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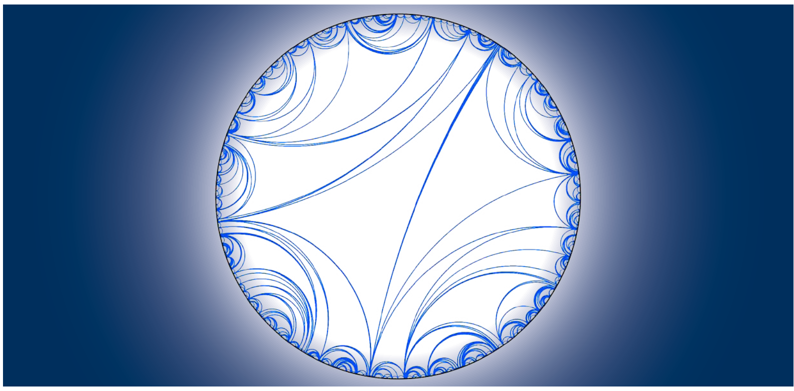

Nikolai Tarkhanov

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A SIMPLE NUMERICAL APPROACH TO THE RIEMANN HYPOTHESIS

N. TARKHANOV

This paper is dedicated to P. M. Gauthier on the occasion of his 70 *th birthday*

Abstract. The Riemann hypothesis is equivalent to the fact that the reciprocal function $1/\zeta(s)$ extends from the interval $(1/2, 1)$ to an analytic function in the quarter-strip $1/2 < \Re s < 1$, $\Im s > 0$. Function theory allows one to rewrite the condition of analytic continuability in an elegant form amenable to numerical experiments.

Contents

Introduction

The Riemann hypothesis is that all zeros of the Riemann zeta function $\zeta(z)$ in the critical strip $0 < \Re z < 1$ belong to the critical line $\Re z = 1/2$. This just amounts to saying that the function $1/\zeta(z)$ extends from the interval $(1/2, 1)$ to an analytic function in the quarter-strip $1/2 < \Re z < 1$, $\Im z > 0$. Note that the restriction of $1/\zeta(s)$ to $(1/2, 1)$ is actually continuous on the closed interval. Hence the function theory allows one to rewrite the condition of analytic continuability in an elegant form which is amenable to numerical experiments. More precisely, one constructs an explicit sequence $\{c_n\}$ of complex numbers, such that the equality $\lim \sqrt[n]{|c_n|} = 1$ is fulfilled if and only if the Riemann hypothesis is true. The numbers c_n are integrals of $1/\zeta(s)$ over the interval $[1/2, 1]$ with explicit weight function depending on n. Computations with the newest versions of Mathematica, Maple and Matlab performed by my diploma students give certain evidence to the fact that the limit is 1 indeed. However, the standard computer programmes are not sufficient to evaluate the sequence $\sqrt[n]{|c_n|}$ with strict accuracy. The numerical data

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obtained in this way testify to $\lim \sqrt[n]{|c_n|} = 1$ not only for $1/\zeta(s)$ but also for other functions (e.g. $1/(s - 3/4 - \imath/2)$) which fail to be analytically extendable to the critical quarter-strip. Thus the paper gives rise to the problem of elaborating an efficient programme which recognises through the behaviour of the sequence $\sqrt[n]{|c_n|}$ those continuous functions on $[1/2, 1]$ which extend to analytic functions in the quarter-strip.

1. The Riemann zeta function

In this section we gather necessary material about the Riemann zeta function. For complete proofs the reader is referred to [Tit51, KV92, Con03].

For complex numbers $s = \Re s + \sqrt{-1}\Im s$ in the half-plane $\Re s > 1$ the Riemann function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

the series converging absolutely and uniformly in each half-plane $\Re s > s_0$ with $s_0 > 1$.

In 1737 Euler proved his product formula which displayed a deep connection of $\zeta(s)$ with the distribution of prime numbers.

Theorem 1.1. *If* $\Re s > 1$ *then*

$$\zeta(s) = \prod_{p} \Big(1 - \frac{1}{p^s}\Big)^{-1},$$

where the product runs over all prime numbers p *(*$p = 1$ *is no prime number).*

In order to extend $\zeta(s)$ to an analytic function on all of $\mathbb{C}$, one uses the analytic extension of the gamma function constructed by Weierstraß. More precisely,

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} \Big(1 + \frac{1}{n}\Big) e^{-s/n}$$

holds for $\Re s > 0$, where γ is the Euler-Mascheroni constant.

