Boundary Value Problems for the Lorentzian Dirac Operator

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Univ.-Diss.
zur Erlangung des akademischen Grades
”doctor rerum naturalium”
(Dr. rer. nat.)
in der Wissenschaftsdisziplin Mathematik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät
Institut für Mathematik
der Universität Potsdam

Ort und Tag der Disputation: Universität Potsdam, 18.02.2022

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Online veröffentlicht auf dem
Publikationsserver der Universität Potsdam:
https://doi.org/10.25932/publishup-54839
https://nbn-resolving.org/urn:nbn:de:kobv:517-opus4-548391

Introduction

The index theorem for elliptic operators on a closed Riemannian manifold by Atiyah and Singer [1] has many applications in analysis, geometry and topology, but it is not suitable for a generalization to a Lorentzian setting.
In the case where a boundary is present Atiyah, Patodi and Singer provide an index theorem for compact Riemannian manifolds [2] by introducing non-local boundary conditions obtained via the spectral decomposition of an induced boundary operator, so called *APS* boundary conditions. In [9], Bär and Strohmaier prove a Lorentzian version of this index theorem for the Dirac operator on a manifold with boundary by utilizing results from [2, 3] and the characterization of the spectral flow by Phillips [14]. In their case the Lorentzian manifold is assumed to be globally hyperbolic and spatially compact, and the induced boundary operator is given by the Riemannian Dirac operator on a spacelike Cauchy hypersurface. Their results show that imposing *APS* boundary conditions for this boundary operator will yield a Fredholm operator with a smooth kernel and its index can be calculated by a formula similar to the Riemannian case.
Back in the Riemannian setting, Bär and Ballmann [12, 13] provide an analysis of the most general kind of boundary conditions that can be imposed on a first order elliptic differential operator that will still yield regularity for solutions as well as Fredholm property for the resulting operator. These boundary conditions can be thought of as deformations to the graph of a suitable operator mapping *APS* boundary conditions to their orthogonal complement.

This thesis aims at applying the boundary conditions found by Bär and Ballmann to a Lorentzian setting to understand more general types of boundary conditions for the Dirac operator, conserving Fredholm property as well as providing regularity results and relative index formulas for the resulting operators. As it turns out, there are some differences in applying these graph-type boundary conditions to the Lorentzian Dirac operator when compared to the Riemannian setting. It will be shown that in contrast to the Riemannian case, going from a Fredholm boundary condition to its orthogonal complement works out fine in the Lorentzian setting. On the other hand, in order to deduce Fredholm property and regularity of solutions for graph-type boundary conditions, additional assumptions for the deformation maps need to be made.

The thesis is organized as follows. In chapter 1 basic facts about Lorentzian and Riemannian spin manifolds, their spinor bundles and the Dirac operator are listed. These will serve as a foundation to define the setting and prove the results of later chapters.
Chapter 2 defines the general notion of boundary conditions for the Dirac operator used in this thesis and introduces the *APS* boundary conditions as well as their graph type deformations. Also the role of the wave evolution operator in finding Fredholm boundary conditions is

analyzed and these boundary conditions are connected to notion of Fredholm pairs in a given Hilbert space.

Chapter 3 focuses on the principal symbol calculation of the wave evolution operator and the results are used to prove Fredholm property as well as regularity of solutions for suitable graph-type boundary conditions. Also sufficient conditions are derived for (pseudo-)local boundary conditions imposed on the Dirac operator to yield a Fredholm operator with a smooth solution space.

In the last chapter 4, a few examples of boundary conditions are calculated applying the results of previous chapters. Restricting to special geometries and/or boundary conditions, results can be obtained that are not covered by the more general statements, and it is shown that so-called *transmission* conditions behave very differently than in the Riemannian setting.

Zusammenfassung

Der Indexsatz für elliptische Operatoren auf geschlossenen Riemannschen Mannigfaltigkeiten von Atiyah und Singer hat zahlreiche Anwendungen in Analysis, Geometrie und Topologie, ist aber ungeeignet für eine Verallgemeinerung auf Lorentz-Mannigfaltigkeiten.
Durch die Einführung nicht-lokaler Randbedingungen, gewonnen aus der Spektralzerlegung eines induzierten Randoperators, beweisen Atiyah, Patodi und Singer (APS) einen Indexsatz für den Fall kompakter Riemannscher Mannigfaltigkeiten mit Rand. Aufbauend auf diesem Resultat und mit Hilfe der Charakterisierung des Spektralflusses durch Philipps gelangen Bär und Strohmaier zu einem Indexsatz für den Dirac-Operator auf global hyperbolischen Lorentz-Mannigfaltigkeiten mit kompakten und raumartigen Cauchy-Hyperflächen. Ihr Ergebnis zeigt unter anderem, dass der Dirac Operator auf solchen Mannigfaltigkeiten und unter APS Randbedingungen ein Fredholm-Operator mit glattem Kern ist und das sein Index sich aus einer zum Riemannschen Fall analogen Formel berechnen lässt.
Zurück im Riemannschen Setup zeigen Bär und Ballmann eine allgemeine Charakterisierung von Randbedingungen für elliptische Differentialoperatoren erster Ordnung die sowohl die Regularität von Lösungen, als auch Fredholm-Eigenschaft des resultierenden Operators garantieren. Die dort entwickelten Randbedingungen können als Deformation auf den Graphen einer geeigneten Abbildung der APS-Randbedingung auf ihr orthogonales Komplement verstanden werden.

Die vorliegende Arbeit hat das Ziel die von Bär und Ballmann beschriebenen Randbedingungen auf den Dirac-Operator von global hyperbolischen Lorentz-Mannigfaltigkeiten zu übertragen um eine allgemeinere Klasse von Randbedingungen zu finden unter denen der resultierende Dirac-Operator Fredholm ist und einen glatten Lösungsraum hat. Weiterhin wird analysiert wie sich derartige Deformation von APS-Randbedingungen auf den Index solcher Operatoren auswirken und wie dieser aus den bekannten Resultaten für den APS-Index berechnet werden kann. Es wird unter anderem gezeigt, dass im Gegensatz zum Riemannschen Fall beim Übergang von Randbedingungen zu ihrem orthogonalen Komplement die Fredholm-Eigenschaft des Operators erhalten bleibt. Andererseits sind zusätzliche Annahme nötig um die Regularität von Lösungen, sowie die Fredholm-Eigenschaft für Graph-Deformationen im Fall von Lorentz-Mannigfaltigkeiten zu erhalten.

Die Arbeit ist dabei wie folgt aufgebaut. In Kapitel 1 werden grundlegende Fakten zu Lorentzschen und Riemannschen Spin-Mannigfaltigkeiten, ihren Spinor-Bündeln und Dirac-Operatoren zusammengetragen. Diese Informationen dienen als Ausgangspunkt zur Definition und Analyse von Randbedingungen in späteren Kapiteln der Arbeit.
Kapitel 2 definiert allgemein den Begriff der Randbedingung wie er in dieser Arbeit verwendet wird und führt zudem den sogenannten "wave-evolution-Operator" ein, der eine wichtige Rolle

im Finden und Analysieren von Fredholm-Randbedingungen für den Dirac-Operator spielen wird. Zuletzt wird der Zusammenhang zwischen Fredholm-Paaren eines Hilbert-Raumes und Fredholm-Randbedingungen für den Dirac-Operator erklärt.
Kapitel 3 beschäftigt sich mit der Berechnung des Hauptsymbols des wave-evolution-Operators und die dort erzielten Resultate werden verwendet um Fredholm-Eigenschaft, sowie Regularität von Lösungen für geeignete Deformationen von APS-Randbedingungen zu beweisen. Weiterhin werden hinreichende Bedingungen für (pseudo-)lokale Randbedingungen abgeleitet, die Fredholm-Eigenschaft und Regularität für den resultierenden Dirac-Operator garantieren.
Kapitel 4 zeigt, aufbauend auf den Ergebnissen der Kapitel 1-3, einige Beispiele von lokalen und nicht-lokalen Randbedingungen für den Dirac-Operator. Unter gewissen Einschränkungen an die Geometrie der zugrunde liegenden Mannigfaltigkeit bzw. den gestellten Randbedingungen können Ergebnisse erzielt werden die in den allgemeineren Resultaten der vorangehenden Kapitel nicht enthalten sind. Zuletzt werden sogenannte Transmission-Bedingungen analysiert und die Unterschiede dieser Randbedingungen zum riemannschen Fall aufgezeigt.

Contents

1 Preliminaries

1.1 Dirac Operator on Globally Hyperbolic Spacetimes

In this section, we will describe the main setting for this thesis and list some crucial facts for the Dirac operator on a Lorentzian manifold. For further introduction to Lorentzian geometry see e.g. [17] and for more information about Dirac operators on Lorentzian spacetimes see [5, 16].

1.1.1 Globally Hyperbolic Manifolds

Assume that $\mathcal{M}$ is an even dimensional oriented and time-oriented Lorentzian spin manifold and that $\mathcal{M}$ is globally hyperbolic, i.e. $\mathcal{M}$ possesses a Cauchy hypersurface $\Sigma \subset \mathcal{M}$. Further we assume that all Cauchy hypersurfaces $\Sigma \subset \mathcal{M}$ are compact and spacelike. In this case by ([18] Theorem 1.2 and also [19] Theorem 1) the manifold $\mathcal{M}$ can be written as

$$\mathcal{M} = \mathbb{R} \times \Sigma$$

where each $\Sigma_t := \{t\} \times \Sigma \subset \mathcal{M}$ is a smooth and spacelike Cauchy hypersurface. Further the metric on $\mathcal{M}$ is then given by $g = -N^2 \mathrm{dt}^2 + g_t$ where $N : \mathcal{M} \to \mathbb{R}$ is smooth and positive, and g_t is a smooth family of Riemannian metrics on Σ.

For the boundary value problems discussed in this thesis, we fix two smooth and spacelike Cauchy hypersurfaces $\Sigma_1, \Sigma_2 \in \mathcal{M}$, where we suppose that Σ_1 lies in the past of Σ_2 and set $M := J^+(\Sigma_1) \cap J^-(\Sigma_2)$. M is a globally hyperbolic spin manifold with boundary and can be written as:

$$M = [t_1, t_2] \times \Sigma$$

where $t_1 < t_2$, $\{t_1\} \times \Sigma = \Sigma_1$ and $\{t_2\} \times \Sigma = \Sigma_2$. The boundary of M is given by $\partial M = \Sigma_1 \dot\cup \Sigma_2$, where both (Σ_1, g_{t_1}) and (Σ_2, g_{t_2}) are closed Riemannian spin manifolds. This is the same setting for boundary conditions used in [9, 10].

1.1.2 Spinor Bundle

Denote the complex spinor bundle of M by $SM \to M$ together with its indefinite inner product $(\cdot, \cdot)$. For Clifford multiplication on M by a tangent vector $\eta \in T_xM$ we write $\gamma(\eta) : S_xM \to S_xM$ and it satisfies

1. $\gamma(\eta)\gamma(\mu) + \gamma(\mu)\gamma(\eta) = -2g(\mu, \eta)$

2. $(\gamma(\eta)\phi, \varphi) = (\phi, \gamma(\eta)\varphi)$

for all $\mu, \eta \in T_xM$, $\phi, \varphi \in S_xM$ and $x \in M$. Let $e_0, e_1, \dots e_n$ be a positively oriented Lorentz-orthonormal tangent frame, where $n + 1 = \dim(M)$. The volume form is defined by $\Gamma = i^{n(n+3)/2}\gamma(e_0)\gamma(e_1)\cdots\gamma(e_n)$ and satisfies $\Gamma^2 = \mathrm{id}_{SM}$. This induces a splitting of the spinor bundle $SM = S^+M \oplus S^-M$ into ± 1 eigenspaces of Γ. By property (1.) of Clifford multiplication on M, and since $\dim(M) = n+1$ is even we have $\gamma(\eta)\Gamma = -\Gamma\gamma(\eta)$, hence S^+M and S^-M are of the same dimension, and Clifford multiplication maps $\gamma(\eta) : S_x^{\pm}M \to S_x^{\mp}M$ for all $\eta \in T_xM$ and $x \in M$.
Let $\Sigma \subset M$ be a smooth spacelike Cauchy hypersurface and by ν denote the past-directed unit timelike vector field along Σ. The spinor bundle of Σ can be naturally identified with the restrictions $S\Sigma = S^{\pm}M|_{\Sigma}$ and the positive definite inner product $\langle \cdot, \cdot \rangle$ of $S\Sigma$ is related to the inner product of SM by $\langle \cdot, \cdot \rangle = (\gamma(\nu)\cdot, \cdot)$. For $X \in T_p\Sigma$, we denote Clifford multiplication on Σ by $\gamma_{\Sigma}(X) : S_x\Sigma \to S_x\Sigma$ and under the above identification it corresponds to $i\gamma(\nu)\gamma(X)$. Further Clifford multiplication on Σ is skew-adjoint, i.e.

$$\langle \gamma_{\Sigma}(X)\xi, \rho \rangle = -\langle \xi, \gamma_{\Sigma}(X)\rho \rangle$$

for all $\xi, \rho \in S_x\Sigma$, $X \in T_x\Sigma$ and $x \in \Sigma$.

1.1.3 Dirac Operator

Let ∇ be the spin connection on SM induced by the Levi-Civita connection of M, then the Dirac operator acting on smooth spinor fields is defined by

$$\mathcal{D} := \sum_{k=0}^{n} \epsilon_k \gamma(e_k)\nabla_{e_k} : C^\infty(M; SM) \to C^\infty(M; SM) \tag{1.1}$$

where $e_0, e_1, \dots e_n$ is a Lorentz-orthonormal tangent frame and $\epsilon_k = g(e_k, e_k) = \pm 1$. With respect to the splitting of SM into ± 1 eigenvalues of Γ the Dirac operator takes the form

$$\mathcal{D} = \begin{pmatrix} 0 & \tilde{D} \\ D & 0 \end{pmatrix} : C^\infty(M; S^+M) \oplus C^\infty(M; S^-M) \to C^\infty(M; S^+M) \oplus C^\infty(M; S^-M)$$

and throughout this thesis we will focus on the operator

$$D : C^\infty(M; S^+M) \to C^\infty(M; S^-M).$$

Along the smooth spacelike Cauchy hypersurface $\Sigma \subset M$ with past-directed timelike unit normal vector field ν the Dirac operator D splits into

$$D = \gamma(\nu)\left(\nabla_\nu + iA_\Sigma - \tfrac{n}{2}H\right) \tag{1.2}$$

where A_Σ denotes the (elliptic) Dirac operator of the closed Riemannian manifold Σ and H the mean curvature of $\Sigma \subset M$.

1.1.4 Well-Posedness of the Cauchy Problem

For any Cauchy hypersurface $\Sigma \subset M$ we define the L^2-scalar product for smooth sections $\phi, \varphi \in C^\infty(\Sigma; S^\pm M)$ by

$$(\phi, \varphi)_{L^2} := \int_\Sigma \langle \phi, \varphi \rangle \mathrm{d}\Sigma$$

and set $L^2(\Sigma; S^\pm M) = \overline{C^\infty(\Sigma; S^\pm M)}^{L^2}$. Since $\mathcal{M}$ is assumed to be spatially compact, the manifold M is compact and we can use any positive definite inner product, e.g. $\langle \cdot, \cdot \rangle = (\gamma(\nu)\cdot, \cdot)$ to define $L^2(M; S^\pm M)$ in the same way as above by integrating over the manifold M. For the last function space needed, consider the norm given by

$$\|\phi\|^2_{FE^0} = \|\phi|_\Sigma\|^2_{L^2} + \|D\phi\|^2_{L^2}$$

and $FE^0(M; D) := \overline{C^\infty(M; S^\pm M)}^{\|\cdot\|_{FE^0}}$ is called the *finite energy* space. By construction the Dirac operator extends to a bounded operator

$$D : FE^0(M; D) \longrightarrow L^2(M; S^- M)$$

and the restriction map to the Cauchy hypersurface can be seen as a bounded operator

$$\mathrm{res}_\Sigma : FE^0(M; D) \longrightarrow L^2(\Sigma; S^+ M).$$

With this we have the following theorem known as well-posedness of the Cauchy problem for the Dirac operator, see ([9] Theorem 2.1 and [15] Chapter IV).

Theorem 1.1.1. *Let M be defined as above and $\Sigma \subset M$ a smooth and spacelike Cauchy hypersurface, then*

$$res_\Sigma \oplus D : FE^0(M; D) \longrightarrow L^2(\Sigma; S^+ M) \oplus L^2(M; S^- M)$$

is an isomorphism of Hilbert spaces.

In particular by restricting this map to the kernel of the Dirac operator we get that

$$\mathrm{res}_\Sigma : \ker(D) \subset FE^0(M; D) \longrightarrow L^2(\Sigma; S^+ M)$$

is an isomorphism and hence the Cauchy problem

$$\begin{cases} D\phi = 0 \\ \phi|_\Sigma = \phi_0, \quad \phi_0 \in L^2(\Sigma; S^+ M) \end{cases}$$

possesses a unique solution $\phi \in FE^0(M; D)$ and is hence well-posed.

1.1.5 Boundary Conditions for D

Recall that $M = J^+(\Sigma_1) \cap J^-(\Sigma_2)$ is globally hyperbolic manifold with boundary $\partial M = \Sigma_1 \dot{\cup} \Sigma_2$ and the restrictions map $\mathrm{res}_{\Sigma_i} : FE^0(M; D) \to L^2(\Sigma_i; S^+M) = L^2(\Sigma_i; S\Sigma_i)$. We make the following definition.

Definition 1.1.2. Under a *boundary condition* for D we understand a pair (B_1, B_2) of closed linear subspaces $B_1 \subset L^2(\Sigma_1; S\Sigma_1)$ and $B_2 \subset L^2(\Sigma_2; S\Sigma_2)$.

By restricting the domain of the Dirac operator to sections satisfying these boundary conditions, i.e.

$$FE^0_{B_1B_2}(M; D) := \{\phi \in FE^0(M; D) : \phi|_{\Sigma_1} \in B_1, \phi|_{\Sigma_2} \in B_2\}$$

we obtain a closed subspace $FE^0_{B_1B_2}(M; D) \subset FE^0(M; D)$. The Dirac operator subject to a boundary condition (B_1, B_2) is then defined to be the Dirac operator restricted to this subspace

$$D_{B_1B_2} := D|_{FE^0_{B_1B_2}(M;D)} : FE^0_{B_1B_2}(M; D) \longrightarrow L^2(M; S^-M).$$

1.2 Dirac Operator on Closed Riemannian Manifolds

The manifold M was assumed to be spatially compact meaning that any smooth and spacelike Cauchy hypersurface $\Sigma \subset M$, in particular the boundary components $\Sigma_0, \Sigma_1 \subset M$, are closed Riemannian manifolds. In order to define boundary conditions for D as in definition 1.1.2, we often make use of the Dirac operators A_0, A_1, appearing in the decomposition of D 1.2, of Σ_0 and Σ_1 respectively. In this chapter, we will collect some basic facts for Dirac operators on closed Riemannian manifolds that will be used throughout this entire thesis. For a further introduction to elliptic operators on closed Riemannian manifolds see e.g. [4].

By definition 1.1 the Dirac operator, in this section denoted by A, is a first order differential operator and as such it extends to a bounded linear operator

$$A : H^k(\Sigma; S\Sigma) \longrightarrow H^{k-1}(\Sigma; S\Sigma)$$

where H^k denotes the *Sobolev space* of k-times weakly differentiable sections. The following results for Sobolev spaces will be crucial for understanding mapping properties of the wave evolution operator and regularity of solutions for D in 3.2 and 3.3.

Theorem 1.2.1 ([4] Theorem 2.5)**.** *Let Σ be a closed Riemannian spin manifold of dimension $\dim(\Sigma) = n$ and $k \in \mathbb{N}$, then for any $s > \frac{n}{2} + k$ there is a continuous embedding*

$$H^s(\Sigma; S\Sigma) \subset C^k(\Sigma; S\Sigma).$$

In particular theorem 1.2.1 states that a section of arbitrarily high Sobolev regularity will automatically be smooth, i.e.

$$\bigcap_k H^k(\Sigma; S\Sigma) = C^\infty(\Sigma; S\Sigma).$$

Theorem 1.2.2 ([4] Theorem 2.6 The Rellich Lemma)**.** *For $k' < k$ the natural embedding*

$$\iota : H^k(\Sigma; S\Sigma) \hookrightarrow H^{k'}(\Sigma; S\Sigma)$$

is compact.

For $x \in \Sigma$, $\xi \in \dot{T}_x^*\Sigma$ choose a smooth function $f : \Sigma \to \mathbb{R}$ such that $\mathrm{d}f_x = \xi$, then the principal symbol of A can be calculated via

$$\begin{aligned}
\sigma_A(\xi)\phi_x &= [A(f\phi) - fA\phi]_x \\
&= \left[-fA\phi - \sum_k \gamma(e_k)\nabla_{e_k}(f\phi)\right]_x \\
&= \left[-fA\phi + \sum_k \gamma(e_k)\left(f\nabla_{e_k}\phi + \mathrm{d}f(e_k)\phi\right)\right]_x \\
&= \left[-fA\phi + \sum_k \left(f\gamma(e_k)\nabla_{e_k}\phi + \gamma(\mathrm{d}f)\phi\right)\right]_x \\
&= [-fA\phi + fA\phi + \gamma(\mathrm{d}f)\phi]_x \\
&= [\gamma(\mathrm{d}f)\phi]_x \\
&= \gamma(\xi)\phi_x
\end{aligned}$$

where $\phi \in C^\infty(\Sigma; S\Sigma)$. Clifford multiplication satisfies $\gamma(\xi)^2 = -\|\xi\|^2$, and since Σ is a Riemannian manifold $\|\xi\| \neq 0$ for all $\xi \in \dot{T}^*\Sigma$, hence $\sigma_A(\xi)$ is invertible for $\xi \neq 0$. A differential operator with this property is called *elliptic* and we have the following result.

Theorem 1.2.3 ([4] Theorem 4.6)**.** *Let Σ be a closed Riemannian spin manifold with Dirac operator A. Then there exists a pseudo-differential operator P of order -1 such that*

$$AP = \mathrm{id} - S' \quad \text{and} \quad PA = \mathrm{id} - S$$

where S and S' are smoothing operators. The operator P is called a ***parametrix*** *for A.*

Theorem 1.2.3 in particular implies that any eigenspinor of A has to be smooth, because if there is a $\phi \in L^2(\Sigma; S\Sigma)$ and $\lambda \in \mathbb{C}$ such that $A\phi = \lambda\phi$, then

$$\begin{aligned}
\phi &= PA\phi + S\phi \\
&= \lambda P\phi + S\phi
\end{aligned}$$

where $P\phi \in H^1(\Sigma; S\Sigma)$ and $S\phi \in C^\infty(\Sigma; S\Sigma)$. Hence $\phi \in H^1(\Sigma; S\Sigma)$ and repeating this argument shows that $\phi \in \bigcap_k H^k(\Sigma; S\Sigma)$ is in fact smooth by theorem 1.2.2.

The last property of the Dirac operator we want to point out in this section is that it satisfies

$$(A\phi, \varphi)_{L^2} = (\phi, A\varphi)_{L^2}$$

for all $\phi, \varphi \in C^\infty(\Sigma; S\Sigma)$. A differential operator with this property is called *(formally) self-adjoint* and we obtain the following decomposition of $L^2(\Sigma; S\Sigma)$ in terms of eigenspaces for A.

Theorem 1.2.4 ([4] Theorem 5.8)**.** *Let* Σ *be a closed riemannian spin manifold with Dirac operator* A*. Then all eigenspaces of* A *are smooth and finite dimensional, further we have that*

$$L^2(\Sigma; S\Sigma) = \bigoplus_{\lambda} E_\lambda(A)$$

is a Hilbert space direct sum decomposition.

2 Boundary Conditions and Fredholm Pairs

2.1 The Wave Evolution Operator

2.1.1 Wave Evolution Operator

Definition 2.1.1. By well-posedness of the Cauchy problem for D the **wave evolution operator** is defined via the commutative diagram:

$$\begin{array}{ccc} & \ker(D) & \\ {\scriptstyle \mathrm{res}_{\Sigma_1}}\swarrow\cong & & \cong\searrow{\scriptstyle \mathrm{res}_{\Sigma_2}} \\ L^2(\Sigma_1, S\Sigma_1) & \xrightarrow{\quad Q \quad} & L^2(\Sigma_2, S\Sigma_2) \end{array}$$

The wave evolution operator plays a crucial role in analyzing kernel, cokernel and Fredholm property of the Dirac operator. A given boundary condition (B_1, B_2) induces splittings of the corresponding L^2 spaces on the boundary, upto the choice of complementary subspaces B_1^c and B_2^c.

$$L^2(\Sigma_i, S\Sigma_i) = B_i \oplus B_i^c$$

Note that a canonical choice, since $L^2(\Sigma_i, S\Sigma_i)$ are Hilbert spaces, would be $B_i^c = B_i^\perp$. In general B_i^c can be any complementary subspace and the sum decomposition need not be orthogonal. With respect to these splittings the wave evolution operator can be written as

$$\begin{pmatrix} Q_{B_2^c B_1} & Q_{B_2^c B_1^c} \\ Q_{B_2 B_1} & Q_{B_2 B_1^c} \end{pmatrix} : B_1 \oplus B_1^c \longrightarrow B_2^c \oplus B_2$$

where $Q_{B_2B_1} := P_{B_2B_2^c} \circ Q \circ P_{B_1B_1^c}$ and similar for the others entries. Here and henceforth $P_{B_iB_i^c}$ denotes the projection map $L^2(\Sigma_i, S\Sigma_i) \to B_i$ induced by the choice of B_i^c and the corresponding sum decomposition.

