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A multifaceted study of marked Gibbs point processes

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Alexander Zass

Più del presente contava il passato, più del possesso il ricordarsene. Di fronte alla memoria, ogni possesso non può apparire che delusivo, banale, insufficiente.

– Giorgio Bassani, *Il giardino dei Finzi–Contini*

These personal histories, old tales from old times that may not be worth remembering. I remember. I must. But who else — to whom can this matter?

– Saul Bellow, *Herzog*

Summary

This thesis focuses on the study of marked Gibbs point processes, in particular presenting some results on their existence and uniqueness, with ideas and techniques drawn from different areas of statistical mechanics: the entropy method from large deviations theory, cluster expansion and the Kirkwood–Salsburg equations, the Dobrushin contraction principle and disagreement percolation.

We first present an existence result for infinite-volume marked Gibbs point processes. More precisely, we use the so-called *entropy method* (and large-deviation tools) to construct marked Gibbs point processes in $\mathbb{R}^d$ under quite general assumptions. In particular, the random marks belong to a general normed space $\mathscr{S}$ and are not bounded. Moreover, we allow for interaction functionals that may be unbounded and whose range is *finite but random*. The entropy method relies on showing that a family of finite-volume Gibbs point processes belongs to sequentially compact entropy level sets, and is therefore tight.

We then present infinite-dimensional Langevin diffusions, that we put in interaction via a Gibbsian description. In this setting, we are able to adapt the general result above to show the existence of the associated infinite-volume measure. We also study its correlation functions via cluster expansion techniques, and obtain the uniqueness of the Gibbs process for all inverse temperatures β and activities z below a certain threshold. This method relies in first showing that the correlation functions of the process satisfy a so-called Ruelle bound, and then using it to solve a fixed point problem in an appropriate Banach space. The uniqueness domain we obtain consists then of the model parameters z and β for which such a problem has exactly one solution.

Finally, we explore further the question of uniqueness of infinite-volume Gibbs point processes on $\mathbb{R}^d$, in the unmarked setting. We present, in the context of repulsive interactions with a hard-core component, a novel approach to uniqueness by applying the discrete Dobrushin criterion to the continuum framework. We first fix a discretisation parameter $a > 0$ and then study the behaviour of the uniqueness domain as $a \longrightarrow 0$. With this technique we are able to obtain explicit thresholds for the parameters z and β, which we then compare to existing results coming from the different methods of cluster expansion and disagreement percolation.

Throughout this thesis, we illustrate our theoretical results with various examples both from classical statistical mechanics and stochastic geometry.

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Introduction

The aim of this thesis is to give an insight into some aspects of the theory of marked Gibbs point processes, namely by providing some answers to the age-old questions of existence and uniqueness of this mathematical object.

Gibbs point processes, and point process theory more generally, lies right at the intersection of Statistical Physics and Probability Theory. This duality has been clear from the start, dating back to the seminal papers of R. L. Dobrushin [28, 30] and O. E. Lanford and D. Ruelle [57] who, in 1968-69, proposed it as a mathematical description of an *equilibrium state* of a physical system consisting of a very large number of interacting components ([42]).

The applications of point processes are not limited, however, to classical (or quantum) physical systems, but are indeed vast ([73, 84]), ranging from economics to biology ([100]), from epidemiology ([27]) to digital imaging and object identifications ([88, 87, 95, 26]), from earthquake modelling ([93, 8]) to criminology ([72]).

Before discussing the results that are included in this work, we look back on the origins of the field to see where these questions originated from and, through some examples, see how the physical field of statistical mechanics connects to the probabilistic theory of point processes.

Equilibrium statistical mechanics

The field of statistical mechanics, initiated in the second half of the 19th century by J. C. Maxwell ([65, 66]), L. E. Boltzmann ([3, 4, 5]), and J. W. Gibbs ([48]), aims at deriving the laws of thermodynamics, the macroscopic behaviour of matter via the microscopic behaviour of its atoms or molecules, as described by the laws of classical (or quantum) mechanics. In this thesis we place ourselves on the grand canonical scale, exemplified below in the setting of a *classical system*.

In the atomic hypothesis matter is thought of as made up of a very large number N of particles of given mass m, located inside a bounded region $\Lambda \subset \mathbb{R}^d$, in interaction via conservative forces. Denote by $\Omega_{\Lambda,N} \coloneqq (\mathbb{R}^d \times \Lambda)^N$, $d \geq 1$, the set of all *microstates*, that is the description obtained by specifying each of the particles position $x_i \in \Lambda$ and momentum $p_i \in \mathbb{R}^d$. The energy of such a system (often called *Hamiltonian*) is

defined by setting

$$H(p_1, \dots, p_N, x_1, \dots, x_N) = \underbrace{\sum_{i=1}^{N} \frac{|p_i|^2}{2m}}_{\text{kinetic energy}} + \underbrace{U(x_1, \dots, x_N)}_{\text{potential energy}}.$$

The dynamics of the system is given by the Newtonian equations

$$\begin{cases} \dot{x}_i = \frac{1}{m} p_i \\ \dot{p}_i = -\nabla_i U(x_1, \dots, x_N), \end{cases}$$

with elastic reflection at the boundary of Λ.

It is clear that because of the enormous number of microscopic variables (recall Avogadro's number of 6.022×10^{23} particles in a mole) one cannot hope for a perfect description of the macroscopic system. Indeed, the basis of statistical mechanics is the idea (due to Boltzmann, [6]) that the state of a large physical system can be described by a probability measure over the set $\Omega_{\Lambda,N}$ of all microstates, as the long-time behaviour of the dynamics (macroscopic observables like free energy, pressure, and entropy) would converge to the *ensemble average* under such an *equilibrium* measure. But what should this probability measure look like? Three probability spaces, *ensembles*, can be used (we do not delve here in the discussion on the equivalence of ensembles, see [43, 44], but only briefly present the different descriptions).

Thanks to the principle of conservation of energy, it is natural to assume such a measure is concentrated on a subset of constant energy level E. This description is the so-called *micro-canonical ensemble* and is given by the collection of all uniform probability distributions $P_{\Lambda,N,E}$ on the high-dimensional submanifold $\Omega_{\Lambda,N,E} = \{(p, x) \in \Omega_{\Lambda,N} : H(p, x) = E\}$. This constraint leads, however, to computational difficulties, as computing constrained integrals over a very high-dimensional manifold is feasible only in special cases, but not in general ([7], Sections 2.1–2.3). It is then actually convenient to remove it by letting the energy vary; this is achieved by fixing the *inverse temperature* thermodynamic parameter $\beta > 0$ instead and introducing a *Boltzmann factor* $e^{-\beta H}$. The *canonical ensemble* is then described by a probability measure

$$P_{\Lambda,N,\beta}(dp, dx) = \frac{\frac{1}{N!} e^{-\beta H(p,x)} dp dx}{Z_{\Lambda,N,\beta}}.$$

on $\Omega_{\Lambda,N}$. The normalisation constant $Z_{\Lambda,N,\beta}$, called the *canonical partition function*, plays a fundamental role in statistical mechanics, as the macroscopic observables of free energy, pressure, and entropy of the system can all be expressed in terms of it.

One may also want to randomise the number N of particles in the system by

adding an exponential weight, via the *chemical potential* μ, in the partition function above. The *grand canonical ensemble* is then described by the following probability measure on the union of phase spaces $\cup_N \Omega_{\Lambda,N}$:

$$P_{\Lambda,\mu,\beta}(dp, dx, N) = \frac{e^{\mu\beta N} P_{\Lambda,N,\beta}(dp, dx)}{\sum_{N=1}^{+\infty} e^{\mu\beta N} Z_{\Lambda,N,\beta}}.$$

Since the terms coming from the momenta and from the positions actually factorise, one can integrate over the momenta to describe the equilibrium *position* of the system. The *configurational* grand canonical ensemble is described by a probability measure

$$P_{\Lambda,z,\beta}(dx, N) \propto z^N e^{-\beta U(x_1,\dots,x_N)} dx_1 \cdots dx_N,$$

where $z = e^{\mu\beta} \left(\frac{2\pi m}{\beta}\right)^{d/2} > 0$ is the *activity* parameter. Note that $P_{\Lambda,z,\beta}$ is absolutely continuous with respect to the Lebesgue measure on $(\mathbb{R}^d)^N$.

In this presentation we take this grand canonical point of view, and identify the potential energy U with the Hamiltonian H.

Point process setting

In the terminology of point process theory, the grand canonical ensemble is described by introducing the space of point *configurations* γ on $\Lambda \subset \mathbb{R}^d$ (σ-finite measures, identified with collections of points in Λ), and considering probability measures on such a space. These measures are called *point processes*, and the most popular one is the Poisson point process, which provides a natural way of regularly distributing points in space.

The *Poisson point process* π_Λ^z of activity measure $z\, dx_\Lambda$, where $z > 0$ and dx_Λ is the Lebesgue measure on Λ, is a probability measure on the space of configurations such that:

- The number of points in Λ under π_Λ^z is a Poisson distributed random variable of mean $z\mathrm{Vol}(\Lambda)$.
- Given the number of points in Λ, the said points are independent and uniformly distributed in Λ.

As before, the interaction is introduced in the system by means of the Boltzmann factor $e^{-\beta H}$, $\beta > 0$, and the resulting probability measure is the *finite-volume Gibbs point process*

$$P_\Lambda(d\gamma) := \frac{1}{Z_\Lambda} e^{-\beta H(\gamma)} \pi_\Lambda^z(d\gamma),$$

with activity z and inverse temperature β, defined over the configurations of points γ in Λ.

In order to model systems with a large number of interacting particles, it is natural

to then consider the thermodynamic limit, that is the limit $\Lambda \uparrow \mathbb{R}^d$. In particular one would like to know whether there exists a *limit state*, a probability measure on the space of configurations that, on any finite volume Λ has the form described above (and when it is unique). Indeed ([90]), *the main problem of equilibrium statistical mechanics is to study the infinite system equilibrium states.* In this sense, the behaviour of the partition function Z_Λ in the thermodynamic limit is of central importance as the free energy, pressure, and entropy can be expressed in terms of its logarithm – for example the pressure $p = \lim_{\Lambda\uparrow\mathbb{R}^d}(\mathrm{Vol}(\Lambda))^{-1}\log(Z_\Lambda)$. Below we discuss conditions for these limits to exist.

Note that the thermodynamic limit is usually considered along a sequence of increasing sets $(\Lambda_n)_n$ that converge to $\mathbb{R}^d$ in the *Van Hove sense* ([99, 39]), that is with $\lim_{n\longrightarrow+\infty}\frac{|\partial\Lambda_n|}{|\Lambda_n|} = 0$. In the prototypical sequence for such a convergence – and the one we use throughout this work – this sets are given by the centred cubes $\Lambda_n = [-n, n)^d$.

Gibbs point processes

Infinite-volume Gibbs point processes are the equilibrium states associated to a (Gibbsian) specification. Characterised by an energy functional H, the activity parameter $z > 0$, and the inverse temperature $\beta > 0$, this large class of point processes is widely used, because of the various types of interaction that it allows for: it can be in the form of a k-body potential, like interacting hard spheres or three-body potentials ([94]); depend on geometric features like Voronoi tessellation or the Area-interaction process (see Figure A); be attractive or repulsive. Standard references in this setting are H.-O. Georgii [42] and D. Ruelle [90].

For many physical models it is natural to consider interactions that come from a sum of *multi-body interactions*: more precisely, the energy functional H can be expressed as the sum $H^{(1)} + \cdots + H^{(k)}$, where for each $1 \le \ell \le k$, the corresponding ℓ-body interaction is

$$H^{(\ell)}(\gamma) := \sum_{\{x_1,\ldots,x_\ell\}\subset\gamma} \phi^{(\ell)}(x_1, \ldots, x_\ell),$$

and $\phi^{(\ell)} : (\mathbb{R}^d)^\ell \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ some (symmetric) ℓ-body potential. Consider, for example, the two-body interaction $H(\gamma) = \sum_{\{x_i,x_j\}\subset\gamma}\phi(x_i, x_j)$. Often, this (pair) potential is taken to be radial, so that it only depends on the distance $|x_i - x_j|$. This can be used to describe a model of indistinguishable *impenetrable spheres* of diameter 1, which is modelled by considering a *hard-core potential*

$$\phi(u) = \begin{cases} +\infty & \text{for } u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

This type of interaction has been, and continues to, be widely studied, see [38].

Gibbs point process theory, however, allows also for interactions of a very different nature. In the framework of geometric interactions, for example, the *Area-interaction model* is an interaction which, while being quite natural from a geometric point of view, is not easily described using any potential.

In this model, proposed in [1] by A. J. Baddeley and M. N. M. van Lieshout, one considers configurations given by circles of radius 1/2 centred in random points of $\mathbb{R}^2$; the energy functional H is proportional to the area (in dimension 2) covered by the configuration γ, i.e. equal to α Area $\left(\cup_{x\in\gamma} B(x, 1/2)\right)$, $\alpha \in \mathbb{R} \setminus \{0\}$. Notice that, for positive values of α, this interaction is attractive, and the typical configurations under the Gibbs point process are those that minimise the total area, exhibiting therefore clustering properties (few but large connected components). On the other hand, for $\alpha < 0$ the interaction is repulsive, and typical configurations exhibit a more ordered pattern (more and smaller connected components).

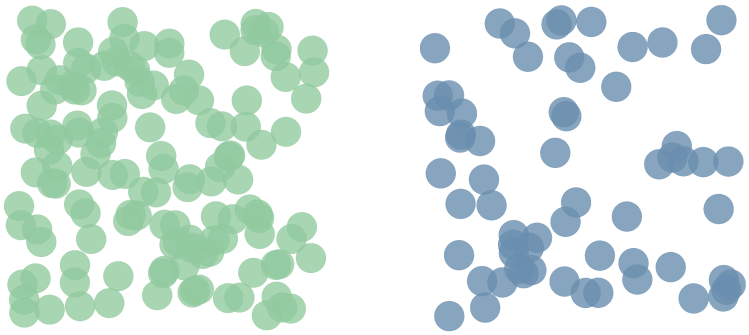

Figure A. Simulations of an attractive ($\alpha > 0$) and repulsive ($\alpha < 0$) Area-interaction model with fixed radius 1/2, same activity z and same $|\alpha|$. Thanks to F. Lavancier for the code that generated the points.

The works of Dobrushin, Lanford and Ruelle show that Gibbs point processes can be described by a set of equations that define their conditional probabilities; we present these equations – referred to as *DLR equations* – in Section 1.3. For an introduction to Gibbs point processes in this spirit, see [20]. For an abstract approach, based on the works of H. Föllmer [36, 37], see the lecture notes [82] by C. Preston.

Chapters 2 and 3 explore, for path-space point processes and classical systems, respectively, the question of how to guarantee the uniqueness of the Gibbs point process. Indeed, while we do not focus on it in this work, a consequence of the DLR description is that a Gibbs point process for a given energy functional H may indeed fail to be unique, which means that the corresponding physical system can have

distinct equilibria. This phenomenon, known as of *phase transition*, has a special physical significance ([22]), but its study – due to its enormous difficulty – is still very incomplete. It will be the subject of future works.

Stable and superstable interactions. We mentioned before how the treatment of the partition function is of crucial importance in statistical mechanics. Indeed, the first question that comes up is whether the grand partition function associated to an energy functional H

$$Z_\Lambda = e^{-z|\Lambda|}\left(1 + \sum_{n=1}^{+\infty} \frac{z^N}{N!} \int_{\Lambda^N} e^{-\beta H(x_1,\dots,x_N)} dx_1 \dots dx_N\right)$$

is well defined: when does this series converge?

In order to solve this problem, it is usual to consider energy functionals H which are *stable*, i.e. there exists a constant $B \geq 0$ such that

$$\forall N \geq 1, \forall x_1, \dots, x_N \in \mathbb{R}^d, \quad H(x_1, \dots, x_N) \geq -BN.$$

This condition is almost always assumed to hold, as it guarantees the convergence of the above series:

$$0 < Z_\Lambda \leq e^{-z|\Lambda|}\left(1 + \sum_{n=1}^{+\infty} \frac{z^N}{N!}|\Lambda|^N e^{\beta BN}\right) = e^{z|\Lambda|(e^{\beta B}-1)} < +\infty,$$

in particular making sense of the definition of the pressure that we gave above (for further discussions on the meaning of the stability conditions and its connection to the existence of the thermodynamic limit, see Section 4.1 of [40]).

An other common assumption is the superstability of the interaction: an energy functional H is said to be *superstable* if there exist constants $B_1, B_2 > 0$ such that, if $\Lambda \subset \mathbb{R}^d$ is a cube of sufficiently large volume,

$$\forall N \geq 1, \forall x_1, \dots, x_N \in \Lambda, \quad H(x_1, \dots, x_N) \geq -B_1 N + B_2 \frac{N^2}{|\Lambda|}.$$

This is satisfied, for example, by the two-body energy functional $H(x_1, \dots, x_N) = \sum_{1\leq i<j\leq N} \phi_{LJ}(|x_i - x_j|)$, where ϕ_{LJ} is the *Lennard–Jones pair potential*

$$\phi_{LJ}(u) = \frac{a}{u^{12}} - \frac{b}{u^6}, \ a, b > 0,$$

pictured in Figure B. In the setting of superstable interactions, the works of Ruelle [91] and Dobrushin [31] were major stepping stones towards answering the question of existence of infinite-volume Gibbs point processes in $\mathbb{R}^d$, as well as providing

first methods for proving uniqueness. Moreover, the superstability assumption is common in the literature, as it allows for a granular control of the density of points in a bounded domain.

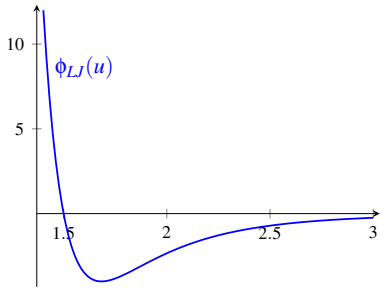

Figure B. The Lennard–Jones pair potential $\phi_{LJ}(u) = 16\big(\big(\frac{3/2}{u}\big)^{12} - \big(\frac{3/2}{u}\big)^{6}\big)$. It explodes in 0, vanishes at $+\infty$, and crosses the x-axis at $x = 3/2$.

On the other hand, in stochastic geometry one deals with interactions that are seldom superstable. Indeed, the Area-interaction model considered earlier is stable but not superstable (see Example 1.1).

When describing the results of the first chapter, we will see how the superstability assumption – quite natural in this Euclidean setting – becomes less so in more complicated spaces. In Chapter 1 we come back to the different (super-) stability assumptions that can be considered and the purposes they serve when proving the existence of an infinite-volume measure.

Lastly, we mention the integral characterisation of Gibbs point processes given by the *GNZ equations*, after H.-O. Georgii, X. X. Nguyen and H. Zessin ([41, 75]). Generalising the Mecke formula of Poisson point processes ([68, 69]), they consist of implicit equations that are in particular very useful in statistical settings ([23, 24, 21]), as they can be solved without computing the partition function.

In Chapter 2 we use the GNZ equations to prove convergence of the Kirkwood–Salsburg equations for the correlation functions.

Marked point processes

Let us come back to the impenetrable-spheres and area-interaction models presented above, at each point $x \in \mathbb{R}^d$ we placed a sphere of fixed radius 1/2. It is natural to think of a situation in which each point is instead assigned a sphere of *random* radius, according to some probability distribution $\mathbf{R}$ on $\mathbb{R}_+$ (see Figure C). This is the typical setting of *marked point processes*: a marked point consists of a location $x \in \mathbb{R}^d$, $d \geq 1$, and a mark m belonging to a general space $\mathscr{S}$. The state space

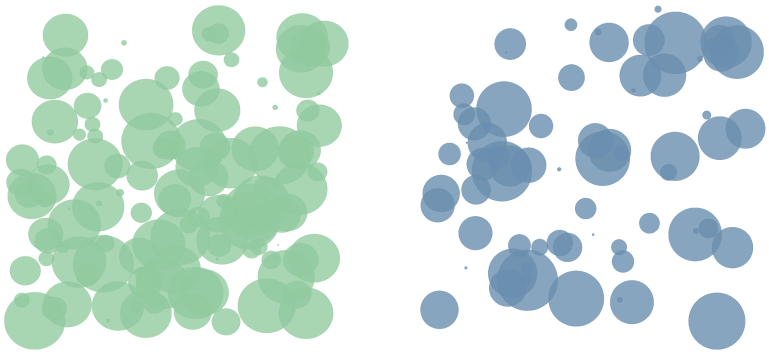

Figure C. Simulations of an attractive ($\alpha > 0$) and repulsive ($\alpha < 0$) Area-interaction model with random radii uniformly distributed in $(0, 1)$, same activity z and same $|\alpha|$.

is then a product space $\mathscr{E} := \mathbb{R}^d \times \mathscr{S}$, and marked point processes are probability measures on the space $\mathscr{M}(\mathscr{E})$ of marked configurations on $\mathscr{E}$.

As before, a finite-volume Gibbs point process can be constructed by considering a perturbation of the Poisson point process π^z_Λ on $\Lambda \times \mathscr{S}$ with intensity measure $z\, dx_\Lambda \otimes \mathbf{R}(dm)$. In this work we are interested in studying the existence and uniqueness, in the thermodynamic limit $\Lambda \uparrow \mathbb{R}^d$, of marked infinite-volume Gibbs point processes.

This thesis is structured as follows: in Chapter 1 we present an existence result for a class of marked Gibbs point processes with unbounded marks and unbounded interaction range. In Chapter 2 we present a point process description for infinite-dimensional Langevin diffusions; in this setting, we first prove the existence of an infinite-volume Gibbs point process in section 2.2, as well as show that a Ruelle bound holds for its correlation functions. After this, in sections 2.3 and 2.4, we present some cluster expansion techniques in order to prove uniqueness of the infinite-volume Gibbs point process. Finally, in Chapter 3, in the setting of unmarked systems, we obtain an explicit uniqueness domain by using a novel approach to the classical Dobrushin criterion.

The three chapters that make up this thesis explore two main themes, those of the existence and uniqueness of an infinite-volume Gibbs point process.

THEME I
EXISTENCE

From the DLR description of marked finite-volume Gibbs point processes, one is then interested in constructing a measure on the full space that, when projected on any finite volume satisfies those same equations. Two issues pop up when looking for such an object: firstly, while the natural starting point is to consider the thermodynamic limit of the finite-volume measure, proving that such a limiting object does indeed exist is far from trivial; secondly, the limiting object could very well not be a Gibbs point process, and showing that it is is an even more delicate matter. In Chapter 1 we present an approach for solving this problem. These results are published in [Z2].

Chapter 1. Marked Gibbs point processes with unbounded interaction: an existence result

Consider a sequence $P_n := P_{\Lambda_n}$, $n \geq 1$, of *finite*-volume Gibbs point processes

$$P_n(d\gamma) := \frac{1}{Z_{\Lambda_n}} e^{-\beta H(\gamma)} \pi^z_{\Lambda_n}(d\gamma)$$

with energy functional H, activity z and inverse temperature β. In Chapter 1 we investigate the question of existence of a corresponding *infinite*-volume point process, and construct, for any z and β, a certain class of continuous marked Gibbs point processes with interactions described by an energy functional H acting both on locations and on marks. This includes, in particular, the case of multi-body potentials, but is indeed a more general framework, useful to treat examples coming from the field of stochastic geometry (as e.g. the area- or the Quermass-interaction model, see Example 1.1).

The entropy method

The originality of our method for constructing an infinite-volume point process consists in the use of the specific entropy as a tightness tool. This relies on the fact that the level sets of the specific-entropy functional are relatively compact in the local convergence topology; see Section 1.3.2. This powerful topological property was first shown in the setting of marked point processes by H.-O. Georgii and H. Zessin in [47].

We briefly present this result here:

Toolbox for the entropy method. Given two probability measures Q and Q' on $\mathscr{M}$, and any finite-volume $\Lambda \subset \mathbb{R}^d$,

- The *relative entropy* of Q with respect to Q' on Λ is defined as

$$I_\Lambda(Q|Q') := \int \log f \; dQ_\Lambda \qquad \text{if } Q_\Lambda \preccurlyeq Q'_\Lambda \text{ with } f := \tfrac{dQ_\Lambda}{dQ'_\Lambda},$$

  otherwise $I_\Lambda(Q|Q') = +\infty$. Here Q_Λ (resp. Q'_Λ) is the image of Q (resp. Q') under the projection $\gamma \mapsto \gamma_\Lambda$.
- The *specific entropy* of Q with respect to Q' is defined as the volumic limit of the relative entropy:

$$\mathcal{I}(Q|Q') = \lim_{n \to +\infty} \frac{1}{|\Lambda_n|} I_{\Lambda_n}(Q|Q').$$

- Let $(\vartheta_\kappa)_{\kappa \in \mathbb{Z}^d}$ be the group of translations in the lattice. For any $a > 0$, the a-entropy level set

$$\mathscr{P}(\mathscr{M})_{\le a} := \left\{ Q \in \mathscr{P}(\mathscr{M}), \text{ stationary under } (\vartheta_\kappa)_{\kappa \in \mathbb{Z}^d} \,:\, \mathcal{I}(Q|\pi^z) \le a \right\}$$

  is relatively compact for the local convergence topology $\tau_{\mathscr{L}}$.

We prove in Proposition 1.13 that the entropy of some sequence of finite-volume Gibbs point processes is uniformly bounded. This sequence is therefore tight, and admits at least one accumulation point.

We also remark that the entropy tool relies mainly on *stability* assumptions of the energy H, without the need for superstability. The usual approach (see e.g. [91]), in fact, uses the superstability condition to precisely control the local density of points; in our framework, we do this thanks to an *equi-integrability* property, which holds on the entropy level sets (see Lemma 1.16). Furthermore, the stability notion we use here is weaker than the classic one of Ruelle, as it includes a term depending on the marks of the configuration. For more details and examples, see Subsection 1.2.3.

The last step of the proof consists in showing that any such accumulation point satisfies the *Gibbsian* property. Since the interaction is not local and not bounded, this property is far from being inherited automatically from the finite-volume approximations, requiring instead an accurate analysis, which is done in Subsection 1.3.4.

Let us mention recent works on the existence of marked Gibbs point processes for particular models. In [18] D. Dereudre proves the existence of the *Quermass-interaction process* as a planar germ-grain model; we draw inspiration from his approach, presenting here an existence result for more general processes, under weaker assumptions. In [14] and [10] the authors treat the case of unbounded marks

in $\mathbb{R}^d$ but restricted to the case of a *finite-range* energy functional which is induced by a *pairwise interaction*.

The novelty of the results presented in this chapter is threefold. Firstly, we do not assume a specific form of the interaction – like pairwise or k-body – but only make assumptions (in Subsection 1.2.3) on the resulting energy functional H itself. In particular, we do not assume superstability of the interaction, but only rely on two stability assumptions ($\mathscr{H}_{st}$) and ($\mathscr{H}_{loc.st}$). Therefore, our results are applicable, for example, in the field of stochastic geometry, as many quite natural energy functionals considered there are stable but not superstable (Example 1.1).

Secondly, the Gibbsian energy functional we consider has an *unbounded range*: it is finite, but random and not uniformly bounded, see Assumption ($\mathscr{H}_r$) – as opposed to models treated for example in [10] which deal with a bounded-range interaction. For a very recent existence proof in the case of infinite-range interaction (without marks) see [25]. Moreover, unlike the hyper-edge interactions presented in [21], we treat the case of interactions which are highly non local: the range of the conditional energy (see Definition 1.5) on a bounded region of an infinite configuration requires knowledge of the whole configuration and cannot be determined only by a local restriction of the configuration.

Lastly, we work with a mark reference distribution whose support is a priori *unbounded* but fulfils a super-exponential integrability condition (see Assumption ($\mathscr{H}_m$)). This requirement, which can unfortunately exclude some measures (for example, Gaussian marks are not allowed), is very much embedded in the method, as it provides the integrability of the functions that induce the local convergence topology.