The right-hand side of this equality is an entire function of s vanishing at the points $s = 0, -1, -2, \ldots$.

Lemma 1.2. *If* $\Re s > 1$ *then*

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \Big(\frac{1}{s(s-1)} + \int_1^{\infty} \Big(x^{s/2-1} + x^{-s/2-1/2}\Big) \sum_{n=1}^{\infty} e^{-\pi n^2 x}\, dx \Big).$$

The lemma shows that the Riemann zeta function extends to a meromorphic function in the whole complex plane with the only pole at $s = 1$ which is simple.

This function vanishes at $s = -2, -4, \ldots$, the other zeros of $\zeta(s)$ are known to lie in the critical strip $0 < \Re s < 1$.

B. Riemann conjectured (1869) that all zeros of $\zeta(s)$ in the critical strip belong to the line $\Re s = 1/2$.

The restriction of $\zeta(s)$ to the critical strip is symmetric with respect to both the critical line $\Re s = 1/2$ and the interval $(0, 1)$ of the real axis. Moreover, it is different from zero for all $s \in [0, 1]$.

Hence the Riemann hypothesis just amounts to saying that $\zeta(s)$ has no zeros in the quarter-strip $1/2 < \Re s < 1$, $\Im s > 0$.

For real $x > 0$, let $\pi(x)$ denote the number of prime numbers p which satisfy $p \leq x$.

B. Riemann showed a formula for the difference

$$\pi(x) - \int_0^x \frac{ds}{\log s}$$

in terms of x and zeros of $\zeta(s)$ lying in the critical strip.

If $\zeta(s)$ has no zeros with $\Re s > s_0$ for some $1/2 \leq s_0 < 1$, then the asymptotic formula

$$\pi(x) = \int_0^x \frac{ds}{\log s} + O(x^{s_0} \log x)$$

holds. The Riemann hypothesis just amounts to this formula with $s_0 = 1/2$.

Some textbooks in complex analysis include the so-called prime number theorem proved independently by J. Hadamard and Ch.-J. de la Vallée-Poussin (1896). It reads $\pi(x) \sim x/\log x$.

2. Analytic continuation in a lune

Denote by $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ the open unit disk with center at the origin in the plane of complex variable w. Let S be a regular curve in $\mathbb{D}$, whose endpoints lie on the unit circle and which does not run through 0 (i.e. $0 \notin S$). The curve S divides the disk $\mathbb{D}$ into two domains and we write G for the subdomain of $\mathbb{D}$ that does not contain the origin 0. In this way we obtain a bounded domain with piecewiese smooth boundary which is referred to as lune. The boundary of G consists of two parts, one of the two is the curve S and the other an arc of the circle $\partial\mathbb{D}$, see Figure 1.

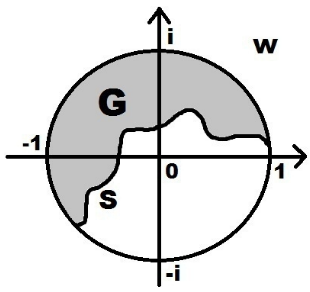

FIG. 1. A basic domain.

In 1926 T. Carleman found a simple formula for analytic continuation in a corner. For this reason the following refined formula is named after him. This formula is well known, see for instance [Aiz93]. Since the proof is very simple we give it for completeness.

Theorem 2.1. *Suppose f is a holomorphic function in G continuous up to the boundary of G. Then*

$$f(w) = \lim_{n\to\infty} \int_S f(w') \Big(\frac{w}{w'}\Big)^n \frac{1}{2\pi\imath} \frac{dw'}{w' - w} \tag{2.1}$$

for all $w \in G$.

Proof. Fix an arbitrary $w \in G$. Since $0 \notin \overline{G}$ and f is holomorphic in G and continuous up to the boundary, the function

$$F(w') := f(w')\Big(\frac{w}{w'}\Big)^n$$

is holomorphic in G and continuous on $\overline{G}$ for all $n = 1, 2, \ldots$. By the integral formula of Cauchy one gets

$$F(w) = \int_{\partial G} F(w') \frac{1}{2\pi \imath} \frac{dw'}{w' - w}$$

for all $w \in G$. Substituting F yields

$$f(w) = \int\limits_{\partial G} f(w') \left(\frac{w}{w'}\right)^n \frac{1}{2\pi \imath} \frac{dw'}{w' - w}.$$

for each $n = 1, 2, \ldots$.