Lemma 2.1.2 ([9] Lemma 2.4)**.** *The wave evolution operator is unitary*

$$Q^*Q = 1|_{L^2(\Sigma_1, S\Sigma_1)} \quad \textit{and} \quad QQ^* = 1|_{L^2(\Sigma_2, S\Sigma_2)}$$

Using unitarity of Q and writing both Q and Q^* in terms of the splitting given by the boundary condition (B_1, B_2) yields the following set of equations:

$$Q^*_{B_1B_2^c} Q_{B_2^cB_1} + Q^*_{B_1B_2} Q_{B_2B_1} = 1|_{B_1} \tag{2.1}$$

$$Q^*_{B_1^cB_2^c}Q_{B_2^cB_1^c} + Q^*_{B_1^cB_2}Q_{B_2B_1^c} = 1|_{B_1^c} \tag{2.2}$$
$$Q_{B_2^cB_1}Q^*_{B_1B_2^c} + Q_{B_2^cB_1^c}Q^*_{B_1^cB_2^c} = 1|_{B_2^c} \tag{2.3}$$
$$Q_{B_2B_1}Q^*_{B_1B_2} + Q_{B_2B_1^c}Q^*_{B_1^cB_2} = 1|_{B_2} \tag{2.4}$$

$$Q^*_{B_1^cB_2^c}Q_{B_2^cB_1} + Q^*_{B_1^cB_2}Q_{B_2B_1} = 0 \tag{2.5}$$
$$Q^*_{B_1B_2^c}Q_{B_2^cB_1^c} + Q^*_{B_1B_2}Q_{B_2B_1^c} = 0 \tag{2.6}$$
$$Q_{B_2B_1}Q^*_{B_1B_2^c} + Q_{B_2B_1^c}Q^*_{B_1^cB_2^c} = 0 \tag{2.7}$$
$$Q_{B_2^cB_1}Q^*_{B_1B_2} + Q_{B_2^cB_1^c}Q^*_{B_1^cB_2} = 0 \tag{2.8}$$

Note that in the case of non-orthogonal splittings of the boundary spaces

$$(Q_{B_2B_1})^* = (P_{B_2B_2^c} \circ Q \circ P_{B_1B_1^c})^* = (P_{B_1B_1^c})^* \circ Q^* \circ (P_{B_2B_2^c})^* \neq Q^*_{B_1B_2}$$

where in the orthogonal case we have $(P_{B_iB_i^\perp})^* = P_{B_iB_i^\perp}$ and equality holds.
In order to relate Fredholm property and index of the operator entries in the splitting of Q to the Dirac operator subject to a given boundary condition, we need the following Lemma, see [13] Proposition A.1

Lemma 2.1.3. *Let H be a Hilbert space, E and F Banach spaces, $L : H \to E$ and $P : H \to F$ bounded linear maps where P is onto. Then $L|_{\ker(P)} : \ker(P) \to E$ is Fredholm of index k if and only if $L \oplus P : H \to E \oplus F$ is Fredholm of index k.*

Theorem 2.1.4. *Let (B_1, B_2) be a boundary condition for D and the splitting of Q as introduced above, then the following are equivalent:*

1. *For some choice of complementary subspaces B_1^c and B_2^c the operator $Q_{B_2^cB_1}$ is Fredholm of index k.*
2. *For any choice of complementary subspaces B_1^c and B_2^c the operator $Q_{B_2^cB_1}$ is Fredholm of index k.*
3. *The Dirac operator $D_{B_1B_2}$ is Fredholm of index k.*

Proof. Applying Lemma 2.1.3 to

$$\begin{aligned} H &= FE^0(M;D) \\ E &= L^2(M, S^-M) \\ F &= B_1^c \oplus B_2^c \\ L &= D \\ P &= (P_{B_1^cB_1} \circ \mathrm{res}_{\Sigma_1}) \oplus (P_{B_2^cB_2} \circ \mathrm{res}_{\Sigma_2}) \end{aligned}$$

shows that $D_{B_1B_2}$ being Fredholm of index k is equivalent to the operator

$$(P_{B_1^cB_1} \circ \mathrm{res}_{\Sigma_1}) \oplus (P_{B_2^cB_2} \circ \mathrm{res}_{\Sigma_2}) \oplus D : FE^0(M;D) \to B_1^c \oplus B_2^c \oplus L^2(M, S^-M)$$

being Fredholm of index k. Now applying Lemma 2.1.3 again, this time with:

$$\begin{aligned} H &= FE^0(M;D) \\ E &= B_1^c \oplus B_2^c \\ F &= L^2(M, S^-M) \\ L &= (P_{B_1^c B_1} \circ \mathrm{res}_{\Sigma_1}) \oplus (P_{B_2^c B_2} \circ \mathrm{res}_{\Sigma_2}) \\ P &= D \end{aligned}$$

and combining with the previous equivalence, it follows that $D_{B_1 B_2}$ is Fredholm of index k if and only if the operator

$$L|_{\ker(P)} = (P_{B_1^c B_1} \circ \mathrm{res}_{\Sigma_1}) \oplus (P_{B_2^c B_2} \circ \mathrm{res}_{\Sigma_2}) : \ker(D) \longrightarrow B_1^c \oplus B_2^c$$

is Fredholm of index k. First we calculate the kernel of this operator:

$$\begin{aligned} \ker(L|_{\ker(P)}) &= \left\{\varphi \in \ker(D) : \ \mathrm{res}_{\Sigma_1}(\varphi) \in B_1, \ \mathrm{res}_{\Sigma_2}(\varphi) \in B_2\right\} \\ &= \left\{\varphi \in \ker(D) : \ \mathrm{res}_{\Sigma_1}(\varphi) \in B_1, \ Q(\mathrm{res}_{\Sigma_1}(\varphi)) \in B_2\right\} \\ &\cong \{u \in B_1 : \ Q(u) \in B_2\} \\ &= \left\{u \in B_1 : \ Q_{B_2^c B_1}(u) = 0\right\} \\ &= \ker(Q_{B_2^c B_1}) \end{aligned}$$

Before looking at the cokernels of those operators we want to show that $\mathrm{im}(Q_{B_2^c B_1})$ is closed if and only if $L|_{\ker(P)}$ has closed image.

$$\begin{aligned} &\mathrm{im}(L|_{\ker(P)}) \\ &= \left\{(x,y) \in B_1^c \oplus B_2^c : \ \exists \varphi \in \ker(D) : \ x = P_{B_1^c B_1}\left(\varphi|_{\Sigma_1}\right), \ y = P_{B_2^c B_2}\left(\varphi|_{\Sigma_2}\right)\right\} \\ &= \left\{(x,y) \in B_1^c \oplus B_2^c : \ \exists \varphi \in \ker(D) : \ x = P_{B_1^c B_1}\left(\varphi|_{\Sigma_1}\right), \ y = P_{B_2^c B_2}\left(Q\left(\varphi|_{\Sigma_1}\right)\right)\right\} \\ &= \left\{(x,y) \in B_1^c \oplus B_2^c : \ y = Q_{B_2^c B_1^c}(x) + Q_{B_2^c B_1}(z) \text{ for some } z \in B_1\right\} \\ &= \left\{\left(x, Q_{B_2^c B_1^c}(x) + Q_{B_2^c B_1}(z)\right) : \ x \in B_1^c, \ z \in B_1\right\} \end{aligned}$$

Now suppose that $\mathrm{im}(Q_{B_2^c B_1})$ is closed and let $x_i \in B_1^c$, $z_i \in B_1$ be sequence such that $(x_i, Q_{B_2^c B_1^c}(x_i) + Q_{B_2^c B_1}(z_i))$ converges to say (x, b). It follows then that $x_i \longrightarrow x \in B_1^c$ and hence $Q_{B_2^c B_1^c}(x_i) \longrightarrow Q_{B_2^c B_1^c}(x)$. Since $\mathrm{im}(Q_{B_2^c B_1})$ is closed and $Q_{B_2^c B_1}(z_i) \longrightarrow b - Q_{B_2^c B_1^c}(x)$ there exists $z \in B_1$ with $Q_{B_2^c B_1}(z) = b - Q_{B_2^c B_1^c}(x)$. This shows that $(x_i, Q_{B_2^c B_1^c}(x_i) + Q_{B_2^c B_1}(z_i)) \longrightarrow (x, Q_{B_2^c B_1^c}(x) + Q_{B_2^c B_1}(z)) \in \mathrm{im}(L|_{\ker(P)})$ meaning that $\mathrm{im}(L|_{\ker(P)})$ is closed.

On the other hand, if $\mathrm{im}(L|_{\ker(P)})$ is closed and $z_i \in B_1$ is a sequence such that $Q_{B_2^c B_1}(z_i)$ converges to say $b \in B_2^c$ then there exist $x \in B_1^c$, $z \in B_1$ with $(0, Q_{B_2^c B_1^c}(0) + Q_{B_2^c B_1}(z_i)) \longrightarrow (x, Q_{B_2^c B_1^c}(x) + Q_{B_2^c B_1}(z))$. So, now we have

$(x, Q_{B_2^c B_1^c}(x) + Q_{B_2^c B_1}(z)) = (0, b)$ and it follows that $x = 0$ and $b = Q_{B_2^c B_1}(z)$ meaning that $\operatorname{im}(Q_{B_2^c B_1})$ is closed.

To calculate the cokernel of $L|_{\ker(P)}$:

$$
\begin{aligned}
&\operatorname{coker}\left(L|_{\ker(P)}\right) \cong \operatorname{im}\left(L|_{\ker(P)}\right)^{\perp} \\
&= \left\{(u,v) \in B_1^c \oplus B_2^c : \langle u, x\rangle + \langle v, Q_{B_2^c B_1^c}(x) + Q_{B_2^c B_1}(z)\rangle = 0 \ \forall x \in B_1^c, z \in B_1\right\} \\
&= \left\{(u,v) \in B_1^c \oplus B_2^c : \langle u + \left(Q_{B_2^c B_1^c}\right)^* (v), x\rangle + \langle \left(Q_{B_2^c B_1}\right)^* (v), z\rangle = 0 \ \forall x \in B_1^c, z \in B_1\right\} \\
&= \left\{\left(-\left(Q_{B_2^c B_1^c}\right)^* (v), v\right) : v \in \ker\left[\left(Q_{B_2^c B_1}\right)^*\right]\right\} \\
&\cong \ker\left[\left(Q_{B_2^c B_1}\right)^*\right] \\
&\cong \operatorname{coker}\left(Q_{B_2^c B_1}\right)
\end{aligned}
$$

Since the choice of complementary subspaces B_1^c and B_2^c was not specified throughout this proof, it follows that $D_{B_1 B_2}$ being Fredholm of index k is equivalent to $Q_{B_2^c B_1}$ being Fredholm of index k for any such choice. Following the same argumentation, we also have that if $Q_{B_2^c B_1}$ is Fredholm of index k for some B_1^c and B_2^c then the same is true for $D_{B_1 B_2}$, hence, the claimed equivalences hold. □

Remark 2.1.5. While Theorem 2.1.4 only states that being Fredholm of a certain index is equivalent for the operators $Q_{B_2^c B_1}$ and $D_{B_1 B_2}$, the proof also shows that those operators have the same kernel and cokernel dimensions. Kernel and cokernel for $Q_{B_2^c B_1}$ can also be directly computed in terms of B_1 and B_2:

$$
\begin{aligned}
\ker\left(Q_{B_2^c B_1}\right) &= \left\{x \in B_1 : P_{B_2^c B_2} \circ Q(x) = 0\right\} \\
&= \{x \in B_1 : Q(x) \in B_2\} \\
&= Q^*(B_2) \cap B_1 \\
&\cong Q(B_1) \cap B_2
\end{aligned}
$$

Corollary 2.1.6. *If $D_{B_1 B_2}$ is Fredholm, then $Q(B_1) \cap B_2$ and $Q(B_1^\perp) \cap B_2^\perp$ are finite dimensional and we have that*

- $\ker(D_{B_1 B_2}) \cong Q(B_1) \cap B_2$
- $\operatorname{coker}(D_{B_1 B_2}) \cong Q(B_1^\perp) \cap B_2^\perp$
- $\operatorname{ind}(D_{B_1 B_2}) = \dim(Q(B_1) \cap B_2) - \dim(Q(B_1^\perp) \cap B_2^\perp)$

Proof. By Theorem 2.1.4 we can choose $B_1^c = B_1^\perp$ and $B_2^c = B_2^\perp$ and since $D_{B_1 B_2}$ is Fredholm by assumption, $Q_{B_2^\perp B_1}$ is also Fredholm. In the proof of 2.1.4 it was already shown that $\ker(D_{B_1 B_2}) = \ker(Q_{B_2^\perp B_1})$ and Remark 2.1.5 now states that $\ker(D_{B_1 B_2}) \cong Q(B_1) \cap B_2$. Again from Theorem 2.1.4 we have that, given $D_{B_1 B_2}$ is Fredholm, $\operatorname{coker}(D_{B_1 B_2}) \cong \ker[(Q_{B_2^\perp B_1})^*]$ and since the splitting was chosen in terms of the orthogonal complements $\ker[(Q_{B_2^\perp B_1})^*] = \ker(Q^*_{B_1 B_2^\perp}) = Q^*(B_2^\perp) \cap B_1^\perp \cong Q(B_1^\perp) \cap B_2^\perp$. □

The proof of 2.1.4 shows that $D_{B_1B_2}$ is Fredholm if and only if the operator

$$L = \left(P_{B_1^cB_1} \circ \mathrm{res}_{\Sigma_1}\right) \oplus \left(P_{B_2^cB_2} \circ \mathrm{res}_{\Sigma_2}\right) : \ker(D) \to B_1^c \oplus B_2^c$$

is Fredholm. Recalculating the image of this operator in the same way, this time in terms of the adjoint operator Q^*, yields:

$$\begin{aligned}
&\mathrm{im}(L|_{\ker(P)}) \\
&= \left\{(x,y) \in B_1^c \oplus B_2^c : \exists \varphi \in \ker(D) : x = P_{B_1^cB_1}\left(\varphi|_{\Sigma_1}\right),\ y = P_{B_2^cB_2}\left(\varphi|_{\Sigma_2}\right)\right\} \\
&= \left\{(x,y) \in B_1^c \oplus B_2^c : \exists \varphi \in \ker(D) : x = P_{B_1^cB_1}\left(Q^*\left(\varphi|_{\Sigma_2}\right)\right),\ y = P_{B_2^cB_2}\left(\varphi|_{\Sigma_1}\right)\right\} \\
&= \left\{(x,y) \in B_1^c \oplus B_2^c : x = Q^*_{B_1^cB_2}(y) + Q^*_{B_1^cB_2}(z),\ \text{for some } z \in B_2\right\} \\
&= \left\{\left(Q^*_{B_1^cB_2^c}(y) + Q^*_{B_1^cB_2}(z), y\right) : y \in B_2^c,\ z \in B_2\right\}.
\end{aligned}$$

Now the rest of the proof can be repeated as above and we get the following proposition:

Proposition 2.1.7. *Let (B_1, B_2) be a boundary condition for D, then $D_{B_1B_2}$ is Fredholm if and only if $Q^*_{B_1^cB_2}$ is Fredholm of the same index for some and then any choice of complementary subspaces B_1^c, B_2^c.*

Now we want to look at what happens to the Fredholm property of the Dirac operator when we replace a boundary condition (B_1, B_2) by its orthogonal complement $(B_1^\perp, B_2^\perp)$, Corollary 2.1.6 suggests that, if the corresponding operator is still Fredholm, its index should only differ by a sign.

Proposition 2.1.8. *Let (B_1, B_2) be a boundary condition for D sucht that $D_{B_1B_2}$ is Fredholm of index k, then $D_{B_1^\perp B_2^\perp}$ is Fredholm of index $-k$.*

Proof. By Theorem 2.1.4 $D_{B_1^\perp B_2^\perp}$ is Fredholm if and only if the Operator $Q_{B_2B_1^\perp}$ is Fredholm of the same index, this is again equivalent to its adjoint $(Q_{B_2B_1^\perp})^* = Q^*_{B_1^\perp B_2}$ being Fredholm. Since $D_{B_1B_2}$ is Fredholm by assumption, we know by Proposition 2.1.7 that $Q^*_{B_1^\perp B_2}$ is Fredholm of the same index and hence, $Q_{B_2B_1^\perp}$ is Fredholm with $\mathrm{ind}(Q_{B_2B_1^\perp}) = -\mathrm{ind}(Q^*_{B_1^\perp B_2}) = -\mathrm{ind}(D_{B_1B_2})$. □

Proposition 2.1.8 shows that going from a boundary condition (B_1, B_2) to its orthogonal complement $(B_1^\perp, B_2^\perp)$ conserves Fredholm property of the corresponding Dirac operator while the index gets a sign. Even though Theorem 2.1.4 suggests that the choice of complementary subspaces for B_1 and B_2 is arbitrary, an analogous statement for going from a boundary condition (B_1, B_2) to some complementary conditions (B_1^c, B_2^c) is false, as the following example illustrates.

Example 2.1.9. Set $M = [0,1] \times S^1$ and $g := -\mathrm{d}t^2 + h$ where h is a fixed Riemannian metric on S^1 such that $\mathrm{vol}(S^1) = 1$. For the so called trivial spin structure of S^1 the Dirac operator A has eigenvalues $\lambda_k = 2\pi k$ where $k \in \mathbb{Z}$, the Dirac operator on M is given by

$$D = \gamma(\nu)\left(\frac{\partial}{\partial t} - iA\right).$$

So solving the Dirac equation $D\varphi = 0$ for initial condition $\varphi|_{\Sigma_1} = u_k$ where u_k is an eigenspinor on Σ_1 with $Au_k = \lambda_k u_k$ gives:

$$\varphi(t, x) = e^{i\lambda_k t} u_k(x).$$

Since $\lambda_k = 2\pi k$ we have $\varphi(1, x) = u_k(x)$, hence, $Q = \mathrm{id}$. Now we define so called *APS boundary conditions* (see section 2.3.1) for D by setting $B_1 = \chi_{(-\infty,0)}(A)$, $B_2 = \chi_{[0,\infty)}(A)$ where χ_I denotes the characteristic function of the Intervall I. In other words, the APS boundary conditions are given by the sum of eigenspaces for the Dirac operator A in the following way

$$B_1 = \bigoplus_{\lambda<0} E_\lambda(A)$$

$$B_2 = \bigoplus_{\lambda\geq 0} E_\lambda(A)$$

and clearly $B_1^\perp = B_2$ by Theorem 1.2.4. Using Theorem 2.1.4 it is easy to check that $D_{B_1B_2}$ is Fredholm, in fact $B_1 \cap B_2 = \{0\} = B_1^\perp \cap B_2^\perp$, hence $\mathrm{ind}(D_{B_1B_2}) = 0$. Now let $G : B_1 \longrightarrow B_2$ be an isomorphism (e.g. by mapping eigenspace to eigenspace) then the graph $\Gamma(G) = \{x + Gx : x \in B_1\}$ is complementary to both B_1 and B_2. On the other hand, $D_{\Gamma(G)\Gamma(G)}$ is clearly not Fredholm since $\Gamma(G) = \Gamma(G) \cap \Gamma(G) = \ker(D_{\Gamma(G)\Gamma(G)})$ is not finite dimensional.

2.1.2 Fredholm Property for Diagonal Terms

Theorem 2.1.4 provides a method to prove Fredholm property for the Dirac operator with a given boundary condition (B_1, B_2), provided one can show that for some choice of complementary subspaces B_1^c and B_2^c the operator $Q_{B_2^cB_1}$ is Fredholm. This again can be done by using the unitarity of Q where equations 2.1 and 2.3 already give potential candidates for right (left) inverses of $Q_{B_2^cB_1}$ modulo compact operators.

Proposition 2.1.10. *Let (B_1, B_2) be a boundary condition for D and B_1^c, B_2^c some choice of complementary subspaces, then the following holds:*

1. *If $Q_{B_2B_1}$ or $Q^*_{B_1B_2}$ is compact, then $D_{B_1B_2}$ has finite dimensional kernel and closed image.*

2. *If additionally $Q_{B_2^cB_1^c}$ or $Q^*_{B_1^cB_2^c}$ is compact, then $D_{B_1B_2}$ is Fredholm and*
   $\mathrm{ind}(D_{B_1B_2}) = \dim(Q(B_1) \cap B_2) - \dim(Q(B_1^\perp) \cap B_2^\perp).$

Proof. By equation 2.1 we have that $Q^*_{B_1B_2^c}Q_{B_2^cB_1} = 1 - Q^*_{B_1B_2}Q_{B_2B_1}$, where $Q^*_{B_1B_2}Q_{B_2B_1}$ is compact if either $Q_{B_2B_1}$ or $Q^*_{B_1B_2}$ is compact, this implies the first statement. The same argument for equation 2.3 together with Corollary 2.1.6 shows the second statement. □

Remark 2.1.11. The Dirac operator $D_{B_1B_2}$ being Fredholm for a given boundary condition (B_1, B_2) and given complementary subspaces B_1^c, B_2^c is not equivalent to the off-diagonal entries in the corresponding splitting of the operator Q being compact.

Example 2.1.12. Take the same setting as in example 2.1.9: $M = \mathbb{R} \times S^1$, $g = -\mathrm{dt}^2 + h$. We write

$$L^2_-(S^1, SS^1) := \chi_{(-\infty,0)}(A)$$

$$L^2_+(S^1, SS^1) := \chi_{[0,\infty)}(A)$$

and let $G : L^2_-(S^1, SS^1) \to L^2_+(S^1, SS^1)$ again be an isomorphism. This time as boundary conditions choose $(B_1, B_2) = (\Gamma(G), L^2_+(S^1, SS^1))$ and $B_1^c = B_2 = L^2_+(S^1, SS^1)$, $B_2^c = B_1 = \Gamma(G)$. Its not hard to check that as long as G is an isomorphism, B_1^c, B_2^c are actually complementary to B_1 and B_2 respectively, and in section 2.3 it is shown that $D_{B_1 B_2}$ is indeed Fredholm.
Now looking at the corresponding splitting of Q and its off-diagonal entries we get:

$$Q_{B_2 B_1}|_{B_1} = Q_{\Gamma(G)\Gamma(G)}|_{\Gamma(G)} = \mathrm{Id} : \Gamma(G) \longrightarrow \Gamma(G)$$

$$Q_{B_2^c B_1^c}|_{B_1^c} = Q_{L^2_+(S^1,SS^1)L^2_+(S^1,SS^1)}|_{L^2_+(S^1,SS^1)} = \mathrm{Id} : L^2_+(S^1, SS^1) \longrightarrow L^2_+(S^1, SS^1)$$

these are certainly both not compact, since $\Gamma(G)$ and $L^2_+(S^1, SS^1)$ are not finite dimensional.

2.2 Fredholm Property for Fredholm Pairs

In this section the notion of *Fredholm pairs* as subspaces of a given Hilbert space is introduced. It will be shown why this setting is suitable for treating boundary conditions and Fredholm property for the Dirac operator and how it can be applied to certain types of deformed boundary conditions called *boundary conditions in graph form* in section 2.3

2.2.1 Fredholm Pairs

Definition 2.2.1. Let H be a Hilbert space and B_1, $B_2 \subset H$ closed linear spaces. The pair (B_1, B_2) is called a **Fredholm pair** if $B_1 \cap B_2$ is finite dimensional and $B_1 + B_2$ is closed and has finite codimension. The number

$$\mathrm{ind}(B_1, B_2) := \dim(B_1 \cap B_2) - \dim(H/(B_1 + B_2))$$

is called the index of the pair.

Remark 2.2.2. For closed subspaces $B_1, B_2 \subset H$ such that $H = B_1 \oplus B_2$ the pair (B_1, B_2) is of course a Fredholm pair. On the other hand, a Fredholm pair (B_1, B_2) of a Hilbert space H can be thought of as being close to a direct sum splitting of H. Meaning that the error, namely $B_1 \cap B_2$ and $H/(B_1 + B_2)$, is "small" i.e. finite dimensional.

The following remark summarizes some elementary properties of Fredholm pairs, that follow immediately from the definition above.

Remark 2.2.3.

1. A pair of subspaces (B_1, B_2) is a Fredholm pair if and only if (B_2, B_1) is a Fredholm pair, and in this case
$$\text{ind}(B_2, B_1) = \text{ind}(B_1, B_2)$$
2. (B_1, B_2) is a Fredholm pair if and only if $(B_1^\perp, B_2^\perp)$ is a Fredholm pair and in this case
$$\text{ind}(B_1, B_2) = -\text{ind}(B_1^\perp, B_2^\perp)$$
3. Let $B_1' \subset B_1$ be a linear subspace such that $\dim(B_1/B_1') < \infty$, then (B_1', B_2) is a Fredholm pair if and only if (B_1, B_2) is a Fredholm pair. In this case
$$\text{ind}(B_1, B_2) = \text{ind}(B_1', B_2) + \dim(B_1/B_1')$$

.

Before Fredholm pairs can be related to the Fredholm property of the Dirac operator with a boundary condition (B_1, B_2), a reformulation of this concept in terms of orthogonal projections is needed. For a proof of the following Lemma see ([6] Lemma 24.3).

Lemma 2.2.4. *A pair of closed linear subspaces $B_1, B_2 \subset H$ of a Hilbert space is a Fredholm pair of index k if and only if the operator*
$$P_{B_2^\perp}|_{B_1} : B_1 \to B_2^\perp$$
is a Fredholm operator of index k. In this case we have $\ker(P_{B_2^\perp}|_{B_1}) = B_1 \cap B_2$ and $\text{coker}(P_{B_2^\perp}|_{B_1}) \cong B_1^\perp \cap B_2^\perp$.

