [14] (introduction to chapter 3) that sheaves of rings of functions are in general not sufficient to describe these objects. This was a motivation in [52] to study the categorical approach and is in particular important when trying to define the "space of all morphisms". Moreover, concepts like *inner morphisms* (see definition 2.5) arise very natural in the categorical setting. Compared to the ringed space language, this approach furthermore resembles the constructions of superspace and supergeometric objects appearing in the physics literature, although this might be not obvious until one gets used to the new language. Finally, it allows to compare all the different concepts for supergeometry that have been developed in a systematic way.

4.1 The categorical approach to algebra and geometry

The general idea of the categorical approach to algebra and geometry is to replace objects (e.g. modules, supermanifolds,...) by functors and morphisms between such objects by natural transformations of the corresponding functors. These ideas are described in detail in [30] and [21], the categorical formulation of supergeometry is discussed in [10], [52] or [44].

In any category[5] $\mathcal{C}$, we can associate to an object X the following functor:

$$\mathrm{Hom}_{\mathcal{C}}(-,X):\mathcal{C}^{op}\longrightarrow \mathrm{Set}\quad \mathrm{Hom}_{\mathcal{C}}(-,X)(Y):=\mathrm{Hom}_{\mathcal{C}}(Y,X)\quad \mathrm{Hom}_{\mathcal{C}}(-,X)(f):=f^* \tag{4.1}$$

Here, Y is also an object in $\mathcal{C}$ and f is a morphism in this category. $\mathrm{Hom}_{\mathcal{C}}(-,X)$ is called the contravariant Hom-functor. Moreover, if $f: X_1 \longrightarrow X_2$ is a morphism in $\mathcal{C}$ then

$$\tau_f:\mathrm{Hom}_{\mathcal{C}}(-,X_1)\longrightarrow \mathrm{Hom}_{\mathcal{C}}(-,X_2)\qquad\qquad (\tau_f)_Y:=f_*$$

clearly defines a natural transformation between the functors. Thus, we associated a functor to each object and a natural transformation to each morphism. The converse direction easily follows from the Yoneda-Lemma (see A.1):

Proposition 4.1 *Let $\mathcal{C}$ be a category, X an object and $f: X_1 \longrightarrow X_2$ a morphism in $\mathcal{C}$ then we have:*

(a) The object X is determined up to a unique isomorphism by the functor $\mathrm{Hom}_{\mathcal{C}}(-,X)$.

(b) The natural transformations $\tau:\mathrm{Hom}_{\mathcal{C}}(-,X_1)\longrightarrow \mathrm{Hom}_{\mathcal{C}}(-,X_2)$ correspond bijectively to the morphisms $X_1 \longrightarrow X_2$.

The proposition says that it is possible to replace objects and morphisms by the corresponding Hom-functors and their natural transformations. Since not every functor $\mathcal{C} \longrightarrow \mathrm{Set}$ is of the form $\mathrm{Hom}_{\mathcal{C}}(-,X)$ for some object X, it is useful to give a name to those functors which represent objects in this sense:

Definition 4.2 ([30], II.3.2) *A functor $F:\mathcal{C}^{op}\longrightarrow \mathrm{Set}$ is called* representable *if there exist an object $X\in\mathcal{C}$ such that F and $\mathrm{Hom}_{\mathcal{C}}(-,X)$ are naturally equivalent.*

There exist certain criteria insuring that a functor is representable (see [59] 10.3 and [52] for supermodules). Since we will not deal with questions of representability, we will not go into these details but refer to the literature. The following example shows that the functorial point of view and the representing objects play an important role already at the superalgebra level:

[5] See Appendix A for the notions from category theory which are used in the following

Example 4.3 (Inner Hom-functors for super vector spaces, see [66] 3.7, [10] §1.6) Let V, W be real super vector spaces. By definition 2.1, $\mathrm{Hom}_{\mathbb{R}}(V,W)$ are the linear maps from V to W which preserve parity. So far, $\mathrm{Hom}_{\mathbb{R}}(V,W)$ is not a super vector space and in fact, the super vector space of all linear maps $V \longrightarrow W$ was defined by (see definition 2.5)

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathbb{R}}(V,W)_{\overline{0}} &:= \{f : V \longrightarrow W \mid f \text{ is linear, preserves parity}\}\\ &\cong \mathrm{Hom}_{\mathbb{R}}(V_{\overline{0}}, W_{\overline{0}}) \oplus \mathrm{Hom}_{\mathbb{R}}(V_{\overline{1}}, W_{\overline{1}})\\ \underline{\mathrm{Hom}}_{\mathbb{R}}(V,W)_{\overline{1}} &:= \{f : V \longrightarrow W \mid f \text{ is linear, interchanges parity}\}\\ &\cong \mathrm{Hom}_{\mathbb{R}}(V_{\overline{0}}, W_{\overline{1}}) \oplus \mathrm{Hom}_{\mathbb{R}}(V_{\overline{1}}, W_{\overline{0}})\end{aligned}$$

We can now see that this ad hoc construction (which proved important to define e.g. dual spaces) arises very naturally in the categorical framework as follows: In ordinary "non super" linear algebra, we have the relation

$$\mathrm{Hom}_{\mathbb{R}}(U, \mathrm{Hom}_{\mathbb{R}}(V,W)) \cong \mathrm{Hom}_{\mathbb{R}}(U \otimes V, W) \tag{4.2}$$

which is functorial in U. Clearly, we wish to preserve this feature in super linear algebra, so we define the functor

$$F : \mathrm{SVec} \longrightarrow \mathrm{Set} \qquad\qquad U \mapsto \mathrm{Hom}_{\mathbb{R}}(U \otimes V, W)$$

where $\mathrm{Hom}_{\mathbb{R}}$ denote the parity preserving morphisms in SVec and $\otimes$ the super tensor product. A representing object for F, if it exists, then satisfies the super version of (4.2). Using (4.2) for the homogeneous components $V_{\overline{i}}, W_{\overline{i}}$, it is then easy to verify that we have

$$\mathrm{Hom}_{\mathbb{R}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{R}}(U, \underline{\mathrm{Hom}}_{\mathbb{R}}(V,W))$$

Thus, a representing object for F (which is unique up to isomorphism) is given by $\underline{\mathrm{Hom}}_{\mathbb{R}}(V,W)$ and this justifies its introduction from an abstract point of view. $\underline{\mathrm{Hom}}_{\mathbb{R}}$ is also know as *inner Hom-functor* or *internal Hom-functor* (see also [30], II.4.23).

Since the elements of the sets $\mathrm{Hom}_{\mathcal{C}}(-,X)$ determine a representing object X completely, the following notion of "point" is introduced:

Definition 4.4 *Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$. For each object S of $\mathcal{C}$, the elements of* $\mathrm{Hom}_{\mathcal{C}}(S,X)$ *are called S-points of X.*

Remark 4.5 Example 4.3 demonstrates the importance of looking at maps from different vector spaces into $\underline{\mathrm{Hom}}_{\mathbb{R}}(V,W)$, i.e. at its different U-points. If we restricted ourselves to $U = \mathbb{R}$, then we would not "see" the odd part of $\underline{\mathrm{Hom}}_{\mathbb{R}}(V,W)$ since parity preservation implies $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \underline{\mathrm{Hom}}_{\mathbb{R}}(V,W)) \cong \mathrm{Hom}_{\mathbb{R}}(V,W) = \underline{\mathrm{Hom}}_{\mathbb{R}}(V,W)_{\overline{0}}$. The odd part becomes only visible, provided we start with a space already containing odd elements, e.g. for $U = \mathbb{R}^{0|1}$, we have $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{0|1}, \underline{\mathrm{Hom}}_{\mathbb{R}}(V,W)) \cong \underline{\mathrm{Hom}}_{\mathbb{R}}(V,W)_{\overline{1}}$ (see also [23] lecture 1). Functoriality is then required to obtain a construction independent of U. The same general principle also clarifies and resolves the problem discussed in the introduction of this section. In fact, we will see that the morphisms Φ considered there only correspond to $\mathbb{R}$-points of the space of all morphisms from $\mathbb{R}^{1|1}$ to N (see 4.26 for the precise definition) and we will not obtain an odd vector field in their component decomposition. However, looking at more general S-points where S is allowed to have a nontrivial odd part, we will see that these odd component fields do exist.

It should be pointed out that regarding supermanifolds (in the sense of definition 3.4) as ringed spaces and as representable functors is equivalent by proposition 4.1. These are just two different points of view and it is possible to switch to the picture which is more appropriate in the situation under consideration. As an example, it is possible in both approaches to define the concept of a super Lie group:

Example 4.6 (Super Lie groups) Following [10], §2.10, a super Lie group can be defined as a supermanifold G, such that all its S-points carry compatible groups structures, i.e. for all supermanifolds S, T and all morphisms $\varphi : S \longrightarrow T$, we have

(a) $\mathrm{Hom}(S, G)$ is a group with unit element e_S, multiplication μ_S and inversion i_S

(b) $\varphi^* : \mathrm{Hom}(T, G) \longrightarrow \mathrm{Hom}(S, G)$ is a group homomorphism.

This approach is valid in any category, the concept is described in detail in appendix A. Moreover, it is also shown there, that this approach is equivalent to the classical definition of a super Lie group (see [66], chapter 7.1), i.e. a supermanifold equipped with a multiplication $\mu : G \times G \longrightarrow G$, an inversion $i : G \longrightarrow G$ and a unit element $e : \mathbb{R}^{0|0} \longrightarrow G$ satisfying certain compatibility relations. Finally, a supermanifold can then also be constructed by giving a functor $\mathrm{BKL}^{op} \longrightarrow \mathrm{Set}$ which satisfies (a) and (b) above and proving its representability afterwards.

Another important example is given by the construction of the total space of a bundle:

Example 4.7 (The total space of a vector bundle) We will follow the construction outlined in [10] (3.2, p.72), see also [15] (p.42). Let $X = (\tilde{X}, \mathcal{O})$ a supermanifold and $\mathcal{E}$ a locally free module of rank $r|s$. Locally, the total space E should then have the structure of $X \times \mathbb{R}^{s|t}$. Recall that the pullback of $\mathcal{E}$ was defined in section 3.3 using ringed space formalism. Define the functor $E : \mathrm{BKL}^{op} \longrightarrow \mathrm{Set}$ by

$$\begin{aligned} E(S) &:= \{(\varphi, s) | \varphi \in \mathrm{Hom}(S, X), s \in (\varphi^* \mathcal{E})_{\overline{0}}\} \\ E(\rho : S \longrightarrow S') &:= ((\varphi', s') \mapsto \rho^*(\varphi', s') := (\varphi' \rho, \rho^* s')) \end{aligned}$$

To prove the representability of E, first let $U^{p|q} \subset \mathbb{R}^{p|q}$ be an open subsupermanifold and $\mathcal{E}$ a free $\mathcal{O}^{p|q}$-module of rank $r|s$. Denoting $S = (\tilde{S}, \Sigma)$, we have $(\varphi^* \mathcal{E})_0 = \Sigma_0^r \oplus \Sigma_1^s$. Moreover, by theorem 3.17, $\mathrm{Hom}(S, U^{p|q}) = (\Sigma_0^p)^{map} \oplus \Sigma_1^q$, where ^{map} indicates, that the even functions have to satisfy the mapping condition. Hence, we have

$$E(S) \cong (\Sigma_0^p)^{map} \oplus \Sigma_1^q \oplus \Sigma_0^r \oplus \Sigma_1^s \cong \mathrm{Hom}(S, U^{p|q} \times \mathbb{R}^{r|s})$$

which is natural in S. An arbitrary supermanifold X of dimension $p|q$ equipped with a locally free module of rank $r|s$ can be covered by such neighborhoods $\{U_i^{p|q} = (U_i, \mathcal{O}_i)\}$. Denoting $U_{ij} = U_i \cap U_j$, we have isomorphisms of functors

$$\mathrm{Hom}(-, U_i|_{U_{ij}} \times \mathbb{R}^{r|s}) \cong E|_{U_{ij}}(-) \cong \mathrm{Hom}(-, U_j|_{U_{ij}} \times \mathbb{R}^{r|s})$$

By proposition 4.1, they induce isomorphisms $f_{ij} : U_i|_{\tilde{U}_{ij}} \times \mathbb{R}^{r|s} \longrightarrow U_j|_{\tilde{U}_{ij}} \times \mathbb{R}^{r|s}$ which satisfy the cocycle conditions by construction and glueing these patches (see [66] section 4.2, p.135 for the details) then yields a supermanifold E of dimension $p + r|q + s$. By construction, E represents the functor $E(-)$, i.e. the isomorphisms $E(S) \cong \mathrm{Hom}(S, E)$ are natural in S.
Finally, it is possible to recover the sections $\mathcal{E}(U)$ from $E(-)$ for some open set $U \subset \tilde{X}$. Let $S = \mathbb{R}^{0|1} \times X|_U$ so that $\Sigma = \bigwedge^\bullet \mathbb{R}^1 \otimes \mathcal{O}(U)$ and consider the morphism $\varphi = pr_{X|_U} : \mathbb{R}^{0|1} \times X|_U \twoheadrightarrow X|_U$. Then, we have

$$(pr_X^* \mathcal{E})_{\bar{0}} = \bigwedge^{ev} \mathbb{R} \otimes \mathcal{E}(U)_{\bar{0}} \;\oplus\; \bigwedge^{odd} \mathbb{R} \otimes \mathcal{E}(U)_{\bar{1}} \cong \mathcal{E}(U)_{\bar{0}} \oplus \mathcal{E}(U)_{\bar{1}}$$

which allows us to identify $\mathcal{E}(U)$ with $\{\, s \,|\, (pr_{X|_U}, s) \in E(\mathbb{R}^{0|1} \times X|_U)\}$.

In general, handling all the S-points of a supermanifold can be difficult because S ranges over a large set of objects and the sets $\mathrm{Hom}_{\mathrm{BKL}}(X, Y)$ are large, they already carry the structure of an infinite dimensional manifold if X and Y are smooth manifolds. Thus, it is desirable to find a smaller class $\mathcal{G}$ of objects in BKL such that the restriction of $\mathrm{Hom}(-, X)$ to $\mathcal{G}$ still determines X in an appropriate sense. Looking at the category of sets first, we easily find, that any set $\{pt\}$ consisting of one arbitrary element is enough:

(a) We have $\mathrm{Hom}_{\mathrm{Set}}(pt, X) \cong X$ for any set X because any such morphism s just corresponds to the image point $s(pt) \in X$. Thus, every set X is determined up to isomorphism by the restriction of $\mathrm{Hom}_{\mathrm{Set}}(-, X)$ to $\mathcal{G} = \{pt\}$.

(b) If $f : X \longrightarrow Y$ is a map between the sets X and Y, then f is clearly determined by all $f \circ s \in \mathrm{Hom}_{\mathrm{Set}}(pt, Y)$ for $s \in \mathrm{Hom}_{\mathrm{Set}}(pt, X)$ because $f \circ s$ just corresponds to the value of f at all the points in X given by s.

The second property states, that different morphisms $f, g : X \longrightarrow Y$ can already be distinguished by looking at the composites $f \circ s$ for all $s \in \mathrm{Hom}_{\mathrm{Set}}(pt, X)$. This property is used to define the general concept of a set of generators[6]:

Definition 4.8 ([59], 10.5.1) *A set $\mathcal{G}$ of objects in a category $\mathcal{C}$ is called generating, if for all $f \neq g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, there is $G \in \mathcal{G}$ and $s \in \mathrm{Hom}_{\mathcal{C}}(G, X)$ such that $fs \neq gs$.*

Even though we will see that the restriction of a functor $\mathrm{Hom}(-, X)$ to a set of generators need not uniquely specify X, it will be important to have a suitable set of generators for BKL:

Definition 4.9 *A superpoint is a $(0|q)$-dimensional supermanifold. The full subcategory of* BKL *consisting of superpoints and their morphisms is denoted by* SPoint*.*

[6]There is a different notion of generator given in [32], 5.2.1. We will not discuss the difference here.

Thus, a superpoint is given by a topological space $\{pt\}$ and a sheaf of superfunctions $\mathcal{O}$ being isomorphic to $C^\infty(pt) \otimes \Lambda_q = \Lambda_q$, where $\Lambda_q = \bigwedge^\bullet \mathbb{R}^q$. Every superpoint of dimension $(0|q)$ is clearly isomorphic (but not canonically) to $\mathbb{R}^{0|q}$. We have the following result (see [52], 2.2.16):

Lemma 4.10 *Let* Gr *be the category of the finite dimensional Grassmann algebras*[7] $\Lambda_n := \bigwedge^\bullet \mathbb{R}^n$ *(morphisms are again assumed to preserve parity). Then we have a natural equivalence of categories* $\mathcal{P} : \mathrm{Gr}^{op} \longrightarrow \mathrm{SPoint}$ *which is defined by*

$$\mathcal{P}(\Lambda_n) := (\{pt\}, \Lambda_n) := \mathcal{P}_n$$
$$\mathcal{P}(\varphi : \Lambda_n \longrightarrow \Lambda_m) := \Phi = (id_{\{pt\}}, \varphi^* : \mathcal{P}_m \longrightarrow \mathcal{P}_n)$$

In particular, a morphism between superpoints $\mathcal{P}_n \longrightarrow \mathcal{P}_m$ is given by the homomorphism of the superalgebras $\Lambda_m \longrightarrow \Lambda_n$ going in the converse direction. These categories provide us with a suitable set of generators :

Theorem 4.11 ([52], 3.3.3) *The set of superpoints* $\{\mathcal{P}_n | n \in \mathbb{N}\}$ *forms a set of generators of the category* BKL *of finite dimensional supermanifolds. We will just say* n*-points instead of* $\mathcal{P}_n$*-points.*

We will only sketch the argument: Let $\Phi_1 \neq \Phi_2 : X \longrightarrow Y$ be two morphisms and φ_1, φ_2 their underlying maps. In case $\varphi_1 \neq \varphi_2$, we can distinguish these maps by some 0-point $s \in \mathrm{Hom}(\mathcal{P}_0, X)$, as described above in the category Set and in particular, we do not need generators of even dimension > 0. In general, we have $\Phi_1^*(g) \neq \Phi_2^*(g) \in \mathcal{O}(U) \cong C^\infty(U) \otimes \Lambda_q$ for some local superfunction g on Y. But then, choosing n large enough, we can find a homomorphism $\psi : \Lambda_q \longrightarrow \Lambda_n$ and $x \in U$ such that $\psi(\Phi_1(f)(x)) \neq \psi(\Phi_2(f)(x))$. ψ and x specify the desired n-point of X. Note however, that we have to take an infinite set of superpoints as generators to allow for arbitrary large odd dimension of X. This is a fundamental difference in comparison with the category of smooth manifolds where there exist a single generator. It is caused by the existence of nilpotent functions.

The following examples in the category BKL or rather Man (the category of smooth manifolds) show, that the term “generator” is in fact a bit misleading:

Example 4.12

(a) Let X be an ordinary smooth manifold, considered to be an object of BKL. The sets $\mathrm{Hom}_{\mathrm{BKL}}(\mathcal{P}_n, X)$ do not determine X as a smooth (super)manifold since the smooth structure of the underlying manifold is not encoded by them. An example are the famous Milnor 7-spheres, where the underlying topological spaces are even homeomorphic but the smooth manifolds are not diffeomorphic.

(b) Following example 3.17 of [1], let $\varphi : X \longrightarrow Y$ be a smooth map between smooth manifolds. We define $\alpha_{\mathcal{P}_n}(f) := ev_{pt}(\varphi f)$ for $f \in \mathrm{Hom}_{\mathrm{BKL}}(\mathcal{P}_n, X)$. Thus, we have a

[7] For convenience, we are choosing one representative for each dimension so Gr is a skeleton of the category of all finite dimensional Grassmann algebras and equivalent to this larger category.

natural transformation $\alpha : \mathrm{Hom}(-,X) \longrightarrow \mathrm{Hom}(-,Y)$ but it is not of the form $f \mapsto \psi f$ for some $\psi \in \mathrm{Hom}(X,Y)$. To see this, note that on functions, α is given by $pr_{\mathbb{R}}(f^* \circ \varphi^*)$ which is not of the form $f^* \circ \psi^*$.
This means that it is not sufficient to define morphisms on the level of generators (see remark 4.18 for the additional requirements).

These simple examples show that some extra structure is needed in order to characterize supermanifolds and their morphisms in terms of functors $\mathrm{Gr} \longrightarrow \mathrm{Set}$. In fact, the concept of generators remains useful if we consider $\mathrm{Hom}(-,M)$ not as a functor into the category of sets but in that of smooth manifolds. In this case, it is again obvious that $\mathrm{Hom}(pt,M)$ determines M up to isomorphism (here: diffeomorphism) and a similar result will hold for supermanifolds. This is the subject of the next section.

4.2 The Molotkov-Sachse approach to supergeometry

The Molotkov-Sachse approach was introduced in [44], [53] and [52] (chapter 3). Very similar concepts are discussed in earlier publications by Leites ([39] 3.3.1 and also [40], 1.3) as well as [60] and used in [1]. Here, functors $\mathrm{Gr} \longrightarrow \mathrm{Set}$ (recall that Gr is defined in lemma 4.10) are the starting point for defining and constructing algebraic and geometric objects. They form a category denoted by $\mathrm{Set}^{\mathrm{Gr}}$ (see appendix A). Additional structure is specified by requiring the functors to take values in suitable categories of modules, manifolds etc. In this way, it is possible to define everything without the need to use the BKL-definition of supermanifolds, this category is in fact rediscovered in the categorical framework at a later point. Furthermore, the concept remains meaningful in infinite dimensions. We will only give a short account of the work in [44] and [53] here and refer to the literature for all the details.

First of all, it is necessary to replace the real numbers by a ring $\overline{\mathbb{R}}$ in the category $\mathrm{Set}^{\mathrm{Gr}}$. That means that for $\Lambda, \Lambda' \in \mathrm{Gr}$ and $\varphi \in \mathrm{Hom}_{\mathrm{Gr}}(\Lambda, \Lambda')$[8], each $\overline{\mathbb{R}}(\Lambda)$ has the structure of a commutative ring and each $\overline{\mathbb{R}}(\varphi)$ is a ring homomorphism. It is defined as follows (see section 3.1 [53]):

$$\overline{\mathbb{R}}(\Lambda) := \Lambda_{\overline{0}} \qquad\qquad \overline{\mathbb{R}}(\varphi) := \varphi|_{\Lambda_{\overline{0}}}$$

It is now possible to define $\overline{\mathbb{R}}$-modules in $\mathrm{Set}^{\mathrm{Gr}}$:

Definition 4.13 ([53], section 3.2 and definition 4.5)
(a) *A functor $\mathcal{M} \in \mathrm{Set}^{\mathrm{Gr}}$ is a $\overline{\mathbb{R}}$-module if all $\mathcal{M}(\Lambda)$ are $\overline{\mathbb{R}}(\Lambda)$-modules and all $\mathcal{M}(\varphi)$ are homomorphism of $\overline{\mathbb{R}}(\Lambda)$-modules.*

(b) *A $\overline{\mathbb{R}}$-module $\mathcal{M}$ is called superrepresentable, if there exist a $\mathbb{R}$-super vector space V, such that $\mathcal{M}$ is isomorphic in $\mathrm{Set}^{\mathrm{Gr}}$ to the functor $\overline{V}$ defined by*

$$\overline{V}(\Lambda) := (\Lambda \otimes_{\mathbb{R}} V)_{\overline{0}} \qquad\qquad \overline{V}(\varphi) := (\varphi \otimes id_V)|_{\overline{V}(\Lambda)}$$

[8] In this chapter, we assume that $\Lambda, \Lambda' \in \mathrm{Gr}$ and $\varphi \in \mathrm{Hom}_{\mathrm{Gr}}(\Lambda, \Lambda')$ without always mentioning it.

Note that not every $\overline{\mathbb{R}}$-module is superrepresentable, see [53], p. 13 equation (38) and below for a counterexample. The superrepresentable modules however will be important since they serve as model spaces for supermanifolds. More precisely, in complete analogy to an open subset of $\mathbb{R}^n$, it is possible to introduce open subfunctors of certain $\overline{\mathbb{R}}$-modules. Thus, it is first necessary to discuss topologic notions. Let Top denote the category of topological spaces and $\mathrm{Top}^{\mathrm{Gr}}$ the category of functors $\mathrm{Gr} \longrightarrow \mathrm{Top}$.

Definition 4.14 ([53], 4.1 and 4.3) *Let F, F' be functors in $\mathrm{Top}^{\mathrm{Gr}}$.*

(a) F' is called a subfunctor of F (denoted $F' \subset F$) if all $F'(\Lambda)$ are topological subspaces of $F(\Lambda)$ and if the inclusions $\{i_\Lambda : F'(\Lambda) \hookrightarrow F(\Lambda)\}_{\Lambda \in \mathrm{Gr}}$ form a natural transformation, i.e. a morphism in $\mathrm{Top}^{\mathrm{Gr}}$.

(b) F' is called open subfunctor of F if $F'(\Lambda) \subset F(\Lambda)$ is an open set for all Λ.

(c) A morphism $\alpha : F' \longrightarrow F$ in $\mathrm{Top}^{\mathrm{Gr}}$ is called open, if there exist an open subfunctor G of F and an isomorphism of functors $\beta : F' \xrightarrow{\sim} G$ s.t. α factorizes as follows:

$$\alpha = F' \xrightarrow{\beta} G \subset F$$

Open morphisms will be used to generalize the notion of a chart in smooth differential geometry, which is, by definition, a continuous map $x^{-1} : V \xrightarrow{\sim} U \subset M$ such that $U \subset M$ and $V \subset \mathbb{R}^n$ are open and $x^{-1} : V \longrightarrow U$ is a homeomorphism. V will be replaced by a superdomain, which can now be defined as open subfunctors of certain superrepresentable modules:

Definition 4.15 ([53], 4.5, 4.6) *A superrepresentable $\overline{\mathbb{R}}$-module $\mathcal{M}$ in $\mathrm{Top}^{\mathrm{Gr}}$ is called Banach, Fréchet or locally convex etc. if all the topological $\mathbb{R}$-vector spaces $\mathcal{M}(\Lambda)$ are Banach, Fréchet or locally convex respectively. In this case, an open subfunctor $F \subset \mathcal{M}$ is called Banach, Fréchet or locally convex superdomain.*

For all functors F in $\mathrm{Top}^{\mathrm{Gr}}$, it is possible to construct the following kind of open subfunctor called a "restriction" (see [53] below definition 4.4). Denote by $\mathrm{pr}_\Lambda : \Lambda \twoheadrightarrow \mathbb{R}$ the canonical projection. Let U be an open subset of the topological space $F(\mathbb{R})$. Then, the restriction of F to U is defined by

$$F|_U(\mathbb{R}) := U \qquad\qquad F|_U(\Lambda) := (F(\mathrm{pr}_\Lambda))^{-1}(U) \subset F(\Lambda)$$

and the action on morphisms is given by the obvious restriction. In general, there can be many open subfunctors but for superdomains, the concept is restrictive:

Proposition 4.16 ([53], 4.8 and 4.9) *Let F be a superdomain (or locally isomorphic to a superdomain, see below). Then every open subfunctor of F is a restriction. In particular, each superdomain is of the form $M|_U$ for a superrepresentable $\overline{\mathbb{R}}$-module M and some open set $U \subset M(\mathbb{R})$.*

The proposition essentially says that an open subfunctor of a superdomain is already determined by its underlying open set. There are no open subfunctors F_1, F_2 which differ only in the sets $F_i(\Lambda)$ for some $\Lambda \neq \mathbb{R}$, so that the shape of an open subfunctor is restricted.

To be able to use superdomains as building blocks for more general spaces, it is necessary to specify a concept of supersmooth morphism among them. For ordinary Banach- and Fréchet-spaces, there exist a well established notion of smooth maps, which is used in the following definition:

Definition 4.17 ([53], 4.10) *Let $M|_U$ and $N|_V$ be Banach- or Fréchet superdomains. A morphism $f : M|_U \longrightarrow N|_V$ in $\mathrm{Top}^{\mathrm{Gr}}$ is called supersmooth provided that*

(a) The map $f(\Lambda) : M|_U(\Lambda) \longrightarrow N|_V(\Lambda)$ is smooth (in the Banach- or Fréchet-sense, respectively) for each Λ.

(b) The derivative $Df(\Lambda) : M|_U(\Lambda) \times M(\Lambda) \longrightarrow N(\Lambda)$ is $\Lambda_{\bar{0}}$-linear in the second entry for all Λ.

Remark 4.18

(a) In the second part of definition 4.17, we used a formulation also appropriate in the situation when the topology of the spaces is only Fréchet. If we are in the Banach category, it is possible to work with maps $g(\Lambda)$ which are smooth as maps among Banach spaces. In this case, the differentials $Dg(\Lambda)$ exist as a bounded linear maps $M(\Lambda) \to N(\Lambda)$ and not only in the weaker form as in (b).

(b) The $\Lambda_{\bar{0}}$-linearity is a crucial and strong requirement. The situation is similar to the definition of complex differentiability, where a map $f : \mathbb{C} \longrightarrow \mathbb{C}$ is supposed to be differentiable as a map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ but in addition, the differential *df* is required to be $\mathbb{C}$-linear. Looking at example 4.12 (b), the morphism $\{pr_\Lambda\}_{\Lambda \in Gr}$ satisfies the first assumption of the definition but it is clearly not $\Lambda_{\bar{0}}$-linear. In fact, it is also discussed in [60] (see [1] lemma 4.8 and theorem 4.5 for details) that this linearity condition ensures that a morphism of functors is induced by an actual morphism of finite dimensional supermanifolds.

In order to define supermanifolds in the categorical setting, it is necessary to define open coverings of functors which then also allow to introduce the notion of local isomorphisms:

Definition 4.19 ([53] 4.2, 4.4) *Let F', F'' be subfunctors of F in $\mathrm{Top}^{\mathrm{Gr}}$.*

(a) The union $F' \cup F''$ is defined by $(F' \cup F'')(\Lambda) := F'(\Lambda) \cup F''(\Lambda)$ and $(F' \cup F'')(\varphi) := F(\varphi)|_{(F' \cup F'')(\Lambda)}$. The intersection $F' \cap F''$ is defined similarly.

(b) An open covering of F is a family $\{u_i : U_i \longrightarrow F\}$ of open functor morphism such that the induced family $\{u_i(\Lambda) : U_i(\Lambda) \longrightarrow F(\Lambda)\}$ is an open covering in the usual sense for each Λ.

(c) A functor F in $\mathrm{Top}^{\mathrm{Gr}}$ is locally isomorphic to superdomains if there exist an open covering $u_i : U_i \longrightarrow F$ s.t. every U_i is a superdomain.

The definition of a supermanifold can now be given in terms of atlases, precisely as in the case of ordinary smooth manifolds. Let Man denote the category of finite dimensional smooth manifolds, of Banach manifolds or Fréchet-manifolds (see [37] or [27] for the latter cases). This choice for Man will then lead to finite dimensional supermanifolds, Banach supermanifolds and Fréchet supermanifolds respectively. Clearly, the superdomains have to be chosen finite dimensional, Banach or Fréchet respectively and we assume that the corresponding choice has been made.

Definition 4.20 ([53], 4.12) *Let X be a functor in* $\mathrm{Man}^{\mathrm{Gr}}$ *and* $\mathbb{A} = \{u_i : U_i \longrightarrow X\}_{i \in I}$ *an open covering of it.* $\mathbb{A}$ *is called an atlas on X and the $\{u_i\}$ are called charts provided that*

(a) Each U_i is a superdomain.

(b) For each pair of indices $i, j \in I$, the fibre product $U_{ij} := U_i \times_X U_j$ in $\mathrm{Man}^{\mathrm{Gr}}$ *carries the structure of a superdomain such that the projections $\Pi_i : U_{ij} \longrightarrow U_i$ and $\Pi_j : U_{ij} \longrightarrow U_j$ are supersmooth.*

Two atlases are called equivalent if their union is again an atlas on X. A supermanifold is a functor $X \in$ $\mathrm{Man}^{\mathrm{Gr}}$ *which is equipped with an equivalence class of atlases.*

The second condition in the definition of an atlas is explained in appendix A in more detail. It makes sure that U_{ij} is indeed a superdomain (which does not follow automatically as in the case of open subsets of $\mathbb{R}^n$, we refer to [53] section 4.4 for the details) and that the coordinate changes are supersmooth. The concept of a supersmooth map between supermanifolds is now introduced in a similar fashion:

Definition 4.21 ([53], 4.14) *Let X, Y be supermanifolds in the sense of definition 4.20. A morphism $f : X \longrightarrow Y$ in* $\mathrm{Man}^{\mathrm{Gr}}$ *is called supersmooth if for all charts $u : U \longrightarrow X$ and $v : V \longrightarrow Y$, the fibre product*

$$\begin{tikzcd}
 & U \times_Y V \arrow[dl, "\Pi_u"'] \arrow[drr, "\Pi_v"] & & \\
U \arrow[r, "u"] & X \arrow[r, "f"] & Y & V \arrow[l, "v"]
\end{tikzcd}$$

carries the structure of a superdomain such that Π_u and Π_v are supersmooth. The set of morphism is denoted by $SC^\infty(X, Y)$, the category defined by supermanifolds and supersmooth morphisms by SMan.

As before in the case of the coordinate changes, the condition on the fibre product $U \times_Y V$ ensures that local representative of f is a supersmooth morphism between superdomains.

It should be pointed out that it is nontrivial, that SPoint is still is a set of generators for the "new" category SMan (theorem 4.11 on applies to BKL). It is in fact true, more precisely, we have the following proposition:

Proposition 4.22 ([53], 4.18) *For each $X \in \mathrm{SMan}$ and each $\Lambda \in Gr$, there is a bijection*

$$X(\Lambda) \cong SC^\infty(\mathcal{P}(\Lambda), X) = \mathrm{Hom}_{\mathrm{SMan}}(\mathcal{P}(\Lambda), X)$$

which is natural in Λ.