The integral on the right-hand side splits into two integrals, the first is over S and the second one over $\partial G \setminus S$. So

$$f(w) = \int_S f(w') \left(\frac{w}{w'}\right)^n \frac{1}{2\pi \imath} \frac{dw'}{w' - w} + \int_{\partial G \setminus S} f(w') \left(\frac{w}{w'}\right)^n \frac{1}{2\pi \imath} \frac{dw'}{w' - w}.$$

On letting $n \to \infty$ one obtains

$$f(w) = \lim_{n \to \infty} \int_S f(w') \left(\frac{w}{w'}\right)^n \frac{1}{2\pi \imath} \frac{dw'}{w' - w} + \lim_{n \to \infty} \int_{\partial G \setminus S} f(w') \left(\frac{w}{w'}\right)^n \frac{1}{2\pi \imath} \frac{dw'}{w' - w},$$

in the case when at least one of the limits exists.

We now show that the second limit exists and is precisely zero. Since $w \in G$ and $w' \in \partial G \setminus S$, we get

$$\Big|\frac{w}{w'}\Big| = \frac{|w|}{1} < 1.$$

It follows that the sequence $(w/w')^n$ converges to zero uniformly in $w' \in \partial G \setminus S$. So the limit of the second integral can be evaluated under the integral sign and one obtains

$$\begin{aligned} \lim_{n \to \infty} \int_{\partial G \setminus S} f(w') \Big(\frac{w}{w'}\Big)^n \frac{1}{2\pi \imath} \frac{dw'}{w' - w} &= \int_{\partial G \setminus S} f(w') \left(\lim_{n \to \infty} \left(\frac{w}{w'}\right)^n\right) \frac{1}{2\pi \imath} \frac{dw'}{w' - w} \\ &= 0, \end{aligned}$$

which establishes the desired formula. □

To our best knowledge, Theorem 2.1 gives the simplest explicit formula of analytic continuation in complex analysis. Based upon this formula, we show a criterion of analytic continuability into G for a function f_0 given on the part S of boundary ∂G. While polynomials of z are dense in the Banach space $C(\overline{S})$, those functions on $\overline{S}$ which extend analytically to G form a subspace of infinite codimension in $C(\overline{S})$. In particular, the continuous functions $f_0 \not\equiv 0$ of compact support in S fail to have analytic continuation to the domain G, which is a consequence of the uniqueness theorem.

Theorem 2.2. *Let $f_0 \in C(\overline{S})$ satisfy $f_0 \not\equiv 0$. In order that there be a holomorphic function $f \in C(G \cup S)$ in G, such that $f(w) = f_0(w)$ for all $w \in S$, it is necessary and sufficient that*

$$\limsup_{n\to\infty} \sqrt[n]{\Big| \int_S \frac{f_0(w')}{(w')^{n+1}} dw' \Big|} = 1. \tag{2.2}$$

Proof. Necessity. Given a nonzero function $f_0 \in C(\overline{S})$, we define the Cauchy-type integral of f_0 by

$$Cf_0\,(w) = \int_S f_0(w')\,\frac{1}{2\pi\imath}\frac{dw'}{w'-w}$$

for $w \notin \overline{S}$. This is a holomorphic function away from the closure of S, and we denote by $C^{\pm}f_0$ the restrictions of Cf_0 to G and $\mathbb{C}\backslash\overline{G}$, respectively. The Sokhotsky-Plemelj formula says that

$$\lim_{\varepsilon\to 0+} \Big(C^+f_0\,(w' + \varepsilon\nu(w')) - C^-f_0\,(w' - \varepsilon\nu(w')) \Big) = f_0(w') \tag{2.3}$$

holds uniformly in w' on compact subsets of S, where $\nu(w')$ is the inward unit normal vector to S at a point $w' \in S$. In particular, if either of the functions $C^{\pm}f_0$ extends continuously to S then so does the other function. The limit in (2.3) is obviously zero, if $w' \in \partial G \setminus \overline{S}$.

Assume that there is a holomorphic function f in G which is continuous up to S and satisfies $f = f_0$ on S. A simple manipulation with the Cauchy integral formula for f shows that the difference $C^+f_0 - f$ extends to a continuous (even C^∞) function on $G \cup S$. Since f is continuous on $G \cup S$, the integral C^+f_0 extends to a continuous function on $G \cup S$. By the above, C^-f_0 extends continuously to $\mathbb{D} \setminus G$, too.