2.2.2 Fredholm Pairs and Boundary Conditions

Getting back to the setting of the Dirac operator on a globally hyperbolic spacetime a boundary condition (B_1, B_2), where $B_i \subset L^2(\Sigma_i; S\Sigma_i)$ can be considered as a pair of closed linear subspaces of a common Hilbert space via the wave evolution operator. More precisely we consider the pairs (B_1, Q^*B_2) and (QB_1, B_2) as pairs of linear subspaces of $L^2(\Sigma_1; S\Sigma_1)$ and $L^2(\Sigma_2; S\Sigma_2)$ respectively. The following Proposition relates the Fredholm property of those pairs to the Fredholm property of the Dirac operator subject to the corresponding boundary conditions.

Proposition 2.2.5. *Let (B_1, B_2) a pair of closed linear subspaces where $B_i \subset L^2(\Sigma_i; S\Sigma_i)$, then the following are equivalent:*

1. *The pair (B_1, Q^*B_2) is Fredholm of index k.*
2. *The pair (QB_1, B_2) is Fredholm of index k.*
3. *The Dirac operator $D_{B_1 B_2}$ is Fredholm of index k.*

Proof. Assume that (QB_1, B_2) is a Fredholm pair, then $QB_1 + B_2 \subset L^2(\Sigma_2; S\Sigma_2)$ is closed. Since Q is an isomorphism $Q^*(QB_1 + B_2) = B_1 + Q^*B_2 \subset L^2(\Sigma_1; S\Sigma_1)$ is also closed. Further we have that $Q(B_1) \cap B_2 \cong B_1 \cap Q^*(B_2)$ and

$$Q(B_1)^\perp \cap B_2^\perp = Q(B_1^\perp) \cap B_2^\perp \cong B_1^\perp \cap Q^*(B_2^\perp) = B_1^\perp \cap Q^*(B_2)^\perp$$

this shows the equivalence of 1. and 2.
Lemma 2.2.4 now states that (QB_1, B_2) being a Fredholm pair of index k is equivalent to the operator

$$P_{B_2^\perp}|_{QB_1} : QB_1 \to B_2^\perp$$

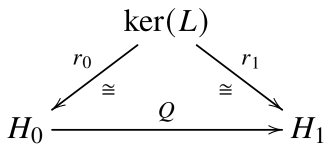
being Fredholm of index k. Since Q is an isomoprhism this is the case if and only if the operator

$$P_{B_2^\perp} \circ Q|_{B_1} = Q_{B_2^\perp B_1} : B_1 \to B_2^\perp$$

is Fredholm of index k, which by Theorem 2.1.4 is again equivalent to the Dirac operator $D_{B_1B_2}$ being Fredholm of the same index. □

Definition 2.2.6. If a pair of linear subspaces $B_1 \subset L^2(\Sigma_1; S\Sigma_1)$, $B_2 = L^2(\Sigma_2; S\Sigma_2)$ is such that $D_{B_1B_2}$ is a Fredholm operator, or equivalently if (QB_1, B_2) is a Fredholm pair, then we call (B_1, B_2) a *Dirac Fredholm pair.*

In fact, the above proposition 2.2.5 follows from a more general result for the following setup: Let E, F, H_0 and H_1 be Hilbert spaces and let $L : E \to F$, $r_j : E \to H_j$ be bounded linear maps. Now, if we assume that $r_j \oplus L : E \to H_j$ is an isomorphism for $j = 0, 1$ then r_j restricts to an isomorphism $r_j : \ker(L) \to H_j$ and we can define a map $Q : H_0 \xrightarrow{\cong} H_1$ by the commutative diagram:



For the setting described above we have the following propostion, for a proof see [11].

Proposition 2.2.7. *Let $B_j \subset H_j$ be closed linear subspaces and assume that $r_0 \oplus r_1 : E \to H_0 \oplus H_1$ is onto, then the following are equivalent:*

1. *The pair $(B_0, Q^{-1}B_1)$ is Fredholm of index k.*
2. *The pair (QB_0, B_1) is Fredholm of index k.*
3. *The operator $(P_{B_0^\perp} \circ r_0) \oplus (P_{B_1^\perp} \circ r_1) \oplus L : E \to B_0^\perp \oplus B_1^\perp \oplus F$ is Fredholm of index k.*
4. *The operator $L : \ker(P_{B_0^\perp} \circ r_0) \cap \ker(P_{B_1^\perp} \circ r_1) \to F$ is Fredholm of index k.*

Remark 2.2.8. By setting $E = \mathrm{FE}^0(M; D)$, $F = L^2(M; S^-M)$ and $H_j = L^2(\Sigma_j; S\Sigma_j)$ as well as $r_j = \mathrm{res}_{\Sigma_j}$ and $L = D$, we obtain the result of proposition 2.2.5. Also it follows from Lemma 2.2.4 that in the proposition above another equivalent statement would be the operator $P_{B_1^\perp}|_{QB_0} : QB_0 \to B_1^\perp$ being Fredholm of index k. While the proposition only states equivalence for Fredholm property and index of the involved operators, a straight forward calculation shows that all these operators in fact have same kernel and cokernel dimensions.

2.3 Boundary Conditions in Graph Form

2.3.1 APS Boundary Conditions

The Atiyah-Patodi-Singer boundary conditions (APS) play an important role in analyzing Fredholm property and index of the Dirac operator on a globally hyperbolic manifold. In [9] Bär and Strohmaier proved Fredholm property for the Dirac operator subject to these boundary conditions as well as an index formula for this case. Later in this section, APS boundary conditions will be used to define so called "boundary conditions in graph form" as deformations of APS boundary conditions, and in section 3.4 we will see how these condtions relate to certain types of local boundary conditions for the Dirac operator.

Definition 2.3.1. Denote by $\chi^{\pm} : \mathbb{R} \to \mathbb{R}$ the characteristic functions of the intervals $(0, \infty)$ and $(-\infty, 0]$ respectively. The operators $P_{\pm} := \chi^{\pm}(A_j) : L^2(\Sigma_j; S\Sigma_j) \to L^2(\Sigma_j; S\Sigma_j)$ are order 0 pseudo-differential orthogonal projectors and we denote their ranges by $L^2_{\pm}(\Sigma_j; S\Sigma_j) := \mathrm{range}(\chi^{\pm}(A_j))$.

- The boundary condition $(B_1, B_2) = (L^2_-(\Sigma_1; S\Sigma_1), L^2_+(\Sigma_2; S\Sigma_2))$ is called *APS boundary condition*
- The boundary condition $(B_1, B_2) = (L^2_+(\Sigma_1; S\Sigma_1), L^2_-(\Sigma_2; S\Sigma_2))$ is called anti-APS or short *aAPS boundary condition*.

Remark 2.3.2. Recall that the Cauchy hypersurfaces $\Sigma_{1/2}$ are closed Riemannian manifolds and their L^2-spinor-spaces split into eigenspaces for the corresponding Dirac operators $A_{1/2}$.

$$L^2(\Sigma_{1/2}; S\Sigma_{1/2}) = \bigoplus_k E_{\lambda_k^{1/2}}(A_{1/2})$$

Where λ_k is an eigenvalue of A and the corresponding eigenspace $E_{\lambda_k}(A)$ is finite dimensional and consists of smooth sections. This means every section $\varphi \in L^2(\Sigma; S\Sigma)$ can be written as a sum over all eigenvalues and corresponding eigensections of A

$$\varphi = \sum_{\lambda} c_\lambda \phi_\lambda$$

where $\phi_\lambda \in E_\lambda(A)$. In this picture APS boundary conditions correspond to deleting all contributions of positive eigenvalues on one hypersurface, while deleting all contributions of non-negative eigenvalues on the other.

$$L^2_-(\Sigma; S\Sigma) = \left\{\varphi \in L^2(\Sigma; S\Sigma) : \varphi = \sum_{\lambda \le 0} c_\lambda \phi_\lambda\right\}$$

$$L^2_+(\Sigma; S\Sigma) = \left\{\varphi \in L^2(\Sigma; S\Sigma) : \varphi = \sum_{\lambda > 0} c_\lambda \phi_\lambda\right\}$$

2.3.2 Generalized APS Boundary Conditions

Similar to the construction of APS boundary conditions, we can choose characteristic functions $\chi^-(a_1)$ and $\chi^+(a_2)$ of intervals $(-\infty, a_1)$ and $[a_2, \infty)$ respectively where $a_1, a_2 \in \mathbb{R}$. The corresponding projection maps $\chi^-(a_1)(A_1)$ and $\chi^+(a_2)(A_2)$ define closed linear subspaces

$$\text{range}\,(\chi^-(a_1)(A_1)) =: L^2_{(-\infty,a_1)}(\Sigma_1; S\Sigma_1)$$

$$\text{range}\left(\chi^+(a_2)(A_2)\right) =: L^2_{[a_2,\infty)}(\Sigma_2; S\Sigma_2)$$

where, just like before, the first one corresponds to deleting all contributions of eigenvalues larger than or equal to a_1, while the second is deleting all contributions of eigenvalues smaller than a_2.

Definition 2.3.3. For $a_1, a_2 \in \mathbb{R}$ a pair of linear subspaces $\left(L^2_{(-\infty,a_1)}(\Sigma_1; S\Sigma_1), L^2_{[a_2,\infty)}(\Sigma_2; S\Sigma_2)\right)$ is called *Generalized APS* or short *gAPS boundary condition*. We will sometimes write $(\text{gAPS}(a_1), \text{gAPS}(a_2))$ to denote these subspaces.

Based on the fact that APS boundary conditions for the Dirac operator yield a Fredholm operator ([9] Theorem 3.3), we can now also prove that generalized APS conditions form Dirac Fredholm pairs. This can be done via the symbol calculus for the wave evolution operator in section 3 or, if we are just interested in proving Fredholm property and deriving a relative index formula, by making use of remark 2.2.3.

Proposition 2.3.4. *For $a_1, a_2 \in \mathbb{R}$ the generalized APS boundary condition $(\text{gAPS}(a_1), \text{gAPS}(a_2))$ is a Dirac Fredholm pair and its index is given by*

$$\text{ind}\,(\text{gAPS}(a_1), \text{gAPS}(a_2)) = \text{ind}_{\text{APS}} + \text{sgn}(a_1)\dim(W_1) - \text{sgn}(a_2)\dim(W_2)$$

where the finite dimensional subspaces W_1, W_2 are given by

$$W_1 := \begin{cases} L^2_{[0,a_1)}(\Sigma_1; S\Sigma_1), & a_1 \geq 0, \\ L^2_{[a_1,0)}(\Sigma_1; S\Sigma_1), & a_1 < 0, \end{cases} \quad \textit{and} \quad W_2 := \begin{cases} L^2_{[0,a_2)}(\Sigma_2; S\Sigma_2), & a_2 \geq 0, \\ L^2_{[a_2,0)}(\Sigma_2; S\Sigma_2), & a_2 < 0. \end{cases}$$

Proof. To apply remark 2.2.3 calculate the quotient spaces

$$\text{gAPS}(a_1)/\text{gAPS}(0) \cong L^2_{[0,a_1)}(\Sigma_1; S\Sigma_1) \quad \text{for } a_1 \geq 0$$
$$\text{gAPS}(0)/\text{gAPS}(a_1) \cong L^2_{[a_1,0)}(\Sigma_1; S\Sigma_1) \quad \text{for } a_1 < 0.$$

Since the Cauchy hypersurfaces Σ_1, Σ_2 are closed Riemannian manifolds the eigenvalues of their corresponding Dirac operators A_1, A_2 are of finite multiplicity and hence these quotients are finite dimensional. This then implies that $(\text{gAPS}(a_1), \text{gAPS}(a_2))$ is a Dirac Fredholm pair, as well as the above formula for the relative index. □

Example 2.3.5. To verify the relative index formula and to get a feeling for the signs that the correction terms come with, it is convenient to recall the basic setup where $M = [0, 1] \times S^1$

and $g = -\mathrm{d}t^2 + h$, where h is a Riemannian metric on S^1 such that $\mathrm{vol}(S^1) = 1$. As mentioned in previous examples, the wave evolution operator in this case is just given by the identity

$$Q = \mathrm{id} : L^2(S^1; SS^1) \to L^2(S^1; SS^1).$$

Setting the boundary condition to generalized APS $(B_1, B_2) = (\mathrm{gAPS}(a_1), \mathrm{gAPS}(a_2))$ for some real parameters, say $a_1 > 0$ and $a_2 < 0$, and calculating the off-diagonal terms in the corresponding splitting of Q (compare section 2.1.2) gives:

$$Q_{B_2 B_1} = P_{B_2} \circ Q \circ P_{B_1} = P_{B_2} \circ P_{B_1}$$

hence, $\mathrm{range}(Q_{B_2 B_1}) = L^2_{[a_2, a_1)}(S^1; SS^1)$ is finite dimensional and $Q_{B_2 B_1}$ is compact. The same argument shows that in this case $Q_{B_1^\perp B_2^\perp}$ is compact and by proposition 2.1.10 and theorem 2.1.4 the Dirac operator $D_{B_1 B_2}$ is then Fredholm. To calculate kernel and cokernel of $D_{B_1 B_2}$ apply corollary 2.1.6

$$\begin{aligned}
\ker(D_{B_1 B_2}) &\cong L^2_{[a_2, a_1)}(S^1; SS^1) = L^2_{[a_2, 0)}(S^1; SS^1) \oplus L^2_{[0, a_1)}(S^1; SS^1) \\
\mathrm{coker}(D_{B_1 B_2}) &\cong L^2_{[a_1, \infty)}(S^1; SS^1) \cap L^2_{(-\infty, a_2)}(S^1; SS^1) = \{0\}
\end{aligned}$$

and for the index we get

$$\mathrm{ind}(D_{B_1 B_2}) = \dim\left(L^2_{(0, a_1]}(S^1; SS^1)\right) + \dim\left(L^2_{[a_2, 0)}(S^1; SS^1)\right)$$

or in the notation of proposition 2.3.4 above

$$\mathrm{ind}\left(D_{B_1 B_2}\right) = \mathrm{sgn}(a_1) \dim\left(W_1\right) - \mathrm{sgn}(a_2)\left(W_2\right).$$

2.3.3 Boundary Conditions in Graph Form

Deformations of APS boundary conditions or boundary conditions in graph form represent deformations of the APS Dirac operator leaving the index unchanged. First we will give a definition of these boundary conditions and then show how they can be useful to analyze Fredholm property for local boundary conditions.

Definition 2.3.6. A pair of closed linear subspaces (B_1, B_2) where $B_i \subset L^2(\Sigma_i; S\Sigma_i)$ is called a *boundary condition in graph form* if there are L^2 orthogonal decompositions

$$L^2(\Sigma_i; S\Sigma_i) = V_i^- \oplus W_i^- \oplus V_i^+ \oplus W_i^+$$

such that

1. W_i^-, W_i^+ are finite dimensional.
2. $V_i^- \oplus W_i^- = L^2_{(-\infty, a_i)}(\Sigma_i; S\Sigma_i)$ and $V_i^+ \oplus W_i^+ = L^2_{[a_i, \infty)}(\Sigma_i; S\Sigma_i)$ for some $a_i \in \mathbb{R}$.

3. There bounded linear maps $G_1 : V_1^- \to V_1^+$ and $G_2 : V_2^+ \to V_2^-$ such that

$$B_1 = \Gamma(G_1) \oplus W_1^+$$

$$B_2 = \Gamma(G_2) \oplus W_2^-$$

where $\Gamma(G_{1/2}) = \{v + G_{1/2}v : v \in V_{1/2}^{\mp}\}$ denotes the graph of $G_{1/2}$.

Remark 2.3.7. For any bounded linear map $G : V^- \to V^+$ as defined above we have that

$$V^- \oplus V^+ = \Gamma(G) \oplus \Gamma(G)^\perp$$

and since for any $w = w^- + w^+ \in \Gamma(G)^\perp \subset V^- \oplus V^+$

$$\begin{aligned} \langle w, v^- + Gv^-\rangle = 0 &\qquad \forall v^- \in V^- \\ \Leftrightarrow \langle w + G^*w, v^-\rangle = 0 &\qquad \forall v^- \in V^- \\ \Leftrightarrow \langle w^- + G^*w^+, v^-\rangle = 0 &\qquad \forall v^- \in V^- \\ \Leftrightarrow w^- = -G^*w^+ & \end{aligned}$$

it follows that $\Gamma(G)^\perp = \Gamma(-G^*)$. With respect to the splitting $V^- \oplus V^+$ the orthogonal projection onto $\Gamma(G)$ is then given by

$$\begin{pmatrix} \mathrm{id} & 0 \\ G & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & -G^* \\ G & \mathrm{id} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathrm{id} + G^*G)^{-1} & (\mathrm{id} + G^*G)^{-1}G^* \\ G(\mathrm{id} + G^*G)^{-1} & G(\mathrm{id} + G^*G)^{-1}G^* \end{pmatrix}$$

see ([13] Lemma 7.7 and Remark 7.8).

The previous remark shows that the orthogonal complement of a boundary condition in graph form is again a boundary condition in graph form, i.e. in the notation of definition 2.3.6 we get

$$B_{1/2}^\perp = \Gamma(-G_{1/2}^*) \oplus W_{1/2}^{\mp}$$

Using proposition 2.1.8 yields the following result in the case of boundary conditions in graph form.

Proposition 2.3.8. *Let $B_1 = \Gamma(G_1) \oplus W_1^+$ and $B_2 = \Gamma(G_2) \oplus W_2^-$ be boundary conditions in graph form, then (B_1, B_2) is a Fredholm pair of index k if and only if $(\Gamma(-G_1)^* \oplus W_1^-, \Gamma(-G_2^*) \oplus W_2^+)$ is a Fredholm pair of index $-k$*

Starting from the fact that the Dirac operator under APS boundary conditions is Fredholm we can now analyze when such a graph type deformation of these conditions again yields a Fredholm operator. See also ([11] Proposition 4.5).

Theorem 2.3.9. *Let (B_1, B_2) be boundary conditions in graph form as defined in definition 2.3.6, then there exists an $\epsilon > 0$ such that the Dirac operator $D_{B_1B_2}$ is Fredholm, provided that*

1. *G_1 or G_2 is compact or*

2. $\|G_1\| \cdot \|G_2\| \leq \epsilon$

In this case the index is given by

$$\operatorname{ind}\left(D_{B_1B_2}\right) = \operatorname{ind}\left(\mathrm{gAPS}(a_1), \mathrm{gAPS}(a_2)\right) + \dim W_1^+ - \dim W_1^- + \dim W_2^- - \dim W_2^+$$

Before we begin the proof of Theorem 2.3.9 note that Bär and Strohmaier already showed in ([9] Lemma 2.6) that for APS boundary conditions and an orthogonal splitting of the L^2 spaces on the boundary

$$L^2(\Sigma_i; S\Sigma_i) = L^2_{(-\infty,0)}(\Sigma_i; S\Sigma_i) \oplus L^2_{[0,\infty)}(\Sigma_i; S\Sigma_i)$$

$$\begin{pmatrix} Q_{--} & Q_{-+} \\ Q_{+-} & Q_{++} \end{pmatrix} : L^2_{(-\infty,0)}(\Sigma_1; S\Sigma_1) \oplus L^2_{[0,\infty)}(\Sigma_1; S\Sigma_1) \to L^2_{(-\infty,0)}(\Sigma_2; S\Sigma_2) \oplus L^2_{[0,\infty)}(\Sigma_2; S\Sigma_2)$$

the off-diagonal terms Q_{-+}, Q_{+-} are compact and hence the Dirac operator is Fredholm under these boundary conditions.

In the case of the above theorem we are working with graph type deformations of generalized APS boundary conditions and proposition 2.3.4 shows that the Dirac operator under these conditions is also Fredholm.

Now while proposition 2.1.10 states that compactness of these off-diagonoal terms in the splitting of Q is sufficient for the Dirac operator being Fredholm, we already saw in remark 2.1.11 that even though the Dirac operator might be Fredholm subject to some boundary conditions, compactness for the off-diagonal terms in the splitting of the wave evolution operator is not necessarily given.

During the proof we will be using the fact that for generalized APS conditions and a corresponding orthogonal splitting of the L^2 spaces on the boundary,

$$\begin{pmatrix} Q_{\mathrm{gAPS}(a_2)^\perp \mathrm{gAPS}(a_1)} & Q_{\mathrm{gAPS}(a_2)\mathrm{gAPS}(a_1)} \\ Q_{\mathrm{gAPS}(a_2)^\perp \mathrm{gAPS}(a_1)^\perp} & Q_{\mathrm{gAPS}(a_2)\mathrm{gAPS}(a_1)^\perp} \end{pmatrix} : \mathrm{gAPS}(a_1) \oplus \mathrm{gAPS}(a_1)^\perp \to \mathrm{gAPS}(a_2)^\perp \oplus \mathrm{gAPS}(a_2)$$

the off-diagonal terms $Q_{\mathrm{gAPS}(a_2)\mathrm{gAPS}(a_1)}$ and $Q_{\mathrm{gAPS}(a_2)^\perp \mathrm{gAPS}(a_1)^\perp}$ are still compact. This can be seen by just a slight modification of the proof by Bär and Strohmaier and will also be explained more detailed in section 3.2 corollary 3.2.2.

proof (of 2.3.9). Start with the case where $W^\pm_{1/2} = \{0\}$ and $V_1^- = L^2_{(-\infty,a_1)}(\Sigma_1; S\Sigma_1)$, $V_2^+ = L^2_{[a_2,\infty)}(\Sigma_2; S\Sigma_2)$. Let $G_1 : V_1^- \to V_1^+$, $G_2 : V_2^+ \to V_2^-$ be bounded linear maps and set $B_{1/2} = \Gamma(G_{1/2})$. We split the wave evolution operator with respect to an orthogonal splitting of the L^2 spaces relative to the graph type boundary conditions on both boundary components:

$$L^2(\Sigma_{1/2}; S\Sigma_{1/2}) = \Gamma(G_{1/2}) \oplus \Gamma(-G^*_{1/2}).$$

Then using lemma 2.2.4 or theorem 2.1.4, we have to show that $Q_{B_2^c B_1} : \Gamma(G_1) \to \Gamma(-G_2^*)$ is a Fredholm operator with the required index. Since the maps

$$\mathrm{id} + G_1 : V_1^- \to \Gamma(G_1)$$

$$\mathrm{id} - G_2^* : V_2^- \to \Gamma(-G_2^*)$$

are isomorphisms, this is equivalent the the operator

$$L = P_{V_2^-} \circ Q_{B_2^c B_1} \circ (\mathrm{id} + G_1) : V_1^- \longrightarrow V_2^-$$

being Fredholm of the same index. Now switching to the splitting $L^2(\Sigma_i; S\Sigma_i) = V_i^- \oplus V_i^+$ and by remark 2.3.7 the operator L can be written explicitly in terms of the maps Q and G_i. To shorten the notation we abreviate for $i = 1, 2$: $[i] := (1 + G_i^* G_i)^{-1}$.

$$\begin{aligned} L &= \begin{pmatrix} \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} [2] & -[2]G_2 \\ -G_2^*[2] & G_2^*[2]G_2 \end{pmatrix} \begin{pmatrix} Q_{V_2^- V_1^-} & Q_{V_2^- V_1^+} \\ Q_{V_2^+ V_1^-} & Q_{V_2^+ V_1^+} \end{pmatrix} \begin{pmatrix} [1] & [1]G_1^* \\ G_1[1] & G_1[1]G_1^* \end{pmatrix} \begin{pmatrix} \mathrm{id} \\ G_1 \end{pmatrix} \\ &= [2] \left(Q_{V_2^- V_1^-} + Q_{V_2^- V_1^+} G_1 + G_2 Q_{V_2^+ V_1^-} + G_2 Q_{V_2^+ V_1^+} G_1 \right) [1](1 + G_1^*). \end{aligned}$$

The maps $[1]$, $[2]$ and $(1+G_1^*)$ are isomorphisms, which means that the operator L is Fredholm if and only if $Q_{V_2^- V_1^-} + Q_{V_2^- V_1^+} G_1 + G_2 Q_{V_2^+ V_1^-} + G_2 Q_{V_2^+ V_1^+} G_1$ is Fredholm, in this case their indices coincide.
Since $V_1^\pm$ and $V_2^\pm$ represent generalized APS boundary conditions, the operators $Q_{V_2^- V_1^+}$ and $Q_{V_2^+ V_1^-}$ are compact by the preceding remark. If either G_1 or G_2 is a compact map then $G_2 Q_{V_2^+ V_1^+} G_1$ is also compact and because $Q_{V_2^- V_1^-}$ is a Fredholm operator L will also be Fredholm of the same index.
If both G_1 and G_2 are not compact, then there exists an $\epsilon > 0$ such that for

$$\|G_2 Q_{V_2^+ V_1^+} G_1\| \leq \|G_2\| \|Q_{V_2^+ V_1^+}\| \|G_1\| \leq \|G_1\| \|G_2\| \leq \epsilon$$

the operator $Q_{V_2^- V_1^-} + G_2 Q_{V_2^+ V_1^+} G_1$ is Fredholm of the same index as $Q_{V_2^- V_1^-}$.
For the general case where $W_{1/2}^\pm \neq \{0\}$, we still have

$$V_1^- \oplus W_1^- = L^2_{(-\infty, a_1)}(\Sigma_1; S\Sigma_1) \qquad \text{and} \qquad V_2^+ \oplus W_2^+ = L^2_{[a_2, \infty)}(\Sigma_2; S\Sigma_2)$$

for some $a_1, a_2 \in \mathbb{R}$ and bounded linear maps $G_1 : V_1^- \to V_1^+$, $G_2 : V_2^+ \to V_2^-$. The graph of such a map $\Gamma(G_1)$ can be considered as the graph of a map $\widetilde{G}_1 : L^2_{(-\infty, a_1)}(\Sigma_1; S\Sigma_1) \to L^2_{[a_1, \infty)}(\Sigma_1; S\Sigma_1)$ just by setting

$$\widetilde{G}_1(v + w) = G_1(v) \quad \text{for} \quad v + w \in V_1^- \oplus W_1^- = L^2_{(-\infty, a_1)}(\Sigma_1; S\Sigma_1)$$

and similarly for $\widetilde{G}_2 : L^2_{[a_2, \infty)}(\Sigma_2; S\Sigma_2) \to L^2_{(-\infty, a_2)}(\Sigma_2; S\Sigma_2)$. The first part then states, provided condition 1. or 2. is satisfied, that $\left(\Gamma(\widetilde{G}_1), \Gamma(\widetilde{G}_2)\right)$ is a Fredholm pair with the same index as $(\mathrm{gAPS}(a_1), \mathrm{gAPS}(a_2))$, and comparing both graphs we get

$$\Gamma(\widetilde{G}_1) / \Gamma(G_1) = \left(\Gamma(G_1) + W_1^-\right) / \Gamma(G_1) = W_1^-$$

and

$$\Gamma(\widetilde{G}_2) / \Gamma(G_2) = \left(\Gamma(G_2) + W_2^+\right) / \Gamma(G_2) = W_2^+.$$

W_1^- and W_2^+ are finite dimensional. Hence by remark 2.2.3, $(\Gamma(G_1), \Gamma(G_2))$ is a Dirac Fredholm pair and

$$\begin{aligned}\operatorname{ind}(\Gamma(G_1), \Gamma(G_2)) &= \operatorname{ind}\left(\Gamma(\widetilde{G}_1), \Gamma(\widetilde{G}_2)\right) - \dim W_1^- - \dim W_2^+ \\ &= \operatorname{ind}(\mathrm{gAPS}(a_1), \mathrm{gAPS}(a_2)) - \dim W_1^- - \dim W_2^+.\end{aligned}$$

For the full boundary condition $B_1 = \Gamma(G_1) \oplus W_1^+$, $B_2 = \Gamma(G_2) \oplus W_2^-$ we then get

$$B_1 / \Gamma(G_1) = W_1^+ \qquad \text{and} \qquad B_2 / \Gamma(G_2) = W_2^-$$

and applying remark 2.2.3 one more time yields

$$\operatorname{ind}(B_1, B_2) = \operatorname{ind}(\mathrm{gAPS}(a_1), \mathrm{gAPS}(a_2)) + \dim W_1^+ - \dim W_1^- + \dim W_2^- - \dim W_2^+.$$

□

To end this section, we want to discuss if the two conditions from theorem 2.3.9 could be dropped or relaxed. Firstly, it is not hard to set up an example to show that deformations via arbitrary bounded linear maps G_1 and G_2 in the sense of boundary conditions in graph form don't necessarily yield Dirac Fredholm pairs.