So far, there is the BKL-construction for finite dimensional supermanifolds and the Molotkov-Sachse approach which also yields a subcategory fSMan of SMan consisting of finite dimensional supermanifolds. To relate both constructions, we follow the construction in chapter 5.1 of [53]. First, it is possible to define a $\overline{\mathbb{R}}$-superalgebra $\mathfrak{R}$ in $\mathrm{Set}^{\mathrm{Gr}}$ by

$$\mathfrak{R}(\Lambda) := \Lambda \qquad\qquad \mathfrak{R}(\varphi) := \varphi$$

which also carries the structure of a finite dimensional supermanifold. Given any other finite dimensional supermanifold X, it is possible to form the morphisms on X with values in $\mathfrak{R}$,

$$SC^\infty(X) := SC^\infty(X, \mathfrak{R})$$

It becomes an $\mathbb{R}$-superalgebra in the sense of definition 2.2 by imbedding $\mathbb{R}$ as the constant morphisms of $SC^\infty(X)$. If $X = \overline{V}$ is a linear supermanifold (i.e. a finite dimensional superdomain represented by the entire super vector space $V = V_{\overline{0}} \oplus V_{\overline{1}}$), it can be shown that $SC^\infty(\overline{V}) \cong C^\infty(V_{\overline{0}}) \otimes_{\mathbb{R}} \bigwedge^\bullet V_{\overline{1}}$ which is precisely the superalgebra of smooth functions used on V in the BKL approach. More general, $\tilde{X} := X(\mathbb{R})$ is a smooth manifold and it can be verified that $\tilde{X} \supset U \mapsto SC^\infty(X|_U)$ is a sheaf of $\mathbb{R}$-superalgebras on $\tilde{X}$, locally isomorphic to $C^\infty(\mathbb{R}^p) \otimes_{\mathbb{R}} \Lambda_q$ (see [53] above theorem 5.1). Thus, X induces a BKL supermanifold $S(X)$ Since a morphism $f : X \longrightarrow Y$ in $SMan$ acts on these sheaves of functions by pullback, $f^* SC^\infty(Y, \mathfrak{R}) \longrightarrow SC^\infty(X, \mathfrak{R})$, it defines a morphism $S(f) : S(X) \longrightarrow S(Y)$. Thus, we obtain a functor $S : fSMan \longrightarrow BKL$ by

$$X \mapsto S(X) := (X(\mathbb{R}), SC^\infty(X)) \qquad\qquad f \mapsto S(f)$$

Theorem 4.23 ([53], 5.1) *The functor $S : \mathrm{fSMan} \longrightarrow \mathrm{BKL}$ is an equivalence of categories.*

Remark 4.24 It should be pointed out that the preceding theorem implies, that in finite dimensions, we can either use the functorial formalism sketched in this section or the BKL approach. The author prefers to use the latter one, e.g. when defining geometric structures on a supermanifold. Nevertheless, it is often useful to think of a supermanifold X as a space defined by its n-points $\mathrm{Hom}_{\mathrm{BKL}}(\mathcal{P}_n, X)$ for $n \in \mathbb{N}$
As mentioned before, the situation is different for the infinite dimensional case. Here, the functorial approach really extends the BKL concept and makes it possible to define a "space of all morphisms" in the next section. Some authors (e.g [10]) prefer to use the entire category BKL for their functorial approach instead of $\mathrm{SPoint} \cong \mathrm{Gr}^{op}$ which has been used by Molotkov and Sachse. On the one hand, this restriction to Gr leads to a much smaller class of objects that have to be taken into account, on the other hand, it is necessary to take care of the $\Lambda_{\overline{0}}$-linearity of the derivatives in all definitions explicitly. In both cases however, the definition of infinite dimensional supermanifolds requires the specification of Banach- or Fréchet-structures on certain functors, because neither SPoint nor BKL contains infinite dimensional structures.

As a result, it is in particular possible to define the total space E of a super vector bundle $\mathcal{E}$ by specifying its n-points. In contrast to example 4.7, the functor V_E is then defined in $\mathrm{Man}^{\mathrm{Gr}}$ and all morphisms are required to be supersmooth. The following example will be used later:

Example 4.25 Let N be a smooth manifold, TN its tangent bundle with corresponding sheaf of sections $\mathcal{T}_N$ and $\Pi(\mathcal{T}_N)$ the parity reversed sheaf. They have rank $dim(N)|0$ and $0|dim(N)$ respectively. Finally let $TN \oplus \Pi(TN)$ denote the bundle obtained from $\mathcal{T}_N \oplus \Pi(\mathcal{T}_N)$ using the functorial construction in example 4.7 which has rank $dim(N)|dim(N)$. Note that this sheaf has odd sections, which are also given by vector fields on N but whose parity is odd by definition. An n-point of $TN \oplus \Pi(TN)$ is an n-point of the base N (i.e. an element $f \in \mathrm{Hom}(\mathcal{P}_n, N) = SC^\infty(\mathcal{P}_n, N)$) and a section σ of the sheaf

$$(\Lambda_n \otimes_f (\mathcal{T}_N \oplus \Pi\mathcal{T}_N))_{\overline{0}} = \Lambda_{n,\overline{0}} \otimes_f (\mathcal{T}_N)_{\overline{0}} \oplus \Lambda_{n,\overline{1}} \otimes_f (\Pi\mathcal{T}_N)_{\overline{1}} \cong \Lambda_{n,\overline{0}} \otimes_f \mathcal{T}_N \oplus \Lambda_{n,\overline{1}} \otimes_f \mathcal{T}_N$$

Note that it becomes obvious here how odd tangent vectors of N arise as sections of $\Lambda_{n,\overline{1}} \otimes_f \mathcal{T}_N$.

Finally, it should be mentioned that it is also possible to relate the Rogers-DeWitt approach (see [9] and [51]) to the functorial one discussed in this section. Since this concept is not used in this work, we will not go into the details but refer to section 5.2 of [53].

4.3 The space of morphisms

One motivation to introduce the functorial approach to supergeometry in this work is given by the need to introduce the “space of all morphisms between two supermanifolds” as already indicated in the introduction to this chapter. Since this space will be infinite dimensional, it has to be constructed as a functor. We will follow the approach in [52], section 7.1.2 - 7.1.4, a similar construction can be found in [61], (p.649 ff). A variant of this approach already appears in Leites' work [39], 3.3.2, who considers families of morphisms instead of functors.

In the category of sets, the set of maps $\mathrm{Hom}_{\mathrm{Set}}(M, N)$ between sets M and N is itself an object of the category Set. We obviously have the relation

$$\mathrm{Hom}_{\mathrm{Set}}(S, \mathrm{Hom}_{\mathrm{Set}}(M, N)) \cong \mathrm{Hom}_{\mathrm{Set}}(S \times M, N) \tag{4.3}$$

which is furthermore natural in S. In other words, the set $\mathrm{Hom}_{\mathrm{Set}}(M, N)$ is a representing object for the functor $S \mapsto \mathrm{Hom}(S \times M, N)$. This situation is analogous to example 4.3, where a similar relation was used to define the inner Hom-object $\underline{\mathrm{Hom}}(V, W)$ for super vector spaces. An adjunction formula like 4.3 can be used to define objects of morphisms between objects in a broad class of categories, see e.g. [30] II.4.23. In particular for M and N smooth, finite dimensional manifolds where M is compact, it is possible to give the structures of infinite dimensional manifolds to spaces of smooth mappings, such that $C^\infty(S \times M, N) = C^\infty(S, C^\infty(M, N))$ holds (see [37], 42.14).

Since SPoint still forms a set of generators for SMan, the space of morphisms between two supermanifolds can be defined as a functor in $\mathrm{Set}^{\mathrm{Gr}}$ in the following way, using the notation of lemma 4.10:

Definition 4.26 ([52], 7.1.3) *Let X and Y be supermanifolds, then the corresponding inner Hom-functor, denoted by $\underline{SC}^\infty(X,Y) : Gr \longrightarrow \mathrm{Set}$, is defined by*

$$\underline{SC}^\infty(X,Y)(\Lambda) := SC^\infty(\mathcal{P}(\Lambda) \times X, Y)$$
$$\underline{SC}^\infty(X,Y)(\varphi) := (f \mapsto f \circ (\mathcal{P}(\varphi) \times id_X))$$

where $\Lambda, \Lambda' \in \mathrm{Gr}$ and $\varphi \in \mathrm{Hom}_{\mathrm{Gr}}(\Lambda, \Lambda')$.

Again, similar concepts also appear in [65] and [39] (if the parameter spaces are taken to be $\mathbb{R}^{0|n}$)

Remark 4.27

(a) For Z another supermanifold, there is a natural way to define a composition $\circ$, which is a natural transformation in $\mathrm{Set}^{\mathrm{Gr}}$ from $\underline{SC}^\infty(X,Y) \times \underline{SC}^\infty(X,Y)$ to $\underline{SC}^\infty(X,Z)$ and defined as follows (see also [52], section 7.1.3):

$$(f \circ_\Lambda g) := (\mathcal{P}(\Lambda) \times X \overset{(id_{\mathcal{P}(\Lambda)}, f)}{\longrightarrow} \mathcal{P}(\Lambda) \times Y \overset{g}{\longrightarrow} Z)$$

where $f \in \underline{SC}^\infty(X,Y)(\Lambda)$ and $g \in \underline{SC}^\infty(Y,Z)(\Lambda)$. It was furthermore shown in [52] (proposition 7.1.4), that $\underline{SC}^\infty(X,X)$ forms a monoid with unit id_X.

(b) Setting $\Lambda = \mathbb{R}$, it is clear that $\underline{SC}^\infty(X,Y)(\mathbb{R}) = SC^\infty(X,Y)$, so that the 0-points of this inner Hom-functor are precisely the morphisms between X and Y in the category of supermanifolds. The n-points for $n > 0$ then contain components which are invisible for $n = 0$ and we will study their structure in the next chapter. In particular, we will see that the "missing" odd vector fields mentioned in the introduction, arise as components of these higher points.

To the author's knowledge, it has not been discussed so far under which conditions the functor $\underline{SC}^\infty(X,Y) \in \mathrm{Set}^{\mathrm{Gr}}$ is representable in SMan for a suitable choice of smoothness in Man. In case that it is not, the functor itself can serve as a generalization for this space. In this work, we will be eventually interested in spaces of solutions of certain differential equations. These will be subfunctors of some $\underline{SC}^\infty(X,Y)$. However, in case of nonlinear equations, these are usually not manifolds. This is nothing special to the super case but already occurs if we look for closed geodesics on a two dimensional flat torus $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. A geodesic starting at some point closes if and only if the slope of the velocity vector in $\mathbb{R}^2$ (i.e. tan(corresponding angle)) is rational. Thus, the set of closed geodesic on T^2 can be identified with a subset of the unit tangent bundle $T^1T^2 = \{X \in TT^2 \mid \|X\| = 1\}$ but it is not a manifold.

Remark 4.28 In the rest of this work, we will use $\underline{SC}^\infty(X,Y)$ as a functor and will not try to give a manifold structure to it. However, this topic should be addressed in future research

because, as discussed in 4.18 (see also example 4.12 b)), the $\Lambda_{\overline{0}}$-linearity is a crucial part of the definition of supersmoothness. Without this property, $\underline{SC}^\infty(X,Y)$ is more a "superset" than a supermanifold.
The component and the subcomponent formalism introduced in section 5.5 and definition 6.20 respectively might provide an ansatz how to equip $\underline{SC}^\infty(X,N)$ with the structure of a Fréchet supermanifold if X is compact and N is a smooth manifold. A n-point $\Phi \in \underline{SC}^\infty(X,N)$ is decomposed into a smooth map $\varphi_\varnothing : \tilde{X} \longrightarrow N$ and a family of vector fields along φ_0. It has been proven in [27], section I.4.1 to I.4.3 that for M,N smooth manifolds and M compact, $C^\infty(M,N)$ is a Fréchet manifold and its tangent bundle is given by all vector fields along smooth maps $M \longrightarrow N$. Hence, it is interesting to study whether there is the possibility to carry this structure over to $\underline{SC}^\infty(X,N)$. Moreover, it would be interesting to investigate whether it is possible to use theorem 5.20 to find a Fréchet structure in more general cases. We will not develop these ideas in this text.

We will close this section with some remarks on vector bundles and their supermanifold of sections. They are only included for the purpose of comparison with some constructions in chapter 6. Therefore, we only give a brief sketch of the ideas.
The notion of a vector bundle can be formulated entirely in the Molotkov-Sachse approach without reference to a locally free sheaf of modules. The details can be found in [52], section 3.8. According to definition 3.8.3 in Sachse's work, a super vector bundle is a functor $E : \mathrm{Gr} \longrightarrow \mathrm{VBun}$ (where VBun is the subcategory of vector bundles in Man) which admits an open covering by trivial super vector bundles. Similar to definition 4.20, it is furthermore required that the changes of trivializations on overlaps are supersmooth and $\overline{\mathbb{R}}$-linear. In particular, this implies that there is supersmooth projection $\pi : E \longrightarrow X$ onto a base supermanifold X. Sections are defined in the usual way:

$$\Gamma(M,E) := \{\sigma : M \longrightarrow E \mid \sigma \text{ supersmooth}, \pi \circ \sigma = id_M\} \subset SC^\infty(M,E) \tag{4.4}$$

Since this set is not sufficient to define a supermanifold of sections, the latter is defined as a functor in $\mathrm{Set}^{\mathrm{Gr}}$ in complete analogy to the definition of $\underline{SC}^\infty(X,Y)$ (see [52], equations (5.3),(5.4)):

$$\underline{\Gamma}(X,E)(\Lambda) := \Gamma(\mathcal{P}(\Lambda) \times X, pr_X^* E) \quad \underline{\Gamma}(X,E)(\varphi : \Lambda \longrightarrow \Lambda') := (\sigma \mapsto \sigma \circ (\mathcal{P}(\varphi) \times id_X))$$

Here, Λ is a Grassmann algebra and pr_X^*E denotes the pullback of E along the projection $\mathrm{pr}_X : \mathcal{P}(\Lambda) \times X \twoheadrightarrow X$ (see [52] section 3.8.1). The functor $\underline{\Gamma}(X,E)$ is actually always representable:

Theorem 4.29 ([52], 5.3.1) *Let $\pi : E \longrightarrow M$ be a super vector bundle and $V := \Gamma(M, E \oplus \Pi E)$. Then V is a super vector space representing $\underline{\Gamma}(X,E)$, i.e. there is an isomorphism*

$$\Gamma(\mathcal{P}(\Lambda) \times X, E) = \underline{\Gamma}(X,E) \cong \overline{V}(\Lambda) = (\Lambda \otimes_{\mathbb{R}} V)_{\overline{0}} \tag{4.5}$$

We will come back to this description of sections in a vector bundle at the begin of chapter 6

5 The fine structure of the space of morphisms

In this chapter, we will determine the geometric structure of the n-points of $\underline{SC}^\infty(X,Y)$ for arbitrary supermanifolds X and Y. This has been done in [52], chapter 7.2.1 and [54], chapter 5.2 for invertible elements of $\underline{SC}^\infty(X,X)$. A similar technique is also discussed in [29]. The approach uses an expansion of elements of $\underline{SC}^\infty(X,Y)(\mathcal{P}_n) \cong \mathrm{Hom}(X \times \mathcal{P}_n, Y)$ w.r.t. the odd parameters in $\mathcal{P}_n$, in other words, the decomposition is done w.r.t. a basis of Λ_n. We will show that the resulting coefficients are super differential operators of suitable degree and parity.
The geometric theory of differential operators on C^∞-manifolds is usually formulated using jet bundles. Here, we only review the very basic constructions at the beginning and refer the reader to [56] or [49] for all the details. Again, it is necessary to work with the rings of functions or sections rather than with bundles itself to generalize these notions to supermanifolds. The general theory is discussed in [36] and we will work out the formalism for supermanifolds and its relation to points of $\underline{SC}^\infty(X,Y)$ in detail. Since differential operators of order ≥ 2 are complicated geometric objects, we will discuss how to decompose them into vector fields in an appropriate way. This methods corresponds to the decomposition into component fields in the physics literature. Finally, we will apply the formalism to the case when the supermanifold is described using a Batchelor bundle.

5.1 Classical formalism of jets and differential operators

Let M be a smooth manifolds and $E \longrightarrow M$ a vector bundle. Let $p \in M$ and (x^i, u^α) bundle coordinates of E near p. Two local sections ω, τ of E near p are called k-equivalent for $k \in \mathbb{N}_0$ if

$$\frac{\partial^{\|I\|}\omega^\alpha}{\partial x^I} = \frac{\partial^{\|I\|}\tau^\alpha}{\partial x^I} \qquad \text{for all } \|I\| \leq k$$

where we used the multi-index notation (cf. remark 3.22 for the definition of $\|\|$). By the chain rule, this is independent of the choice of the bundle coordinates and hence, we have well defined equivalence classes denoted by $jet^k_p(\omega)$. We can then form

$$Jet^k(E) := \{jet^k_p(\omega) \mid p \in M,\ \omega \text{ section near } p\}$$

which obviously projects to M. It can be shown (see [56], 6.2.7) that this defines a smooth bundle over M which becomes a vector bundle by

$$jet^k_p(\omega) + jet^k_p(\tau) := jet^k_p(\omega + \tau) \qquad \lambda jet^k(\omega) := jet^k_p(\lambda\omega)$$

using the vector bundle operations of E. Intuitively speaking, $jet^k_p(\omega)$ not only captures the coordinates of p and the fibre value $\omega(p) \in E|_p$ but also the values of all derivatives up to order k. Clearly, a section ω of E induces a section $p \mapsto jet^k_p(\omega)$ of $Jet^k(E)$, so that we obtain a mapping

$$jet^k : \Gamma(E) \longrightarrow \Gamma(Jet^k(E))$$

However, not every section of $Jet^k(E)$ is obtained in this way because the derivatives of a given section of E obviously depending on lower order derivatives whereas they are independent for a general section of $Jet^k(E)$.
The fact that $Jet^k(E)$ encodes derivatives up to order k allows us to define (linear) differential operators of order $\leq k$ since such an operator is locally given as a $C^\infty(M)$-linear combination of derivatives up to order k. More formally, let $F \longrightarrow M$ be another vector bundle, and $\sigma : Jet^k(E) \longrightarrow F$ a morphism of vector bundles. Then (see [56], 6.2.22)

$$\sigma \circ jet^k : \Gamma(E) \longrightarrow \Gamma(F)$$

defines a differential operator from E to F. For example, on a Riemannian manifold, let $E = F = M \times \mathbb{R}$ so that sections of the bundles are just functions on M. Fixing local coordinates $\{x^i\}$ on M, the coordinates of $\xi := jet^2_p(f)$ are given by $(x^i(\xi) := x^i(p), v(p) := f(p), v_{,i}(\xi) := \frac{\partial f}{\partial x^i}(p), v_{,ij}(\xi) := \frac{\partial^2 f}{\partial x^i \partial x^j}(p))_{i,j=1,2}$. In these coordinates, we can define a morphism

$$\sigma : Jet^2(M \times \mathbb{R}) \longrightarrow M \times \mathbb{R} \qquad (x^i, v_{,i}, v_{,ij})_{i,j=1,2} \mapsto g^{ij} v_{,ij} - g^{ij}\Gamma^k_{ij} v_{,k}$$

Then, σ clearly induces the Riemannian Laplace operator on functions. More general, a linear differential operator of order $\leq k$ is a linear map $D : \Gamma(E) \longrightarrow \Gamma(F)$ which factors over jet^k:

$$\begin{tikzcd} \Gamma(E) \arrow[r, "jet^k"] \arrow[dr, "D"'] & \Gamma(Jet^k(E)) \arrow[d, "\sigma_D"] \\ & \Gamma(F) \end{tikzcd} \tag{5.1}$$

The map σ_D is called the (total) symbol of the differential operator D.

The last paragraph already indicates, how the theory of jets and differential operator should be carried over to super vector bundles: The staring point is the locally free sheaf of sections defining the bundle. It is then possible to give purely algebraic definitions of linear differential operators and jets which remain related to each other by a property similar to that given in diagram (5.1).

5.2 Algebraic theory of super differential operators and -jets

We will now discuss the theory of linear differential operators and jet modules on supermanifolds by reformulating and generalizing the concepts of the preceding chapter in the algebraic language. In contrast to the classical approach, the algebraic approach starts with the definition of the differential operators as the more fundamental concept. General references for the algebraic approach are [36] chapter 1, [47] chapter 9 and [25] chapter 6.2 for supermodules.

In the following, let A be a supercommutative $\mathbb{R}$-superalgebra. In principle, we can allow for arbitrary supercommutative rings of scalars, but for sake of simplicity, we choose $\mathbb{R}$. A will take the role of the ring of functions, so the typical example is $A = \mathcal{O}(U)$ where $\mathcal{O}$ is the sheaf of superfunctions on some supermanifold. P, Q will denote supermodules over A or some other superring which will be indicated if necessary. Here, the typical example will be the module of local sections in a super vector bundle. Furthermore, when speaking of linear maps, we will indicate by a subscript, with respect to which ring of scalars the linearity is to be understood. For example, as defined in chapter 2, $\underline{\mathrm{Hom}}_{\mathbb{R}}(P, Q)$ is the space of $\mathbb{R}$-linear mappings from P to Q which carries a natural $\mathbb{Z}_2$-grading denoted by $|\cdot|$. Following the exposition in [36] chapter 1 §1 for the commutative case, we introduce some notation:

Definition 5.1 *Let $\Delta \in \underline{\mathrm{Hom}}_{\mathbb{R}}(P, Q)$, then we introduce*

(a) a left and a right A-module structure on $\underline{\mathrm{Hom}}_{\mathbb{R}}(P, Q)$ by defining

$$(a \cdot \Delta)(p) := a\Delta(p) \qquad\qquad (\Delta \cdot a)(p) := \Delta(ap)$$

(b) the commutator of the preceding module structures by defining

$$\delta_a\Delta := a \cdot \Delta - (-1)^{|a||\Delta|}\Delta \cdot a$$

The abstract definition of a super differential operator is now as follows:

Definition 5.2 *A map $\Delta \in \underline{\mathrm{Hom}}_{\mathbb{R}}(P, Q)$ is called linear differential operator from P to Q of order $\leq\ k \in \mathbb{N}_0$ and parity $|\Delta| \in \mathbb{Z}_2$ if the following identity is satisfied:*

$$\forall a_0, \dots a_k \in A: \quad \delta_{a_0} \circ \cdots \circ \delta_{a_k}\Delta = 0$$

The set of all differential operators of order $\leq k$ from P to Q is denoted by $\mathrm{Diff}^k(P, Q)$

The following remarks are obvious:

Remark 5.3

(a) $\mathrm{Diff}^k(P, Q)$ inherits from $\underline{\mathrm{Hom}}_{\mathbb{R}}(P, Q)$ a left, a right and an A-bi- supermodule structure. The modules are denoted by $\mathrm{Diff}^k(P, Q)$, $\mathrm{Diff}^k_+(P, Q))$ and $\mathrm{Diff}^k_{(+)}(P, Q)$. We will not mention the module structures explicitly unless there is the danger of confusion.

(b) A differential operator of order zero is just an $\mathbb{R}$-linear map, which supercommutes with A, i.e. which is A-linear. Thus, we have $\mathrm{Diff}^0(P, Q) = \underline{\mathrm{Hom}}_A(P, Q)$.

(c) More general, Δ is a differential operator of order $\leq k$ iff for all $a_0, \dots, a_{k-1}$, the map $\delta_{a_0} \circ \cdots \circ \delta_{a_{k-1}}\Delta$ is A-linear.

We are going to give some examples:

Example 5.4

(a) **Super derivations and first order operators** Let $P = Q = A$ so that we consider differential operators acting on "functions". Let $\Delta \in \mathrm{Diff}^1(A, A)$. We may decompose Δ as follows (see also [55], p.57/58):

$$\Delta = (\Delta - \Delta(1)) + \Delta(1)$$

where $\Delta(1) \in A$ is considered to be a multiplication operator so that we have $\Delta(1) \in \mathrm{Diff}^0(A, A)$. A simple calculation shows:

$$(\Delta - \Delta(1))(ab) = (\Delta - \Delta(1))(a)b + (-1)^{|a||\Delta|}a(\Delta - \Delta(1))(b)$$

Thus, $\Delta - \Delta(1)$ is a superderivation of parity $|\Delta|$ on A. In case $A = \mathcal{O}(U)$, this says that a first order operator can be decomposed into a super vector field and a multiplication operator. This decomposition is in fact a direct sum because an A-linear derivation Δ satisfies $\Delta(1) = 0$ but A-linearity then implies $\Delta(f) = \Delta(1)f = 0$. Thus, we obtain the following exact sequence which splits by the preceding argument:

$$0 \longrightarrow \mathrm{Der}(\mathcal{O}(U)) \longrightarrow \mathrm{Diff}^1(\mathcal{O}(U), \mathcal{O}(U)) \longrightarrow \mathcal{O}(U) \longrightarrow 0$$

where we used $\mathcal{O}(U) \cong \underline{\mathrm{Hom}}_{\mathcal{O}(U)}(\mathcal{O}(U), \mathcal{O}(U)) = \mathrm{Diff}^0(\mathcal{O}(U), \mathcal{O}(U))$. This is the Spencer sequence of degree 1 ([36], (1.1)). A similar decomposition for higher order operators does in general not exist.

(b) **Vector fields along a morphism** Let $\Phi : X = (\tilde{X}, \mathcal{O}) \longrightarrow Y := (\tilde{Y}, \mathcal{R})$ be a morphism of supermanifolds with underlying smooth map $\varphi : \tilde{X} \longrightarrow \tilde{Y}$. Let $P = A = \varphi^*\mathcal{R}(U)$ and $Q = \mathcal{O}(U)_\Phi$ the module obtained by restricting to the ring of scalars to A as defined in 2.21. Then, an element $\Delta \in \mathrm{Diff}^1(\varphi^*\mathcal{R}(U), \mathcal{O}(U)_\Phi)$ without constant term (i.e. $\Delta(1) = 0$) satisfies

$$\Delta(gg') = \Delta(g) \cdot g' + (-1)^{|g||\Delta|}g \cdot \Delta(g') = \Delta(g)\Phi(g') + (-1)^{|g||\Delta|}\Phi(g)\Delta(g')$$

Thus, elements of $\mathrm{Diff}^1(\varphi^*\mathcal{R}(U), \mathcal{O}(U)_\Phi)$ without constant term generalize the classical notion of vector fields along a map. Finally, note that sections in $\mathcal{R}(V)$ canonically induce sections of $\varphi^*\mathcal{R}(\varphi^{-1}(V))$. In this way, elements of $\mathrm{Diff}^1(\varphi^*\mathcal{R}(U), \mathcal{O}(U)_\Phi)$ also act on superfunctions on Y.

The second example offers a nice possibility to define differential operators on a super manifold along a morphism, just by choosing the appropriate module structures (cf. part (b) of the preceding example for the notation):

Definition 5.5 *Let $\Phi : X = (\tilde{X}, \mathcal{O}) \longrightarrow Y = (\tilde{Y}, \mathcal{R})$ be a morphism of supermanifolds with underlying smooth map $\varphi : \tilde{X} \longrightarrow \tilde{Y}$, P a $\mathcal{R}$- module on Y and Q an $\mathcal{O}$-module on X. Then, a linear differential operator from P to Q along Φ of degree $\leq k$ is an element of* $\mathrm{Diff}^k(\varphi^*P(U), Q(U)_\Phi)$.

The following proposition clarifies the structure of differential operators along Φ:

Proposition 5.6 *Let Φ, P and Q satisfy the assumptions of the preceding definition and assume that P is locally free. Then we have for all $U \subset \tilde{X}$:*

$$\mathrm{Diff}^k(\varphi^*P(U), Q(U)_\Phi) \cong Q(U) \otimes_\Phi \mathrm{Diff}^k(\varphi^*P(U), \varphi^*\mathcal{R}(U))$$

Before proving the proposition, we work out its meaning in more detail: We may identify $\Delta \in \mathrm{Diff}^k(\varphi^*\mathcal{R}(U), \mathcal{O}(U)_\Phi)$ with $\sum_i f_i \otimes_\Phi \Delta_i$ for suitable $\Delta_i \in \mathrm{Diff}^k(\varphi^*\mathcal{R}(U), \varphi^*\mathcal{R}(U))$ and $f_i \in \mathcal{O}(U)$. If now $g \in \varphi^*\mathcal{R}(U)$, the action of Δ on g is given by (cf. Proposition 3.33)

$$\Delta(g) = \textstyle\sum_i f_i \otimes_\Phi \Delta_i(g) = f_i \Phi(\Delta_i(g))$$

This means that a differential operator along Φ is essentially given by an $\mathcal{O}$-linear combination of differential operators on Y of the same order composed with the morphism Φ. Again, this coincides with the well known decomposition of a vector field ξ along a smooth map $\varphi : M \longrightarrow N$ on ordinary manifolds, used in the proof of proposition 3.26.

Proof of proposition 5.6 We will see in 5.15 that both sides are locally free sheaves. Thus, it is sufficient to prove the statement for open sets of the form $U = \varphi^{-1}(V)$ such that $P(V)$ is a free module, since these sets cover $\tilde{X}$.
First, there is an isomorphism of left $\varphi^*\mathcal{R}(U)$-supermodules $\varphi^*\mathcal{R}(U) \otimes_\Phi Q(U) \cong Q(U)_\Phi$ given by $r \otimes q \mapsto \Phi(r)q$ whose inverse is $q \mapsto 1 \otimes q$. Using this and proposition 5.22 (b)[9], we have

$$\begin{aligned}
\mathrm{Diff}^k(\varphi^* P(U), Q(U)_\Phi) &\cong \mathrm{Diff}^k(\varphi^* P(U), \varphi^*\mathcal{R}(U) \otimes_\Phi Q(U)) \\
&\cong \underline{\mathrm{Hom}}_{\varphi^*\mathcal{R}}(Jet^k \varphi^* P(U), \varphi^*\mathcal{R}(U) \otimes_\Phi Q(U))
\end{aligned}$$

Now, $\varphi^* P(U)$ is a free module by lemma 3.24 and hence, we have (see [6] II.4.2.2 (ii))

$$\begin{aligned}
\underline{\mathrm{Hom}}_{\varphi^*\mathcal{R}}(Jet^k \varphi^* P(U), \varphi^*\mathcal{R}(U) \otimes_\Phi Q(U)) &\cong \underline{\mathrm{Hom}}_{\varphi^*\mathcal{R}}(Jet^k \varphi^* P(U), \varphi^*\mathcal{R}(U)) \otimes_\Phi Q(U) \\
&\cong \mathrm{Diff}^k(\varphi^* P(U), \varphi^*\mathcal{R}(U)) \otimes_\Phi Q(U)
\end{aligned}$$

In the last step, we again used proposition 5.22. The statement now follows from the supercommutativity of the super tensor product.

□

In the following, we fix some notations for multi-indices and permutations, which do not coincide with the common conventions. They will be used to discuss some product rules:

For $n \in \mathbb{N}$, we use the abbreviation $\underline{n}$ to denote the tuple $(1, \dots, n)$ as well as the set $\{1, \dots, n\}$. As usual, a multi-index is an element $I = (i_1, \dots, i_l) \in \bigcup_{k \in \mathbb{N}_0} \mathbb{N}_0^k$ where we set $\mathbb{N}_0^0 := \{\varnothing\}$. The number $l \in \mathbb{N}_0$ s.t. $I \in \mathbb{N}_0^l$ is also called the length of I. A graded multi-index of type $(p|q)$ is a multi-index I of length $l = p + q$ together with a partition $\underline{l} = \{1, \dots, p+q\} = \mathrm{ev}(I) \sqcup \mathrm{odd}(I)$. Writing $I = (i_1, \dots, i_{p+q})$, an index i_k is called even if $k \in \mathrm{ev}(I)$ and odd if $k \in \mathrm{odd}(I)$. Note that this definition simply assigns a parity to each of the i_k, it does not take into account whether i_k is even or odd as an integer. For a graded multi-index I of type $(p|q)$ as above, we will abbreviate

$$\begin{aligned}
\|I\| &= \text{length of } I = p + q \\
|I| &= \text{number of odd indices in } I = \#\mathrm{odd}(I) = q
\end{aligned} \tag{5.2}$$

$|I| \mod 2\mathbb{Z}$ is called the parity of the index.

[9] Although proposition 5.22 appears later in this work, it is clearly independent of the statement which is proven here.

A subindex J of I is a multi-index of length $\|J\| \leq \|I\|$, together with a strictly monotonic increasing map $\iota_J : \underline{\|J\|} = \{1, \ldots, \|J\|\} \longrightarrow \underline{\|I\|} = \{1, \ldots, \|I\|\}$ such that $j_k = i_{\iota_J(k)}$. This means that each element of J has to appear as an element of I in the order prescribed by J. We will usually omit ι_J and just write $J \subset I$. Finally, a graded subindex J of a graded multi-index I is a subindex, such that the parity of an element of J coincides with the parity of its image in I, i.e. $\iota_J(\mathrm{ev}(J)) \subset \mathrm{ev}(I)$ and $\iota_J(\mathrm{odd}(J)) \subset \mathrm{odd}(I)$.
Given an index I of type (p, q) and a subindex $J \subset I$ of type (p', q'), we define the complementary index of J in I as $J^c := I \setminus J = (i_k \mid k \notin \iota_J(\underline{\|J\|}))$ which is the index obtained from I by removing all the elements of J. Clearly, J^c inherits from I the structure of a graded multi-index of type $(p - p', q - q')$ and is a graded subindex of I.
Now, given a graded index I of type (p, q), any element $\sigma \in S_{p+q}$ defines a new index of the same type by permuting the entries. For any graded subindex $J \subset I$, we may form the new index (J, J^c) which has the type of I. Introducing the permutations

$$\sigma(J, J^c) := \begin{pmatrix} 1 \cdots p+q \\ \iota_J(\underline{\|J\|}) \ , \ \underline{p+q} \setminus \iota_J(\underline{\|J\|}) \end{pmatrix}^{-1} \qquad \sigma^{\mathrm{odd}}(J, J^c) := \begin{pmatrix} odd(I) \\ odd(J) odd(J^c) \end{pmatrix}^{-1},$$

it is clear that $\sigma(J, J^c)$ transforms (J, J^c) back to I. Its sign will be denoted by $\mathrm{sign}(J, J^c)$. Finally, $\mathrm{osign}(J, J^c) := \mathrm{sign}(\sigma^{\mathrm{odd}}(J, J^c))$ only counts transpositions in $\sigma(J, J^c)$ which correspond to permuting odd elements of I.
Let I be a graded multi-index of type (p, q) and let $a_{i_1}, \ldots, a_{i_{p+q}}$ be pure elements of A such that $|a_{i_k}| = 1$ if and only if $k \in odd(I)$. We then write

$$a_I := a_{i_1} \cdots a_{i_{p+q}} \qquad\qquad \delta_I := \delta_{a_{i_1}} \circ \cdots \circ \delta_{a_{i_{p+q}}} \tag{5.3}$$

We clearly have $|a_I| = |I|$ and as usual, these notations are extended to non-homogeneous elements by multilinearity. Note that a_I is not the common mult-index monomial!