Consider the function

$$F(w) = \begin{cases} C^+f_0\,(w) - f(w), & \text{if} \quad w \in G \cup S, \\ C^-f_0\,(w), & \text{if} \quad w \in \mathbb{D} \setminus \overline{G}, \end{cases}$$

in the disk $\mathbb{D}$. This function is holomorphic in $\mathbb{D} \setminus S$ and continuous on all of $\mathbb{D}$, for $C^+f_0 - f = C^-f_0$ on S, which is due to Sokhotsky-Plemelj formula (2.3). From the Morera theorem we easily deduce that F is actually holomorphic in the unit disk $\mathbb{D}$. Hence, the Taylor series of this function around the origin converges in all of $\mathbb{D}$. The series looks like

$$F(w) = \sum_{n=0}^{\infty} c_n w^n \tag{2.4}$$

for $|w| < 1$, where

$$c_n = \frac{1}{2\pi\imath} \int_S \frac{f_0(w')}{(w')^{n+1}}\, dw',$$

for $F = C^-f_0$ nearby the origin. From the Cauchy-Hadamard formula for the convergence radius of power series we readily conclude that $\limsup \sqrt[n]{|c_n|} \leq 1$. If this limit is less than 1, then the series (2.4) converges in a a disk about the origin of radius greater than 1. Hence, C^-f_0 extends to a holomorphic function in a neighbourhood of the closure of S, and so does Cf_0. On applying the Sokhotsky-Plemelj formula once again we see that $f_0 \equiv 0$ on S, a contradiction. This establishes (2.2), as desired.

Sufficiency. To prove the converse theorem, let f_0 be a continuous function on the closure of S satisfying (2.2). By assumption, the integral C^-f_0 is holomorphic in a disk of sufficiently small radius $\varepsilon > 0$ around the origin (take $\varepsilon < \operatorname{dist}(0, S)$). Hence, $C^- f_0$ expands in this small disk as a power series whose coefficients are given by

$$c_n = \frac{1}{2\pi\imath} \int_S \frac{f_0(w')}{(w')^{n+1}}\, dw',$$

cf. (2.4). Condition (2.2) forces the power series (2.4) to actually converge in the unit disk $\mathbb{D}$ to a holomorphic function F. By the uniqueness theorem, the integral $C^- f_0$ extends holomorphically to all of $\mathbb{D}$, and this analytic continuation is F. Hence it follows that the integral $C^+ f_0$ extends to a continuous function on $G \cup S$. We now set

$$f(w) := C^+ f_0\,(w) - F(w)$$

for $w \in G \cup S$, thus obtaining a holomorphic function in G which is continuous up to S and satisfies $f(w) = f_0(w)$ for all $w \in S$, as desired. □

3. A Carleman formula for a half-disk

The upper half-disk $\mathbb{D}' = \{z \in \mathbb{C} : |z| < 1, \Im(z) > 0\}$ is a canonical domain of the lune type corresponding to $S = (-1, 1)$. Since $0 \in (-1, 1)$, formula (2.1) is no longer applicable. To this end one needs a transformation $w = h(z)$ which maps $\mathbb{D}'$ conformally onto a lune like that in formula (2.1). We look for such a transformation in the group of fractional affine automorphisms of the unit disk $\mathbb{D}$. These have the form

$$h(z) = e^{\imath\varphi}\, \frac{z - a}{\bar{a}z - 1}$$

for $z \in \mathbb{D}$, with $\varphi \in [0, 2\pi)$ and $|a| < 1$. We pose two additional conditions on h, namely

1) $h(0) = t\imath$, where $t \in (0, 1)$;
2) $h(\imath) = \imath$.

The desired transformation is illustrated in Figure 2.

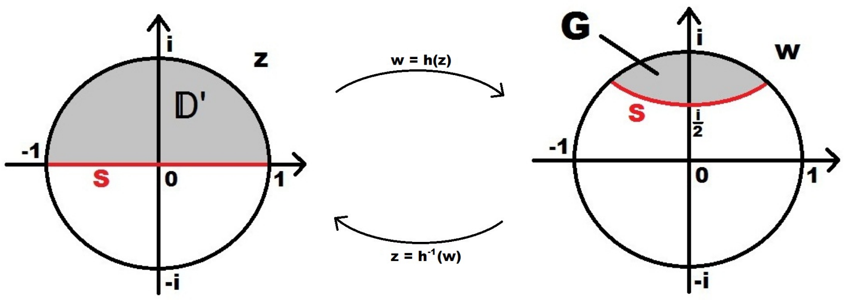

Fig. 2. A conformal mapping of $\mathbb{D}'$ onto a lune for $t = 1/2$.