Example 2.3.10. Set $M = [0, 1] \times S^1$ and $g = -\mathrm{dt}^2 + h$ where h is such that $\mathrm{vol}(S^1) = 1$. For the trivial spin structure of S^1 the Dirac operator A has eigenvalues $\lambda_k = 2\pi k$ for $k \in \mathbb{Z}$ and the wave evolution operator is given by $Q = \mathrm{id}$ as discussed in example 2.1.9.
Further let $\varphi_k \in L^2(S^1; SS^1)$ be the eigensection $A\varphi_k = \lambda_k \varphi_k$, $\|\varphi_k\| = 1$ and define a bounded linear map G by setting:

$$\begin{aligned} G : L^2(S^1; SS^1) &\to L^2(S^1; SS^1) \\ \varphi_k &\mapsto \varphi_{-k}. \end{aligned}$$

Now to get graph type boundary conditions, we choose $V_{1/2}^- = L^2_{(-\infty,0)}(S^1; SS^1)$, $V_{1/2}^+ = L^2_{(0,\infty)}(S^1; SS^1)$, $W_{1/2}^+ = \ker(A)$, $W_{1/2}^- = \{0\}$ and

$$G_1 := G|_{V_1^-} : V_1^- \longrightarrow V_1^+ \qquad \text{and} \qquad G_2 := G|_{V_2^+} : V_2^+ \longrightarrow V_2^-.$$

In this case we have $G_1 G_2 = G_2 G_1 = \mathrm{id}$ and both graphs actually coincide $\Gamma(G_1) = \Gamma(G_2)$.

For the Dirac operator with boundary conditions $(B_1, B_2) = (\Gamma(G_1), \Gamma(G_2))$ to be Fredholm a necessary condition given by corollary 2.1.6 would be that $Q(\Gamma(G_1)) \cap \Gamma(G_2) = \Gamma(G_1) \cap \Gamma(G_2)$ is finite dimensional. Since this is clearly not the case here $D_{B_1 B_2}$ can't be Fredholm.
Theorem 2.3.9 still tells us that rescaling say $G_1 \rightsquigarrow \gamma \cdot G_1$ by a sufficiently small factor γ, gives a Dirac Fredholm pair. In this particular case it is not hard to see that any $\gamma \neq 1$ does the trick.

Now we also want to give an example for a boundary condition in graph form where the norm product $\|G_1\|\|G_2\|$ is "not small", but the compactness assumption from theorem 2.3.9 still provides that the resulting Dirac operator is Fredholm.

Example 2.3.11. Take the same setting as example 2.3.10, let $\tilde{V}_2^+ \subset V_2^+$ be a finite dimensional subspace and set

$$G_1 := G|_{V_1^-} : V_1^- \longrightarrow V_1^+$$
$$G_2 : V_2^+ \longrightarrow V_2^- \quad \text{s.t.} \quad G_2|_{\tilde{V}_2^+} = G|_{\tilde{V}_2^+} \quad \text{and} \quad G_2|_{(\tilde{V}_2^+)^\perp} = 0.$$

For this case, we have that $\Gamma(G_1) \cap \Gamma(G_2) = \Gamma(G_2|_{\tilde{V}_2^+})$ is finite dimensional, since by definition G_2 is a finite rank operator and hence compact, we get Dirac Fredholm pair $(\Gamma(G_1), \Gamma(G_2))$ by theorem 2.3.9.
The norm product $\|G_1\|\|G_2\| = 1$ remains unchanged and in fact by again rescaling $G_2 \rightsquigarrow \gamma \cdot G_2$, it can be made arbitrary large without changing the fact that G_2 is compact and without losing Fredholm property for $D_{\Gamma(G_1)\Gamma(G_2)}$.

3 The Symbol of Q

The wave evolution operator discussed in section 2.1 is known to be a Fourier integral operator. Bär and Strohmaier used this fact in [9] to prove compactness of the off-diagonal terms in a splitting with respect to APS boundary conditions by calculating the principal symbol of Q and composing with the APS projectors. For completeness, we will repeat the calculation of the principal symbol of Q as a Fourier integral operator and give a simplified version of the result. For a detailed proof see [9] Lemma 2.6 and for general information about Fourier integral operators see e.g [7] or [8].

3.1 The Principal Symbol of Q

Remark 3.1.1. The wave evolution operator is given by the formula

$$Q = \mathrm{Res}_{t_2} \circ \tilde{D} \circ \mathcal{T} \circ \beta$$

where Res_{t_2} denotes the restriction map onto Σ_2, $\tilde{D} : C^\infty(M; S^-M) \to C^\infty(M; S^+M)$ the Dirac operator and β Clifford multiplication by the unit, normal, past-directed, timelike vector field.

The operator $\mathcal{T} : C^\infty(\Sigma_1; S\Sigma_1) \to C^\infty(M; S^-M)$ denotes the solution operator to the initial value problem

$$D\tilde{D}u = 0, \qquad u|_{\Sigma_1} = 0, \qquad (\nabla_\nu u)\,|_{\Sigma_1} = f.$$

The operators Res_{t_1} and $\mathcal{T}$ are Fourier integral operators and their canonical relations are described in ([9] Theorem A.1). The canonical relation of Q as the composition of those operators, also explained in [9], is given by

$$C_Q = \{((y,\eta),(x,\xi)) \in \dot{T}^*\Sigma_1 \times T^*\Sigma_2 \mid (y,\eta) \sim (x,\xi)\}$$

where the relation "∼" can be described as follows: If $(y,\eta) \in \dot{T}^*\Sigma_1$ then there are precisely two lightlike co-vectors $\tilde{\eta}^\pm = \mp\|\eta\|\nu^\flat + \eta \in T^*_yM$ such that $\tilde{\eta}^\pm|_{\Sigma_1} = \eta$. The lightlike geodesic $\gamma^\pm_\eta$ with $\gamma^\pm_\eta(0) = y$, $\dot{\gamma}^\pm_\eta(0) = \tilde{\eta}^\pm$ (here tangent and co-tangent vectors are identified via the metric g) intersects Σ_2 at exactly one point, say $\gamma^\pm_\eta(s^\pm) = x^\pm(y,\eta) \in \Sigma_2$ and $\dot{\gamma}^\pm_\eta(s^\pm) = \tilde{\xi}^\pm(y,\eta) \in T^*_{x^\pm}M$ again gives a lightlike co-vector. Then $((y,\eta),(x,\xi)) \in C_Q$ if and only if $(x,\xi) = (x^\pm, \tilde{\xi}^\pm|_{\Sigma_2})$.

Before calculating the principal symbol of Q, it is useful to relate Clifford multiplication on M to the principal symbol of the APS projectors $P_\pm$, since they naturally appear in this calculation and will allow us to write the result in a short and simplified form.

Lemma 3.1.2. *Let $\Sigma_t \subset M$ be a Cauchy hypersurface and $A_t : C^\infty(\Sigma_t; S\Sigma_t) \to C^\infty(\Sigma_t; S\Sigma_t)$ its Dirac operator and let $P_t^\pm : L^2(\Sigma_t; S\Sigma_t) \to L^2_\pm(\Sigma_t; S\Sigma_t)$ be the APS projectors onto the positive/negative spectral part of A_t, then the following maps coincide:*

$$\begin{aligned} p_\pm(\xi)\beta : S^-M|_{\Sigma_t} &\longrightarrow S^+M|_{\Sigma_t} \\ \mp\tfrac{1}{2}\|\xi\|^{-1}\gamma(\tilde{\xi}^\pm) : S^-M|_{\Sigma_t} &\longrightarrow S^+M|_{\Sigma_t} \end{aligned}$$

where $\xi \in \dot{T}^*\Sigma_t$, $\tilde{\xi}^\pm := \mp\|\xi\|\nu^\flat + \xi$, γ denotes Clifford multiplication on M and $p_\pm$ is the principal symbol of $P^\pm$.

Proof. The APS projectors on Σ_t can be calculated through the Dirac operator A_t via $P_\pm = \frac{1}{2}(1 \pm \frac{A}{|A|})$. The principal symbol of A_t is given by Clifford multiplication on Σ_t, $\sigma_{A_t}(\xi) = \gamma_t(\xi)$ and hence,

$$\begin{aligned} p_\pm(\xi)\beta &= \tfrac{1}{2}\left(1 \pm \tfrac{i\gamma_t(\xi)}{\|\xi\|}\right)\beta \\ &= \tfrac{1}{2}\|\xi\|^{-1}\left(\|\xi\|\beta \pm \beta\gamma(\xi)\beta\right) \\ &= \tfrac{1}{2}\|\xi\|^{-1}\left(\|\xi\|\beta \mp \gamma(\xi)\right) \\ &= \mp\tfrac{1}{2}\|\xi\|^{-1}\left(\mp\|\xi\|\beta + \gamma(\xi)\right) \\ &= \tfrac{1}{2}\|\xi\|^{-1}\gamma(\tilde{\xi}^\pm). \end{aligned}$$

□

Lemma 3.1.3. *The principal symbol of the wave evolution operator Q is given by*

$$\begin{aligned} q(y,\eta,x^\pm,\xi^\pm) &\cong p_\pm(\xi^\pm)\beta\,\Gamma^\pm_{(y,\eta,x^\pm,\xi^\pm)}\beta \\ &= \Gamma^\pm_{(y,\eta,x^\pm,\xi^\pm)}p_\pm(\eta) \end{aligned}$$

for $((y,\eta),(x^\pm,\xi^\pm)) \in C_Q$. Here "$\cong$" means equality up to a scalar factor and $\Gamma^\pm_{(y,\eta,x^\pm,\xi^\pm)}$ denotes parallel transport along the lightlike geodesic $\gamma^\pm_\eta$ (see Remark 3.1.1) connecting y and $x^\pm$.

Note that multiplication by β is understood as a map $S^\pm M \to S^\mp M$, while parallel transport $\Gamma^\pm_{(y,\eta,x^\pm,\xi^\pm)} : S^\pm_y M \to S^\pm_{x^\pm} M$ conserves the chirality splitting. Identifying $S^+M|_{\Sigma_t} \cong S\Sigma_t$ to apply $p_\pm$, we get that $q(y,\eta,x^\pm,\xi^\pm)$ in fact yields a map $S_y\Sigma_1 \to S_{x^\pm}\Sigma_2$.

Proof. The wave evolution operator is given by $Q = \mathrm{Res}_{t_2} \circ \tilde{D} \circ \mathcal{T}_{t_1} \circ \beta$ and its principal symbol can be calculated as the composition of symbols

$$q(y,\eta,x^\pm,\xi^\pm) = \mathrm{res}_{t_2}(\tilde{\xi}^\pm) \circ \sigma_{\tilde{D}}(\tilde{\xi}^\pm) \circ \tau_{t_1}(y,\eta,x^\pm,\tilde{\xi}^\pm) \circ \beta \;\text{ for }\; (y,\eta,x^\pm,\xi^\pm) \in C_Q.$$

Using [9] Theorem A.1 and neglecting scalar factors we get:

$$\begin{aligned} q(y,\eta,x^\pm,\xi^\pm) &\cong \gamma(\tilde{\xi}^\pm)\Gamma^\pm_{(y,\eta,x^\pm,\xi^\pm)}\beta \\ &\overset{\text{Lemma 3.1.2}}{\cong} p_\pm(\xi^\pm)\beta\,\Gamma^\pm_{(y,\eta,x^\pm,\xi^\pm)}\beta \end{aligned}$$

proving the first line of the statement. To get to the second part, we use the fact that Clifford multiplication is compatible with parallel transport meaning that

$$\gamma(\tilde{\xi}^{\pm})\Gamma^{\pm}_{(y,\eta,x^{\pm},\xi^{\pm})} = \Gamma^{\pm}_{(y,\eta,x^{\pm},\xi^{\pm})}\gamma(\tilde{\eta}).$$

Using this compatibility and lemma 3.1.2 again shows that

$$\begin{aligned} q(y,\eta,x^{\pm},\xi^{\pm}) &\cong \Gamma^{\pm}_{(y,\eta,x^{\pm},\xi^{\pm})}\gamma(\tilde{\eta})\beta \\ &\cong \Gamma^{\pm}_{(y,\eta,x^{\pm},\xi^{\pm})}p_{\pm}(\eta)\beta\beta \\ &= \Gamma^{\pm}_{(y,\eta,x^{\pm},\xi^{\pm})}p_{\pm}(\eta). \end{aligned}$$

□

3.2 APS Boundary Conditions

The wave evolution operator $Q : L^2(\Sigma_1; S\Sigma_1) \to L^2(\Sigma_2; S\Sigma_2)$ is an isomorphism (see section 2.1) while the APS projectors $P^{\pm}$ are pseudo differential operators of order zero, with principal symbol calculated above. In order to show compactness for off-diagonal terms in an orthogonal splitting for APS boundary conditions, Bär and Strohmaier ([9] Lemma 2.6) calculated the principal symbol of the composition P^+QP^- (and P^-QP^+) showing that it vanishes, meaning those operators map $H^k(\Sigma_1; S\Sigma_1) \to H^{k+1}(\Sigma_2; S\Sigma_2)$ continuously and are hence compact. By proposition 2.1.10 this then implies that the corresponding Dirac operator D_{APS} is Fredholm.

Theorem 3.2.1 (Bär, Strohmaier)**.** *The Dirac operator with APS boundary conditions,* $B_1 = P^-\left(L^2(\Sigma_1; S\Sigma_1)\right)$ *and* $B_2 = P^-\left(L^2(\Sigma_2; S\Sigma_2)\right)$*, is Fredholm.*

The original proof of this statement can be found in ([9] Theorem 3.2). Since all necessary symbol calculations have already been done above, the proof will be repeated here for completeness.

Proof. The principal symbol of the composition $P^+QP^- =: Q_{+-}$ can be calculated as the composition of symbols

$$q_{+-}(y,\eta,x^{\pm},\xi^{\pm}) = p_+(\xi^{\pm})q(y,\eta,x^{\pm},\xi^{\pm})p_-(\eta).$$

Substituting Lemma 3.1.3 for the principal symbol of Q, we get

$$\begin{aligned} q_{+-}(y,\eta,x^{\pm},\xi^{\pm}) &\cong p_+(\xi^{\pm})p_{\pm}(\xi)\beta\,\Gamma^{\pm}_{(y,\eta,x^{\pm},\xi^{\pm})}\beta p_-(\eta) \\ &= p_+(\xi^{\pm})\Gamma^{\pm}_{(y,\eta,x^{\pm},\xi^{\pm})}p_{\pm}(\eta)p_-(\eta) \end{aligned}$$

which have to be shown to vanish for both possible relations between the points (y,η) and $(x^{\pm},\xi^{\pm})$. For the pair (y,η), (x^+,ξ^+) we have

$$p_+(\xi^{\pm})\Gamma^{+}_{(y,\eta,x^{+},\xi^{+})}\underbrace{p_+(\eta)p_-(\eta)}_{=0} = 0$$

while for the pair (y,η), (x^-,ξ^-) we get

$$\underbrace{p_+(\xi^-)p_-(\xi^-)}_{=0}\beta\,\Gamma^-_{(y,\eta,x^-,\xi^-)}\beta p_-(\eta)=0$$

and hence $q_{+-}(y,\eta,x^\pm,\xi^\pm)=0$. □

From the above statement, it is actually easy to get Fredholm property for generalized APS boundary conditions, already proved in proposition 2.3.4 as well. Since for any $a\in\mathbb{R}$ the projectors $P^-: L^2(\Sigma;S\Sigma)\to L^2_-(\Sigma;S\Sigma)$ and $P^-_a: L^2(\Sigma;S\Sigma)\to L^2_{(-\infty,a)}(\Sigma;S\Sigma)$ only differ by a smoothing operator $\mathrm{Im}(P^- - P^-_a)=L^2_{[a,0)}(\Sigma;S\Sigma)$ for $a<0$ or $\mathrm{Im}(P^- - P^-_a)=L^2_{[0,a)}(\Sigma;S\Sigma)$ for $a\geq 0$. This implies that the principal symbols p^- and p^-_a coincide, and hence the proof above, showing compactness for off-diagonal terms of Q, can be repeated unchanged.

Corollary 3.2.2. *Let $a_1,a_2\in\mathbb{R}$ and $B_1=L^2_{(-\infty,a_1)}(\Sigma_1;S\Sigma_1)$, $B_2=L^2_{[a_2,\infty)}(\Sigma_2;S\Sigma_2)$ be generalized APS boundary conditions, then the off-diagonal terms of Q in the orthogonal splitting of the boundary spaces, $P^+_{a_2}QP^-_{a_1}$ and $P^-_{a_2}QP^+_{a_1}$ are compact and the Dirac operator $D_{B_1B_2}$ is Fredholm.*

At this point the above corollary provides no new information compared with propostion 2.3.4. However, the mapping property $P^+_{a_2}QP^-_{a_1}: H^k(\Sigma_1;S\Sigma_1)\to H^{k+1}(\Sigma_2;S\Sigma_2)$ becomes relevant for discussing regularity of solutions in the following section.

3.3 Regularity for Boundary Conditions in Graph Form

So far, we only discussed Fredholm property and relative index formulas for the Dirac operator under certain generalizations of the APS boundary conditions. In ([9] Theorem 3.5), it is also proved that for APS boundary conditions the solution space $\ker(D_{\mathrm{APS}})\subset FE^0_{\mathrm{APS}}(M;D)$ consists of smooth sections only. In this section, we will explain the strategy of this proof and see to what extend it can be generalized to the boundary conditions introduced in this thesis.

3.3.1 The Kernel of $Q_{B_2^cB_1}$

For any boundary condition $B_1\subset L^2(\Sigma_1;S\Sigma_1)$, $B_2\subset L^2(\Sigma_2,S\Sigma_2)$ and any choice of complementary subspaces B_1^c, B_2^c, such that $L^2(\Sigma_i,S\Sigma_i)=B_i\oplus B_i^c$, the kernel of the Dirac operator $D_{B_1B_2}$ subject to these boundary conditions can be identified with $\ker(Q_{B_2^cB_1})=B_1\cap Q^*(B_2)$ via the restriction map onto Σ_1. This identification is independent of whether (B_1,B_2) is a Dirac Fredholm pair, i.e. whether the kernels of $D_{B_1B_2}$ and $Q_{B_2^cB_1}$ are finite dimensional or not. Now we want to relate regularity of the kernel of $\ker(Q_{B_2^cB_1})$ to that of the corresponding Dirac operator. This is done by the following Lemma see also [9] corollary 2.7.

Lemma 3.3.1. *Let (B_1,B_2) be a boundary condition for D, then with respect to any splitting $L^2(\Sigma_i;S\Sigma_i)=B_i\oplus B_i^c$, we have*

$$\ker(Q_{B_2^cB_1})\subset C^\infty(\Sigma_1;S\Sigma_1)\Leftrightarrow\ker(D_{B_1B_2})\subset C^\infty(M;S^+M)$$

Proof. Since the restriction $\mathrm{res}_{t_1} : FE^0(M;D) \to L^2(\Sigma_1; S\Sigma_1)$ maps $\ker(D_{B_1B_2}) \to \ker(Q_{B_2^cB_1})$ isomorphically, we have that

$$\ker(D_{B_1B_2}) \subset C^\infty(M; S^+M) \Rightarrow \ker(Q_{B_2^cB_1}) \subset C^\infty(\Sigma_1; S\Sigma_1).$$

For the other direction note that if $\varphi \in \ker(Q_{B_2^cB_1})$ is smooth, then the corresponding solution to the Dirac equation ϕ, where $D\phi = 0$ and $\phi|_{\Sigma_1} = \varphi$, lies in $FE^s(M;D)$, as defined in [9] chapter 2, for any $s \in \mathbb{R}$ by well-posedness of the Cauchy problem for D ([9] Theorem 2.1.). In particular, ϕ can be considered as a section

$$\begin{aligned} M = [t_1, t_2] \times \Sigma &\to S^+M \\ \phi &\to \phi(t,x) \in S^+_{(t,x)}M \end{aligned}$$

that is smooth in the second argument and continuous in the first. Now the Dirac equation for ϕ implies that

$$\nabla_\nu \phi = (-iA_t + \tfrac{n}{2}H)\phi$$

and since H and coefficients of A_t depend smoothly on t, the right-hand side is of the same regularity as ϕ, implying that $\phi(t,x)$ is actually C^1 in the first slot. Differentiating this equation again and iterating the argument, shows that $\phi(t,x)$ depends smoothly on t and is then contained in $C^\infty(M; S^+M)$. □

Remark 3.3.2. Note that in Lemma 3.3.1 the choice of complementary subspaces B_1^c, B_2^c is not specified and in fact the statement holds for any such choice. This is due to remark 2.1.5 and the fact that the kernel of $Q_{B_2^cB_1}$ can be expressed as

$$\ker(Q_{B_2^cB_1}) = Q(B_1) \cap B_2 = B_1 \cap Q^*(B_2)$$

and is hence independent of the complements B_1^c, B_2^c.

However, having this freedom of choice in the splittings of $L^2(\Sigma_1; S\Sigma_1)$ and $L^2(\Sigma_2; S\Sigma_2)$ will turn out to be useful and simplify proving regularity for graph-type boundary conditions in section 3.3.2.

Remark 3.3.3. In previous sections we discussed possible ways to obtain Fredholm property, in particular finite kernel dimension, for the Dirac operator subject to some boundary condition. So far no statement about regularity for solutions, namely elements of the kernel of D, to the Dirac equation was made, and in fact $D_{B_1B_2}$ having finite dimensional kernel for some boundary condition does not imply regularity in any way. Conversely, since $Q_{B_2^cB_1}$ is a bounded operator, its kernel is a closed subspace of $L^2(\Sigma_1; S\Sigma_1)$ and if it is smooth it will always automatically be finite dimensional as well. This means that a smooth solution space $\ker(D_{B_1B_2}) \subset C^\infty(M; S^-M)$ for the Dirac operator subject to some boundary condition already implies for its kernel to be finite dimensional.

3.3.2 Graph-Type Boundary Conditions

In order to show smoothness for the kernel of the Dirac operator $D_{B_1B_2}$, it suffices to show smoothness of $\ker(Q_{B_2^cB_1})$ by Lemma 3.3.1. Note that this correlation is independent of the choice of complementary subspaces B_1^c and B_2^c, and if it is true for some complements, it is true for any such choice. This freedom of choosing complementary subspaces to a given boundary condition (B_1, B_2) translates to the choice of projection maps $L^2(\Sigma_i; S\Sigma_i) \to B_i$ and will prove useful when treating graph-type boundary conditions in this section. For now note that at this point, we already have all the necessary ingredients to conclude smoothness of $\ker(D_{\mathrm{gAPS}(a_1)\mathrm{gAPS}(a_2)})$ for generalized APS boundary conditions (and in particular for APS boundary conditions).