The main thread of our approach is the reduction of the general marked point process to kind of germ-grain model, where two marked points $(x_1, m_1), (x_2, m_2) \in \mathbb{R}^d \times \mathscr{S}$ do not interact as soon as the two balls with centre x_i and radius $\|m_i\|$, $i = 1, 2$, do not intersect. The framework we work in requires the introduction of a notion of *tempered configurations* (see Section 1.2.2) in order to better control the support of the Gibbs point process we construct. In this way, the growth rate of the marks of faraway points is bounded. In Section 1.3.3 we see that this procedure is justified by the fact that the constructed infinite-volume Gibbs point process is actually concentrated on the subset of tempered configurations.

Chapter 2. Gibbs point processes on path space: existence

Having obtained this general existence result, we were interested in seeing how it could be applied to different and diverse settings. What caught our attention was the possibility of considering point processes on path spaces. When I was offered to write a contribution for the Proceedings of the international conference in Armenia on *Stochastic and analytic methods in Mathematical Physics*, we decided it was the

right opportunity to explore this topic. In [Z3], we applied our previous existence result to the path space setting, along with some first results and a conjecture on the uniqueness of the associated Gibbs point process. This was then rewritten almost from scratch and now makes up the rigorous presentation of infinitely many Langevin diffusions in interaction of Chapter 2. These results are contained in [Z4], which has been submitted for publication.

Through the lens of Gibbs point process theory, we see a diffusion – starting in $x \in \mathbb{R}^d$ and with displacement $\big(m(s),\ s \in [0,1]\big)$ – as a marked point $\mathbf{x} = (x, m) \in \mathscr{E} := \mathbb{R}^d \times C_0$, where C_0 is the space of continuous paths $\big(m(s), s \in [0,1]\big)$ starting at $m(0) = 0$. On this state space we then consider a pair potential Φ that acts on both the starting points and the trajectories of the marked points. This leads to a Gibbsian energy functional H, with (finite but) not uniformly bounded interaction range, for which the questions of existence and uniqueness of Gibbs point processes are far from trivial. In particular, we note how the random marks are a priori unbounded.

In this setting we can start from interactions which are common for classical systems in $\mathbb{R}^d$, like the Lennard–Jones pair potential, and use them to describe interactions between paths instead. We remark, however, that the typical potentials that we consider (see Example 2.1) need a hard-core repulsion near the origin in order to satisfy the stability conditions that are required in the method. Such an assumption also appears in Chapter 3, in order to control the number of paths starting in a given cube in $\mathbb{R}^d$.

In Section 2.2 we tackle the existence question, via the Dobrushin–Lanford–Ruelle description of Gibbs point processes. Under some stability assumptions for H, we are able to prove (in Theorem 2.1) the existence of at least one infinite-volume Gibbs point process on path space P^z with energy functional H, for any activity z and inverse temperature β, by applying the entropy method presented for the general marked setting in the previous chapter.

Moreover, we also show that, for any $N \geq 1$, the N-point correlation function ρ_N of these Gibbs point processes satisfy a (point-dependent) *Ruelle bound* of the following form: there exists a function $\mathbf{c} : \mathscr{E} \longrightarrow \mathbb{R}_+$ such that, for almost any finite path configuration $(\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathscr{E}^N$,

$$\rho_N(\mathbf{x}_1, \dots, \mathbf{x}_N) \leq \prod_{i=1}^{N} \mathbf{c}(\mathbf{x}_i).$$

The uniqueness question is tackled next, and spiritually belongs to the second theme of this thesis.

THEME II
UNIQUENESS

As mentioned before, the DLR description leaves open the question of (non-) uniqueness of the infinite-volume Gibbs point process. While in Chapter 1 we were able to prove the existence of an infinite-volume measure for a large class of processes, a different, more specific, approach is needed when tackling the uniqueness question. Indeed, we focus on particular models to show that under certain assumptions there exists only one infinite-volume Gibbs point process. This notwithstanding, we are hopeful the techniques and assumptions of Sections 2.3 and 2.4 – presented here making use of the specificity of the path space properties – could be adapted to different marked settings.

We do not explore here the topic of phase transition, but mention that the question of non-uniqueness of the Gibbs point process is of major interest and very few results are known. In particular, the existing literature mainly deals with coloured (multi-species) models like the *Widom–Rowlinson* model; see for instance [9, 22]. In these works, phase transition is proved by showing that one species "dominates" the others when the activity of the points is large enough.

Chapter 2. Gibbs point processes on path space: uniqueness

In Sections 2.3 and 2.4 we present, as a novel result, an explicit activity domain where uniqueness of the Gibbs point process holds. This is obtained with the approach of cluster expansion and the Kirkwood–Salsburg equations – a method which was first developed for lattice systems in the 1980s (see e.g. [63]) and then extended to the continuous case (see e.g. [64, 74]). These results are also contained in [Z4].

The cluster expansion approach to uniqueness

In the case of unmarked continuous point processes, the technique relies on considering a series expansion of the correlation functions. As presented by D. Ruelle in [90], one first shows that the correlation functions of a Gibbs point process can be expressed as an absolutely converging series of cluster terms, and then proves uniqueness by considering a system of integral equations – the so-called Kirkwood–Salsburg equations – that the correlation functions satisfy. In fact, these equations can be reformulated as a fixed-point problem for an operator $\mathbf{K}_z$ in an appropriately chosen Banach space, having therefore a unique solution.

The cluster expansion approach is actually well adapted to the marked setting. Indeed, S. Poghosyan and D. Ueltschi develop, in [80], abstract techniques that can be used both in the classical and in the marked setting, under assumptions of so-called modified-regularity of the interaction. These assumptions and techniques

are further developed in [81] by S. Poghosyan and H. Zessin, proving uniqueness of infinite-volume Gibbs point processes for potentials satisfying a certain stability condition (which they refer to as *Penrose stability*). Some similar result is presented by S. Jansen in [51], but making strong use of the repulsive nature of the interaction she considers. These techniques can be restrictive in our setting of unbounded marks (see Example 2.4), so we use here a different approach: inspired by the work [56] of T. Kuna, our approach relies on some tree-graph estimates, that allow to prove a Ruelle bound for the correlation functions of infinite-volume Gibbs point processes.

A key point, presented in Section 2.3 under a different set of assumptions than that of Section 2.2, consists in using cluster expansion to obtain a Ruelle bound for the correlation functionals of a Gibbs point process of activity z. In particular, we show that, under an additional regularity assumption for the interaction potential Φ, there exists an activity threshold $\mathfrak{z}_{\mathrm{Ru}}(\beta) > 0$ such that, for any $z \in (0, \mathfrak{z}_{\mathrm{Ru}}(\beta))$, the correlation functions $\rho_N^{(P)}$ of any Gibbs point process P with activity z and inverse temperature β satisfy a Ruelle bound as above, but where $\mathbf{c}$ is uniformly bounded: for almost any $(\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathscr{E}^N$, $\rho_N^{(P)}(\mathbf{x}_1, \dots, \mathbf{x}_N) \leq \mathbf{c}^N$. This shows that the correlation functions belong to a certain Banach space $\mathbb{X}_{\mathbf{c}}$.

In Section 2.4, after showing that there exists an activity threshold $\mathfrak{z}_{\mathrm{crit}}(\beta) > 0$ such that, for any $z \in (0, \mathfrak{z}_{\mathrm{crit}}(\beta))$, the norm of the Kirkwood–Salsburg operator $\mathbf{K}_z$ in $\mathbb{X}_{\mathbf{c}}$ is bounded by 1, we show that the associated equations have a unique solution and obtain the following uniqueness domain (in Theorem 2.2): for any $\beta > 0$ and $z \in (0, \mathfrak{z}_{\mathrm{crit}}(\beta))$, there exists a unique infinite-volume Gibbs point process P with activity z and inverse temperature β associated to the energy functional H.

Chapter 3. An explicit Dobrushin uniqueness region for Gibbs point processes with repulsive interactions

Cluster expansion is far from the only technique that can be used to prove uniqueness of an infinite-volume Gibbs point process. Indeed, different methods exist, and usually one tries to find the one which yields the larger activity domain for the model at hand.

Already for lattice systems, the uniqueness question is one of main interest in the community, and different arguments and methods exist, including Peierls' argument [78], the criterion proposed by Dobrushin in the pioneering paper [29], the previously discussed cluster expansion (see [67, 90]), a characterisation due to J. L. Lebowitz and A. Martin Löf [59], and disagreement percolation (see, for example, [46, 97, 98]). No one method is a priori stronger than the others: it is a question of finding which is better adapted to the specific model one wishes to prove uniqueness of the Gibbs point process for.

This consideration holds true when passing from the lattice to the continuous

framework, where the uniqueness question is even more delicate. It is with this in mind that P. Houdebert and I set out to find a class of potentials for which we could not only prove a uniqueness result with a new technique, but also compare our uniqueness domains with those pre-existing in the literature. The three techniques we analyse in this setting – namely the *Dobrushin contraction criterion*, *cluster expansion*, and *disagreement percolation* – all work under different assumptions, and yield different parameter domains in which uniqueness holds. It is therefore generally complicated to compare their efficacy.

In Chapter 3 we present a uniqueness result for a class of Gibbs point processes in $\mathbb{R}^d$, $d \geq 2$, where the Gibbsian interaction is given by a non-negative pair potential ϕ. In the setting of Gibbs point processes that are parametrised by an activity parameter $z > 0$ and an inverse temperature $\beta > 0$, we placed particular focus on obtaining an explicit uniqueness domain of the parameters z, β. The results of this chapter are contained in [Z1], which has been submitted for publication.

We remark how the intuition that, for given β, uniqueness is achieved for activities z small enough has actually been disproved for the specific case of the Widom–Rowlinson model with random radii with heavy tails, see [22].

The Dobrushin contraction method

In the main result of this chapter, Theorem 3.1, we provide a set of (simple to test) assumptions which lead to a uniqueness result for small activity. The strength of this result lies in the explicit nature of the uniqueness domain it yields, i.e. the following parameter region for z, β:

$$\left\{ z, \beta > 0 : \; z < \left(\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\beta \phi(x,y)} dy \right)^{-1} \right\},$$

and in the simplicity of its proof, which makes use of the Dobrushin criterion. In fact, we show that the celebrated original uniqueness criterion from [29] – very general, but presented only for discrete Gibbs models – can be applied to continuum models by first decomposing the space $\mathbb{R}^d$ into disjoint cubes of some side length $a > 0$. For different values of $a > 0$ one then obtains different uniqueness regions which are not explicit and therefore not easily comparable. However, by letting this discretisation parameter a tend to 0, we were able to obtain the explicit uniqueness bound above.

The more restricting requirement for our theorem is the assumption (A1), introduced in Section 3.2.1, that the potential ϕ have a hard-core component close to the origin. This condition is key in the technique, as it allows us to restrict the admissible boundary conditions to those having at most one point in each small cube, which in turn simplifies the computations when taking the limit $a \longrightarrow 0$.

Since the probability of having more than one point in a small cube vanishes with its size, we conjecture that it could be possible for Theorem 3.1 to still be valid

without this hard-core assumption. In Section 3.2.3 we discuss further on the assumptions used in Theorem 3.1.

The comparison of our uniqueness region with the one that can be obtained from existing works using respectively cluster expansion ([51]) and disagreement percolation ([49]) is included in Section 3.2.5. What transpires from this comparison is that Theorem 3.1 yields a larger uniqueness region than what can be obtained via cluster expansion [34, 51]. Furthermore, as expected, for β small enough, the result is also better than the one obtained from disagreement percolation, yielding a larger range of possible activities z for which uniqueness holds.

THINKING BACK
AND LOOKING AHEAD

The work documented in this thesis has drawn from a wide range of techniques and methods, ranging more than 150 years. It is therefore no surprise that, while we have presented quite a few different ideas, discussing when and how one is better adapted than another, there are many others that I did not explore in depth during my time as a PhD student.

The *elephant in the room* that we managed to look away from during this exposition is the *variational principle*: we already mentioned the GNZ equations, but another interesting and historically relevant characterisation of Gibbs point processes – tightly related to a large deviations principle ([47]) – is that they are the unique minimisers of the free excess energy ([19]). This has been rigorously established so far only under strong locality assumptions on the interaction (or compactness of the mark space), so the question of how to extend this theory to a more general marked space is definitely something I look forward to exploring in the future.

A second large project that I would like to undertake is the possibility of generalising the existence result presented in Chapter 1 to interaction ranges that have a more general dependence on the full configuration (cf. Assumption 1.1). This would require a much more precise control of the number of points in a given region, which could be obtained, for example, thanks to some intrinsic geometrical information of the model (perhaps building a bridge with the hyperedge potentials, [21, 50]).

In Chapter 3 we gave an overview of several methods of proving uniqueness of a Gibbs point process. In this direction there are at least two things I plan to look at: as we focused here on providing a common framework where we could compare the different methods, the assumptions on the interaction potential are not necessarily optimal, and could hopefully be weakened to obtain a more general result. In connection to this, I would like to try and extend the technique to a marked space

setting – for example the path space of Chapter 2, and see if perhaps it is better adapted than cluster expansion.

On the topic of Gibbs point processes on path space, D. Dereudre showed in [17] the equivalence between the law of an infinite-dimensional interacting SDE with Gibbsian initial law, and a Gibbs point process on the path space with a certain energy functional. It is therefore a natural question to ask whether a Gibbs point process with energy functional H as in Chapter 2 is indeed the law of infinite dimensional interacting SDE. Using Malliavin derivatives, D. Dereudre proved that Gibbs point processes with *regular* H are the law of SDEs with a certain non-markovian drift. The existence and uniqueness results on path space presented here could therefore be useful in obtaining a criterion for solving infinite-dimensional SDEs. We note that this was explored in the lattice case in [70, 13] (for an example of superstable interactions in this setting, see [62]).

I am hopeful that the methods and techniques I learned during these years can be adapted and extended to many more settings, and I am eager to expand my obviously still small knowledge in the huge field of Gibbs point processes.

Come stabilire il momento esatto in cui comincia una storia? Tutto è sempre cominciato già prima. La prima riga della prima pagina di ogni romanzo rimanda a qualcosa che è già successo fuori del libro. Oppure la vera storia è quella che comincia dieci pagine più avanti e tutto ciò che precede è solo un prologo.

– Italo Calvino, *Se una notte d'inverno un viaggiatore*

Ah non, par example! The intellectual imagination! With me all or not at all, Non Serviam.

– James Joyce, *Ulysses*

Marked Gibbs point processes with unbounded interaction: an existence result

1

We construct marked Gibbs point processes in $\mathbb{R}^d$ under quite general assumptions. Firstly, we allow for interaction functionals that may be unbounded and whose range is not assumed to be uniformly bounded. Indeed, our typical interaction admits an a.s. *finite but random* range. Secondly, the random marks – attached to the locations in $\mathbb{R}^d$ – belong to a general normed space $\mathscr{S}$. They are not bounded, but their law should admit a super-exponential moment. The approach used here relies on the so-called *entropy method* and large-deviation tools in order to prove tightness of a family of finite-volume Gibbs point processes.

1.1 Point-measure formalism

The point configurations considered here live in the product state space $\mathscr{E} := \mathbb{R}^d \times \mathscr{S}$, $d \geq 1$, where $\left(\mathscr{S}, \|\cdot\|\right)$ is a general normed space: each point *location* in $\mathbb{R}^d$ has an associated *mark* belonging to $\mathscr{S}$. The location space $\mathbb{R}^d$ is endowed with the Euclidean norm $|\cdot|$, and the associated Borel σ-algebra $\mathscr{B}(\mathbb{R}^d)$; we denote by $\mathscr{B}_b(\mathbb{R}^d) \subset \mathscr{B}(\mathbb{R}^d)$ the set of bounded Borel subsets of $\mathbb{R}^d$. A set Λ belonging to $\mathscr{B}_b(\mathbb{R}^d)$ will often be called a *finite volume*. We denote by $\mathscr{B}(\mathscr{S})$ the Borel σ-algebra on $\mathscr{S}$.

The set of point measures (or *configurations*) on $\mathscr{E}$ is denoted by $\mathscr{M}$; it consists of the integer-valued, σ-finite measures γ on $\mathscr{E}$:

$$\mathscr{M} := \left\{ \gamma = \textstyle\sum_i \delta_{\mathbf{x}_i} \,:\, \mathbf{x}_i = (x_i, m_i) \in \mathbb{R}^d \times \mathscr{S}, \forall \Lambda \in \mathscr{B}_b(\mathbb{R}^d),\ \mathrm{Card}(\{ i \,:\, x_i \in \Lambda\}) < +\infty \right\}.$$

We endow $\mathscr{M}$ with the canonical σ-algebra generated by the family of local counting functions on $\mathscr{M}$,

$$\gamma = \sum_i \delta_{(x_i, m_i)} \longmapsto \mathrm{Card}(\{ i \,:\, x_i \in \Lambda,\ m_i \in A\}),\ \Lambda \in \mathscr{B}_b(\mathbb{R}^d),\ A \in \mathscr{B}(\mathscr{S}).$$

We denote by $\underline{o}$ the zero point measure whose support is the empty set. Since, in the framework developed in this work, we only consider *simple* point measures, we identify them with the subset of their atoms:

$$\gamma \equiv \left\{ \mathbf{x}_1, \ldots, \mathbf{x}_n, \ldots \right\} \subset \mathscr{E}.$$

For a point configuration $\gamma \in \mathscr{M}$ and a fixed set $\Lambda \subset \mathbb{R}^d$, we denote by γ_Λ the restriction of the point measure γ to the set $\Lambda \times \mathscr{S}$:

$$\gamma_\Lambda := \gamma \cap (\Lambda \times \mathscr{S}) = \sum_{\{i\colon x_i \in \Lambda\}} \delta_{(x_i, m_i)}.$$

Given two configurations $\gamma, \xi \in \mathscr{M}$, we denote their *concatenation* (or union) by $\gamma\xi := \gamma \cup \xi$.

A *functional* is a measurable $\mathbb{R} \cup \{+\infty\}$-valued map defined on $\mathscr{M}$. We introduce specific notations for some of them: the mass of a point measure γ is denoted by $|\gamma|$. It corresponds to the number of its atoms if γ is simple.

We also denote by $\mathfrak{m}$ the supremum of the size of the marks of a configuration:

$$\mathfrak{m}(\gamma) := \sup_{(x,m)\in\gamma} \|m\|, \quad \gamma \in \mathscr{M}.$$

The integral of a fixed function $f : \mathscr{E} \longrightarrow \mathbb{R}$ under the measure $\gamma \in \mathscr{M}$ – when it exists – is denoted by

$$\langle \gamma, f \rangle := \int f d\gamma = \sum_{\mathbf{x}\in\gamma} f(\mathbf{x}).$$

For a finite volume Δ, we call *local* or more precisely *Δ-local*, any functional F satisfying

$$F(\gamma) = F(\gamma_\Delta), \quad \gamma \in \mathscr{M}.$$

We also define the set of finite point measures on $\mathscr{E}$:

$$\mathscr{M}_f := \left\{ \gamma \in \mathscr{M} \ : \ |\gamma| < +\infty \right\}.$$

Moreover, for any bounded subset $\Lambda \subset \mathbb{R}^d$, $\mathscr{M}_\Lambda$ is the subset of $\mathscr{M}_f$ consisting of the point measures whose support is included in $\Lambda \times \mathscr{S}$:

$$\mathscr{M}_\Lambda := \left\{ \gamma \in \mathscr{M} \ : \ \gamma = \gamma_\Lambda \right\} \subset \mathscr{M}_f.$$

Let $\mathscr{P}(\mathscr{M})$ denote the set of probability measures (or *point processes*) on $\mathscr{M}$.
We write $\mathbb{N}^*$ for the set of non-zero natural numbers $\mathbb{N} \setminus \{0\}$. The open ball in $\mathbb{R}^d$ centred in $y \in \mathbb{R}^d$ with radius $r \in \mathbb{R}_+$ is denoted by $B(y, r)$.

1.2 Gibbsian setting

1.2.1 Mark reference distribution

The mark associated to any point of a configuration is random. We assume that the reference mark distribution $\mathbf{R}$ on $\mathscr{S}$ is such that its image under the map $m \mapsto \|m\|$

is a probability measure ρ on $\mathbb{R}_+$ that admits a super-exponential moment, in the following sense:
($\mathscr{H}_m$) There exits $\delta > 0$ such that

$$\int_{\mathbb{R}_+} e^{\ell^{d+2\delta}} \rho(d\ell) < +\infty. \tag{1.1}$$

Throughout this chapter, the parameter δ is fixed.

Remark. The probability measure ρ is the density of a positive random variable X such that $X^{\frac{2}{d}+\varepsilon}$ is subgaussian for some $\varepsilon > 0$ (see e.g. [52], [60]).

1.2.2 Tempered configurations

We introduce the concept of *tempered configuration*. For such a configuration γ, the number of its points in any finite volume Λ, $|\gamma_\Lambda|$, should grow sublinearly w.r.t. the volume, while its marks should grow as a fraction of it. More precisely, we define the space $\mathscr{M}^{\text{temp}}$ of tempered configurations as the following increasing union

$$\mathscr{M}^{\text{temp}} := \bigcup_{\mathfrak{t}\in\mathbb{N}} \mathscr{M}^{\mathfrak{t}},$$

where

$$\mathscr{M}^{\mathfrak{t}} = \left\{ \gamma \in \mathscr{M} \ : \ \forall l \in \mathbb{N}^*, \ \langle \gamma_{B(0,l)}, f \rangle \le \mathfrak{t}\, l^d \ \text{ for } f(x,m) := 1 + \|m\|^{d+\delta} \right\}. \tag{1.2}$$

We now prove some properties satisfied by tempered configurations.

Lemma 1.1. The mark associated to a point in a tempered configuration is asymptotically negligible with respect to the norm of the said point: any tempered configuration $\gamma \in \mathscr{M}^{\text{temp}}$ satisfies

$$\lim_{l\to+\infty} \frac{1}{l} \mathfrak{m}(\gamma_{B(0,l)}) = 0.$$

Proof. Let $\gamma \in \mathscr{M}^{\mathfrak{t}}$, $\mathfrak{t} \ge 1$. From (1.2), recalling that $\mathfrak{m}(\gamma) = \sup_{(x,m)\in\gamma} \|m\|$, we get that, for all $l \ge 1$,

$$\mathfrak{m}(\gamma_{B(0,l)}) \le \left(\mathfrak{t} l^d\right)^{1/d+\delta} = \frac{(\mathfrak{t} l^d)^{1/d+\delta}}{l} l.$$

Define, for any $\eta \in (0,1)$,

$$l_1(\mathfrak{t},\eta) := \left(\frac{\mathfrak{t}}{\eta^{d+\delta}}\right)^{1/\delta}. \tag{1.3}$$

Then, if $l \ge l_1(\mathfrak{t},\eta)$,

$$\frac{\mathfrak{m}(\gamma_{B(0,l)})}{l} \le \frac{(\mathfrak{t} l^d)^{1/d+\delta}}{l} \le \eta \in (0,1), \tag{1.4}$$

and the Lemma is proved. □

Lemma 1.2. Let $\gamma \in \mathscr{M}^{\mathfrak{t}}, \mathfrak{t} \geq 1$, and define $\mathfrak{l}(\mathfrak{t}) := \frac{1}{2}\, l_1(\mathfrak{t}, \frac{1}{2})$, where l_1 is defined by (1.3). Then, for all $l \geq \mathfrak{l}(\mathfrak{t})$, the following implication holds:

$$\mathbf{x} = (x, m) \in \gamma_{B(0,2l+1)^c} \implies B(x, \|m\|) \cap B(0, l) = \emptyset.$$

Proof. Let $\gamma \in \mathscr{M}^{\mathfrak{t}}$ and $(x, m) \in \gamma$ such that $|x| \geq 2l + 1$.

By definition of $l_1(\mathfrak{t}, \frac{1}{2})$, since $(x, m) \in \gamma_{B(0,\lceil x \rceil)}$,

$$|x| - \|m\| \overset{(1.4)}{\geq} |x| - \tfrac{1}{2}\lceil |x| \rceil \geq \tfrac{1}{2}|x| - \tfrac{1}{2} \geq l.$$ □

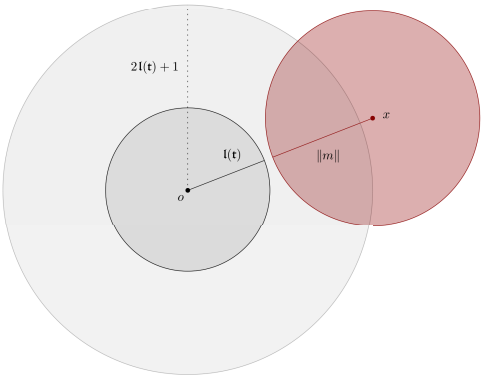

Figure 1.1. For $(x, m) \in \gamma \in \mathscr{M}^{\mathfrak{t}}$, $\mathfrak{t} \geq 1$, such that $|x| \geq 2\,\mathfrak{l}(\mathfrak{t}) + 1$, $B(x, \|m\|)$ does not intersect $B\big(0, \mathfrak{l}(\mathfrak{t})\big)$.

The assertion of Lemma 1.2 is illustrated in Figure 1.1. Define the germ-grain set Γ of a configuration γ as usual by

$$\Gamma := \bigcup_{(x,m)\in\gamma} B(x, \|m\|) \subset \mathbb{R}^d,$$

where the point x is the *germ* and the ball $B(0, \|m\|)$ is the *grain*. Lemma 1.2 then implies that, for tempered configurations, only a finite number of balls of their germ-grain set can intersect a fixed bounded subset of $\mathbb{R}^d$. This remark will be very useful when defining the range of the interaction in (1.9).

1.2.3 Energy functionals and finite-volume Gibbs point processes

For a fixed finite volume $\Lambda \subset \mathbb{R}^d$, we consider, as reference marked point process, the Poisson point process π^z_Λ on $\mathscr{E}$ with intensity measure $z\, dx_\Lambda \otimes \mathbf{R}(dm)$. The coefficient z is a positive real number, dx_Λ is the Lebesgue measure on Λ, and the

probability measure $\mathbf{R}$ on $\mathscr{S}$ was introduced in Subsection 1.2.1. In this model, since the spatial component of the intensity measure is diffuse, the configurations are a.s. simple. Moreover, the random marks of different points of the configuration are independent random variables.

We recall briefly the definition of a *Poisson point process* (for more details see, for example, [15, 55, 58]):

i. For every bounded set $\Lambda \subset \mathbb{R}^d$, the number of points in $\Lambda \times \mathscr{S}$ under π^z_Λ is given by a Poisson random variable with parameter $z dx_\Lambda(\Lambda)$.
ii. Given the number of points in a bounded set Λ, the locations x of said points are independent and uniformly distributed in Λ, while their marks m are independently distributed according to the probability measure $\mathbf{R}$.

To model and quantify a possible interaction between the point locations and the marks of a configuration, one introduces the general notion of energy functional.

Definition 1.3. An *energy* functional H is a translation-invariant measurable functional on the space of finite configurations

$$H : \mathscr{M}_f \longrightarrow \mathbb{R} \cup \{+\infty\}.$$

We use the convention $H(\underline{o}) = 0$.

Configurations with infinite energy will be negligible with respect to Gibbs point processes.

Definition 1.4. For $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, the *finite-volume Gibbs point process with free boundary condition* with energy functional H, activity $z > 0$ and inverse temperature $\beta > 0$ is the probability measure P_Λ on $\mathscr{M}$ defined by

$$P_\Lambda(d\gamma) := \frac{1}{Z_\Lambda} e^{-\beta H(\gamma_\Lambda)} \pi^z_\Lambda(d\gamma). \tag{1.5}$$

The normalisation constant Z_Λ is called *partition function.* We will see in Lemma 1.7 why this quantity is well defined under the assumptions we work with.

Notice how π^z_Λ – and therefore P_Λ – is actually concentrated on $\mathscr{M}_\Lambda$, the finite point configurations with atoms in Λ.

Remark. In this chapter we omit the dependence on z and β from the notations of the Gibbs point process, as the results presented here are valid for, and indeed do not depend on, any value of the intensity and inverse temperature parameters.