Using these notations, we have the following rules (see [47] 9.58 for smooth manifolds):

Lemma 5.7 *Let I be a graded multi-index, $a, b, a_i \in A$ and $\Delta' \in \underline{\mathrm{Hom}}_{\mathbb{R}}(P, Q)$, $\Delta \in \underline{\mathrm{Hom}}_{\mathbb{R}}(Q, R)$. Then we have*

(a) $\quad \delta_a \circ \delta_b = (-1)^{|a||b|} \delta_b \circ \delta_a$

(b) $\quad \delta_I(\Delta \circ \Delta') = \sum_{J \subset I} (-1)^{|\Delta||a^{J^c}|} (-1)^{\mathrm{osign}(J, J^c)} \delta_J(\Delta) \circ \delta_{J^c}(\Delta')$

(c) $\quad \delta_I(\Delta)(b) = (-1)^{\|I\|} \sum_{J \subset I} (-1)^{\|J\|} (-1)^{|\Delta||a^{J^c}|} (-1)^{\mathrm{osign}(J, J^c)} a^J \Delta(a^{J^c} b)$

Proof The first statement is clear. Part (b) and (c) are proven by induction on $\|I\|$. We start with (b). For $\|I\| = 0$, there is nothing to show and the case $\|I\| = 1$ is essentially the definition of δ. For the induction step, we write $I = (i, I')$, fix some $a_i, a_{i_0}, \ldots, a_{i_k}$ and by using the induction hypothesis and the already proven case for $I = \{i\}$, we obtain:

$$\begin{aligned}
\delta_I(\Delta \circ \Delta') &= \delta_{a_i} \sum_{J' \subset I'} (-1)^{|\Delta||J'^c|}(-1)^{\mathrm{osign}(J',J'^c)}\delta_{J'}(\Delta) \circ \delta_{J'^c}(\Delta') \\
&= \sum_{J' \subset I'} (-1)^{|\Delta||J'^c|}(-1)^{\mathrm{osign}(J',J'^c)}\delta_{a_i} \circ \delta_{J'}(\Delta) \circ \delta_{J'^c}(\Delta') \\
&+ \sum_{J' \subset I'} (-1)^{|\Delta||J'^c|}(-1)^{(|J'|+|\Delta|)|a_i|}(-1)^{\mathrm{osign}(J',J'^c)}\delta_{J'}(\Delta) \circ \delta_{a_i} \circ \delta_{J'^c}(\Delta')
\end{aligned} \tag{5.4}$$

Now, we reorganize the indices in the two summands as follows:

- *1.summand:* We set $J = (i, J')$ which implies $J^c = J'^c$ and

$$\begin{aligned}
|J| = |J'| + |a_i|, \quad |J^c| = |J'^c|, &\quad \|J\| = \|J'\| + 1 \\
(-1)^{|\Delta||J^c|} &= (-1)^{|\Delta||J'^c|} \\
(-1)^{\mathrm{osign}(J,J^c)} &= (-1)^{\mathrm{osign}(J',J'^c)}
\end{aligned}$$

- *2.summand:* We set $J = J'$ which implies $J^c = (i, J'^c)$ and

$$\begin{aligned}
|J| = |J'|, \quad |J^c| = |J'^c| + |a_i|, &\quad \|J\| = \|J'\| \\
(-1)^{|\Delta||J^c|} &= (-1)^{|\Delta||a_i|+|\Delta||J'^c|} \\
(-1)^{\mathrm{osign}(J,J^c)} &= (-1)^{|J'||a_i|}(-1)^{\mathrm{osign}(J',J'^c)}
\end{aligned}$$

Hence, the first sum becomes the sum over all $J \subset I$ containing the first index i whereas the second one runs over all subindices not containing i. Substituting into (5.4), we find

$$\delta_I(\Delta \circ \Delta') = \sum_{J \subset I} (-1)^{|\Delta||J^c|}(-1)^{\mathrm{osign}(J,J^c)}\delta_J(\Delta) \circ \delta_{J^c}(\delta')$$

which proves b).
c) is proven similarly. Again, the cases $\|I\| = 0, 1$ are trivial or follow directly from the definition. For the induction, we proceed as above, so let $I = (i, I')$. We get, using the induction hypothesis and the definition of δ_{a_i}

$$\begin{aligned}
(\delta_I \Delta)(b) &= (\delta_{a_i}(-1)^{\|I'\|} \sum_{J' \subset I'} (-1)^{\|J'\|}(-1)^{|\Delta||J'^c|}(-1)^{\mathrm{osign}(J',J'^c)} a^{J'} \Delta(a^{J'^c}))(b) \\
&= (-1)^{\|I'\|} \sum_{J' \subset I'} (-1)^{\|J'\|}(-1)^{|\Delta||J'^c|}(-1)^{\mathrm{osign}(J',J'^c)} a_i a^{J'} \Delta(a^{J'^c} b) \\
&- (-1)^{|a_i|(|I'|+|\Delta|)}(-1)^{\|I'\|} \sum_{J' \subset I'} (-1)^{\|J'\|}(-1)^{|\Delta||J'^c|}(-1)^{\mathrm{osign}(J',J'^c)} a^{J'} \Delta(a^{J'^c} a_i b)
\end{aligned}$$

Now, we can treat the two summands as in the proof of b). Inserting the different expressions for the signs in the last equation, we finally arrive at

$$\delta_I(\Delta)(b) = (-1)^{\|I\|} \sum_{J \subset I} (-1)^{\|J\|}(-1)^{|\Delta||a^{J^c}|}(-1)^{\mathrm{osign}(J,J^c)} a^J \Delta(a^{J^c} b)$$

□

The last identity leads to a useful equivalent characterization of the property of being a differential operator of order $\leq s$:

Corollary 5.8 *Let $\Delta \in \underline{\mathrm{Hom}}_{\mathbb{R}}(P, Q)$ and $s \in \mathbb{N}$, then Δ is a differential operator of order $\leq s$ if an only if for all $a_0, \ldots, a_s \in A$ and $p \in P$*

$$\Delta(a_0 \cdots a_s p) = -\sum_{\varnothing \neq J \subset \underline{s+1}} (-1)^{\|J\|}(-1)^{|\Delta||a^J|}(-1)^{\mathrm{osign}(J,J^c)} a^J \Delta(a^{J^c} p)$$

If $P = A$, i.e. the differential operator acts on "functions", we can say more. In the following statement, the number of factors appearing in the product rule is reduced by one in comparison with the general statement of the last corollary, where we have $s+1$ elements of A and an element $p \in P$.

Corollary 5.9 *$\Delta \in \underline{\mathrm{Hom}}_{\mathbb{R}}(A, Q)$ is a differential operator of order $\leq s$ if and only if for all $a_1, \ldots, a_s, a_{s+1}$, we have*

$$\sum_{J \subset \underline{s+1}} (-1)^{\|J\|}(-1)^{|\Delta||a^{J^c}|}(-1)^{\mathrm{osign}(J,J^c)} a^J \Delta(a^{J^c}) = 0$$

This equation is equivalent to the following product formula:

$$\Delta(a_1 \cdots a_{s+1}) = -\sum_{\varnothing \neq J \subset \underline{s+1}} (-1)^{\|J\|}(-1)^{|\Delta||a^J|}(-1)^{\mathrm{osign}(J,J^c)} a^J \Delta(a^{J^c}) \tag{5.5}$$

Proof By remark 5.3 (c), Δ is a differential operator if and only if $\delta_{a_1} \cdots \delta_{a_s} \Delta$ is A-superlinear, i.e. for all $a_1, \ldots, a_s, a_{s+1}$, we have:

$$(\delta_{a_1} \cdots \delta_{a_s} \Delta)(a_{s+1}) - (\delta_{a_1} \cdots \delta_{a_s} \Delta)(1) a_{s+1} = 0$$

Rearranging the sum with the techniques used in the proof of lemma 5.7 then gives the result. □

Composing differential operators of order k and l, we expect that the result is again a differential operator of order $\leq k+l$. In fact, applying δ_I with $\|I\| = k+l+1$ to their composition and using part (b) of lemma 5.7, we see that one of the two factors $\delta_J(\Delta)$ and $\delta_{J^c}(\Delta')$ must vanish since either $\|J\| \geq k+1$ or $\|J^c\| \geq l+1$. Thus, we have

Corollary 5.10 *Let $\Delta \in \mathrm{Diff}^k(P, Q)$ and $\Delta' \in \mathrm{Diff}^l(Q, R)$, then $\Delta' \circ \Delta \in \mathrm{Diff}^{k+l}(P, Q)$ and $|\Delta' \circ \Delta| = |\Delta'| + |\Delta|$.*

Next, we will show that this concept of differential operators coincides with the more common approach using local coordinates. Parts of the results have already been stated in [35], chapter 8, without proofs. We will only sketch parts of the proof here, generally following the strategy which is used in chapter 4.5 of [8] to show that $Der_{\mathcal{O}}$ forms a locally free sheaf of $\mathcal{O}$-modules. We will restrict ourselves to the case $\mathrm{Diff}^k(\mathcal{O}, \mathcal{O})$ on a supermanifold $(\tilde{X}, \mathcal{O})$, the result can

be generalized in the obvious way to locally free $\mathcal{O}$-modules.
Let $U \subset \tilde{X}$ be a coordinate neighbourhood with coordinates $(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q) =: (\xi^i)$. Denote the corresponding coordinate vector fields by $\{\partial/\partial\xi^i\}$. Using the characterization given in corollary 5.9, repeated application of the Leibniz rule for these vector fields establishes the following statement:

Lemma 5.11 *Let $I = (I_{\bar{0}}, I_{\bar{1}})$ be a graded multi-index where $I_{\bar{0}} = (i_1, \ldots, i_p) \in \mathbb{N}_0^p$ and $I_{\bar{1}} = (\alpha_1, \ldots, \alpha_q) \in \{0,1\}^q$. Then each expression of the form*

$$\frac{\partial^{\|I\|}}{\partial\xi^I} := \frac{\partial^{i_1}}{\partial(x^1)^{i_1}} \circ \cdots \circ \frac{\partial^{\alpha_q}}{\partial(\theta^q)^{\alpha_q}}$$

is an element of $\mathrm{Diff}^{\|I\|}(\mathcal{O}(U), \mathcal{O}(U))$ *whose parity is given by* $\|I_{\bar{1}}\| \mod 2$.

Note that at this point, we use multi-indices in the familiar way (e.g. known from Taylor polynomials). Let $K = (K_{\bar{0}}, K_{\bar{1}}) \in \mathbb{N}_0^p \times \{0,1\}^q$ be another graded multi-index and ξ^K the corresponding monomial defined by the coordinate functions. It is easy to see that we have

$$\frac{\partial^{\|I\|}\xi^K}{\partial\xi^I} = \begin{cases} \frac{K!}{I!}(-1)^{\lfloor\|I_{\bar{1}}\|/2\rfloor}\mathrm{osign}(I, I^c)\xi^{I^c} & \text{for } I \subset K \\ 0 & \text{for } I \subsetneq K \end{cases}$$

where the complementary subindex I^c is defined w.r.t. K and $K! = k_1! \cdots k_{p+q}!$ etc. Note that for $\|I\| = \|K\|$, the expression vanishes unless $K = I$. If superfunctions $f_I \in \mathcal{O}(U)$ are chosen for each $\|I\| \leq s$, it follows from lemma 5.11 that

$$\Delta := \sum_{\|I\| \leq s} f^I \frac{\partial}{\partial\xi^I} \tag{5.6}$$

defines a differential operator of order $\leq s$ which in general is not of pure parity. Applying it to the monomials $1, \xi^i, \ldots, \xi^K, \ldots \in \mathcal{O}(U)$, we obtain a recursive procedure for the determination of the coefficients f_I:

$$\begin{aligned} f^{\varnothing} &= \Delta(1) \\ f^i &= \Delta(\xi^i) - f^{\varnothing}\xi^i = \Delta(\xi^i) - \Delta(1)\xi^i \\ &\;\vdots \\ f^K &= (-1)^{\lfloor\|K_{\bar{1}}\|/2\rfloor}\Big(\Delta(\xi^K) - \sum_{\|I\|<\|K\|} f_I \frac{\partial^{\|K\|}\xi^K}{\partial\xi^I}\Big) \\ &\;\vdots \end{aligned} \tag{5.7}$$

Thus, by induction on $\|K\|$, the coefficients f_I are uniquely determined by Δ if the sum in (5.6) runs over ordered multi-indices $I = (I_{\bar{0}}, I_{\bar{1}})$ as used in lemma 5.11. In fact, a representation of the form (5.6) does always exist so we obtain

Proposition 5.12 ([35], 2.9.1) *The $\mathcal{O}(U)$ module $\mathrm{Diff}^s(\mathcal{O}(U),\mathcal{O}(U))$ is free, a basis is given by* $\{ \frac{\partial^{\|I\|}}{\partial \xi^I} \mid I \in \mathbb{N}_0^p \times \{0,1\}^q$ *a multi-index s.t.* $\|I\| \leq s\}$

Proof Uniqueness of the decomposition was already discussed. For the existence, we follow the strategy used in the proof of proposition 4.38 in [8]. Given $\Delta \in \mathrm{Diff}^s(\mathcal{O}(U),\mathcal{O}(U))$, we define its coefficients by setting $\Delta^{\varnothing} = \delta(1)$ and defining Δ^I for $\|I\| \geq 1$ recursively using (5.7). The difference $D := \Delta - \sum_{\|I\|\leq s} \Delta^I \frac{\partial}{\partial \xi^I}$ is then a differential operator of order $\leq s$ and we have to show $D = 0$.

We claim that D vanishes on all monomials ξ^K for $\|K\| \leq s$. Obviously, $D(1) = \Delta(1) - \Delta^{\varnothing} = 0$. For $0 < \|K\| \leq s$, we find using the definition (5.7)

$$\begin{aligned} D(\xi^K) &= \Delta(\xi^K) - \Delta^I \frac{\partial}{\partial \xi^I} - \sum_{\|I\|<\|K\|} \Delta^I \frac{\partial^{\|I\|}\xi^K}{\partial \xi^I} \\ &= \Delta(\xi^K) - (-1)^{\lfloor \|K_{\bar{1}}\|/2 \rfloor}\Big(\Delta(\xi^K) - \sum_{\|I\|<\|K\|} f_I \frac{\partial^{\|K\|}\xi^K}{\partial \xi^I}\Big)(-1)^{\lfloor \|K_{\bar{1}}\|/2 \rfloor} - \sum_{\|I\|<s} \Delta^I \frac{\partial^{\|I\|}\xi^K}{\partial \xi^I} \\ &= \; 0 \end{aligned}$$

By corollary 5.8, this property extends to monomials of order $> s$ so that by linearity, D vanishes on all polynomials of the coordinate functions.

Finally, let $f \in \mathcal{O}(U)$ be an arbitrary superfunction and $u \in U$. We can choose a polynomial p (of degree $\leq q+s$) s.t. $f - p \in I_u(U)^{q+s+1}$ (see [8], lemma 4.13 b)). By the product rule 5.9 and $D \in \mathrm{Diff}^s(\mathcal{O}(U),\mathcal{O}(U))$, we clearly have $D(I_u(U)^{q+s+1}) \subset I_u(U)^{q+1}$. $D(p) = 0$ then implies $D(f) = D(f-p) \in I_u(U)^{q+1}$ and since this is valid for all $u \in U$, we have $D(f) = 0$ by lemma 3.8.

□

We are now going to discuss the global structure of $\mathrm{Diff}^s(\mathcal{O},\mathcal{O})$ by proving that the assignment

$$U \mapsto \mathrm{Diff}^s(\mathcal{O}(U),\mathcal{O}(U))$$

defines a sheaf of $\mathcal{O}(U)$-modules on X. One of the most important properties of differential operators is that of locality:

Definition 5.13 *Let $U \subset X$ be an open set and $A \in \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{O}(U),\mathcal{O}(U))$. Then A is a local operator iff for all $V \subset U$ open and $f \in \mathcal{O}(U)$, we have:* $\rho^U_V(f) = 0 \implies \rho^U_V(Af) = 0$

In particular, a local operator A induces maps $A_x : \mathcal{O}_x \longrightarrow \mathcal{O}_x$ of the stalks $\mathcal{O}_x$ at $x \in U$, given by $[f]_x \mapsto [Af]_x$ where f can always be chosen to be an element of $\mathcal{O}(U)$. As in the case of differential operators on smooth manifolds, we have

Proposition 5.14 *A differential operator $\Delta \in \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{O}(U),\mathcal{O}(U))$ is local.*

Proof Sketch We will follow the proof of [8], lemma 4.34. Let $V \subset U$ and $f \in \mathcal{O}(U)$ s.t. $\rho^U_V(f) = 0$. Given $x \in V$, we can choose (see corollary 3.12) open neighbourhoods $V_x \subset W_x \subset V$ of x and a cutoff function $\varphi_x \in \mathcal{O}(U)$ satisfying $\overline{W_x} \subset V$, $\rho^U_{V_x}(\varphi_x) = 0$ and

$\rho^U_{U\setminus\overline{W_x}}(\varphi_x) = 1$. We then also have $\varphi_x f = f$ and the product rule (5.5) applied to $\varphi_x f$ implies $\rho^U_{V_x}(\Delta(f)) = 0$. This is true for every $x \in V$ and the result follows from the sheaf properties of $\mathcal{O}$.

□

Consider now the sheaf $\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{O}, \mathcal{O})$ which is defined by $\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{O}, \mathcal{O})(U) = \underline{\mathrm{Hom}}(\mathcal{O}|_U, \mathcal{O}_U)$. Here, the right hand side is the set of morphisms of sheaves between the restricted sheaves $\mathcal{O}|_U$. It is well known that an element of $A \in \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{O}(U), \mathcal{O}(U))$ is induced by an element of $\underline{\mathrm{Hom}}(\mathcal{O}|_U, \mathcal{O}_U)$ iff A is local[10]. This can be seen most easily in the sheaf space picture: By locality we already have a map on stalks and it is straight forward to verify, that it is also open and hence continuous (see also [62], 4.12). It is clear that the maps Δ_x induced by a differential operator Δ (of order $\leq k$) on stalks are also differential operators (of order $\leq k$). Thus, a morphism glued from local data which are differential operators (of order $\leq k$) are differential operators (of order $\leq k$) again and we obtain that $\mathrm{Diff}^k(\mathcal{O}, \mathcal{O})$ is a subsheaf of $\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{O}, \mathcal{O})$. In fact, an analogue of Peetre's theorem should hold so that the sheaves are equal but we will not discuss it. Taking into account proposition 5.12, we eventually conclude:

Theorem 5.15 $\mathrm{Diff}^s(\mathcal{O}, \mathcal{O})$ *is a locally free sheaf of* $\mathcal{O}$*-modules.*

Using proposition 5.6, the preceding results can be carried over immediately to differential operators along morphisms:

Corollary 5.16 *Let* $\Phi : X \longrightarrow Y$ *be a morphism of supermanifolds. Then the differential operators of order* $\leq s$ *along* Φ *form a locally free sheaf.*

5.3 The structure of higher points

In this chapter, we will determine the structure of the higher points of $\underline{SC}^\infty(X, Y)$ where $X = (\tilde{X}, \mathcal{O})$ and $Y = (\tilde{Y}, \mathcal{R})$ are two supermanifolds. Let $\Phi^{(n)} \in \underline{SC}^\infty(X, Y)(\mathcal{P}_n)$, so by definition $\Phi^{(n)} : \varphi^*\mathcal{R} \longrightarrow \bigwedge_n \otimes \mathcal{O}$ where $\varphi : \tilde{X} \longrightarrow \tilde{Y}$ is the underlying smooth map. We will write $\Phi^{(n)}$ instead of $\Phi^{(n)*}$ and omit the reference to open sets. Recall that $\Phi^{(n)}$ is even by definition. Denoting the generators of Λ_n by $\eta_1, \ldots, \eta_n$ and choosing $g \in \varphi^*\mathcal{R}$, we can decompose $\Phi^{(n)}$ as (cf. (5.3))

$$\Phi^{(n)}(g) = \sum_{I \subset \underline{n}} \eta_I \Phi^{(n)}_I(g) \qquad \text{where for } I \subset \underline{n} : \quad \Phi^{(n)}_I : \varphi^*\mathcal{R} \longrightarrow \mathcal{O} \tag{5.8}$$

where $\underline{n}$ (and hence also I) are considered to be purely odd multi-indices in accordance with the fact, that all η^i are odd. Since $\Phi^{(n)}$ is an even object, we obviously have

$$|\Phi^{(n)}_I| = |I| \qquad\qquad |\Phi^{(n)}_I(g)| = |I| + |g|$$

[10] In general, the assignment $U \mapsto \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{O}(U), \mathcal{O}(U))$ can not be equipped with the structure of a sheaf of $\mathbb{R}$-vector spaces. If this were a sheaf and U_1, U_2 two disjoint, non empty open sets, then we would have $\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{O}(U_1 \sqcup U_2), \mathcal{O}(U_1 \sqcup U_2)) \cong \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{O}(U_1), \mathcal{O}(U_1)) \oplus \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{O}(U_2), \mathcal{O}(U_2))$. This is in general not true.

By definition, $\Phi^{(n)}$ is a unital homomorphism of $\mathbb{R}$-superalgebras and it is this property which determines the structure of the coefficients $\Phi_I^{(n)}$. From this and equation (5.8), we immediately obtain

Lemma 5.17

(a) $\Phi_\varnothing^{(n)}$ *is a morphism of superalgebras* $\varphi^*\mathcal{R} \longrightarrow \mathcal{O}$, *in other words, we have a morphism of supermanifolds* $\Phi_\varnothing^{(n)} : X \longrightarrow Y$.

(b) For $\|I\| > 0$, *we have* $\Phi_I^{(n)}(1) = 0$.

Hence, the structure of $\Phi_\varnothing^{(n)}$ is completely understood and it remains do discuss $\Phi_I^{(n)}$ for $\|I\| > 0$ in greater detail. To formulate the combinatorics, we will have to deal systematically with decompositions of multi-indices and to this end, we introduce the following notation: Let $I \subset \underline{n}$ be a multi-index and $k \in \mathbb{N}$, then we write

$$I = I_1 \sqcup \cdots \sqcup I_k$$

if I is the union of pairwise disjoint subindices $I_j \subset I$ **which are allowed to be empty**. These partitions are ordered, that is, two partitions which only differ in the order of the I_j are considered to be different. The partition itself is also denoted by $(I_1, \dots, I_k)$. Next, we say that two partitions $I_1 \sqcup \cdots \sqcup I_k$ and $\tilde{I}_1 \sqcup \cdots \sqcup \tilde{I}_l$ of I are equivalent (or essentially equal), denoted by $(I_1, \dots, I_k) \sim (\tilde{I}_1, \dots, \tilde{I}_l)$, if the nonempty subindices on both sides are the same and appear in the same order. We can finally introduce the following notations:

$$\begin{aligned}
\mathcal{D}_n(I) &:= \{(I_1, \dots, I_k) | I = I_1 \sqcup \cdots \sqcup I_k,\ k \le n\} \\
\mathcal{D}_n^{ess}(I) &:= \mathcal{D}_n(I)/\sim \\
\mathrm{L}(I_1, \dots, I_k) &:= k \\
\mathrm{L}^{ess}((I_1, \dots, I_k)) &:= \#\{I_j | I_j \neq \varnothing\} \\
\mathrm{L}^{\pm}((I_1, \dots, I_k)) &:= \#\{I_j |\ \|I_j\| \text{ is even (odd)}\}
\end{aligned}$$

We say that for $D \in \mathcal{D}_n(I)$, $\mathrm{L}^{\mathrm{ess}}(D)$ is the essential length of D since only the number of nonempty subindices is counted. Hence, this notion is also well defined for classes in $\mathcal{D}_n^{ess}(I)$. Also note that $\mathrm{sign}(I_1, \dots, I_k)$, which was introduced before, only depends on the equivalence class in $D_n^{ess}(I)$ and not on the partition itself, too. We have the following easy combinatorial lemma:

Lemma 5.18 *Let* $(I_1, \dots, I_k) = D \in \mathcal{D}_n(I)$ *and* $1 \le \alpha, \beta \le k$. *Then*

$$\prod_{\alpha<\beta} (-1)^{\|I_\alpha\|\|I_\beta\|} = (-1)^{\binom{L^-(D)}{2}}$$

Proof There is a contribution of -1 to the product, if both, $\|I_\alpha\|$ and $\|I_\beta\|$, are odd numbers. Hence the number of -1 occurring in the product is equal to the number of (unordered) pairs of subindices which have an odd length, which in turn is given by $\binom{L^-(I)}{2}$. □

We can now start the discussion of the properties of the components $\Phi_I^{(n)}$:

Lemma 5.19 *Let $I \subset \underline{n}$ be a multi-index and $g_1, \dots, g_n \in \varphi^*\mathbb{R}$. For $\Phi^{(n)}, \Phi_I^{(n)}$ as before, we have*

$$\Phi_I(g_1 \cdots g_n) = \sum_{I_1 \sqcup \cdots \sqcup I_n = I} \operatorname{sign}(I_1, \dots, I_n) \prod_{1 \leq \alpha < \beta \leq n} (-1)^{|I_\beta|(|I_\alpha| + |g_\alpha|)} \prod_{k=1\dots n} \Phi_{I_k}^{(n)}(g_k)$$

Proof By definition, we have $\Phi^{(n)}(g_1 \cdots g_n) = \Phi^{(n)}(g_1) \cdots \Phi^{(n)}(g_n)$. Using the expansion from (5.8), we obtain

$$\sum_{I \subset \underline{n}} \eta_I \Phi_I(g_1 \cdots g_n) = \Big(\sum_{I_1 \subset \underline{n}} \eta_{I_1} \Phi_{I_1}(g_1)\Big)\Big(\sum_{I_2 \subset \underline{n}} \eta_{I_2} \Phi_{I_2}(g_2)\Big) \cdots \Big(\sum_{I_n \subset \underline{n}} \eta_{I_n} \Phi_{I_n}(g_n)\Big)$$

We now rearrange the product on the right side and collect all the terms contributing to the η_I-summand. Clearly, these are precisely given by all the decompositions $I = I_1 \sqcup \cdots \sqcup I_n$. Furthermore, each summand has to be rearranged in the way, that all η are to the left of the Φ-factors. In other words, η^{I_β} has to be interchanged with all $\Phi_{I_\alpha}^{(n)}(g_\alpha)$ for $\alpha < \beta \in \underline{n}$, yielding a factor $\prod_{\alpha<\beta}(-1)^{|I_\beta|(|I_\alpha|+|g_\alpha|)}$ for each summand. Finally, each product of ηs is rearranged as $\eta^{I_1} \cdots \eta^{I_n} = \eta^I \operatorname{sign}(I_1, \dots, I_n)$. Collecting all the summands and signs proves the lemma.

□

Basing on these product-type rules, we can now prove, that all Φ_I are in fact differential operators:

Theorem 5.20 *Let $\Phi^{(n)}$ be a n-point of $\underline{SC}^\infty(X, Y)$ and let $\Phi_I^{(n)}$ ($\|I\| > 0$) be one of the coefficients of the expansion* (5.8)*. Then $\Phi_I^{(n)} \in \underline{\mathrm{Hom}}_{\mathbb{R}}(\varphi^*\mathcal{R}, \mathcal{O}_{\Phi_\varnothing^{(n)}})$ is a linear super differential operator of degree $\leq \|I\|$ and of parity $\|I\| \mod 2$ along the morphism $\Phi_\varnothing^n$. In particular, $\Phi_I^{(n)}$ are sections of a super vector bundle on X.*

Proof It was already observed in lemma 5.17 that $\Phi_\varnothing^{(n)}$ is in fact a morphism and it is also obvious from the definition, that all $\Phi_I^{(n)}$ are $\mathbb{R}$-linear. It remains to verify the property of being differential operator along $\Phi_\varnothing^{(n)}$. We use the characterization of corollary 5.9, i.e for $g_1, \dots, g_{\|I\|+1} \in \varphi^*\mathcal{R}$, we have to show

$$\sum_{J \subsetneq \underline{\|I\|+1}} (-1)^{\|J\|}(-1)^{|I||J^c|} \mathrm{osign}(J, J^c) \Phi_\varnothing^{(n)}(g^J) \Phi_I^{(n)}(g^{J^c}) = 0$$

taking into account that $\varphi^*\mathcal{R}$ acts on $\mathcal{O}$ via $\Phi_\varnothing^{(n)}$ and that there is no contribution for $J = \underline{\|I\|+1}$ by lemma 5.17 b). Using the product formula of lemma 5.19 with n replaced by

$\|J^c\|$, we have to show

$$0 = \sum_{J \subsetneq \underline{\|I\|+1}} (-1)^{\|J\|}(-1)^{|I||J^c|}\mathrm{osign}(J,J^c)\Phi^{(n)}_{\varnothing}(g^J) \times \qquad (5.9)$$
$$\left\{ \sum_{I_1 \sqcup \cdots \sqcup I_{\|J^c\|} = I} \mathrm{sign}(I_1,\ldots,I_{\|J^c\|}) \prod_{1\le\alpha<\beta\le\|J^c\|} (-1)^{|I_\beta|(|I_\alpha|+|g_{J^c_\alpha}|)} \prod_{k=1\ldots\|J^c\|} \Phi^{(n)}_{I_k}(g_{J^c_k}) \right\}$$

Here J^c_k denotes the k-th entry of the multi-index J^c. Expanding this sum, all summands are of the form $\pm\Phi^{(n)}_{I_1}(g_1)\cdots\Phi^{(n)}_{I_{\|I\|+1}}(g_{\|I\|+1})$ where the multi-indices I_k are possibly empty. The strategy of the proof is now to collect all those summands which belong to a specific combination of multi-indices I_j and argument functions g_k and show, that already the resulting partial sum vanishes. If this holds for any combination of multi-indices and functions, then obviously the total sum in (5.9) vanishes.

In a first step, we will simplify the product of factors (-1). To this end, let $D := (I_1,\ldots,I_{\|J^c\|})$. Note that

$$(-1)^{\|I\||J^c|} \prod_{1\le\alpha<\beta\le\|J^c\|} (-1)^{\|I_\beta\||g_{J^c_\alpha}|} = \prod_{1\le\beta\le\alpha\le\|J^c\|} (-1)^{\|I_\beta\||g_{J^c_\alpha}|}$$

Using this identity and lemma 5.18, we can compute

$$\begin{aligned} S_{D,J} &:= (-1)^{\|J\|}(-1)^{\|I\||J^c|}\mathrm{osign}(J,J^c)\Phi^{(n)}_{\varnothing}(g^J)\mathrm{sign}(D)\prod_{1\le\alpha<\beta\le\|J^c\|}(-1)^{\|I_\beta\|(\|I_\alpha\|+|g_\alpha|)} \prod_{k=1\ldots\|J^c\|}\Phi^{(n)}_{I_k}(g_{J^c_k}) \\ &= (-1)^{\|J\|}\mathrm{osign}(J,J^c)\mathrm{sign}(D)(-1)^{\binom{L^-(D)}{2}}\prod_{1\le\beta\le\alpha\le\|J^c\|}(-1)^{\|I_\beta\||g_{J^c_\alpha}|} \quad \Phi^{(n)}_{\varnothing}(g^J)\prod_{k=1\ldots\|J^c\|}\Phi^{(n)}_{I_k}(g_{J^c_k}) \end{aligned}$$

Now we rearrange the sums in (5.9) with respect to the class of index D in $\mathcal{D}^{ess}_{\|I\|+1}(I)$ and the length of these indices. We obtain

$$\begin{aligned} \sum_{J\subsetneq\underline{\|I\|+1}} \sum_{Z\in\mathcal{D}_{\|J^c\|}(I)} S_{D,J} &= \sum_{j=1}^{\|I\|+1} \sum_{\substack{J\subsetneq\underline{\|I\|+1} \\ \|J^c\|=j}} \sum_{Z\in\mathcal{D}_j(I)} S_{D,J} \\ &= \sum_{\mathcal{E}\in D^{ess}_{\|I\|+1}(I)} \sum_{j=L^{ess}(\mathcal{E})}^{\|I\|+1} \sum_{\substack{J\subsetneq\underline{\|I\|+1} \\ \|J^c\|=j}} \sum_{\substack{D\in\mathcal{E} \\ L(D)=j}} S_{D,J} \end{aligned}$$

After inserting the expression for $S_{D,J}$, observing that $\mathrm{sign}(D)$ and $L^-(D)$ only depend on D's class in $D^{ess}_{\|I\|+1}(I)$ and using $\|J\| = \|I\|+1-\|J^c\|$, we obtain

$$\sum_{\mathcal{E}\in D^{ess}_{\|I\|+1}(I)} sign(\mathcal{E})(-1)^{\binom{L^-(\mathcal{E})}{2}}(-1)^{\|I\|+1} \tag{5.10}$$
$$\times \sum_{j=L^{ess}(\mathcal{E})}^{\|I\|+1} \sum_{\substack{J\subsetneq \underline{\|I\|+1}\\ \|J^c\|=j}} \sum_{\substack{D\in\mathcal{E}\\ L(D)=j}} (-1)^j \mathrm{osign}(J,J^c) \prod_{1\le\beta\le\alpha\le\|J^c\|}(-1)^{\|I_\beta\||g_{J^c_\alpha}|} \quad \Phi^{(n)}_{\varnothing}(g^J)\prod_{k=1\ldots\|J^c\|}\Phi^{(n)}_{I_k}(g_{J^c_k})$$

Recalling that the multi-indices J are ordered by definition, it is clear that two partitions of I, which are not essentially equal, can not produce the same combinations of I_j and g_k in the summands of (5.10). Thus, we fix a class $\mathcal{E}_0 \in D^{ess}_{\|I\|+1}(I)$, choose a multi-index $H = (h_1, \ldots, h_{\|H\|}) \subset \underline{\|I\|+1}$ of length $\|H\| = L^{ess}(\mathcal{E}_0)$ and collect all the summands of the form $\pm\Phi^{(n)}_{\varnothing}(g^{\underline{\|I\|+1}\setminus H})\Phi^{(n)}_{I_1}(h_1)\cdots\Phi^{(n)}_{I_{\|H\|}}(h_{\|H\|})$. We make the following observations:

(a) The summands in (5.10) depend only on combination of I_j and g_k. To see this, we set $G := \underline{\|I\|+1} \setminus H = (\gamma_1, \ldots, \gamma_{\|G\|})$ and arrange the products in an increasing order $\Phi^{(n)}_{\varnothing}(g_{\gamma_1})\cdots\Phi^{(n)}_{\varnothing}(g_{\gamma_{\|G\|}})\Phi^{(n)}_{I_1}(g_{h_1})\cdots\Phi^{(n)}_{I_{\|H\|}}(g_{h_{\|H\|}})$. This involves permuting g_ks with I_js and other g_ls. First, each $g_{J^c_\alpha}$ is interchanged with all the I_β for $\beta < \alpha$ which each time gives a factor $(-1)^{\|I_\beta\||g_{J^c_\alpha}|}$. Second, we clearly have $\mathrm{osign}(J,J^c)g^Jg^{J^c} = \mathrm{osign}(G,H)g^Gg^H$ so that we can read off the sign resulting from rearranging the functions g. Summarizing, we find that for a fixed combination of I_j and g_k, all the corresponding summands have the form

$$(-1)^j\mathrm{osign}(G,H) \prod_{\substack{1\le\beta\le\alpha\le\|J^c\|\\ J^c_\alpha\in H}}(-1)^{\|I_\beta\||g_{J^c_\alpha}|} \quad \Phi^{(n)}_{\varnothing}(g_{\gamma_1})\cdots\Phi^{(n)}_{I_{\|H\|}}(g_{h_{\|H\|}}) =: (-1)^j s_{\mathcal{E}_0,H}$$

Since the value of the product only depends on the choices of $\mathcal{E}_0$ (which encodes the structure of the indices $I_k \neq \varnothing$) and H, i.e. on the choice of the combination of I_j and g_k, the same holds for the summands $s_{\mathcal{E}_0,H}$.

(b) Given a combination of I_j and g_k specified by $\mathcal{E}_0$ and H as above, a multi-index J does not contribute to it if $H \not\subset J^c$. If $H \subset J^c$, then there is precisely one $D \in \mathcal{E}_0$ such that the resulting summand contributes to the combination in question. By the first observation, all these contributions for different $J^c \supset H$ have in fact the same value. Denoting $j := \|J^c\|$, the number of these contributions is then given by

$$\binom{\|I\|+1-\|H\|}{\|J^c\|-\|H\|} = \binom{\|I\|+1-L^{ess}(\mathcal{E}_0)}{j-L^{ess}(\mathcal{E}_0)}$$

because next to the mandatory elements of H, J^c contains another $\|J^c\|-\|H\|$ elements, which are to be chosen out of $\underline{\|I\|+1}\setminus H$.

Using both observations, we can finally write down the contribution to (5.10) as well as (5.9) of all the summands for some fixed $\mathcal{E}_0$ and $H \subset \underline{\|I\|+1}$ such that $\|H\| = L^{ess}(\mathcal{E}_0)$:

$$\begin{aligned}
&\sum_{j=L^{ess}(\mathcal{E})}^{\|I\|+1} \binom{\|I\|+1-L^{ess}(\mathcal{E}_0)}{j-L^{ess}(\mathcal{E}_0)} (-1)^j s_{\mathcal{E}_0,H} \\
&= s_{\mathcal{E}_0,H}(-1)^{L^{ess}(\mathcal{E}_0)} \sum_{j=0}^{\|I\|+1-L^{ess}(\mathcal{E}_0)} (-1)^j \binom{\|I\|+1-L^{ess}(\mathcal{E}_0)}{j} \\
&= 0
\end{aligned}$$

The last equality follows immediately from the Binomial theorem. Since this is true for any combination of I_j and g_k, i.e. any choice of $\mathcal{E}_0$ and H, the total sum in (5.10) vanishes which proves the theorem.