Proposition 3.3.4. *Let $a_1, a_2 \in \mathbb{R}$ and $(B_1, B_2) = (\mathrm{gAPS}(a_1), \mathrm{gAPS}(a_2))$ generalized APS boundary conditions, then* $\ker(Q_{B_2^cB_1}) \subset C^\infty(\Sigma_1; S\Sigma_1)$.

Proof. In corollary 3.2.2, it is shown that for generalized APS boundary conditions the off-diagonal terms in the splitting of the wave evolution operator $Q_{B_2B_1}$ and $Q_{B_2^cB_1^c}$ map $H^k(\Sigma_1; S\Sigma_1) \to H^{k+1}(\Sigma_2; S\Sigma_2)$. Using this mapping property together with the unitarity of Q we have for $u \in \ker(Q_{B_2^cB_1})$ (Lemma 2.1.2):

$$u = Q^*_{B_1B_2} Q_{B_2B_1} u.$$

Since $Q^*_{B_1B_2}$ is at most of order 0, this implies that $u \in C^\infty(\Sigma_1; S\Sigma_1)$. □

In particular the above proposition holds for APS boundary conditions, i.e. $a_1 = a_2 = 0$ (see definition 2.3.3) and smoothness for the solution space of the Dirac operator with generalized APS boundary conditions follows directly from lemma 3.3.1.

Corollary 3.3.5. *Let $a_1, a_2 \in \mathbb{R}$ and $(B_1, B_2) = (\mathrm{gAPS}(a_1), \mathrm{gAPS}(a_2))$ generalized APS boundary conditions, then* $\ker(D_{B_1B_2}) \subset C^\infty(M; S^+M)$.

In order to make the symbol calculus of Q work for boundary conditions in graph form, we need to make some further assumptions on the deformation maps G_1 and G_2, defining the boundary conditions in definition 2.3.6. First, remember that theorem 2.3.9 only guarantees Fredholm property for the resulting Dirac operator if either G_1 or G_1 is compact or if the norm product $\|G_1\|\|G_2\|$ is small, and we have also seen counterexamples for non-compact deformations where not even the kernel of $D_{B_1B_2}$ is finite dimensional for certain graph-type boundary conditions. Due to this fact, for the following statement compactness of one of the deformation maps will still be required, since having smooth kernel would otherwise directly imply it also is finite dimensional, see remark 3.3.3.
Further, we will require the maps G_1, G_2 to be pseudo-differential operators so that they have a well defined principal symbol that can then be used to calculate principal symbols of projection maps onto the boundary condition, and eventually the symbols of $Q_{B_2B_1}$ and $Q^*_{B_1^cB_2^c}$ as compositions of those operators.

Theorem 3.3.6. *Let B_1, B_2 be boundary conditions in graph form as defined in definition 2.3.6. Further assume that G_1 and G_2 are pseudo-differential operators of order zero where G_1 or G_2 is compact. Then $D_{B_1B_2}$ is Fredholm and* $\ker(D_{B_1B_2}) \subset C^\infty(M; S^+M)$.

Proof. Step 1: Assume that the boundary conditions are given by graphs $B_1 = \Gamma(G_1)$, $B_2 = \Gamma(G_2)$ for bounded linear maps

$$G_1 : L^2_-(\Sigma_1; S\Sigma_1) \to L^2_+(\Sigma_1; S\Sigma_1),$$

$$G_2 : L^2_+(\Sigma_2; S\Sigma_2) \to L^2_-(\Sigma_2; S\Sigma_2)$$

and that G_1 is compact. Then define the following projection maps onto the boundary conditons

$$P_{\Gamma(G_1)} := P_- + G_1 P_- : L^2(\Sigma_1; S\Sigma_1) \to \Gamma(G_1)$$

$$P_{\Gamma(G_2)} := P_+ + G_2 P_+ : L^2(\Sigma_2; S\Sigma_2) \to \Gamma(G_2)$$

where $P_\pm$ denote the APS projectors onto $L^2_\pm(\Sigma_{1/2}; S\Sigma_{1/2})$. The maps $P_{\Gamma(G_i)}$ are indeed projections, since $P^2_{\Gamma(G_1)} = (P_- + G_1 P_-)^2 = P_-^2 + P_- G_1 P_- + G_1 P_-^2 + G_1 P_- G_1 P_- = P_- + G_1 P_-$ where $P_- G_1 P_- = 0$ since G_1 maps $L^2_-(\Sigma_1; S\Sigma_1) \to L^2_+(\Sigma_1; S\Sigma_1)$ and similarly for $P_{\Gamma(G_2)}$. For the kernel we get:

$$\begin{aligned}\ker(P_{\Gamma(G_1)}) &= \{x \in L^2(\Sigma_1; S\Sigma_1) : P_- x = -G_1 P_- x\} \\ &= \{x \in L^2(\Sigma_1; S\Sigma_1) : P_- x = 0\} \\ &= L^2_+(\Sigma_1; S\Sigma_1)\end{aligned}$$

and similarly

$$\ker(P_{\Gamma(G_2)}) = L^2_-(\Sigma_2; S\Sigma_2).$$

Since $P_\pm$ and $G_{1/2}$ are pseudo-differential operators of order zero, the projections $P_{\Gamma(G_1)}, P_{\Gamma(G_2)}$ are also pseudo of order zero with principal symbols

$$p_{\Gamma(G_1)} = p_- + g_1 p_- = p_-$$

$$p_{\Gamma(G_2)} = p_+ + g_2 p_+$$

because G_1 is assumed to be compact and where $p_\pm$, $g_{1/2}$ denote the principal symbols of $P_\pm$ and $G_{1/2}$ respectively. Making use of lemma 3.1.3, we can calculate the principal symbol of the operator $Q_{B_2 B_1} = P_{\Gamma(G_2)} Q P_{\Gamma(G_1)}$ to get

$$q_{B_1 B_2} = p_{\Gamma(G_2)} q p_{\Gamma(G_1)} \cong (p_+ + g_2 p_+) p_\pm \beta \Gamma^\pm \beta p_- = (p_+ + g_2 p_+) \Gamma^\pm p_\pm p_-$$

where $p_+ p_- = 0$ for the first possible sign and $(p_+ + g_2 p_+) p_-$ vanishes for the second one. In conclusion we have $q_{B_2 B_1} = 0$ and hence $Q_{B_2 B_1}$ is compact. Now by the same reasoning as in proposition 3.3.4 follows $\ker(Q_{B_2^c B_1}) \subset C^\infty(\Sigma_1; S\Sigma_1)$ and lemma 3.3.1 implies that also $\ker(D_{B_1 B_2}) \subset C^\infty(M; S^+ M)$.
For the case where G_1 is the non-compact mapping while G_2 is compact, the same arguments can be applied to the operator $Q^*_{B_1 B_2}$ instead. The principal symbol of this operator then vanishes for the same reason as above and we still have $\ker(Q_{B_2^c B_1}) \subset C^\infty(\Sigma_1; S\Sigma_1)$. Note here that in proposition 3.3.4 it is sufficient for one of the operators $Q_{B_2 B_1}$ and $Q^*_{B_1 B_2}$ to improve regularity, since both always are of order zero at most.

Step 2: Suppose we have a splitting of the boundary spaces of the form

$$L^2(\Sigma_1; S\Sigma_1) = L^2_{(-\infty,a_1)}(\Sigma_1; S\Sigma_1) \oplus L^2_{[a_1,\infty)}(\Sigma_1; S\Sigma_1)$$

$$L^2(\Sigma_2; S\Sigma_2) = L^2_{(-\infty,a_2)}(\Sigma_2; S\Sigma_2) \oplus L^2_{[a_2,\infty)}(S\Sigma_2; S\Sigma_2)$$

where $a_1, a_2 \in \mathbb{R}$ and G_1, G_2 are maps

$$G_1 : L^2_{(-\infty,a_1)}(\Sigma_1; S\Sigma_1) \to L^2_{[a_1,\infty)}(\Sigma_1; S\Sigma_1)$$

$$G_2 : L^2_{(-\infty,a_2)}(\Sigma_2; S\Sigma_2) \to L^2_{[a_2,\infty)}(S\Sigma_2; S\Sigma_2).$$

The boundary conditions are still given as graphs of these maps, i.e. $B_1 = \Gamma(G_1) \subset L^2(\Sigma_1; S\Sigma_1)$ and $B_2 = \Gamma(G_2) \subset L^2(\Sigma_2; S\Sigma_2)$. Projection maps onto these boundary conditions are given by

$$P_{B_1} = P_{(-\infty,a_1)} + G_1 P_{(-\infty,a_1)}$$
$$P_{B_2} = P_{[a_2,\infty)} + G_2 P_{[a_2,\infty)}$$

Here for some interval $I \subset \mathbb{R}$, $P_I : L^2(\Sigma; S\Sigma) \to L^2_I(\Sigma; S\Sigma)$ denotes the orthogonal projection onto the corresponding spectral subspace.
Suppose that $a_1 \leq 0$ then define the map $\widetilde{G}_1 : L^2_-(\Sigma_1; S\Sigma_1) \to L^2_+(\Sigma_1; S\Sigma_1)$ by

$$\widetilde{G}_1 := P_+ G_1 P_{(-\infty,a_1)}$$

and define a boundary condition by $\widetilde{B}_1 = \Gamma(\widetilde{G}_1)$. As a projector onto $\widetilde{B}_1$, we can use $P_{\widetilde{B}_1} = P_- + \widetilde{G}_1 P_-$ similar to the projections used in Step 1. The difference

$$\begin{aligned}
P_{\widetilde{B}_1} - P_{B_1} &= P_- + \widetilde{G}_1 P_- - \left(P_{(-\infty,a_1)} + G_1 P_{(-\infty,a_1)}\right) \\
&= P_- - P_{(-\infty,a_1)} + \widetilde{G}_1 P_- - G_1 P_{(-\infty,a_1)} \\
&= P_{[a_1,0)} + P_+ G_1 P_{(-\infty,a_1)} - G_1 P_{(-\infty,a_1)} \\
&= P_{[a_1,0)} + P_+ G_1 P_{(-\infty,a_1)} - \left(P_+ + P_{[a_1,0)}\right) G_1 P_{(-\infty,a_1)} \\
&= P_{[a_1,0)} - P_{[a_1,0)} G_1
\end{aligned}$$

is then compact because $P_{[a_1,0)}$ is a compact operator and hence, the principal symbols $p_{B_1} = p_{\widetilde{B}_1}$ coincide.
Suppose that $a_1 > 0$, then we set

$$\begin{aligned}
\widetilde{G}_1 &:= G_1 P_- : L^2_-(\Sigma_1; S\Sigma_1) \to L^2_+(\Sigma_1; S\Sigma_1) \\
\widetilde{B}_1 &= \Gamma(\widetilde{G}_1) \\
P_{\widetilde{B}_1} &= P_- + G_1 P_-
\end{aligned}$$

and the difference

$$\begin{aligned}
P_{\widetilde{B}_1} - P_{B_1} &= P_- + \widetilde{G}_1 P_- - \left(P_{(-\infty,a_1)} + G_1 P_{(-\infty,a_1)}\right) \\
&= P_- - P_{(-\infty,a_1)} + \widetilde{G}_1 P_- - G_1 P_{(-\infty,a_1)}
\end{aligned}$$

$$= -P_{[0,a_1)} + G_1\left(P_- - P_{(-\infty,a_1)}\right)$$
$$= -P_{[0,a_1)} - G_1 P_{[0,a_1)}$$

is again a compact operator such that the principal symbols $p_{B_1} = p_{\widetilde{B}_1}$ coincide. The same arguments can be applied to the second boundary component to show that $p_{B_2} = p_{\widetilde{B}_2}$ for any $a_2 \in \mathbb{R}$.
Calculating the principal symbol of $Q_{B_2B_1}$, we get

$$q_{B_2B_1} = p_{B_2} q p_{B_1} = p_{\widetilde{B}_2} q p_{\widetilde{B}_1} = q_{\widetilde{B}_2\widetilde{B}_1} = 0$$

and consequently a smooth solution space $\ker(D_{B_1B_2}) \subset C^\infty(M; S^+M)$ by Step 1 of the proof.

Step 3: For the general case of boundary conditions in graph form, we have splittings of the boundary spaces

$$L^2(\Sigma_1; S\Sigma_1) = V_1^- \oplus W_1^- \oplus V_1^+ \oplus W_1^+$$
$$L^2(\Sigma_2; S\Sigma_2) = V_2^- \oplus W_2^- \oplus V_2^+ \oplus W_2^+$$

where $V_i^- \oplus W_i^- = L^2_{(-\infty,a_i)}(\Sigma_i; S\Sigma_i)$, $V_i^+ \oplus W_i^+ = L^2_{[a_i,\infty)}(\Sigma_i; S\Sigma_i)$ $(i = 1, 2)$ for some $a_1, a_2 \in \mathbb{R}$ and the $W_{1/2}^\pm$ are smooth and finite dimensional. The boundary conditions are given as

$$B_1 = \Gamma(G_1) \oplus W_1^+$$

$$B_2 = \Gamma(G_2) \oplus W_2^-$$

where

$$G_1 : V_1^- \to V_1^+$$
$$G_2 : V_2^+ \to V_2^-.$$

We define projection maps onto the graphs of G_1 and G_2 by

$$P_{\Gamma(G_1)} := P_{V_1^-} + G_1 P_{V_1^-}$$

$$P_{\Gamma(G_2)} := P_{V_2^+} + G_2 P_{V_2^+}$$

and projections onto the boundary conditions B_1 and B_2 by

$$P_{B_1} := P_{\Gamma(G_1)} + P_{W_1^+}$$

$$P_{B_2} := P_{\Gamma(G_2)} + P_{W_2^-}.$$

Similar to Step 2, further we can find maps $\widetilde{G}_1$ and $\widetilde{G}_2$ acting on the generalized APS boundary spaces by setting

$$\widetilde{G}_1 := G_1 P_{V_1^-} : L^2_{(-\infty,a_1)}(\Sigma_1; S\Sigma_1) \to L^2_{[a_1,\infty)}(\Sigma_1; S\Sigma_1)$$

$$\widetilde{G}_2 := G_2 P_{V_2^+} : L^2_{[a_2,\infty)}(\Sigma_2; S\Sigma_2) \to L^2_{(-\infty,a_2)}(\Sigma_2; S\Sigma_2)$$

and define corresponding projection maps

$$P_{\Gamma(\widetilde{G}_1)} := P_{(-\infty,a_1)} + \widetilde{G}_1 P_{(-\infty,a_1)}$$

$$P_{\Gamma(\widetilde{G}_2)} := P_{[a_2,\infty)} + \widetilde{G}_2 P_{[a_2,\infty)}.$$

Calculating the difference of these projections yields

$$\begin{aligned} P_{\Gamma(\widetilde{G}_1)} - P_{B_1} &= P_{(-\infty,a_1)} + \widetilde{G}_1 P_{(-\infty,a_1)} - P_{\Gamma(G_1)} - P_{W_1^+} \\ &= P_{(-\infty,a_1)} + \widetilde{G}_1 P_{(-\infty,a_1)} - P_{V_1^-} - G_1 P_{V_1^-} - P_{W_1^+} \\ &= P_{W_1^-} - P_{W_1^+} + G_1 P_{V_1^-} P_{(-\infty,a_1)} - G_1 P_{V_1^-} \\ &= P_{W_1^-} - P_{W_1^+} \end{aligned}$$

hence a compact operator, since $W^{\pm}_{1/2}$ are assumed to be smooth and finite dimensional. The analogous statement is true on the second boundary component, because

$$\begin{aligned} P_{\Gamma(\widetilde{G}_2)} - P_{B_2} &= P_{[a_2,\infty)} + \widetilde{G}_2 P_{[a_2,\infty)} - P_{\Gamma(G_2)} - P_{W_2^-} \\ &= P_{[a_2,\infty)} + \widetilde{G}_2 P_{[a_2,\infty)} - P_{V_2^+} - G_2 P_{V_2^+} - P_{W_2^-} \\ &= P_{W_2^+} - P_{W_2^-} + G_2 P_{V_2^+} P_{[a_2,\infty)} - G_2 P_{V_2^+} \\ &= P_{W_2^+} - P_{W_2^-} \end{aligned}$$

which is again a compact operator. In conclusion, we have that on the principal symbol level the projections $p_{B_1} = p_{\Gamma(\widetilde{G}_1)}$, $p_{B_2} = p_{\Gamma(\widetilde{G}_2)}$ and the maps $\widetilde{G}_1$, $\widetilde{G}_2$ are of the same type already discussed during Step 2 of the proof. Combining all the steps, then yields $\ker(D_{B_1B_2}) \subset C^\infty(M; S^+M)$ for the general case of graph type boundary conditions. □

Theorem 3.3.6 makes use of the fact that the mapping property $Q_{B_2B_1} : H^k(\Sigma_1; S\Sigma_1) \to H^{k+1}(\Sigma_2; S\Sigma_2)$ implies smoothness of the $\ker(Q_{B_2^cB_1})$ and consequently, also of $\ker(D_{B_1B_2})$ as stated in lemma 3.3.1 and proposition 3.3.4. Note that while regularity for $\ker(Q_{B_2^cB_1})$ and $\ker(D_{B_1B_2})$ are always equivalent (for any choice of complementary subspaces B_1^c, B_2^c) by lemma 3.3.1, the inverse implication on the mapping properties of $Q_{B_2B_1}$ (i.e. compactness of this operator) is not true in general, but rather depends on the choice of complements. This can be easily seen by looking at the ultrastatic case (compare example 2.1.9):

Let $M = [0,1] \times S^1$ and $g = -\mathrm{d}t^2 + h$ a metric such that $\mathrm{vol}(S^1) = 1$ and $Q = \mathrm{id}$. Assume we have a pseudo-differential operator mapping

$$G_2 : L^2_{[0,\infty)}(S^1; SS^1) \to L^2_{(-\infty,0)}(S^1; SS^1)$$

and define the boundary conditions $B_1 = L^2_{(-\infty,0)}(S^1; SS^1)$, $B_2 = \Gamma(G_2)$. By theorem 2.3.9 we know that $D_{B_1B_2}$ is a Fredholm operator (here $G_1 \equiv 0$ is compact) and by theorem 3.3.6 we have that $\ker(Q_{B_2^cB_1})$ is smooth (for any choice of complementary subspaces B_1^c, B_2^c) and consequently $\ker(D_{B_1B_2}) \subset C^\infty(M; S^+M)$. This is because for $B_1^c = L^2_{[0,\infty)}(S^1; SS^1)$ and

$B_2^c = L^2_{(-\infty,0)}(S^1;SS^1)$ the proof of theorem 3.3.6 shows that the operator $Q_{B_2B_1}$ is compact mapping $H^k(S^1;SS^1) \to H^{k+1}(S^1;SS^1)$.
Alternatively, we could also choose $B_2^c = \Gamma(G_2)^\perp = \Gamma(-G_2^*)$, of course without changing any results on Fredholm property or regularity of the Dirac operator $D_{B_1B_2}$. However trying to apply the methods used to proof theorem 3.3.6, requires calculating the principal symbol of $Q_{B_2B_1} = P_{\Gamma(G_2)}QP_- = P_{\Gamma(G_2)}P_-$, where the orthogonal projection onto $\Gamma(G_2)$ is given by remark 2.3.7:

$$\begin{aligned} P_{\Gamma(G_2)}P_- &= (\mathrm{id} + G_2^*G_2)^{-1}P_+ + (\mathrm{id} + G_2^*G_2)^{-1}G_2^*P_- \\ &= (\mathrm{id} + G_2^*G_2)^{-1}\left(P_+ + G_2^*P_-\right). \end{aligned}$$

Since $(\mathrm{id} + G_2^*G_2)^{-1}$ is an isomorphism, the operator is compact if and only if $\left(P_+ + G_2^*P_-\right)$ is compact. Hence we would have to show that

$$\left(p_+ + g_2^*p_-\right) = 0$$

for the principal symbols of $P_\pm$ and G_2, which would imply that $g_2^* = g_2 = 0$ making G_2 itself a compact operator.

3.4 (Pseudo-)Local Boundary Conditions

The methods used in section 3.3.2 can in principle also be applied to the general case of pseudo-local boundary conditions, if one can show compactness for the corresponding off-diagonal terms in the splitting of the wave evolution operator. In this section we want to derive a principal symbol equation that guarantees Fredholm property and smoothness of solutions for the corresponding Dirac operator and look at examples of pseudo-local boundary conditions satisfying this equation.

3.4.1 Pseudo-Local Boundary Conditions

Definition 3.4.1. We call a closed linear subspace $B_i \subset L^2(\Sigma_i;S\Sigma_i)$ a (pseudo-)local boundary condition if there is a projection $P_B : L^2(\Sigma_i;S\Sigma_i) \to B$ that is a (pseudo-)differential operator of order zero.

Having a (pseudo-)local boundary condition, or more precisely a (pseudo-)local projector onto the boundary condition, allows using the principal symbol calculation (compare section 3.3.2) for the operators $Q_{B_2B_1}$, $Q^*_{B_1B_2}$ to analyze Fredholm property and regularity for the corresponding Dirac operator. The following proposition gives a sufficient condition for a general pseudo-local boundary condition to imply those properties for the Dirac operator.

Proposition 3.4.2. *Let* (B_1, B_2) *be pseudo-local boundary conditons for* D *with order zero pseudo-differential projection maps* $P_{B_i} : L^2(\Sigma_i;S\Sigma_i) \to B_i$. *Then the Dirac operator* $D_{B_1B_2}$ *is Fredholm and* $\ker(D_{B_1B_2}) \subset C^\infty(M;S^+M)$, *provided that*

$$p_{B_2}(\xi^\pm)p_\pm(\xi^\pm)\beta\,\Gamma^\pm_{(y,\eta,x^\pm,\xi^\pm)}\beta p_{B_1}(\eta) = p_{B_2}(\xi^\pm)\Gamma^\pm_{(y,\eta,x^\pm,\xi^\pm)}p_\pm(\eta)p_{B_1}(\eta) = 0$$

for all $((y,\eta),(x^\pm,\xi^\pm)) \in C_Q$. *Here* p_{B_1}, p_{B_2} *denote the principal symbols of* P_{B_1} *and* P_{B_2} *respectively.*

Proof. By proposition 2.1.10 and proposition 3.3.4, it is sufficient to show that the operator $Q_{B_2B_1}$ has vanishing principal symbol, i.e. maps $H^k(\Sigma_1; S\Sigma_1) \to H^{k+1}(\Sigma_2, S\Sigma_2)$ and is hence compact. Since the principal symbol of $Q_{B_2B_1} = P_{B_2}QP_{B_1}$ is given by composition of symbols we can use Lemma 3.1.3 to get

$$q_{B_2B_1} = p_{B_2}qp_{B_1} = p_{B_2}(\xi^\pm)p_\pm(\xi^\pm)\beta\Gamma^\pm_{(y,\eta,x^\pm,\xi^\pm)}\beta p_{B_1}(\eta) = p_{B_2}(\xi^\pm)\Gamma^\pm_{(y,\eta,x^\pm,\xi^\pm)}p_\pm(\eta)p_{B_1}(\eta).$$

□

Even though proposition 3.4.2 in principle provides a way to check for Fredholm property and regularity of $D_{B_1B_2}$ given a pseudo-local boundary condition, it might be hard to do so in practise. This is due to the appearance of the parallel transport $\Gamma^\pm_{(y,\eta,x^\pm,\xi^\pm)}$ connecting (y,η) and $(x^\pm,\xi^\pm)$ via a lightlike geodesic and is required to analyze the principal symbol and the canonical relation C_Q itself. While specific assumptions on the geometry of M could help dealing with this parallel transport, its mapping properties are not obvious for a general geometry.
The proof of the statement that D_{APS} is a Fredholm operator (see theorem 3.2.1) relies on the appearance of $p_\pm$ in the principal symbol of the wave evolution operator Q (see section 3.1), where $p_\pm$ denote the principal symbols of the APS projectors $P_\pm$. There, the fact that $P_+P_- = P_-P_+ = 0$ and consequently $p_+p_- = p_-p_+ = 0$ yields compactness for the off-diagonal terms in the splitting of Q. Actually the statement for the principal symbols alone would be sufficient for the proof to go through without having $P_+P_- = P_-P_+ = 0$ on the L^2-projection level.