The process π^z_Λ extends naturally to an infinite-volume measure π^z; the question we explore in this work is how to do the same for P_Λ. The first step in order to define an infinite-volume Gibbs point process is to be able to consider the energy of configurations with infinitely many points. In order to do this, we approximate any

(tempered) configuration γ by a sequence of finite ones $(\gamma_{\Lambda_n})_n$. Using a terminology that goes back to Föllmer [35], we introduce the following

Definition 1.5. For $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, the *conditional energy of γ on Λ given its environment* is the functional H_Λ defined, on the tempered configurations, as the following limit:

$$H_\Lambda(\gamma) = \lim_{n\to\infty} \Big(H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n\setminus\Lambda}) \Big), \quad \gamma \in \mathscr{M}^{\mathrm{temp}}, \tag{1.6}$$

where $\Lambda_n := [-n, n)^d$ is an increasing sequence of centred cubes of volume $(2n)^d$, converging to $\mathbb{R}^d$.

Remarks.
i. Notice that the conditional energy of finite configurations confined in Λ coincides with their energy: $H_\Lambda(\gamma_\Lambda) \equiv H(\gamma_\Lambda)$. In general, however, the conditional energy $H_\Lambda(\gamma)$ of an infinite configuration γ does not reduce to $H(\gamma_\Lambda)$ because of the possible interaction between (external) points of γ_{Λ^c} and (internal) points of γ_Λ. In other words, the conditional energy is possibly not a local functional. In this paper, we are interested in this general framework.
ii. Indeed, we will work with energy functionals H for which the limit in (1.6) is stationary, i.e. reached for a finite n (that depends on γ). Assumption $(\mathscr{H}_r)$ below ensures this property.
iii. Since π^z_Λ only charges configurations in Λ, P_Λ can be equivalently defined as

$$P_\Lambda(d\gamma) = \frac{1}{Z_\Lambda} e^{-\beta H_\Lambda(\gamma)} \pi^z_\Lambda(d\gamma).$$

The key property of such conditional energy functionals is the following additivity; the proof of this lemma is analogous to the one in [18], Lemma 2.4, that works in the more specific setting of Quermass-interaction processes.

Lemma 1.6. The family of conditional energy functionals is additive, i.e. for any $\Lambda \subset \Delta \in \mathscr{B}_b(\mathbb{R}^d)$, there exists a measurable function $\phi_{\Lambda,\Delta} : \mathscr{M}^{\mathrm{temp}} \longrightarrow \mathbb{R}$ such that

$$H_\Delta(\gamma) = H_\Lambda(\gamma) + \phi_{\Lambda,\Delta}(\gamma_{\Lambda^c}), \quad \gamma \in \mathscr{M}^{\mathrm{temp}}. \tag{1.7}$$

Assumption 1.1. Let us now describe the framework of our study, by considering for the energy functional H a global stability assumption $(\mathscr{H}_{st})$, a range assumption $(\mathscr{H}_r)$ and a locally-uniform stability assumption $(\mathscr{H}_{loc.st})$:

$(\mathscr{H}_{st})$ There exists a constant $B \geq 0$ such that the following *stability* inequality holds

$$H(\gamma) \geq -B\langle \gamma, 1 + \|m\|^{d+\delta} \rangle, \quad \gamma \in \mathscr{M}_f. \tag{1.8}$$

($\mathscr{H}_r$) Fix $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$. For any $\gamma \in \mathscr{M}^{\mathfrak{t}}$, $\mathfrak{t} \geq 1$, there exists a positive finite number $\mathfrak{r} = \mathfrak{r}(\gamma, \Lambda)$ such that

$$H_\Lambda(\gamma) = H\big(\gamma_{\Lambda \oplus B(0,\mathfrak{r})}\big) - H\big(\gamma_{\Lambda \oplus B(0,\mathfrak{r}))\setminus\Lambda}\big), \tag{1.9}$$

where $\Lambda \oplus B(0,\mathfrak{r}) := \big\{ x \in \mathbb{R}^d \,:\, \exists y \in \Lambda,\ |y - x| \leq \mathfrak{r} \big\}$. Equivalently, the limit in (1.6) is already attained at the smallest $n \geq 1$ such that $\Lambda_n \supset \Lambda \oplus B(0,\mathfrak{r})$. Indeed, one can choose

$$\mathfrak{r}(\gamma, \Lambda) = 2\,\mathfrak{l}(\mathfrak{t}) + 2\,\mathfrak{m}(\gamma_\Lambda) + 1.$$

($\mathscr{H}_{loc.st}$) Fix $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$. For any $\mathfrak{t} \geq 1$ there exists a constant $B' = B'(\Lambda, \mathfrak{t}) \geq 0$ such that the following *stability* of the conditional energy holds, uniformly for all $\xi \in \mathscr{M}^{\mathfrak{t}}$:

$$H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c}) \geq -B' \langle \gamma_\Lambda, 1 + \|m\|^{d+\delta} \rangle, \quad \gamma_\Lambda \in \mathscr{M}_\Lambda. \tag{1.10}$$

Remarks. *i.* Notice how the stability assumption ($\mathscr{H}_{st}$) is weaker than the usual Ruelle stability $H(\gamma) \geq -B|\gamma| = -B\langle \gamma, 1 \rangle$, for the presence of the mark-dependent negative term $-B\langle \gamma, \|m\|^{d+\delta} \rangle$.

ii. Two points $\mathbf{x} = (x, m), \mathbf{y} = (y, n) \in \mathscr{E}$ of a configuration γ are not in interaction whenever $B(x, \|m\|) \cap B(y, \|n\|) = \emptyset$, so that Assumption (1.9) has the following interpretation: there is no influence from the points of $\gamma_{(\Lambda \oplus B(0,\mathfrak{r}))^c}$ on the points of γ_Λ: $H_\Lambda(\gamma) = H_\Lambda(\gamma_{\Lambda \oplus B(0,\mathfrak{r})})$. Therefore, the *range* of the energy H_Λ at the configuration γ is smaller than $\mathfrak{r}(\gamma, \Lambda)$, which is *finite but random* since it depends on γ. This range may not be uniformly bounded when γ varies.

Lemma 1.7. Under assumptions ($\mathscr{H}_{st}$) and ($\mathscr{H}_m$) the partition function Z_Λ is well defined, that is, it is finite and positive.

Proof. We estimate:

$$\begin{aligned} Z_\Lambda &\geq \pi^z_\Lambda(\underline{o}) = e^{-z|\Lambda|} > 0; \\ Z_\Lambda &= \int e^{-\beta H(\gamma_\Lambda)} \pi^z_\Lambda(d\gamma) \overset{(1.8)}{\leq} \int e^{\beta B \langle \gamma_\Lambda, 1 + \|m\|^{d+\delta} \rangle} \pi^z_\Lambda(d\gamma) \\ &\leq e^{-z|\Lambda|} \exp\Big\{ e^{\beta B} z|\Lambda| \int_{\mathbb{R}_+} e^{\beta B \ell^{d+\delta}} \rho(d\ell) \Big\} \overset{(1.1)}{<} +\infty. \end{aligned}$$

□

We provide here examples of energy functionals on marked configurations, which satisfy the assumptions above. In Chapter 2 we describe, in the context of interacting

diffusions, a further example of a pair interaction that acts on both locations and marks of a configuration, where the mark space is a path space.

Example 1.1 (Geometric multi-body interaction in $\mathbb{R}^2$)**.** Consider the marked-point state space $\mathscr{E} = \mathbb{R}^2 \times \mathbb{R}_+$, and recall that one can associate, to any finite configuration $\gamma = \{(x_1, m_1), \dots, (x_N, m_N)\}$, $N \geq 1$, the germ-grain set

$$\Gamma = \bigcup_{i=1}^{N} B(x_i, m_i) \subset \mathbb{R}^2.$$

Consider, as reference mark measure, a measure $\mathbf{R}$ on $\mathbb{R}_+$ satisfying $(\mathscr{H}_m)$, that is, there exists $\delta > 0$ such that

$$\int_{\mathbb{R}_+} e^{\ell^{2+2\delta}} \mathbf{R}(d\ell) < +\infty.$$

The *Quermass energy functional* H^Q (see [54]) is defined as any linear combination of area, perimeter, and Euler-Poincaré characteristic functionals:

$$H^Q(\gamma) = \alpha_1 \operatorname{Area}(\Gamma) + \alpha_2 \operatorname{Per}(\Gamma) + \alpha_3 \chi(\Gamma), \ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}.$$

Notice how this interaction, depending on the values of the parameters α_i, can be attractive or repulsive. It is difficult (and not useful) to decompose this multi-body energy functional as the sum of several k-body interactions. The functional H^Q satisfies assumptions $(\mathscr{H}_{st})$, $(\mathscr{H}_r)$, and $(\mathscr{H}_{loc.st})$. Indeed, it even satisfies the following stronger conditions:

- There exists a constant $B \geq 0$ such that, for any finite configuration γ,

$$|H^Q(\gamma)| \leq B\langle \gamma, 1 + \|m\|^2 \rangle. \qquad (\textit{two-sided stability})$$

- For any $\Lambda \in \mathscr{B}_b(\mathbb{R}^2)$ and $\mathbf{t} \geq 1$, there exists $B'(\Lambda, \mathbf{t}) \geq 0$ such that, for any $\gamma \in \mathscr{M}$, $\xi \in \mathscr{M}^{\mathbf{t}}$,

$$|H^Q_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})| \leq B'(\Lambda, \mathbf{t})\langle \gamma_\Lambda, 1 + \|m\|^2 \rangle. \qquad (\textit{two-sided loc. stability})$$

Under these conditions – stronger than ours – the existence for the Quermass-interaction model was proved in [18]; notice that H^Q is not superstable.

For more examples of geometric interactions, see [20].

Example 1.2 (Two-body interactions)**.**

i. Interacting hard spheres of random radii. On $\mathscr{E} = \mathbb{R}^d \times \mathbb{R}_+$, consider a model of hard balls centred at points x_i, of random radii m_i distributed according to a measure $\mathbf{R}$ satisfying Assumption $(\mathscr{H}_m)$. The hard-core energy functional of

a finite configuration $\gamma = \{(x_1, m_1), \dots (x_N, m_N)\}$, $N \geq 1$, is given by

$$H(\gamma) = \sum_{1 \leq i < j \leq N} (+\infty)\, \mathbb{1}_{\{B(x_i, m_i) \cap B(x_j, m_j) \neq \emptyset\}},$$

with the convention $+\infty \cdot 0 = 0$.

ii. Non-negative pair interaction. On $\mathscr{E} = \mathbb{R}^d \times \mathscr{S}$, consider any energy functional H of the form $H(\gamma) = \sum_{1 \leq i < j \leq N} \Phi(\mathbf{x}_i, \mathbf{x}_j)$, where

$$\Phi(\mathbf{x}_i, \mathbf{x}_j) = \phi(|x_i - x_j|)\, \mathbb{1}_{\{|x_i - x_j| \leq \|m_i\| + \|m_j\|\}},$$

where ϕ is non-negative and null at 0.

In both cases, since H is a non-negative functional, it satisfies $(\mathscr{H}_{st})$ and $(\mathscr{H}_{loc.st})$. It is also easy to see that, by construction, the range assumption $(\mathscr{H}_r)$ also holds.

1.2.4 Local topology

We endow the space of point measures with the topology of local convergence (see [45], [47]), defined as the weak* topology induced by a class of functionals on $\mathscr{M}$ which we now introduce.

Definition 1.8. A functional F is called *tame* if there exists a constant $c > 0$ such that

$$|F(\gamma)| \leq c \left(1 + \langle \gamma, 1 + \|m\|^{d+\delta} \rangle\right), \quad \gamma \in \mathscr{M}.$$

We denote by $\mathscr{L}$ the set of all tame and local functionals. The topology $\tau_{\mathscr{L}}$ of *local convergence* on $\mathscr{P}(\mathscr{M})$ is then defined as the weak* topology induced by $\mathscr{L}$, i.e. the smallest topology on $\mathscr{P}(\mathscr{M})$ under which all the mappings $P \mapsto \int F\, dP$, $F \in \mathscr{L}$, are continuous.

1.3 Construction of an infinite-volume Gibbs point process

Let us first precise the terminology (see [42]).

Definition 1.9. Let H be an energy functional satisfying the three assumptions $(\mathscr{H}_{st})$, $(\mathscr{H}_r)$, and $(\mathscr{H}_{loc.st})$. We say that a probability measure P on $\mathscr{M}$ is an *infinite-volume Gibbs point process* with energy functional H, activity $z > 0$ and inverse temperature $\beta > 0$, denoted $P \in \mathscr{G}_{z,\beta}(H)$, if for every finite volume $\Lambda \subset \mathbb{R}^d$ and for any measurable, bounded and local functional $F : \mathscr{M} \longrightarrow \mathbb{R}$, the following identity (called *DLR equation* after Dobrushin–Lanford–Ruelle) holds under P:

$$\int_{\mathscr{M}} F(\gamma)\, P(d\gamma) = \int_{\mathscr{M}} \int_{\mathscr{M}_\Lambda} F(\gamma_\Lambda \xi_{\Lambda^c})\, \Xi_\Lambda(\xi, d\gamma)\, P(d\xi), \qquad (\mathrm{DLR})_\Lambda$$

where Ξ_Λ, called the *Gibbsian probability kernel associated to* H, is defined on $\mathscr{M}_\Lambda$ by

$$\Xi_\Lambda(\xi, d\gamma) := \frac{e^{-\beta H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})}}{Z_\Lambda(\xi)} \pi^z_\Lambda(d\gamma), \tag{1.11}$$

where $Z_\Lambda(\xi) := \int_{\mathscr{M}_\Lambda} e^{-\beta H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})} \pi^z_\Lambda(d\gamma)$.

Remarks. *i.* The probability kernel $\Xi_\Lambda(\xi, \cdot)$ is not necessarily well-defined for any $\xi \in \mathscr{M}$. In Lemma 1.17, we will show that this is the case when we restrict it to the subspace $\mathscr{M}^{\text{temp}}$.

ii. The map $\xi \mapsto \Xi_\Lambda(\xi, d\gamma)$ is a priori *not local* since $\xi \mapsto H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})$ may depend on the full configuration ξ_{Λ^c}.

iii. The renormalisation factor $Z_\Lambda(\xi)$ – when it exists – only depends on the external configuration ξ_{Λ^c}. Therefore $\Xi_\Lambda(\xi, \cdot) \equiv \Xi_\Lambda(\xi_{\Lambda^c}, \cdot)$.

We can now state the main result of this chapter:

Theorem 1.1. Let H be an energy functional satisfying assumptions $(\mathscr{H}_m)$, $(\mathscr{H}_{st})$, $(\mathscr{H}_r)$, and $(\mathscr{H}_{loc.st})$. For any $z > 0$ and $\beta > 0$, there exists at least one infinite-volume Gibbs point process P with energy functional H, activity z and inverse temperature β.

Moreover, it is supported on the tempered configurations: $P \in \mathscr{G}^{\text{temp}}_{z,\beta}(H)$.

This section is structured as follows:

1.3.1 We define a sequence of stationarised finite-volume Gibbs point processes $(\bar{P}_n)_n$.

1.3.2 We use uniform bounds on the entropy to show the convergence, up to a subsequence, to an infinite-volume measure $\bar{P}$.

1.3.3 We prove, using an ergodic property, that $\bar{P}$ carries only the space of tempered configurations.

1.3.4 Noticing that, for any fixed $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, $\bar{P}_n$ does not satisfy $(\text{DLR})_\Lambda$, we introduce a new sequence $(\hat{P}_n)_n$ asymptotically equivalent to $(\bar{P}_n)_n$ but satisfying $(\text{DLR})_\Lambda$. We use appropriate approximations, by localising the interaction, to show that also $\bar{P}$ satisfies $(\text{DLR})_\Lambda$.

1.3.1 A stationarised sequence

In this subsection, we extend each finite-volume point process $P_n := P_{\Lambda_n}$, $\Lambda_n = [-n, n)^d$, defined on $\mathscr{M}_{\Lambda_n}$ to a probability measure $\bar{P}_n$ on the full space $\mathscr{M}$, invariant under lattice-translations.

We start with the following

Lemma 1.10. There exists a constant a_1 such that

$$\forall n \geq 1, \quad J_n := \int_{\mathscr{M}} \langle \gamma, 1 + \|m\|^{d+\delta} \rangle P_n(d\gamma) \leq a_1 |\Lambda_n|. \tag{1.12}$$

Proof. We partition the space of configurations $\mathscr{M}_{\Lambda_n}$ in three sets:

$$\mathscr{M}_{\Lambda_n}^{(1)} := \{\gamma \in \mathscr{M}_{\Lambda_n} : \langle \gamma, 1 + \|m\|^{d+\delta} \rangle \leq a_{11}|\Lambda_n|\},$$
$$\mathscr{M}_{\Lambda_n}^{(2)} := \{\gamma \in \mathscr{M}_{\Lambda_n} : \langle \gamma, 1 + \|m\|^{d+\delta} \rangle > a_{11}|\Lambda_n|,\ |\gamma| > a_{12}|\Lambda_n|\},$$
$$\mathscr{M}_{\Lambda_n}^{(3)} := \{\gamma \in \mathscr{M}_{\Lambda_n} : \langle \gamma, 1 + \|m\|^{d+\delta} \rangle > a_{11}|\Lambda_n|,\ |\gamma| \leq a_{12}|\Lambda_n|\},$$

for some constants a_{11}, a_{12} which will be fixed later. Therefore, the integral J_n can be written as the sum of three integrals, $J_n^{(1)}, J_n^{(2)}, J_n^{(3)}$, resp. over each of these sets.

The first term is straightforward:

$$J_n^{(1)} := \int_{\mathscr{M}_{\Lambda_n}^{(1)}} \langle \gamma_{\Lambda_n}, 1 + \|m\|^{d+\delta} \rangle P_n(d\gamma) \leq a_{11}|\Lambda_n|.$$

For the second term,

$$\begin{aligned}
J_n^{(2)} &:= \int_{\mathscr{M}_{\Lambda_n}^{(2)}} \langle \gamma_{\Lambda_n}, 1 + \|m\|^{d+\delta} \rangle P_n(d\gamma) \\
&\overset{(1.5)}{\leq} \int_{\mathscr{M}_{\Lambda_n}} \mathbb{1}_{\{|\gamma_{\Lambda_n}| > a_{12}|\Lambda_n|\}} \langle \gamma, 1 + \|m\|^{d+\delta} \rangle \frac{1}{Z_{\Lambda_n}} e^{-\beta H(\gamma_{\Lambda_n})} \pi^z_{\Lambda_n}(d\gamma) \\
&\overset{(1.8)}{\leq} \frac{e^{-z|\Lambda_n|}}{Z_{\Lambda_n}} \sum_{k=a_{12}|\Lambda_n|}^{+\infty} \frac{(z|\Lambda_n|)^k}{k!} \int_{\mathscr{S}^k} e^{\beta B \sum_{i=1}^k (1+\|m_i\|^{d+\delta})} \sum_{j=1}^k (1 + \|m_j\|^{d+\delta}) \mathbf{R}(dm_1) \dots \mathbf{R}(dm_k) \\
&\leq \sum_{k=a_{12}|\Lambda_n|}^{+\infty} \frac{(z|\Lambda_n|)^k}{k!}\ k \left(\int (1 + \ell^{d+\delta}) e^{\beta B(1+\ell^{d+\delta})} \rho(d\ell) \right) \left(\int e^{\beta B(1+\ell^{d+\delta})} \rho(d\ell) \right)^{k-1}.
\end{aligned}$$

Using (1.1), we are able to find a constant b_1 such that

$$\int (1 + \ell^{d+\delta})\, e^{\beta B(1+\ell^{d+\delta})} \rho(d\ell) \leq b_1.$$

We then get

$$J_n^{(2)} \le \sum_{k=a_{12}|\Lambda_n|}^{+\infty} \frac{(zb_1|\Lambda_n|)^k}{k!} k \le \sum_{k=a_{12}|\Lambda_n|}^{+\infty} \frac{(2zb_1|\Lambda_n|)^k}{k!} \le e^{2zb_1|\Lambda_n|}\mathbb{P}\Big(S_{|\Lambda_n|} \ge a_{12}|\Lambda_n|\Big),$$

for a sequence $(S_m)_{m\ge 1}$ of Poisson random variables with parameter $2zb_1 m$. Recalling the Cramér–Chernoff inequality (cf. [11])

$$\mathbb{P}(\tfrac{1}{m}S_m \ge a_{12}) \le e^{-mL^*(a_{12})},$$

where $L^*(x) = 2zb_1 + x\log x - x(1+\log(2zb_1))$ is the Legendre transform associated to the Poisson random variable of parameter $2zb_1$, we can choose a_{12} large enough, so that $\log a_{12} \ge 1 + \log(2zb_1)$. Thus $L^*(a_{12}) \ge 2zb_1$, and we get that $J_n^{(2)} \le 1$.

For the third term,

$$\begin{aligned}
J_n^{(3)} &\overset{(1.5)}{=} \int_{\mathscr{M}_{\Lambda_n}} \mathbb{1}_{\{\langle\gamma,1+\|m\|^{d+\delta}\rangle > a_{11}|\Lambda_n|,\ |\gamma|\le a_{12}|\Lambda_n|\}} \langle \gamma, 1+\|m\|^{d+\delta}\rangle \frac{1}{Z_{\Lambda_n}} e^{-\beta H(\gamma_{\Lambda_n})} \pi^z_{\Lambda_n}(d\gamma) \\
&\le \frac{e^{-z|\Lambda_n|}}{Z_{\Lambda_n}} \sum_{k=0}^{a_{12}|\Lambda_n|} \frac{(z|\Lambda_n|)^k}{k!} \int_{\mathscr{S}^k} \mathbb{1}_{\{\sum_{i=1}^k (1+\|m_i\|^{d+\delta}) > a_{11}|\Lambda_n|\}} \\
&\qquad\qquad e^{\beta B\sum_{i=1}^k (1+\|m_i\|^{d+\delta})} \sum_{j=1}^k (1+\|m_j\|^{d+\delta}) \mathbf{R}(dm_1)\dots\mathbf{R}(dm_k) \\
&\le \sum_{k=0}^{a_{12}|\Lambda_n|} \frac{(z|\Lambda_n|)^k}{k!} \int_{\mathbb{R}_+^k} \mathbb{1}_{\{\sum_{i=1}^k (1+\ell_i^{d+\delta}) > a_{11}|\Lambda_n|\}} \\
&\qquad\qquad e^{\beta B\sum_{i=1}^k (1+\ell_i^{d+\delta})} \sum_{j=1}^k (1+\ell_j^{d+\delta}) \rho(d\ell_1)\dots\rho(d\ell_k).
\end{aligned}$$

Applying the Cauchy–Schwarz inequality, we find:

$$\begin{aligned}
J_n^{(3)} \le & \sum_{k=0}^{a_{12}|\Lambda_n|} \frac{(z|\Lambda_n|)^k}{k!} \sqrt{\rho^{\otimes k}\Big(\sum_{i=1}^k \big(1+\ell_i^{d+\delta}\big) > a_{11}|\Lambda_n|\Big)} \\
& \sqrt{\int_{\mathbb{R}_+^k} e^{2\beta B\sum_{i=1}^k (1+\ell_i^{d+\delta})} \Big(\sum_{j=1}^k (1+\ell_j^{d+\delta})\Big)^2 \rho(d\ell_1)\dots\rho(d\ell_k)} \\
\le & \sqrt{\rho^{\otimes a_{12}|\Lambda_n|}\Big(\sum_{i=1}^{a_{12}|\Lambda_n|} \big(1+\ell_i^{d+\delta}\big) > a_{11}|\Lambda_n|\Big)} \\
& \sum_{k=0}^{a_{12}|\Lambda_n|} \frac{(z|\Lambda_n|)^k}{k!} \sqrt{k^2 \int (1+\ell^{d+\delta})^2 e^{2\beta B(1+\ell^{d+\delta})} \rho(d\ell) \left(\int e^{2\beta B(1+\ell^d)}\rho(d\ell)\right)^{k-1}}.
\end{aligned}$$

Using (1.1), there exists a positive constant b_2 such that

$$\int (1 + \ell^{d+\delta})^2 e^{2\beta B(1+\ell^{d+\delta})} \rho(d\ell) \leq b_2.$$

Thus

$$\begin{aligned} J_n^{(3)} &\leq \sqrt{\rho^{\otimes a_{12}|\Lambda_n|}\Big(\sum_{i=1}^{a_{12}|\Lambda_n|} \big(1 + \ell_i^{d+\delta}\big) > a_{11}|\Lambda_n|\Big)} \sum_{k=0}^{a_{11}|\Lambda_n|} \frac{(z\sqrt{b_2}|\Lambda_n|)^k}{k!} k \\ &\leq \sqrt{\rho^{\otimes a_{12}|\Lambda_n|}\Big(\sum_{i=1}^{a_{12}|\Lambda_n|} \big(1 + \ell_i^{d+\delta}\big) > a_{11}|\Lambda_n|\Big)}\, e^{2z\sqrt{b_2}|\Lambda_n|}. \end{aligned}$$

Using again the Cramér–Chernoff inequality, we can choose a_{11} large enough such that $\bar{L}^*$, the Legendre transform of the image measure of ρ by $\ell \mapsto 1 + \ell^{d+\delta}$, satisfies $\bar{L}^*(a_{11}) \geq 4z\sqrt{b_2}$ (since it is stricly increasing on the positive half-line). Thus

$$\rho^{\otimes a_{12}|\Lambda_n|}\Big(\sum_{i=1}^{a_{12}|\Lambda_n|} \big(1 + \ell_i^{d+\delta}\big) > a_{11}|\Lambda_n|\Big) \leq e^{-4z\sqrt{b_2}|\Lambda_n|},$$

which yields $J_n^{(3)} \leq 1$.
Putting it all together, the claim of Lemma 1.10 follows with $a_1 := a_{11} + 2$. □

We start by considering the probability measure $\tilde{P}_n$ on $\mathscr{M}$, under which the configurations in the disjoint blocks $\Lambda_n^\kappa := \Lambda_n + 2n\kappa$, $\kappa \in \mathbb{Z}^d$, are independent, with identical distribution P_n. We then build the empirical field associated to the probability measure $\tilde{P}_n$, i.e. the sequence of lattice-stationarised probability measures

$$\bar{P}_n = \frac{1}{(2n)^d} \sum_{\kappa \in \Lambda_n \cap \mathbb{Z}^d} \tilde{P}_n \circ \vartheta_\kappa^{-1}, \tag{1.13}$$

where ϑ_κ is the translation on $\mathbb{R}^d$ by the vector $\kappa \in \mathbb{Z}^d$.

Remarks. *i.* As usual we identify the translation ϑ_κ on $\mathbb{R}^d$ with the image of a point measure under such translation.

ii. So constructed, the probability measure $\bar{P}_n$ is invariant under $(\vartheta_\kappa)_{\kappa \in \mathbb{Z}^d}$.

iii. An upper bound similar to (1.12) holds also under $\bar{P}_n$:

$$\exists a_2 > 0, \forall n \geq 1, \quad \int_{\mathscr{M}} \langle \gamma_{\Lambda_n}, 1 + \|m\|^{d+\delta} \rangle \bar{P}_n(d\gamma) \leq a_2 |\Lambda_n|.$$

Moreover, using stationarity and the fact that the covering $\Lambda_n = \bigcup_\kappa \Lambda_1^\kappa$ con-

tains n^d terms,

$$\begin{aligned}\int_{\mathscr{M}} \langle \gamma_{\Lambda_1}, 1 + \|m\|^{d+\delta} \rangle \bar{P}_n(d\gamma) &= \int_{\mathscr{M}} \frac{1}{n^d} \sum_{\kappa} \langle \gamma_{\Lambda_1^\kappa}, 1 + \|m\|^{d+\delta} \rangle \bar{P}_n(d\gamma) \\ = \frac{1}{n^d} \int_{\mathscr{M}} \langle \gamma_{\Lambda_n}, 1 + \|m\|^{d+\delta} \rangle \bar{P}_n(d\gamma) &\leq \frac{1}{n^d} (2n)^d a_2 = 2^d a_2.\end{aligned} \tag{1.14}$$

As we will see in the following subsection, in order to prove that $(\bar{P}_n)_n$ admits an accumulation point, it is enough to prove that all elements of the sequence belong to the same entropy level set.