$\square$

The preceding theorem discusses the decomposition of a given element of $\underline{SC}^\infty(X,Y)(\mathcal{P}_n)$ into its components which have been proven to be super differential operators. Conversely, we can ask whether each family of super differential operators parameterized by multi-indices $I \subset \underline{n}$,

$$\Delta_I^{(n)} : \varphi^*\mathcal{R} \longrightarrow \mathcal{O}$$

of order $\|I\|$ and parity $\|I\| \mod 2$ defines an element of $\underline{SC}^\infty(X,Y)(\mathcal{P}_n)$ by setting $\Phi^{(n)} = \sum_I \eta^I \Delta_I^{(n)}$ in analogy to equation (5.8). This is clearly not true, because by lemma 5.19, the differential operators have to fulfill certain compatibility relations. However, it is not satisfying to describe higher points of $\underline{SC}^\infty(X,Y)$ by families of differential operators of suitable order and parity being subject to the relations of lemma 5.19. In fact, we just used these relations to prove, that the coefficients $\Phi_I^{(n)}$ are differential operators so that imposing the product rules of the lemma on a family of differential operators is redundant. We should separate the property of being a differential operator from the algebraic compatibility relations from the lemma. As announced in section 5.1, this can be done using jets. We will give a short review of the algebraic approach to this concept following [36], I.2, to complete the discussion of higher points of $\underline{SC}^\infty(X,Y)$.

As before, let A be a unital supercommutative $\mathbb{R}$-super algebra and P a module over A. We can then form the tensor product $A \otimes_\mathbb{R} P$ and have an induced A-module structure on this space by setting $a'(a \otimes p) := (a'a) \otimes p$. Following [55] 3.2.6 and below, we set

Definition 5.21

(a) For $a' \in A$, let $\delta^{a'} : A \otimes_\mathbb{R} P \longrightarrow A \otimes_\mathbb{R} P$ the ($\mathbb{R}$-linear) endomorphism defined by

$$\delta^{a'}(a \otimes p) := a'a \otimes p - (-1)^{|a'||a|} a \otimes a'p$$

(b) *For $k \in \mathbb{N}_0$, the module of k-jets is defined by*

$$Jet^k(P) := A \otimes_{\mathbb{R}} P/M_{k+1} \quad \text{where } M_{k+1} := span_A\{\delta^{a_0} \cdots \delta^{a_k}(x) | x \in A \otimes_{\mathbb{R}} P, a_i \in A\}$$

(c) *The natural map $jet^k : P \longrightarrow Jet^k(P)$ is defined by*

$$jet^k(p) := 1 \otimes p + M_{k+1}$$

We will not discuss the relation to the classical concept of jet bundles (see e.g. [47] 11.46 ff), but collect some important properties of $Jet^k(P)$.

Proposition 5.22 *Let P, Q be A-modules and $k \in \mathbb{N}_0$, then we have:*

(a) *$jet^k \in \mathrm{Diff}^k(P, Jet^k(P))_{\overline{0}}$, in words: jet^k is an even super differential operator of order $\leq k$. Its image generates $Jet^k(P)$.*

(b) *For each super differential operator $\Delta \in \mathrm{Diff}^k(P,Q)_{\overline{i}}$ ($i \in \mathbb{Z}_2$), there is a unique morphism $\sigma(\Delta) \in \underline{\mathrm{Hom}}(Jet^k(P), Q)_{\overline{i}}$ such that the following diagram commutes:*

$$\begin{tikzcd} P \arrow[rr, "jet^k"] \arrow[dr, "\Delta"'] & & Jet^k(P) \arrow[dl, "\sigma(\Delta)"] \\ & Q & \end{tikzcd}$$

This universal property characterizes $Jet^k(P)$ up to isomorphism. Using the language of definition 4.2, this statement says, that $Jet^k(P)$ is a representing object for the covariant functor $Q \mapsto \mathrm{Diff}^s(P,Q)_{\overline{0}}$.

We are not giving the proof of this elementary properties but refer to [36] I.2. The map $\sigma(\Delta) \in \underline{\mathrm{Hom}}(Jet^k(P), Q)$ thus contains all the information about the Δ:

Definition 5.23 *The map $\sigma(\Delta) \in \underline{\mathrm{Hom}}(Jet^k(P), Q)$ is called the total symbol of the differential operator $\Delta \in \mathrm{Diff}^k(P,Q)$.*

As an application, we let $l \geq k$ and set $\Delta = jet^l : P \longrightarrow Jet^l(P) =: Q$ which is a differential operator of order $\leq l$ by part a) of the preceding proposition. By proposition 5.22 (a), jet^k is also a differential operator of order $\leq l$. Applying part b) of the proposition, we get the following lemma

Lemma 5.24 *For $k \leq l$, there is a unique map $jet^{l,k} \in \mathrm{Hom}_A(Jet^l(P), Jet^k(P))$ such that $jet^k = jet^{k,l} \circ jet^l$. For $k \leq l \leq m$, we furthermore have $jet^{m,l} \circ jet^{l,k} = jet^{m,k}$.*

This statement can of course also be proven directly by observing that $M_l \subset M_k$ ($l \geq k$) which leads to the desired map. A special situation arises, if we choose $A = P$, i.e. we look at differential operators acting on functions. In this case, $A \otimes_{\mathbb{R}} A$ is not only a left (and right) A-module but also an $\mathbb{R}$-superalgebra by $(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b||a'|}(aa') \otimes (bb')$. Defining the submodules M_k as above, we have

Lemma 5.25 *For all $k \in \mathbb{N}_0$, $M_k \subset A \otimes_{\mathbb{R}} A$ is a graded ideal. In particular, $Jet^{k-1}(A) = \otimes_{\mathbb{R}} A/M_k$ carries a natural structure of a $\mathbb{R}$-superalgebra and we have $jet^k(ab) = jet^k(a)jet^k(b)$.*

Proof We only have to show that M_k is a graded ideal, the rest follows from the usual construction of the quotient algebra and of the definition $jet^k(a) = [1 \otimes a]$. But it is easily verified that

$$(a \otimes b)\delta^c(a' \otimes b') = (-1)^{(|a|+|b|)|c|}\delta^c((aa') \otimes (bb'))$$

which proves that M_k is indeed an ideal. Moreover, it is easy to see that we have

$$(\delta^c(a \otimes b))_i = \sum_{j+k+l \equiv i \mod 2} \delta^{c_j}(a_k \otimes b_l)$$

which proves that for $x \in M_k$, we have $x_0, x_1 \in M_k$, too. Thus, M_k is a graded ideal.

□

Remark 5.26 It should be noted, that the multiplicativity of jet^k does not mean, that this map is A-linear (and hence a 0-order operator). In detail, both relations read

$$\begin{aligned} jet^k(bb') &= jet^k(b)jet^k(b') &\quad\Leftrightarrow\quad& [1 \otimes bb'] = [1 \otimes b][1 \otimes b'] \\ jet^k(ab) &= a\ jet^k(b) &\quad\Leftrightarrow\quad& [1 \otimes ab] = [a \otimes b] \end{aligned}$$

Whereas the first relation is always true, the second relation precisely generates M_1 so that A-linearity of jet^k is only satisfied for $k = 0$.

The splitting $\Delta = \sigma(\Delta) \circ jet^k$ from proposition 5.22 (b) allows us in some sense to separate Δ's property of being a differential operator from its algebraic properties. This is what we need to discuss the question mentioned above, which families of differential operators define elements of $\underline{SC}^\infty(X, Y)$:

Let $\Phi^{(n)} \in \underline{SC}^\infty(X, Y)(\mathcal{P}_n)$ and write $\Phi^{(n)} = \sum_I \eta^I \Phi_I^{(n)}$ as in (5.8). By theorem 5.20, we know that $\Phi_I^{(n)}$ are differential operators of order $\|I\|$ and parity $\|I\| \mod 2$. We can now factorize $\Phi_I^{(n)} =: \sigma(\Phi_I^{(n)}) \circ jet^{\|I\|}$ where $\sigma_I^{(n)} := \sigma(\Phi_I^{(n)}) \in \underline{\mathrm{Hom}}(Jet^{\|I\|}(\varphi^*\mathcal{R}), \mathcal{O})_{\|I\| \mod 2}$ is the symbol of the differential operator, which contains in particular all information about the parity. Since $\|I\| \leq n$ by definition, we can use the maps $jet^{n,\|I\|}$ from lemma 5.24 to define the following trivial extensions of all symbols to the space $Jet^n(\varphi^*\mathcal{R})$:

$$\begin{aligned} \hat{\sigma}_I^{(n)} &:= \sigma(\Phi_I^{(n)}) \circ jet^{n,\|I\|} : Jet^n(\varphi^*\mathcal{R}) \longrightarrow \mathcal{O} \qquad \text{and} \\ \hat{\Phi}^{(n)} &:= \sum_I \eta^I \hat{\sigma}_I^{(n)} : Jet^n(\varphi^*\mathcal{R}) \longrightarrow \Lambda_n \otimes \mathcal{O} \end{aligned} \tag{5.11}$$

Note that, by lemma 5.24, we obviously have

$$\hat{\Phi} \circ jet^n = \sum_I \eta^I \sigma(\Phi_I^{(n)}) \circ jet^{\|I\|} = \Phi^{(n)} \tag{5.12}$$

By slight abuse of notation, we call $\hat{\Phi}^n$ the total symbol of $\Phi^{(n)}$. It can be used to characterize those families of differential operators, which give rise to a n-point of $\underline{SC}^\infty(X,Y)$

Theorem 5.27 *Every element $\Phi^{(n)} \in \underline{SC}^\infty(X,Y)(\mathcal{P}_n)$ determines a unique morphism $\Phi^{(n)}_\varnothing : \varphi^*\mathcal{R} \to \mathcal{O}$ and a unique family of differential operators $\{\Phi^{(n)}_I\}$ of order $\|I\|$ and parity $\|I\| \mod 2$ along $\Phi^{(n)}_\varnothing$, such that the total symbol $\hat{\Phi}^{(n)}$ defined in (5.11) is a homomorphism of superalgebras. Conversely, given a morphism $\Psi_\varnothing : X \longrightarrow Y$ and a family $\{\Psi_J\}_{\varnothing \neq J \subset \underline{n}}$ of differential operators along $\Psi_\varnothing$, such that $\hat{\Psi}$ is a homomorphism of superalgebras, then (5.12) defines a n-point of $\underline{SC}^\infty(X,Y)$. These constructions are inverse to each other.*

Proof For the first part, we only have to prove that $\hat{\Phi}^{(n)} \in \mathrm{Hom}_{SAlg}(Jet^n(\varphi^*\mathcal{R}), \Lambda_n \otimes \mathcal{O})$. Since the parity of $\Phi^{(n)}_I$ is given by $\|I\| \mod 2$, it is clear that $\hat{\Phi}$ is preserves the grading. Choosing elements $jet^n(f), jet^n(g) \in Jet^n(\varphi^*\mathcal{R})$ and using lemma 5.25 and equation (5.12), we calculate

$$\hat{\Phi}^{(n)}(jet^n(fg)) = \Phi^{(n)}(fg) = \Phi^{(n)}(f)\Phi^{(n)}(g) = \hat{\Phi}^{(n)}(jet^n(f))\hat{\Phi}^{(n)}(jet^n(g))$$

where we used the fact, that $\Phi^{(n)}$ is a homomorphism. Since the elements $jet^n(f)$ generate $Jet^n(\varphi^*\mathcal{R})$, we obtain that $\hat{\Phi}^{(n)}$ is a homomorphism.
Conversely, given $\Psi_\varnothing$ and the family $\{\Psi_I\}$, we set

$$\Psi^{(n)} := \hat{\Psi}^{(n)} \circ jet^n : \varphi^*\mathcal{R} \longrightarrow Jet^n(\varphi^*\mathcal{R}) \longrightarrow \textstyle\bigwedge_n \otimes \mathcal{O}$$

which is a homomorphism of superalgebras because jet^n and $\hat{\Psi}^{(n)}$ are. Finally, the decomposition into components is clearly unique since $\{\eta^I\}_{I \subset \underline{n}}$ is a basis of the Grassmann algebra. This proves that both constructions are inverse to each other.

□

5.4 Application to the Batchelor picture

In the Batchelor picture (see theorem 3.9), we can use the theory derived in the previous section to obtain a description of morphisms using only notions from ordinary smooth differential geometry. The results are not crucial for subsequent sections, so the discussion will be less detailed. The author is grateful to Gregor Weingart for drawing attention to this point of view.

Let $E \longrightarrow M$ and $F \longrightarrow N$ two vector bundles over smooth manifolds which define the supermanifolds $X := (M, \Gamma(-, \bigwedge^\bullet E))$ and $Y := (N, \Gamma(-, \bigwedge^\bullet F))$ in the Batchelor picture. The specification of the bundles E, F equips X and Y with the following extra structure:

(a) Since $\Gamma(M, \bigwedge^0 E) = C^\infty(M)$, we have a canonical function factor.

(b) Since $\Gamma(M, \bigwedge^1 E) \cong \mathcal{J}/\mathcal{J}^2$ where $\mathcal{J} \subset \Gamma(M, \bigwedge^\bullet E)$ is the ideal of nilpotent elements, we have a canonical identification of the quotient with an actual subspace of the ring of superfunctions. Moreover, the $\mathbb{Z}$-grading of $\Gamma(\bigwedge^\bullet E) \cong \bigoplus_k \Gamma(\bigwedge^k E)$ induces the $\mathbb{Z}_2$-grading of the super algebra.

It can now be used to discuss the structure of ordinary morphisms $\Phi : X \longrightarrow Y$, i.e. of 0-points of $\underline{SC}^\infty(X,Y)$. By definition, Φ is given by $\varphi \in C^\infty(M,N)$ and a morphism

$$\Phi^* : \varphi^*\Gamma(-,\bigwedge^\bullet F) \longrightarrow \Gamma(-,\bigwedge^\bullet E)$$

of sheaves of superalgebras. Since it preserves the $\mathbb{Z}_2$-grading, we clearly have

$$\Phi(\varphi^*\Gamma(-,\bigwedge^{ev} F)) \subset \Gamma(-,\bigwedge^{ev} E) \qquad \Phi(\varphi^*\Gamma(-,\bigwedge^{odd} F)) \subset \Gamma(-,\bigwedge^{odd} E) \tag{5.13}$$

In particular, for $f \in C^\infty(N) \subset \Gamma(N,\bigwedge^{ev} F)$, the image $\Phi(f)$ will in general have a non-trivial component in $\Gamma(M,\bigwedge^{\geq 2} E)$. Hence the fact that $\Phi(f\sigma) = \Phi(f)\Phi(\sigma)$ for $f \in C^\infty(M)$ and $\sigma \in \Gamma(N,\bigwedge^\bullet F)$ does not imply, that Φ is given by a homomorphism of smooth vector bundles $\varphi^*\bigwedge^\bullet F \longrightarrow \bigwedge^\bullet E$. Its structure is more complicated and again involves differential operators now acting on smooth functions in $C^\infty(N)$ and smooth sections of F, respectively.

To see this, it is sufficient to work locally and to choose a basis of local sections $\{e^i\}_{i=1,\ldots,rk(E)}$ of E which corresponds to a choice of an odd coordinate system. Locally, the supermanifold X now has the form $X \cong M \times \mathcal{P}_{rk(E)}$ and using the expansion

$$\Phi(\sigma) = \sum_{I \subset \underline{rk(E)}} \Phi_I(\sigma)e^I \qquad \text{for } \sigma \in \Gamma(\bigwedge^\bullet F),$$

theorem 5.20 implies that the maps Φ_I are differential operators acting on $\Gamma(N,\bigwedge^\bullet F)$ with values in $C^\infty(M)$ of order $\|I\|$. However, more can be said. In particular, the value for the order of the differential operators is not optimal, Φ is already determined by operators of lower order (see proposition 5.29).We need the following lemma to get a refined description:

Lemma 5.28 *Let $\Delta : \Gamma(\bigwedge^\bullet F) \longrightarrow \Gamma(\bigwedge^\bullet E)$ be a differential operator of order $\leq k$ along $\varphi : M \longrightarrow N$ and $e \in \Gamma(\bigwedge^\bullet E)$. Then, the map*

$$\Delta \cdot e : \Gamma(\bigwedge^\bullet F) \longrightarrow \Gamma(\bigwedge^\bullet E) \qquad \Delta \cdot e(\sigma) := (-1)^{|e||\sigma|}(\Delta(\sigma))e$$

is also a differential operator of order $\leq k$. The same holds for $C^\infty(N)$-submodules of $\Gamma(\bigwedge^\bullet F)$.

Proof For $f \in C^\infty(N)$ and $\sigma \in \Gamma(\bigwedge^\bullet F)$, we find

$$(\delta_f(\Delta\cdot e))(\sigma) = (-1^{|\sigma||e|})\big(f\Delta(\sigma)e) - \Delta(f\sigma)e\big) = (-1^{|\sigma||e|})\big((\delta_f\Delta)(\sigma)e\big) = \big((\delta_f\Delta)\cdot e\big)(\sigma)$$

taking into account that f is even. Now Δ is of order $\leq k$ if $\delta_{f_0}\cdots\delta_{f_k}\Delta = 0$ for all $f_i \in C^\infty(N)$. But the previous computation then clearly implies that $\delta_{f_0}\cdots\delta_{f_k}(\Delta\cdot e) = 0$, which means that $\Delta\cdot e$ is a differential operator of order $\leq k$, too.

□

We can now discuss the refined structure of Φ taking advantage of the extra structure given by the choice of Batchelor bundles. Recall that $\mathrm{Diff}^k(\varphi^*\Gamma(F),\Gamma(E)_\varphi))$ denotes the module of differential operators along φ acting on sections of F with values in sections of E.

Proposition 5.29 *Let $(M, \Gamma(-, \bigwedge^\bullet E))$ and $(N, \Gamma(-, \bigwedge^\bullet F))$ be supermanifolds in the Batchelor picture and $r := \lfloor rk(E)/2 \rfloor$. The morphisms $\Phi : (M, \Gamma(-, \bigwedge^\bullet E)) \longrightarrow (N, \Gamma(-, \bigwedge^\bullet F))$ of supermanifolds are in one to one correspondence with triples (φ, D_0, D_1) where $\varphi : M \longrightarrow N$ is a smooth map and D_0, D_1 are differential operators along φ,*

$$D_0 \in \mathrm{Diff}^r(\varphi^* C^\infty_N, \Gamma(\textstyle\bigwedge^{ev\geq 2} E)_\varphi) \qquad\qquad D_1 \in \mathrm{Diff}^r(\varphi^*\Gamma(F), \Gamma(\textstyle\bigwedge^{odd} E)_\varphi)$$

satisfying the following compatibility conditions for $f, g \in C^\infty_N$ and $\sigma \in \Gamma(F)$:

$$\begin{aligned} D_0(fg) &= \varphi^* f D_0(g) + D_0(f)\varphi^* g + D_0(f) D_0(g) \\ D_1(f\sigma) &= \varphi^* f D_1(\sigma) + D_0(f) D_1(\sigma) \end{aligned} \tag{5.14}$$

More precisely, the correspondence is given by

$$\varphi \cong pr^0 \circ \Phi|_{C^\infty_N} \qquad\qquad D_0 \cong pr^{ev\ \geq 2} \circ \Phi|_{C^\infty_N} \qquad\qquad D_1 \cong \Phi|_{\Gamma(F)}$$

where $pr^k : \Gamma(\bigwedge^\bullet E) \twoheadrightarrow \Gamma(\bigwedge^k E)$ are the obvious projections.

Proof From the universal property of the exterior algebra (see [6], III.7.1.1) and the fact that morphisms preserve the grading, it follows that Φ is determined by its restriction to $C^\infty(N) \oplus \Gamma(F) = \Gamma(\bigwedge^{\leq 1} F) \subset \Gamma(\bigwedge^\bullet F)$. Conversely, a map $\varphi : \Gamma(\bigwedge^{\leq 1} F) \longrightarrow \Gamma(\bigwedge^\bullet E)$ which is multiplicative in the sense

$$\varphi(fg) = \varphi(f)\varphi(g) \qquad\qquad \varphi(f\sigma) = \varphi(f)\varphi(\sigma) \tag{5.15}$$

for $f \in C^\infty_N, \sigma \in \Gamma(F)$ extends uniquely to a multiplicative map $\Gamma(\bigwedge^\bullet F) \longrightarrow \Gamma(\bigwedge^\bullet E)$. By (5.13), $\Phi|_\Gamma(\bigwedge^{\leq 1} F)$ can be decomposed into $\varphi^* : C^\infty_N \longrightarrow C^\infty_M$, $D_0 : C^\infty_N \longrightarrow \Gamma(\bigwedge^{ev\geq 2} E)$ and $D_1 : \Gamma(F) \longrightarrow \Gamma(\bigwedge^{odd} E)$. It is the easy to see that the compatibility conditions (5.14) are equivalent to the multiplicativity property. Moreover, $\varphi : C^\infty_N \longrightarrow C^\infty_M$ is multiplicative and hence defines a smooth map $M \longrightarrow N$.

It remains to show that property (5.15) implies that D_0 and D_1 are differential operators of order r along φ. Writing $\Phi = \sum_I \Phi_I e^I$ and recalling that $r = \lfloor rk(E)/2 \rfloor$, we can rearrange this sum by factoring out even powers e^I and we obtain

$$\Phi = \sum_{J \subset \underline{2r}} (A_J + B_J) e^J$$

where A_J has values in C^∞_M and B_J in $\Gamma(E)$ respectively. The symbols $\{e^J | J \subset \underline{2r}\}$ form a basis of the algebra Λ^{ev}_{2r}. As a vector space, the latter is isomorphic to Λ_r but since it only contains even elements, this is not true on the level of algebras. Thus we can not apply theorem 5.20 literally but we have to adapt the proof by removing all signs arising from $\mathbb{Z}_2$-parity only keeping those occurring in the definition of even differential operators. We will not give the details here but only conclude that coefficients $(A_J + B_J)$ are even differential operators of order $\leq r$ since we do the expansion w.r.t. Λ_r (without parity !). Since both summands of $\Gamma(\bigwedge^{\leq 1} E) = C^\infty_M \oplus \Gamma(E)$ are C^∞_M-submodules, A_J and B_J are already differential operators separately. Finally, we have $D_0 = \sum_{\|J\|\geq 2} A_J e^J$ and $D_1 = \sum_J B_J e^J$ on the coordinate neighbourhood so that D_0 and D_1 are differential operators of order $\leq r$ along φ by lemma 5.28.

$\square$

We close this subsection with slightly different points of view:

Remark 5.30

(a) It is possible to include φ as a component of a new operator D_0' with values in $\Gamma(\bigwedge^{ev} E)$, so that φ is given implicitly by it. However, the concept seems to be more satisfying if φ is recorded separately and D_0 and D_1 are considered to be corrections to it arising from the superworld.

(b) It is possible to omit restricting Φ to $\Gamma(\bigwedge^{\leq 1} F)$. In that case, Φ is described by the map φ and two differential operators $D_0 : \Gamma(\bigwedge^{ev} F) \longrightarrow \Gamma(\bigwedge^{ev\geq 2} E)$, $D_1 : \Gamma(\bigwedge^{odd} F) \longrightarrow \Gamma(\bigwedge^{odd} E)$ satisfying a condition similar to (5.14).

(c) As in theorem 5.27 and (5.12), we can factor the differential operators D_0 and D_1 over $Jet^r(\tilde{X})$ and $Jet^r(F)$, respectively. In this cases, the compatibility relations (5.14) translate into a similar properties for the total symbols σ_0 and σ_1.

5.5 Component formalism

Theorems 5.20 and 5.27 provide a complete descriptions of n-points of $\underline{SC}^\infty(X,Y)$ for arbitrary supermanifolds X, Y and $n \in \mathbb{N}_0$. However, the result has certain drawbacks:

(a) The components appearing in the composition are differential operators of arbitrary order. In contrast to vector fields (i.e. differential operators of order ≤ 1 without 0.-order term), these higher order terms are difficult to use in the tensor calculus of differential geometry. For instance, it is not immediately clear how to contract these fields with curvature tensors.

(b) To construct a n-point of $\underline{SC}^\infty(X,Y)$, we not only have to construct a family of differential operators but we also have to ensure, that the corresponding symbol maps define a homomorphism of super algebras. These constraints are in general quite complicated.

One possible approach, which actually resolves both drawbacks, is the introduction of component fields of a morphism (or superfield in physicists' terminology). This idea has been used for a long time in mathematical and physical literature (see [68], III and IV, [7], 2.4.4 or [11] §1.2) Roughly speaking, it consists in fixing some odd coordinate system and performing a sort of Taylor expansion with respect to the powers of the odd coordinates. The component fields are the resulting coefficients, obtained by differentiating the morphism w.r.t. odd coordinates of X (not w.r.t. odd parameters in Λ_n). On a general curved, topologically nontrivial supermanifold, we can not just use ordinary derivatives but have to take into account covariant derivatives w.r.t. some connection, this is the approach used in [11]. We will adapt this concept to points of $\underline{SC}^\infty(X,Y)$ and prove, that each point uniquely determines a set of component fields and vice versa.

To define the components in a precise way, we fix a connection ∇ on $\mathcal{T}_Y$. Moreover, we choose odd coordinates on X which means that in general, we have to work locally. To simplify notation, we may assume that we have a $p|q$-dimensional supermanifold X with globally defined odd coordinates $\theta^1 \ldots, \theta^q \in \mathcal{O}(X)_{\overline{1}}$ which remain fixed in this section. In the Batchelor picture, this means that we describe the supermanifold by a trivializable vector bundle $E \longrightarrow \tilde{X}$ and $\theta^1, \ldots, \theta^q$ define a trivialization of it. Furthermore, denote by $\iota : \tilde{X} \longrightarrow X$ the canonical embedding from 3.6 which is given by the map setting all θ^α to zero. Since we have to deal with higher points of X and $\underline{SC}^\infty(X, Y)$, it is convenient to extend ι to $\bigwedge^\bullet \mathbb{R}^n \otimes \mathcal{O}_X$ by acting as the identity on the factor $\bigwedge^\bullet \mathbb{R}^n$:

$$\iota : \mathcal{P}_n \times \tilde{X} \longrightarrow \mathcal{P}_n \times X \qquad\qquad \iota^* : \Lambda_n \otimes \mathcal{O}_X \longrightarrow \Lambda_n \otimes C^\infty(\tilde{X})$$

We now define the components in analogy to [10]. To this end, we introduce the notation $\overline{I} := (i_k \cdots, i_1)$ for an arbitrary multi-index $I = (i_1 \cdots i_k)$ and abbreviate $\nabla_I^\Phi := \nabla^\Phi_{\frac{\partial}{\partial \theta^{i_1}}} \cdots \nabla^\Phi_{\frac{\partial}{\partial \theta^{i_k}}}$. The components are now defined as follows:

Definition 5.31 *Let X, Y be supermanifolds and $n \in \mathbb{N}_0$. The components of an element $\Phi^{(n)} \in \underline{SC}^\infty(X, Y)(\mathcal{P}_n)$ (w.r.t to the fixed odd coordinates and the connection ∇) are given by*

$$\begin{aligned} \varphi^{(n)} &:= \iota^* \Phi^{(n)} && \\ \psi_\alpha^{(n)} &:= \iota^* d\Phi^{(n)}\left(\tfrac{\partial}{\partial \theta^\alpha}\right) && \text{for } (\alpha = 1, \ldots, q) \\ \psi_A^{(n)} &:= \iota^* \nabla^{\Phi^{(n)}}_{\overline{A \setminus \alpha_1}} d\Phi\left(\tfrac{\partial}{\partial \theta^{\alpha_1}}\right) && \text{for } A = (\alpha_1, \ldots, \alpha_k) \subset \underline{q},\ \|A\| > 1. \end{aligned}$$

A similar definition may be used for vector fields along an element of $\underline{SC}^\infty(X, Y)(\mathcal{P}_n)$:

Definition 5.32 *Let $\Phi^{(n)} \in \underline{SC}^\infty(X, Y)$ and let V be a vector field along $\Phi^{(n)}$ (i.e. a section of $\Phi^{(n)*}\mathcal{T}_Y$). Then, the components of V (w.r.t. the fixed coordinates and the connection ∇) are defined as follows:*

$$\begin{aligned} V_\varnothing &:= \iota^* V && \\ V_A &:= \iota^* \nabla^{\Phi^{(n)}}_{\overline{A}} V && \text{for } A \subset \underline{q},\ \|A\| > 0 \end{aligned}$$

From the discussion in section 3.3 and 3.6, we immediately obtain

Lemma 5.33 *Let $\Phi^{(n)}$ and V as in definitions 5.31 and 5.32. Then $\varphi^{(n)}$ is a morphism of supermanifolds from $\tilde{X}$ to Y and all components ψ_α, ψ_A and V_A are vector fields along φ.*

We will start by proving the correspondence between vector fields and and their components because it will be convenient to use it in the proof for the morphisms. To do so, we need to introduce a right inverse j for ι. We choose a function factor $C \subset \mathcal{O}_X$ and define j^* using the inclusion of the rings of functions:

$$j^* : \Lambda_n \otimes C \hookrightarrow \Lambda_n \otimes \mathcal{O}$$

Recall that the choice of a function factor is by definition the choice of an embedding $C^\infty(\tilde{X}) \cong C \hookrightarrow \mathcal{O}_X$, which is, in contrast to the projection $\mathcal{O} \longrightarrow C^\infty(\tilde{X})$, not naturally given. Hence, j^* must invoke this choice, however, the statement we prove is independent of it. Having fixed odd coordinates $\{\theta^1 \dots, \theta^q\}$ and the function factor C, we write $\tilde{X}^{p|q}$ for the supermanifold specified by the function algebra $C[\theta^1, \dots, \theta^q]$, which is isomorphic to $C^\infty(\tilde{X}) \otimes \Lambda^\bullet \mathbb{R}^q$. We can also define restriction and extension to a subset $I = (i_1, \dots, i_k) \subset \underline{q}$ of the odd coordinate directions:

$$\begin{aligned} \iota_I : \mathcal{O} = C[\theta^1, \dots, \theta^q] &\longrightarrow C[\theta^{i_1^c}, \dots, \theta^{i_l^c}] \qquad \text{where } I^c = (i_1^c, \dots, i_k^c) \\ j_I : C &\longrightarrow C[\theta^{i_1}, \dots, \theta^{i_k}] \end{aligned} \tag{5.16}$$

Clearly, $\Lambda_n \otimes \mathcal{O}$ becomes an $\Lambda_n \otimes C$ module via j^* and since $C \subset \mathcal{O}_0$, we may commute elements of C with any other elements. In analogy to the operation $\bigwedge^\bullet \mathbb{R}^n \otimes C^\infty(\tilde{X}) \otimes_{\iota^*} -$, which restricts a vector field along $\Phi \in \underline{SC}^\infty(X, Y)(\mathcal{P}_n)$ to a vector field along $\Phi_0 := \iota_* \Phi$, we can use j^* to extend a vector field along Φ_0 to one along Φ. To this end, we set

$$j^* \Phi_0^* \mathcal{T}_Y := \bigwedge\nolimits^\bullet \mathbb{R}^n \otimes C\theta^1 \cdots \theta^q \otimes_{j^*} \Phi_0^* \mathcal{T}_Y$$

This construction obviously generalizes to the case when we only extend by some odd directions to $C[\theta^{i_1}, \dots, \theta^{i_k}]$.

Lemma 5.34 *We have $j^* \Phi_0^* \mathcal{T}_Y \cong \bigwedge^\bullet \mathbb{R}^n \otimes C\theta^1 \cdots \theta^q \otimes_{j^* \Phi_0} \mathcal{T}_Y$. Moreover, we have a well defined embedding*

$$j^* \Phi_0^* \mathcal{T}_Y \longrightarrow \Phi^* \mathcal{T}_Y \qquad \theta^1 \cdots \theta^q f^i \otimes_{j^* \Phi_0^*} \frac{\partial}{\partial y^i} \mapsto \theta^1 \cdots \theta^q f^i \otimes_\Phi \frac{\partial}{\partial y^i}$$

In particular, sections of $j^* \Phi_0^* \mathcal{T}_Y$ can be canonically identified with vector fields along Φ and we will always do so.

Proof Since $\bigwedge^\bullet \mathbb{R}^n \otimes C\theta^1 \cdots \theta^q$ is a $\bigwedge^\bullet \mathbb{R}^n \otimes C$-supermodule, we can form the product given in the definition. The transitivity rule from proposition 2.25 (a) yields the first isomorphism. To prove the existence of the embedding, we consider

$$\Lambda_n \otimes C\theta^1 \cdots \theta^q \times \mathcal{T}_Y \longrightarrow \Phi^* \mathcal{T}_Y \qquad (f^i, \tfrac{\partial}{\partial y^i}) \mapsto f^i \otimes_\Phi \frac{\partial}{\partial y^i} \tag{5.17}$$

For $g \in \Lambda_n \otimes C\theta^1 \cdots \theta^q$ and $f \in \mathcal{R}$, we clearly have $g\Phi^*(f) = g j^* \Phi_0^*(f)$ since all θ^α already appear in g. Thus, the map in (5.17) is $\mathcal{R}$-bilinear and induces a map on $\Lambda_n \otimes C\theta^1 \cdots \theta^q \otimes_{j^* \Phi_0^*} \mathcal{T}_Y$. Since $\{\frac{\partial}{\partial y^i}\}$ is a local basis for the module $\mathcal{T}_Y$, it easily follows that this map is injective.

□

Remark 5.35 Note that the preceding embedding is well defined because all odd coordinates necessarily appear in the elements of $\Lambda_n \otimes C\theta^1 \cdots \theta^q$. There is no corresponding embedding of $\Lambda_n \otimes C\theta^{i_1} \cdots \theta^{i_k}$ for $k < q$.

In the next lemma, we collect some simple rules for calculation.

Lemma 5.36 *Let $V \in \Phi_0^* \mathcal{T}_Y$ and $I, J \subset \underline{q}$.*

(a) We have $\iota_I \iota_J = \iota_J \iota_I = \iota_{I \cup J}$.