An easy computation shows that there is only one automorphism of the unit disk satisfying 1) and 2). This is

$$h(z) = \frac{z + t\imath}{1 - t\imath z}. \tag{3.1}$$

Corollary 3.1. *Let f be a holomorphic function in the half-disk $\mathbb{D}'$ continuous up to the boundary of $\mathbb{D}'$. Then*

$$f(z) = \lim_{n\to\infty} \int_{-1}^{1} f(z') \Big(\frac{h(z)}{h(z')} \Big)^n \frac{1}{2\pi\imath} \frac{dh(z')}{h(z') - h(z)} \tag{3.2}$$

for all $z \in \mathbb{D}'$.

Proof. Set $F(w) := f\left(h^{-1}(w)\right)$ for all $w \in G$. Since $z = h^{-1}(w)$ maps the domain G conformally onto the half-disk $\mathbb{D}'$ and f is holomorphic in $\mathbb{D}'$, the function F is holomorphic in G. Furthermore, $z = h^{-1}(w)$ extends to a homeomorphism of the closure of G onto that of $\mathbb{D}'$. Hence, F is continuous up to the boundary of G. By formula (2.1),

$$F(w) = \lim_{n\to\infty} \int_S F(w') \Big(\frac{w}{w'} \Big)^n \frac{1}{2\pi\imath} \frac{dw'}{w' - w}$$

for all $w \in G$. On substituting $w = h(z)$ and $w' = h(z')$ and taking into account that S is the image of $(-1, 1)$ by h, we arrive at

$$F(h(z)) = \lim_{n\to\infty} \int_{-1}^{1} F(h(z')) \Big(\frac{h(z)}{h(z')} \Big)^n \frac{1}{2\pi\imath} \frac{dh(z')}{h(z') - h(z)}$$

for all $z \in \mathbb{D}'$, as desired. □

Replacing the function h in (3.2) by its expression (3.1) we write the formula for analytic continuation from the interval $(-1, 1)$ into the half-disk in explicit form. More precisely,

$$f(z) = \lim_{n\to\infty} \Big(\frac{z + t\imath}{1 - t\imath z} \Big)^n \int_{-1}^{1} f(z') \Big(\frac{1 - t\imath z'}{z' + t\imath} \Big)^n \frac{1 - t\imath z}{1 - t\imath z'} \frac{1}{2\pi\imath} \frac{dz'}{z' - z}$$

for all $z \in \mathbb{D}'$. Recall that t is any number in the interval $(0, 1)$. For $t = 0$ we recover formula (2.1), however, this value is prohibited, for the integrand becomes singular at $z' = 0$.

Under conformal map (3.1) Theorem 2.2 is also traced back to conditions of analytic continuability of functions from the interval $(-1, 1)$ to the upper half-disk $\mathbb{D}'$. We actually rewrite the same invariant object in other holomorphic coordinates. Let f_0 be a continuous function on $[-1, 1]$. Then $F_0 := f_0 \circ h^{-1}$ is a continuous function on S, the image of $(-1, 1)$ by $w = h(z)$. This is a regular curve in $\mathbb{D} \setminus \{0\}$ with endpoints

$$\pm \frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2}\, \imath$$

on the unit circle. Obviously, f_0 extends to a holomorphic function f in $\mathbb{D}'$, which is continuous up to $(-1, 1)$, if and only if F_0 extends to a holomorphic functions $F := f \circ h^{-1}$ in G continuous up to S. By Theorem 2.2 F_0 extends analytically to G if and only if

$$\lim_{n\to\infty} \sqrt[n]{|c_n|} = 1,$$

where

$$\begin{aligned} c_n &= \frac{1}{2\pi\imath}\int_S \frac{F_0(w)}{w^{n+1}}\,dw \\ &= \frac{1}{2\pi\imath}\int_{-1}^{1} \frac{f_0(z)}{(h(z))^{n+1}}dh(z) \\ &= \frac{1}{2\pi\imath}\int\limits_{-1}^{1} f_0(z)\left(\frac{1-t\imath z}{z+t\imath}\right)^n \frac{1-t^2}{(z+t\imath)(1-t\imath z)}\,dz \end{aligned} \tag{3.3}$$

for $n = 0, 1, \ldots$, because $dh(z) = \dfrac{1-t^2}{(1-t\imath z)^2}dz$.