1.3.2 Entropy bounds

Let us now introduce the main tool of our study, the specific entropy of a (stationary) probability measure on $\mathscr{M}$.

Definition 1.11. Given two probability measures Q and Q' on $\mathscr{M}$, and any finite-volume $\Lambda \subset \mathbb{R}^d$, the *relative entropy* of Q' with respect to Q on Λ is defined as

$$I_\Lambda(Q|Q') := \begin{cases} \displaystyle\int \log f \; dQ_\Lambda & \text{if } Q_\Lambda \preccurlyeq Q'_\Lambda \text{ with } f := \frac{dQ_\Lambda}{dQ'_\Lambda}, \\ +\infty & \text{otherwise,} \end{cases}$$

where Q_Λ (resp. Q'_Λ) is the image of Q (resp. Q') under the projection $\gamma \mapsto \gamma_\Lambda$.

As usual,

Definition 1.12. The *specific entropy* of Q with respect to Q' is defined by

$$\mathcal{I}(Q|Q') = \lim_{n \to +\infty} \frac{1}{|\Lambda_n|} I_{\Lambda_n}(Q|Q').$$

From now on, the reference measure Q' will be the marked Poisson point process π^z with intensity measure $z\,dx \otimes \mathbf{R}(dm)$. In this case, the specific entropy of a point process Q with respect to π^z is always well defined if Q is stationary under the lattice translations $(\vartheta_\kappa)_{\kappa \in \mathbb{Z}^d}$. Moreover, recall that for any $a > 0$, the a-entropy level set

$$\mathscr{P}(\mathscr{M})_{\leq a} := \left\{ Q \in \mathscr{P}(\mathscr{M}),\ \text{stationary under } (\vartheta_\kappa)_{\kappa \in \mathbb{Z}^d} \,:\, \mathcal{I}(Q|\pi^z) \leq a \right\}$$

is relatively compact for the topology $\tau_{\mathscr{L}}$, as proved in [47].

Proposition 1.13. There exists a constant $a_3 > 0$ such that,

$$\forall n \geq 1, \quad \mathcal{I}(\bar{P}_n|\pi^z) \leq a_3$$

where $\bar{P}_n \in \mathscr{P}(\mathscr{M})$ is the empirical field defined by (1.13).

Proof. Since the map $Q \mapsto \mathcal{I}(Q|\pi^z)$ is affine, it holds

$$\begin{aligned}\mathcal{I}(\bar{P}_n|\pi^z) &= \frac{1}{(2n)^d} \sum_{\kappa \in \mathbb{Z}^d \cap \Lambda_n} \mathcal{I}(\tilde{P}_n \circ \vartheta_\kappa^{-1}|\pi^z) \\ &= \mathcal{I}(\tilde{P}_n|\pi^z) = \lim_{m \to +\infty} \frac{1}{|2m\Lambda_n|} I_{2m\Lambda_n}(\tilde{P}_n|\pi^z) \\ &= \lim_{m \to +\infty} \frac{1}{(2m)^d|\Lambda_n|}(2m)^d \, I_{\Lambda_n}(P_n|\pi^z) = \frac{1}{|\Lambda_n|} I_{\Lambda_n}(P_n|\pi^z).\end{aligned}$$

Using the stability of the energy functional, we find

$$I_{\Lambda_n}(P_n|\pi^z) = -\beta \int H(\gamma) P_n(d\gamma) - \log(Z_{\Lambda_n}) \overset{(1.8)}{\leq} \beta B \int \langle \gamma, 1 + \|m\|^{d+\delta} \rangle P_n(d\gamma) + z|\Lambda_n|.$$

From Lemma 1.10, we know that inequality (1.12) holds. Defining $a_3 := \beta B a_1 + z$, we conclude that, uniformly in $n \geq 1$, $\mathcal{I}(\bar{P}_n|\pi^z) \leq a_3$. □

From the above proposition we deduce that the sequence $(\bar{P}_n)_{n \geq 1}$ belongs to the relatively compact set $\mathscr{P}(\mathscr{M})_{\leq a_3}$. It then admits at least one converging subsequence which we will still denote by $(\bar{P}_n)_{n \geq 1}$ for simplicity. The limit measure of this subsequence, here denoted by $\bar{P}$, is stationary under the translations $(\vartheta_\kappa)_{\kappa \in \mathbb{Z}^d}$. We will prove in what follows that $\bar{P}$ is (one of) the infinite-volume Gibbs point process we are looking for.

1.3.3 Support of the infinite-volume limit measure

We now justify the introduction of a set of tempered configurations as the right support of each of the probability measures $\bar{P}_n$, $n \geq 1$, as well as of the constructed limit probability measure $\bar{P}$.

Proposition 1.14. The measures $\bar{P}_n$, $n \geq 1$, and the limit measure $\bar{P}$ are all supported on the tempered configurations, i.e.

$$\forall n \geq 1, \quad \bar{P}_n\big(\mathscr{M}^{\text{temp}}\big) = \bar{P}\big(\mathscr{M}^{\text{temp}}\big) = 1.$$

Proof. Let us show that, for $\bar{P}$ (resp. $\bar{P}_n$)-a.e. $\gamma \in \mathscr{M}$, there exists $\mathfrak{t} = \mathfrak{t}(\gamma) \geq 1$ such that

$$\sup_{l \in \mathbb{N}^*} \frac{1}{l^d} \langle \gamma_{B(0,l)}, 1 + \|m\|^{d+\delta} \rangle \leq \mathfrak{t}. \tag{1.15}$$

From (1.14), we know that

$$\forall n \geq 1, \quad \int \langle \gamma_{[-1,1)^d}, 1 + \|m\|^{d+\delta} \rangle \bar{P}_n(d\gamma) \leq 2^d a_2. \tag{1.16}$$

Since the integrand is a tame local function, the same inequality remains true when passing to the limit:

$$\int \langle \gamma_{[-1,1)^d}, 1 + \|m\|^{d+\delta} \rangle \bar{P}(d\gamma) \leq 2^d a_2.$$

The integrability of $\langle \gamma_{[-1,1)^d}, 1 + \|m\|^{d+\delta} \rangle$ under $\bar{P}$ (resp. $\bar{P}_n$) is precisely what we need in order to apply the ergodic theorem in [75]. Doing so yields the following spatial asymptotics, where we have $\bar{P}$ (resp. $\bar{P}_n$)-a.s. convergence to the conditional expectation under $\bar{P}$ (resp. $\bar{P}_n$) with respect to the σ-field $\mathscr{J}$ of $(\vartheta_\kappa)_{\kappa \in \mathbb{Z}^d}$- invariant sets:

$$\lim_{l \to +\infty} \frac{1}{|B(0,l)|} \langle \gamma_{B(0,l)}, 1 + \|m\|^{d+\delta} \rangle = \frac{1}{2^d} \mathbb{E}_{\bar{P}} \left[\langle \gamma_{[-1,1)^d}, 1 + \|m\|^{d+\delta} \rangle \mid \mathscr{J} \right]$$

(resp. $\mathbb{E}_{\bar{P}_n}$). This implies that, $\bar{P}$ (resp. $\bar{P}_n$)-a.s.,

$$\lim_{l \to +\infty} \frac{1}{|B(0,l)|} \langle \gamma_{B(0,l)}, 1 + \|m\|^{d+\delta} \rangle < +\infty$$

so that, a fortiori, (1.15) holds, and the proposition is proved. □

In Subsection 1.3.4, in order to prove Gibbsianity of the limit measure, we need more: a uniform estimate of the support of the measures $\bar{P}_n$, $n \geq 1$. For this reason, we introduce the increasing family $(\underline{\mathscr{M}}^l)_{l \in \mathbb{N}^*}$ of subsets of $\mathscr{M}^{\mathrm{temp}}$, defined by

$$\underline{\mathscr{M}}^l := \left\{ \gamma \in \mathscr{M}^{\mathrm{temp}} \, : \, \forall k \in \mathbb{N}^*, \, k \geq l, \, \forall (x,m) \in \gamma_{B(0,2k+1)^c}, \, B(x, \|m\|) \cap B(0,k) = \emptyset \right\}.$$

Notice that, thanks to Lemma 1.2, for any $\mathbf{t} \geq 1$, $\mathscr{M}^{\mathbf{t}} \subset \underline{\mathscr{M}}^{\mathbf{l}(\mathbf{t})}$ (see Figure 1.1).

Proposition 1.15. For any $\varepsilon > 0$, there exists $l \geq 1$ such that

$$\forall n \geq 1, \quad \bar{P}_n(\underline{\mathscr{M}}^l) \geq 1 - \varepsilon.$$

Proof. We want to find $l \geq 1$ such that

$$\bar{P}_n \Big(\sup_{k \geq l} \sup_{(x,m) \in \gamma_{B(0,2k+1)^c}} \frac{\|m\|}{|x|} \geq \frac{1}{2} \Big) \leq \varepsilon. \tag{1.17}$$

For any $\kappa = (\kappa^1, \dots, \kappa^d) \in \mathbb{Z}^d$, let $D_\kappa = [\kappa^1, \kappa^1 + 1) \times \cdots \times [\kappa^d, \kappa^d + 1) \subset \mathbb{R}^d$. We list all

the elements of $\mathbb{Z}^d$ by a sequence $(\kappa_i)_{i\in\mathbb{N}} \subset \mathbb{Z}^d$ that forms a spiral, starting at 0; in particular, there exist constants $a, b > 0$ (depending on the dimension d), such that $i\,a \le |\kappa_i|^d \le i\,b$. We can then compute, for any $l \ge 1$,

$$\begin{aligned}
&\sum_{\substack{\kappa\in\mathbb{Z}^d:\\ |\kappa|\ge 2l}} \bar{P}_n\big(\mathfrak{m}(\gamma_{D_\kappa}) \ge \tfrac{1}{2}|\kappa|\big) = \sum_{\substack{i\ge 1:\\ |\kappa_i|\ge 2l}} \bar{P}_n\big(\mathfrak{m}(\gamma_{D_{\kappa_i}}) \ge \tfrac{1}{2}|\kappa_i|\big)\\
&\le \sum_{i\ge (2l)^d/b} \bar{P}_n\big(\mathfrak{m}(\gamma_{D_{\kappa_i}}) \ge \tfrac{1}{2}|\kappa_i|\big) \le \sum_{i\ge (2l)^d/b} \bar{P}_n\big(\mathfrak{m}(\gamma_{D_{\kappa_i}})^d \ge \frac{a}{2^d} i\big)\\
&\le \sum_{i\ge (2l)^d/b} \bar{P}_n\Big(\underbrace{\frac{2^d}{a} \sum_{(x,m)\in\gamma_{D_0}} (1+\|m\|^d)}_{=:\,F(\gamma)} \ge i\Big)\\
&\le \mathbb{E}_{\bar{P}_n}\big[\mathbb{1}_{\{F(\gamma)\ge (2l)^d/b\}} F(\gamma)\big]\\
&= \frac{2^d}{a}\mathbb{E}_{\bar{P}_n}\Big[\mathbb{1}_{\{\sum_{(x,m)\in\gamma_{D_0}}(1+\|m\|^d)\ge \frac{a}{b} l^d\}} \sum_{(x,m)\in\gamma_{D_0}} (1+\|m\|^d)\Big].
\end{aligned}$$

To control this expression, recall the following result (due to H.-O. Georgii and H. Zessin), which proves that point configurations in $\mathbb{R}^d$ with marks in a complete, separable metric space, satisfy a local equi-integrability property on entropy level sets, with respect to the marks:

Lemma 1.16 ([47], Lemma 5.2)**.** For any measurable non-negative function $f : \mathscr{S} \longrightarrow \mathbb{R}_+$ and for every $a > 0$ and $\Delta \in \mathscr{B}_b(\mathbb{R}^d)$,

$$\lim_{N\to\infty} \sup_{P\in\mathscr{P}(\mathscr{M})_{\le a}} \mathbb{E}_P\Big[\mathbb{1}_{\{\langle \gamma_\Delta, f\rangle \ge N\}} \langle \gamma_\Delta, f\rangle\Big] = 0.$$

Applying this result to the sequence $(\bar{P}_n)_n$, with $f(x,m) = 1 + \|m\|^d$ and $\Delta = D_0$, we have that, for any $\varepsilon > 0$, there exists $l \ge 1$ large enough, such that

$$\forall n \ge 1, \quad \bar{P}_n\Big(\sup_{\kappa\in\mathbb{Z}^d,\ |\kappa|\ge 2l} \frac{1}{|\kappa|}\mathfrak{m}(\gamma_{D_\kappa}) \ge \frac{1}{2}\Big) \le \varepsilon.$$

For any $(x,m) \in \gamma_{B(0,2k+1)^c}$, with $k \ge l$, there exists $\kappa \in \mathbb{Z}^d$ with $|\kappa| \ge 2k$, such that $(x,m) \in \gamma_{D_\kappa}$; since then $\frac{\|m\|}{|x|} \le \frac{\|m\|}{|\kappa|}$, we find that (1.17) holds, and the claim follows. □

Remark. One could have thought that such a uniform estimate held in $\mathscr{M}^{\mathfrak{t}}$, for some $\mathfrak{t} \ge 1$, but this is not the case; we thank one of the referees for pointing how this would not work. In order to have the uniform estimate, we had then to enlarge the set of tempered configurations by introducing $\underline{\mathscr{M}}^l$ instead.

1.3.4 The limit measure is Gibbsian

We are now ready to prove that the infinite-volume point process $\bar{P}$ we have constructed satisfies the Gibbsian property.

Lemma 1.17. Consider the Gibbsian kernel Ξ_Λ defined by (1.11). It satisfies:
(i) For any $\xi \in \mathscr{M}^{\mathrm{temp}}$, $\Xi_\Lambda(\xi, d\gamma)$ is well defined: $Z_\Lambda(\xi) < +\infty$;
(ii) For any Λ-local tame functional F on $\mathscr{M}$, the map $\xi \mapsto \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, d\gamma)$ defined on $\mathscr{M}^{\mathrm{temp}}$ is measurable.
(iii) The family $(\Xi_\Lambda)_{\Lambda\in\mathscr{B}_b(\mathbb{R}^d)}$ satisfies a *finite-volume compatibility condition*, in the sense that, for any ordered finite-volumes $\Lambda \subset \Delta$,

$$\int_{\mathscr{M}_{\Delta\setminus\Lambda}} \Xi_\Lambda(\zeta_{\Delta\setminus\Lambda}\xi_{\Delta^c}, d\gamma_\Lambda)\, \Xi_\Delta(\xi_{\Delta^c}, d\zeta_{\Delta\setminus\Lambda}) = \Xi_\Delta(\xi_{\Delta^c}, d(\gamma_\Lambda\zeta_{\Delta\setminus\Lambda})). \tag{1.18}$$

Proof. (i) We have to show that, for any $\xi \in \mathscr{M}^{\mathrm{temp}}$, $0 < Z_\Lambda(\xi) < +\infty$. Lemma 1.7 dealt with the free boundary condition case, so this followed from the stability assumption (1.8). Since $H_\Lambda(\gamma_\Lambda\xi_{\Lambda^c}) \neq H(\gamma_\Lambda)$, this now follows in the same way from (1.10).
(ii) The measurability of the map $\xi \mapsto \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, d\gamma)$ follows from the measurability of $\xi \mapsto H(\gamma_\Lambda\xi_{\Lambda^c})$ and $\xi \mapsto Z_\Lambda(\xi)$.
(iii) The compatibility of the family $(\Xi_\Lambda)_\Lambda$ follows, as in [82], from the additivity (1.7) of the conditional energy functional.

□

We now state the main result of this section:

Proposition 1.18. The probability measure $\bar{P}$ is an infinite-volume Gibbs point process with energy functional H.

Proof. Since $\bar{P}$ is concentrated on the tempered configurations, we have to check that, for any finite-volume Λ, the following DLR equation is satisfied under $\bar{P}$:

$$\int_{\mathscr{M}^{\mathrm{temp}}} F(\gamma)\, \bar{P}(d\gamma) = \int_{\mathscr{M}^{\mathrm{temp}}} \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, d\gamma)\, \bar{P}(d\xi),$$

where F is a measurable, bounded and Λ-local functional.

Fix $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$. We would like to use the fact that its finite-volume approximations $(\bar{P}_n)_n$ satisfy $(\mathrm{DLR})_\Lambda$; but since they are lattice-stationary and periodic, this is not true. To overcome this difficulty, we use some approximation techniques, articulated in the following three steps:

i. An equivalent sequence: We introduce a new sequence $(\hat{P}_n)_n$ and show it is asymptotically equivalent to $(\bar{P}_n)_n$

ii. **A cut-off kernel:** We introduce a cut off of the Gibbsian kernel by a local functional

iii. **Gibbsianity of the limit measure:** We use estimations via the cut-off kernel to prove that $\bar{P}$ satisfies $(\mathrm{DLR})_\Lambda$.

---

i. **An equivalent sequence:** We introduce a modified sequence of measures $(\hat{P}_n)_n$ satisfying $(\mathrm{DLR})_\Lambda$ and having the same asymptotic behaviour as $(\bar{P}_n)_n$: for every $n \geq 1$, consider

$$\hat{P}_n = \frac{1}{|\Lambda_n|} \sum_{\substack{\kappa \in \Lambda_n \cap \mathbb{Z}^d : \\ \vartheta_\kappa(\Lambda_n) \supset \Lambda}} P_n \circ \vartheta_\kappa^{-1}.$$

Since the above sum is not taken over all $\kappa \in \Lambda_n \cap \mathbb{Z}^d$, $\hat{P}_n$ is not a probability measure. Moreover, $\hat{P}_n$ is bounded from above by $\bar{P}_n$, in the sense that, for any measurable $A \subset \mathscr{M}$, $\hat{P}_n(A) \leq \bar{P}_n(A)$.
We introduce the index $i_0 \in \mathbb{N}$ as the smallest $n \geq 1$ such that Λ is contained in the box Λ_n. Using the compatibility of the kernels (1.18), since $\Lambda \subset \Lambda_n$. For every $n \geq i_0$, the measure $\hat{P}_n$ satisfies $(\mathrm{DLR})_\Lambda$.
The sequences $(\hat{P}_n)_n$ and $(\bar{P}_n)_n$ are locally asymptotically equivalent, in the sense that, for every tame Λ-local functional G in $\mathscr{L}$,

$$\lim_{n \to \infty} \left| \int G(\gamma) \hat{P}_n(d\gamma) - \int G(\gamma) \bar{P}_n(d\gamma) \right| = 0.$$

In particular, asymptotically $\hat{P}_n$ is a probability measure, i.e. for any $\varepsilon > 0$, we can find n_0 such that

$$\forall n \geq n_0, \quad \hat{P}_n(\mathscr{M}) \geq 1 - \varepsilon. \tag{1.19}$$

Indeed: let G be a tame Λ-local functional in $\mathscr{L}$ as in Definition 1.8, and set

$$\delta_1 := \left| \int_{\mathscr{M}^{\mathrm{temp}}} G(\gamma)\, \hat{P}_n(d\gamma) - \int_{\mathscr{M}^{\mathrm{temp}}} G(\gamma)\, \bar{P}_n(d\gamma) \right|.$$

We then have

$$\begin{aligned} \delta_1 &= \frac{1}{(2n)^d} \left| \sum_{\substack{\kappa \in \Lambda_n \cap \mathbb{Z}^d : \\ \vartheta_\kappa(\Lambda_n) \supset \Lambda}} \int G(\gamma) P_n \circ \vartheta_\kappa^{-1}(d\gamma) - \sum_{\kappa \in \Lambda_n \cap \mathbb{Z}^d} \int G(\gamma) \tilde{P}_n \circ \vartheta_\kappa^{-1}(d\gamma) \right| \\ &\leq \frac{1}{(2n)^d} \sum_{\substack{\kappa \in \Lambda_n \cap \mathbb{Z}^d : \\ \vartheta_\kappa(\Lambda_n) \not\supseteq \Lambda}} \left| \int G(\gamma) \tilde{P}_n \circ \vartheta_\kappa^{-1}(d\gamma) \right|, \end{aligned}$$

and since G is local and tame,

$$\delta_1 \le \frac{c}{(2n)^d} \sum_{\substack{\kappa\in\Lambda_n\cap\mathbb{Z}^d:\\ \vartheta_\kappa(\Lambda_n)\not\supseteq\Lambda}} \int \Big(1 + \langle \gamma_\Lambda, 1 + \|m\|^{d+\delta}\rangle\Big)\tilde{P}_n\circ\vartheta_\kappa^{-1}(d\gamma).$$

As $\Lambda \subset \Lambda_{i_0}$, the number of $\kappa \in \Lambda_n \cap \mathbb{Z}^d$ such that $\vartheta_\kappa(\Lambda_n) \not\supseteq \Lambda$ is

$$\mathrm{Card}\Big\{\, \kappa \in \Lambda_n \cap \mathbb{Z}^d \;:\; \vartheta_\kappa(\Lambda_n) \not\supseteq \Lambda \Big\} \le i_0 2d(2n-1)^{d-1},$$

since: for Λ to be moved out of Λ_n, one of the components of κ should be larger than $n - i_0$ (i_0 options for this); there are $2d$ directions Λ can be moved *through* Λ_n; and the other $d-1$ components of $\kappa \in \Lambda_n \cap \mathbb{Z}^d$ are left free ($2n-1$ options). Calling $c' := i_0 dc$, we find

$$\delta_1 \le \frac{c'}{n} + \frac{c}{(2n)^d} \sum_{\substack{\kappa\in\Lambda_n\cap\mathbb{Z}^d:\\ \vartheta_\kappa(\Lambda_n)\not\supseteq\Lambda}} \int \langle \gamma_\Lambda, 1 + \|m\|^{d+\delta}\rangle \tilde{P}_n\circ\vartheta_\kappa^{-1}(d\gamma).$$

Now, for any $a_4 > 0$ (which will be fixed later), we split the above integral over the set $\Big\{ \sum_{(x,m)\in\gamma_\Lambda}(1+\|m\|^{d+\delta}) \ge a_4 \Big\}$ and its complement. We obtain

$$\begin{aligned}
\delta_1 &\le \frac{c'}{n} + \frac{a_4 c'}{n} \\
&\qquad + \frac{c}{(2n)^d} \sum_{\substack{\kappa\in\Lambda_n\cap\mathbb{Z}^d:\\ \vartheta_\kappa(\Lambda_n)\not\supseteq\Lambda}} \int_{\{\langle\gamma_\Lambda, 1+\|m\|^{d+\delta}\rangle \ge a_4\}} \langle \gamma_\Lambda, 1 + \|m\|^{d+\delta}\rangle \tilde{P}_n\circ\vartheta_\kappa^{-1}(d\gamma) \\
&\le \underbrace{\frac{(1+a_4)c'}{n}}_{\text{first term}} + \underbrace{c\int_{\{\langle\gamma_\Lambda, 1+\|m\|^{d+\delta}\rangle \ge a_4\}} \langle \gamma_\Lambda, 1 + \|m\|^{d+\delta}\rangle \bar{P}_n(d\gamma)}_{\text{second term}}.
\end{aligned}$$

Fix $\varepsilon > 0$; for $n \ge \frac{2(1+a_4)c'}{\varepsilon}$, the first term is smaller than $\varepsilon/2$.
To control the second term, we apply Lemma 1.16, to the sequence $(\bar{P}_n)_n$, and the function $f(x,m) = 1 + \|m\|^{d+\delta}$; we find $a_4 > 0$ such that the second term is smaller than $\varepsilon/2$, uniformly in n, and conclude the proof of this step.

ii. A cut-off kernel: We know that $\hat{P}_n$ satisfies $(\mathrm{DLR})_\Lambda$, i.e. for any Λ-local and bounded functional F

$$\int F(\gamma)\, \hat{P}_n(d\gamma) = \int \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, d\gamma)\, \hat{P}_n(d\xi).$$

If $\xi \mapsto \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, d\gamma)$ were a local functional, we would be able to con-

clude simply by taking the limit in n on both sides of the above expression, since $\bar{P} = \lim_n \hat{P}_n$ for the topology of local convergence. But this is not the case because of the unboundedness of the range of the interaction. We are then obliged to consider some approximation tools.

To that aim, we introduce a (Δ, m_0)-*cut off* of the Gibbsian kernels $\Xi_\Lambda(\xi, d\gamma)$, which takes into account only the points of ξ belonging to a finite volume Δ and having marks smaller than m_0.

Definition 1.19. Let $\Delta \in \mathscr{B}_b(\mathbb{R}^d)$ with $\Delta \supset \Lambda$. The (Δ, m_0)-cut off Ξ_Λ^{Δ,m_0} of the Gibbsian kernel Ξ_Λ is defined as follows:
for every measurable, Λ-local and bounded functional $G : \mathscr{M}_\Lambda \to \mathbb{R}$,

$$\int_{\mathscr{M}_\Lambda} G(\gamma)\Xi_\Lambda^{\Delta,m_0}(\xi, d\gamma) := \frac{1}{Z_\Lambda^{\Delta,m_0}(\xi_{\Delta\setminus\Lambda})} \int_{\mathscr{M}_\Lambda} \mathbb{1}_{\{\mathfrak{m}(\gamma)\le m_0\}} G(\gamma) e^{-\beta H_\Lambda(\gamma_\Lambda \xi_{\Delta\setminus\Lambda})} \pi_\Lambda^z(d\gamma),$$

where $Z_\Lambda^{\Delta,m_0}(\xi_{\Delta\setminus\Lambda})$ is the normalisation constant.

Remarks. *i.* Ξ_Λ^{Δ,m_0} is well defined since the normalisation constant Z_Λ^{Δ,m_0} is positive and finite:

$$0 < e^{-z|\Lambda|} \le Z_\Lambda^{\Delta,m_0}(\xi_{\Delta\setminus\Lambda}) < +\infty.$$

ii. The functional

$$\xi \mapsto \int_{\mathscr{M}_\Lambda} G(\gamma)\, \Xi_\Lambda^{\Delta,m_0}(\xi, d\gamma)$$

is now local and bounded, since the supremum norm of G is bounded.

We now show that Ξ_Λ^{Δ,m_0} is a uniform local approximation of the Gibbsian kernel Ξ_Λ, i.e.
For any $\varepsilon > 0, \mathfrak{t} \ge 1$, for any measurable, Λ-local and bounded functional F, there exist $\underline{m}_0 > 0$ and $\underline{\Delta} \supset \Lambda$ such that, for any $m_0 \ge \underline{m}_0$ and $\Delta \supset \underline{\Delta}$, we have

$$\sup_{\xi \in \mathscr{M}^{\mathfrak{t}}} \left| \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta,m_0}(\xi, d\gamma) - \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, d\gamma) \right| \le \varepsilon. \tag{1.20}$$

Indeed, let $\xi \in \mathscr{M}^{\mathfrak{t}}$. First notice that, since $H_\Lambda(\gamma_\Lambda \xi_{\Delta\setminus\Lambda}) = H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})$ as soon as $\Delta \supseteq \Lambda \oplus B\big(0, 2\mathfrak{I}(\mathfrak{t}) + 2\mathfrak{m}(\gamma_\Lambda) + 1\big)$ then $e^{-H_\Lambda(\gamma_\Lambda \xi_{\Delta\setminus\Lambda})} - e^{-H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})} = 0$ on the set of configurations $\{\gamma : \mathfrak{m}(\gamma_\Lambda) \le m_0 \text{ and } \Lambda \oplus B\big(0, 2\mathfrak{I}(\mathfrak{t}) + 2\mathfrak{m}(\gamma_\Lambda) + 1\big) \subset \Delta\}$.