Corollary 3.4.3. *Let* (B_1, B_2) *be (pseudo-)local boundary conditions for* D*, then the Dirac operator* $D_{B_1B_2}$ *is Fredholm and* $\ker(D_{B_1B_2}) \subset C^\infty(M; S^+M)$ *provided that*

1. $p_+p_{B_1} = 0$ *and* $p_{B_2}p_- = 0$*, i.e.* $P_+P_{B_1}$ *and* $P_{B_2}P_-$ *are compact, or*
2. $p_-p_{B_1} = 0$ *and* $p_{B_2}p_+ = 0$*, i.e.* $P_-P_{B_1}$ *and* $P_{B_2}P_+$ *are compact.*

Proof. Since the principal symbol combinations $p_+p_{B_1}$ and $p_{B_2}p_-$ or $p_-p_{B_1}$ and $p_{B_2}p_+$ vanish by assumption, we get for the principal symbol in the off-diagonal term of Q

$$\underbrace{p_{B_2}(\xi^+)p_+(\xi^+)}_{=0}\beta\Gamma^+_{(y,\eta,x^+,\xi^+)}\beta p_{B_1}(\eta) = 0 \quad\text{and}\quad p_{B_2}(\xi^-)\Gamma^-_{(y,\eta,x^-,\xi^-)}\underbrace{p_-(\eta)p_{B_1}(\eta)}_{=0} = 0$$

or

$$\underbrace{p_{B_2}(\xi^-)p_-(\xi^-)}_{=0}\beta\Gamma^-_{(y,\eta,x^-,\xi^-)}\beta p_{B_1}(\eta) = 0 \quad\text{and}\quad p_{B_2}(\xi^+)\Gamma^+_{(y,\eta,x^+,\xi^+)}\underbrace{p_+(\eta)p_{B_1}(\eta)}_{=0} = 0.$$

The claim then follows from proposition 3.4.2. □

Corollary 3.4.3 basically states that (pseudo-)local boundary conditions still yield a Fredholm operator with a smooth solution space as long as these boundary conditions are "close" to the APS (or aAPS) boundary conditions, in the sense that $P_+P_{B_1}$ and $P_{B_2}P_-$ are compact (close to APS conditions) or $P_-P_{B_1}$ and $P_{B_2}P_+$ are compact (close to aAPS conditions). Note that choosing (pseudo-)local boundary conditions (B_1, B_2) in a way such that $P_+P_{B_1}$ and $P_{B_2}P_-$ are compact, implies that $(B_1, L^2_+(\Sigma_1; S\Sigma_1))$ and $(B_2, L^2_-(\Sigma_2; S\Sigma_2))$ might not be complements anymore but are still Fredholm pairs. Compare section 2.2, where Fredholm pairs were introduced as being "close" to direct sum splitting of a Hilbert space. However, it is certainly not enough for a boundary condition (B_1, B_2) that $(B_1, L^2_+(\Sigma_1; S\Sigma_1))$ and $(B_2, L^2_-(\Sigma_2; S\Sigma_2))$ are Fredholm pairs for the Dirac operator $D_{B_1B_2}$ to be Fredholm. This can also be seen by looking at the ultra static example discussed at the end of the previous section:

If we have $M = [t_1, t_2] \times S^1$, $g = -\mathrm{dt}^2 + h$ such that $Q = \mathrm{id}$ and any isomorphism

$$G : L^2_{(-\infty,0)}(S^1; SS^1) \to L^2_{[0,\infty)}(S^1; SS^1)$$

we can define boundary conditions

$$B_1 := \Gamma(G) \quad \text{and} \quad B_2 := \Gamma(G^{-1}).$$

Then we have splittings $L^2(S^1; SS^1) = \Gamma(G_1) \oplus L^2_{[0,\infty)}(S^1; SS^1) = L^2_{(-\infty,0)}(S^1; SS^1) \oplus \Gamma(G^{-1})$, and by remark 2.2.2 $(B_1, L^2_+(\Sigma_1; S\Sigma_1))$ and $(B_2, L^2_-(\Sigma_2; S\Sigma_2))$ are certainly Fredholm pairs. On the other hand, $D_{B_1B_2}$ cannot be a Fredholm operator, since by corollary 2.1.6 we have $\ker(D_{B_1B_2}) \cong Q(B_1) \cap B_2 = \Gamma(G)$, which is infinite dimensional.

While this example shows that being a Fredholm pair with APS conditions is not "close" enough for corollary 3.4.3 to hold, boundary conditions satisfying the assumptions can be found by changing the Dirac operators A_1, A_2 of the boundary components Σ_1 and Σ_2 to any operators adapted to D in the sense of [12] section 2.2 (see also [13]).

Corollary 3.4.4. *Let* $M = [t_1, t_2] \times \Sigma$, $g = -N^2\mathrm{dt} + g_t$ *be our usual setup of a globally hyperbolic spin manifold with compact spacelike Cauchy hypersurfaces* $\Sigma_t = \{t\} \times \Sigma$ *with boundary* $\Sigma_1 \dot\cup \Sigma_2$.
By A_1, A_2, *we denote the Dirac operators* $A_i : C^\infty(\Sigma_i; S\Sigma_i) \to C^\infty(\Sigma_i; S\Sigma_i)$ *of* Σ_1 *and* Σ_2 *respectively.*
For any hermitian maps $h_i \in End(S\Sigma_i)$ *we define operators*

$$A_i(h_i) := A_i + h_i$$

and boundary conditions

$$(B_1^\pm(h_1), B_2^\mp(h_2)) := (\chi_\pm(A_1(h_1)), \chi_\mp(A_2(h_2))).$$

Then the Dirac operator $D_{B_1^\pm(h_1)B_2^\mp(h_2)}$ *is Fredholm with*

$$\ker(D_{B_1^\pm(h_1)B_2^\mp(h_2)}) \subset C^\infty(M; S^+M) \quad \textit{and} \quad \mathrm{ind}\left(D_{B_1^-(h_1)B_2^+(h_2)}\right) = -\mathrm{ind}\left(D_{B_1^+(h_1)B_2^-(h_2)}\right).$$

Proof. Note that the Dirac operators A_i are first order elliptic, self-adjoint operators on compact Riemannian manifolds Σ_i. By adding a hermitian map $h_i \in \mathrm{End}(S\Sigma_i)$, the resulting operator $A_i(h_i)$ remains elliptic and self-adjoint. This guarantees that $\mathrm{spec}(A_i(h_i)) \subset \mathbb{R}$ is real and discrete and $L^2(\Sigma_i; S\Sigma_i)$ splits as a direct sum of eigenspaces for $A_i(h_i)$ (compare 1.2.4). Orthogonal projections onto the boundary conditions are given by

$$P_{\chi^\pm(A_i(h_i))} = \frac{1}{2}\left(1 \pm \frac{A_i(h_i)}{|A_i(h_i)|}\right)$$

and since the operators $A_i(h_i)$ and A_i only differ by a zero order term, their principal symbols $\sigma_{A_i(h_i)} = \sigma_{A_i}$ conincide. Hence, the principal symbols of the projection maps

$$p_{\chi^\pm(A_i(h_i))} = p_\pm$$

are the same as those of the APS projectors $P_\pm$. Then calculating the principal symbol of $Q_{B_2B_1}$ yields the same result as for APS boundary conditions

$$q_{B_2B_1} = q_{\pm\mp} = 0,$$

the same is true for $Q_{B_2^\perp B_1^\perp}$. Hence, the operators $Q_{B_2B_1}$, $Q_{B_2^\perp B_1^\perp}$ are compact mapping $H^k(\Sigma_1; S\Sigma_1) \to H^{k+1}(\Sigma_2; S\Sigma_2)$ and the statement follows from proposition 2.1.10 and proposition 3.3.4. □

Remark 3.4.5. The Definition of boundary conditions $(B_1^\pm(h_1), B_2^\mp(h_2)) := (\chi_\pm(A_1(h_1)), \chi_\mp(A_2(h_2)))$ in corollary 3.4.4 could as well have been made with characteristic functions for $\chi_{(-\infty,a_i)}, \chi_{[a_i,\infty)}$ for intervals $(-\infty, a_i), [a_i, \infty) \subset \mathbb{R}$ and $a_i \in \mathbb{R}$, similar to the definition of gAPS boundary conditions as generalizations of APS conditions (see section 2.3.2).
This shift in the "cut" of the spectrum is already included in the choice of maps h_i, meaning that

$$\chi_{(-\infty,a_i)}(A_i(h_i)) = \chi_-(A_i(h_i - a_i))$$
$$\chi_{[a_i,\infty)}(A_i(h_i)) = \chi_+(A_i(h_i - a_i)).$$

3.4.2 Local Boundary Conditions

Local boundary conditions were already defined as a subclass of pseudo-local boundary conditions in definition 3.4.1, an equivalent definition can be given as follows:

Definition 3.4.6. Let $\mathcal{B}_1 \subset S\Sigma_1$ and $\mathcal{B}_2 \subset S\Sigma_1$ be smooth subbundles, then we call $(B_1, B_2) := (L^2(\Sigma_1; \mathcal{B}_1), L^2(\Sigma_2; \mathcal{B}_2))$ local boundary conditions for D.

There is a connection between local boundary conditions for the Dirac operator as defined above, and boundary conditions in graph form discussed in section 3.3.2. The following Lemma gives sufficient conditions for a local boundary condition to be expressable as a graph over APS boundary conditions, see also [12] Theorem 7.20 and [11] Proposition 4.9.

Lemma 3.4.7. *Let $\mathcal{B}_1 \subset \Sigma_1$, $\mathcal{B}_2 \subset \Sigma_2$ be smooth subbundles and $(B_1, B_2) = (L^2(\Sigma_1; \mathcal{B}_1), L^2(\Sigma_2, \mathcal{B}_2))$ the corresponding local boundary condition. Then the following statements are equivalent:*

(i) *(B_1, B_2) can be written as boundary conditions in graph form, i.e. there exist L^2-orthogonal splittings*

$$L^2(\Sigma_i; S\Sigma_i) = V_i^- \oplus W_i^- \oplus V_i^+ \oplus W_i^+ \quad (i = 1, 2)$$

as in definition 2.3.6 and bounded linear maps $G_1 = V_1^- \to V_1^+$ and $G_2 : V_2^+ \to V_2^-$ such that $B_1 = W_1^+ \oplus \Gamma(G_1)$ and $B_2 = W_2^- \oplus \Gamma(G_2)$.

(ii) *For $i = 1, 2$, for every $x \in \Sigma_i$ and $\xi \in T_x\Sigma_i$, $\xi \neq 0$, the fiberwise orthogonal projection map $P_{(\mathcal{B}_i)_x} : (S\Sigma_i)_x \to (\mathcal{B}_i)_x$ restricts to an isomorphism from the eigenspace to the negative ($i = 1$) or positive ($i = 2$) eigenvalue of $i\sigma_{A_i}(\xi)$ onto $(\mathcal{B}_i)_x$.*

Proof. First note that the fiberwise projections $P_{(\mathcal{B}_i)_x}$ mentioned above extend to orthogonal projection maps

$$P_{\mathcal{B}_i} : L^2(\Sigma_i; S\Sigma_i) \to L^2(\Sigma_i; \mathcal{B}_i)$$

and by ([12] Thm 7.20) statement (ii) is then equivalent to the operators

$$P_{\mathcal{B}_1} - P_{[a,\infty)}(\Sigma_1) : L^2(\Sigma_1; S\Sigma_1) \to L^2(\Sigma_1; S\Sigma_1)$$

$$P_{\mathcal{B}_2} - P_{(-\infty,a)}(\Sigma_2) : L^2(\Sigma_2; S\Sigma_2) \to L^2(\Sigma_2; S\Sigma_2)$$

being Fredholm for some and then any $a \in \mathbb{R}$. Here $P_{[a,\infty)}(\Sigma_1)$ and $P_{(-\infty,a)}(\Sigma_2)$ denote the orthogonal projections onto $L^2_{[a.\infty)}(\Sigma_1; S\Sigma_1)$ and $L^2_{(-\infty,a)}(\Sigma_2; S\Sigma_2)$.

To show that (i) implies (ii) note that given statement (i) is true, we have that $B_1 = W_1^+ \oplus \Gamma(G_1) = L^2(\Sigma_1; \mathcal{B}_1)$ and hence, $P_{B_1} = P_{\mathcal{B}_1}$ for the orthogonal projection maps. First assume that $W_1^- = W_1^+ = \{0\}$ and $V_1^- = L^2_{(-\infty,a]}(\Sigma_1; S\Sigma_1)$ for some $a \in \mathbb{R}$. By remark 2.3.7, with respect to the splitting $L^2(\Sigma_1; S\Sigma_1) = V_1^- \oplus V_1^+$, the projection maps $P_{B_1} = P_{\Gamma(G_1)}$ and $P_{V_1^+}$ are then given by

$$P_{V_1^+} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix} \quad \text{and} \quad P_{B_1} = \begin{pmatrix} \mathrm{id} & 0 \\ G & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & -G^* \\ G & \mathrm{id} \end{pmatrix}^{-1}$$

and we get

$$\begin{aligned} P_{B_1} - P_{V_1^+} &= \left[\begin{pmatrix} \mathrm{id} & 0 \\ G & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ G_1 & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & -G_1^* \\ G & \mathrm{id} \end{pmatrix} \right] \begin{pmatrix} \mathrm{id} & -G_1^* \\ G_1 & \mathrm{id} \end{pmatrix}^{-1} \\ &= \left[\begin{pmatrix} \mathrm{id} & 0 \\ G_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ G_1 & \mathrm{id} \end{pmatrix} \right] \begin{pmatrix} \mathrm{id} & -G_1^* \\ G_1 & \mathrm{id} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathrm{id} & 0 \\ 0 & -\mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & -G_1^* \\ G_1 & \mathrm{id} \end{pmatrix}^{-1} \end{aligned}$$

which is an isomorphism and in particular a Fredholm operator.
Now for the general case of a graph type boundary condition, we have an orthogonal splitting

$$L^2(\Sigma_1; S\Sigma_1) = V_1^- \oplus W_1^- \oplus V_1^+ \oplus W_1^+$$

where $W_1^\pm$ are finite dimensional and the deformation map G_1 maps

$$G_1 : V_1^- \to V_1^+.$$

Again, by remark 2.3.7 we have that

$$V_1^+ \oplus V_1^+ = \Gamma(G_1) \oplus \Gamma(G_1)^\perp = \Gamma(G_1) \oplus \Gamma(-G_1^*)$$

where all the splittings are still orthogonal. Hence, the orthogonal projection on a boundary condition $B_1 = \Gamma(G_1) \oplus W_1^+$ can be written as

$$P_{B_1} = P_{\Gamma(G_1)} + P_{W_1^+}.$$

Further we know by definition 2.3.6 that $L^2_{[a,\infty)}(\Sigma_1; S\Sigma_1) = V_1^+ \oplus W_1^+$ and hence

$$P_{[a,\infty)}(\Sigma_1) = P_{V_1^+} + P_{W_1^+}$$

and the difference $P_{B_1} - P_{[a,\infty)}(\Sigma_1)$ is then given by

$$P_{B_1} - P_{[a,\infty)}(\Sigma_1) = P_{\Gamma(G_1)} - P_{V_1^+}.$$

By the calculation above this restricts to an isomorphism $V_1^- \oplus V_1^+ \to V_1^- \oplus V_1^+$ and since we have $L^2(\Sigma_1; S\Sigma_1) = V_1^- \oplus W_1^- \oplus V_1^+ \oplus W_1^+$ where $W_1^\pm$ are finite dimensional, it is still a Fredholm operator mapping $L^2(\Sigma_1; S\Sigma_1) \to L^2(\Sigma_1; S\Sigma_1)$.
Now to prove that (ii) implies (i), we start with a local boundary condition $\mathcal{B}_1 \subset S\Sigma_1$ on Σ_1 and construct a decomposition of $L^2(\Sigma_1; S\Sigma_1)$ and a deformation map G_1 as in definition 2.3.6. Since (ii) is assumed to be satisfied, we have a Fredholm operator $P_{B_1} - P_+$, where $B_1 = L^2(\Sigma_1; \mathcal{B}_1)$ and P_+ denotes the APS projector onto the non-negative part of the spectrum.
Now set

$$W_1^+ := B_1 \cap L^2_{[0,\infty)}(\Sigma_1; S\Sigma_1) \qquad V_1^+ := (W_1^+)^\perp \cap L^2_{[0,\infty)}(\Sigma_1; S\Sigma_1)$$
$$V_1^- := P_-(B_1) \qquad W_1^- := (V_1^-)^\perp \cap L^2_{(-\infty,0)}(\Sigma_1; S\Sigma_1)$$

By construction, we then have $V_1^- \oplus W_1^- = L^2_{(-\infty,0)}(\Sigma_1; S\Sigma_1)$ and $V_1^+ \oplus W_1^+ = L^2_{[0,\infty)}(\Sigma_1; S\Sigma_1)$.
To show that $W_1^\pm$ are finite dimensional for $w \in W_1^+$, we have

$$(P_{B_1} - P_+)(w) = w - w = 0$$

by definition of W_1^+, hence $W_1^+ \subset \ker(P_{B_1} - P_+)$ which is finite dimensional since $P_{B_1} - P_+$ is Fredholm. For $w \in W_1^-$ and $x \in L^2(\Sigma_1; S\Sigma_1)$ we calculate

$$\langle w, (P_{B_1} - P_+)(x)\rangle_{L^2} = \langle w, \underbrace{P_{B_1}(x)}_{\in V_1^-}\rangle_{L^2} - \langle w, \underbrace{P_+(x)}_{\in L^2_{[0,\infty)}(\Sigma_1; S\Sigma_1)}\rangle_{L^2} = 0$$

where we used that by construction $W_1^- \subset (V_1^-)^\perp$ as well as $W_1^- \subset L^2_{(-\infty,0)}(\Sigma_1; S\Sigma_1)$. This shows that W_1^- is contained in $\mathrm{ran}(P_{B_1} - P_+)^\perp$ which is again finite dimensional since $P_{B_1} - P_+$ is a Fredholm operator.
To construct the deformation map G_1, we define another subspace $U := (W_1^+)^\perp \cap B_1$, then we get $B_1 = W_1^+ \oplus ((W_1^+)^\perp \cap B_1)$ since $W_1^+ \subset B_1$ by definition and consequently $B_1 = W_1^+ \oplus U$. The map $P_-|_{B_1} : B_1 \to V_1^-$ is surjective by construction and has kernel

$$\begin{aligned}\ker(P_-|_{B_1}) &= \{x \in B_1 :\ P_- x = 0\}\\ &= \{x \in B_1 :\ x \in L^2_{[0,\infty)}(\Sigma_1; S\Sigma_1)\}\\ &= B_1 \cap L^2_{[0,\infty)}(\Sigma_1; S\Sigma_1)\\ &= W_1^+\end{aligned}$$

hence, $P_-|_U : U \to V_1^-$ is an isomorphism. Further, we have that $(W_1^+)^\perp = V_1^+ \oplus L^2_{(-\infty,0)}(\Sigma_1; S\Sigma_1)$. Because $V_1^+ \subset \mathrm{ran}(P_+)$ and $L^2_{(-\infty,0)}(\Sigma_1; S\Sigma_1) \subset \ker(P_+)$, it follows that P_+ maps $(W_1^+)^\perp \to V_1^+$ and since $U \subset (W_1^+)^\perp$, it also maps $U \to V_1^+$. The composition

$$G_1 : V_1^- \xrightarrow{(P_-|_U)^{-1}} U \xrightarrow{P_+} V_1^+$$

is then a bounded linear map. Since $B_1 = W_1^+ \oplus U$ as mentioned above, all that is left to show is that $U = \Gamma(G_1)$ to conclude the proof. This can be seen as follows:

$$\begin{aligned}\Gamma(G_1) &= \{x + G_1 x \mid x \in V_1^-\}\\ &= \{P_- u + G_1 P_- u \mid u \in U\}\\ &= \{P_- u + P_+(P_-|_U)^{-1} P_- u \mid u \in U\\ &= \{(P_- + P_+)u \mid u \in U\}\\ &= U\end{aligned}$$

A boundary condition B_2 on Σ_2 can be treated similarly by interchanging $V_1^\pm$ for $V_2^\mp$, $W_1^\mp$ for $W_2^\pm$ and $P_\mp$ for $P_\pm$. □

Since Lemma 3.4.7 shows how local boundary conditions on Σ_1 and/or Σ_2 can be related to graph type boundary conditons discussed in 3.3.2, we want to look at an example of a local boundary condition satisfying the assumptions of the Lemma.

Example 3.4.8. (Chirality Conditions) Let Σ be a closed Riemannian spin manifold with Dirac operator $A : \Gamma S\Sigma \to \Gamma S\Sigma$ and suppose we have a smooth unitary involution $h \in \Gamma\mathrm{End}(S\Sigma)$ sucht that

$$h \circ \sigma_A(\xi) = -\sigma_A(\xi) \circ h \quad \forall \xi \in T^*\Sigma.$$

where σ_A denotes the principal symbol of A. Since h is unitary, it is diagonalizable and because $h^2 = \mathrm{id}$ we know that it has eigenvalues ± 1 splitting $S\Sigma = E_{-1}(h) \oplus E_1(h)$ into an orthogonal direct sum of the corresponding subbundles.
Now let $\xi \in \dot{T}_x\Sigma$ for $x \in \Sigma$, then $(i\sigma_A(\xi))^2 = \|\xi\|^2$, $(i\sigma_A(\xi))^* = \pm i\sigma_A(\xi)$ and hence

$S_x\Sigma = E_{-1}(h)_x \oplus E_1(h)_x = E_{-\|\xi\|}(i\sigma_A(\xi)) \oplus E_{\|\xi\|}(i\sigma_A(\xi))$ splits into the direct sum of the $\pm\|\xi\|$ eigenspaces of $i\sigma_A(\xi)$. Since $i\sigma_A(\xi)$ and h anti-commute, we see that $\frac{i\sigma_A(\xi)}{\|\xi\|}$ restricts to a map $E_{\pm 1}(h)_x \to E_{\mp 1}(h)_x$ and because $\left(\frac{i\sigma_A(\xi)}{\|\xi\|}\right)^2 = \mathrm{id}$ we have that $E_{-1}(h)_x \cong E_1(h)_x$ are actually isomorphic. In the same way it follows that $E_{-\|\xi\|}(i\sigma_A(\xi)) \cong E_{\|\xi\|}(i\sigma_A(\xi))$ and hence all of these eigenspaces are of the same dimension, namely half the dimension of $S\Sigma_x$.
The fiberwise orthogonal projections onto the eigenspaces $E_{\pm 1}(h)_x$ are given by $\pi_\pm(x) = \frac{1}{2}(1 \pm h_x)$ and for $\phi \in E_{-\|\xi\|}(i\sigma_A(\xi))$ we calculate:

$$\begin{aligned}\frac{i\sigma_A(\xi)}{\|\xi\|}\pi_\pm(x)\phi &= \frac{i\sigma_A(\xi)}{\|\xi\|}\frac{1}{2}(1 \pm h_x)\phi \\ &= \frac{1}{2}(1 \mp h_x)\frac{i\sigma_A(\xi)}{\|\xi\|}\phi \\ &= -\pi_\mp(x)\phi\end{aligned}$$

hence

$$\pi_\pm(x)\phi = 0 \Leftrightarrow \pi_-(x)\phi = 0 \wedge \pi_+(x)\phi = 0 \Leftrightarrow \phi = 0.$$

Thus, the restrictions $\pi_\pm(x)|_{E_{-\|\xi\|}(i\sigma_A(\xi))}$ are injective and for dimensional reasons they are then isomorphisms $\pi_\pm(x)|_{E_{-\|\xi\|}(i\sigma_A(\xi))} : E_{-\|\xi\|}(i\sigma_A(\xi)) \to E_{\pm 1}(h)_x$. By the same calculations one can show that the projections $\pi_\pm(x)$ also yield isomorphisms when restricted to $E_{\|\xi\|}(i\sigma_A(\xi))$.
So in conclusion, we have shown that the local boundary conditions $\mathcal{B}^- := L^2(\Sigma; E_{-1}(h))$ and $\mathcal{B}^+ := L^2(\Sigma; E_1(h))$ satisfy condition (ii) from lemma 3.4.7 and can be considered as boundary conditions in graph form as in definition 2.3.6.

Remark 3.4.9. Condition (ii) in Lemma 3.4.7 actually reads differently for each of the boundary components Σ_1 and Σ_2. This is due to considering graph-type boundary conditions as deformations of APS boundary conditions, i.e. B_1 as a graph over $L^2_{(-\infty,0)}(\Sigma_1; S\Sigma_1)$ and B_2 as a graph over $L^2_{[0,\infty)}(\Sigma_2; S\Sigma_2)$. Of course, one could equally well consider graph-type deformations of aAPS boundary conditions by interchanging the roles of Σ_1 and Σ_2 in 3.4.7 (ii), as proposition 2.3.8 already suggests.
However, the example above shows that for chirality conditions defined by a unitary involution $h_i \in \mathrm{End}(S\Sigma_i)$ anti-commuting with σ_{A_i} both $\mathcal{B}_i^-$ and $\mathcal{B}_i^+$ can be considered as graph-type deformations of $L^2_{(-\infty,0)}(\Sigma_i; S\Sigma_i)$ and $L^2_{[0,\infty)}(\Sigma_i; S\Sigma_i)$ simultaneously. We will come back to this fact and how it affects Fredholm property of the resulting Dirac operator when discussing an example for such a unitary involution map in chapter 4.

To end this chapter, we will deduce Fredholm property and regularity of the solution space of the resulting Dirac operator for local graph-type boundary conditions by applying the results of theorems 2.3.9 and 3.3.6.