(b) If $I \cap J = \varnothing$, we can identify the following elements in $\theta^{I \cup J} \Phi_0^* \mathcal{T}_Y$:

$$\theta^I \otimes_{j_I} (\theta^J \otimes_{j_J} V) \cong \theta^I \theta^J \otimes_{I \cup J} V \cong (-1)^{\|I\| \|J\|} \theta^J \otimes_{j_J} (\theta^I \otimes_{j_I} V)$$

(c) We have $\iota_I^* \nabla_{\overline{I}}^{j_I \Phi_0} (\theta^I \otimes_{j_I} V) = V$

(d) For $I \cap J = \varnothing$ and $W \in j_J^* \Phi_0^* \mathcal{T}_Y$, we can identify

$$\theta^I \otimes_{j_I} \nabla_J^{j_J \Phi_0} W \cong (-1)^{\|I\| \|J\|} \nabla_J^{j_{I \cup J} \Phi_0} (\theta^I \otimes_{j_I} W)$$

Proof Part a) and b) are obvious. Part c) follows from the fact, that terms containing θs are set to zero by ι_I^*. Hence, applying Leibniz rule, the only term which contributes to the left hand side is the one where all the derivatives act on the θ^I-factor. Applying ι_I^* then clearly gives X. The last statement is obtained by observing, that differentiating θ^I w.r.t. one of the θ^j-directions ($j \in J$) is always zero since $I \cap J = \varnothing$. More precisely, writing $Y = Y^m \otimes_{j_J \Phi_0} \frac{\partial}{\partial y}$, and assuming for simplicity $\|J\| = 1,\ J = (\alpha)$, we arrive at

$$\begin{aligned} \nabla_\alpha^{j_{(I,\alpha)} \Phi_0} (\theta^I \otimes_{j_I} Y) = (-1)^{\|I\|} \theta^I \frac{\partial Y^m}{\partial \theta^\alpha} \otimes_{j_{(I,\alpha)} \Phi_0} \frac{\partial}{\partial y^m} + \\ (-1)^{\|I\| + \|I\| |y^n| + |Y^m| (1 + |y^n|)} \theta^I Y^m \frac{\partial j_{(I,\alpha)} \Phi_0^n}{\partial \theta^\alpha} \otimes_{j_{(I,\alpha)} \Phi_0} \nabla_{y^n} \frac{\partial}{\partial y^m} \end{aligned}$$

Since $\frac{\partial (j_{(I,\alpha)} \Phi_0^n)}{\partial \theta^\alpha} = j_I \frac{\partial (j_\alpha \Phi_0^n)}{\partial \theta^\alpha}$, lemma 5.36 b) shows, that this expression equals $(-1)^{\|I\|} \theta^I \otimes_{j_I} \nabla_\alpha^{j_\alpha \Phi_0} Y$. The general statement the follows by induction on $\|J\|$.

□

We can now state and prove the reconstruction theorem for vector fields:

Theorem 5.37 *Let $\Phi^{(n)} \in \underline{SC}^\infty(X, Y)$ and $\{W_I\}_{I \subset \underline{q}}$ a family of vector fields along $\varphi^{(n)} := \iota^* \Phi^{(n)}$. Then, there exist a unique vector field V along $\Phi^{(n)}$, such that its components V_I, defined in 5.32, are given by the family $\{W_I\}$. More precisely, we have the following reconstruction formula:*

$$V = \sum_{I \subset \underline{q}} \nabla_{I^c}^\Phi \left(\theta^1 \cdots \theta^q \otimes_{j^*} \iota^* \nabla_{\overline{I}}^\Phi X \right) \operatorname{sign}(\overline{I^c}, I) \tag{5.18}$$

Proof We proceed by induction on the odd dimension q of X. The case $q = 0$ is trivial. In the case $q = 1$, it is clear that $\operatorname{sign}(\overline{I^c}, I) = 1$, so we have to prove

$$V = \nabla_\theta^\Phi (\theta \otimes_{j^*} \iota^* V) + \theta \otimes_{j^*} \iota^* (\nabla_\theta^\Phi X) \tag{5.19}$$

$$W_0 = \iota^* (\nabla_\theta^\Phi (\theta \otimes_{j^*} W_0) + \theta \otimes_{j^*} W_1) \tag{5.20}$$

$$W_1 = \iota^* \nabla_\theta^\Phi (\nabla_\theta^\Phi (\theta \otimes_{j^*} W_0) + \theta \otimes_{j^*} W_1) \tag{5.21}$$

for any $W_0, W_1 \in \varphi^{(n)*}\mathcal{T}_Y$. We omit the proof of (5.19), it can be easily verified by using the identification of lemma 5.34 and the definition of ∇^Φ. (5.20) follows immediately from lemma 5.36 (c). To prove the third part, we denote by $O(\theta)$ an expression, which contains a factor θ. Using coordinate expansion, we compute:

$$\begin{aligned}
&\iota^*\nabla^\Phi_\theta(\nabla^\Phi_\theta(\theta \otimes_{j^*} W_0) + \theta \otimes_{j^*} W_1 \\
&\qquad = \iota^*\Big((-1)^{|y^k||V_0^i|}\tfrac{\partial \Phi^k}{\partial \theta} W_0^i \otimes_\Phi \nabla_k \tfrac{\partial}{\partial y^i} + (-1)^{|y^k||\theta V_0^i|}\tfrac{\partial}{\partial \theta}(\tfrac{\partial \Phi^k}{\partial \theta}\theta V_0^i) \otimes_\Phi \nabla_k \tfrac{\partial}{\partial y^i} \\
&\qquad\quad + \tfrac{\partial}{\partial \theta}(\theta V_1^j) \otimes_\Phi \tfrac{\partial}{\partial y^j} + O(\theta)\Big) \\
&\qquad = \iota^*(W_1^j \otimes_{\varphi^{(n)}} \tfrac{\partial}{\partial y^j}) = W_1
\end{aligned}$$

To do the induction step $q \to q+1$, recall that we have a fixed set of global odd coordinates and that all tensor products used in this section are super tensor products. Thus, we have

$$\Lambda_n \otimes \mathcal{O} \cong \Lambda_n \otimes C[\theta^0, \ldots, \theta^q] \cong \Lambda_{n+q} \otimes C[\theta^0] \tag{5.22}$$

This means that we can view $\Lambda_n \otimes \mathcal{O}$ either as function algebra of $\mathcal{P}_n \times X = \mathcal{P}_n \times \tilde{X}^{p|1+q}$ or of $\mathcal{P}_{n+q} \times \tilde{X}^{p|1}$ as long as the coordinates $\{\theta^\alpha\}$ are not changed. To the last configuration, we can apply (5.19) since this equation holds for arbitrary n. We will first prove (5.18) and then show, that the components are indeed given by the prescribed vector fields.
Denoting by $\iota_0 : \Lambda_n \otimes C[\theta^0, \ldots, \theta^q] \longrightarrow \Lambda_n \otimes C[\theta^1, \ldots, \theta^q]$ the restriction map w.r.t. the first odd coordinate θ^0 and similarly, by $j_0 : \Lambda_n \otimes C[\theta^1, \ldots, \theta^q] \hookrightarrow \Lambda_n \otimes C[\theta^0, \ldots, \theta^q]$ the inclusion, (5.19) yields

$$V = \nabla^\Phi_{\theta^0}(\theta^0 \otimes_{j_0^*} \iota_0^* V) + \theta^0 \otimes_{j_0^*} \iota_0^* \nabla^\Phi_{\theta^0} V \tag{5.23}$$

The two expressions $\iota_0^* X$ and $\iota_0^* \nabla^\Phi_{\theta^0} X$ are vector fields along $\iota_0^*\Phi : \mathcal{R} \longrightarrow \Lambda_n \otimes C[\theta^1, \ldots, \theta^q]$ and we can apply the induction hypothesis to them:

$$\begin{aligned}
\iota_0^* V &= \sum_{J \subset \underline{q}} \nabla^{\iota_0\Phi}_{J^c}\left(\theta^1 \cdots \theta^q \otimes_{j_q^*} \iota_q^* \nabla^{\iota_q\Phi}_{\overline{J}} V\right) sign(\overline{J^c}, J) \\
\iota_0^* \nabla^\Phi_{\theta^0} V &= \sum_{J \subset \underline{q}} \nabla^{\iota_0\Phi}_{J^c}\left(\theta^1 \cdots \theta^q \otimes_{j_q^*} \iota_q^* \nabla^{\iota_q\iota_0\Phi}_{\overline{J}} \iota_0^* \nabla^{\iota_0\Phi} V\right) sign(\overline{J^c}, J)
\end{aligned}$$

Here, ι_q and j_q denote the projection and embedding for the remaining odd variables $\theta^1, \ldots, \theta^q$. Inserting both expressions in (5.23), and simplifying using lemma 5.36, we obtain

$$\begin{aligned}
V = &\sum_{J \subset \underline{q}} \nabla^\Phi_{(0,J^c)}\left(\theta^0 \ldots \theta^q \otimes_{j^*} \iota^* \nabla^\Phi_{\overline{J}} V\right) \mathrm{sign}(\overline{J^c}, J)(-1)^{\|J^c\|} \\
&+ \sum_{J \subset \underline{q}} \nabla^\Phi_{J^c}\left(\theta^0 \ldots \theta^q \otimes_{j^*} \iota^* \nabla^\Phi_{(\overline{J},0)} V\right) \mathrm{sign}(\overline{J^c}, J)(-1)^{\|J^c\|}
\end{aligned}$$

The first part of this sum corresponds to all the indices $I \subset \{0, \ldots, q\}$ not containing 0. Setting $I := J$, we have $I^c = (0, J^c)$ and $\mathrm{sign}(\overline{I^c}, I) = (-1)^{\|J^c\|} sign(\overline{J^c}, J)$. Analogously, the

second part contains precisely those indices I containing 0 and we find the same relations for the signs. This proves the formula.
Let now $\{W_I\}_{I\subset\{0,\dots,q\}}$ be vector fields along $\varphi^{(0)}$. By induction hypothesis, we obtain unique vector fields W', W" along $\iota_0\Phi$, given by (5.18), such that their components are given by $\{W_J | J\subset \underline{q}\}$ and $\{W_{(0,J)} | J\subset \underline{q}\}$ respectively. Viewing W', W'' as vector fields on $\mathcal{P}_{n+q}\times\tilde{X}$ by the identification (5.22), we obtain a unique vector field on $\mathcal{P}_n\times\tilde{X}^{p|1+q}\cong\mathcal{P}_{n+q}\times\tilde{X}^{p|1}$ along Φ:

$$W := \nabla^{\Phi}_{\theta^0}(\theta^0\otimes_{j_0} W') + \theta^0\otimes_{j_0} W'' = \sum_{I\subset\{0,\dots,q\}} \nabla^{\Phi}_{I}(\theta^0\cdots\theta^q\otimes_j W_{\overline{I}})$$

By (5.20) and (5.21), its components, considered as a vector field on $\mathcal{P}_{n+q}\times\tilde{X}^{p|1}$, are W' and W''. Using lemma 5.36, this leads by construction of W to the following components w.r.t the coordinates $\{\theta^0,\dots,\theta^q\}$ for $J\subset\underline{q}$:

$$\begin{aligned}\iota^*\nabla^{\Phi}_{\overline{J}}W &= \iota_q^*\nabla^{\iota_0\Phi}_{\overline{J}}\iota_0^*W = \iota_q^*\nabla^{\iota_0\Phi}_{\overline{J}}W' = W_{\overline{J}}\\ \iota^*\nabla^{\Phi}_{(\overline{J},0)} &= \iota_q\nabla^{\iota_0\Phi}_{\overline{J}}\iota_0^*\nabla^{\Phi}_{\theta^0}W = \iota_q^*\nabla^{\iota_0\Phi}_{\overline{J}}W'' = W_{(\overline{J},0)}\end{aligned}$$

This shows that the components of W are given by the prescribed $\{\mathcal{W}_I\}$ and finishes the proof.

□

We will now use the reconstruction theorem for vector fields to prove the corresponding result for points of $\underline{SC}^\infty(X,Y)$:

Theorem 5.38 *Every $\Phi\in\underline{SC}^\infty(X,Y)$ gives rise to components defined in 5.31, which uniquely determine Φ. Moreover, for any morphism $\Psi_\varnothing$ and vector fields Ψ_I along $\Psi_\varnothing$ ($\varnothing\neq I\subset\underline{q}$), there is an element $\Phi\in\underline{SC}^\infty(X,Y)$ such that its components are given by $\{\Psi_\varnothing,\Psi_I\}$.*

Proof We will prove the statement using induction on q. The case $q=0$ is again trivial. Before doing the induction step $q\to q+1$, we will discuss the central part of the argument that will be used.
For $g\in\mathcal{R}$, we decompose $\Phi(g)$ with respect to powers of the coordinates $\{\theta^0,\dots,\theta^q\}$ (this decomposition is similar but not equal to that in (5.8)):

$$\begin{aligned}\Phi(g) &= \sum_{I\subset\{0\dots q\}}\theta^I\Phi_I(g) = \sum_{I\subset\underline{q}}\theta^I\Phi_I(g) + \sum_{I\subset\underline{q}}\theta^0\theta^I\Phi_{0,I}(g)\\ &= j_0(\iota_0\Phi(g)) + \theta^0\otimes_{j_0}\iota_0^*d\Phi(\tfrac{\partial}{\partial\theta^0})(g) \qquad (5.24)\end{aligned}$$

The last equality follows directly from the definition of $d\Phi(\frac{\partial}{\partial\theta^0})$. Thus, Φ is decomposed in $j_0*\iota_0*\Phi$ and $\theta^0\otimes_{j_0^*}\iota_0^*d\Phi(\frac{\partial}{\partial\theta^0})$, which is a vector field along $j_0*\iota_0*\Phi$ and even by definition. Consequently, $\iota_0^*d\Phi(\frac{\partial}{\partial\theta^0})$ is a vector field along $\iota_0*\Phi$ and odd. Conversely, given a morphism $\Phi_0:\tilde{X}^{p|q-1}\longrightarrow Y$ and an odd vector field $\xi\in\Phi_0^*\mathcal{T}_Y$, we can form

$$\Phi':\mathcal{R}\longrightarrow\Lambda_n\otimes C[\theta^0,\dots,\theta^q] \qquad \Phi'(g) := j_1^*\Phi_0(g) + \theta^0\otimes_{j_0}\xi(g) \qquad (5.25)$$

Using the nilpotency of θ^0, it is straightforward to verify, that $\Phi'(fg) = \Phi'(f)\Phi'(g)$ which shows that Φ' defines an element of $\underline{SC}^\infty(X,Y)(\mathcal{P}_n)$.

We now do the induction step: Let $\Phi : \tilde{X}^{p|1+q} \longrightarrow Y$ be decomposed into $j_0 * \iota_0 * \Phi$ and $\theta^0 \otimes_{j_0} \iota_0^* d\Phi(\frac{\partial}{\partial\theta^0})$ as in (5.24). By induction hypothesis, $\iota_0^*\Phi : \tilde{X}^{p|q} \longrightarrow Y$ (and hence $j_0^*\iota_0^*\Phi$) is determined by all the components of the form

$$\begin{aligned} \iota_q^*\iota_0^*\Phi &= \iota^*\Phi && (5.26)\\ \iota_q^* d(\iota_0^*\Phi)(\tfrac{\partial}{\partial\theta^\alpha}) &= \iota^* d\Phi(\tfrac{\partial}{\partial\theta^\alpha}) && \text{where } \alpha = 1,\dots,q\\ \iota_q^*\nabla^{\iota_0\Phi}_{\underline{J}\setminus j_1}\iota_0^* d\Phi(\tfrac{\partial}{\partial\theta^{j_1}}) &= \iota^*\nabla^{\Phi}_{\underline{J}\setminus j_1}(d\Phi(\tfrac{\partial}{\partial\theta^{j_1}})) && \text{where } J = (j_1,\dots,j_k) \subset \underline{q} \end{aligned}$$

where we have used lemma 5.36 to rearrange terms. By theorem 5.37, the vector field $\theta^0 \otimes_{j_0} \iota_0^* d\Phi(\frac{\partial}{\partial\theta^0})$ along Φ is determined by its components $\iota^*\nabla^{\Phi}_{\underline{J}}\iota_0^* d\Phi(\frac{\partial}{\partial\theta^0})$ for $J \subset \{0,1,\dots,q\}$. Here, we only have to take into account the case where $0 \in J$ since all other components are zero by definition of ι^*, so we may write $J = (0, I)$. Hence, the vector field is determined by the components

$$\begin{aligned} \iota^*(\theta^0 \otimes_{j_0} \iota_0^* d\Phi(\tfrac{\partial}{\partial\theta^0})) &= \iota^* d\Phi(\tfrac{\partial}{\partial\theta^0}) && (5.27)\\ \iota^*\nabla^{\Phi}_{(\underline{J},0)}(\theta^0 \otimes_{j_0} \iota_0^* d\Phi(\tfrac{\partial}{\partial\theta^0})) &= \iota^*\nabla^{\Phi}_{\underline{J}} d\Phi(\tfrac{\partial}{\partial\theta^0}) && \text{where } J \subset \underline{q} \end{aligned}$$

Together, (5.26) and (5.27) contain precisely the components of Φ as defined in definition 5.31.

Conversely, given $\{\Psi_I\}_{I\subset\{0,1,\dots\}}$, by induction hypothesis, the components $\{\Psi_I\}_{I\subset\underline{q}}$ form $\Phi' \in \underline{SC}^\infty(\tilde{X}^{p|q}, Y)(\mathcal{P}_n)$. The remaining components are of the form $\Psi_I = \Psi_{(0,J)}$ and vector fields along $\Psi_\varnothing = \iota_q^*\Phi'$. Since $J \subset \underline{q}$, this family of vector fields define a unique vector field Φ'' along Φ' by theorem 5.37. Thus, in accordance with (5.25), we can define an element of $\underline{SC}^\infty(\tilde{X}^{p|1+q}, Y)(\mathcal{P}_n)$ by

$$\Phi := j_0^*\Phi' + \theta^0 \otimes_{j^0} \Phi''$$

It remains to show that the components Φ_I, obtained from (5.26) and (5.27), are in fact given by $\{\Psi_I\}_{I\subset\{0,1,\dots,q\}}$. For $0 \notin I$, these are obviously given by the corresponding components of Φ', which in turn are given by $\{\Psi_I\}_{I\subset\underline{q}}$ by induction. For $I = (0, J)$, we obtain the components $\iota^*\nabla^{\Phi}_{(\underline{J})} d\Phi(\frac{\partial}{\partial\theta^0}) = \iota_q^*\nabla^{\Phi'}_{\underline{J}}\Phi''$ and these are precisely the terms $\{\Psi_I\}_{0\in I}$ by construction and theorem 5.37. This completes the proof.

□

5.6 Components on general supermanifolds

So far, component fields have been introduced w.r.t. globally defined odd coordinates, in other words, we assumed that "the" Batchelor bundle is globally trivializable (this condition clearly does not depend on the specific choice of the bundle). To be able to apply the component formalism for an arbitrary supermanifold $X = (\tilde{X}, \mathcal{O})$, we may work on coordinate neighbourhoods $U \subset \tilde{X}$ and try to obtain global statements or globally well defined objects (e.g. by glueing) at the end. However, this is not a promising approach to solve variational problems. Even the existence of closed geodesics is usually proven by global variational methods (cf. section 6.2), it is in general impossible to obtain these results by solving the geodesic equation locally and gluing the segments to obtain a closed geodesic. Thus, we proceed in a different fashion: We enlarge the supermanifold X to a suitable $\mathbb{X}$ where the latter admits global odd coordinates. Then we discuss and solve the problem on $\mathbb{X}$ and restrict back to X at the end. Here we will only discuss the foundations and work out the details for the variational problems in later chapters.

We need the following well known result (see e.g. [31], III.5.7 and III.5.8 for the topological case and [22], 2.2.2 for an argument in the smooth complex case):

Theorem 5.39 *Let M be a smooth manifold and $E \longrightarrow M$ a $\mathbb{K}$-vector bundle on M ($\mathbb{K} = \mathbb{R}, \mathbb{C}$). Then there exist a $\mathbb{K}$-vector bundle $E' \longrightarrow M$ such that the vector bundle $E \oplus E'$ is trivializable.*

Given a supermanifold $X = (\tilde{X}, \mathcal{O})$, we choose a Batchelor bundle $E \longrightarrow \tilde{X}$ and denote the (non-canonical) associated superdiffeomorphism by $\Phi_E : (\tilde{X}, \Gamma(\bigwedge^\bullet E)) \longrightarrow X$. Using theorem 5.39, we furthermore choose an inverse bundle $E' \longrightarrow \tilde{X}$ and defined the enlarged supermanifold by

$$\mathbb{X} := (\tilde{X}, \Gamma(\textstyle\bigwedge^\bullet(E \oplus E'))) \tag{5.28}$$

We now have a well defined map of vector bundles of rank $2^{rk(E)+rk(E')}$

$$\textstyle\bigwedge^\bullet E \otimes \bigwedge^\bullet E' \longrightarrow \bigwedge^\bullet(E \oplus E') \qquad\qquad (e \otimes e') \mapsto e \wedge e'$$

It is easily seen to be injective by inserting elements of a basis and since the ranks of the bundles on the left and the right hand side are equal, this map is an isomorphism. Moreover, if $\otimes$ denotes the super tensor product, this also defines an isomorphism of super algebra bundles. For the sheaves of super functions, this implies:

$$\Gamma(\textstyle\bigwedge^\bullet E) \otimes \Gamma(\bigwedge^\bullet E') \cong \Gamma(\bigwedge^\bullet E \otimes \bigwedge^\bullet E') \cong \Gamma(\bigwedge^\bullet(E \oplus E'))$$

where the last isomorphism is induced by the wedge product. Note that this is not the product in the category of supermanifolds (cf. proposition 3.10) since $\mathbb{X}$ still has the same underlying manifold as X. Now let $\tilde{pr} : \Gamma(\bigwedge^\bullet E') \twoheadrightarrow \Gamma(\bigwedge^0 E') = C^\infty(\tilde{X})$ denote the canonical projection. Using the isomorphism $\Gamma(\bigwedge^\bullet E) \otimes C^\infty(\tilde{X}) \cong \Gamma(\bigwedge^\bullet E)$, we obtain an inclusion $\iota_{\mathbb{X}} : X \hookrightarrow \mathbb{X}$ and a projection $\pi_{\mathbb{X}} : \mathbb{X} \twoheadrightarrow X$ defined on the level of sheaves as follows:

$$\begin{aligned} \iota_{\mathbb{X}} :\ & \Gamma(\textstyle\bigwedge^\bullet(E \oplus E')) \xrightarrow{\wedge^{-1}} \Gamma(\bigwedge^\bullet E) \otimes \Gamma(\bigwedge^\bullet E') \xrightarrow{id \otimes \tilde{pr}} \Gamma(\bigwedge^\bullet E) \otimes C^\infty(\tilde{X}) \xrightarrow{\sim} \Gamma(\bigwedge^\bullet E) \xrightarrow{\Phi_E^{-1}} \mathcal{O} \\ \pi_{\mathbb{X}} :\ & \mathcal{O} \xrightarrow{\Phi_E} \Gamma(\textstyle\bigwedge^\bullet E) \xrightarrow{id \otimes 1} \Gamma(\bigwedge^\bullet E) \otimes \Gamma(\bigwedge^\bullet E') \xrightarrow{\wedge} \Gamma(\bigwedge^\bullet(E \oplus E')) \end{aligned} \tag{5.29}$$

In both cases, the underlying map of smooth manifolds is of course given by $id_{\tilde{X}}$.

We now compare this extension of X with the notation introduced in section 5.5. Let $U \subset \tilde{X}$ be an open set such that there exist odd coordinate systems $\theta^1, \ldots, \theta^q$ and $\theta^{q+1}, \ldots, \theta^{q+rk(E')}$ on $X|_U$ and $(U, \Gamma(U, \bigwedge^\bullet E'))$, respectively. We obtain an odd coordinate system $\theta^1, \ldots, \theta^{q+rk(E')}$ for $\mathbb{X}$ over U. In (5.16) and below, we discussed the restriction to and the prolongation by a subset of odd coordinates. By construction we have

Proposition 5.40 *Under the identifications given by the coordinate systems on $U \subset \tilde{X}$ the inclusion $\iota_{\mathbb{X}}$ corresponds to $\iota_{(q+1,\ldots,q+rk(E'))}$ and the projection $\pi_{\mathbb{X}}$ to $j_{(q+1,\ldots,q+rk(E'))}$. In particular, we can apply all the calculation rules, established for ι and j in chapter 5.5, to $\iota_{\mathbb{X}}$ and $\pi_{\mathbb{X}}$.*

Concluding this chapter, one important feature of the construction should be emphasized:

Remark 5.41 All the construction in subsection 5.6 left the underlying manifold $\tilde{X}$ untouched. We only added odd coordinate directions given by an additional vector bundle over the same smooth manifold $\tilde{X}$. Loosely speaking one might say that only part of a linear structure was changed whereas the underlying nonlinear part remained unchanged. This will prove to be crucial in chapter 6 because it will enable us to extend a variational problem to $\mathbb{X}$ and prove that its solutions, restricted to X, solve the original problem. This method would fail if we had also enlarged $\tilde{X}$, e.g. by embedding it into some $\mathbb{R}^N$.

6 Variational Problems on Supermanifolds

In this section, we will apply the previously discussed formalism to study variational problems on supermanifolds. We will not discuss a general theory of calculus of variation using jet theory (see e.g. [45] for results not using the functorial language) but focus on two concrete examples for functionals arising in geometry. For $(N, \langle,\rangle)$ a Riemannian manifold, we consider

(a) The energy functional for curves $\Phi : \mathbb{R}^{1|1} \longrightarrow (N, \langle,\rangle)$ (and $S^{1|1} \longrightarrow (N, \langle,\rangle)$, where $S^{1|1}$ is a supermanifold defined by a Batchelor bundle $E \to S^1$ of $\mathrm{rk}(E) = 1$) given by

$$E_1(\Phi) := \int_{\mathbb{R}^{1|1}} dtd\theta \left\langle \dot{\Phi}, d\Phi(D) \right\rangle \qquad \text{where} \qquad D := \tfrac{\partial}{\partial\theta} - \theta \tfrac{\partial}{\partial t} \tag{6.1}$$

(b) An energy functional for maps $(X, G) \longrightarrow (N, \langle,\rangle)$ from a compact super Riemannian manifold X of dimension $m|2$ into N given by (cf. definition 3.46 of vol_G)

$$E_2(\Phi) := \tfrac{1}{2} \int_X vol_G \langle d\Phi, d\Phi\rangle \tag{6.2}$$

The first functional is introduced and discussed e.g. in [11] §1.3 and [61]. A theory of solutions is sketched in this case but not all the details of the proof are given. We will reproduce this result (theorem 6.24) and extend it to maps defined on $S^{1|1}$. In the latter case, the situation is more involved. The second functional is a generalization of the energy functional which is used in the theory of harmonic maps (cf section 6.2) and we will again show the existence of solutions under assumptions of the geometry of the target space. A similar functional has been considered in [48] and [33], the authors study similar functionals for maps into Lie groups and symmetric spaces. We will first clarify the notion of a "critical point" for functionals in super geometry and shortly review some results from the theory of harmonic maps as well as linear elliptic equations. Both are needed in the approach that we will take to find critical points. It consists in first decomposing the equations of motions into its components and then applying the theory for PDE on ordinary Riemannian manifolds. This procedure also reveals the geometric meaning of the super equations of motion. We do not try to set up variational methods (e.g. minimizing the functional or using a heat flow) on the supermanifold itself, this topic lies outside the scope of this work and might be addressed in future research.

6.1 Functionals and critical points

If M and N are smooth manifolds, a critical point φ_c of the energy functional

$$E : C^\infty(M, N) \longrightarrow \mathbb{R} \qquad\qquad \varphi \mapsto E(\varphi) := \int_M \|d\varphi\|^2 dvol_M$$

is usually viewed as a map, namely $\varphi_c : M \longrightarrow N$ which has a certain regularity. On the other hand, it is of course possible to view φ_c as a point (or an element) of a space $\underline{Crit}$ of solutions which is a subspace of $C^\infty(M, N)$ or of some suitable Sobolev space. Both points of view are equivalent but the second one is more natural in the context of super geometry.

Here, the space of smooth maps is given by the functor $\underline{SC}^\infty(X,Y)$ defined in definition 4.26 and the subspace of critical points is then given by a subfunctor

$$\underline{Crit} \subset \underline{SC}^\infty(X,Y) \in \mathrm{Set}^{\mathrm{Gr}}$$

It may be defined by equations of motion and is sometimes presentable by a supermanifold, but this is not always true. By definition, such a functor has higher points $\underline{Crit}(\Lambda_n)$ which are morphisms $\mathcal{P}_n \times X \longrightarrow Y$ in the category of supermanifolds. Critical 0-points are just morphisms $X \longrightarrow Y$ but a general critical n-point is not of this form but depends on the "odd parameters" contained in Λ_n. It is this dependency which gives rise to many of the interesting features of supergeometry as the existence of odd vector fields (see example 4.25) on an ordinary smooth manifold. Since $\underline{Crit}$ is a functor, it does not only act on the objects Λ_n but also on morphisms $\rho : \Lambda_n \longrightarrow \Lambda_m$, i.e. we have maps induced by $\underline{SC}^\infty(X,Y)$

$$\underline{Crit}(\Lambda_n) \subset \underline{SC}^\infty(X,Y)(\Lambda_n) \overset{\underline{Crit}(\rho)}{\longrightarrow} \underline{Crit}(\Lambda_m) \subset \underline{SC}^\infty(X,Y)(\Lambda_m)$$

satisfying the usual functoriality properties. This implies that, whenever there is a critical Λ_n-point $\varphi^{(n)}$ and a morphism $\rho : \Lambda_n \longrightarrow \Lambda_m$, then $\underline{Crit}(\rho)(\varphi^{(n)})$ must be a critical Λ_m-point. This demonstrates the necessity to construct the whole functor of critical points (or a subfunctor of it) and not only single critical points.
In a similar fashion, we have to extend the notion of a functional. Instead of being a map $C^\infty(M,N) \longrightarrow \mathbb{R}$, it is now a map (or natural transformation) of functors in $\mathrm{Man}^{\mathrm{Gr}}$ (or at least $\mathrm{Set}^{\mathrm{Gr}}$),

$$E : \underline{SC}^\infty(X,Y) \longrightarrow \overline{\mathbb{R}}$$

This means that E has to be a morphism in the sense of definition 4.21, i.e. compatible with all morphisms $\rho : \Lambda_n \longrightarrow \Lambda_m$. As mentioned in 4.28, we should also check $\Lambda_{n,\overline{0}}$-linearity of derivatives but since the smooth structure on $\underline{SC}^\infty(X,Y)$ has not been defined yet, we will restrict ourselves to the discussion of functoriality.
Applying these concepts to the category of ordinary smooth manifolds, they reduce to the notions familiar from geometric analysis because there is just one generator $\{pt\}$ (as discussed for the category of sets above definition 4.8) and one morphism $id_{\{pt\}} : \{pt\} \longrightarrow \{pt\}$ among generators.

Next, we give formal definitions for the structure described above following [61]. The two functionals in (6.1) and (6.2) are formally defined for ordinary morphisms of supermanifolds. However, they should be maps of functors. This means that geometric objects, defined on BKL-supermanifolds in chapter 3, have to be extended to $\mathcal{P}_n \times X$ and its tangent sheaf to be able to rigorously define the actions and compute equations of motion. The sheaf of functions of $\mathcal{P}_n \times X$ is given by $\Lambda_n \hat{\otimes} \mathcal{O} = \Lambda_n \otimes_{\mathbb{R}} \mathcal{O}$. Moreover, we have a canonical projection $pr_X : \mathcal{P}_n \times X \twoheadrightarrow X$ given on functions by $pr_X^* f = 1 \otimes_{\mathbb{R}} f$ and similarly for $pr_{\mathcal{P}_n}$. For the tangent sheaves, we have

$$\mathcal{T}_{\mathcal{P}_n \times X} = pr_{\mathcal{P}_n}^* \mathcal{T}_{\mathcal{P}_n} \oplus \mathrm{pr}_X^* \mathcal{T}_X = \mathcal{O} \otimes_{\mathbb{R}} \mathcal{T}_{\mathcal{P}_n} \;\oplus\; \Lambda_n \otimes_{\mathbb{R}} \mathcal{T}_X$$

We choose the following extensions for the relevant objects:

Definition 6.1 *Let Λ be a finite dimensional Grassmann algebra. Then we do the following extensions by Λ-super linearity:*

(a) An $\mathbb{R}$-super vector space V is extended to the Λ-module $\Lambda \otimes_{\mathbb{R}} V$, $v \in V$ is mapped to $1 \otimes v$. In the same way, elements of V^* are extended by Λ-super linearity to $\Lambda \otimes_{\mathbb{R}} V^*$.

(b) An $\mathcal{O}$-module $\mathcal{E}$ on a supermanifold X is embedded into the $\Lambda \otimes_{\mathbb{R}} \mathcal{O}$-module by $\mathrm{pr}_X^* \mathcal{E}$ by $e \mapsto 1 \otimes_{pr_X} e$.

Note that the second part of this definition is special case of the first part. Both are consistent since $(\Lambda \otimes_{\mathbb{R}} \mathcal{O}) \otimes_{pr_X} \otimes \mathcal{E} \cong \Lambda \otimes_{\mathbb{R}} \mathcal{E}$. The following examples will be used later:

Example 6.2

(a) A vector field $\xi \in \mathcal{T}_X$ is extended to $1 \otimes \xi \in \mathrm{pr}_X^* \mathcal{T}_X \subset \mathcal{T}_{\mathcal{P}_n \times X}$. The derivation defined in this way is in particular $\Lambda_{\overline{0}}$-linear, i.e. $1 \otimes \xi(\lambda g) = \lambda\, 1 \otimes \xi(g)$.

(b) A metric $\langle .,. \rangle$ on some vector bundle $\mathcal{E}$ over X is an element of $\mathcal{E} \otimes_{\mathcal{O}} \mathcal{E}$. It is hence mapped to $1 \otimes \langle , \rangle \in pr_X^*(\mathcal{E} \otimes_{\mathcal{O}} \mathcal{E}) \cong (pr_X^* \mathcal{E}^*) \otimes_{\Lambda \otimes \mathcal{O}} (pr_X^* \mathcal{E}^*)$, which is precisely the Λ-bi-super linear extension.

(c) Similarly, a linear connection $\nabla : \mathcal{E} \longrightarrow \Omega^1 \otimes_{\mathcal{O}} \mathcal{E}$ is extended to a Λ-super linear connection $\mathrm{pr}_X^* \mathcal{E} \longrightarrow \mathrm{pr}_X^* \Omega^1 \otimes_{\Lambda_n \otimes \mathcal{O}} \mathrm{pr}_X^* \mathcal{E}$.

(d) The Berezinian $Ber(\Omega^1)$ is extended to $\mathrm{pr}_X^* Ber(\Omega^1)$, so that the "volume forms" take coefficients in $\Lambda \otimes_{\mathbb{R}} \mathcal{O}$ but the generators are still given by elements of Ω^1. Thus, there is no integration with respect to the directions of $\mathcal{P}_n$ but rather a Λ-super linear extension of the $\mathbb{R}$-linear functional $\int : \Gamma(Ber(\Omega^1)) \longrightarrow \mathbb{R}$ to $\int : \Gamma(\mathrm{pr}_X^* Ber(\Omega^1)) \longrightarrow \Lambda$.

Remark 6.3 The preceding construction shows what was called "hybrid approach" in the introduction. We used the functorial formalism to define the infinite dimensional spaces $\underline{SC}^\infty(X, N)$, which can not be defined as a ringed spaces. However, all the geometric objects on the finite dimensional (super) manifolds are treated using ringed space language. This of course requires to extend these objects to the Λ-parameters in a natural way which is given by Λ-super linearity. Since we are also dealing with odd objects (e.g. the odd vector field D), we have to require Λ-super linearity rather than $\Lambda_{\overline{0}}$-linearity, which is sufficient in the functorial construction in section 4.2.