Considering the difference between both partition functions, we obtain

$$\begin{aligned}\left|Z_\Lambda^{\Delta,m_0}(\xi_{\Delta\setminus\Lambda}) - Z_\Lambda(\xi_{\Lambda^c})\right| &= \left|\int \left(\mathbb{1}_{\{\mathfrak{m}(\gamma_\Lambda)\le m_0\}} e^{-\beta H_\Lambda(\gamma_\Lambda\xi_{\Delta\setminus\Lambda})} - e^{-\beta H_\Lambda(\gamma_\Lambda\xi_{\Lambda^c})}\right)\pi_\Lambda^z(d\gamma)\right| \\ &\le \int \mathbb{1}_{\{\mathfrak{m}(\gamma_\Lambda)>m_0\}\cup\{\Lambda\oplus B\left(0,2\mathfrak{l}(\mathfrak{t})+2\mathfrak{m}(\gamma_\Lambda)+1\right)\not\subseteq\Delta\}}\left(e^{-\beta H_\Lambda(\gamma_\Lambda\xi_{\Delta\setminus\Lambda})} + e^{-\beta H_\Lambda(\gamma_\Lambda\xi_{\Lambda^c})}\right)\pi_\Lambda^z(d\gamma) \\ &\overset{(1.10)}{\le} \int \mathbb{1}_{\{\mathfrak{m}(\gamma_\Lambda)>m_0\}\cup\{\Lambda\oplus B\left(0,2\mathfrak{l}(\mathfrak{t})+2\mathfrak{m}(\gamma_\Lambda)+1\right)\not\subseteq\Delta\}} 2e^{\beta B'\langle\gamma_\Lambda,1+\|m\|^{d+\delta}\rangle}\pi_\Lambda^z(d\gamma).\end{aligned}$$

Notice that this upper bound does *not* depend on ξ anymore. Thanks to the integrability assumption (1.1), by dominated convergence this implies that the map

$$\xi \mapsto Z_\Lambda^{\Delta,m_0}(\xi_{\Delta\setminus\Lambda}) - Z_\Lambda(\xi_{\Lambda^c})$$

converges to 0 as $m_0 \uparrow \infty$ and $\Delta \uparrow \mathbb{R}^d$ uniformly in $\xi \in \mathscr{M}^{\mathfrak{t}}$. Similarly,

$$\begin{aligned}\xi \mapsto \int_{\mathscr{M}_\Lambda} \mathbb{1}_{\{\mathfrak{m}(\gamma_\Lambda)\le m_0\}} F(\gamma_\Lambda) e^{-\beta H_\Lambda(\gamma_\Lambda\xi_{\Delta\setminus\Lambda})}\pi_\Lambda^z(d\gamma) \\ - \int_{\mathscr{M}_\Lambda} F(\gamma_\Lambda) e^{-\beta H_\Lambda(\gamma_\Lambda\xi_{\Lambda^c})}\pi_\Lambda^z(d\gamma)\end{aligned}$$

converges to 0 as $m_0 \uparrow \infty$ and $\Delta \uparrow \mathbb{R}^d$, uniformly in $\xi \in \mathscr{M}^{\mathfrak{t}}$. This concludes the proof of this step: we can find $\underline{m}_0 = \underline{m}_0(\varepsilon, \mathfrak{t})$ and $\underline{\Delta} = \underline{\Delta}(\varepsilon, \mathfrak{t})$ such that (1.20) holds for any $m_0 \ge \underline{m}_0$ and $\Delta \supset \underline{\Delta}$.

iii. Gibbsianity of the limit measure: To prove the Gibbsianity of $\bar{P}$ we have to check that

$$\delta_2 := \left|\int_{\mathscr{M}^{\mathrm{temp}}} \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, d\gamma)\, \bar{P}(d\xi) - \int_{\mathscr{M}^{\mathrm{temp}}} F(\gamma)\, \bar{P}(d\gamma)\right|$$

vanishes.

We first show that for large enough $\mathfrak{t} \ge 1$, the sets $\mathscr{M}^{\mathfrak{t}}, \mathscr{M}^{\mathfrak{l}(\mathfrak{t})}$ are close to the support of the measures $\bar{P}$, $\bar{P}_n$, and $\hat{P}_n$: let m_0 and Δ be large enough in the above sense, and satisfy $\Delta \supset \Lambda \oplus B\big(0, 2\mathfrak{l}(\mathfrak{t}) + 2m_0 + 1\big)$. Thanks to the results on the supports of $\bar{P}$ (Proposition 1.14) and $\bar{P}_n$ (Proposition 1.15), we can find $a_5 > 0$, independent of n, such that, for any m_0 and $\mathfrak{t}$ larger than a_5, and all $n \ge 1$,

$$\bar{P}(\mathscr{M}^{\mathfrak{t}}) \ge 1-\varepsilon,\ \bar{P}_n(\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}) \ge 1-\varepsilon,\ \bar{P}_n\Big(\big\{\gamma \in \mathscr{M} \,:\, \mathfrak{m}(\gamma_\Lambda) \le m_0\big\}\Big) \overset{(1.4)}{\ge} 1-\varepsilon. \quad (1.21)$$

Since, by construction, $\bar{P}_n$ dominates $\hat{P}_n$, using (1.19) yields, for $n \geq n_0$,

$$\hat{P}_n(\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}) \geq 1 - 2\varepsilon, \quad \hat{P}_n\Big(\big\{\gamma \in \mathscr{M} \;:\; \mathfrak{m}(\gamma_\Lambda) \leq m_0\big\}\Big) \geq 1 - 2\varepsilon. \tag{1.22}$$

The following steps deal with the estimation of δ_2: using (1.21), we have that (w.l.o.g. $\|F\|_\infty \leq 1$)

$$\delta_2 \leq \underbrace{\|F\|_\infty \bar{P}\big((\mathscr{M}^{\mathfrak{t}})^c\big)}_{\leq \varepsilon} + \underbrace{\left|\int_{\mathscr{M}^{\mathfrak{t}}} \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, d\gamma)\bar{P}(d\xi) - \int_{\mathscr{M}^{\mathfrak{t}}} F(\gamma)\bar{P}(d\gamma)\right|}_{=:\,\delta_{21}}.$$

Using the estimates of (1.20), we have

$$\begin{aligned}\delta_{21} &\overset{(1.20)}{\leq} \varepsilon + \left|\int_{\mathscr{M}^{\mathfrak{t}}} \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta,m_0}(\xi, d\gamma)\bar{P}(d\xi) - \int_{\mathscr{M}^{\mathfrak{t}}} F(\gamma)\bar{P}(d\gamma)\right| \\ &\overset{(1.21)}{\leq} 2\varepsilon + \underbrace{\left|\int_{\mathscr{M}^{\mathrm{temp}}} \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta,m_0}(\xi, d\gamma)\bar{P}(d\xi) - \int_{\mathscr{M}^{\mathrm{temp}}} F(\gamma)\bar{P}(d\gamma)\right|}_{=:\,\delta_{22}}.\end{aligned}$$

By construction, the functional $\xi \mapsto \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta,m_0}(\xi, d\gamma)$ is local; thus the local convergence of $(\hat{P}_n)_n$ to $\bar{P}$ implies that there exists $n_1 \in \mathbb{N}^*$ such that, for $n \geq n_1$, both estimates hold:

$$\left|\int_{\mathscr{M}^{\mathrm{temp}}} \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta,m_0}(\xi, d\gamma)\bar{P}(d\xi) - \int_{\mathscr{M}^{\mathrm{temp}}} \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta,m_0}(\xi, d\gamma)\hat{P}_n(d\xi)\right| \leq \varepsilon,$$

$$\left|\int_{\mathscr{M}^{\mathrm{temp}}} F(\gamma)\bar{P}(d\gamma) - \int_{\mathscr{M}^{\mathrm{temp}}} F(\gamma)\hat{P}_n(d\gamma)\right| \leq \varepsilon.$$

Therefore,

$$\delta_{22} \leq 2\varepsilon + \underbrace{\left|\int_{\mathscr{M}^{\mathrm{temp}}} \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta,m_0}(\xi, d\gamma)\hat{P}_n(d\xi) - \int_{\mathscr{M}^{\mathrm{temp}}} F(\gamma)\hat{P}_n(d\gamma)\right|}_{=:\,\delta_{23}}.$$

Now δ_{23} can be further decomposed:

$$\begin{aligned}\delta_{23} &\overset{(1.22)}{\leq} 2\varepsilon + \underbrace{\left|\int_{\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}} \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta,m_0}(\xi, d\gamma)\hat{P}_n(d\gamma) - \int_{\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}} \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, d\gamma)\hat{P}_n(d\xi)\right|}_{=:\,\delta_{24}} \\ &\quad + \left|\int_{\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}} \int_{\mathscr{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, d\gamma)\,\hat{P}_n(d\xi) - \int_{\mathscr{M}^{\mathrm{temp}}} F(\gamma)\hat{P}_n(d\gamma)\right|.\end{aligned}$$

We can estimate δ_4 by conditioning

$$\begin{aligned}
\delta_{24} &= \left| \int_{\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}} \int_{\mathscr{M}_\Lambda} F(\gamma) \Big(\Xi_\Lambda^{\Delta, m_0}(\xi, d\gamma) - \Xi_\Lambda(\xi, d\gamma) \Big) \hat{P}_n(d\gamma) \right| \\
&= \left| \int_{\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}} \int_{\mathscr{M}_\Lambda} F(\gamma) \Big(\Xi_\Lambda^{\Delta, m_0}(\xi, d\gamma) \hat{P}_n(d\xi) \right. \\
&\qquad - \Xi_\Lambda\big(\xi, d\gamma \mid \{\gamma : \mathfrak{m}(\gamma) \le m_0\}\big) \big(1 - \Xi_\Lambda\big(\xi, \{\gamma' : \mathfrak{m}(\gamma') > m_0\}\big)\big) \\
&\qquad\quad \left. + \Xi_\Lambda\big(\xi, d\gamma \mid \{\gamma : \mathfrak{m}(\gamma) > m_0\}\big)\, \Xi_\Lambda\big(\xi, \{\gamma' : \mathfrak{m}(\gamma') > m_0\}\big) \Big) \hat{P}_n(d\xi) \right| \\
&\le \left| \int_{\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}} \int_{\mathscr{M}_\Lambda} F(\gamma) \Big(\Xi_\Lambda^{\Delta, m_0}(\xi, d\gamma) - \Xi_\Lambda\big(\xi, d\gamma \mid \{\gamma : \mathfrak{m}(\gamma) \le m_0\}\big) \Big) \hat{P}_n(d\xi) \right| \\
&\qquad + 2 \int_{\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}} \Xi_\Lambda\big(\xi, \{\gamma' : \mathfrak{m}(\gamma') > m_0\}\big) \hat{P}_n(d\xi).
\end{aligned}$$

The first term in the above inequality vanishes if the two kernels coincide on $\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}$, which is the case since $\Lambda \oplus B\big(0, 2\mathfrak{l}(\mathfrak{t}) + 2\mathfrak{m}(\gamma_\Lambda) + 1\big) \subset \Delta$. Since $\hat{P}_n$ satisfies $(\mathrm{DLR})_\Lambda$, for the second term we have

$$\int_{\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}} \Xi_\Lambda\big(\xi, \{\gamma' : \mathfrak{m}(\gamma') > m_0\}\big) \hat{P}_n(d\xi) \le \hat{P}_n\big(\{\gamma' : \mathfrak{m}(\gamma'_\Lambda) > m_0\}\big) \overset{(1.22)}{\le} 2\varepsilon.$$

We then have $\delta_{24} \le 4\varepsilon$. Putting it all together,

$$\begin{aligned}
\delta_2 &\le 11\,\varepsilon + \left| \int_{\underline{\mathscr{M}}^{\mathfrak{l}(\mathfrak{t})}} \int_{\mathscr{M}_\Lambda} F(\gamma) \Xi_\Lambda(\xi, d\gamma)\, \hat{P}_n(d\xi) - \int_{\mathscr{M}^{\mathrm{temp}}} F(\gamma) \hat{P}_n(d\gamma) \right| \\
&\overset{(1.22)}{\le} 13\,\varepsilon + \left| \int_{\mathscr{M}^{\mathrm{temp}}} \int_{\mathscr{M}_\Lambda} F(\gamma) \Xi_\Lambda(\xi, d\gamma)\, \hat{P}_n(d\xi) - \int_{\mathscr{M}^{\mathrm{temp}}} F(\gamma) \hat{P}_n(d\gamma) \right| = 13\,\varepsilon,
\end{aligned}$$

since $\hat{P}_n$ satisfies $(\mathrm{DLR})_\Lambda$. Thanks to the arbitrariness of $\varepsilon > 0$, we can conclude that also $\bar{P}$ satisfies $(\mathrm{DLR})_\Lambda$.

In conclusion, $\bar{P}$ satisfies $(\mathrm{DLR})_\Lambda$ for any finite volume Λ, so Proposition 1.18 – and consequently Theorem 1.1 – is proved: for any $z > 0$ and $\beta > 0$, $\bar{P} \in \mathscr{G}_{z,\beta}(H)$. □

The following chapter presents an application to infinite-dimensional interacting diffusions.

Gibbs point processes on path space: existence, cluster expansion and uniqueness

2

We study a class of infinite-dimensional diffusions under Gibbsian interactions, in the context of marked point configurations: the starting points belong to $\mathbb{R}^d$, and the marks are the paths of Langevin diffusions. We use the entropy method to prove existence of an infinite-volume Gibbs point process and use cluster expansion tools to provide an explicit activity domain in which uniqueness holds.

2.1 The setting

We consider infinitely-many independent gradient diffusions and add a dependence between them by introducing an *interaction energy* in the context of marked Gibbs point processes. In this setting, we adopt the DLR description and set up the existence and uniqueness questions that are explored in the later sections.

2.1.1 Infinite-dimensional free system of Langevin dynamics

The basic mathematical object of this work is the following Langevin dynamics on $\mathbb{R}^d$:

$$dX(s) = dB(s) - \frac{1}{2}\nabla V\big(X(s)\big)\,ds, \quad s \in [0,1], \tag{2.1}$$

where B is a standard $\mathbb{R}^d$-valued Brownian motion, and $V : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a smooth potential satisfying, outside of some compact subset of $\mathbb{R}^d$,

$$\exists \delta', b_1, b_2 > 0, \quad V(x) \geq b_1|x|^{d+\delta'} \text{ and } \Delta V(x) - \frac{1}{2}|\nabla V(x)|^2 \leq -b_2|x|^{2+2\delta'}. \tag{2.2}$$

It is a known result (see e.g. [86]) that, under these conditions, there exists a unique solution to the SDE (2.1), which generates an *ultracontractive* semigroup (see [53, 16]). Moreover, for any $\delta < \delta'/2$,

$$\mathbb{E}\left[e^{\sup_{s\in[0,1]}|X(s)-X(0)|^{d+2\delta}}\right] < +\infty. \tag{2.3}$$

For the rest of this work, let $\delta > 0$ be fixed.

2.1.2 The system with Gibbsian interaction

Consider now that any (continuous) path $\mathbf{x}$ on $[0,1]$ can be decomposed into its initial location x and a (shifted) path m starting from 0. In other words, we identify $\mathbf{x}$ with the pair $(x, m) \in \mathscr{E} := \mathbb{R}^d \times C_0$, where C_0 is the space of continuous paths on $[0,1]$ starting at 0. The space C_0, endowed with the norm $\|m\|$ given by the maximum displacement of the trajectory m, that is $\|m\| := \sup_{s\in[0,1]}|m(s)|$, is a normed space.

On C_0, we consider the measure $\mathbf{R}$, given by the law of the process X solution of (2.1) starting at $X(0) = 0$. Notice that, thanks to (2.3), for any $\delta < \delta'/2$,

$$\int_{C_0} e^{\|m\|^{d+2\delta}} \mathbf{R}(dm) < +\infty. \tag{2.4}$$

We consider point measures on the product state space $\mathscr{E}$. More precisely, we take the following product measure on $\mathscr{E}$:

$$\lambda(dx, dm) = dx \otimes \mathbf{R}(dm).$$

We denote by $\mathscr{M}$ the space of simple point measures (*configurations*) on $\mathscr{E}$, i.e. of all σ-finite measures of the form

$$\gamma = \sum_i \delta_{\mathbf{x}_i},\ \mathbf{x}_i = (x_i, m_i) \in \mathscr{E},\ \text{with } \mathbf{x}_i \neq \mathbf{x}_j \text{ if } i \neq j.$$

Since the configurations are simple, we identify them with the subset of their atoms:

$$\gamma \equiv \left\{ \mathbf{x}_1, \dots, \mathbf{x}_n, \dots \right\} \subset \mathscr{E}.$$

Moreover, for two disjoint configurations $\gamma, \xi \in \mathscr{M}$, we denote by $\gamma\xi$ their concatenation: $\gamma\xi := \gamma \cup \xi$. For $\gamma \in \mathscr{M}$, $|\gamma|$ denotes the number of its points; $\mathscr{M}_f \subset \mathscr{M}$ is the subset of *finite* configurations, i.e. with $|\gamma| < +\infty$. We denote by $\underline{o}$ the configuration supported on the empty set.

For any $\Lambda \subset \mathbb{R}^d$, $\mathscr{M}_\Lambda \subset \mathscr{M}$ denotes the subset of point measures with support in $\Lambda \times C_0$, and $\gamma_\Lambda := \gamma \cap (\Lambda \times C_0)$. Let $\mathscr{B}(\mathbb{R}^d)$ denote the Borel σ-algebra on $\mathbb{R}^d$, and $\mathscr{B}_b(\mathbb{R}^d)$ the set of bounded Borel subsets of $\mathbb{R}^d$, which we often call *finite volumes*. For $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, $|\Lambda|$ denotes its volume.

We denote by $\mathscr{P}(\mathscr{M})$ (resp. $\mathscr{P}(\mathscr{M}_\Lambda)$) the set of probability measures (or *point processes*) on $\mathscr{M}$ (resp. $\mathscr{M}_\Lambda$). Finally, let $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

We consider the following measure (of infinite mass):

Definition 2.1. Fix $z > 0$. We define the measure $\tilde{\pi}^z = 1 + \sum_{N=1}^{+\infty} \frac{z^N}{N!} \lambda^{\otimes N}$ on $\mathscr{M}_f$.

For any finite volume Λ, we consider as reference probability measure the

marked Poisson point process $\pi^z_\Lambda \in \mathscr{P}(\mathscr{M}_\Lambda)$ with intensity parameter z, defined by renormalising the restriction $\tilde{\pi}^z_\Lambda$ of $\tilde{\pi}^z$ to $\mathscr{M}_\Lambda$ as follows:

$$\pi^z_\Lambda(d\gamma) = e^{-z|\Lambda|}\tilde{\pi}^z_\Lambda(d\gamma).$$

As a modification of the Poisson point process, we introduce an interaction between the paths by considering the finite-volume Gibbs point process associated to an energy functional H. More precisely:

Definition 2.2. An *energy functional* $H : \mathscr{M}_f \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a measurable functional on the set of finite configurations, with $H(\underline{o}) = 0$ by convention. In this chapter we consider the energy of a finite number $N \geq 1$ of paths to be defined, for any $\gamma = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \in \mathscr{M}_f$, by the sum of a self-interaction term and a pair-potential term:

$$H(\gamma) := \sum_{i=1}^{N} \Psi(\mathbf{x}_i) + \beta \sum_{1 \leq i < j \leq N} \Phi(\mathbf{x}_i, \mathbf{x}_j) \in \mathbb{R} \cup \{+\infty\}, \tag{2.5}$$

where $\beta > 0$ is the *inverse temperature*.

We denote the *pair-interaction* component of the energy as

$$E_\Phi(\gamma) := \sum_{1 \leq i < j \leq N} \Phi(\mathbf{x}_i, \mathbf{x}_j),$$

and the *conditional energy* of any path $\mathbf{x} \in \mathscr{E}$ given any $\xi \in \mathscr{M}$ as

$$E_\Phi(\mathbf{x} \,|\, \xi) := \sum_{\mathbf{y} \in \xi} \Phi(\mathbf{x}, \mathbf{y}).$$

Note that this infinite sum is not always well defined (see Assumption 2.2).

Finally, for any $\gamma \in \mathscr{M}$, let

$$E_\Phi(\gamma \,|\, \xi) := \sum_{\mathbf{x} \in \gamma} E_\Phi(\mathbf{x} \,|\, \xi).$$

be the conditional energy of the configuration γ given the configuration ξ.

We specify later a growth condition on the self potential Ψ, and consider different sets of assumptions on the pair potential $\Phi : \mathscr{E} \times \mathscr{E} \longrightarrow \mathbb{R} \cup \{+\infty\}$.

Definition 2.3. Let H be an energy functional as in (2.5). For any $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, the free-boundary-condition *finite-volume Gibbs point process* on Λ with energy functional H, activity $z > 0$ and inverse temperature $\beta > 0$ is the probability

measure $P_\Lambda^{z,\beta}$ on $\mathscr{M}_\Lambda$ defined by

$$P_\Lambda^{z,\beta}(d\gamma) := \frac{1}{Z_\Lambda^z} e^{-\beta H(\gamma_\Lambda)}\, \pi_\Lambda^z(d\gamma), \tag{2.6}$$

where the *partition function* Z_Λ^z is the renormalisation constant.

In this work we investigate the existence and uniqueness of an infinite-volume Gibbs point process, in the following sense:

Definition 2.4. Let H be an energy functional as in (2.5). A probability measure P on $\mathscr{M}$ is said to be an *infinite-volume Gibbs point process* with energy functional H, activity $z > 0$ and inverse temperature $\beta > 0$, denoted $P \in \mathscr{G}_{z,\beta}(H)$, if it satisfies, for any $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$ and any positive, bounded, and measurable functional $F : \mathscr{M} \longrightarrow \mathbb{R}$, the following *DLR equation*

$$\int_{\mathscr{M}} F(\gamma)\, P(d\gamma) = \int_{\mathscr{M}} \frac{1}{Z_\Lambda^z(\xi)} \int_{\mathscr{M}_\Lambda} F(\gamma_\Lambda \xi_{\Lambda^c}) e^{-\beta\left(H(\gamma_\Lambda)+E_\Phi(\gamma_\Lambda|\xi_{\Lambda^c})\right)} \pi_\Lambda^z(d\gamma)\, P(d\xi), \tag{DLR}$$

where the partition function $Z_\Lambda^z(\xi)$ depends on the boundary condition ξ.

A concept that will help in showing that such an infinite-volume measure exists is that of tempered configuration. For such a configuration γ, the number $|\gamma_\Lambda|$ of its points in any finite volume Λ, should grow sublinearly w.r.t. the volume, while the norm $\|m\|$ of its marks should grow as a fractional power of it. More precisely,

Definition 2.5. The set of *tempered path configurations* is given by the increasing union $\mathscr{M}^{\mathrm{temp}} := \bigcup_{\mathfrak{t}\in\mathbb{N}^*} \mathscr{M}^{\mathfrak{t}}$, where

$$\mathscr{M}^{\mathfrak{t}} := \Big\{ \gamma \in \mathscr{M} \,:\, \forall l \in \mathbb{N}^*, \sum_{\substack{(x,m)\in\gamma \\ |x|\le l}} (1 + \|m\|^{d+2\delta}) \le \mathfrak{t} l^d \Big\}. \tag{2.7}$$

We denote by $\mathscr{G}_{z,\beta}^{\mathrm{temp}}(H) := \mathscr{G}_{z,\beta}(H) \cap \mathscr{P}(\mathscr{M}^{\mathrm{temp}})$ the set of *tempered Gibbs point processes*, i.e. those whose support is included in the tempered configurations.

In what follows we show that a tempered Gibbs point process associated to infinitely-many interacting Langevin dynamics exists as soon as the interaction energy satisfies some quite natural assumptions.

Assumption 2.1 (Self interaction growth and stability).
($\mathscr{H}_{\mathrm{self}}$) The self potential $\Psi : \mathscr{E} \longrightarrow \mathbb{R} \cup \{+\infty\}$ acting on each path is bounded

from below by the opposite of a power of its maximum displacement, i.e.

$$\exists A_\Psi > 0 \,:\, \inf_{x\in\mathbb{R}^d} \Psi(x,m) \geq -A_\Psi \|m\|^{d+\delta}. \tag{2.8}$$

($\mathscr{H}_{\text{st.}}$) The pair potential $\Phi : \mathscr{E}\times\mathscr{E} \longrightarrow \mathbb{R}\cup\{+\infty\}$ between two paths is a symmetric functional that satisfies the following *stability* condition: there exists a constant $B_\Phi \geq 0$ such that for any finite configuration $\gamma = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \in \mathscr{M}_f$,

$$E_\Phi(\gamma) = \sum_{1\leq i<j\leq N} \Phi(\mathbf{x}_i, \mathbf{x}_j) \geq -B_\Phi N. \tag{2.9}$$

2.2 Existence of a Gibbs point process of diffusions

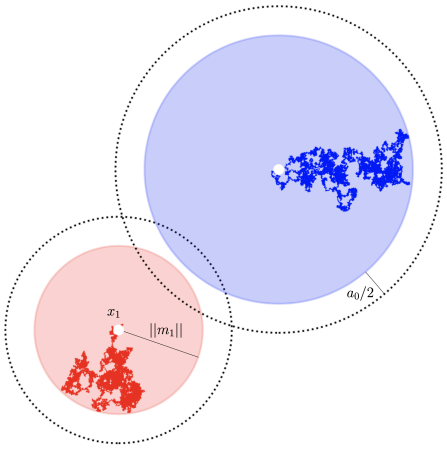

Figure 2.1. Two interacting paths of a Langevin diffusion in $\mathbb{R}^2$. Each circle is centred in the starting point, while the radii of the coloured circles correspond to their maximum displacement in the time interval $[0,1]$; the dotted circles represent the security distance $a_0/2$ introduced in (2.10).

The proof of the existence of an infinite-volume Gibbs point process that we describe here makes use of the specific entropy functional as a tightness tool, as in the general approach presented in Chapter 1. In order for our path model of interacting Langevin diffusions to fit the setting of the aforementioned paper, in this section we consider energy functionals H that satisfy, in addition to Assumption 2.1, the following:

Assumption 2.2 (Range and local stability). The pair potential $\Phi : \mathscr{E}\times\mathscr{E} \longrightarrow \mathbb{R}\cup\{+\infty\}$ is such that

($\mathscr{H}_{\text{r}}$) Two diffusions $\mathbf{x}_i, \mathbf{x}_j$ do not interact whenever they start *too far away*: there exists a constant $a_0 \geq 0$ such that

$$\Phi(\mathbf{x}_i, \mathbf{x}_j) = 0 \text{ whenever } |x_i - x_j| > a_0 + \|m_i\| + \|m_j\|. \tag{2.10}$$

($\mathscr{H}_{\text{loc.st}}$) There exists a constant $\bar{B}_\Phi \geq 0$ such that, for any path $\mathbf{x} \in \mathscr{E}$, for any configuration $\xi \in \mathscr{M}^{\text{temp}}$,

$$E_\Phi(\mathbf{x} \mid \xi) = \sum_{\mathbf{y} \in \xi} \Phi(\mathbf{x}, \mathbf{y}) \geq -\bar{B}_\Phi(1 + \|m\|^{d+\delta}). \tag{2.11}$$

Remark. We briefly comment on these two assumptions: the expression in (2.11) is well defined since, as we will see in the proof of Theorem 2.1, the range assumption ($\mathscr{H}_{\text{r}}$) implies that the infinite sum of the conditional energy of $\mathbf{x}$ given ξ is actually given by a finite (random) number of terms.

It is easy to show that the following Lemma holds for the support of any Gibbs point process:

Lemma 2.6. For any activity $z > 0$ and inverse temperature $\beta > 0$, any infinite-volume Gibbs point process $P \in \mathscr{G}^{\text{temp}}_{z,\beta}(H)$ is supported on configurations with locally finite energy, that is configurations $\gamma \in \mathscr{M}^{\text{temp}}$ such that, for any $\Delta \in \mathscr{B}_b(\mathbb{R}^d)$, $E_\Phi(\gamma_\Delta) < +\infty$. Note that this is true also whenever Φ takes infinite values.