Corollary 3.2. *Let $f_0 \in C[-1,1]$ be a nonzero function. In order that there be a holomorphic function f in $\mathbb{D}'$ continuous up to $(-1,1)$ and satisfying $f = f_0$ on $(-1,1)$, it is necessary and sufficient that* $\limsup \sqrt[n]{|c_n|} = 1$, *where c_n are given by (3.3).*

Using the triangle inequality one finds readily an explicit estimate from above for the limit in question. Namely,

$$\begin{aligned} |c_n| &\leq \Big(\sup_{z\in(-1,1)} \frac{1}{|h(z)|}\Big)^{n+1} \frac{1}{2\pi}\int_{-1}^{1} |f_0(z)|\,|dh(z)| \\ &\leq \Big(\frac{1}{|h(0)|}\Big)^{n+1} \frac{1}{2\pi}\int_{-1}^{1} |f_0(z)|\,\frac{1-t^2}{1+t^2z^2}\,dz \end{aligned}$$

for all $n = 0, 1, \ldots$, the last inequality being a consequence of the fact that the modulus of $h(z)$ takes on its global infimum in $(-1,1)$ at the point $z = 0$, as is easy to see from Figure 2. The right-hand side here is a constant multiple of $|h(0)|^{-n}$, and so

$$\limsup \sqrt[n]{|c_n|} \leq \frac{1}{|h(0)|} = \frac{1}{t}. \tag{3.4}$$

4. Reduction of the Riemann hypothesis

Arguing as in Section 3, we look for a conformal mapping $z = k(v)$ of the critical quarter-strip $\mathbb{H}' := \{v \in \mathbb{C} : \Re v \in (1/2, 1),\ \Im v > 0\}$ onto the half-disk $\mathbb{D}'$. For this purpose we need several lemmata.

Lemma 4.1. *The function $z = \tan v$ maps the strip $-\pi/4 < \Re v < \pi/4$ conformally onto the unit disk $\mathbb{D}$.*

Proof. As is well known, the function $z = \tan v$ maps the strip $-\pi/2 < \Re v < \pi/2$ conformally into the complex plane $\mathbb{C}$. It remains to specify the image of the strip $-\pi/4 < \Re v < \pi/4$ by this mapping. Given any v in the strip $-\pi/4 < \Re v < \pi/4$, we get

$$|\tan v| = \sqrt{\frac{e^{2\Im v} - 2\cos(2\Re v) + e^{-2\Im v}}{e^{2\Im v} + 2\cos(2\Re v) + e^{-2\Im v}}}.$$

By assumption, $-\pi/2 < 2\Re v < \pi/2$, whence $\cos(2\Re v) > 0$ and $\cos(2\Re v) = 0$ if and only if either $\Re v = -\pi/4$ or $\Re v = \pi/4$. So the quotient under the root sign is less than 1 in the open strip and it is equal to 1 if and only if $\Re v = -\pi/4$ or $\Re v = \pi/4$, as desired. $\square$

From Lemma 4.1 we further deduce

Lemma 4.2. *The function $z = \tan v$ maps the half-strip*

$$\{v \in \mathbb{C} : -\pi/4 < \Re v < \pi/4,\ \Im v > 0\}$$

conformally onto the half-disk $\mathbb{D}'$. The image of the interval $(-\pi/4, \pi/4)$ by this mapping is the interval $(-1, 1)$.

Proof. Using the Euler formula one easily obtains

$$\tan v = \frac{2\sin(2\Re v) + \imath\,(e^{2\Im v} - e^{-2\Im v})}{e^{2\Im v} + 2\cos(2\Re v) + e^{-2\Im v}}.$$

If $\Im v > 0$, then $e^{2\Im v} > e^{-2\Im v}$, and so $\Im \tan v > 0$, which shows the first part of the lemma. For the second part we assume $v \in (-\pi/4, \pi/4)$, so v is real. Then $z = \tan v$ is also real. Since the function $\tan v$ is strongly monotonic increasing on the interval $(-\pi/2, \pi/2)$ and $\tan(-\pi/4) = -1$ and $\tan(\pi/4) = 1$, the assertion is clear. $\square$

In order to construct a conformal mapping of the half-disk $\mathbb{D}'$ onto the critical quarter-strip $\mathbb{H}'$, it suffices to take the composition of $z = \tan v$ with an affine transformation of the v-plane. That is

$$z := k(v) = \tan \pi\Big(v - \frac{3}{4}\Big) \tag{4.1}$$

with inverse $v = k^{-1}(z) := \dfrac{3}{4} + \dfrac{1}{\pi}\arctan z$, see Figure 3.