It is interesting to compare the choices made above with the fully functorial treatment. For the first functional defined on $X = \mathbb{R}^{1|1}$, it would be necessary to replace $D := \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial t}$ and $\frac{\partial}{\partial t}$ by n-points of $\underline{\Gamma}(X, TX)$, the volume element $dtd\theta$ by n-points of $\underline{\Gamma}(X, Ber(\Omega_X^1))$ etc. for all $n \in \mathbb{N}_0$. By equation (4.5), we have

$$\underline{\Gamma}(X, TX)(\Lambda_n) \cong (\Lambda_n \otimes_{\mathbb{R}} \Gamma(X, TX \oplus \Pi TX))_{\overline{0}}$$

Since $D \in \Gamma(X, \Pi TX)$ and $\frac{\partial}{\partial t} \in \Gamma(X, TX)$, there is a natural choice for the higher n-points given by $1 \otimes D$ and $1 \otimes \frac{\partial}{\partial t}$. This is precisely the choice made in definition 6.1 b). Similarly, $\mathbb{R}$-super vector spaces and $\mathbb{R}$-linear forms are replaced by the $\overline{\mathbb{R}}$-super modules represented

by them which corresponds to the choice in part a) of the definition. In this sense, the hybrid approach which is used in this work coincides with a fully functorial construction of the functionals.
It should be pointed out that choosing different extensions which depend in a nontrivial way on the parameters $\eta^i \in \Lambda_n$ would eventually break the functoriality of the resulting functionals E. The choice made here, where the only true dependence of E on the parameters in Λ_n is given through the fields Φ, is the only reasonable one.

Using these canonical extensions, we obtain

Proposition 6.4 *The functionals E_1 and E_2, extended to functionals (denoted by the same symbols) $E_1 : \underline{SC}^\infty(\mathbb{R}^{1|1}, N) \longrightarrow \overline{\mathbb{R}}$ and $\underline{SC}^\infty(X, N) \longrightarrow \overline{\mathbb{R}}$ indeed define natural transformations.*

Proof We only give the argument for E_1, naturality of E_2 can be proved in a similar way. The actions only depend on $\lambda \in \Lambda_n$ through $\Phi \in \underline{SC}^\infty(\mathbb{R}^{1|1}, N)$, so we are mainly interested in the coefficient functions in $\Lambda_n \otimes_{\mathbb{R}} \mathcal{O}^{1|1}$ occurring in the various modules. Since D is odd and $\frac{\partial}{\partial t}$ is even, we have

$$d\Phi(\tfrac{\partial}{\partial t}) = \textstyle\sum_i (\lambda_0^i f^i + \lambda_1^i g^i \theta) \otimes e_i \qquad d\Phi(D) = \textstyle\sum_i (\mu_1^i \tilde{f}^i + \mu_0^i \tilde{g}^i \theta) \otimes e_i$$

where $\{e_i\}$ is a local orthonormal frame, $\lambda_{0/1}^i, \mu_{0/1}^i \in \Lambda_{\overline{0}/\overline{1}}$ and f^i, g^i are smooth functions on $\mathbb{R}$. Taking into account that the Berezin integration picks the θ-coefficient of the integrand, we have $E_1(\Phi) = \int_{\mathbb{R}} dt \sum_i (-\lambda_1^i g^i \mu_1^i \tilde{f}^i + \lambda_0^i f^i \mu_0^i \tilde{g}^i)$. For $\varphi : \Lambda_n \longrightarrow \Lambda_m$, let $\hat{\Phi} := \underline{SC}^\infty(\varphi)\Phi$. Since φ acts only on the Λ_n-coefficients, we obtain

$$\begin{aligned} E(\hat{\Phi}) &= \int_{\mathbb{R}} dt \sum_i (-\varphi(\lambda_1^i) g^i \varphi(\mu_1^i) \tilde{f}^i + \varphi(\lambda_0^i) f^i \varphi(\mu_0^i) \tilde{g}^i) \\ &= \int_{\mathbb{R}} dt \sum_i \varphi(-\lambda_1^i g^i \mu_1^i \tilde{f}^i + \lambda_0^i f^i \mu_0^i \tilde{g}^i) \\ &= \varphi(E(\Phi)) \end{aligned}$$

because φ is linear.

□

Critical points are defined as in the classical setting, following [10], p.651, (ii). Again, we work functorially over Grassmann algebras:

Definition 6.5 *Let X, Y be supermanifolds and X compact.*

(a) A variation of $\Phi \in \underline{SC}^\infty(X, Y)(\Lambda)$ is an element $\Psi \in \underline{SC}^\infty(\mathbb{R} \times X, Y)(\Lambda)$ such that $\Psi \circ r = \Phi$. Here r denotes the morphism $\mathcal{P}_n \times X \cong \mathcal{P}_n \times \{0\} \times X \longrightarrow \mathcal{P}_n \times \mathbb{R} \times X$ which is induced by the evaluation map $f \mapsto f(0)$ on $C^\infty(\mathbb{R})$.

(b) *A critical n-point of a functional $E : \underline{SC}^\infty(X,Y) \longrightarrow \overline{\mathbb{R}}$ is an n-point $\Phi \in \underline{SC}^\infty(X,Y)(\Lambda_n)$ such that for each morphism $\varphi : \Lambda_n \longrightarrow \Lambda_m$ and for each variation of Ψ of $\underline{SC}^\infty(X,Y)(\varphi)\Phi$, we have*

$$\tfrac{d}{dt}\Big|_0 E(\Psi) = 0$$

The corresponding space of critical n-points will be denoted $\underline{Crit}(E)(\Lambda_n) \subset \underline{SC}^\infty(X,Y)(\Lambda_n)$.

Since $\underline{SC}^\infty(X,Y)$ is a functor, it induces an action of morphisms $\Lambda_n \longrightarrow \Lambda_m$ on $\underline{Crit}(E)(\Lambda_n)$. In fact, the definition of critical points makes sure that $\underline{Crit}(E)$ is a functor:

Proposition 6.6 *Let E be a functional as in the preceding definition. Then, the assignment*

$$\Lambda_n \mapsto \underline{Crit}(E)(\Lambda_n) \qquad (\varphi : \Lambda_n \longrightarrow \Lambda_m) \mapsto \underline{Crit}(E)(\varphi) := \underline{SC}^\infty(X,Y)(\varphi)$$

defines a subfunctor of $\underline{SC}^\infty(X,Y)$ in $\mathrm{Set}^{\mathrm{Gr}}$.

Proof We only have to check that $\underline{Crit}(E)(\varphi)(\underline{Crit}(E)(\Lambda_n)) \subset \underline{Crit}(E)(\Lambda_m)$. But this is trivial since the condition in 6.5 b) is invariant under the action of $SC^\infty(X,Y)(\varphi)$ by definition.

□

In subsequent sections, we will characterize the critical points of some concrete functionals by equations of motion. Using these, it is also possible to check the functor property of $\underline{Crit}$ by showing directly, that a solution of the equation is mapped to another one under $\underline{Crit}(E)(\varphi)$.

6.2 Harmonic maps and elliptic theory

We will give a very brief survey on some results on harmonic maps. There is a large amount of literature available on this subject, see e.g. [16], [17], [18] or the books [69], [58] and [41] Let (M,g) and (N,h) be two Riemannian manifolds. The energy of a smooth map $\varphi \in C^\infty(M,N)$ is the defined as by the expression

$$E(\varphi) := \tfrac{1}{2}\int_M \|d\varphi\|^2 vol_g \tag{6.3}$$

which is finite if M is assumed to be compact. The critical point s of this functional can be obtained using calculus of variations and one obtains

$$\varphi \text{ is a critical point} \qquad \Longleftrightarrow \qquad \tau(\Phi) = tr(\nabla d\varphi) = 0$$

Here ∇ denotes the Levi-Civita connection on N, pulled back along φ to M. τ is called the tension field and is a vector field along φ.

Definition 6.7 *The critical points of the functional* (6.3), *characterized by $\tau(\varphi) = 0$, are called harmonic maps.*

Remark 6.8

(a) In case $M = \mathbb{R}$ or $M = S^1$, the functional (6.3) gives the usual energy of a curve and its critical points are clearly (closed) geodesics. Hence, harmonic maps are a higher dimensional generalization of the theory of geodesics.

(b) The equation $\tau(\varphi) = 0$ is not linear, more precisely, it is a semilinear elliptic equation.

Since the differential equation for harmonic maps is not linear, the standard theory for elliptic equations can not be applied. Nevertheless, under certain assumptions on the geometry of N, existence of critical points could be proven by using heat equation techniques:

Theorem 6.9 (Eells & Sampson, [19], 11) *Let M and N be compact and let N have non-positive Riemannian sectional curvature. Let $\varphi : M \longrightarrow N$ be a continuous map. Then there exist a harmonic map $\varphi' : M \longrightarrow N$ which is homotopic to φ and satisfies $E(\varphi') \leq E(\varphi)$.*

Remark 6.10

(a) The compactness assumption on N can be replaced by the weaker requirement, that N is complete an satisfies a certain growth condition (see [19], 10). We will not use this more general form.

(b) Given a continuous map $f : M \longrightarrow N$, there can be several harmonic maps in the homotopy class given by f. One of the easiest examples is given by maps $S^1 \longrightarrow T^2$ (flat torus), where there are infinitely many closed geodesics homotopic to each other. In general, under the hypothesis that N is complete and of nonpositive sectional curvature, there is a result due to Hartman ([28], p. 674, statement (E)) saying that two harmonic mappings φ, φ' homotopic to each other are smoothly homotopic through a family of harmonic maps. Moreover, if there is a point $n \in \varphi(M)$ such that the curvature is negative there, then φ is unique in its homotopy class unless it is a closed geodesic ([28], p.675, statement (H)).

The existence of closed geodesics is actually valid without the assumption on the curvature of N, see [64], theorem 3.4.2 or [34], theorems 2.1.4 and 2.1.6:

Theorem 6.11 *Let N be compact. Then there is a closed geodesic in every homotopy class of closed curves, which attains the minimum of the energy in this class.*

We are now going to discuss some central results of the theory of elliptic partial differential equations which will be applied in subsequent parts of this chapter. We will follow [38], chapter III.1, II.4 and III.5. Let $E \longrightarrow M$ and $F \longrightarrow M$ be two vector bundles over M, equipped with Riemannian (or Hermitian) metrics. We then have the vector space $\Gamma(E)$ of smooth sections and its completion $L^2(E)$ with respect to the scalar product induced by the bundle metrics. We consider linear differential operators

$$P : \Gamma(E) \longrightarrow \Gamma(F)$$

These have been abstractly defined in section 5.2 and it was shown that each operator of order $\leq m \in \mathbb{N}$ can be written w.r.t. local coordinates x on M and local trivializations on E

and F in the form

$$P = \sum_{|\alpha|\leq m} A^\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{6.4}$$

The total symbol $\sigma(P)$ was defined in 5.23. It can be shown (see [38], p.167 and 168) that the part of $\sigma(P)$ associated to the derivatives of highest degree m, given by the coefficients $\{A^\alpha\}_{|\alpha|=m}$ in equation (6.4), defines a well defined section of $\mathrm{Sym}^m TM \otimes \mathrm{Hom}(E,F)$:

Definition 6.12 ([38], III.1.2) *The section $\sigma^p(P) \in \Gamma(\mathrm{Sym}^m TM \otimes \mathrm{Hom}(E,F))$ is called the principal symbol of P.*

Since there is a canonical isomorphism of $\mathrm{Sym}^m T_x X$ with the space of polynomial functions on T_x^*M of degree $\leq m$, we may evaluate $\sigma^p(P)$ for any fixed $\xi \in T_x^*M$ and obtain a linear map $\sigma_\xi^p(P) : E_x \longrightarrow F_x$.

Definition 6.13 ([38], III.1.3) *A linear differential operator is called elliptic if for each $\xi \in T^*M \setminus \{0\}$, the linear map $\sigma_\xi^p(P)$ is invertible.*

The following result is a consequence from Fredholm theory and is of fundamental importance for the theory of linear elliptic equations on vector bundles:

Theorem 6.14 ([38], III.5.5) *Let $P : \Gamma(E) \longrightarrow \Gamma(E)$ be a linear, elliptic, self-adjoint operator over a compact Riemannian manifold. Then there is a direct sum decomposition which is orthogonal w.r.t. the L^2-scalar product:*

$$\Gamma(E) = ker(P) \oplus im(P)$$

This implies the following corollary for inhomogeneous, elliptic equation:

Corollary 6.15 *Let P be as in theorem 6.14. The inhomogeneous equation $Pu = f$ has a solution if and only if f is L^2-orthogonal to $ker(P)$. In this case, the space of solutions is given by $u_0 + ker(P)$ for any solution u_0 of the inhomogeneous equation.*

Moreover, the eigenspaces of P which include in particular its kernel have the following nice properties:

Theorem 6.16 ([38], II.5.8) *Let $P : \Gamma(E) \longrightarrow \Gamma(E)$ be a self-adjoint elliptic differential operator of order $m > 0$ over a compact manifold. Then each eigenspace E_λ of P is finite-dimensional and consists of smooth sections, Moreover, the eigenvalues are real, form a discrete set and the corresponding eigenspaces furnish a complete orthonormal system for $L^2(E)$, i.e. $L^2(E) = \bigoplus_\lambda E_\lambda$.*

The following example will be used in the following :

Example 6.17 Let ∇ be a connection on E. The connection Laplacian is the linear differential operator of order ≤ 2 given by

$$\nabla^*\nabla : \Gamma(E) \longrightarrow \Gamma(E)$$

Its principal symbol is given by $\sigma^p_\xi(\nabla^*\nabla) = \|\xi\|^2$ ([38] p.154) so that the operator is clearly elliptic. Moreover, it can be shown ([38] II.8.1 and following) that it is essentially self-adjoint on complete Riemannian manifolds, i.e. it extends uniquely to a self-adjoint operator $\nabla^*\nabla L^2(E) \longrightarrow L^2(E)$.
This notion can be extended to generalized Laplacians, which are defined ([5]) to be differential operators of order ≤ 2 on $\Gamma(E)$ such that $\sigma^p_\xi(L) = \|\xi\|^2$ is satisfied. It can be shown ([5], proposition 2.5) that for each generalized Laplacian, there exist a connection ∇ such that

$$L = \nabla^*\nabla + F$$

where F is a differential operator on $\Gamma(E)$ of order zero, i.e. a endomorphism field. If the endomorphism field is fibrewise self-adjoint and the base manifold M is compact, this in particular implies that L extends to a self adjoint, elliptic operator on $L^2(E)$ and theorems 6.14 to 6.16 can be applied.

6.3 Supergeodesics

We consider the functional for supergeodesics given in (6.1) on $\mathbb{R}^{1|1}$. These are supercurves in N rather than geodesics in a supermanifold X, see [24] for a discussion of the latter subject. Using the extensions to n-points discussed in 6.1, we write $d\Phi(D) =: D\Phi$ for $\Phi \in \underline{SC}^\infty(\mathbb{R}^{1|1}, N)$ and there is no ambiguity of sign since Φ is even. The functional reads

$$E : \underline{SC}^\infty(\mathbb{R}^{1|1}, N) \longrightarrow \underline{\mathbb{R}} \qquad \Phi \mapsto \int_{\mathbb{R}^{1|1}} \tfrac{1}{2} dt d\theta \left\langle \dot{\Phi}, D\Phi \right\rangle \tag{6.5}$$

Here, the integrand depends on the choice of coordinates t, θ whereas the functional is independent of it. The functional also implicitly depends on the odd parameters in Λ_n. Moreover, there is a certainly global coordinate system on $\mathbb{R}^{1|1}$ but there need not be a global odd coordinate θ on $S^{1|1}$. For sake of simplicity, we assume that the Batchelor bundle is trivializable to be able to define component fields. We obtain the following equation of motion (compare [61], (2.7)).

Theorem 6.18 *Let $n \in \mathbb{N}_0$. A n-point Φ of $\underline{SC}^\infty(\mathbb{R}^{1|1}, N)$ is a critical point of the functional E from* (6.5) *if and only if it satisfies the following equation of motion:*

$$\nabla^\Phi_t D\Phi = 0 \tag{6.6}$$

Here, ∇^Φ is induced by the Levi-Civita-connection as in remark 3.54. Again, this equation has to be interpreted functorially in the Grassmann algebras Λ_n.

The component fields of Φ (see definition 5.31) read

$$\varphi := \iota^*\Phi \in \underline{SC}^\infty(\mathbb{R}, N) \qquad \psi := \iota^* d\Phi(\tfrac{\partial}{\partial\theta}) \in \mathcal{T}_{\mathbb{R}^{1|1}}$$

It should be noted that in [10] (p.651 (b)), the component field $\psi = \iota^* d\Phi(D)$ is defined in a slightly different fashion. However, since ι^* annihilates each section containing a factor θ, it is easy to see that for any point Φ of $\underline{SC}^\infty(\mathbb{R}^{1|1}, N)$ and any vector field V along Φ,

$$\iota^* d\Phi(D) = \iota^* d\Phi(\tfrac{\partial}{\partial\theta}) \qquad \iota^*\nabla^\Phi_D V = \iota^*\nabla^\Phi_{\frac{\partial}{\partial\theta}} V$$

Thus, we can use $\frac{\partial}{\partial\theta}$ instead of D and obtain the same component expressions. A similar argument would apply in the situation with several odd coordinates θ^α. We will use the coordinate basis for the component decomposition but of course leave the action unchanged. The component equations of motion are given by:

Proposition 6.19 ([10] p.653, 654) *The point Φ is a critical point for the energy functional if and only if φ and ψ satisfy the following equations*

$$\nabla_t^\varphi \dot{\varphi} = \tfrac{1}{2} R(\psi, \psi)\dot{\varphi} \qquad\qquad \nabla_t^\varphi \psi = 0 \tag{6.7}$$

Conversely, each pair (φ, ψ) satisfying these equations defines a critical point Φ.

The component equations of motion are obtained by computing the components of the expression $\nabla_t^\Phi D\Phi = 0$ using 3.54. We omit the details, they can be found in [10], p.653 and 654. It is then clear from theorem 5.37 that (6.6) is satisfied if and only if (6.7) are satisfied. Furthermore, using again theorem 5.38, there is a critical field Φ for each critical pair of component fields φ, ψ.

These equations still depend on the odd parameters in some Λ_n since we are working functorially over the Grassmann algebras. Hence, it is impossible to apply techniques from geometric analysis directly at this stage. However, the superpoints $\mathcal{P}_n$ are supermanifolds in their own right. Since an n-point $\varphi^{(n)}$ is just a morphism of supermanifolds $\varphi^{(n)} : \mathcal{P}_n \times \mathbb{R} \longrightarrow N$, we can apply the formalism of theorems 5.38 and 5.37 again and do an expansion w.r.t. the odd coordinates of $\mathcal{P}_n$. Similarly, ψ is a vector field along $\varphi^{(n)}$ and can be decomposed using 5.37. In this way, we obtain maps and vector fields defined entirely in the setting of smooth differential geometry.

Definition 6.20 *Let $\eta_1, \ldots, \eta_n$ be a set of generators for Λ_n and let $\iota : \mathcal{P}_0 \hookrightarrow \mathcal{P}_n$ denote*[11] *the canonical map given by $\iota^* : \Lambda_n \twoheadrightarrow \mathbb{R}$. For $\varnothing \neq A \subset \underline{n}$, the components*

$$\varphi_\varnothing^{(n)} \in C^\infty(\mathbb{R}, N) \qquad \varphi_\alpha^{(n)}, \varphi_A^{(n)} \in \Gamma(\varphi_\varnothing^{(n)*}TN) \qquad \psi_\varnothing^{(n)}, \psi_A^{(n)} \in \Gamma(\varphi_\varnothing^{(n)*}TN)$$

given by definitions 5.31 and 5.32 applied to the coordinates η instead of θ will be called subcomponents of $\varphi^{(n)}$ and $\psi^{(n)}$. We will usually write $\varphi_{\underline{0}}^{(n)}$ instead of $\varphi_\varnothing^{(n)}$.

Remark 6.21

(a) In the preceding definition, the superscript $^{(n)}$ was included to point out that there are subcomponents for each $n \in \mathbb{N}_0$. If $\varphi^{(n)}$ and $\varphi^{(m)}$ are two points of $\underline{SC}^\infty(\mathbb{R}, N)$, there is in general no relation between, say $\varphi_{\underline{0}}^{(n)}$ and $\varphi_{\underline{0}}^{(m)}$. We will drop the superscript $^{(n)}$ whenever there is no danger of confusion to get a more convenient notation.

(b) Since $\varphi^{(n)}$ is even, it contains only summands with an even power of ηs. Thus, the subcomponents belong to an index of odd length, i.e. $\varphi_\alpha^{(n)}, \varphi_{\alpha\beta\gamma,\ldots}^{(n)}$ are always zero because

[11] We use the same character ι for the morphism sending the θs and the ηs to zero since there is no danger of confusion.

at least one η is left after taking the (covariant) derivatives which is the annihilated by ι^*. Similarly, all the even subcomponents $\psi_{\underline{0}}^{(n)}, \psi_{\alpha\beta}, \ldots$ must vanish. On the whole there are 2^{n-1} even subcomponents of $\varphi^{(n)}$ and 2^{n-1} odd ones of $\psi^{(n)}$.

We will give the explicit expressions for the subcomponent equations arising from 6.19 for $n = 0, 1, 2, 3$ and describe the general structure afterwards, the proof of 6.23 also indicates the type calculations that have to be done here. By the following remark, it is enough to compute one equation for each n:

Remark 6.22 The equations (6.7) are functorial in Λ_n. For $I = \{i_1, \ldots, i_m\} \subset \underline{n}$, let $\iota_I : \Lambda_n \longrightarrow \Lambda_{n-\|I\|}$ be the projection morphism as defined in (5.16) with θ replaced by η. Applying ι_I to an n-point of the equation of motion thus yields an $n - \|I\|$-point of it. In this way, every subcomponent equation at level n already occurs at a level $m < n$ except the one for the top degree field $\varphi_{\underline{n}}^{(n)} = \varphi_{(1\ldots n)}^{(n)}$ (or $\psi_{(1\ldots n)}^{(n)}$, in case n is odd). Hence, for each n, it is sufficient to compute the $(1\ldots n)$-subcomponent. More formally

$$\iota^* \nabla_{I^c}^{\varphi^{(n)}} (\nabla_t^{\varphi^{(n)}} \dot{\varphi}^{(n)} - \tfrac{1}{2} R(\psi^{(n)}, \psi^{(n)}) \dot{\varphi}^{(n)}) = \iota_{I^c}^* \nabla_{I^c}^{\iota_I \varphi^{(n)}} \iota_I^* (\nabla_t^{\varphi^{(n)}} \dot{\varphi}^{(n)} - \tfrac{1}{2} R(\psi^{(n)}, \psi^{(n)}) \dot{\varphi}^{(n)})$$

(and similarly for the other equation). The right hand side is the highest degree component of $\iota_I^* (\nabla_t^{\varphi^{(n)}} \dot{\varphi}^{(n)} - \frac{1}{2} R(\psi^{(n)}, \psi^{(n)}) \dot{\varphi}^{(n)})$ which is a $(n - \|I\|)$-point of (6.7).

For the lowest values of n, we obtain the following fields and they have to satisfy the following equations:

n=0 The only nontrivial subcomponent field is $\varphi_{\underline{0}}^{(0)}$:

$$\nabla_t^{\varphi_{\underline{0}}^{(0)}} \dot{\varphi}_{\underline{0}}^{(0)} = 0 \tag{6.8}$$

n=1 The nontrivial subcomponent fields are $\varphi_{\underline{0}}^{(1)}, \psi_1^{(1)}$:

$$\nabla_t^{\varphi_{\underline{0}}^{(1)}} \psi_1^{(1)} = 0 \tag{6.9}$$

n=2 The nontrivial subcomponent fields are $\varphi_{\underline{0}}^{(2)}, \psi_\alpha^{(2)}, \varphi_{12}^{(2)}$ for $\alpha = 1, 2$:

$$\nabla_t^{\varphi_{\underline{0}}^{(2)}} \nabla_t^{\varphi_{\underline{0}}^{(2)}} \varphi_{12}^{(2)} + R(\varphi_{12}^{(2)}, \dot{\varphi}_{\underline{0}}^{(2)}) \dot{\varphi}_{\underline{0}}^{(2)} = 2R(\psi_1^{(2)}, \psi_2^{(2)}) \varphi_{\underline{0}}^{(2)} \tag{6.10}$$

n=3 The nontrivial subcomponent fields are $\varphi_{\underline{0}}^{(3)}, \psi_\alpha^{(3)}, \varphi_{\alpha\beta}^{(3)}, \psi_{123}^{(3)}$ for $\alpha = 1, 2, 3$, $\alpha < \beta \leq 3$:

$$\nabla_t^{\varphi_{\underline{0}}^{(3)}} \psi_{123}^{(3)} = R(\varphi_{13}^{(3)}, \dot{\varphi}_{\underline{0}}^{(3)}) \psi_2^{(3)} - R(\varphi_{12}^{(3)}, \dot{\varphi}_{\underline{0}}^{(3)}) \psi_3^{(3)} - R(\varphi_{23}^{(3)}, \dot{\varphi}_{\underline{0}}^{(3)}) \psi_1^{(3)}$$

The general structure is stated in the following proposition:

Proposition 6.23 *Let $n \in \mathbb{N}^+$. Omitting the index $^{(n)}$, the equations of motions for the highest degree subcomponent fields $\varphi_{\underline{n}}$ (for n even) and $\psi_{\underline{n}}$ (for n odd) are given by*

$$\nabla_t^{\varphi_{\underline{0}}}\nabla_t^{\varphi_{\underline{0}}}\varphi_{\underline{n}} + R(\varphi_{\underline{n}}, \dot{\varphi}_{\underline{0}})\dot{\varphi}_{\underline{0}} = F_n(\varphi_A, \psi_B) \tag{6.11}$$

$$\nabla_t^{\varphi_{\underline{0}}}\psi_{\underline{n}} = G_n(\varphi_A, \psi_B) \tag{6.12}$$

where F and G are multilinear functions of the subcomponents φ_A, ψ_B for $\|A\|, \|B\| < n$ depending in a complicated way on the curvature of (N, g) and its (higher) derivatives.
Note that the left hand side of the first equation is just the Jacobi-operator, applied to the vector field $\varphi_{\underline{n}}$ in case that $\varphi_{\underline{0}}$ is a geodesic.

Proof For $n > 0$ even, we have to rewrite the equation

$$\iota^*\nabla^{\varphi}_{\eta^n}\cdots\nabla^{\varphi}_{\eta^1}\nabla^{\varphi}_t\dot{\varphi} = \iota^*\nabla^{\varphi}_{\eta^n}\cdots\nabla^{\varphi}_{\eta^1}(R(\psi,\psi)\dot{\varphi}).$$

To obtain $\varphi_{\underline{n}}$, all the η-derivatives have to be moved to φ. The right hand side can not contain $\varphi_{\underline{n}}$ at all because moving all derivatives to φ would imply that the corresponding summand still contains η-factors resulting from the odd fields ψ. Thus, all these summands are annihilated by ι but there are of course contributions to $F(\varphi_A, \psi_B)$.
Since the connection is torsion-free, we have $\nabla^{\varphi}_{\eta^1}\dot{\varphi} = \nabla^{\varphi}_t d\varphi(\frac{\partial}{\partial\eta^1})$. Thus, using the definition of the curvature tensor R, the left hand side reads:

$$\iota^*\nabla^{\varphi}_{\eta^n}\cdots\nabla^{\varphi}_{\eta^1}\nabla^{\varphi}_t\dot{\varphi} = \iota^*\nabla^{\varphi}_{\eta^n}\cdots\nabla^{\varphi}_{\eta^2}\big(R(d\varphi(\tfrac{\partial}{\partial\eta^1}),\dot{\varphi})\dot{\varphi} + \nabla^{\varphi}_t\nabla^{\varphi}_t d\varphi(\tfrac{\partial}{\partial\eta^1}\big) \tag{6.13}$$

Differentiating the curvature term, there is precisely one summand containing $\varphi_{\underline{n}}$ which is obtained by collecting all derivatives at $d\varphi(\frac{\partial}{\partial\eta^1})$. After evaluation of ι^*, it yields $R(\varphi_{\underline{n}}, \dot{\varphi}_{\underline{0}})\dot{\varphi}_{\underline{0}}$. In the second summand in (6.13), we successively interchange the η- and the t-derivatives. The resulting curvature terms can not contain $\nabla^{\varphi}_{\eta^n}\cdots\nabla^{\varphi}_{\eta^2}d\varphi(\frac{\partial}{\partial\eta^1})$ and are collected in $F(\varphi_A, \psi_B)$, the only summand containing it is $\nabla^{\varphi}_t\nabla^{\varphi}_t\nabla^{\varphi}_{\eta^n}\cdots\nabla^{\varphi}_{\eta^2}d\varphi(\frac{\partial}{\partial\eta^1})$. Applying ι^* yields $\nabla_t^{\varphi_{\underline{0}}}\nabla_t^{\varphi_{\underline{0}}}\varphi_{\underline{n}}$ which proves the first equation. The equation for $\psi_{\underline{n}}$ is obtained in a similar fashion.

□

The following result is already mentioned in [10] (p.655) and [23], lecture 2 although no rigorous proof is given there. Moreover, super parallel transport, which is part of the problem, was discussed in detail in [15].

Theorem 6.24 *Let (N, h) be a complete Riemannian manifold and fix $t_0 \in \mathbb{R}$. Then, the functor of critical points of the functional 6.5 can be identified as*

$$\underline{Crit} \cong TN \oplus \Pi(TN)$$

The isomorphism depends on the choice of the reference parameter t_0

Proof Equation (6.8) is the equation for a geodesic on N. Since N is assumed to be complete, for each tuple $(p \in N, v \in T_pN)$, there is a unique geodesic starting at time t_0 at the point p in direction of v which is defined on all of $\mathbb{R}$. Equation (6.11) is a linear ordinary differential equation of order two for $\varphi_{\underline{n}}$ on the vector bundle $\varphi_{\underline{0}}^*TN$. It has solutions

defined on all of $\mathbb{R}$ which are uniquely defined by a point $p = \varphi_0(t_0) \in N$ and two vectors $v_1, v_2 \in T_pN$. Similarly, a solution for 6.12) is uniquely determined by p and one vector $v \in T_pN$.
As discussed in 4.25, an n-point of $TN \oplus \Pi TN$ consists of an n-point $f : \mathcal{P}_n \longrightarrow N$ and an even section $\sigma = (\sigma', \sigma'')$ of $f^*(TN \oplus \Pi TN)$. The idea is that f and σ' provide the initial conditions at t_0 for the first equation of (6.7) and σ'' those for the second. We will make that precise by giving an inductive definition of an isomorphism $\underline{Crit}(E_1)(\Lambda_n) \cong (TN \oplus \Pi TN)(\Lambda_n)$ making use of the decomposition into subcomponents which are defined for all ordered multi-indices $I \subset \underline{n}$.
For $I = \underline{0}$, the subcomponent equation is the geodesic equation. Its solutions are in bijection with the pair $(f_{\underline{0}}, \sigma'_{\underline{0}})$ since the first is a map $\{pt\} \longrightarrow N$, i.e. a point in $p \in N$ and the second is a tangent vector in T_pN. The $\underline{0}$-components of σ'' are clearly zero.
For $I = (\alpha),\ \alpha = 1, \dots, n$, the I-subcomponents of f and σ' are zero. There are n subcomponents $\sigma''_\alpha \in T_pN$, which correspond to the initial conditions for the first order equations (6.9) for $\alpha = 1, \dots, n$. This sets up a bijection for $\|I\| = 1$.
For $I = (\alpha, \beta)$ with $1 \leq \alpha < \beta \leq n$, the subcomponents of σ'' vanish and we have $f_{(\alpha\beta)}, \sigma'_{(\alpha\beta)} \in T_pN$. These two vectors are mapped to $\varphi_{\alpha\beta}(t_0)$ and $\nabla_t^{\varphi_{\underline{0}}}\varphi_{\alpha\beta}(t_0)$, which uniquely determine the solution $\varphi_{(\alpha\beta)}$ since it satisfies a second order equation. Note that the right hand side of (6.10) depend on ψ_α and ψ_β, which have already been identified in the step $\|I\| = 1$. Thus, we established the bijection for $\|I\| = 2$.
Using induction on $\|I\|$, we construct an bijection for the whole set of $n-points$. For $\|I\| \in 2\mathbb{N}$, we have two new vectors $f_I, \sigma''_I \in T_pN$ for each I. On the other hand, the subcomponent φ_I satisfies the second order equation (6.11) whose solutions is specified by $\varphi_I(t_0)$ and $\nabla_t^{\varphi_{\underline{0}}}\varphi_I(t_0)$. For $\|I\| \in 2\mathbb{N}+1$, there is only one new vector σ''_I which gives the initial condition for the field ψ_I which satisfies the first order equation (6.12). In both cases, the right hand sides only depend on subcomponent fields, which were already completely determined in an earlier step.
Finally, even though we chose a set of generators $\eta^1, \dots, \eta^n$ for Λ_n, the bijection is functorial by proposition 6.6 which completes the proof.

□

The situation for the same functional defined on $S^{1|1}$ is more complicated. The reason is, that we are now looking for closed geodesics and vector fields along them, which means that these fields have to close smoothly after running around the geodesic. Already the set of closed geodesics will have a complicated structure for a generic geometry and the coupled equations for higher subcomponent fields makes things worse. We will offer a geometric description of the resulting functor although no closed description in terms of a representing object (as $TM \oplus \Pi TN$ above) is available.

Let $\eta^1, \dots, \eta^n$ be generators of Λ_n and identify $\Lambda_{n-1} \cong \mathbb{R}[\eta^1, \dots, \eta^{n-1}]$. Thus, we have two morphisms of Grassmann algebras, a projection and a lift:

$$pr_n : \Lambda_n \twoheadrightarrow \Lambda_{n-1} \qquad\qquad lf_n : \Lambda_{n-1} \hookrightarrow \Lambda_n$$

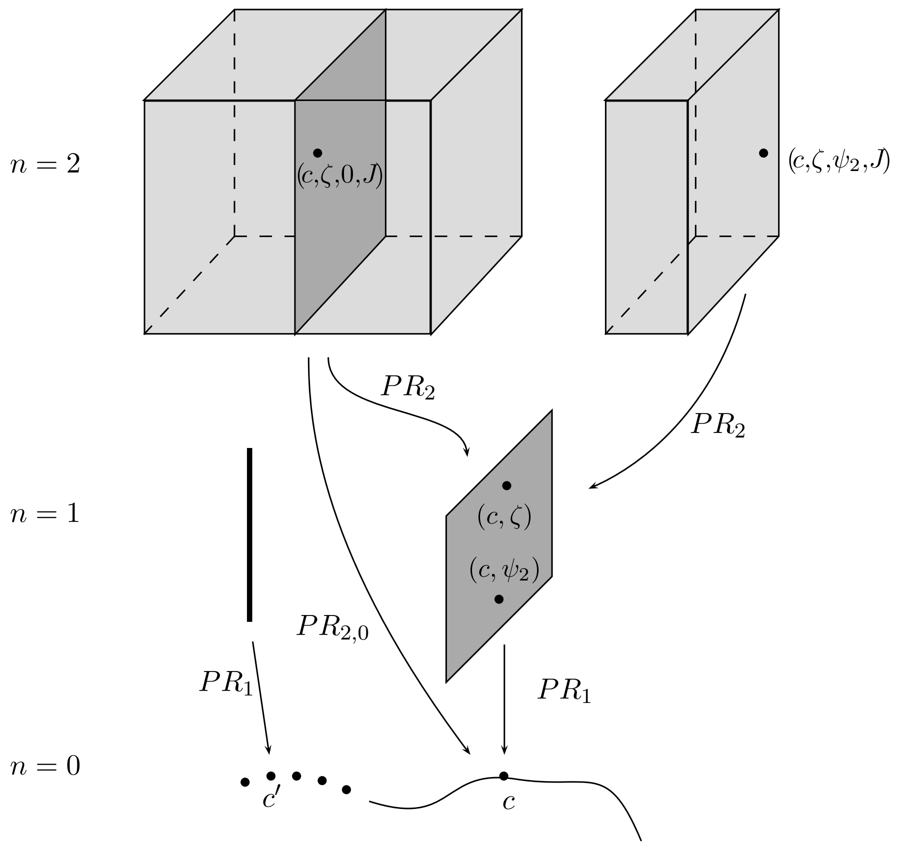
By proposition 6.6, there are induced maps

$$\begin{aligned} PR_n &:= \underline{Crit}(E)(pr_n) : \underline{Crit}(E)(\Lambda_n) \longrightarrow \underline{Crit}(E)(\Lambda_{n-1}) \\ LF_n &:= \underline{Crit}(E)(lf_n) : \underline{Crit}(E)(\Lambda_{n-1}) \longrightarrow \underline{Crit}(E)(\Lambda_n) \end{aligned}$$

Since the whole functor of critical points is given by all of its n-points, this suggest the following geometric structure:
The critical points of E form a tower of sets which at each n have a bundle structure as follows: At each level n, the set $\underline{Crit}(E)(\Lambda_n)$ forms the total space over the base $\underline{Crit}(E)(\Lambda_{n-1})$ where the projection is given by PR_n. The maps LF_n provide sections of the bundle structures for each n.