Example 2.1. Consider the following class of interactions, described by a path pair potential of the form

$$\Phi(\mathbf{x}_i, \mathbf{x}_j) = \Big(\int_0^1 \phi(|x_i - x_j + m_i(s) - m_j(s)|) ds \Big) \mathbb{1}_{[0, a_0 + \|m_i\| + \|m_j\|]}(|x_i - x_j|), \tag{2.12}$$

with ϕ given by the sum of two potentials on $\mathbb{R}_+$: $\phi = \phi_{hc} + \phi_l$, where

- The potential ϕ_{hc} is pure *hard core* at some diameter $R > 0$, that is

$$\phi_{hc}(u) = (+\infty)\mathbb{1}_{[0,R)}(u).$$

- The potential ϕ_l satisfies a stability property, i.e. there exists a constant $B_\phi \geq 0$ such that, for any admissible configuration $\{y_1, \dots, y_N\}$, $N \geq 1$, the following holds (see [90], paragraph 3.2.5):

$$\sum_{i=1}^{N} \phi_l(|y_i|) \geq -2B_\phi, \tag{2.13}$$

  where a finite configuration $\{y_1, \dots, y_N\} \subset \mathbb{R}^d$, $N \geq 1$, is called *admissible* if, for any pair $y_i \neq y_j$, $\phi(|y_i - y_j|) < +\infty$.

Note how the coefficient a_0 here plays the role of a *sensitivity parameter* (see Figure 2.1): if the pair potential ϕ is repulsive (i.e. positive), then a_0 can take any finite positive value. If instead ϕ is attractive (i.e. negative) on some region, a_0 should be chosen in such a way that ϕ remains attractive on $[a_0, +\infty)$: $\phi(u) \leq 0$ if $u \geq a_0$ (see Figure 2.2). We now show that this class of potentials satisfy Assumption 2.2.

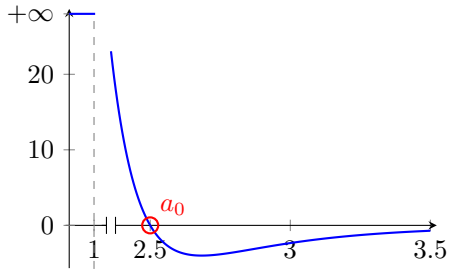

Figure 2.2. A shifted Lennard–Jones potential $\phi_{LJ}(u-1) = 16\left(\left(\frac{3/2}{u-1}\right)^{12} - \left(\frac{3/2}{u-1}\right)^{6}\right)$ with hard core diameter $R = 1$; it is always negative after $a_0 = 2.5$, and explodes as $x \longrightarrow 1^+$.

Proof. Firstly, thank to the previous Lemma, we can actually restrict our study to the admissible configurations. It is easy to see that the stability (2.9) of the potential Φ holds with $B_\Phi = B_\phi$. Moreover, setting $\mathfrak{l}(\mathfrak{t}) := 2^{\frac{d+\delta}{\delta}-1}\mathfrak{t}^{\frac{1}{\delta}}$, one can see that the range of the interaction is bounded by

$$\mathfrak{r}(\gamma, \Lambda) = 2\mathfrak{l}(\mathfrak{t}) + 2\sup_{\mathbf{x}\in\gamma_\Lambda}\|m\| + 1 + a_0,$$

i.e. for any $\mathbf{x} = (x, m) \in \mathscr{E}$ and $\xi \in \mathscr{M}^{\mathfrak{t}}$, $\mathfrak{t} \geq 1$, setting $\Delta := B(x, \mathfrak{r}(\gamma, \Lambda))$, the conditional energy $E_\Phi(\mathbf{x} \,|\, \xi \setminus \{x\})$ of $\mathbf{x}$ given ξ is actually given by $E_\Phi\left(\mathbf{x} \,|\, \xi_{\Delta\setminus\{x\}}\right)$: it is a finite sum, and is bounded from below by $-2B_\phi$.

$$\begin{aligned} E_\Phi\left(\mathbf{x} \,|\, \xi_{\Delta\setminus\{x\}}\right) &= \left(\int_0^1 \sum_{\mathbf{x}_i\in\xi_{\Delta\setminus\{x\}}} \phi(|x - x_i + m(s) - m_i(s)|)ds\right)\mathbb{1}_{\{|x-x_i|\leq a_0+\|m\|+\|m_i\|\}} \\ &\geq -2B_\phi. \end{aligned}$$

Notice how, under these conditions, the trajectories of two interacting paths $\mathbf{x}_1 = (x_1, m_1)$ and $\mathbf{x}_2 = (x_2, m_2)$ are allowed to intersect, but at each time s the paths keep at a distance of at least R; the hard-core component, indeed, imposes $|x_1 + m_1(s) - x_2 + m_2(s)| \geq R$ for any $s \in [0, 1]$. □

A particular case: Let ϕ be given by the sum of a hard-core component and a shifted Lennard–Jones potential, i.e.

$$\phi(u) = \phi_{hc}(u) + \phi_{LJ}(u - R)\mathbb{1}_{[R,+\infty)}(u),\ u \in \mathbb{R}_+,$$

where $\phi_{LJ}(u) = \frac{a}{u^{12}} - \frac{b}{u^6}$, $a, b > 0$. Pictured in Figure 2.2 is an example with $R = 1$. We remark that this potential has a non-integrable growth in a neighbourhood of its hard core component; in particular, it does not satisfy Assumption 2.3 below, which is used for the uniqueness proof.

Example 2.2. One can consider a class of *translation-invariant* pair potentials. More precisely, let Φ be invariant by translation: $\Phi(\mathbf{x}_i, \mathbf{x}_j) = \Phi(\mathbf{x}_i - \mathbf{x}_j)$, with

$$\Phi(\mathbf{x}) = \Big(\int_0^1 \phi(|x + m(s)|) ds \Big) \mathbb{1}_{\{|x| \le a_0 + \|m\|\}},$$

where ϕ is given by the above sum of a hard-core component and a shifted Lennard–Jones potential.

Definition 2.7. Consider a configuration $\gamma \in \mathscr{M}$. For any $N \ge 1$, its *factorial measure of order N* is given by

$$\gamma^{(N)}(d\mathbf{x}_1, \dots, d\mathbf{x}_N) := \gamma(d\mathbf{x}_1)(\gamma \setminus \{\mathbf{x}_1\})(d\mathbf{x}_2) \dots (\gamma \setminus \{\mathbf{x}_1, \dots, \mathbf{x}_{N-1}\})(d\mathbf{x}_N).$$

By taking the expectation under a point process P we obtain its *N-th factorial moment measure*: a measure $\alpha_N^{(P)}$ on $\mathscr{E}^N$ defined by

$$\alpha_N^{(P)}(\cdot) := \mathbb{E}_P[\gamma^{(N)}(\cdot)].$$

For any point process P, one can consider, for any $N \ge 1$, its N-point correlation function, defined as the Radon–Nikodym derivative of its N-th factorial moment measure $\alpha_N^{(P)}$ with respect to the product measure $(z\sigma)^{\otimes N}$, where

$$\sigma(d\mathbf{x}) := e^{-\Psi(\mathbf{x})} \lambda(d\mathbf{x}).$$

Proposition 2.8 ([75])**.** Let $P \in \mathscr{G}_{z,\beta}(H)$, $z > 0$, $\beta > 0$. Its *N-point correlation function* admits, for $\sigma^{\otimes N}$-almost all $(\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathscr{E}^N$, the following representation:

$$\rho_N^{(P)}(\mathbf{x}_1, \dots, \mathbf{x}_N) = e^{-\beta E_\Phi(\mathbf{x}_1, \dots, \mathbf{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\mathbf{x}_1, \dots, \mathbf{x}_N \mid \xi)} P(d\xi), \tag{2.14}$$

as soon as this expression is well defined.

Remark. Note that $\rho_N^{(P)}(\cdot)$ is a symmetric function, as for any $(\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathscr{E}^k$ and any permutation $\{i_1, \dots, i_N\}$, $\rho_N^{(P)}(\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_N}) = \rho_N^{(P)}(\mathbf{x}_1, \dots, \mathbf{x}_N)$.

We can now state our existence result:

Theorem 2.1. Let H be an energy functional as in (2.5), satisfying Assumptions 2.1+2.2. For any $z > 0$ and $\beta > 0$, there exists at least one infinite-volume tempered Gibbs point process $P^{z,\beta} \in \mathscr{G}_{z,\beta}^{\mathrm{temp}}(H)$. Moreover, for any $N \ge 1$, the N-point correlation function of $P^{z,\beta}$ exists and can be written as in (2.14).

Proof. Let $z > 0$, $\beta > 0$. In order to apply the existence result of Theorem 1.1 to this path space context, we show that a stability condition holds both for the energy of a finite configuration and for the conditional energy, and that the random interaction range is finite (possibly unbounded). These are the conditions $(\mathscr{H}_{\mathrm{st.}})$, $(\mathscr{H}_{\mathrm{r}})$, and $(\mathscr{H}_{\mathrm{loc.st}})$ of Assumption 1.1.

Step 1. We start by noting that (2.13) implies that the potential ϕ – defined on the location space $\mathbb{R}^d$ – is *stable* in the sense of Ruelle (see [91]), with stability constant B_ϕ, i.e.

$$\forall N \geq 1,\ \forall \{y_1, \dots, y_N\} \subset \mathbb{R}^d, \qquad \sum_{1 \leq i < j \leq N} \phi(|y_i - y_j|) \geq -B_\phi N.$$

The conditions (2.8) and (2.9) – on the self interaction and pair potential, respectively, yield the following *stability* for the energy of a finite number of paths:

$$\forall \gamma \in \mathscr{M}_f, \quad H(\gamma) \geq -(B_\phi \vee A_\Psi)\big(|\gamma| + \sum_{(x,m)\in\gamma} \|m\|^{d+\delta}\big).$$

Step 2. We now focus on analysing the *range of the interaction*: we show that for any tempered configuration $\gamma \in \mathscr{M}^{\mathfrak{t}}$, $\mathfrak{t} \geq 1$, and for any finite volume Λ, there exists a positive number $\mathfrak{r} = \mathfrak{r}(\gamma, \Lambda)$ such that

$$E_\Phi(\mathbf{x} \,|\, \xi) = \sum_{\substack{\mathbf{y}\in\xi \\ 0<|y-x|\leq \mathfrak{r}}} \Phi(\mathbf{x}, \mathbf{y}). \tag{2.15}$$

Set $\mathfrak{l}(\mathfrak{t}) := 2^{\frac{d+\delta}{\delta}-1}\mathfrak{t}^{\frac{1}{\delta}}$. Using the definition of tempered configurations, one has that, for all $l \geq \mathfrak{l}(\mathfrak{t})$ and for any $\mathbf{x} \in \gamma \in \mathscr{M}^{\mathfrak{t}}$ such that $|x| > 2l + 1 + a_0$,

$$|x| - \|m\| \overset{(2.7)}{\geq} |x| - \frac{1}{2}\lceil |x| \rceil \geq l + a_0.$$

Thanks to condition $(\mathscr{H}_{\mathrm{r}})$, this means that the range of the interaction is bounded by

$$\mathfrak{r}(\gamma, \Lambda) = 2\mathfrak{l}(\mathfrak{t}) + 2\sup_{\mathbf{x}\in\gamma_\Lambda}\|m\| + 1 + a_0.$$

Step 3. Fix $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, and consider, for $\gamma \in \mathscr{M}$ and $\xi \in \mathscr{M}^{\mathrm{temp}}$, the conditional energy of γ_Λ given ξ_{Λ^c}, that is:

$$H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c}) := H(\gamma_\Lambda) + E_\Phi(\gamma_\Lambda \,|\, \xi_{\Lambda^c}).$$

Thanks to (2.15), denoting $\Delta := \Lambda \oplus B(0, \mathfrak{r}(\gamma, \Lambda))$, we have

$$E_\Phi(\gamma_\Lambda \,|\, \xi_{\Lambda^c}) = E_\Phi(\gamma_\Lambda \,|\, \xi_{\Delta\setminus\Lambda}) = \sum_{\mathbf{x}_i\in\gamma_\Lambda} \sum_{\mathbf{x}_j\in\xi_{\Delta\setminus\Lambda}} \Phi(\mathbf{x}_i, \mathbf{x}_j). \tag{2.16}$$

It is unfortunately not true – as used instead in Section 4 of [Z2] – that we can control the cardinality of the second sum, i.e. the number of points of ξ_Δ, *uniformly in* γ. On the other hand, thanks to Lemma 2.6, we can assume that ξ_Δ is of finite energy, and therefore use (2.13) to estimate

$$\begin{aligned}
&\sum_{\mathbf{x}_i\in\gamma_\Lambda}\sum_{\mathbf{x}_j\in\xi_{\Delta\setminus\Lambda}}\Phi(\mathbf{x}_i,\mathbf{x}_j)\\
&\quad=\int_0^1\sum_{\mathbf{x}_i\in\gamma_\Lambda}\sum_{\mathbf{x}_j\in\xi_{\Delta\setminus\Lambda}}\phi(|x_i-x_j+m_i(s)-m_j(s)|)ds\,\mathbb{1}_{\{|x_i-x_j|\le a_0+\|m_i\|+\|m_j\|\}}\\
&\quad\overset{(2.13)}{\ge}\int_0^1\sum_{\mathbf{x}_i\in\gamma_\Lambda}-2B_\phi\ge-2B_\phi|\gamma_\Lambda|.
\end{aligned}$$

Together with the stability of $\gamma_\Lambda\mapsto H(\gamma_\Lambda)$, this yields the following lower bound for the conditional energy:

$$H_\Lambda(\gamma_\Lambda\xi_{\Lambda^c})\ge-(A_\Psi\vee 2B_\phi)\sum_{\mathbf{x}\in\gamma_\Lambda}(1+\|m\|^{d+\delta}),$$

Having checked the three conditions ($\mathscr{H}_{\text{st.}}$), ($\mathscr{H}_{\text{r}}$), and ($\mathscr{H}_{\text{loc.st}}$), we can now apply Theorem 1.1: there exists an infinite-volume Gibbs measure $P^{z,\beta}\in\mathscr{G}^{\text{temp}}_{z,\beta}(H)$.

The correlation functions of a Gibbs point process can be written as in (2.14) whenever the term $e^{-\sum_i\sum_{\mathbf{y}\in\xi}\Phi(\mathbf{x}_i,\mathbf{y})}$ is well defined. Thanks to (2.11), this is indeed the case, as we have $\sum_{\mathbf{y}\in\xi}\Phi(\mathbf{x},\mathbf{y})\ge-\bar{B}_\Phi(1+\|m\|^{d+\delta})$. □

Proposition 2.9. For any $N\ge 1$, the N-point correlation function $\rho_N^{(P^{z,\beta})}$ of any Gibbs point process $P^{z,\beta}$ constructed above satisfy a *Ruelle bound*: for $\sigma^{\otimes N}$-almost all $(\mathbf{x}_1,\dots,\mathbf{x}_N)\in\mathscr{E}^N$,

$$\rho_N^{(P^{z,\beta})}(\mathbf{x}_1,\dots,\mathbf{x}_N)\le\prod_{i=1}^N\mathbf{c}(\mathbf{x}_i),\tag{2.17}$$

where $\mathbf{c}(x,m):=\exp\left(\beta B_\Phi+\beta\bar{B}_\Phi(1+\|m\|^{d+\delta})\right)$.

Proof. Putting together (2.9) and (2.11), we estimate

$$\rho_N^{(P^{z,\beta})}(\mathbf{x}_1,\dots,\mathbf{x}_N)\le e^{\beta B_\Phi N}\int_{\mathscr{M}^{\text{temp}}}e^{\beta\sum_{i=1}^N\bar{B}_\Phi(1+\|m_i\|^{d+\delta})}P^{z,\beta}(d\xi),$$

yielding the desired bound. □

Example 2.1 (continued). For the class of potentials described in Example 2.1, the

Ruelle bound holds uniformly in $(\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathscr{E}^{\otimes N}$, and is of the form

$$\rho_N^{(P^{z,\beta})}(\mathbf{x}_1, \dots, \mathbf{x}_N) \leq e^{3\beta B_\phi N}. \tag{2.18}$$

2.3 Ruelle bounds for correlation functions

Suppose you have a pair potential Φ – not necessarily satisfying the assumptions of the previous section – and that you already have an infinite-volume Gibbs point process $P^{z,\beta} \in \mathscr{G}_{z,\beta}(H)$, not necessarily constructed as above. In this section – under an additional regularity condition (Assumption 2.3) – we use tools from cluster expansion (see, for example, [89, 91]) to find a domain of activity $(0, \mathfrak{z}_{\mathrm{Ru}}(\beta))$ such that, for any $z \in (0, \mathfrak{z}_{\mathrm{Ru}}(\beta))$, the correlation functions of $P^{z,\beta}$ exist and satisfy a Ruelle bound.

An important tool is given by the Ursell kernel (see the work by R.A. Minlos and S. Poghosyan in [71]), introduced in Subsection 2.3.2. As this method requires for the correlation functions to have a specific representation, we assume a priori here that, for any $N \geq 1$, the expression (2.14) for the N-point correlation function $\rho_N^{(P)}$ of any $P \in \mathscr{G}_{z,\beta}(H)$ is well defined.

2.3.1 Correlation functions

While we have so far decomposed the energy functional in (2.5) into self- and pair-interaction terms, in order to set ourselves in the framework of cluster expansion – that typically deals exclusively with pair interactions – in what follows we include the self-interaction term in the reference measure, and define, for $z > 0$, the measure

$$\tilde{\pi}^{z\sigma} = \sum_{N=0}^{+\infty} \frac{z^N}{N!} \sigma^{\otimes N},$$

and the corresponding Poisson point process $\pi^{z\sigma}$. The finite-volume Gibbs point process $P_\Lambda^{z,\beta}$ defined in (2.6) on $\mathscr{M}_\Lambda$ can then be equivalently defined using $\pi^{z\sigma}$ and just the pair interaction $E_\Phi(\gamma) = \sum_{\{\mathbf{x},\mathbf{y}\} \subset \gamma} \Phi(\mathbf{x}, \mathbf{y})$ (in place of the full energy functional H):

$$P_\Lambda^{z,\beta}(d\gamma) = \frac{1}{Z_\Lambda^{z\sigma}} e^{-\beta E_\Phi(\gamma_\Lambda)} \pi_\Lambda^{z\sigma}(d\gamma),$$

where $Z_\Lambda^{z\sigma}$ is the normalisation constant.

As we already mentioned, the proof of the uniqueness of the Gibbs point process revolves around the study of its correlation functions, which we now introduce. We start by introducing a finite-volume correlation function induced by the interaction Φ:

Definition 2.10. Let $z > 0$, $\beta > 0$. For any finite volume $\Lambda \subset \mathbb{R}^d$, the *finite-volume correlation function* $\rho_\Lambda^{(z,\beta)}$ in Λ (with free boundary condition) is given, for any $\gamma \in \mathscr{M}_\Lambda$, by

$$\rho_\Lambda^{(z,\beta)}(\gamma) = \frac{1}{\tilde{Z}_\Lambda^{z\sigma}} \int_{\mathscr{M}_\Lambda} e^{-\beta E_\Phi(\xi\gamma)} \tilde{\pi}_\Lambda^{z\sigma}(d\xi),$$

where $\tilde{Z}_\Lambda^{z\sigma}$ is the normalisation constant.

Remark. Note that, from the stability (2.9) of the pair potential Φ, there exists a functional

$$\mathbf{i} : \mathscr{M}_f \setminus \{\underline{o}\} \longrightarrow \mathscr{E}$$

such that for any non-empty path configuration γ there exists a path $\mathbf{i}(\gamma) \in \gamma$ where the sum of its interactions with the other paths in γ is bounded from below:

$$\forall \gamma \in \mathscr{M}_f \setminus \{\underline{o}\}, \quad E_\Phi\left(\mathbf{i}(\gamma) \,|\, \gamma \setminus \{\mathbf{i}(\gamma)\}\right) \geq -2B_\Phi. \tag{2.19}$$

As a consequence, Φ is bounded from below by $-2B_\Phi$:

$$\inf \Phi(\mathbf{x}, \mathbf{y}) \geq -2B_\Phi. \tag{2.20}$$

In Example 2.1 below we make use of (2.20), while (2.19) is used in Proposition 2.17.

In the following we fix the inverse temperature parameter $\beta > 0$, and consider energy functionals H such that, additionally to Assumption 2.1, the following holds:

Assumption 2.3 (regularity)**.** The pair potential Φ satisfies the following uniform *regularity condition* (for some, and therefore any, $\beta > 0$):

$$C(\beta) := \sup_{\mathbf{x} \in \mathscr{E}} \int_{\mathscr{E}} |e^{-\beta\Phi(\mathbf{x},\mathbf{y})} - 1| \sigma(d\mathbf{y}) < +\infty.$$

Example 2.1 (continued)**.** Suppose the potential $\phi = \phi_{hc} + \phi_l$ is *integrable* outside of the hard core, that is

$$\|\phi\|_{R_+} := \int_R^{+\infty} |\phi_l(u)| u^{d-1} du < +\infty,$$

then Assumption 2.3 holds. Indeed, since for any $x \in (-\infty, +\infty]$,

$$|e^{-x} - 1| \leq x^- e^{x^-} + (1 - e^{-x^+}) \leq |x| e^{x^-}, \tag{2.21}$$

where $x^- := \max(0, -x)$ and $x^+ := \max(0, x)$ are the negative and positive part of x, respectively, we have

$$|e^{-\beta\Phi} - 1| \leq \beta |\bar{\Phi}| e^{2\beta B_\phi},$$

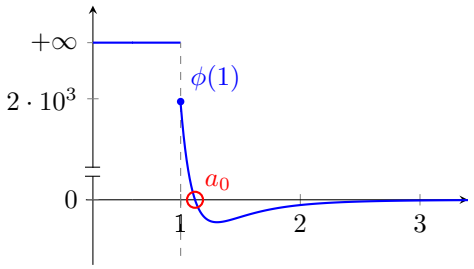

Figure 2.3. The sum of a hard-core potential ϕ_{hc} and the Lennard–Jones potential ϕ_{LG}. The integrable component ϕ_{LG} of the potential has a maximum in $\phi_{LG}(1)$.

where we denote by $\bar{\Phi}$ the *truncated pair potential*, defined, for any $\mathbf{x}, \mathbf{y} \in \mathscr{E}$, by

$$\bar{\Phi}(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \text{if } \Phi(\mathbf{x}, \mathbf{y}) = +\infty \\ \Phi(\mathbf{x}, \mathbf{y}) & \text{otherwise.} \end{cases}$$

Let $\bar{\phi}(u) := \mathbb{1}_{\{u<R\}} + \phi(u)\mathbb{1}_{\{u\geq R\}}$. Using the above bound, we can estimate, for any $\mathbf{x}_1 \in \mathscr{E}$,

$$\begin{aligned}
&\int_{\mathscr{E}} |e^{-\beta\Phi(\mathbf{x}_1,\mathbf{x}_2)} - 1|\sigma(d\mathbf{x}_2) \leq e^{2\beta B_\phi} \int_{\mathscr{E}} \beta|\bar{\Phi}(\mathbf{x}_1, \mathbf{x}_2)|\, \sigma(d\mathbf{x}_2) \\
&\leq e^{2\beta B_\phi} \int_{\mathscr{E}} \int_0^1 \beta|\bar{\phi}(x_2 + m_2(s) - x_1 - m_1(s))|ds\, \mathbb{1}_{\{|x_2-x_1|\leq a_0+\|m_2\|+\|m_1\|\}}\, \sigma(d\mathbf{x}_2) \\
&\overset{(2.8)}{\leq} e^{2\beta B_\phi} \int_{C_0} \int_0^1 \int_{\mathbb{R}^d} \beta|\bar{\phi}(x_2 + m_2(s) - x_1 - m_1(s))|\, dx_2\, ds\, e^{A_\Psi \|m_2\|^{d+\delta}} \mathbf{R}(dm_2) \\
&\leq e^{2\beta B_\phi} \beta \left(b_d R^d + \|\phi\|_{R_+}\right) \int_{C_0} e^{A_\Psi \|m_2\|^{d+\delta}} \mathbf{R}(dm_2),
\end{aligned}$$

which is finite thanks to the ultra-contractivity assumption, see (2.4).

A particular case: Suppose the potential ϕ is given by the sum of a hard-core potential ϕ_{hc} in $[0, R)$ and the Lennard–Jones potential $\phi_l \equiv \phi_{LJ}$ in $[R, +\infty)$. In particular, it is finite in $[R, +\infty)$, with maximum $\phi_l(R)$. Pictured in Figure 2.3 is an example with $R = 1$.

2.3.2 Cluster expansion: Ursell kernel and tree-graph estimates

In this subsection, after introducing the *Ursell kernel*, we use it to rewrite the correlation functions of a Gibbs point process and – following an approach inspired by [12, 56] – use *tree-graph estimates* to obtain a Ruelle bound for them. Our innovation comes from being able to obtain that the correlation functions of any Gibbs point process satisfy a Ruelle bound with the same constant $\mathbf{c}_z$, uniformly in the finite volume, therefore yielding uniqueness in the set of tempered Gibbs point processes.

We consider here *undirected connected graphs*. For any non-empty set $A \subset \mathbb{R}^d$, a *graph* G on A is given by a pair (V, E): the vertex set V is a subset of A, and the set of edges is a subset of $\{\{x, y\} \subset A : x \neq y\}$. Indeed, for a graph $G = (V, E)$ on A, we write $\{x, y\} \in G$ to denote the edge $xy \in E$ between two vertices $x, y \in V$. A *tree* T is a connected graph without loops. We also introduce the following notations:

- $C_n(A)$ denotes the set of all undirected connected graphs with n vertices belonging to A.
- $\mathcal{T}(A)$ denotes the set of all *trees* on A.

Note that the notion of graph $G = (V, E) \in C_n(A)$ does not depend on the possible orderings of the points of the vertex set $V = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$. Moreover, when there is no risk of confusion, we identify a graph G on $\{x_1, \dots, x_n\}$ with the corresponding one on the index set $\{1, \dots, n\} \in \mathbb{N}$ (i.e. where the edge $\{x_i, x_j\}$ corresponds with the edge $\{i, j\}$, see Figure 2.4).

When using these notations on a finite configuration $\gamma \subset \mathbb{R}^d \times C_0$, with an abuse of notations, we write $C_n(\gamma)$ as shorthand for $C_n(\mathrm{proj}_{\mathbb{R}^d}(\gamma))$ (analogously for $\mathcal{T}$).

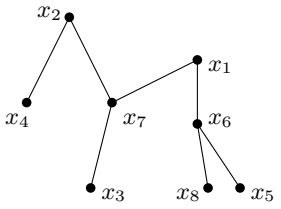

Figure 2.4. Example of a treee $T \in \mathcal{T}(A)$, where $A = \{x_1, \dots, x_8\} \subset \mathbb{R}^2$. It can be equivalently described by placing the points of A on the vertices of a tree $\tilde{T}$ on $\{1, \dots, 8\} \in \mathbb{N}$. More precisely, $\tilde{T}$ on $\{1, \dots, 8\}$ is constructed by placing an edge $\{i, j\} \in \tilde{T}$ if and only if there is an edge $\{x_i, x_j\} \in T$.