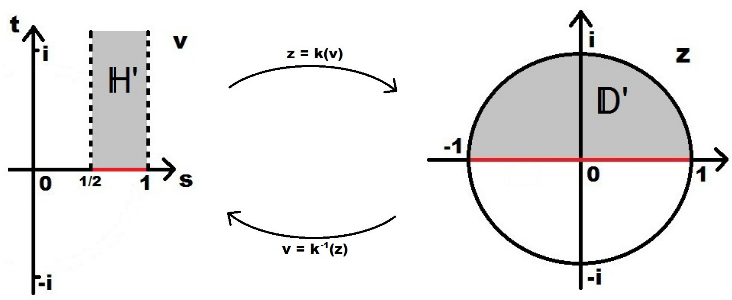

FIG. 3. A conformal mapping of $\mathbb{H}'$ onto $\mathbb{D}'$.

Lemma 4.3. *If f is a holomorphic function in the critical quarter-strip $\mathbb{H}'$, then the composition $f \circ k^{-1}$ is a holomorphic function in the upper half-disk $\mathbb{D}'$. If moreover the function f is continuous up to the interval $(1/2, 1)$, then $f \circ k^{-1}$ is continuous up to $(-1, 1)$.*

Proof. The proof is obvious by the above. We formulate this lemma for convenience of references. $\square$

The function $z = k(v)$ maps the segment $\Re v = 1$, $\Im v \geq 0$ homeomorphically onto the quarter-circle $|z| = 1$, $\arg z \in [0, \pi/2)$ and the segment $\Re v = 1/2$, $\Im v \geq 0$ homeomorphically onto the quarter-circle $|z| = 1$, $\arg z \in (\pi/2, \pi]$. The inverse mapping $v = k^{-1}(z)$ extends continuously to the entire boundary of the upper half-disk $\mathbb{D}'$ except for the north pole $z = \imath$ where k^{-1} blows up. Thus, the mapping $v = k^{-1}(z)$ might be used to compactify the closure of $\mathbb{H}'$ by adding a "point at infinity" to it. A function f on the closure of $\mathbb{H}'$ is said to be continuous on such a one-point compactification of the closure of $\mathbb{H}'$ if the composition $f \circ k^{-1}$ is continuous on the closure of $\mathbb{D}'$. A Carleman-type formula with integration over the interval $[1/2, 1]$ still holds for functions f holomorphic in $\mathbb{H}'$ and continuous on the compactification of $\overline{\mathbb{H}}'$. However, the Riemann zeta function does not belong to this class, so we shall not discuss the Carleman formula for holomorphic functions in $\mathbb{H}'$. On the other hand, the criterion of analytic continuability does not require any continuity on the compactification of the closure of $\mathbb{H}'$. Therefore, it extends to holomorphic functions in the critical quarter-strip in much the same way as Corollary 3.2.

Theorem 4.4. *Let f_0 a continuous function on the interval $[1/2, 1]$. In order that there be a holomorphic function f in $\mathbb{H}'$, such that f is continuous up to the open interval $(1/2, 1)$ and coincides with f_0 on this interval, it is necessary and sufficient that*

$$\limsup \sqrt[n]{|c_n|} = 1,$$

where

$$c_n = \frac{1}{2\pi\imath} \int\limits_{-1}^{1} f_0\Big(\frac{3}{4} + \frac{1}{\pi} \arctan z\Big) \Big(\frac{1 - t\imath z}{z + t\imath}\Big)^n \frac{1 - t^2}{(z + t\imath)(1 - t\imath z)}\, dz. \tag{4.2}$$

Proof. By assumption,

$$F_0(z) := f_0\Big(\frac{3}{4} + \frac{1}{\pi} \arctan z\Big)$$

is a continuous function of $z \in [-1, 1]$. By Lemma 4.3, it extends to a holomorphic function in the half-disk $\mathbb{D}'$ continuous up to $(-1, 1)$ if and only if $f_0(v)$ extends to a holomorphic function in the quarter-strip $\mathbb{H}'$ continuous up to $(1/2, 1)$. The theorem now follows from Corollary 3.2. □

In Section 1 we have done an alternative formulation of the Riemann hypothesis. We now make it more precise.