Note that this structure is not natural since it depends on the choice of parameters η^i. In general, there are no natural maps between $\underline{Crit}(E)(\Lambda_n)$ and $\underline{Crit}(E)(\Lambda_m)$ since there is no distinguished morphism $\Lambda_n \longrightarrow \Lambda_m$. The only exceptions are $n = 0$ or $m = 0$, i.e. each n-point has a naturally defined 0-base point and conversely, there is only one way to map a 0-point into the set of n-points. We will denote these natural maps by $PR_{n,0}$ and $LF_{0,n}$.

Despite the lack of naturalness, this structure is helpful to picture the functor of critical points. We will picture it for $n = 0, 1, 2$ (see description below the picture):



$\underline{Crit}(E)(\Lambda_0)$ consists of the set of all closed geodesics in N. It can again be identified with a subset of TN but in general, it does not have a nice submanifold structure. Each fibre $PR_1^{-1}(c)$ for some closed geodesic c is given by the space of solutions of the first order linear homogeneous differential equation (6.9), i.e. by the parallel vector fields along c. Its dimension is between 1 and $dim(n)$, since there is certainly always the parallel vector field $\dot{c}$. However, depending on the geometry, this dimension may depend on the base point (in the picture, the fibre above c' has smaller dimension than that over c). Thus, we arrive at a decomposition of $\underline{Crit}(E)(\Lambda_1)$ into vector spaces but this does not provide a nice bundle structure since the fibres even need not be isomorphic.
For $n = 2$, we fix a point $(c, \zeta) \in \mathcal{P}_1^{-1}(c)$ and analyze the fibre $PR_2^{-1}(c, \zeta)$. Each 2-point of this fibre has subcomponents $(c, \zeta, \psi_2, \varphi_{12})$ where ψ_2 is another parallel vector field along c. φ_{12} has to satisfy (6.10), which is inhomogeneous and needs not have a solution at all. From corollary 6.15, we conclude that it has a solution if and only if $\langle J, R(\zeta, \psi_2)\dot{c}\rangle_{L^2(S^1)} = 0$. In that case, the space of solutions for fixed c, ζ and ψ_2 is an affine space modelled on the vector space $\mathcal{J}(c)$ of Jacobi fields along c, which is determined by c alone. Hence, we can write the fibre as

$$PR_2^{-1}(c, \zeta) = \bigsqcup_{\substack{\psi_2 \in PR_1^{-1}(c) \\ \mathcal{J}(c) \perp R(\zeta, \psi_2)\dot{c}}} G^{\mathcal{J}}(R(\zeta, \psi_2)\dot{c}) + \mathcal{J}(c) \tag{6.14}$$

where $G^{\mathcal{J}}$ is the Greens operator for the Jacobi operator. In the picture, this fibre has been pictured as a grey block, which possibly contains “gaps” where the inhomogeneous equation has no solution, but which always contains $\mathcal{J}(c)$ as a subspace (dark grey) for $\psi_2 = 0$. In particular, there is a always a solution in this subspace.

Parts of this structure persist for larger n but it gets more complicated since the number of subcomponent fields increase as 2^n. We have the following structure:

Proposition 6.25 *Let $\Lambda_n \subset \Lambda_{n+1}$ be given as above. Denote by $(\varphi^{(n)}, \psi^n)$ and $(\varphi^{(n+1)}, \psi^{n+1})$ the critical points. Then:*

(a) $PR_n((\varphi^{(n+1)}$ and $\psi^{(n+1)}))$ is the critical point at level n, whose subcomponent fields $\varphi_I^{(n+1)}, \psi_J^{(n+1)}$ are annihilated for indices I, J containing $n+1$ and left unchanged otherwise. Hence, the fibres as well as the base at level n have dimension $\leq 2^{n-1}$ (in those cases when there is a well defined dimension).

(b) $LF_n((\varphi^{(n)}, \psi^{(n)}))$ is the critical point at level $n+1$, whose subcomponent fields $\varphi_I^{(n+1)}$ and $\psi_J^{(n+1)}$ are defined to be zero for indices I, J containing $n+1$ whereas the other components are copied from $(\varphi^{(n)}, \psi^{(n)})$. The image of LF_n is contained in the solution space of the homogeneous parts of (6.11) *and* (6.12) *respectively.*

The statements follow from the fact, that pr_{n+1} and lf_{n+1} set η^n to zero or extend by this condition respectively. To see that the image of LF_n indeed consists of solutions of the homogeneous equations, we study the right hand sides of (6.12) and (6.11). By the proof of proposition 6.23, each summand of F_n and G_n contains a subcomponent field whose index contains $n+1$. Thus, all summands are set to zero.

Summarizing, we have

Theorem 6.26 *Let (N,h) be a compact Riemannian manifold. Then there exist solutions to equation (6.6) in the sense that for each n, the set $\underline{Crit}(E)(\Lambda_n)$ is nonempty and these sets behave functorially. More precisely, let $p \in \underline{Crit}(E)(\Lambda_{n-1})$ and $c := P_{n-1,0}(p)$ the underlying closed geodesic, then*

- *For $n \in 2\mathbb{N}$, the fibre $PR_n^{-1}(p)$ is a disjoint union of affine spaces modelled on $\mathcal{J}(c)$.*
- *For $n \in 2\mathbb{N}+1$, the fibre $PR_n^{-1}(p)$ is a disjoint union of affine spaces modelled on $\mathcal{V}(c)$.*

Here, the spaces of Jacobi fields $\mathcal{J}(c)$ and of parallel vector fields $\mathcal{V}(c)$ only depend on the lowest order subcomponent, i.e. the underlying closed geodesic.

Proof By theorem 6.11, there exist a closed geodesic in every homotopy class of curves. Iterated application of the lifts LF_n $(n = 1, 2, \ldots)$ yields critical n-points for each n. The decomposition of the fibres $PR_n^{-1}(p)$ was given for $n = 2$ in (6.14) and the argument can be easily generalized to $n > 2$.

□

The following example illustrates the dependence of $\underline{Crit}$ on the geometry of the target space:

Example 6.27 Let $N = T^n$ be the standard flat torus. Then, the equations (6.11) and (6.12) reduce to $\nabla_t^{\varphi_0}\nabla_t^{\varphi_0}\varphi_{\underline{n}}$ and $\nabla_t^{\varphi_0}\psi_{\underline{n}} = 0$ respectively. In particular, these equations are homogeneous, so that solutions always exist. If c is a closed geodesic in T^n, each vector at $T_{c(0)}N$ gives rise to a parallel vector field along c by parallel transport. However, on a flat manifold, the set of Jacobi fields along c is given by

$$\mathcal{J}(c) = \{J = V + tW | V, W \text{ are parallel vector fields along } c\}$$

It is clear that only the choice $W = 0$ yields a Jacobi field closing smoothly, in other words, only parallel Jacobi fields solve (6.11) globally. Thus, $\underline{Crit}$ is only a subfunctor of $TN \oplus \Pi TN$. Indeed, denoting a n-point of $TN \oplus \Pi TN$ by $(f, (\sigma', \sigma''))$ as in the proof of theorem 6.24, all the subcomponet contributions σ'_I for $\|I\| > 0$ must vanish since they give the initial value for φ_I. We can describe $\underline{Crit}$ as follows: We can identify the set of closed geodesics with a subset $cg \subset T^1T^n$ of the unit tangent bundle and define a subfunctor of TN by $CG(\Lambda_n) = \mathrm{Hom}_{\mathrm{Set}}(\mathcal{P}_0, cg)$ so that there are no nontrivial higher points. Then, as in the proof of 6.24, we find $\underline{Crit} \cong CG \times_{(N,\underline{0})} \Pi TN$, i.e.

$$\underline{Crit}(\Lambda_n) \cong \{(c, (f, \sigma''))| c \in cg, f \in SC^\infty(\mathcal{P}_n, N), \sigma \in f^*\Pi TN \text{ such that } c(0) = f_{\underline{0}}\}$$

This functor is no longer represented by a supermanifold but can still be described in geometric terms. This simple description is no longer available, if N is not assumed to be flat.

It is not difficult to see that it is possible to construct a metric on T^n for $n \geq 3$ such that there exist a closed geodesic c, parallel vector fields X, Y and a Jacobi field ξ along c, such that $(\xi, R(X,Y)\dot{c})_{L^2(S^1)} \neq 0$. In this case, already the inhomogeneous equation (6.10) has no solution for this special choice of c, X, Y.

Remark 6.28

(a) It should be pointed out that the existence of solutions mainly depends on the existence of the solution for the 0-subcomponent equation. The remaining equations are linear equations, they can be interpreted as an infinitesimal correction to the nonlinear geodesic equation from the super world.

(b) It might be interesting to look for a generalization of the energy functional to super curves defined on $\mathbb{R}^{1|q}$. For $q = 1$, the space of critical points is basically determined by closed geodesics and parallel vector fields as well as Jacobi fields along them. For $q > 1$, it might be possible to use such a functional to explore higher order variations of geodesics or the energy of smooth curves.

(c) It might be interesting to study $\underline{Crit} \subset \underline{SC}^\infty(S^{1|1}, N)$ if N is a (locally) symmetric space. Examples for small n indicate, that the existence of solutions of (6.11) and (6.12) can be understood in these cases, although a general statement has still to be shown. This might be a starting point for further research on the relationship between symmetry and curvature properties of the target space and the structure of $\underline{Crit}$.

6.4 Superharmonic maps

We consider the functional from (6.2) given by the extensions to n-points discussed in 6.1:

$$E : \underline{SC}^\infty(X,N) \longrightarrow \underline{\mathbb{R}} \qquad\qquad \Phi \mapsto \int_X vol_G \, \langle d\Phi, d\Phi \rangle \tag{6.15}$$

(X, G) is assumed to be compact and super Riemannian (see definition 3.34). The product $\langle d\Phi, d\Phi \rangle$ is induced by the metrics $\langle , \rangle$ on N and G on X as usual. Again, the integrand depends implicitly on the odd parameters in Λ_n but we will simply write Φ instead of $\Phi^{(n)}$ to improve the readability of some rather long equations.

Choosing a frame $\{e_i\}$ such that the metric has the normal form from proposition 3.39, we obtain the following equations of motion:

Theorem 6.29 *A n-point Φ of $\underline{SC}^\infty(X,N)$ is critical if and only if*

$$0 = \tau(\Phi) := str_G(\nabla^\Phi d\Phi) := \sum_i (\nabla^\Phi_{e_\mu} d\Phi)(J(e_\mu)) \tag{6.16}$$

PROOF Let Φ be an n-point of $\underline{SC}^\infty(X,N)$ and $\rho : \Lambda_n \longrightarrow \Lambda_m$ a morphism of Grassmann algebras. Let $\tilde{\Phi} := \underline{SC}^\infty(\rho)\Phi = \Phi \circ (\mathcal{P}(\rho) \times id_M)$. By definition 6.5, Φ is critical if for all m, all ρ and all variations $\tilde{\Phi}_t$ of the resulting point $\tilde{\Phi}$, we have $\frac{d}{dt}|_0 E(\tilde{\Psi}_t) = 0$.

By lemma 3.44, the order of t-differentiation and Berezin integration can be interchanged. Fixing an orthosymplectic frame $\{e_i\}$ and using J as in (3.9) ff., we get[12]

$$\begin{aligned} \frac{d}{dt}\Big|_t E(\tilde{\Phi}_t) &= \int_{\mathcal{P}_m\times M} vol_G \tfrac{d}{dt}|_0 \left\langle d\tilde{\Phi}, d\tilde{\Phi}\right\rangle \\ &= 2\int_{\mathcal{P}_m\times M} \textstyle\sum_i \left\langle \nabla_t^{\tilde{\Phi}_t}(d\tilde{\Phi}_t(e_i)), d\tilde{\Phi}_t(Je_i)\right\rangle \end{aligned}$$

Introducing the vector field $W_t := \left\langle \nabla_{e_i}^{\tilde{\Phi}_t}((-1)^{|e_j|}\left\langle d\tilde{\Phi}_t(\partial_t), d\tilde{\Phi}_t(e_j)\right\rangle Je_j), Je_i\right\rangle$, computation of its divergence yields

$$div(W_t) = \left\langle \nabla_t^{\tilde{\Phi}_t} d\tilde{\Phi}_t(e_i), d\tilde{\Phi}_t(Je_i)\right\rangle + \left\langle d\tilde{\Phi}_t(\partial_t), (\nabla_{e_i}^{\tilde{\Phi}_t} d\tilde{\Phi}_t)(Je_i)\right\rangle$$

By the divergence theorem, there is no contribution from $div(W_t)$, integrated over M. Restricting to $t = 0$, we finally obtain

$$\frac{d}{dt}\Big|_t E(\tilde{\Phi}_t) = -\int_{\mathcal{P}_m\times M} vol_G \left\langle \tilde{V}, \tau(\tilde{\Phi})\right\rangle$$

where $\tilde{V} := d\tilde{\Phi}_t(\partial_t)|_{t=0}$ is the variational vector field. Introducing local coordinates on N and denoting the coefficient of the metric on the target by $h_{\mu\nu}$, the integrand is given by $\tilde{V}(\tilde{\Phi}(y^\mu))\tau(\tilde{\Phi})(\tilde{\Phi}(y^\nu))\tilde{\Phi}h_{\mu\nu}$. Since the components of $\tilde{V}$ are arbitrary, we obtain $\tau(\tilde{\Phi}) = 0$. Finally, every morphism ρ of Grassmann algebras induces the map

$$\hat{\rho} := (\rho \otimes id_{\mathcal{O}}) \otimes id_{\mathcal{T}_N} : (\Lambda_n \otimes_{\mathbb{R}} \mathcal{O}) \otimes_\Phi \mathcal{T}_N \longrightarrow (\Lambda_m \otimes_{\mathbb{R}} \mathcal{O}) \otimes_{\tilde{\Phi}} \mathcal{T}_N$$

It is clear that this extension of ρ commutes with the covariant derivative etc. so that we obtain: Φ is critical if and only if, for all $\rho : \Lambda_n \longrightarrow \Lambda_m$, we have $\hat{\rho}(\tau(\Phi)) = \tau(\tilde{\Phi}) = 0$. This is clearly equivalent to $\tau(\Phi) = 0$.

$\square$

$\tau(\Phi)$ is a vector field along Φ and following the strategy used for supergeodesics, we use components to reformulate (6.16). We will furthermore make the following assumptions on (X, G) to obtain a manageable set of component equations:

(a) We will assume $dim(X) = m|2$. Recall that the odd dimension has to be an even number by lemma 3.36 (c). The number of component fields on a general supermanifold of dimension $p|q$ is 2^q so that already $q = 4$ leads to 16 superfields and extremely complex equations. Some general properties of the component equations and the functor of critical points will nevertheless be valid in general.

(b) We will assume that X admits global odd coordinates θ^1, θ^2, i.e. that any Batchelor bundle $E \longrightarrow \tilde{X}$ is trivializable. This condition is necessary to obtain globally defined component fields, it will be removed at the end. Note that the even coordinates fields will in general only be defined locally.

[12]The calculation follows that in [69], 1.2.3. However, the calculation is slightly more complicated due to the super signs and due to the fact that we do have a frame parallel at a "point".

(c) We will assume that G is block diagonal, i.e. we have $\mathcal{T}_{X,\overline{0}} \perp \mathcal{T}_{X,\overline{1}}$. This is always true on the level of super tangent spaces by lemma 3.36 but in order to simplify calculations, we assume that this holds for G itself.

Since an odd coordinate system is precisely a frame of a Batchelor bundle, we can always absorb a change of base, given by a matrix in $Gl(2, C^\infty(\tilde{X}))$, into the odd coordinates. Then, the block of G corresponding to $\mathcal{T}_{X,\overline{1}}$ has normal form up to order θ^α and there are only terms left which are proportional to $\theta^1\theta^2$. Moreover, the block corresponding to $\mathcal{T}_{X,\overline{0}}$ must be a Riemannian metric up to order θ^α so that we have the following lemma:

Lemma 6.30 *Under the above assumptions, G and G^{-1} have the form*

$$G = \left(\begin{array}{c|cc} G_0 & 0 & 0 \\ \hline 0 & 0 & -1 - h\theta^1\theta^2 \\ 0 & 1 + h\theta^1\theta^2 & 0 \end{array}\right) \quad G^{-1} = \left(\begin{array}{c|cc} G_0^{-1} & 0 & 0 \\ \hline 0 & 0 & 1 - h\theta^1\theta^2 \\ 0 & -1 + h\theta^1\theta^2 & 0 \end{array}\right) \tag{6.17}$$

with respect to a coordinate frame $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial \theta^1}, \frac{\partial}{\partial \theta^2}\}$. We have $h \in C^\infty(\tilde{X})$ and G_0 is an invertible matrix with entries in $\mathcal{O}_{\overline{0}}$ which may be written as

$$G_0 = g + \hat{g}\theta^1\theta^2 \qquad G_0^{-1} = g^{-1} - g^{-1}\hat{g}g^{-1}\theta^1\theta^2$$

for a Riemannian metric g on $\tilde{X}$ and a symmetric bilinear form $\hat{g}$ on $T\tilde{X}$.

Before giving the component equations, we note two covariant derivatives on (X, G) which can be obtained from Koszul's formula:

$$\nabla^X_{\partial_\mu} \frac{\partial}{\partial x^\nu} = \Gamma^i_{\mu\nu}\partial_i + \big(\hat{\Gamma}^i_{\mu\nu} - \Gamma^k_{\mu\nu}\hat{g}_{kj}g^{ji}\big)\theta^1\theta^2\frac{\partial}{\partial x^i} + \tfrac{1}{2}\hat{g}_{\mu\nu}(\theta^1\frac{\partial}{\partial\theta^1} - \theta^2\frac{\partial}{\partial\theta^2}) \tag{6.18}$$

$$\nabla^X_{\theta^2}\frac{\partial}{\partial\theta^1} = -\tfrac{1}{2}\theta^1\theta^2 grad_g(h) - h(\theta^1\frac{\partial}{\partial\theta^1} + \theta^2\frac{\partial}{\partial\theta^2})$$

Here, $\Gamma^i_{\mu\nu}$ denote the Christoffel symbols of the Levi-Civita connection associated to g whereas $\hat{\Gamma}^i_{\mu\nu} := \frac{1}{2}(\frac{\partial\hat{g}_{\nu\rho}}{\partial x^\mu} + \frac{\partial\hat{g}_{\mu\rho}}{\partial x^\nu} - \frac{\partial\hat{g}_{\mu\nu}}{\partial x^\rho})g^{\rho i}$. Using this, we also find the following super Lie bracket expression for $\alpha = 1, 2$:

$$[\nabla^X_{\partial_\mu}\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial\theta^\alpha}] = (-1)^\alpha\big(\hat{\Gamma}^i_{\mu\nu} - \Gamma^k_{\mu\nu}\hat{g}_{kj}g^{ji}\big)\theta^{3-\alpha}\frac{\partial}{\partial x^i} + (-1)^\alpha\tfrac{1}{2}\hat{g}_{\mu\nu}\frac{\partial}{\partial\theta^\alpha} \tag{6.19}$$

$$\tag{6.20}$$

Here, the components of $\Phi = \Phi^{(n)}$ are

$$\varphi = \iota^*\Phi \qquad \psi_\alpha = \iota^* d\Phi(\tfrac{\partial}{\partial\theta^\alpha}) \quad (\alpha = 1, 2) \qquad \xi = \iota^*\nabla^\Phi_{\theta^2} d\Phi(\tfrac{\partial}{\partial\theta^1}),$$

and we obtain the following component equations:

Theorem 6.31 *Φ is a critical point of the energy functional if and only if the following component hold:*

$$\tau(\varphi) = -2\xi \tag{6.21}$$

$$\begin{aligned}\nabla^{\varphi*}\nabla^{\varphi}\psi_\alpha &= tr_g R^N(\psi_\alpha, d\varphi(\cdot))d\varphi(\cdot) - 2(-1)^\alpha R^N(\psi_1,\psi_2)\psi_\alpha \\ &\quad + (-1)^\alpha \tfrac{1}{2} tr_g(\hat{g})\psi_\alpha + 2h\psi_\alpha\end{aligned} \tag{6.22}$$

$$\begin{aligned}\nabla^{\varphi*}\nabla^{\varphi}\xi &= R^N(\psi_2,\tau(\varphi))\psi_1 + 2R(\psi_1,\xi)\psi_2 + tr_g R(\xi, d\varphi(\cdot))d\varphi(\cdot) \\ &\quad + 2tr_g R^N(\psi_2, d\varphi(\cdot))\nabla^{\varphi}_{\cdot}\psi_1 - 2tr_g R^N(\psi_1, d\varphi(\cdot))\nabla^{\varphi}_{\cdot}\psi_2 \\ &\quad + tr_g(\nabla^N_{\psi_2}R^N)(\psi_1, d\varphi(\cdot))d\varphi(\cdot) + tr_g(\nabla^N_{d\varphi(\cdot)}R^N)(\psi_2, d\varphi(\cdot))\psi_1 \\ &\quad + 2(\nabla^N_{\psi_2}R^N)(\psi_1,\psi_2)\psi_1 \\ &\quad + 2h\xi + d\varphi(grad_g(h)) - \langle\nabla^{\varphi}d\varphi, \hat{g}\rangle - \tfrac{1}{2}tr_g(\hat{g})\xi - g^{\mu\nu}(\hat{\Gamma}^i_{\mu\nu} - \Gamma^k_{\mu\nu}\hat{g}_{kj}g^{ji})d\varphi(\tfrac{\partial}{\partial x^i})\end{aligned} \tag{6.23}$$

Conversely, each set of component equations defines a critical point Φ. Here tr_g denotes the metric trace on the underlying Riemannian manifold $(\tilde{X}, g)$ and $\langle\nabla^{\varphi}d\varphi, \hat{g}\rangle$ the contraction of $\nabla^{\varphi}d\varphi$ and $\hat{g}$ on $\mathrm{Sym}^2 T^*\tilde{X}$ which yields a vector field along φ.

Remark 6.32 At a first glance, the third equation shows an unexpected asymmetry in ψ_1 and ψ_2. While the $\mathrm{tr}_g R$-terms show a supersymmetric behavior under exchange of ψ_1 and ψ_2, this seems to be wrong for the $\mathrm{tr}_g\nabla R$-terms. However, using the second and the derivative of the first Bianchi identity, it is not difficult to show that we have

$$\begin{aligned}&tr_g(\nabla^N_{\psi_2}R^N)(\psi_1, d\varphi(\cdot))d\varphi(\cdot) + tr_g(\nabla^N_{d\varphi(\cdot)}R^N)(\psi_2, d\varphi(\cdot))\psi_1 = \\ &\qquad \tfrac{1}{2}\big(tr_g(\nabla^N_{\psi_2}R^N)(\psi_1, d\varphi(\cdot))d\varphi(\cdot) - tr_g(\nabla^N_{\psi_1}R^N)(\psi_2, d\varphi(\cdot))d\varphi(\cdot)\big) \\ &\qquad + \tfrac{1}{2}\big(tr_g(\nabla^N_{d\varphi(\cdot)}R^N)(d\varphi(\cdot),\psi_1)\psi_2 - tr_g(\nabla^N_{d\varphi(\cdot)}R^N)(d\varphi(\cdot),\psi_2)\psi_1\big)\end{aligned}$$

The right hand side is indeed supersymmetric in ψ_1 and ψ_2 and a similar expression can be obtained for the summand $(\nabla^N_{\psi_2}R^N)(\psi_1,\psi_2)\psi_1$.

Proof of theorem 6.31 We have to compute the expressions $\iota^*\tau(\Phi), \iota^*\nabla^{\Phi}_{\theta^\alpha}\tau(\Phi)$ for $\alpha = 1, 2$ and $\iota^*\nabla^{\Phi}_{\theta^2}\nabla^{\Phi}_{\theta^1}\tau(\Phi)$. Using the special form (6.17) of the metric, we find

$$\tau(\Phi) = str(\nabla^{\Phi}d\Phi) = G^{\mu\nu}(\nabla^{\Phi}_{\partial_\mu}d\Phi)(\partial_\nu) - 2G^{21}(\nabla^{\Phi}_{\theta^2}d\Phi)(\tfrac{\partial}{\partial\theta^1}) \tag{6.24}$$

$\mu, \nu = 1, \ldots, m$ will label only the even coordinates in the course of this proof.
Equation (6.21) is obtained by rearranging $\iota^*\tau(\Phi) = 0$ using (6.24), the form of the metric given in lemma 6.30 and the definition of the components. To prove (6.22) for $\alpha = 1$, we obtain from lemma 6.30 that $\iota^*\frac{\partial G^{ij}}{\partial\theta^\alpha} = 0$ and hence

$$\iota^*\tau(\Phi) = \underbrace{g^{\mu\nu}\iota^*\nabla_{\theta^\alpha}\big(\nabla^{\Phi}_{\mu}d\Phi(\tfrac{\partial}{\partial x^\nu}) - d\Phi(\nabla_\mu\tfrac{\partial}{\partial x^\nu})\big)}_{=:I} - 2\underbrace{\iota^*\nabla^{\Phi}_{\theta^\alpha}\big(\nabla^{\Phi}_{\theta^2}d\Phi(\tfrac{\partial}{\partial\theta^1}) - d\Phi(\nabla_{\theta^2}\tfrac{\partial}{\partial\theta^1})\big)}_{=:II}$$

We deal with both summands separately. Using the fact that ∇^Φ is free of torsion and the curvature expressions in remark (3.54), we have

$$\begin{aligned} I &= g^{\mu\nu}\iota^*\big(R^N(d\Phi(\tfrac{\partial}{\partial\theta^1}), d\Phi(\tfrac{\partial}{\partial x^\mu}))d\Phi(\tfrac{\partial}{\partial x^\nu}) + \nabla^\Phi_\mu\nabla^\Phi_\nu d\Phi(\tfrac{\partial}{\partial\theta^1}) - \nabla^\Phi_{\nabla_\mu\partial_\nu} d\Phi(\tfrac{\partial}{\partial x^\nu}) - d\Phi([\tfrac{\partial}{\partial\theta^1}, \nabla_\mu\tfrac{\partial}{\partial x^\nu}])\big) \\ &= tr_g R^N(\psi_1, d\varphi(\cdot))d\varphi(\cdot) - \nabla^{\varphi*}\nabla^\varphi\psi_1 - \tfrac{1}{2}tr_g(\hat{g})\psi_1 \end{aligned}$$

In the last step, we used the expression (6.19) for the Lie bracket. Taking into account that $\nabla^\Phi_{\theta^1}(\frac{\partial}{\partial\theta^1}) = 0$ since ∇^Φ has no torsion, we obtain

$$\begin{aligned} II &= -\iota^*\big(R^N(d\Phi(\tfrac{\partial}{\partial\theta^1}), d\Phi(\tfrac{\partial}{\partial\theta^2}))d\Phi(\tfrac{\partial}{\partial\theta^1}) - \nabla^\Phi d\Phi(\nabla_{\theta^2}\tfrac{\partial}{\partial\theta^1})\big) \\ &= -R^N(\psi_1,\psi_2)\psi_1 - h\psi \end{aligned}$$

In the last step, we used (6.18). Combining I and II yields the second equation of motion, the generalization to $\alpha = 2$ is straightforward.
The third equation is obtained in the same way, the expression used above is obtained by using Bianchi identities to simplify expressions involving curvature. We omit the rather long calculations.

□

Remark 6.33 It is tempting to define the component decomposition using the adapted frame $\{e_i\}$ instead of coordinate vector fields. In general, it is possible to define components with respect to some general odd frame for $\mathcal{T}_{\bar{1}}$. However, looking at the preceding proof, it is clear that at each point, where the definition of curvature or torsion is used, additional terms containing Lie brackets will occur. It was impossible so far to find an appropriate adapted frame which circumvents this problem and up to now, a coordinate frame seems to be the most convenient choice to define components. In particular, no straightforward generalization of the left invariant fields $D_\alpha = \frac{\partial}{\partial\theta^\alpha} - \Gamma^\mu_{\alpha\beta}\theta^\alpha\frac{\partial}{\partial x^\mu}$ (cf. [11], chapter 1) could be found. First, the even coordinate fields are in general only defined locally but to be able to define global component fields, global odd frame fields are needed. Second, even in cases when these fields can be defined globally, it seems to be impossible to find a suitable set of fields D_α, adapted to the geometry of (X, G) in a canonical way, which leads to easier component equations than those given above.

Again, all the objects occurring in (6.21) to (6.23) still depend on the odd parameters of the function ring Λ_n of the superpoint $\mathcal{P}_n$. Following the strategy used for supergeodesics, we decompose w.r.t. the odd parameters and obtain a set of subcomponent equations for each $n \in \mathbb{N}$. As described before, functoriality implies that it is only necessary to give the equations for the "new" top component field at each level n. For $n \in 2\mathbb{N}+1$, this is a single equation arising from (6.22), for $n \in 2\mathbb{N}$ we have a coupled system of 2 equations arising from (6.21) and (6.23). We will only give the lowest contributions for $n = 0, 1$ in detail:

n=0 The second equation yield a trivial contribution $0 = 0$, so we are left with two equations for the 0-component $\varphi_{\underline{0}}^{(0)}, \xi_{\underline{0}}^{(0)}$, respectively:

$$\tau(\varphi_{\underline{0}}^{(0)}) = -2\xi_{\underline{0}}^{(0)} \tag{6.25}$$

$$\begin{aligned}\nabla^{\varphi_{\underline{0}}^{(0)}*}\nabla^{\varphi_{\underline{0}}^{(0)}}\xi_{\underline{0}}^{(0)} &= tr_g R^N(\xi_{\underline{0}}^{(0)}, d\varphi_{\underline{0}}^{(0)}(\cdot))d\varphi_{\underline{0}}^{(0)}(\cdot) + 2h\xi_{\underline{0}}^{(0)} - \tfrac{1}{2}tr_g(\hat{g})\xi_{\underline{0}}^{(0)} \\ &+ d\varphi_{\underline{0}}^{(0)}(grad_g(h)) - \langle \nabla^{\varphi_{\underline{0}}^{(0)}} d\varphi_{\underline{0}}^{(0)}, \hat{g}\rangle - g^{\mu\nu}(\hat{\Gamma}^i_{\mu\nu} - \Gamma^k_{\mu\nu}\hat{g}_{kj}g^{ji})d\varphi_{\underline{0}}^{(0)}(\tfrac{\partial}{\partial x^i})\end{aligned} \tag{6.26}$$

n=1 We have the subcomponent fields $\varphi_{\underline{0}}^{(1)}$, $\psi_{\alpha,1}^{(1)}$ and $\xi_{\underline{0}}^{(1)}$ and one new subcomponent equation obtained by differentiating (6.22) w.r.t. the coordinate η of $\mathcal{P}_1$:

$$0 = \nabla^{\varphi_{\underline{0}}^{(1)}*}\nabla^{\varphi_{\underline{0}}^{(1)}}\psi_{\alpha,1}^{(1)} + \textstyle\sum_i R^N(\psi_{\alpha,1}^{(1)}, d\varphi_{\underline{0}}^{(1)}(e_i))d\varphi_{\underline{0}}^{(1)}(e_i) \qquad (\alpha = 1, 2)$$

For $n > 1$, the equations have the following general structure

Proposition 6.34 *For $n \in 2\mathbb{N}^+$, the equations of motion for the highest degree subcomponent fields $\varphi_{\underline{n}}, \xi_{\underline{n}}$ are given by the system*

$$\nabla^{\varphi_{\underline{0}}*}\nabla^{\varphi_{\underline{0}}}\varphi_{\underline{n}} + 2\xi_{\underline{n}} = F_n(\varphi_A, \psi_{\alpha,B}, \xi_C) \tag{6.27}$$

$$\begin{aligned}&\nabla^{\varphi_{\underline{0}}*}\nabla^{\varphi_{\underline{0}}}\xi_{\underline{n}} - tr_g R^N(\xi_{\underline{n}}, d\varphi_{\underline{0}}(\cdot))d\varphi_{\underline{0}}(\cdot) \\ &-tr_g R^N(\xi_{\underline{0}}, \nabla^{\varphi_{\underline{0}}}_{\cdot}\varphi_{\underline{n}})d\varphi_{\underline{0}}(\cdot) - tr_g R^N(\xi_{\underline{0}}, d\varphi_{\underline{0}}(\cdot))\nabla^{\varphi_{\underline{0}}}_{\cdot}\varphi_{\underline{n}} \\ &-2h\xi_n - \nabla^{\varphi_{\underline{0}}}_{(grad_g h)}\varphi_{\underline{n}} + \left\langle(\nabla^{\varphi_{\underline{0}}})^2_{\cdot,\cdot}\varphi_{\underline{n}}, \hat{g}\right\rangle \\ &+\tfrac{1}{2}tr_g(\hat{g})\xi_{\underline{n}} + g^{\mu\nu}(\hat{\Gamma}^i_{\mu\nu} - \Gamma^k_{\mu\nu}\hat{g}_{kj}g^{ji})\nabla^{\varphi_{\underline{0}}}_{x^i}\varphi_{\underline{n}} = G_n(\varphi_A, \psi_{\alpha,B}, \xi_C)\end{aligned}$$

For $n \in 2\mathbb{N} + 1$, the equation of motion for the highest degree subcomponent field $\psi_{\alpha,\underline{n}}$ ($\alpha = 1, 2$) is given by

$$\nabla^{\varphi_{\underline{0}}*}\nabla^{\varphi_{\underline{0}}}\psi_{\alpha,\underline{n}} - tr_g R^N(\psi_{\alpha,\underline{n}}, d\varphi_{\underline{0}}(\cdot))d\varphi_{\underline{0}}(\cdot) - ((-1)^\alpha \tfrac{1}{2}tr_g\hat{g} + 2h)\psi_{\alpha,\underline{n}} = H_n(\varphi_A, \psi_{\alpha,B}, \xi_C) \tag{6.28}$$

Again, F, G and H are multilinear functions of the subcomponents $\varphi_A, \psi_{\alpha,B}, \xi_C$ such that $\|A\|, \|B\|, \|C\| < n$, depending in a complicated way on the curvature of (N, g) and its (higher) derivatives.