Definition 2.11. For any two measurable functionals $F, G : \mathscr{M}_f \to \mathbb{R}$, define their $*$-*product* by

$$(F * G)(\gamma) := \sum_{\xi \subset \gamma} F(\gamma \setminus \xi) G(\xi), \quad \gamma \in \mathscr{M}_f.$$

with identity $1^*(\gamma) := \mathbb{1}_{\{\gamma = \underline{o}\}}$. The space of measurable functionals with this operation is an algebra $\mathscr{A}$. Moreover, the set

$$\mathscr{A}_0 := \{F \in \mathscr{A} : F(\underline{o}) = 0\}$$

is an ideal of $\mathscr{A}$. The *exponential* and *logarithm operators* are defined by

$$\exp^* F := \sum_{n \geq 0} \tfrac{1}{n!} F^{*n}, \quad \log^*(1^* + F) := \sum_{n \geq 1} \tfrac{(-1)^{n-1}}{n} F^{*n}.$$

Definition 2.12 (Ursell function and kernel)**.** We introduce the two following notions:

- The *Ursell function* $k : \mathscr{M}_f \longrightarrow \mathbb{R}$ is a functional on finite configurations, defined by setting

$$k(\gamma) := \log^*(e^{-\beta E_\Phi})(\gamma), \quad \gamma \in \mathscr{M}_f.$$

  Equivalently ([12], Proposition 4.3), $k(\underline{o}) = 0$ and, for any γ with $|\gamma| = n \geq 1$,

$$k(\gamma) = \sum_{G \in \mathcal{C}_n(\gamma)} \prod_{\{\mathbf{x},\mathbf{y}\} \in G} \left(e^{-\beta\Phi(\mathbf{x},\mathbf{y})} - 1\right).$$

- The *Ursell kernel* $\bar{k} : \mathscr{M}_f \times \mathscr{M}_f \longrightarrow \mathbb{R}$ is defined on disjoint configurations by

$$\bar{k}(\gamma, \xi) := \left[\exp^*(-k) * (e^{-\beta E_\Phi})(\gamma, \cdot)\right](\xi), \quad \gamma, \xi \in \mathscr{M}_f,\ \gamma \cap \xi = \underline{o}.$$

The Ursell kernel relates to the Ursell function as follows:

Lemma 2.13 ([12], Lemma 4.6)**.** For any finite configuration $\gamma \neq \underline{o}$,

$$\forall \mathbf{x} \in \gamma, \quad \bar{k}(\{\mathbf{x}\}, \gamma \setminus \{\mathbf{x}\}) = k(\gamma).$$

Moreover, it provides a new expression for the correlation functions:

Lemma 2.14 ([12], Proposition 4.5)**.** Let $\gamma \in \mathscr{M}_\Lambda$, $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$. If $\int_{\mathscr{M}_\Lambda} |k(\xi)| \tilde{\pi}^{z\sigma}(d\xi) < +\infty$, then

$$\rho_\Lambda^{(z,\beta)}(\gamma) = \int_{\mathscr{M}_\Lambda} \bar{k}(\gamma, \xi) \tilde{\pi}^{z\sigma}(d\xi). \tag{2.22}$$

Lemma 2.15 ([12], Remark 4.8)**.** The Ursell kernel $\bar{k}$ is the unique solution of the so-called non-integrated Kirkwood–Salsburg equation

$$\begin{cases} \bar{k}(\gamma, \xi) = e^{-\beta \sum_{\mathbf{y} \in \gamma \setminus \{\mathbf{x}\}} \Phi(\mathbf{x},\mathbf{y})} \sum_{\eta \subset \xi} k_\eta(\mathbf{x}) \bar{k}\big((\gamma \setminus \{\mathbf{x}\})\eta, \xi \setminus \eta\big) \\ \bar{k}(\underline{o}, \xi) = \mathbb{1}_{\{\xi = \underline{o}\}}, \end{cases} \tag{2.23}$$

where $k_\eta(\mathbf{x}) := \prod_{\mathbf{y} \in \eta} \left(e^{-\beta\Phi(\mathbf{x},\mathbf{y})} - 1\right)$, and $\mathbf{x} \in \gamma$ is chosen arbitrarily.

We now introduce a second functional Q, which satisfies a similar equation to (2.23), dominates the Ursell kernel, and its simpler expression allows for more convenient computations.

Definition 2.16. Consider a functional Q on $\mathscr{M}_f \times \mathscr{M}_f$ defined as follows: for any $\xi \in \mathscr{M}_f$, $Q(\underline{o}, \xi) = \mathbb{1}_{\{\xi = \underline{o}\}}$, and for any $\gamma = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$, $N \geq 1$,

$$Q(\gamma, \xi) := \sum_{\substack{\xi_1, \dots, \xi_N \subset \xi \\ \xi_i \cap \xi_j = \underline{o}\ \forall i \neq j}} Q(\{\mathbf{x}_1\}, \xi_1) \cdots Q(\{\mathbf{x}_N\}, \xi_N),$$

where

$$\begin{cases} Q(\{\mathbf{x}\}, \xi) := e^{2\beta B_\Phi(|\xi|+1)} \sum_{T \in \mathcal{T}(\{x\} \cup \xi)} \prod_{\{\mathbf{y}_1, \mathbf{y}_2\} \in T} |e^{-\beta \Phi(\mathbf{y}_1, \mathbf{y}_2)} - 1| & \text{if } \xi \neq \underline{o} \\ Q(\{\mathbf{x}\}, \underline{o}) = e^{2\beta B_\Phi}. \end{cases} \tag{2.24}$$

Proposition 2.17 ([12], Proposition 4.10)**.** The functional Q defined above is the unique solution of

$$\begin{cases} Q(\gamma, \xi) = e^{2\beta B_\Phi} \sum_{\eta \subset \xi} |k_\eta(\mathbf{i}(\gamma))| Q(\gamma \setminus \mathbf{i}(\gamma) \cup \eta, \xi \setminus \eta) \\ Q(\underline{o}, \xi) = \mathbb{1}_{\{\xi = \underline{o}\}}, \end{cases}$$

where the functional $\mathbf{i}$ was defined in (2.19).

Corollary 2.18 ([12], Proposition 4.11)**.** For any $\gamma = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$, $N \geq 1$, and $\xi \in \mathscr{M}_f$ such that $\gamma \cap \xi = \underline{o}$, we have

$$\begin{aligned} |\bar{k}(\gamma, \xi)| &\leq Q(\gamma, \xi) \\ &= \sum_{\substack{\xi_1, \dots, \xi_N \subset \xi \\ \xi_i \cap \xi_j = \underline{o}\ \forall i \neq j}} Q(\{\mathbf{x}_1\}, \xi_1) \dots Q(\{\mathbf{x}_N\}, \xi_N), \end{aligned}$$

and

$$|k(\gamma)| \leq e^{2\beta B_\Phi |\gamma|} \sum_{T \in \mathcal{T}(\gamma)} \prod_{\{\mathbf{x}_i, \mathbf{x}_j\} \in T} |e^{-\Phi(\mathbf{x}_i, \mathbf{x}_j)} - 1|.$$

Lemma 2.19. For any finite volume $\Lambda \subset \mathbb{R}^d$ and $N \geq 1$, for λ-a.a. $\mathbf{x} \in \mathscr{E}$,

$$\begin{aligned} &\int_{(\Lambda \times C_0)^N} Q(\{\mathbf{x}\}, \{\mathbf{y}_1, \dots, \mathbf{y}_N\}) \sigma(d\mathbf{y}_1) \cdots \sigma(d\mathbf{y}_N) \\ &\qquad \leq e^{2\beta B_\Phi (N+1)} C(\beta)^{N-1} (N+1)^{N-1} \int_{\Lambda \times C_0} |e^{-\beta \Phi(\mathbf{x}, \mathbf{y})} - 1| \sigma(d\mathbf{y}). \end{aligned}$$

Proof. Using (2.24), we rewrite the l.h.s. as

$$e^{2\beta B_\Phi (N+1)} \sum_{T \in \mathcal{T}([N+1])} \underbrace{\int_{(\Lambda \times C_0)^N} \prod_{\{i,j\} \in T} |e^{-\beta \Phi(\mathbf{y}_i, \mathbf{y}_j)} - 1| \, \sigma(d\mathbf{y}_1) \cdots \sigma(d\mathbf{y}_N)}_{=: I_N},$$

where we set $\mathbf{y}_{N+1} := \mathbf{x}$, and $[N+1] := \{1, \dots, N+1\}$. We estimate I_N by induction on $N \geq 1$:

- For $N = 1$,
$$I_1 = \int_{\Lambda \times C_0} |e^{-\beta \Phi(\mathbf{x}, \mathbf{y}_1)} - 1| \sigma(d\mathbf{y}_1).$$
- For the inductive step, assume that, for all $T \in \mathcal{T}([N])$,
$$\int_{(\Lambda \times C_0)^{N-1}} \prod_{\{i,j\} \in T} |e^{-\beta \Phi(\mathbf{y}_i, \mathbf{y}_j)} - 1| \bigotimes_{i=1}^{N-1} \sigma(d\mathbf{y}_i) \leq C(\beta)^{N-2} \int_{\Lambda \times C_0} |e^{-\beta \Phi(\mathbf{y}_N, \mathbf{y})} - 1| \sigma(d\mathbf{y}).$$
- Let $T \in \mathcal{T}([N+1])$ be given, and root it in $\mathbf{y}_{N+1}$. There exists then an edge $\{j_1, j_2\} \in T$, where $\mathbf{y}_{j_1}$ is a leaf, and $\mathbf{y}_{j_1} \neq \mathbf{y}_{j_{N+1}}$. We obtain

$$\begin{aligned}
&\int_{(\Lambda \times C_0)^N} \prod_{\{i,j\} \in T} |e^{-\beta \Phi(\mathbf{y}_i, \mathbf{y}_j)} - 1| \bigotimes_{i=1}^{N} \sigma(d\mathbf{y}_i) \\
&= \int_{(\Lambda \times C_0)^{N-1}} \underbrace{\int_{\Lambda \times C_0} |e^{-\beta \Phi(\mathbf{y}_{j_1}, \mathbf{y}_{j_2})} - 1| \, \sigma(d\mathbf{y}_{j_1})}_{\leq C(\beta)} \prod_{\{i,j\} \in T \setminus \{\{j_1, j_2\}\}} |e^{-\beta \Phi(\mathbf{y}_i, \mathbf{y}_j)} - 1| \bigotimes_{\substack{i=1 \\ i \neq j_1}}^{N} \sigma(d\mathbf{y}_i) \\
&\leq C(\beta) \int_{(\Lambda \times C_0)^{N-1}} \prod_{\{i,j\} \in T \setminus \{\{j_1, j_2\}\}} |e^{-\beta \Phi(\mathbf{y}_i, \mathbf{y}_j)} - 1| \bigotimes_{\substack{i=1 \\ i \neq j_1}}^{N} \sigma(d\mathbf{y}_i).
\end{aligned}$$

We can then use the inductive step to prove the assertion.

Moreover,

$$e^{2\beta B_\Phi (N+1)} \sum_{T \in \mathcal{T}([N+1])} I_N \leq e^{2\beta B_\Phi (N+1)} \sum_{T \in \mathcal{T}([N+1])} C(\beta)^{N-1} \int_{\Lambda \times C_0} |e^{-\beta \Phi(\mathbf{y}_{N+1}, \mathbf{y})} - 1| \sigma(d\mathbf{y}),$$

and the claim follows, since the number of elements of $\mathcal{T}([N+1])$ is $(N+1)^{N-1}$ (see Theorem 4.1.3 of [76]). □

Lemma 2.20. Define the threshold activity

$$\mathfrak{z}_{\mathrm{Ru}}(\beta) := (C(\beta)e^{2\beta B_\Phi+1})^{-1} > 0, \tag{2.25}$$

and let $z < \mathfrak{z}_{\mathrm{Ru}}(\beta)$. For any finite volume $\Lambda \subset \mathbb{R}^d$, for $\tilde{\pi}^{z\sigma}$-a.a. $\gamma \in \mathscr{M}_f$, if $|\gamma| = N \geq 1$,

$$\int_{\mathscr{M}_\Lambda} |\bar{k}(\gamma,\xi)|\tilde{\pi}^{z\sigma}_\Lambda(d\xi) \leq \mathbf{c}_z^N,$$

where

$$\mathbf{c}_z := e^{2\beta B_\Phi}\left(1 + \frac{e}{\sqrt{2\pi}}\log\left(\frac{1}{1 - z/\mathfrak{z}_{\mathrm{Ru}}(\beta)}\right)\right) < +\infty. \tag{2.26}$$

Moreover, for any $z < \mathfrak{z}_{\mathrm{Ru}}(\beta)$,

$$\int_{\mathscr{M}_\Lambda} |k(\xi)|\tilde{\pi}^{z\sigma}_\Lambda(d\xi) < +\infty. \tag{2.27}$$

Proof. Let $\gamma = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$. From Corollary 2.18,

$$\int_{\mathscr{M}_\Lambda} |\bar{k}(\gamma,\xi)|\tilde{\pi}^{z\sigma}_\Lambda(d\xi) \leq \prod_{i=1}^N \int_{\mathscr{M}_\Lambda} Q(\{\mathbf{x}_i\},\xi)\tilde{\pi}^{z\sigma}_\Lambda(d\xi).$$

Thanks to Lemma 2.19,

$$\begin{aligned}
\int_{\mathscr{M}_\Lambda} Q(\{\mathbf{x}_i\},\xi)\tilde{\pi}^{z\sigma}_\Lambda(d\xi) &= \sum_{N=0}^{+\infty}\frac{z^N}{N!}\int_{(\Lambda\times C_0)^N} Q(\{\mathbf{x}_i\},\{\mathbf{y}_1,\dots,\mathbf{y}_N\})\,\sigma(d\mathbf{y}_1)\cdots\sigma(d\mathbf{y}_N) \\
&= e^{2\beta B_\Phi} + \sum_{N=1}^{+\infty}\frac{z^N}{N!}e^{2B_\phi(N+1)}C(\beta)^{N-1}(N+1)^{N-1}\underbrace{\int_{\Lambda\times C_0}|e^{-\beta\Phi(\mathbf{x},\mathbf{y})}-1|\sigma(d\mathbf{y})}_{\leq C(\beta)} \\
&\leq e^{2\beta B_\Phi}\left(1 + \frac{e}{\sqrt{2\pi}}\sum_{N=1}^{+\infty}\frac{(z\,C(\beta)e^{2\beta B_\Phi+1})^N}{N^{3/2}}\right) \leq e^{2\beta B_\Phi}\left(1 + \frac{e}{\sqrt{2\pi}}\sum_{N=1}^{+\infty}\frac{(z\,C(\beta)e^{2\beta B_\Phi+1})^N}{N}\right),
\end{aligned}$$

where, in the third step, we used the inequality $(N+1)^{N-1} \leq \frac{1}{\sqrt{2\pi}}e^{N+1}\frac{N!}{(N+1)^{3/2}}$, which is a consequence of Stirling's formula: for any $n \geq 0$,

$$\sqrt{2\pi}n^{n+1/2}e^{-n}e^{1/(12n+1)} \leq n! \Longrightarrow n^{n-2} \leq \frac{1}{\sqrt{2\pi}}e^n\frac{(n-1)!}{n^{3/2}}.$$

For $z < (C(\beta)e^{2\beta B_\Phi+1})^{-1} =: \mathfrak{z}_{\mathrm{Ru}}(\beta)$, the above series converges, and we obtain

$$\int_{\mathscr{M}_\Lambda} Q(\{\mathbf{x}_i\},\xi)\tilde{\pi}^{z\sigma}_\Lambda(d\xi) \leq e^{2\beta B_\Phi}\left(1 + \frac{e}{\sqrt{2\pi}}\log\left(\frac{1}{1 - z/\mathfrak{z}_{\mathrm{Ru}}(\beta)}\right)\right) =: \mathbf{c}_z.$$

By using Corollary 2.18, and proceeding similarly to the proof of Lemma 2.19, we obtain that, for $z < \mathfrak{z}_{\mathrm{Ru}}(\beta)$,

$$\int_{\mathscr{M}_\Lambda} |k(\xi)| \tilde{\pi}_\Lambda^{z\sigma}(d\xi) < +\infty.$$

□

Remark. Note that $\mathbf{c}_z$ depends on z but is uniform in Λ; moreover, $\mathbf{c}_0 = e^{2\beta B_\Phi}$.

2.3.3 A Ruelle bound for correlation functions

As a consequence of (2.27), we can use the representation (2.22) of the correlation function

$$\rho_\Lambda^{(z,\beta)}(\gamma) = \int_{\mathscr{M}_\Lambda} \bar{k}(\gamma, \xi) \tilde{\pi}_\Lambda^{z\sigma}(d\xi),$$

and use the above tree-graph estimates to obtain the following Ruelle bound:

Proposition 2.21. Let $\beta > 0$ and $\mathfrak{z}_{\mathrm{Ru}}(\beta)$ as defined in (2.25). For a pair potential Φ satisfying Assumptions 2.1+2.3, for any activity $z \in (0, \mathfrak{z}_{\mathrm{Ru}}(\beta))$ and any finite volume $\Lambda \subset \mathbb{R}^d$, the finite-volume correlation function $\rho_\Lambda^{(z,\beta)}$ satisfies, for $\tilde{\pi}^{z\sigma}$-a.a. $\gamma \in \mathscr{M}_\Lambda$,

$$\rho_\Lambda^{(z,\beta)}(\gamma) \le \mathbf{c}_z^{|\gamma|}, \tag{2.28}$$

where the constant $\mathbf{c}_z$ is defined in (2.26). Moreover, a similar bound holds for the N-point correlation functions of any $P \in \mathscr{G}_{z,\beta}(H)$: for any $z \in (0, \mathfrak{z}_{\mathrm{Ru}}(\beta))$, for any $N \ge 1$, for $\sigma^{\otimes N}$-almost all $\{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathscr{E}^N$

$$\rho_N^{(P)}(\mathbf{x}_1, \dots, \mathbf{x}_N) \le \mathbf{c}_z^N. \tag{2.29}$$

Proof. Fix $z < \mathfrak{z}_{\mathrm{Ru}}(\beta)$. The first statement is an immediate consequence of Lemma 2.20. Moreover, as the right hand side of (2.28) does not depend on Λ, this bound also holds in the limit as $\Lambda \uparrow \mathbb{R}^d$, so for the limiting correlation function $\rho_f^{(z)}(\gamma) := \int_{\mathscr{M}_f} \bar{k}(\gamma, \xi) \tilde{\pi}^{z\sigma}(d\xi)$, $\gamma \in \mathscr{M}_f$.

For the second statement, consider $\gamma = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$. It is known (see [81], Lemmas 12 and 15), that the limiting correlation functional $\rho_f^{(z)}(\gamma)$ coincides with the correlation function $\rho_N^{(P)}(\gamma)$ whenever the expression in (2.14) is well defined. As this is true thanks to (2.14), the Ruelle bound (2.29) holds for any $P \in \mathscr{G}_{z,\beta}(H)$. □

2.4 Uniqueness via the Kirkwood–Salsburg equations

We are in the following situation: we have an infinite-volume Gibbs point process $P \in \mathscr{G}_{z,\beta}(H)$ associated to a potential Φ (not necessarily constructed as in Section

2.2) and whose correlation functions satisfy a Ruelle bound, and wish to understand whether it is indeed the unique such process associated to Φ and with activity z.

In this section we assume that, additionally to Assumption 2.1, the correlation functions of any $P \in \mathscr{G}_{z,\beta}(H)$ can be represented as in (2.14):

$$\rho_N^{(P)}(\mathbf{x}_1, \dots, \mathbf{x}_N) = e^{-\beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N \,|\, \xi)} P(d\xi).$$

The uniqueness proof is structured as follows: we prove that the correlation functions of a Gibbs point process satisfy the Kirkwood–Salsburg equations. Moreover, thanks to the Ruelle bounds, these correlation functions belong to an appropriate Banach space, where these equations have at most one solution. From this, we obtain the uniqueness of the Gibbs point process P.

2.4.1 The Kirkwood–Salsburg equations

The key of this part is to show that the correlation functions $(\rho_N^{(P)})_N$ of any $P \in \mathscr{G}_{z,\beta}(H)$ solve, for all $N \geq 1$, for $\sigma^{\otimes(N+1)}$-almost all $(\mathbf{x}_0, \dots, \mathbf{x}_N) \in \mathscr{E}^{N+1}$, the sequence of *Kirkwood–Salsburg equations*

$$\begin{aligned}\rho_{N+1}^{(P)}(\mathbf{x}_0, \dots, \mathbf{x}_N) = e^{-\beta E_\Phi(\mathbf{x}_0 \,|\, \mathbf{x}_1,\dots,\mathbf{x}_N)} \Big(\rho_N^{(P)}(\mathbf{x}_1, \dots \mathbf{x}_N) \\ + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^{k} (e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1) \rho_{N+k}^{(P)}(\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{y}_1, \dots, \mathbf{y}_k) \sigma^{\otimes k}(d\mathbf{y}_1, \dots, d\mathbf{y}_k) \Big),\end{aligned} \tag{KS}_z$$

where, by convention, $\rho_0^{(P)} = 1$.

Note that the different nature of the Ruelle bounds of Section 2.2 and 2.3 – the former allows for a dependence on the marks of the N points that the latter does not – requires two different approaches. We first treat, in Subection 2.4.2, the simpler case where the Ruelle bound holds for a constant $\mathbf{c} > 0$; in Subection 2.4.3 we consider the situation in which the Ruelle bound holds for a positive function $\mathbf{c} : \mathscr{E} \longrightarrow \mathbb{R}_+$. Accordingly, the Banach space in which we prove uniqueness is defined as follows:

Definition 2.22. The Banach space $\mathbb{X}_\mathbf{c}$ is the set of all sequences $r = (r_N)_N$ such that

$$\exists \mathfrak{b}_r \geq 0 \,:\, \forall N \geq 1,\ |r_N(\mathbf{x}_1, \dots, \mathbf{x}_N)| \leq \mathfrak{b}_r \prod_{i=1}^{N} \mathbf{c}(\mathbf{x}_i),$$

endowed with the norm $\|r\|_\mathbf{c}$ equal to the smallest such $\mathfrak{b}_r$.

Note that, in the case of $\mathbf{c} > 0$ constant, the right hand side reads $\mathfrak{b}_r \mathbf{c}^N$.

We can then interpret the Kirkwood–Salsburg equations as an operator acting on the Banach space $\mathbb{X}_{\mathbf{c}}$.

Definition 2.23. Consider the *Kirkwood–Salsburg operator* $\mathbf{K}_z$, $z > 0$, acting on $\mathbb{X}_{\mathbf{c}}$, given by

$$
\begin{aligned}
&(\mathbf{K}_z r)_1(\mathbf{x}_0) = \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^{k} (e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1) r_k(\mathbf{y}_1, \dots, \mathbf{y}_k)\, \sigma^{\otimes k}(d\mathbf{y}_1, \dots, d\mathbf{y}_k)\Big);\\
&(\mathbf{K}_z r)_{N+1}(\mathbf{x}_0, \dots, \mathbf{x}_N) = e^{-\beta \sum_{i=1}^{N} \Phi(\mathbf{x}_0,\mathbf{x}_i)} \Big(r_N(\mathbf{x}_1, \dots \mathbf{x}_N) \qquad (2.30)\\
&+ \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^{k} (e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1) r_{N+k}(\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{y}_1, \dots, \mathbf{y}_k)\, \sigma^{\otimes k}(d\mathbf{y}_1, \dots, d\mathbf{y}_k)\Big),\ N \geq 1.
\end{aligned}
$$

The Kirkwood–Salsburg equations$(\mathrm{KS})_z$ can now be rewritten as the following fixed-point problem in the Banach space $\mathbb{X}_{\mathbf{c}}$:

$$r = \mathbf{K}_z r + \underline{1}_z,$$

where $\underline{1}_z = (\underline{1}_{z,N})_N$ is given by $\underline{1}_{z,1}(\mathbf{x}_1) = 1$, $\underline{1}_{z,N} = 0$ for $N \geq 2$.

2.4.2 The case of uniform Ruelle bounds

In this subsection, we work with energy functionals E_Φ and activities $z > 0$ such that Assumptions 2.1+2.3+2.4 hold. We consider the case of a Ruelle bound that holds for a constant $\mathbf{c}$, *uniformly* in the points $\mathbf{x}_1, \dots, \mathbf{x}_N$, that is:

Assumption 2.4 (Uniform Ruelle bound)**.** Assume there exists a constant $\mathbf{c} > 0$ such that, for any $P \in \mathscr{G}_{z,\beta}(H)$, for any $N \geq 1$, for $\sigma^{\otimes N}$-almost all $\{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathscr{E}^N$, its correlation function $\rho_N^{(P)}$ satisfy, uniformly in $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$, the following Ruelle bound:

$$\rho_N^{(P)}(\mathbf{x}_1, \dots, \mathbf{x}_N) \leq \mathbf{c}^N. \qquad (2.31)$$

Example 2.3. In what we have seen above, this holds

- For any $z > 0$, for Φ as in Example 2.1, $P = P^{z,\beta}$ (see (2.18)), with $\mathbf{c} = e^{3\beta B_\phi}$.
- For $z \in (0, \mathfrak{z}_{\mathrm{Ru}}(\beta))$, under Assumption 2.3, with $\mathbf{c} = \mathbf{c}_z$ as defined in (2.26).

Proposition 2.24. Let $z > 0$, $\beta > 0$. Under Assumptions 2.1+2.3+2.4, the correlation functions $(\rho_N^{(P)})_N$ of any $P \in \mathscr{G}_{z,\beta}(H)$ solve, for all $N \geq 1$, for $\sigma^{\otimes(N+1)}$-almost all $(\mathbf{x}_0, \dots, \mathbf{x}_N) \in \mathscr{E}^{N+1}$, the *Kirkwood–Salsburg equation* $(\mathrm{KS})_z$ defined above.

Proof. Thanks to the stability of Φ, we can define $\mathbf{i}$ as in (2.19), and assume, without loss of generality, that $\mathbf{x}_0 = \mathbf{i}(\gamma)$.

We note first that the absolute convergence of the right hand side of (2.30) is guaranteed by the Ruelle bound and the Ruelle regularity condition. Indeed,

$$\sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^{k} |e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1| \rho^{(P)}_{N+k}(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_1,\dots,\mathbf{y}_k)\, \sigma^{\otimes k}(d\mathbf{y}_1,\dots,d\mathbf{y}_k)\Big)$$
$$\leq \sum_{k=1}^{+\infty} \frac{z^k}{k!} C(\beta) \mathbf{c}_z^{N+k} = \mathbf{c}_z^N \sum_{k=1}^{+\infty} \frac{(z\mathbf{c}_z C(\beta))^k}{k!} \leq \mathbf{c}_z^N e^{z\mathbf{c}_z C(\beta)}.$$

Consider the $(N+1)$-point correlation function of a Gibbs point process P:

$$\rho^{(P)}_{N+1}(\mathbf{x}_0,\dots,\mathbf{x}_N) = e^{-\beta E_\Phi(\mathbf{x}_0,\dots,\mathbf{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\mathbf{x}_0,\dots,\mathbf{x}_N\,|\,\xi)} P(d\xi)$$
$$= e^{-\beta E_\Phi(\mathbf{x}_0\,|\,\mathbf{x}_1,\dots,\mathbf{x}_N)} e^{-\beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\mathbf{x}_0\,|\,\xi)} e^{-\beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N\,|\,\xi)} P(d\xi).$$

Using the factorial measure $\xi^{(k)}$, we have the following expansion:

$$e^{-\beta E_\Phi(\mathbf{x}_0\,|\,\xi)} = 1 + \sum_{k=1}^{+\infty} \frac{1}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} (e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1)\, \xi^{(k)}(d\mathbf{y}_1,\dots,d\mathbf{y}_k),$$

which is indeed absolutely convergent, since using the GNZ equations (see [75]) one has:

$$\int_{\mathscr{M}} \left(1 + \sum_{k=1}^{+\infty} \frac{1}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} |e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1|\, \xi^{(k)}(d\mathbf{y}_1,\dots,d\mathbf{y}_k) \right) P(d\xi)$$
$$\overset{\text{(GNZ)}}{=} 1 + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} |e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1|\, e^{-\beta E_\Phi(\mathbf{y}_1,\dots,\mathbf{y}_k)}$$
$$\int_{\mathscr{M}} e^{-\beta E_\Phi(\mathbf{y}_1,\dots,\mathbf{y}_k\,|\,\xi)} P(d\xi)\, \sigma^{\otimes k}(d\mathbf{y}_1,\dots,d\mathbf{y}_k) \tag{2.32}$$
$$= 1 + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} |e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1|\, \rho^{(P)}_k(\mathbf{y}_1,\dots,\mathbf{y}_k) \sigma^{\otimes k}(d\mathbf{y}_1,\dots,d\mathbf{y}_k)$$
$$\overset{(2.31)}{\leq} 1 + \sum_{k=1}^{+\infty} \frac{(z\mathbf{c})^k}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} |e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1|\, \sigma^{\otimes k}(d\mathbf{y}_1,\dots,d\mathbf{y}_k) \leq e^{z\mathbf{c}C(\beta)} < +\infty,$$

where in the last line we used the Ruelle bound and the regularity assumption 2.3.