Lemma 4.5. *The Riemann hypothesis is true if and only if there exists a holomorphic function in $\mathbb{H}'$ which is continuous up to $(1/2, 1)$ and equal to $1/\zeta(s)$ for all $s \in (1/2, 1)$.*

Proof. Since the Riemann zeta function does not vanish on the interval $[1/2, 1]$ and has a simple pole at $s = 1$, the function $f_0(s) := 1/\zeta(s)$ is continuous on $[1/2, 1]$ and has a simple zero at $s = 1$. If $\zeta(s) \neq 0$ for all $s \in \mathbb{H}'$, then the function $1/\zeta(s)$ is actually holomorphic in the whole critical quarter-strip. Hence, the Riemann hypothesis just amounts to the fact that f_0 extends to a holomorphic function in $\mathbb{H}'$ continuous up to $(1/2, 1)$. □

Corollary 4.6. *The Riemann hypothesis holds if and only if* $\limsup \sqrt[n]{|c_n|} = 1$, *where*

$$c_n = \frac{1}{2\pi\imath} \int_{-1}^{1} \frac{1}{\zeta\Big(\dfrac{3}{4} + \dfrac{1}{\pi} \arctan z\Big)} \Big(\frac{1 - t\imath z}{z + t\imath}\Big)^n \frac{1 - t^2}{(z + t\imath)(1 - t\imath z)}\, dz. \tag{4.3}$$

Proof. This follows immediately from Lemma 4.5 and Theorem 4.4. □

The idea of this approach goes back at least as far as André Weil who first proved the analogue of the Riemann hypothesis for general curves over finite fields in 1942. As but one part of his proof was to show that the logarithmic derivative of the zeta function has no poles in the "critical strip." Weil proved crucial estimates for the radius of convergence of the power series in which the logarithmic derivative expands around the origin. For the Riemann zeta function arguments of Weil were recovered by Aizenberg et al. in [AAL99]. However, the main formula of [AAL99] is not correct.

5. Numerical experiments

Formula (4.2) can be rewritten in the form

$$c_n = \frac{1}{2\pi\imath} \int_S e^{n(-\log w)} f_0\Big((h \circ k)^{-1}(w)\Big) \frac{dw}{w},$$

where $S \subset \mathbb{D}$ is the image of $(-1, 1)$ by $w = h(z)$. Since the function $-\log w$ has no saddle points in the complex plane, the saddle-point method does not apply to construct asymptotics of c_n as $n \to \infty$. Hence, Theorem 4.4 seems to be of purely numerical interest.

My graduate students A. Bühmann (2009) and M. Albinus (2010) evaluated numerically several terms $\sqrt[n]{|c_n|}$ for the functions $1/\zeta(s)$ and $1/(s - (3/4 + \imath))$ on the interval $[1/2, 1]$. Obviously, the latter function does not extend analytically to all of $\mathbf{H}'$. Hence, the limit $\limsup \sqrt[n]{|c_n|}$ for this function is larger than 1 (but $\le 1/t$). Several numerical values of sequence $\sqrt[n]{|c_n|}$ corresponding to $t = 1/2$ are given in Figure 4.

Computations with the newest versions of Mathematica, Maple and Matlab give certain evidence to the fact that for the function $1/\zeta(s)$ the limit of $\sqrt[n]{|c_n|}$ is 1 indeed. However, on replacing $1/\zeta(z)$ by other functions (e.g. $1/(s-(3/4+\imath))$) which fail to be analytically extendable to the critical quarter-strip computer simulations still suggest that $\limsup \sqrt[n]{|c_n|} = 1$. So the standard computer programmes don't allow one to specify by means of Theorem 4.4 those continuous functions on $[1/2, 1]$ which extend analytically to the critical quarter-strip, cf. [AAL99]. A severe difficulty consists in rough evaluations of integrals depending on a parameter. For large values of the parameter the graph of the integrand function in (4.2) fills in a rectangle. The problem arising is to elaborate an effective programme for numerical evaluation of the limit $\limsup \sqrt[n]{|c_n|}$, where c_n are given by formula (4.2). This features once again the transcendental character of the Riemann problem on zeros of zeta function.

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	$1/\zeta(s)$	$1/(s-(3/4+\imath))$
n = 1	0,5361	1,5046
n = 2	0,7322	1,1659
n = 3	0,8124	1,0265
n = 4	0,8557	0,9274
n = 5	0,8827	0,8374
n = 6	0,9013	0,7391
n = 7	0,9148	0,6070
n = 8	0,9250	0,5941
n = 9	0,9330	0,7636
n = 10	0,9395	0,8512
n = 11	0,9449	0,8960
n = 12	0,9493	0,9159
n = 13	0,9531	0,9195
n = 14	0,9564	0,9110
n = 15	0,9592	0,8928
n = 16	0,9617	0,8677

FIG. 4. A computer simulation of $\sqrt[n]{|c_n|}$ for $t = 1/2$.

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