The proof uses the same method which already used to show proposition 6.23. We will not give the details. The subcomponent equations have the following structure:

Proposition 6.35 *The subcomponent equations at level $n \geq 1$ are inhomogeneous partial differential equations of order ≤ 2 for the subcomponent fields with multi-index $\underline{n}$. The inhomogeneity depends on the subcomponent fields up to level $n - 1$ and the geometric data of (X, G) and $(N, \langle,\rangle)$.*

- *For $n \in 2\mathbb{N}_0+1$, equation (6.28) is defined by a linear elliptic operator acting on sections of $\varphi_{\underline{0}}^* TN$ which is selfadjoint.*

- *For $n \in 2\mathbb{N}_0$, equations (6.27) are given by a linear elliptic operator action on sections of $\varphi_{\underline{0}}^*(TN) \oplus \varphi_{\underline{0}}^*(TN)$, which is however not necessarily selfadjoint.*

PROOF The general statement concerning the structure of the inhomogeneity follows from proposition 6.34 or rather the proof of proposition 6.23. Since $\nabla^*\nabla$ is linear, elliptic and selfadjoint (see example 6.17) and $X \mapsto tr_g R^N(X, d\varphi_{\underline{0}}(\cdot))d\varphi_{\underline{0}}(\cdot)$ is linear, of order 0 and selfadjoint since $\tilde{X}$ is compact, we obtain the statement for n odd.
For n even, the principle symbol of the differential operator is determined by the map $(\varphi_{\underline{n}}\xi_{\underline{n}}) \mapsto (\nabla^{\varphi_{\underline{0}}*}\nabla^{\varphi_{\underline{0}}}\varphi_{\underline{n}}, \nabla^{\varphi_{\underline{0}}*}\nabla^{\varphi_{\underline{0}}}\xi_{\underline{n}} + \langle(\nabla^{\varphi_{\underline{0}}})^2_{\cdot,\cdot}\varphi_{\underline{n}}, \hat{g}\rangle)$ so that it is invertible. Thus, we have linear and elliptic operator.

□

Remark 6.36

(a) The operator in equation (6.27) is selfadjoint if we assume that G and $\langle,\rangle$ are flat. However, it is not sufficient to assume G is flat since the properties of the term $tr_g R^N(\xi_{\underline{0}}, \nabla^{\varphi_{\underline{0}}}_{\cdot}\varphi_{\underline{n}})d\varphi_{\underline{0}}(\cdot) + tr_g R^N(\xi_{\underline{0}}, d\varphi_{\underline{0}}(\cdot))\nabla^{\varphi_{\underline{0}}}_{\cdot}\varphi_{\underline{n}}$ depend on the curvature of the target geometry.

(b) Proposition 6.35 suggests that it might be possible to set up an inductive procedure to determine the functor of critical points. In analogy to the situation in section 6.3, we still have a system of nonlinear equation at level $n = 0$ and linear elliptic systems at level $n > 0$. In the latter case, the space of solutions still decomposes into affine spaces modelled over a finite dimensional vector space (this is true for all elliptic operators) but the characterization of the existence of solutions given in corollary 6.15 is in general not applicable.

We will finally discuss the existence of critical points under special circumstances. Again, it is in general impossible to prove that the functor of critical points is representable by a supermanifold due to the nonlinearity of (6.25), already its 0-points do not form a smooth manifold if they exist at all. In analogy to the discussion in section 6.3 it would be possible to introduce a bundle picture where critical $n+1$-points form the total space over the critical n-points. We will not repeat it here but be content with discussing the existence of solutions under severe restrictions.

The subcomponent equations (6.25) and (6.26) at level $n = 0$ form a coupled system of nonlinear differential equations and it is not clear whether they admit solutions at all. We will assume that

$$\hat{g} = 0 \qquad\qquad h = 0 \tag{6.29}$$

in other words, we assume that the super Riemannian metric G is just given by the ordinary Riemannian metric g on $T\tilde{X}$ and that it is flat in the odd directions. The subcomponent equations now take the form

$$\tau(\varphi_{\underline{0}}^{(0)}) = -2\xi_{\underline{0}}^{(0)} \qquad \nabla^{\varphi_{\underline{0}}^{(0)}*}\nabla^{\varphi_{\underline{0}}^{(0)}}\xi_{\underline{0}}^{(0)} = tr_g R(\xi_{\underline{0}}^{(0)}, d\varphi_{\underline{0}}^{(0)}(\cdot))d\varphi_{\underline{0}}^{(0)}(\cdot) \tag{6.30}$$

and it is easy to observe that a harmonic map $\varphi_{\underline{0}}^{(0)}$, together with $\xi_{\underline{0}}^{(0)} = 0$ solve the system.

Theorem 6.37 *Assume that the super Riemannian metric satisfies* (6.29) *and that* $(N, \langle,\rangle)$ *is a compact Riemannian manifold of nonpositive curvature. Then, there exist solutions to* (6.16) *in the sense that for each* n*, the set* $\underline{Crit}(E)(\Lambda_n)$ *is nonempty and these sets behave functorially.*

Under the strong assumptions, the theorem is easy to show because it follows from theorem 6.9 that there are critical 0-points of the form $(\varphi_0, \xi_0 = 0)$ where φ_0 is a harmonic map. The trivial extensions to higher points (i.e. setting all higher subcomponent fields to 0) then form critical n-points for all $n \in \mathbb{N}$. We will not try to determine the structure of the complicated equations 6.34.

The equations (6.30) for $n = 0$ have the following interesting property:

Proposition 6.38 *Let* X *be a compact and let* $(N, \langle,\rangle)$ *be a Riemannian manifold of non-positive section curvature. If* $\varphi_{\underline{0}}, \xi_{\underline{0}}$ *satisfy* (6.30)*, then we already have* $\xi_{\underline{0}} = 0$.

PROOF Multiplying the differential equation on $\xi_{\underline{0}}$ by $\xi_{\underline{0}}$ and integrating it yields

$$\int_{\tilde{X}} \|\nabla^{\varphi_{\underline{0}}} \xi_{\underline{0}}\|^2 = \sum_i \int_{\tilde{X}} \|\xi_{\underline{0}} \wedge d\varphi_{\underline{0}}(e_i)\|^2 K^N(\xi_{\underline{0}} \wedge d\varphi_{\underline{0}}(e_i)) \leq 0$$

which implies $\nabla^{\varphi_{\underline{0}}} \xi_{\underline{0}} = 0$. Thus, $\xi_{\underline{0}}$ is parallel and in particular, $\|\xi_{\underline{0}}\| =: C \in \mathbb{R}$. We define the form $\alpha \in \Omega^2(X^{red})$ by $\alpha(X) := \langle d\varphi_{\underline{0}}(X), \xi_{\underline{0}} \rangle$ and compute its divergence:

$$\begin{aligned} div\alpha &= \textstyle\sum_i \partial_{e_i} \langle d\varphi_{\underline{0}}(e_i), \xi_{\underline{0}} \rangle - \langle d\varphi_{\underline{0}}(\nabla_{e_i} e_i), \xi_{\underline{0}} \rangle \\ &= \langle \tau(\varphi_{\underline{0}}), \xi_{\underline{0}} \rangle + \textstyle\sum_i \langle d\varphi_{\underline{0}}(e_i), \nabla^{\varphi_{\underline{0}}}_{e_i} \xi_{\underline{0}} \rangle \\ &= -2 \langle \xi_{\underline{0}}, \xi_{\underline{0}} \rangle = -2C^2 \end{aligned}$$

Since $\tilde{X}$ is closed, the divergence theorem implies $0 = -2C^2 vol(M)$ which in turn implies $\xi_{\underline{0}} = 0$.

□

Remark 6.39 This result is interesting for two reasons: First, it reduces the subcomponent equations (6.30) at $n = 0$, which provide a coupled system of nonlinear partial differential equations, to the case of the problem of finding harmonic maps.
Second, the result again demonstrates the necessity to introduce the "space" $\underline{SC}^\infty(X, N)$ and look for the functor of critical points. If we had only been looking for critical morphisms in the category BKL, i.e. 0-points of $\underline{SC}^\infty(X, N)$, we could have expected one new contribution from the super world given by $\xi_{\underline{0}}^{(0)}$ which is not there in the setting of classical harmonic maps. However, proposition 6.38 shows that this contribution vanishes under the assumptions on G and $\langle,\rangle$ so that in the BKL-approach to morphisms, we are left with a smooth harmonic map. In this sense, looking at higher points is crucial to obtain new (and interesting) contributions from the super setting.

Finally, based on the method introduced in section 5.6, we now show how to generalize the existence theorem to general supermanifolds which do not admit a global odd coordinate system. Let $E \longrightarrow \tilde{X}$ be a Batchelor bundle for X. Since there exists a super Riemannian metric, E is a complex bundle by corollary 3.41. By theorem 5.39, there is a complex bundle $E' \longrightarrow \tilde{X}$ such that $E \oplus E'$ is trivial and using the corollary again, we find a super Riemannian metric $G'_{\overline{1}}$ on the new odd dimensions given by E' which can be chosen to be flat. Thus, we have a super Riemannian manifold $(\mathbb{X}, \mathbb{G})$ where X is defined in (5.28) and it admits global coordinates.
Assume there exist critical n-points $\overline{\Phi}$ of $\underline{SC}^{\infty}(\mathbb{X}, N)$ for the energy functional defined on the supermanifold $(\mathbb{X}, \mathbb{G})$. These are characterized by the equation of motion $\tau(\overline{\Phi}) = 0$ by theorem 6.29. Using the inclusion and projection defined in (5.29), we obtain natural transformations

$$\iota_{\mathbb{X}}^* : \underline{SC}^{\infty}(\mathbb{X}, N) \longrightarrow \underline{SC}^{\infty}(X, N) \qquad\qquad \pi_{\mathbb{X}}^* : \underline{SC}^{\infty}(X, N) \longrightarrow \underline{SC}^{\infty}(\mathbb{X}, N)$$

which are given by $(id_{\mathcal{P}_n} \times \iota_{\mathbb{X}})^*$ and $(id_{\mathcal{P}_n} \times \pi_{\mathbb{X}})^*$ on n-points, respectively. We then have the following correspondence of critical points:

Proposition 6.40 *Let $\overline{\Phi}$ be a critical n-point for the energy functional defined on $(\mathbb{X}, \mathbb{G})$, then $\Phi := (id_{\mathcal{P}_n} \times \iota_{\mathbb{X}})^* \overline{\Phi}$ is critical for the functional on (X, G). Conversely, if Φ is critical, then also $(id_{\mathcal{P}_n} \times \pi_{\mathbb{X}})^* \Phi$ is critical.*

PROOF We will suppress the first factor of $(id_{\mathcal{P}_n} \times \iota_{\mathbb{X}})^*$ and just write $\iota_{\mathbb{X}}^*$ for the action on functions as well as on vector fields. We may choose a local orthonormal frame for (X, G) and extend it to one of $(\mathbb{X}, \mathbb{G})$, both will be denoted by $\{e_i\}$.
By 6.29, $\tau(\overline{\Phi}) = \sum_i (\nabla^{\overline{\Phi}}_{e_i} d\overline{\Phi})(Je_i) = \sum_i \nabla^{\overline{\Phi}}_{e_i} d\overline{\Phi}(Je_i) - d\overline{\Phi}(\nabla_{e_i} Je_i) = 0$. Applying $\iota_{\mathbb{X}}^*$ to the second summand, we obtain $\iota_{\mathbb{X}}^* \sum_i d\overline{\Phi}(\nabla_{e_i} Je_i) = \sum_i d(\iota_{\mathbb{X}}^* \overline{\Phi})(d\iota_{\mathbb{X}}^*(\nabla_{e_i} Je_i)) = \sum_i d\Phi(\nabla_{e_i} Je_i)$. On the left hand side, we are only summing over the frame of X since $d\iota_{\mathbb{X}}^*$ annihilates tangent vectors arising from the additional directions in $\mathbb{X}$. Using a coordinate expansion as e.g. in the proof of lemma 5.36 (d), we can again use the argument that $d\iota_{\mathbb{X}}^*$ annihilates the extra directions and obtain: $\iota_{\mathbb{X}}^* \sum_i \nabla^{\overline{\Phi}}_{e_i} d\overline{\Phi}(Je_i) = \sum_i \nabla^{\Phi}_{e_i} d\Phi(Je_i)$. Using these identities, we can calculate the tension field of Φ:

$$\tau(\Phi) = \sum_i \nabla^{\Phi}_{e_i} d\Phi(Je_i) - d\Phi(\nabla_{e_i} Je_i) = \iota_{\mathbb{X}}^* \big(\sum_i \nabla^{\overline{\Phi}}_{e_i} d\overline{\Phi}(Je_i) - d\overline{\Phi}(\nabla_{e_i} Je_i)\big) = \iota_{\mathbb{X}}^* \tau(\overline{\Phi}) = 0$$

Thus, Φ is critical. The converse direction is obtained in a similar fashion.

□

Since each critical n-point $\overline{\Phi}$ induces a critical n-point $\Phi := (id_{\mathcal{P}_n} \times \iota_{\mathbb{X}})^* \overline{\Phi} \in \underline{SC}^{\infty}(X, N)(\Lambda_n)$, we have shown the assumption concerning the existence of globally defined odd coordinates can be removed.

A Appendix: Elements of category theory

Let $\mathcal{C}$ be a category. We denote by $Obj(\mathcal{C})$ its class of objects and by $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ the morphisms between the objects X, Y. We denote by $\mathcal{C}^{op}$ the opposite category, i.e. the category defined by

$$Obj(\mathcal{C}^{op}) := Obj(\mathcal{C}) \qquad\qquad \mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(Y,X)$$

Thus, $\mathcal{C}^{op}$ is obtained from $\mathcal{C}$ by reversing arrows.

For two categories $\mathcal{C}$ and $\mathcal{D}$, the functor category $\mathcal{D}^{\mathcal{C}}$ is defined as follows:

$$\begin{aligned} Obj(\mathcal{D}^{\mathcal{C}}) &:= \{ \text{ functors } \mathcal{C} \longrightarrow \mathcal{D}\} \\ \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F,G) &:= \{ \text{ natural transformations } F \longrightarrow G\} \end{aligned}$$

Here, a natural transformation from a functor F to another functor G is a family of morphisms $\{\eta_X\}_{X\in Obj(\mathcal{C})}$ such that the following diagram commutes for all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd} \tag{A.1}$$

The class of all natural transformations $F \longrightarrow G$ will also be denoted by $\mathrm{Nat}(F,G)$ Strictly speaking, it is necessary to discuss some set theoretic subtleties at this point. We will not do this but refer to [59], chapter 3.

Two functors F, G are called isomorphic, if there exist a natural transformation $\eta : F \longrightarrow G$ such that each $\eta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ is an isomorphism (i.e. there exist an inverse morphism). In this case, η is called a natural equivalence. This is also often expressed by saying that a bijection

$$F(X) \cong G(X) \qquad\qquad X \in Obj(\mathcal{C})$$

is functorial in X. Thus, "being functorial" in X means that there is an isomorphism η (or more general a natural transformation in case bijectivity is not assumed) which relates $F(X)$ and $G(X)$ according to (A.1).

Let *Set* denote the category of sets. For any category $\mathcal{C}$, we have the following two functors :

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(-,X) : \mathcal{C}^{op} \longrightarrow \mathrm{Set} \quad & \mathrm{Hom}_{\mathcal{C}}(-,X)(Y) := \mathrm{Hom}_{\mathcal{C}}(Y,X) \quad & \mathrm{Hom}_{C}C(-,X)(f) := f^* \\ \mathrm{Hom}_{\mathcal{C}}(X,-) : \mathcal{C} \longrightarrow \mathrm{Set} \quad & \mathrm{Hom}_{\mathcal{C}}(X,-)(Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y) \quad & \mathrm{Hom}_{C}C(X,-)(g) := g_* \end{aligned}$$

Proposition A.1 (Yoneda Lemma) *Let $\mathcal{C}$ be a category and $F : \mathcal{C} \longrightarrow Sets$ a functor and A an object of $\mathcal{C}$. Then, there is a bijection*

$$\begin{aligned} Y : Nat(\mathrm{Hom}(A, -), F) &\longrightarrow F(A) \\ \tau &\mapsto \tau_A(1_A) \end{aligned} \tag{A.2}$$

See [59], 4.2 for a proof. We have the following corollaries:

Corollary A.2 *Let A, A' be objects in $\mathcal{C}$, then the following map yields a bijection:*

$$\begin{aligned} Y^{-1} \ : \ \mathrm{Hom}(A', A) &\longrightarrow \mathrm{Hom}_{Set^{\mathcal{C}}}(\mathrm{Hom}(A, -), \mathrm{Hom}(A', -)) \\ f &\mapsto f^* \end{aligned}$$

In other words, we have a bijective correspondence $\tau \leftrightarrow f$ of natural transformations of Hom*-functors and morphisms in $\mathcal{C}$.*

Corollary A.3 *Let τ be a natural transformation and f a morphism related by $f = Y(\tau)$, then τ is a natural equivalence iff f is an isomorphism. In other words :* $\mathrm{Hom}(A, -)$ *and* $\mathrm{Hom}(A', -)$ *are naturally equivalent if and only if A and A' are isomorphic.*

The Yoneda lemma and its corollaries allow us to replace an object in the category $\mathcal{C}$ by the functor $\mathrm{Hom}_{\mathcal{C}}(-, X)$ representing it. Similarly, a morphisms in $\mathcal{C}$ can be replace by a natural transformation between the representing functors. For S another object in $\mathbb{C}$, the elements of $\mathrm{Hom}_{\mathcal{C}}(S, X)$ are also called "S-points of X" since they characterize X in the way described before.

Following [30], II.3.9 and II.3.10, we give an example how this method can be used to define group objects in a category (e.g. super Lie groups in some category of supermanifolds)

Example A.4 Assume that $\mathcal{C}$ contains a final object E (i.e. the set $\mathrm{Hom}_{\mathcal{C}}(X, E)$ contains precisely one morphism for every X) and that it admits finite products. We say that an object X carries a group structure if there exist three morphisms

$$\begin{aligned} \mu : X \times X &\longrightarrow X && \text{(multiplication law)} \\ i : X &\longrightarrow X && \text{(inverse map)} \\ e : E &\longrightarrow G && \text{(identity)} \end{aligned}$$

which satisfy certain compatibility relations expressing that μ is associative, that i is the left inversion and the e is a left identity. For instance, the identity $\mu \circ (\mu, id_X) = \mu \circ (id_X, \mu)$ means that the multiplication is associative and there are similar requirements on i and e to make sure that they describe a unit element and the inversion, respectively (see [30] II.3.10 for a diagrammatic expression for these relations). These morphisms and compatibility relations form a generalization of the usual definition of a group to an arbitrary category.

Alternatively, a group object can be defined as an object X such that all its S-points $\mathrm{Hom}(S, X)$ carry compatible group structures (see [30] II.3.9). More precisely, this means

(a) Each set $\mathrm{Hom}(S, X)$ is an ordinary group with unit element e_S, multiplication law μ_S and inversion law i_S.

(b) For any morphism $\varphi : S \longrightarrow T$ in $\mathcal{C}$, the map $\varphi^* : \mathrm{Hom}(T, X) \longrightarrow \mathrm{Hom}(S, X)$ is a group homomorphism in the usual sense.

Both approaches are equivalent : By composition, the morphisms μ, i, e clearly induce operations on the S-points $\mathrm{Hom}(S, X)$. In this way, the sets $\mathrm{Hom}(S, X)$ become groups and the group structures are clearly compatible in the way described above. Conversely, using Yoneda's lemma, the functoriality of the multiplication laws μ_S in S implies the existence of a morphism $\mu : X \times X \longrightarrow X$. Since all μ_S are associative, the same holds for μ. The morphisms i and e are obtained in a completely analogous way from the i_S and e_S.

Thus, replacing objects by functors and morphisms by natural transformations clearly increases the layer of abstraction but on the other hands, it allows us to carry over many constructions well known for sets (with some extra structure) to general categories in an elegant and natural way.

We will finally discuss fibre products and their generalization to categories since this concept is needed in the main part of this thesis. Let M, N, S be sets and $m : M \longrightarrow S, n : N \longrightarrow S$ be maps. The fibre product of m and n is then defined as a set $M \times_S N$, together with a canonical map $m \times_S n : M \times_S N \longrightarrow S$:

$$M \times_S N := \{(x, y) \in M \times N | m(x) = n(y)\} \tag{A.3}$$

$$m \times_S n := ((x, y) \mapsto m(x) = n(y)) \tag{A.4}$$

It is often simply denoted by $M \times_S N$ and generalizes constructions like the intersection $M \cap N$ of two subsets ($m, n \hookrightarrow S$ are the inclusions), the preimage $m^{-1}(N)$ (m a map, $n : N \hookrightarrow S$ the inclusion) and of course the ordinary product (see [30] II.5).
The notion can be generalized to arbitrary categories:

Definition A.5 *Let X, Y, S be objects in a category $\mathcal{C}$ and $x : X \longrightarrow S$, $y : Y \longrightarrow S$ two morphisms. A fibre product $X \times_S Y$ is an object in $\mathcal{C}$, together with a morphism $X \times_S Y \longrightarrow S$ which represents the following functor $\mathcal{C}^O \longrightarrow \mathrm{Set}$:*

$$T \mapsto \mathrm{Hom}(T, X) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}(T, Y) \tag{A.5}$$

and defined on morphisms by pullback.

Some remarks are in order:

Remark A.6

(a) The morphisms x, y clearly induce maps $x_* : \mathrm{Hom}(T, X) \longrightarrow \mathrm{Hom}(T, S)$ and $y_* : \mathrm{Hom}(T, Y) \longrightarrow \mathrm{Hom}(T, S)$. These maps are used for m, n in (A.3) to define the product $\times_{\mathrm{Hom}(T,S)}$.

(b) The family of maps $\{x_* \times_{\mathrm{Hom}(T,S)} y_*\}_T$ is natural in T. If the functor (A.5) is representable by $X \times_S Y$, this family then induces the morphism $X \times_S Y \longrightarrow S$ by the Yoneda lemma.

(c) A fibre product need not exist but if it does, it is uniquely defined by the functor up to isomorphism.

(d) A fibre product comes along with canonical projections $\Pi_X : X \times_S Y \longrightarrow X$ and $\Pi_Y : Y \times_S Y \longrightarrow Y$ which are obtained by taking the obvious projections on the level of S-points, i.e. on the level of sets.

There is a second equivalent way to define a fibre product over S as an ordinary product in a new category $\mathcal{C}_S$, whose objects are all morphisms $\mathrm{Hom}(X, S)$ (X arbitrary) and whose morphisms are all commutative triangles over S (see [30], p.81). We will not use this formulation but use the first definition to discuss the fibre product of two superdomains. We use the notation of definition 4.20.

Example A.7 (Fibre products of superdomains) Let X be a functor in $\mathrm{Man}^{\mathrm{Gr}}$ and $\mathbb{A} = \{u_i : U_i \longrightarrow X\}$ an open covering of it, where each U_i is a superdomain. Thus, by definition 4.14, each u_i factors as

$$u_i : U_i \xrightarrow{\tilde{u}_i} V_i \subset X$$

where $\tilde{u}_i : U_i \longrightarrow V_i$ is an isomorphism of an open subfunctor of X. By definition, we obtain for a Λ-point (Λ a finite dimensional Grassmann algebra) of the fibre product $U_i \times_X U_j$:

$$\begin{aligned}(U_i \times_X U_j)((\Lambda)) &= U_i(\Lambda) \times_{X(\Lambda)} U_j(\Lambda) \\ &= \{(p_i, p_j) \in U_i(\Lambda) \times U_j(\Lambda) | u_i(\Lambda)(p_i) = u_j(\Lambda)(p_j)\} \\ &= (u_i(\Lambda) \times u_j(\Lambda))^{-1}(V_i(\Lambda) \cap V_j(\Lambda))\end{aligned}$$

In definition 4.20, this set is required to be a superdomain. Taking into account the projections $\Pi_i : U_i \times_X U_j \longrightarrow U_i$ and $\Pi_i : U_i \times_X U_j \longrightarrow U_i$, we obtain the following diagram

$$\begin{tikzcd} & (u_i(\Lambda) \times u_j(\Lambda))^{-1}(V_i(\Lambda) \cap V_j(\Lambda)) \arrow[dl, "\Pi_i(\Lambda)"'] \arrow[dr, "\Pi_j(\Lambda)"] & \\ U_i(\Lambda) \arrow[rr] & & U_j(\Lambda) \end{tikzcd}$$

Thus, requiring Π_i and Π_j to be supersmooth means that the horizontal arrow at the bottom of the diagram, which is given by $u_j(\Lambda) \circ u_i(\Lambda)^{-1}$, is required to be supersmooth. In other words, coordinate transitions are required to be supersmooth.

References

[1] L. Balduzzi, C. Carmeli, R. Fioresi, *A Comparison of the functors of points of Supermanifolds*, arXiv:0902.1824v2

[2] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez, *The geometry of supermanifolds.* Mathematics and its Applications, 71. Kluwer Academic Publishers Group, Dordrecht, 1991.

[3] M. Batchelor, *The structure of supermanifolds.* Trans. Amer. Math. Soc. 253 (1979), 329338.

[4] F. A. Berezin, *Introduction to superanalysis.* Mathematical Physics and Applied Mathematics, 9. D. Reidel Publishing Co., Dordrecht, 1987.

[5] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Grundlehren der mathematischen Wissenschaften 298, Springer-Verlag, Berlin, 1992

[6] N. Bourbaki, *Elements of Mathematics, Algebra I, Chapters 1-3*, Springer-Verlag, New York, 1989

[7] I. L. Buchbinder, S. M. Kuzenko, *Ideas and methods of supersymmetry and supergravity. Or a walk through superspace.* Revised edition. Studies in High Energy Physics Cosmology and Gravitation. IOP Publishing, Ltd., Bristol, 1998.

[8] F. Constantinescu, H. F. de Groote, *Geometrische und algebraische Methoden der Physik: Supermannigfaltigekeiten und Virasoro-Algebren*, B.G. Teubner Stuttgart, 1994

[9] B. DeWitt, *Supermanifolds.* Second edition. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1992.

[10] P. Deligne, J. Morgan, *Notes on Supersymmetry (following J.Bernstein)*, in: Quantum fields and strings: a course for mathematicians. Vol. 1, Edited by P. Deligne et.al., American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999.

[11] P. Deligne, D. Freed, *Supersolutions*, in: Quantum fields and strings: a course for mathematicians. Vol. 1, Edited by P. Deligne et.al., American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999.

[12] V. Cortés, *A new construction of homogeneous quaternionic manifolds and related geometric structures.* Mem. Amer. Math. Soc. 147 (2000), no. 700.

[13] V. Cortés, C. Mayer, T. Mohaupt, F. Saueressig, *Special geometry of Euclidean supersymmetry. I. Vector multiplets.* J. High Energy Phys. 2004, no. 3, 028.

[14] A. Douady, *Le probleme des modules pour les sous-espaces analytiques compacts d'un espace analytique donne*. Ann. Inst. Fourier (Grenoble) 16 1966 fasc. 1, 195.

[15] F. Dumitrescu, *Superconnections and Parallel Transport*, PhD thesis, online available at http://etd.nd.edu/ETD-db/theses/available/etd-07212006-131339/. See also arXiv:0711.2766.

[16] J. Eells, L. Lemaire, *A report on harmonic maps*. Bull. London Math. Soc. 10 (1978), no. 1, 168.

[17] J. Eells, L. Lemaire, *Another report on harmonic maps*. Bull. London Math. Soc. 20 (1988), no. 5, 385524.

[18] J. Eells, L. Lemaire, *Selected topics in harmonic maps*. CBMS Regional Conference Series in Mathematics, 50. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1983.

[19] J. Eells, J. H. Sampson, *Harmonic mappings of Riemannian manifolds*. Amer. J. Math. 86 1964 109160.

[20] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995

[21] D. Eisenbud, J. Harris, *The geometry of schemes*, Graduate Texts in Mathematics 197. Springer-Verlag, 2000

[22] H. Figueroa, J. M. Gracia-Bondía, J. C. Varilly, *Elements of noncommutative geometry*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Boston, Inc., Boston, MA, 2001.

[23] D. Freed, *Five lectures on supersymmetry*. American Mathematical Society, Providence, RI, 1999.

[24] S. Garnier, T. Wurzbacher, *The geodesic flow on a Riemannian supermanifold*, arXiv:1107.1815

[25] G. Giachetta, L. Mangiarotti, G. Sardanashvily, *Geometric and algebraic topological methods in quantum mechanics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[26] O. Goertsches, *Riemannian supergeometry*. Math. Z. 260 (2008), no. 3, 557 593.

[27] R. S. Hamilton, *The inverse function theorem of Nash and Moser*. Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 1, 65222.

[28] P. Hartman, *On homotopic harmonic maps*. Canad. J. Math. 19 1967 673687.

[29] F. Hélein, *A representation formula for maps on supermanifolds*, arXiv:math-ph/0603045

[30] S. I. Gelfand, Y. I. Manin, *Methods of homological algebra.* Second edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[31] D. Husemöller, *Fibre bundles.* Third edition. Graduate Texts in Mathematics, 20. Springer-Verlag, New York, 1994.

[32] M. Kashiwara, P. Schapira, *Categories and sheaves.* Grundlehren der Mathematischen Wissenschaften 332. Springer-Verlag, Berlin, 2006.

[33] I. Khemar, *Supersymmetric harmonic maps into symmetric spaces.* J. Geom.Phys. 57 (2007), no. 8, 16011630.

[34] W. Klingenberg, *Lectures on closed geodesics.* Grundlehren der Mathematischen Wissenschaften, Vol. 230. Springer-Verlag, Berlin-New York, 1978.

[35] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization.* Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975), pp. 177306. Lecture Notes in Math., Vol. 570, Springer, Berlin, 1977.

[36] I. S. Krasil'shchik, V. V. Lychagin, A. M. Vinogradov, *Geometry of jet spaces and nonlinear partial differential equations.* Translated from the Russian by A. B. Sosinskii. Advanced Studies in Contemporary Mathematics, 1. Gordon and Breach Science Publishers, New York, 1986.

[37] A. Kriegl, P. Michor, *The convenient setting of global analysis.* Mathematical Surveys and Monographs, 53. American Mathematical Society, Providence, RI, 1997.

[38] H. B. Lawson, M.-L. Michelsohn, *Spin geometry.* Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989.

[39] D. A. Leites, *Introduction to the theory of supermanifolds.* Uspekhi Mat. Nauk 35 (1980), no. 1(211), 357, 255.

[40] D. A. Leites, *Quantization and supermanifolds.* Supplement 3 in: F. A. Berezin, M. A. Shubin, *The Schrdinger equation.* Mathematics and its Applications (Soviet Series), 66. Kluwer Academic Publishers Group, Dordrecht, 1991.

[41] F. Lin, C. Wang, *The analysis of harmonic maps and their heat flows.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[42] Y. I. Manin, *Gauge field theory and complex geometry.* Translated from the 1984 Russian original by N. Koblitz and J. R. King. Second edition. With an appendix by Sergei Merkulov. Grundlehren der Mathematischen Wissenschaften 289. Springer-Verlag, Berlin, 1997.

[43] D. McDuff, D. Salamon, *Introduction to symplectic topology.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[44] V. Molotkov, *Infinite-dimensional $\mathbb{Z}_2^k$-supermanifolds*, ICTP preprints, IC/84/183, 1984. Scanned version available at http://ccdb4fs.kek.jp/cgi-bin/img/allpdf?198506284

[45] J. Monterde, J. Muñoz Masqué, *Variational Problems on Graded Manifolds*, Contemporary Mathematics, Volume 132, 551-571, American Mathematical Society, Providence, RI, 1992.

[46] J. Monterde, O. A. Sánchez-Valenzuela, *The exterior derivative as a Killing vector field.* Israel J. Math. 93 (1996), 157170.

[47] J. Nestruev, *Smooth manifolds and observables.* Joint work of A. M. Astashov, A. B. Bocharov, S. V. Duzhin, A. B. Sossinsky, A. M. Vinogradov and M. M. Vinogradov. Translated from the 2000 Russian edition by Sossinsky, I. S. Krasil'schik and S. V. Duzhin. Graduate Texts in Mathematics, 220. Springer-Verlag, New York, 2003.

[48] F. O'Dea, *Supersymmetric Harmonic Maps into Lie Groups*, arXiv:hep-th/0112091v1

[49] J.-F. Pommaret, *Systems of partial differential equations and Lie pseudogroups.* With a preface by Andre Lichnerowicz. Mathematics and its Applications, 14. Gordon & Breach Science Publishers, New York, 1978.

[50] S. Rempel, T. Schmitt, *Pseudodifferential operators and the index theorem on supermanifolds.* Math. Nachr. 111 (1983), 153175.

[51] A. Rogers, *Supermanifolds.* Theory and applications. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[52] C. Sachse, *Global Analytic Approach to Super Teichmueller Spaces*, arXiv:0902.3289

[53] C. Sachse, *A Categorical Formulation of Superalgebra and Supergeometry*, arXiv:0802.4067

[54] C. Sachse, C. Wockel, *The diffeomorphism supergroup of a finite-dimensional supermanifold*, arXiv:0904.2726

[55] G. Sardanashvily, *Lectures on Differential Geometry of Modules and Rings*, arXiv:0910.1515

[56] D. J. Saunders, *The geometry of jet bundles.* London Mathematical Society Lecture Note Series, 142. Cambridge University Press, Cambridge, 1989.

[57] T. Schmitt, *Super differential geometry*, Report 05/84 des IMath der Akad. der Wiss. der DDR, 1984

[58] R. Schoen, S. T. Yau, *Lectures on harmonic maps.* Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997.

[59] H. Schubert, *Kategorien I*, Heidelberger Taschenbücher, Band 65, Springer-Verlag, Berlin-New York 1970

[60] A. S. Shvarts, *On the definition of superspace.* Teoret. Mat. Fiz. 60 (1984), no. 1, 3742.

[61] *Solution to FP2* (p.649 to 656), in: Quantum fields and strings: a course for mathematicians. Vol. 1, Edited by P. Deligne et. al., American Mathematical Society, Providence, RI, Institute for Advanced Study (IAS), Princeton, NJ, 1999.

[62] B. R. Tennison, *Sheaf theory.* London Mathematical Society Lecture Note Series, No. 20. Cambridge University Press, Cambridge, England-New York-Melbourne, 1975.

[63] G. M. Tuynman, *Supermanifolds and supergroups.* Basic theory. Mathematics and its Applications, 570. Kluwer Academic Publishers, Dordrecht, 2004.

[64] H. Urakawa, *Calculus of variations and harmonic maps.* Translated from the 1990 Japanese original by the author. Translations of Mathematical Monographs, 132. American Mathematical Society, Providence, RI, 1993.

[65] A. A. Voronov, *Maps of supermanifolds*, Teoret. Mat. Fiz. 60 (1984), no. 1, 43-48.

[66] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction.* Courant Lecture Notes in Mathematics, 11. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2004.

[67] R. O. Wells, *Differential analysis on complex manifolds.* Third edition. With a new appendix by Oscar Garcia-Prada. Graduate Texts in Mathematics, 65. Springer, New York, 2008.

[68] J. Wess, J. Bagger, *Supersymmetry and supergravity.* Second edition. Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1992.

[69] Y. Xin, *Geometry of harmonic maps.* Progress in Nonlinear Differential Equations and their Applications, 23. Birkhäuser Boston, Inc., Boston, MA, 1996.