Corollary 3.4.10. *Let $\mathcal{B}_1 \subset S\Sigma_1$ and $\mathcal{B}_2 \subset S\Sigma_2$ be smooth subbundles and $B_1 = L^2(\Sigma_1; \mathcal{B}_1)$, $B_2 = L^2(\Sigma_2; \mathcal{B}_2)$ the corresponding local boundary conditions. Further assume that condition (ii) from lemma 3.4.7 is satisfied for both $\mathcal{B}_1$ and $\mathcal{B}_2$, we write*

$$APS_i^- := L^2_{(-\infty,0)}(\Sigma_i; S\Sigma_i)$$

and

$$APS_i^+ := L^2_{[0,\infty)}(\Sigma_i; S\Sigma_i)$$

for the APS boundary subspaces. Then the Dirac operators $D_{B_1 APS_2^+}$ *and* $D_{APS_1^- B_2}$ *are Fredholm with index*

$$\mathrm{ind}(D_{B_1 APS_2^+}) = \mathrm{ind}(D_{APS}) + dim(W_1^+) - dim(W_1^-)$$

$$\mathrm{ind}(D_{APS_1^- B_2}) = \mathrm{ind}(D_{APS}) + dim(W_2^-) - dim(W_2^+).$$

Where the correction terms are given by

$$W_1^+ := B_1 \cap APS_1^+ \qquad W_1^- := (P_-(B_1))^\perp \cap APS_1^-$$

$$W_2^- := B_2 \cap APS_2^- \qquad W_2^+ := (P_+(B_2))^\perp \cap APS_2^+.$$

Proof. Since condition (ii) from Lemma 3.4.7 is satisfied for subbundles $\mathcal{B}_1, \mathcal{B}_2$ by assumption, we can apply the Lemma to get splittings

$$L^2(\Sigma_i; S\Sigma_i) = V_i^- \oplus W_i^- \oplus V_i^+ \oplus W_i^+$$

where

$$V_i^- \oplus W_i^- = \mathrm{APS}_i^- \quad \text{and} \quad V_i^+ \oplus W_i^+ = \mathrm{APS}_i^+$$

together with bounded linear maps

$$G_1 : V_1^- \to V_1^+ \quad \text{and} \quad G_2 : V_2^+ \to V_2^-$$

such that

$$B_1 = \Gamma(G_1) \oplus W_1^+ \quad \text{and} \quad B_2 = \Gamma(G_2) \oplus W_2^-$$

are boundary conditions in graph form. Setting (B_1, APS_2^+) or (APS_1^-, B_2) as boundary conditions, condition 1 from theorem 2.3.9 holds, since either $G_1 = 0$ or $G_2 = 0$ and is hence compact. The corrections terms for the index formula also given in theorem 2.3.9 can then be taken directly from lemma 3.4.7 above. □

In order to get regularity for the solution space with theorem 3.3.6 more information on the deformation maps G_1, G_2 is needed. However, in the case of *chirality conditions* explained in example 3.4.8 the deformation map can be easily calculated and the regularity theorem applies. This will be shown in the next chapter.

4 Examples and Applications

As a first application of the results for graph-type boundary conditions and their interplay with local boundary conditions, we will discuss so called *chirality* conditions, introduced in example 3.4.8. These boundary conditions are defined via unitary fields of involutions h_1, h_2 along the boundaries Σ_1, Σ_2 and restricting to the ± 1 eigenspaces of these maps.

4.1 Chirality Conditions

We start with a special case of the involution maps h_i, assuming they are anti-commuting with the Dirac operators A_i instead of just anti-commuting with their principal symbol. This setting is more restrictive than just asking for this condition to hold on principal symbol level. For example it directly implies that the spectrum of A_i has to be symmetric.

Corollary 4.1.1. *Let $h_i \in End(S\Sigma_i)$ be smooth unitary such that*

$$h_i \circ A_i = -A_i \circ h_i.$$

where A_i denotes the Dirac operator of the boundary component Σ_i. Then for $B_1 := L^2(\Sigma_1; E_1(h_1))$ and $B_2 := L^2(\Sigma_2; E_{-1}(h_2))$ the Dirac operators $D_{B_1 APS_2^+}$ and $D_{APS_1^- B_2}$ are Fredholm and for their indices we have

$$\mathrm{ind}(D_{B_1 \mathrm{APS}_2^+}) + \mathrm{ind}(D_{B_1^\perp \mathrm{APS}_2^+}) = 2 \cdot \mathrm{ind}(D_{\mathrm{APS}}) + \dim\ker(A_1)$$

$$\mathrm{ind}(D_{\mathrm{APS}_1^- B_2}) + \mathrm{ind}(D_{\mathrm{APS}_1^- B_2^\perp}) = 2 \cdot \mathrm{ind}(D_{\mathrm{APS}}) - \dim\ker(A_2).$$

Additionally, the solution spaces

$$\ker(D_{B_1 APS_2^+}) \subset C^\infty(M; S^+M) \quad \textit{and} \quad \ker(D_{APS_1^- B_2}) \subset C^\infty(M; S^+M)$$

consist of smooth sections only.

Proof. The statement of the resulting operators being Fredholm and the relative index formula follows directly form corollary 3.4.10, since B_1 and B_2 can be written as graph-type boundary conditions by lemma 3.4.7. To prove regularity for the kernels of those operators, we calculate the deformation maps G_1 and G_2. Since h_i and A_i anti-commute by assumption, we have for $\phi_\lambda \in E_\lambda(A_i)$

$$A_i(h_i\phi_\lambda) = -h_i(A_i\phi_\lambda) = -\lambda(h_i\phi_\lambda)$$

and since $h_i^2 = 1$, we get that h_i restricts to an isomorphism $h_i : E_\lambda(A_i) \to E_{-\lambda}(A_i)$ between eigenspaces for the Dirac operator A_i. Since $L^2(\Sigma_i; S\Sigma_i)$ splits into eigenspaces of the Dirac operator, we have that

$$L^2(\Sigma_i; S\Sigma_i) = \ldots \oplus E_{-\lambda_2}(A_i) \oplus E_{-\lambda_1}(A_i) \oplus \ker(A_i) \oplus E_{\lambda_1}(A_i) \oplus E_{\lambda_2}(A_i) \oplus \ldots$$

and consequently

$$L^2(\Sigma_i; S\Sigma_i) = \ldots \oplus E_{-\lambda_2}(A_i) \oplus E_{-\lambda_1}(A_i) \oplus \ker(A_i) \oplus h_i\left(E_{-\lambda_1}(A_i)\right) \oplus h_i\left(E_{-\lambda_2}(A_i)\right) \oplus \ldots$$

Thus, we can write any section $\varphi \in L^2(\Sigma_i; S\Sigma_i)$ as a linear combination of eigensections for A_i in the following way

$$\varphi = \sum_{k=1}^{\infty}(\alpha_k \phi_{-\lambda_k} + \beta_k h_i \phi_{-\lambda_k}) + \phi_0$$

where the $\phi_{-\lambda_k}$ denote eigensections of A_i to eigenvalues $-\lambda_k$ (multiplicity counted) and $\phi_0 \in \ker(A_i)$. Now we require φ to be a -1 eigensection for h_i simultaneously and get

$$\begin{aligned} -\sum_{k=1}^{\infty}(\alpha_k \phi_{-\lambda_k} + \beta_k h_i \phi_{-\lambda_k}) - \phi_0 &= h_i\left(\sum_{k=1}^{\infty}(\alpha_k \phi_{-\lambda_k} + \beta_k h_i \phi_{-\lambda_k})\right) + h_i\phi_0 \\ &= \sum_{k=1}^{\infty}(\alpha_k h_i \phi_{-\lambda_k} + \beta_k \phi_{-\lambda_k}) + h_i\phi_0 \end{aligned}$$

and hence, if φ is a -1 eigensection for h_i, we have that $\alpha_k = -\beta_k$ and $h_i\phi_0 = -\phi_0$:

$$\begin{aligned} \varphi &= \phi_0 + \sum_{k=1}^{\infty}(\alpha_k \phi_{-\lambda_k} - \alpha_k h_i \phi_{-\lambda_k}) \\ &= \phi_0 + \sum_{k=1}^{\infty}\alpha_k(1 - h_i)\phi_{-\lambda_k} \qquad (4.1) \end{aligned}$$

On the other hand, any φ of this form derived above is a -1 eigensection for h_i and thus, $E_{-1}(h_i)$ can be written as a graph type deformation in the following way. Since h_i maps $\ker(A_i) \to \ker(A_i)$, we can define

$$W_i := E_{-1}(h_i|_{\ker(A_i)}) \qquad W_i^{\perp} := (W_i)^{\perp_{\ker(A_i)}}$$

and get

$$\ker(A_i) = W_i \oplus W_i^{\perp}$$

where the direct sum is orthogonal. Hence, we have an L^2-orthogonal splitting of the boundary space given by

$$L^2(\Sigma_i; S\Sigma_i) = L^2_{(-\infty,0)}(\Sigma_i; S\Sigma_i) \oplus W_i \oplus W_i^{\perp} \oplus L^2_{(0,\infty)}(\Sigma_i; S\Sigma_i).$$

By setting

$$\begin{aligned} V_i^- &:= L^2_{(-\infty,0)}(\Sigma_i; S\Sigma_i) & V_i^+ &:= W_i^{\perp} \oplus L^2_{(0,\infty)}(\Sigma_i; S\Sigma_i) \\ W_i^- &:= \{0\} & W_i^+ &:= W_i \end{aligned}$$

this yields a sum decomposition

$$L^2(\Sigma_i; S\Sigma_i) = V_i^- \oplus W_i^- \oplus V_i^+ \oplus W_i^+$$

where $V_i^- \oplus W_i^- = L^2_{(-\infty,0)}(\Sigma_i; S\Sigma_i)$ and $V_i^+ \oplus W_i^+ = L^2_{[0,\infty)}(\Sigma_i; S\Sigma_i)$ as required in definition 2.3.6. For the deformation map we choose

$$G_i := -h_i|_{V_i^-}$$

to get $L^2(\Sigma_i; E_{-1}(h_i)) = \Gamma(G_i) \oplus W_i^+$ by (4.1). Hence, $L^2(\Sigma_i; E_{-1}(h_i))$ is a graph-type boundary condition where the deformation map $G_i = -h_i$ is given by a pseudo-differential operator of order zero and the statement follows by theorem 3.3.6. A similar construction shows the statement for the +1 eigenspace of h_i.
To calculate the index of those operators, theorem 2.3.9 states:

$$\begin{aligned} \operatorname{ind}(D_{B_1 \mathrm{APS}_2^+}) &= \operatorname{ind}(D_{\mathrm{APS}}) + \dim(W_1^+) - \dim(W_1^-) \\ &= \operatorname{ind}(D_{\mathrm{APS}}) + \dim(E_{-1}(h_1|_{\ker(A_1)})) - 0 \end{aligned}$$

and

$$\operatorname{ind}(D_{B_1^\perp \mathrm{APS}_2^+}) = \operatorname{ind}(D_{\mathrm{APS}}) + \dim(E_1(h_1|_{\ker(A_1)})).$$

Hence, by combining the two we get

$$\begin{aligned} \operatorname{ind}(D_{B_1 \mathrm{APS}_2^+}) + \operatorname{ind}(D_{B_1^\perp \mathrm{APS}_2^+}) &= 2 \cdot \operatorname{ind}(D_{\mathrm{APS}}) + \dim(E_{-1}(h_1|_{\ker(A_1)})) + \dim(E_1(h_1|_{\ker(A_1)})) \\ &= 2 \cdot \operatorname{ind}(D_{\mathrm{APS}}) + \dim\ker(A_1) \end{aligned}$$

and $\operatorname{ind}(D_{\mathrm{APS}_1^- B_2}) + \operatorname{ind}(D_{\mathrm{APS}_1^- B_2^\perp})$ is calculated by a similar argument. □

Now, to allow for a more general situation, we assume that the involution map h_i and the Dirac operator A_i only anti-commute on principal symbol level and reprove the statement using the result above.

Corollary 4.1.2. *Let $h_i \in End(S\Sigma_i)$ be smooth, unitary involutions, such that*

$$h_i \circ \sigma_{A_i}(\xi) = -\sigma_{A_i}(\xi) \circ h_i \quad \forall \xi \in T^*\Sigma_i$$

where σ_{A_i} denotes the principal symbol of the Dirac operator A_i, i.e. $\sigma_{A_i}(\xi)$ corresponds to Clifford multiplication by the co-vector ξ. As boundary conditions we define

$$B_i^\pm := L^2(\Sigma_i; E_{\pm 1}(h_i))$$

the L^2-closure of the subbundles corresponding to eigenvalues ± 1 of h_i. Then the Dirac operators

$$D_{B_1^\pm APS_2^+},\ D_{B_1^\pm APS_2^-},\ D_{APS_1^+ B_2^\pm},\ D_{APS_1^- B_2^\pm}$$

are Fredholm operators and have smooth kernels.

Remark 4.1.3. Note that all the boundary conditions in corollary 4.1.2 are pairs of APS conditions $\mathrm{APS}_i^\pm$ and eigenspaces of the unitary involutions $L^2(\Sigma_i; E_{\pm 1}(h_i))$. Since these eigenspaces of h_i can be considered as graph type deformations of either APS^+ or APS^-, the order of the pairings is not really important and the proof of the statement remains basically the same. For simplicity, we will prove the statement for the Dirac operator $D_{\mathrm{APS}_1^- B_2^+}$. The other three combinations then follow by a similar argument.

Proof. In contrast to the proof of corollary 4.1.1, now it is not clear how to write the boundary conditions $B_i^{\pm}$ as deformations of APS boundary conditions for the Dirac operators A_i, since the anti-commutation now only holds on the principal symbol level and not necessarily for the operators themselves. However, we can switch from the Dirac operators A_i to *adapted* operators $\widetilde{A_i}$ on the hypersurfaces that we define by

$$\widetilde{A_i} := A_i - \tfrac{1}{2}\{h_i, A_i\}h_i$$

where $\{h_i, A_i\} = h_iA_i + A_ih_i$ denotes the anti-commutator. The principal symbol of the anti-commutator term computes to:

$$\begin{aligned} \sigma_{\{h_i,A_i\}} &= \sigma_{h_iA_i} + \sigma_{A_ih_i} \\ &= h_i\sigma_{A_i} + \sigma_{A_i}h_i \\ &= h_i\sigma_{A_i} - h_i\sigma_{A_i} \\ &= 0 \end{aligned}$$

where we used that the operators are assumed to anti-commute on the principal symbol level. This shows that in fact the added term $\frac{1}{2}\{h_i, A_i\}h_i$ is of order 0 as a whole, since h_i is an order zero operator. Note that $\widetilde{A_i}$ is still an elliptic operator on the Riemannian manifold Σ_i and, since h_i is assumed to be unitary and satisfy $h_i^2 = 1$, it is still formally self-adjoint. Hence, the spectrum of $\widetilde{A_i}$ is real, discrete, and its smooth and finite dimensional eigenspaces form a basis for L^2 exactly like for the Dirac operator A_i.

$$L^2(\Sigma_i; S\Sigma_i) = \dots E_{\lambda_{-2}}(\widetilde{A_i}) \oplus E_{\lambda_{-1}}(\widetilde{A_i}) \oplus \ker(\widetilde{A_i}) \oplus E_{\lambda_1}(\widetilde{A_i}) \oplus E_{\lambda_2}(\widetilde{A_i}) \dots$$

Now we can check that h_i and the adapted operator $\widetilde{A_i}$ actually anti-commute on the operator level as well:

$$\begin{aligned} \widetilde{A_i}h_i &= A_ih_i - \tfrac{1}{2}(A_ih_i + h_iA_i)h_i^2 \\ &= \tfrac{1}{2}A_ih_i - \tfrac{1}{2}h_iA_i \\ &= \tfrac{1}{2}A_ih_i + \tfrac{1}{2}h_iA_i - h_iA_i \\ &= -h_iA_i + \tfrac{1}{2}h_i(A_ih_i + h_iA_i)h_i \\ &= -h_i\widetilde{A_i}. \end{aligned}$$

By the proof of corollary 3.4.4, we get that $B_i^{\pm}$ are graph-type deformations of APS (or aAPS) boundary conditions for $\widetilde{A_i}$ and hence, projections $P_{B_i^{\pm}}$ onto $B_i^{\pm}$ can be chosen such that $P_{B_i^{\pm}}\widetilde{P_i^-}$ or $P_{B_i^{\pm}}\widetilde{P_i^+}$ is compact, where $\widetilde{P_i^{\pm}}$ denotes the APS projectors for $\widetilde{A_i}$.
This is due to the proof of corollary 4.1.2, where these projection maps were constructed such that $\mathrm{im}(P_{B_i^{\pm}}\widetilde{P_i^{\pm}}) \subset \ker(\widetilde{A_i})$ making it finite dimensional and hence, the combination of operators compact and smoothing.
Now since the adapted operators were obtained by adding only zero order terms to A_i, the principal symbols $\sigma_{A_i} = \sigma_{\widetilde{A_i}}$ coincide and the same is true for the principal symbols of the APS projectors $\widetilde{p_i^{\pm}} = p_i^{\pm}$.

For now fix the boundary condition to the pair $(\mathrm{APS}_1^-, B_2^+)$. Then, in order to prove the statement by lemma 3.3.1 and proposition 2.1.10, it suffices to show that the principal symbols of $Q_{B_2^+\mathrm{APS}_1^-}$ and $Q^*_{\mathrm{APS}_1^+(B_2^+)^c}$ vanish. Note that the projection map $P_{B_2^+(B_2^+)^c}$ onto $B_2^+ = L^2(\Sigma_2; E_1(h_2))$ is chosen such that

$$\ker(P_{B_2^+(B_2^+)^c}) = (B_2^+)^c = \chi^-(\widetilde{A}_2).$$

The composition of this map with the APS projector $P_{B_2^+(B_2^+)^c} \circ P_2^-$ yields a compact map, since on principal symbol level we get

$$p_{B_2^+(B_2^+)^c} \circ p_2^- = p_{B_2^+(B_2^+)^c} \circ \widetilde{p_2^-} = 0.$$

Lemma 3.1.3 then shows the compactness of $Q_{B_2^+\mathrm{APS}_1^-}$ and $Q^*_{\mathrm{APS}_1^+(B_2^+)^c}$ and hence the statement. The other combinations of boundary conditions follow by a similar argument through treating the chirality condition as a deformation of the appropriate APS subspace and repeating the principal symbol calculation for the off-diagonal terms of the wave evolution operator Q. □

So far, we can only apply chirality conditions to one component of the boundary $\Sigma_1 \sqcup \Sigma_2$, while leaving APS boundary conditions, or a compact deformation of APS, on the other part. This is due to theorem 2.3.9 and examples 2.3.10 and 2.3.11, showing that deforming APS boundary conditions on both ends of the boundary simultaneously, will in general not yield a Fredholm operator anymore. In the following section, we will discuss special cases where two-sided deformations, i.e. chirality conditions on both ends, can be seen to still yield Fredholm operators.

4.2 Warped products

In an earlier example 2.3.10, we saw that the assumptions of theorem 2.3.9 cannot just be dropped without potentially losing Fredholm property of the resulting operator. On the other hand, two-sided and non-compact deformations of boundary condition can still yield Fredholm operators (with smooth kernels). In 2.3.10 this was done by directly computing the wave evolution operator Q and defining boundary conditions accordingly, now we want to show that similar computations are still possible in a more general geometric setting.

Assume the spacetime M is given by a cylinder over the compact Riemannian manifold (Σ, h)

$$M = [0, 1] \times \Sigma$$

and that the metric on M is

$$g = -\mathrm{dt}^2 + f(t)h$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth and positive function. Any spinor field $\psi_0 \in \Gamma(S\Sigma_0)$ can then be extented to a spinor field on M via parallel transport along t-lines:

$$\psi(t) := \tau_0^t \psi_0 \in \Gamma(S\Sigma_t).$$

Here $\tau_{t_1}^{t_2}$ denotes point wise parallel transport from Σ_{t_1} to Σ_{t_2} along the t-axis. With this setup we have the following theorem (see [16])

Theorem 4.2.1 (Bär, Gauduchon, Moroianu)**.** *For any smooth spinor field ψ_0 on Σ_0 we have*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \tau_t^{t_0} A_t \tau_{t_0}^t \psi_0 = -\frac{1}{2}\frac{\dot{f}(t_0)}{f(t_0)} A_{t_0}\psi_0.$$

The plan is now to use this result to solve the Dirac equation on M and eventually calculate the wave evolution operator. For this to work, we first need to verify that the above identity indeed holds at any point $s \in [0,1]$.

Corollary 4.2.2. *For any smooth spinor field $\psi \in \Gamma(S\Sigma_0)$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tau_t^{t_0} A_t \tau_{t_0}^t \psi_0\right) = -\frac{1}{2}\frac{\dot{f}(t)}{f(t)} \tau_t^{t_0} A_t \tau_{t_0}^t \psi_0.$$

Proof.

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} \tau_t^{t_0} A_t \tau_{t_0}^t \psi_0 &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} \tau_s^{t_0}\tau_t^{s} A_t \tau_s^t \tau_{t_0}^s \psi_0 \\ &= \tau_s^{t_0} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} \tau_t^{s} A_t \tau_s^t \tau_{t_0}^s \psi_0 \\ &= -\frac{1}{2}\frac{\dot{f}(s)}{f(s)} \tau_s^{t_0} A_s \tau_{t_0}^s \psi_0\end{aligned}$$

□

With corollary 4.2.2, we can make an ansatz $\tau_t^{t_0} A_t \tau_{t_0}^t \psi_0 = \frac{1}{\sqrt{f(t)}}\tilde{\psi}_0$ for some $\tilde{\psi}_0 \in \Gamma(S\Sigma_0)$ such that $A_{t_0}\psi_0 = \frac{1}{\sqrt{f(t_0)}}\tilde{\psi}_0$ and hence, we get $\tau_t^{t_0} A_t \tau_{t_0}^t \psi_0 = \sqrt{\frac{f(t_0)}{f(t)}} A_{t_0}\psi_0$. Shifting the first parallel transport to the right side of the equation then yields

$$A_t \tau_{t_0}^t \psi_0 = \sqrt{\tfrac{f(t_0)}{f(t)}}\, \tau_{t_0}^t A_{t_0}\psi_0.$$

Now suppose we choose $\psi_0 \in \Gamma(S\Sigma_0)$ to be an eigenspinor for A_{t_0}, i.e. $A_{t_0}\psi_0 = \lambda\psi_0$ for some $\lambda \in \mathrm{spec}(A_{t_0})$. The above identity then reduces to

$$A_t \tau_{t_0}^t \psi_0 = \sqrt{\tfrac{f(t_0)}{f(t)}}\, \lambda \tau_{t_0}^t \psi_0 \tag{4.2}$$

showing that parallel transport of an eigenspinor $\psi_0 \in \Gamma(S\Sigma_0)$ to an eigenvalue λ results in an eigenspinor $\tau_{t_0}^t\psi_0 \in \Gamma(S\Sigma_t)$ to the eigenvalue $\lambda(t) := \sqrt{\frac{f(t_0)}{f(t)}}\lambda$.
To calculate the wave evolution operator, we can make use of the splitting for the Dirac operator on M given by 1.2:

$$\mathcal{D}\psi = -\beta\left(\nabla_\nu + iA_t - \tfrac{n}{2}H_t\right)\psi$$

which for our case simplifies to

$$\mathcal{D}\psi = -\beta\left(\nabla_{\partial_t} + iA_t - \tfrac{n}{2}\tfrac{\dot{f}(t)}{f(t)}\right)\psi$$

where $n = \dim(\Sigma)$. Now we want to solve the Cauchy problem

$$\mathcal{D}\psi = 0 \tag{4.3}$$
$$\psi|_{\Sigma_0} = \psi_0 \tag{4.4}$$

where ψ_0 is an eigenspinor for A_0 and to the eigenvalue λ. Again, we make an ansatz by setting $\psi = h(t)\tau_0^t\psi_0$ for some smooth function h. Applying the Dirac operator to this ansatz yields:

$$-\beta\left(\nabla_{\partial_t} + iA_t - \frac{n}{2}\frac{\dot{f}(t)}{f(t)}\right)h(t)\tau_0^t\psi_0 = -\beta\left(\dot{h}(t) + ih(t)\lambda\sqrt{\frac{f(0)}{f(t)}} - \frac{n}{2}\frac{\dot{f}(t)}{f(t)}h(t)\right)\tau_{t_0}^t\psi_0$$

where we used that

$$\nabla_{\partial_t}\tau_0^t\psi_0 = 0 \qquad \text{and} \qquad A_t\tau_0^t\psi_0 = \sqrt{\frac{f(0)}{f(t)}}\lambda\tau_0^t\psi_0.$$

The Cauchy problem then reduces to solving the following ODE for the function $h : [0,1] \to \mathbb{C}$:

$$\begin{aligned}\dot{h}(t) &= \left(-i\lambda\sqrt{\frac{f(0)}{f(t)}} + \frac{n}{2}\frac{\dot{f}(t)}{f(t)}\right)h(t)\\ h(0) &= 1\end{aligned}$$

Again we can make an ansatz $h(t) = \exp(u(t))$ where $u(t)$ is such that

$$\begin{aligned}\dot{u}(t) &= -i\lambda\sqrt{\frac{f(0)}{f(t)}} + \frac{n}{2}\frac{\dot{f}(t)}{f(t)}\\ u(0) &= 0.\end{aligned}$$

This equation will be solved by

$$u(t) = -i\lambda\sqrt{f(0)}\int_0^t \frac{1}{\sqrt{f(\xi)}}\mathrm{d}\xi + \frac{n}{2}\log(f(t)) - \frac{n}{2}\log(f(0))$$

and plugging this in back into our ansatz for $h(t)$ yields

$$\begin{aligned}h(t) &= \exp\left(-i\lambda\sqrt{f(0)}\int_0^t \frac{1}{\sqrt{f(\xi)}}\mathrm{d}\xi + \frac{n}{2}\log(f(t)) - \frac{n}{2}\log(f(0))\right)\\ &= \sqrt{\frac{f(t)}{f(0)}}^n \exp\left(-i\lambda\sqrt{f(0)}\int_0^t \frac{1}{\sqrt{f(\xi)}}\mathrm{d}\xi\right).\end{aligned}$$

In conclusion, the solution to the Cauchy problem 4.3 is given by

$$\psi_t = \sqrt{\frac{f(t)}{f(0)}}^n \exp\left(-i\lambda\sqrt{f(0)}\int_0^t \frac{1}{\sqrt{f(\xi)}}\mathrm{d}\xi\right)\tau_0^t\psi_0$$

and consequently, the wave evolution operator maps

$$Q\psi_0 = \sqrt{\frac{f(1)}{f(0)}}^n \exp\left(-i\lambda\sqrt{f(0)}\int_0^1 \frac{1}{\sqrt{f(\xi)}}\mathrm{d}\xi\right)\tau_0^1\psi_0 \tag{4.5}$$

$$\text{where } A_1 Q\psi_0 = \sqrt{\tfrac{f(0)}{f(1)}}\lambda Q\psi_0 = \lambda(1)Q\psi_0. \tag{4.6}$$

Since both $L^2(\Sigma_0; S\Sigma_0)$ and $L^2(\Sigma_1; S\Sigma_1)$ split into direct sums of eigenpaces for their respective Dirac operators A_0, A_1, we get the following mapping properties for Q:

$$\begin{array}{ccccccccccc}
L^2(\Sigma_0;S\Sigma_0) = & \dots E_{\lambda_{-2}}(A_0) & \oplus & E_{\lambda_{-1}}(A_0) & \oplus & \ker(A_0) & \oplus & E_{\lambda_1}(A_0) & \oplus & E_{\lambda_2}(A_0)\dots \\
\big\downarrow Q & \big\downarrow\cong & & \big\downarrow\cong & & \big\downarrow\cong & & \big\downarrow\cong & & \big\downarrow\cong \\
L^2(\Sigma_1;S\Sigma_1) = & \dots E_{\lambda_{-2}(1)}(A_1) & \oplus & E_{\lambda_{-1}(1)}(A_1) & \oplus & \ker(A_1) & \oplus & E_{\lambda_1(1)}(A_1) & \oplus & E_{\lambda_2(1)}(A_1)\dots
\end{array}$$

where $\lambda_k(1) = \sqrt{\frac{f(0)}{f(1)}}\lambda_k$. With this knowledge about the wave evolution operator, we can start to construct and analyze boundary conditions for the Dirac operator, first take a look at the chirality conditions from section 4.1.

Example 4.2.3 (Chirality Conditions)**.** Let $h_0 \in \mathrm{End}(S\Sigma_0)$ be smooth and unitary such that

$$h_0 \circ A_0 = -A_0 \circ h_0$$

then we can define $h_1 \in \mathrm{End}(S\Sigma_1)$ by setting

$$h_1 := \tau_0^1 h_0 \tau_1^0.$$

First we can check that

$$\begin{aligned}
h_1^2 &= (\tau_0^1 h_0 \tau_1^0)^2 \\
&= \tau_0^1 h_0 \tau_1^0 \tau_0^1 h_0 \tau_1^0 \\
&= \tau_0^1 h_0^2 \tau_1^0 \\
&= \tau_0^1 \tau_1^0 \\
&= 1
\end{aligned}$$

hence h_1 is an involution. To check that h_1 also anti-commutes with the Dirac operator A_1, we apply to an eigenspinor $\phi_{\lambda(1)}$, $A_1\phi_{\lambda(1)} = \lambda(1)\phi_{\lambda(1)}$ of A_1 and calculate:

$$\begin{aligned}
A_1 h_1 \phi_{\lambda(1)} &= A_1 \underbrace{\tau_0^1 \underbrace{h_0 \underbrace{\tau_1^0 \phi_{\lambda(1)}}_{\in E_{\lambda(0)}(A_0)}}_{\in E_{-\lambda(0)}(A_0)}}_{\in E_{-\lambda(1)}(A_1)} \\
&= -\lambda(1) h_1 \phi_{\lambda(1)} \\
&= -h_1 A_1 \phi_{\lambda(1)}.
\end{aligned}$$

Define boundary conditions for D by

$$B_0 = E_{-1}(h_0) = \Gamma(-h_0) \oplus E_{-1}(h_0|_{\ker(A_0)})$$

$$B_1 = E_1(h_1) = \Gamma(h_1) \oplus E_1(h_1|_{\ker(A_1)})$$

where we view h_i as maps $L^2_{(-\infty,0)}(\Sigma_i; S\Sigma_i) \to L^2_{(0,\infty)}(\Sigma_i; S\Sigma_i)$, compare equation 4.1. Corollary 4.1.1 now states that combinations $(Q(\mathrm{APS}_0^{\pm}), B_1)$ and $(Q(B_0), \mathrm{APS}_1^{\mp})$ yield Fredholm pairs and consequently, the Dirac operators subject to these boundary conditions will also be Fredholm. In the geometric setting of this section, the combinations (B_0, B_1) and $(B_0^{\perp}, B_1^{\perp})$ will also yield Fredhom operators. This can be seen by looking at the mapping properties of Q established above.

Before the boundary conditions can be written in the style of 4.1, we need to fix orthonomal bases for $L^2(\Sigma_0; S\Sigma_0)$ and $L^2(\Sigma_1; S\Sigma_1)$. This can be done in the following way:

Start with an orthonormal System on $L^2(\Sigma_0; S\Sigma_0)$ consisting of eigenspinors to negative eigenvalues and elements of the kernel of A_1

$$\dots \phi_{-\lambda_3}, \phi_{-\lambda_2}, \phi_{-\lambda_1}, \underbrace{\phi_0^1, \dots, \phi_0^k}_{\dim\ker(A_0)}.$$

Since h_0 anti-commutes with A_0, we get eigenspinors for positive eigenvalues of A_0 by applying h_0 to the eigenspinors of negative eigenvalues. Furthermore, these elements will still form an orthonormal system, because h_0 is also assumed to be a unitary involution. Hence

$$\dots \phi_{-\lambda_3}, \phi_{-\lambda_2}, \phi_{-\lambda_1}, \underbrace{\phi_0^1, \dots, \phi_0^k}_{\dim\ker(A_0)}, h_0\phi_{-\lambda_1}, h_0\phi_{-\lambda_2}, h_0\phi_{-\lambda_3}, \dots$$

defines an orthonormal basis for $L^2(\Sigma_0; S\Sigma_0)$. Now to fix a basis on $L^2(\Sigma_1; S\Sigma_1)$ we make use of the mapping properties of Q establisehd above. Namely Q maps eigenspace to eigenspace isomorphically and Q is always unitary, hence we get an orthonormal basis on the second boundary component by setting

$$\dots Q\phi_{-\lambda_3}, Q\phi_{-\lambda_2}, Q\phi_{-\lambda_1}, \underbrace{Q\phi_0^1, \dots, Q\phi_0^k}_{\dim\ker(A_1)}, Qh_0\phi_{-\lambda_1}, Qh_0\phi_{-\lambda_2}, Qh_0\phi_{-\lambda_3}, \dots$$

where $A_1 Q\phi_{-\lambda} = -\sqrt{\frac{f(0)}{f(1)}}\lambda Q\phi_{-\lambda}$, $A_1 Q\phi_0^l = 0$ and $A_1 Qh_0\phi_{-\lambda} = \sqrt{\frac{f(0)}{f(1)}}\lambda Qh_0\phi_{-\lambda}$. Note that this construction is compatible with our definition of h_1 in the following sense:

If we start with an orthonormal system on $L^2(\Sigma_1; S\Sigma_1)$ given by

$$\dots Q\phi_{-\lambda_3}, Q\phi_{-\lambda_2}, Q\phi_{-\lambda_1}, \underbrace{Q\phi_0^1, \dots, Q\phi_0^k}_{\dim\ker(A_1)}$$

and then proceed similar to the construction on Σ_0 by applying h_1 to get an orthonormal basis

$$\dots Q\phi_{-\lambda_3}, Q\phi_{-\lambda_2}, Q\phi_{-\lambda_1}, \underbrace{Q\phi_0^1, \dots, Q\phi_0^k}_{\dim\ker(A_1)}, h_1Q\phi_{-\lambda_3}, h_1Q\phi_{-\lambda_2}, h_1Q\phi_{-\lambda_1}, \dots$$

the result remains the same. This is because by definition of h_1 and 4.5, we have

$$\begin{aligned} h_1 Q\phi_{-\lambda} &= \tau_0^1 h_0 \tau_1^0 \sqrt{\frac{f(1)}{f(0)}}^n \exp\left(-i\lambda\sqrt{f(0)}\int_0^1 \frac{1}{\sqrt{f(\xi)}}\mathrm{d}\xi\right)\tau_0^1\phi_{-\lambda} \\ &= \sqrt{\frac{f(1)}{f(0)}}^n \exp\left(-i\lambda\sqrt{f(0)}\int_0^1 \frac{1}{\sqrt{f(\xi)}}\mathrm{d}\xi\right)\tau_0^1 h_0 \tau_1^0 \tau_0^1 \phi_{-\lambda} \\ &= \sqrt{\frac{f(1)}{f(0)}}^n \exp\left(-i\lambda\sqrt{f(0)}\int_0^1 \frac{1}{\sqrt{f(\xi)}}\mathrm{d}\xi\right)\tau_0^1 h_0 \phi_{-\lambda} \\ &= Qh_0\phi_{-\lambda}. \end{aligned}$$

Now we can use 4.1 to write down the boundary conditions B_0 and B_1 in the following way:

$$\psi \in B_0 \Leftrightarrow \psi = \underbrace{\phi_0}_{\in E_{-1}(h_0|_{\ker(A_0)})} + \sum_{k=1}^{\infty} \alpha_k (1-h_0)\phi_{-\lambda_k}$$

$$\psi \in B_1 \Leftrightarrow \psi = \underbrace{\tilde{\phi}_0}_{\in E_1(h_1|_{\ker(A_1)})} + \sum_{k=1}^{\infty} \tilde{\alpha}_k (1+h_1)Q\phi_{-\lambda_k}.$$

Then we apply the wave evolution operator to an element of B_0

$$\begin{aligned} \psi \in B_0 \Rightarrow Q\psi &= Q\phi_0 + \sum_{k=1}^{\infty} \alpha_k (Q\phi_{-\lambda_k} - Qh_0\phi_{-\lambda_k}) \\ &= Q\phi_0 + \sum_{k=1}^{\infty} \alpha_k (Q\phi_{-\lambda_k} - h_1 Q\phi_{-\lambda_k}) \\ &= Q\phi_0 + \sum_{k=1}^{\infty} \alpha_k (1-h_1)Q\phi_{-\lambda_k} \end{aligned}$$

and see that

$$Q\psi \in B_1 \Leftrightarrow \alpha_k = 0 \; \forall k \text{ and } Q\phi_0 \in E_1(h_1|_{\ker(A_1)}).$$

Assuming $Q\phi_0 \in E_1(h_1|_{\ker(A_1)})$, we get

$$\begin{aligned} Q\phi_0 &= h_1 Q\phi_0 \\ &= Qh_0\phi_0 \\ &= -Q\phi_0 \end{aligned}$$

hence $\phi_0 = 0$. In conclusion, this calculation shows that $Q(B_0) \cap B_1 = \{0\}$ and since $B_0^\perp = E_1(h_0)$, $B_1^\perp = E_{-1}(h_1)$, the same computation also yields that $Q(B_0^\perp) \cap B_1^\perp = \{0\}$. In fact $Q(B_0)^\perp = B_1$ making $(Q(B_0), B_1)$ a Fredholm pair. Finally, proposition 2.2.5 states that $D_{B_0 B_1}$ is a Fredholm operator (in this case of index 0).

Remark 4.2.4. The above example shows a case of boundary conditions obtained by non-compact deformations, via the maps h_0 and h_1, of APS conditions on both boundary components, which is not generally covered by theorem 2.3.9. For the particular choice of boundary conditions above, the index of the resulting operator is 0. But the same computations can be used to show that by using h_0 and h_1 as deformation maps for generalized APS conditions on Σ_0 and Σ_1 respectively, i.e.

$$B_0 = \Gamma(-h_0|_{L^2_{(-\infty,-a_0)}(\Sigma_0;S\Sigma_0)})$$

$$B_1 = \Gamma(h_1|_{L^2_{(a_1,\infty)}(\Sigma_1;S\Sigma_1)})$$

where $a_0, a_1 > 0$, to obtain a non-trivial index for the resulting Dirac operator. In the notation above we would get

$$\mathrm{ind}(D_{B_0B_1}) = -\dim(L^2_{[-a_0,a_1]}(\Sigma_0; S\Sigma_0)).$$

To end this section we show another possibility to construct two-sided, non-compact deformations of APS boundary conditions that will still preserve Fredholm property for the resulting Dirac operator.

Example 4.2.5. In the same setting used throughout this section we can write down the eigenvalues (multiplicity repeated) for the Dirac operator A_1 (in an ascending order):

$$\ldots \lambda_{-3}, \lambda_{-2}, \lambda_{-1}, \underbrace{0, \ldots, 0}_{\dim\ker(A_1)}, \lambda_1, \lambda_2, \lambda_3 \ldots$$

and choose a corresponding set of normalized eigenspinors

$$\ldots \phi_{\lambda_{-3}}, \phi_{\lambda_{-2}}, \phi_{\lambda_{-1}}, \underbrace{\phi_0^1, \ldots, \phi_0^k}_{\dim\ker(A_1)}, \phi_{\lambda_1}, \phi_{\lambda_2}, \phi_{\lambda_3}, \ldots$$

where in this case the spectrum does not need to be symmetric as in the previous example. This set of eigenspinors forms a basis for $L^2(\Sigma_1; S\Sigma_1)$ and we define a map $G_1 : L^2_{(-\infty,0)}(\Sigma_1; S\Sigma_1) \to L^2_{[0,\infty)}(\Sigma_1; S\Sigma_1)$ by setting

$$G_1(\phi_{\lambda_{-l}}) = \begin{cases} 0 & l \text{ even} \\ \phi_{\lambda_l} & l \text{ odd} \end{cases}.$$

Now we write down eigenvalues of A_2

$$\ldots \tilde{\lambda}_{-3}, \tilde{\lambda}_{-2}, \tilde{\lambda}_{-1}, \underbrace{0, \ldots, 0}_{\dim\ker(A_2)}, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3 \ldots$$

where $\tilde{\lambda}_l = \frac{f(0)}{f(1)}\lambda_l$ and corresponding eigenspinors

$$\ldots \tilde{\phi}_{\tilde{\lambda}_{-3}}, \tilde{\phi}_{\tilde{\lambda}_{-2}}, \tilde{\phi}_{\tilde{\lambda}_{-1}}, \underbrace{\tilde{\phi}_0^1, \ldots, \tilde{\phi}_0^k}_{\dim\ker(A_2)}, \tilde{\phi}_{\tilde{\lambda}_1}, \tilde{\phi}_{\tilde{\lambda}_2}, \tilde{\phi}_{\tilde{\lambda}_3}, \ldots$$

where we set $\tilde{\phi}_{\tilde{\lambda}_l} = Q\phi_{\lambda_l}$ and define a deformation map $G_2 : L^2_{[0,\infty)}(\Sigma_2; S\Sigma_2) \to L^2_{(-\infty,0)}(\Sigma_2; S\Sigma_2)$ by

$$G_2(\tilde{\phi}_{\tilde{\lambda}_l}) = \begin{cases} 0 & l \text{ odd} \\ \tilde{\phi}_{\tilde{\lambda}_{-l}} & l \text{ even} \end{cases}.$$

The boundary conditions we want to examine are now given by the graphs of the maps we defined above $B_1 = \Gamma(G_1)$, $B_2 = \Gamma(G_2)$. Since both G_1 and G_2 are non-compact, Fredholm property for the Dirac operator is not automatically ensured by theorem 2.3.9, since the wave evolution operator has already been calculated above, we can use this result to compare these boundary conditions anyway.

An element of the boundary condition B_1 is given by

$$\psi \in B_1 \Leftrightarrow \psi = \sum_{k=1}^{\infty} \left(\alpha_k \left(\phi_{\lambda_{-(2k-1)}} + \phi_{\lambda_{2k-1}} \right) + \beta_k \phi_{\lambda_{2k}} \right)$$

and applying the wave evolution operator to such an element yields

$$\begin{aligned} \psi \in Q(B_1) \Leftrightarrow \psi &= \sum_{k=1}^{\infty} \left(\alpha_k \left(Q\phi_{\lambda_{-(2k-1)}} + Q\phi_{\lambda_{2k-1}} \right) + \beta_k Q\phi_{\lambda_{2k}} \right) \\ &= \sum_{k=1}^{\infty} \left(\alpha_k \left(\tilde{\phi}_{\tilde{\lambda}_{-(2k-1)}} + \tilde{\phi}_{\tilde{\lambda}_{2k-1}} \right) + \beta_k \tilde{\phi}_{\tilde{\lambda}_{2k}} \right) \end{aligned}$$

.

On the other hand, an element of the boundary condition B_2 can be written as

$$\psi \in B_2 \Leftrightarrow \psi = \underbrace{\tilde{\phi}_0}_{\in \ker(A_2)} + \sum_{k=1}^{\infty} \left(\tilde{\alpha}_k \left(\tilde{\phi}_{\lambda_{2k}} + \tilde{\phi}_{\lambda_{-2k}} \right) + \tilde{\beta}_k \tilde{\phi}_{\lambda_{2k-1}} \right)$$

note here that $(\tilde{\phi}_{\tilde{\lambda}_{-(2k-1)}} + \tilde{\phi}_{\tilde{\lambda}_{2k-1}})$, $(\tilde{\phi}_{\lambda_{2l}} + \tilde{\phi}_{\lambda_{-2l}})$, $\tilde{\phi}_{\tilde{\lambda}_{2m}}$ and $\tilde{\phi}_{\lambda_{2n-1}}$ are linear independent for all $k, l, m, n \in \mathbb{N}$. Hence $Q(B_1) \cap B_2 = \{0\}$, and in the same way we get $Q(B_1^\perp) \cap B_2^\perp = \{0\}$. Proposition 2.2.5 then states that $D_{B_1 B_2}$ is a Fredholm operator of index 0. As in the previous example this construction can be extended to forming graphs over generalized APS boundary conditions to construct Fredholm operators of any given index.

4.3 Transmission Conditions

In this section we assume the following setting: $M = [t_1, t_2] \times \Sigma$ is a globally hyperbolic spin manifold of even dimension with compact and spacelike Cauchy hypersurfaces, and the metric $g = -N^2 \mathrm{dt} + h_t$ is such that $h_{t_1} = h_{t_2}$. Hence, the closed Riemannian manifolds $(\Sigma_{t_1}, h_{t_1}) \cong (\Sigma_{t_2}, h_{t_2})$ can be naturally identified along with their spinor bundles $S\Sigma_{t_1} \cong S\Sigma_{t_2}$. Instead of defining a boundary condition for D by choosing closed subspaces $B_1 \subset L^2(\Sigma_{t_1}; S\Sigma_{t_1})$ and $B_2 \subset L^2(\Sigma_{t_2}; S\Sigma_{t_2})$, we want to define it as a closed subspace $B \subset L^2(\Sigma_{t_1}; S\Sigma_{t_1}) \times L^2(\Sigma_{t_2}; S\Sigma_{t_2})$ of the product. In this particular setting, we can proceed as follows

$$(\varphi, \phi) \in B \Leftrightarrow \varphi = \phi$$

i.e. a section $\psi \in \mathrm{FE}^0(M; S^-M)$ satisfies the boundary condition B if and only if $\psi|_{\Sigma_{t_1}} = \psi|_{\Sigma_{t_2}}$. This boundary condition is called *Transmission conditions* and Bär and Ballmann have shown that in a Riemannian setting, it can be understood as a graph-type deformation of APS boundary conditions making D_B a Fredholm operator with $\mathrm{ind}(D_B) = \mathrm{ind}(D_{\mathrm{APS}})$, see [13] example 7.28. For our Lorentzian setting the situation is different and we get the following statement.

Proposition 4.3.1. *Let* $(M, g) = ([t_1, t_2] \times \Sigma, -N^2 \mathrm{dt} + h_t)$ *such that* $h_{t_1} = h_{t_2}$ *and* $B \subset L^2(\Sigma_{t_1}; S\Sigma_{t_1}) \times L^2(\Sigma_{t_2}; S\Sigma_{t_2})$ *where* $(\varphi, \phi) \in B \Leftrightarrow \varphi = \phi$*, then if* D_B *is a Fredholm operator its index is* $\mathrm{ind}(D_B) = 0$.

Proof. First we use Lemma 2.1.3 with

$$\begin{aligned} H &= FE^0(M; D) \\ E &= L^2(M; S^-M) \\ F &= B^\perp \\ L &= D \\ P &= P_{B^\perp} \circ (\mathrm{res}_{\Sigma_{t_1}} \times \mathrm{res}_{\Sigma_{t_2}}) \end{aligned}$$

to get that D_B is Fredholm of index k if and only if the operator

$$D \oplus \left(P_{B^\perp} \circ (\mathrm{res}_{\Sigma_{t_1}} \times \mathrm{res}_{\Sigma_{t_2}}) \right) : FE^0(M; D) \longrightarrow L^2(M; S^-M) \oplus B^\perp$$

is Fredholm of index k. Applying the Lemma again with

$$\begin{aligned} H &= FE^0(M; D) \\ E &= B^\perp \\ F &= L^2(M, S^-M) \\ L &= P_{B^\perp} \circ (\mathrm{res}_{\Sigma_{t_1}} \times \mathrm{res}_{\Sigma_{t_2}}) \\ P &= D \end{aligned}$$

yields that this equivalent to the operator

$$L|_{\ker(P)} := \left(P_{B^\perp} \circ (\mathrm{res}_{\Sigma_{t_1}} \times \mathrm{res}_{\Sigma_{t_2}}) \right)\Big|_{\ker(D)} : \ker(D) \longrightarrow B^\perp$$

being Fredholm of index k. To calculate the kernel of this operator:

$$\begin{aligned} \ker(L|_{\ker(P)}) &= \{\psi \in \ker(D) : \ P_{B^\perp}(\psi|_{\Sigma_{t_1}}, \psi|_{\Sigma_{t_2}}) = 0\} \\ &= \{\psi \in \ker(D) : \ (\psi|_{\Sigma_{t_1}}, \psi|_{\Sigma_{t_2}}) \in B\} \end{aligned}$$

$$
\begin{aligned}
&= \{\psi \in \ker(D) :\ \psi|_{\Sigma_{t_1}} = \psi|_{\Sigma_{t_2}}\} \\
&\cong \{\phi \in L^2(\Sigma_{t_1}; S\Sigma_{t_1}) :\ \phi = Q\phi\} \\
&= \ker(Q-1).
\end{aligned}
$$

Assuming that D_B and equivalently $L|_{\ker(P)}$ is Fredholm, we can calculate its cokernel as isomorphic to the orthogonal complement of its image. Note that any element $(\varphi, \phi) \in L^2(\Sigma_{t_1}; S\Sigma_{t_1}) \times L^2(\Sigma_{t_2}; S\Sigma_{t_2})$ can be decomposed as

$$
(\varphi, \phi) = \tfrac{1}{2}(\underbrace{(\varphi, \varphi) + (\phi, \phi)}_{\in B} + \underbrace{(\varphi - \phi, \phi - \varphi)}_{\in B^\perp})
$$

and consequently

$$
\begin{aligned}
\operatorname{im}(L|_{\ker(P)}) &= \{(\varphi - \phi, \phi - \varphi) \in L^2(\Sigma_{t_1}; S\Sigma_{t_1}) \times L^2(\Sigma_{t_2}; S\Sigma_{t_2}) :\ \phi = Q\varphi\} \\
&= \{(\varphi - Q\varphi, Q\varphi - \varphi) :\ \varphi \in L^2(\Sigma_{t_1}; S\Sigma_{t_1})\}.
\end{aligned}
$$

For the cokernel we then get

$$
\begin{aligned}
\operatorname{coker}(L|_{\ker(P)}) &\cong \{(\mu, -\mu) \in B^\perp :\ \langle \mu, \varphi - Q\varphi\rangle + \langle -\mu, Q\varphi - \varphi\rangle = 0\ \forall \varphi \in L^2(\Sigma_{t_1}; S\Sigma_{t_1})\} \\
&= \{(\mu, -\mu) \in B^\perp :\ \langle \mu, \varphi - Q\varphi\rangle = 0\ \forall \varphi \in L^2(\Sigma_{t_1}; S\Sigma_{t_1})\} \\
&= \{(\mu, -\mu) \in B^\perp :\ \langle \mu - Q^*\mu, \varphi\rangle = 0\ \forall \varphi \in L^2(\Sigma_{t_1}; S\Sigma_{t_1})\} \\
&\cong \ker(Q^* - 1) \\
&= \ker(Q - 1)
\end{aligned}
$$

where we used unitarity of Q for the last equality. In conclusion, we have that if D_B is Fredholm, then $\ker(Q-1)$ is finite dimensional and

$$
\begin{aligned}
\operatorname{ind}(D_B) &= \operatorname{ind}(L|_{\ker(P)}) \\
&= \dim(\ker(L|_{\ker(P)})) - \dim(\operatorname{coker}(L|_{\ker(P)})) \\
&= \dim\ker(Q-1) - \dim\ker(Q-1) \\
&= 0.
\end{aligned}
$$

□

Remark 4.3.2. While it was shown in [13] that in a Riemannian setting transmission conditions will always yield a Fredholm operator with the same index as APS, in the Lorentzian setting the Dirac operator D_B with transmission conditions need not be Fredholm at all. This can be seen easily by looking back at example 2.1.9 where the manifold was constructed such that $Q = \mathrm{id}$ and consequently, $\ker(Q-1)$ is infinite dimensional.

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