We can then exchange summation over k and integration over $\mathscr{M}$, yielding

$$
\begin{aligned}
&e^{-\beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\mathbf{x}_0\,|\,\xi)} e^{-\beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N\,|\,\xi)} P(d\xi) \\
&= e^{-\beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N\,|\,\xi)}\, P(d\xi) \\
&\quad + \sum_{k=1}^{+\infty} \frac{1}{k!} \int_{\mathscr{M}} \int_{\mathscr{E}^k} e^{-\beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N) - \beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N\,|\,\xi)} \prod_{j=1}^{k} (e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1)\, \xi^{(k)}(d\mathbf{y}_1,\dots,d\mathbf{y}_k)\, P(d\xi) \\
&= \rho_N^{(P)}(\mathbf{x}_1,\dots,\mathbf{x}_N) \\
&\quad + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int_{\mathscr{E}^k} e^{-\beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_1,\dots,\mathbf{y}_k)} \prod_{j=1}^{k} (e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1) \\
&\qquad\qquad \int_{\mathscr{M}} e^{-\beta E_\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_1,\dots,\mathbf{y}_k\,|\,\xi)}\, P(d\xi)\, \sigma^{\otimes k}(d\mathbf{y}_1,\dots,d\mathbf{y}_k) \\
&= \rho_N^{(P)}(\mathbf{x}_1,\dots,\mathbf{x}_N) \\
&\quad + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} (e^{-\beta\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1) \rho_{N+k}^{(P)}(\mathbf{x}_1,\dots,\mathbf{x}_N,\mathbf{y}_1,\dots,\mathbf{y}_k)\, \sigma^{\otimes k}(d\mathbf{y}_1,\dots,d\mathbf{y}_k),
\end{aligned}
$$

and concluding the proof. □

Proposition 2.25. Under Assumptions 2.1+2.3+2.4, for any $z > 0$, $\beta > 0$, and any $\mathbf{c} > 0$, the Kirkwood–Salsburg operator $\mathbf{K}_z$ is a bounded operator in $\mathbb{X}_{\mathbf{c}}$.

Moreover, there exists a critical threshold

$$
0 < \mathfrak{z}_{\mathrm{crit}}(\beta) := \inf\{z > 0 \,:\, \mathbf{c}_z^{-1} e^{2\beta B_\Phi + z\mathbf{c}_z C(\beta)} > 1\} \le \mathfrak{z}_{\mathrm{Ru}}(\beta) \tag{2.33}
$$

such that, for any $z \in (0, \mathfrak{z}_{\mathrm{crit}}(\beta))$, $\mathbf{K}_z$ is a contraction in $\mathbb{X}_{\mathbf{c}_z}$, where $\mathbf{c}_z$ is defined in (2.26), and $\mathfrak{z}_{\mathrm{Ru}}(\beta)$ is defined in (2.25). For such activities there exists then at most one solution of $(\mathrm{KS})_z$ in $\mathbb{X}_{\mathbf{c}_z}$.

Proof. For any $r \in \mathbb{X}_{\mathbf{c}}$, with $\|r\|_{\mathbf{c}} \le 1$, we estimate

$$
\begin{aligned}
|(\mathbf{K}_z r)_{N+1}(\mathbf{x}_0,\dots,\mathbf{x}_N)| \le\ & e^{-\sum_{i=1}^{N} \Phi(\mathbf{x}_0-\mathbf{x}_i)} \Big(\mathbf{c}^N \\
& + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^{k} |e^{-\Phi(\mathbf{x}_0,\mathbf{y}_j)} - 1| \mathbf{c}^{N+k}\, \sigma^{\otimes k}(d\mathbf{y}_1,\dots,d\mathbf{y}_k) \Big).
\end{aligned}
$$

By stability of Φ, we have

$$\begin{aligned}|(\mathbf{K}_z r)_{N+1}(\mathbf{x}_0,\dots,\mathbf{x}_N)| &\le e^{2\beta B_\Phi}\mathbf{c}^N\Big(1+\sum_{k=1}^{+\infty}\frac{(z\mathbf{c})^k}{k!}\int\prod_{j=1}^{k}|e^{-\Phi(\mathbf{x}_0,\mathbf{y}_j)}-1|\,\sigma^{\otimes k}(d\mathbf{y}_1,\dots,d\mathbf{y}_k)\Big)\\ &\le e^{2\beta B_\Phi}\mathbf{c}^N\Big(1+\sum_{k=1}^{+\infty}\frac{(z\mathbf{c})^k}{k!}C(\beta)^k\Big)\\ &= \mathbf{c}^{N+1}\mathbf{c}^{-1}e^{2\beta B_\Phi+zC(\beta)\mathbf{c}}.\end{aligned}$$

The Kirkwood–Salsburg operator is then bounded in $\mathbb{X}_{\mathbf{c}}$: $|||\mathbf{K}_z|||_{\mathbf{c}} \le \mathbf{c}^{-1}e^{2\beta B_\phi+zC(\beta)\mathbf{c}}$.

Consider now $\mathbf{c}_z$ as defined in (2.26), and set $f(z) := \frac{e^{2\beta B_\phi+z\mathbf{c}_zC(\beta)}}{\mathbf{c}_z}$. We have $f(0)=1$ and

$$f'(z) = e^{2\beta B_\Phi}\frac{e^{z\mathbf{c}_zC(\beta)}}{\mathbf{c}_z^2}\big(C(\beta)(\mathbf{c}_z^2+z\mathbf{c}_z'\mathbf{c}_z-\mathbf{c}_z'\big).$$

so that $f'(0)<0$. Indeed,

$$\operatorname{sign} f'(0) = \operatorname{sign}\big(C(\beta)\mathbf{c}_0^2-\mathbf{c}_0'\big) = \operatorname{sign}\big(C(\beta)e^{4\beta B_\Phi}(1-e^2/\sqrt{2\pi})\big) = -1.$$

(see Figures 2.5 and 2.6), The set $\{z>0 : \mathbf{c}_z^{-1}e^{2\beta B_\Phi+z\mathbf{c}_zC(\beta)}<1\}$ is then non-empty, and defining

$$\mathfrak{z}_{\mathrm{crit}}(\beta) := \inf\{z>0 : \mathbf{c}_z^{-1}e^{2\beta B_\Phi+z\mathbf{c}_zC(\beta)}>1\},$$

we have that, for any $z<\mathfrak{z}_{\mathrm{crit}}(\beta)$, the norm of $\mathbf{K}_z$ in $\mathbb{X}_{\mathbf{c}_z}$ is smaller than 1, so that it is a contraction in $\mathbb{X}_{\mathbf{c}_z}$.

Finally, note that, since $\lim_{z\to\mathfrak{z}_{\mathrm{Ru}}(\beta)^-}\mathbf{c}_z=+\infty$ and $\mathbf{c}_z^{-1}e^{2\beta B_\Phi+z\mathbf{c}_zC(\beta)}=+\infty$ for $z\ge\mathfrak{z}_{\mathrm{Ru}}(\beta)$, we have that $\mathfrak{z}_{\mathrm{crit}}(\beta)\le\mathfrak{z}_{\mathrm{Ru}}(\beta)$. □

Example 2.1 (continued). Consider a potential Φ in the class of Example 2.1. The Ruelle bound is satisfied for $\mathbf{c}=e^{3\beta B_\phi}$ (see (2.18)). For such a value of $\mathbf{c}$, the Kirkwood–Salsburg operator $\mathbf{K}_z$ on $\mathbb{X}_{\mathbf{c}}$ is a contraction as soon as $\mathbf{c}^{-1}e^{2\beta B_\Phi+z\mathbf{c}C(\beta)}<1$, that is for $z<\beta B_\phi(C(\beta)e^{3\beta B_\phi})^{-1}$.

2.4.3 The case of non-uniform Ruelle bounds

In this subsection we allow for a weaker notion of stability, in particular, we work under the following weakening of Assumption 2.1:

Assumption 2.1′ (Weak stability). Consider an energy functional H as in (2.5), where the self-potential Ψ satisfies (2.8), but for which the stability condition (2.9) of the pair potential Φ is replaced by a weaker one:
($\mathscr{H}_{\text{w.st.}}$) The pair potential between two paths is given by a symmetric functional

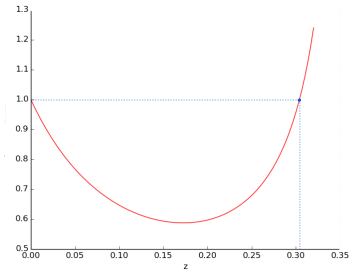

Figure 2.5. Let $B_\phi = 0$ (i.e. a repulsive potential), $\beta = 1$, $C(1) = 1$. Plot of $z \mapsto \mathbf{c}_z^{-1} e^{z\mathbf{c}_z}$. The curve explodes as z approaches $\mathfrak{z}_{Ru}(1) = 1/e \simeq 0.37$, and the uniqueness domain is $(0, \mathfrak{z}_{\text{crit}}(1))$, where $\mathfrak{z}_{\text{crit}}(1) \simeq 0.304$.

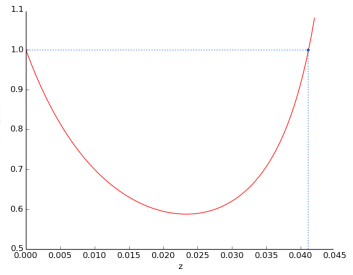

Figure 2.6. Let $B_\phi = 1$, $\beta = 1$, $C(1) = 1$. Plot of $z \mapsto \mathbf{c}_z^{-1} e^{2+z\mathbf{c}_z}$. The curve explodes as z approaches $\mathfrak{z}_{Ru}(1) \simeq 0.05$, and the uniqueness domain is $(0, \mathfrak{z}_{\text{crit}}(1))$, where $\mathfrak{z}_{\text{crit}}(1) \simeq 0.041$.

$\Phi : \mathscr{E} \times \mathscr{E} \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that, for any $\{\mathbf{x}_0, \dots, \mathbf{x}_N\} \subset \mathscr{E}$, there exist a function $\mathbf{b} : \mathscr{E} \longrightarrow \mathbb{R}_+$ and some $\mathbf{x} \in \{\mathbf{x}_0, \dots, \mathbf{x}_N\}$ (w.l.o.g. $\mathbf{x}_0$) with

$$\sum_{i=1}^{N} \Phi(\mathbf{x}_0, \mathbf{x}_i) \geq -\mathbf{b}(\mathbf{x}_0). \tag{2.34}$$

Remark. We know from (2.19) that ($\mathscr{H}_{\text{w.st.}}$) holds whenever Φ is a stable potential. Conversely, if Φ satisfies ($\mathscr{H}_{\text{w. st.}}$) for a constant $\mathbf{b} \equiv B_\Phi$, then it is also stable for that same constant.

While in the previous subection we assumed that the pair potential Φ satisfied a *uniform* regularity condition (Assumption 2.3), here we work with potentials Φ that satisfy the following *weighted* regularity condition (see [80]):

Assumption 2.3′ (Weighted regularity). There exist a function $\mathbf{a} : \mathscr{E} \longrightarrow \mathbb{R}_+$ and a critical activity $\mathfrak{z}_{\text{crit}}(\beta) > 0$ such that, for any $\mathbf{x} \in \mathscr{E}$,

$$\mathfrak{z}_{\text{crit}}(\beta) \int e^{\mathbf{a}(\mathbf{y})+\mathbf{b}(\mathbf{y})} |e^{-\beta\Phi(\mathbf{x},\mathbf{y})} - 1| \, \sigma(d\mathbf{y}) \leq \mathbf{a}(\mathbf{x}), \tag{2.35}$$

with

$$\int_{\mathscr{E}} e^{\mathbf{a}(\mathbf{x})+\mathbf{b}(\mathbf{x})} \sigma(d\mathbf{x}) < +\infty. \tag{2.36}$$

Let $\mathbf{a}$ and $\mathbf{b}$ as above. We also assume that the correlation functions satisfy a Ruelle bound of the following form:

Assumption 2.4′ (Non-uniform Ruelle bound). For any $P \in \mathcal{G}_{z,\beta}(H)$, for any $N \geq 1$, for $\sigma^{\otimes N}$-almost all $\{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathscr{E}^N$, the following holds:

$$\rho_N^{(P)}(\mathbf{x}_1, \dots, \mathbf{x}_N) \leq \prod_{i=1}^{N} e^{\mathbf{a}(\mathbf{x}_i)+\mathbf{b}(\mathbf{x}_i)}.$$

Example 2.4. Consider a potential $\phi = \phi_{hc} + \phi_l$, satisfying Assumptions 2.1 and 2.2, given by the sum of a hard core potential (with hard core diameter $R > 0$) and a bounded potential ϕ_l, on $[R, +\infty)$:

$$\exists M_\phi > 0 \ : \ \phi_l(u) \leq M_\phi \ \forall u \geq R.$$

In particular, we recall from (2.11) that there exists a constant $\bar{B}_\Phi \geq 0$ such that, for any $\mathbf{x} = (x, m) \in \mathscr{E}$, for any $\xi \in \mathscr{M}^{\mathrm{temp}}$,

$$E_\Phi(\mathbf{x} \,|\, \xi) \geq -\bar{B}_\Phi(1 + \|m\|^{d+\delta}). \tag{2.37}$$

We show here that there exist functions $\mathbf{a}$ and $\mathbf{b}$, and a threshold activity $\mathfrak{z}_{\mathrm{crit}}(\beta) > 0$ such that Assumptions 2.3′ and 2.4′ hold for any $z \in (0, \mathfrak{z}_{\mathrm{crit}}(\beta))$.

Proof. Using (2.21), we have $|e^{-\beta\Phi} - 1| \leq \beta|\bar{\Phi}|e^{2\beta B_\Phi}$, and the weighted regularity condition follows as soon as

$$ze^{2\beta B_\Phi} \int_{\mathscr{E}} e^{\mathbf{a}(\mathbf{x}_2)} \int_0^1 \beta|\bar{\phi}(x_2 + m_2(s) - x_1 - m_1(s))| ds \, \mathbb{1}_{\{|x_2-x_1| \leq a_0 + \|m_2\| + \|m_1\|\}} \, \sigma(d\mathbf{x}_2) \leq \mathbf{a}(\mathbf{x}_1).$$

Considering a function $\mathbf{a}$ of the form $\mathbf{a}(x, m) = \mathbf{a}(m) = A(1 + \|m\|^{d+\delta})$, for some constant $A > 0$ to be determined, and recalling that $\Psi(x, m) \geq -A_\Psi\|m\|^{d+\delta}$, this reduces to

$$z\beta e^{2\beta B_\Phi} \int_{C_0} e^{A(1+\|m_2\|^{d+\delta})} \int_{\mathbb{R}^d} \int_0^1 \Big|\phi\big(|x_1 + m_1(s) - x_2 - m_2(s)|\big)\Big| ds \mathbb{1}_{\{|x_1-x_2| \leq a_0 + \|m_1\| + \|m_2\|\}} \\ dx_2 \, e^{A_\Psi\|m_2\|^{d+\delta}} \mathbf{R}(dm_2) \leq A(1 + \|m_1\|^{d+\delta}).$$

Estimating the left hand side leads to:

$$z\beta e^{2\beta B_\Phi} \int_{C_0} \left(b_d R^d + M_\phi k_d b_d (a_0^d + \|m_1\|^d + \|m_2\|^d)\right) e^{A(1+\|m_2\|^d)+A_\Psi\|m_2\|^{d+\delta}} \mathbf{R}(dm_2),$$

where k_d is such that $(x + y + z)^d \leq k_d(x^d + y^d + z^d)$, and b_d the volume of the unit ball in $\mathbb{R}^d$. Setting

$$v_A := \int e^{A(1+\|m\|^{d+\delta})+A_\Psi\|m\|^{d+2\delta}} \mathbf{R}(dm),$$

which is finite thanks to the definition of the measure R, we can fix A by the following (note that $A \geq \bar{B}_\Phi$; the reason for this choice will be apparent shortly):

$$A := \sup_{u \geq 0} \frac{\bar{B}_\Phi(1 + u^{d+\delta}) \vee b_d \left(R^d + M_\phi k_d (a_0^d + u^d + 1)\right)}{1 + u^{d+\delta}} < +\infty,$$

so that the regularity assumption is satisfied for $\mathbf{a}(x, m) = A(1 + \|m_1\|^{d+\delta})$, $\mathbf{b} \equiv 2B_\Phi$, and

$$\mathfrak{z}_{\mathrm{crit}}(\beta) := (v_A \beta e^{2\beta B_\Phi})^{-1}.$$

Note that, as the value of the integral v_A is not easily controlled, the threshold activity $\mathfrak{z}_{\mathrm{crit}}(\beta) > 0$ may be very small.

From the representation (2.14) of the correlation functions and (2.37), the Ruelle bound of Assumption 2.4′ follows as well, as $A \geq \bar{B}_\Phi$ by construction. □

It is easy to see that an analogous to Proposition 2.24 holds also for the case of non-uniform Ruelle bound. Indeed, the computations are the same, except for using Assumption 2.3′ to prove the absolute convergence of the series in (2.32).

Proposition 2.26. Let Φ such that Assumptions 2.1′+2.3′+2.4′ hold, and set $\mathbf{c}(\mathbf{x}) := e^{\mathbf{a}(\mathbf{x})+\mathbf{b}(\mathbf{x})}$. For any $\beta > 0$ and $z \in (0, \mathfrak{z}_{\mathrm{crit}}(\beta))$, the operator $\mathbf{K}_z$ is a contraction in $\mathbb{X}_{\mathbf{c}}$. For such activities there exists then at most one solution of $(\mathrm{KS})_z$ in $\mathbb{X}_{\mathbf{c}}$.

Proof. For any $r \in \mathbb{X}_{\mathbf{c}}$, with $\|r\|_{\mathbf{c}} \leq 1$, we estimate

$$\begin{aligned}
|(\mathbf{K}_z r)_{N+1}(\mathbf{x}_0, \dots, \mathbf{x}_N)| &\leq e^{-\sum_{i=1}^N \Phi(\mathbf{x}_0 - \mathbf{x}_i)} \Bigg(\prod_{i=1}^N \mathbf{c}(\mathbf{x}_i) \\
&\quad + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^k |e^{-\Phi(\mathbf{x}_0, \mathbf{y}_j)} - 1| \prod_{i=1}^N \mathbf{c}(\mathbf{x}_i) \prod_{j=1}^k \mathbf{c}(\mathbf{y}_j)\, \sigma^{\otimes k}(d\mathbf{y}_1, \dots, d\mathbf{y}_k) \Bigg) \\
&\overset{(2.34)}{\leq} e^{\mathbf{b}(\mathbf{x}_0)} \prod_{i=1}^N \mathbf{c}(\mathbf{x}_i) \Bigg(1 + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^k \mathbf{c}(\mathbf{y}_j) |e^{-\Phi(\mathbf{x}_0, \mathbf{y}_j)} - 1|\, \sigma^{\otimes k}(d\mathbf{y}_1, \dots, d\mathbf{y}_k) \Bigg) \\
&= e^{\mathbf{b}(\mathbf{x}_0)} \prod_{i=1}^N \mathbf{c}(\mathbf{x}_i) \Bigg(1 + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^k e^{\mathbf{a}(\mathbf{y})+\mathbf{b}(\mathbf{y})} |e^{-\Phi(\mathbf{x}_0, \mathbf{y}_j)} - 1|\, \sigma^{\otimes k}(d\mathbf{y}_1, \dots, d\mathbf{y}_k) \Bigg) \\
&\overset{(2.35)}{\leq} e^{\mathbf{b}(\mathbf{x}_0)} \prod_{i=1}^N \mathbf{c}(\mathbf{x}_i) \sum_{k=0}^{+\infty} \frac{(z/\mathfrak{z}_{\mathrm{crit}}(\beta))^k \mathbf{a}^k(\mathbf{x}_0)}{k!} \\
&= e^{\mathbf{b}(\mathbf{x}_0)} \prod_{i=1}^N \mathbf{c}(\mathbf{x}_i) e^{\mathbf{a}(\mathbf{x}_0) z/\mathfrak{z}_{\mathrm{crit}}(\beta)} < \prod_{i=0}^N \mathbf{c}(\mathbf{x}_i).
\end{aligned}$$

The Kirkwood–Salsburg operator is then a contraction: $|||\mathbf{K}_z|||_{\mathbf{c}} < 1$. □

2.4.4 Uniqueness domain

We can now state the main result of this section. Recall that, both for the uniform and the non-uniform Ruelle bound setting we have a critical threshold $\mathfrak{z}_{\mathrm{crit}}(\beta) > 0$ such that for any $z \in (0, \mathfrak{z}_{\mathrm{crit}}(\beta))$, the Kirkwood–Salsburg operator $\mathbf{K}_z$ is a contraction in $\mathbb{X}_{\mathbf{c}}$. In the former case $\mathbf{c} > 0$ is a constant, while in the latter it is a non-negative function $\mathbf{c} : \mathscr{E} \longrightarrow \mathbb{R}_+$.

Theorem 2.2. Let H be an energy functional as in (2.5), satisfying either Assumptions 2.1+2.3+2.4 or Assumptions 2.1′+2.3′+2.4′. For any $\beta > 0$ and $z \in (0, \mathfrak{z}_{\mathrm{crit}}(\beta))$, there exists at most one infinite-volume Gibbs point process P in $\mathscr{G}_{z,\beta}(H)$.

Proof. Let $\beta > 0$, $z \in (0, \mathfrak{z}_{\mathrm{crit}}(\beta))$, and consider two Gibbs point processes $P, \hat{P} \in \mathscr{G}_{z,\beta}(H)$.

(i) We know from Proposition 2.24, that the correlation functions $\rho^{(P)}$ and $\rho^{(\hat{P})}$ both satisfy the Kirkwood–Salsburg equations $(\mathrm{KS})_z$.

(ii) By assumption, $\rho^{(P)}$ and $\rho^{(\hat{P})}$ satisfy a Ruelle bound for the same $\mathbf{c}$, and are therefore both elements of $\mathbb{X}_{\mathbf{c}}$.

(iii) For $z < \mathfrak{z}_{\mathrm{crit}}(\beta)$, $(\mathrm{KS})_z$ has a unique solution, so that the correlation functions of $\hat{P}$ – and therefore its factorial moment measures $(\alpha_N^{(\hat{P})})_N$ – must coincide with those of P.

(iv) For any $N \geq 1$ and any bounded $\Gamma \subset \mathscr{E}$, we compute

$$\begin{aligned} \alpha_N^{(P)}(\Gamma^N) &= \mathbb{E}\left[|\gamma_\Gamma|\big(|\gamma_\Gamma| - 1\big) \dots \big(|\gamma_\Gamma| - N + 1\big)\right] \\ &= \int_{\Gamma^N} \rho_N(\mathbf{x}_1, \dots, \mathbf{x}_N) z^N \sigma(d\mathbf{x}_1) \dots \sigma(d\mathbf{x}_N) \\ &\leq \int_{\Gamma^N} \prod_{i=1}^{N} (z\mathbf{c}(\mathbf{x}_i)) \sigma(d\mathbf{x}_1) \dots \sigma(d\mathbf{x}_N) = (z c_\Gamma)^N, \end{aligned} \tag{2.38}$$

with $c_\Gamma := \int_{\Gamma^N} \mathbf{c}(\mathbf{x}) \sigma(d\mathbf{x})$. We have used here the fact that the Ruelle bound holds either for $\mathbf{c}$ constant (under Assumption 2.4) or integrable (under Assumptions 2.3′+2.4′). We can then conclude that $P = \hat{P}$ (see [61, 58]); in other words, $\mathscr{G}_{z,\beta}(H) = \{P\}$. □

Example 2.1 (continued). Consider here a potential $\phi = \phi_{hc}$ with a pure hard core at some diameter $R > 0$, i.e. $\phi_l \equiv 0$. Taking $a_0 = R$ in the range Assumption 2.2 yields a path potential Φ (stable, with stability constant $B_\phi = 0$) of the form

$$\Phi(\mathbf{x}_1, \mathbf{x}_2) = (+\infty)\, \mathbb{1}_{[0,R)}\Big(\inf_{s \in [0,1]} |x_1 + m_1(s) - x_2 - m_2(s)| \Big).$$

Under this interaction, two Langevin diffusions are forbidden from coming closer than R to each other, at any given time $s \in [0, 1]$.

For such a choice of Φ – which satisfies Assumptions 2.1+2.2+2.3+2.4 – the Gibbs point process $P^{z,\beta}$ constructed in Theorem 2.1 is the unique element of $\mathscr{G}^{\mathrm{temp}}_{z,\beta}(H)$.

An explicit Dobrushin uniqueness region for Gibbs point processes with repulsive interactions

3

In this chapter we consider so-called classical systems, i.e. unmarked point configurations in $\mathbb{R}^d$, with an interaction coming from a pair potential. In particular, we study the uniqueness question and provide an explicit uniqueness region for the Gibbs point process.

The chapter is organised as follows: in Subsection 3.1 we introduce the formalism used in this work. In Subsection 3.2.1 we introduce the assumptions needed and state the uniqueness theorem, which is then proved in Subsection 3.2.2. In Subsection 3.2.3 we comment on the assumptions and give possible generalisations to our work. In Subsections 3.2.4 and 3.2.5 we discuss the optimality of our result, and compare it to existing results coming from cluster expansion and disagreement percolation.

3.1 The setting

In this work we consider point configurations in $\mathbb{R}^d$, $d \geq 2$. Though the notations are mostly analogous to those presented in the previous chapters, we present them here for clarity.

3.1.1 The configuration space

We endow $\mathbb{R}^d$ with the usual Euclidean distance $|\cdot|$ and Borel σ-algebra $\mathscr{B}(\mathbb{R}^d)$, and set the configuration space $\mathscr{M}$ to be the set of σ-finite configurations γ on $\mathbb{R}^d$, i.e. measures of the form $\gamma = \sum_i \delta_{x_i}$, with $\mathrm{Card}(\{i : x_i \in \Lambda\}) < \infty$ for any bounded Borel set $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$ (here γ_Λ the restriction of γ to γ_Λ). As the configurations we consider are *simple*, i.e. with no overlapping points, we also denote a configuration $\gamma = \sum_i \delta_{x_i}$ by the subset of $\mathbb{R}^d$ on which it is supported: $\gamma = \{x_1, \dots, x_n, \dots\} \subset \mathbb{R}^d$. Consequently, $\gamma_\Lambda = \gamma \cap \Lambda$. We write $\gamma'\gamma := \gamma' \cup \gamma$ for the *concatenation* (or union) of two configurations.

We endow $\mathscr{M}$ with the usual σ-algebra $\mathscr{F}$ generated by the counting functions on bounded Borel sets, $\gamma \mapsto \mathrm{Card}(\gamma_\Lambda)$, $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$. For any $\Lambda \subset \mathbb{R}^d$, $\mathscr{M}_\Lambda \subset \mathscr{M}$ denotes the subset of configurations supported on Λ (and by $\mathscr{F}_\Lambda$ the corresponding σ-algebra). On the space $\mathscr{M}$, we consider the probability measure π^z given by the distribution of the homogeneous Poisson point process with intensity $z > 0$.

3.1.2 Interactions and Gibbs point processes

We add an interaction on the Poisson point process by considering an energy functional H coming from a pair potential. More precisely, let ϕ be a symmetric *non-negative* (i.e. repulsive) *pair potential*

$$\phi : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}_+ \cup \{+\infty\},$$

and define the *energy* of a finite configuration $\gamma \in \mathscr{M}_f$ as the pair interaction

$$H(\gamma) = E_\phi(\gamma) := \sum_{\{x,y\}\subset\gamma} \phi(x,y). \tag{3.1}$$

For any bounded set $\Lambda \subset \mathbb{R}^d$, the conditional energy of γ in Λ given its exterior is then defined by setting, for any $\xi \in \mathscr{M}$,

$$E_\phi\left(\gamma_\Lambda \,|\, \xi_{\Lambda^c}\right) := \sum_{x\in\gamma_\Lambda} \sum_{y\in\xi_{\Lambda^c}} \phi(x,y).$$

Since the potential ϕ is non-negative, this quantity is always well-defined. In Subsection 3.2.1 we provide the more precise assumptions that are needed for the main result of this work, but we remark now that we do not assume translation invariance.

We recall in this setting the definition of (infinite-volume) Gibbs point process:

Definition 3.1. Let H be an energy functional as in (3.1). A probability measure P on $\mathscr{M}$ is said to be a *Gibbs point process* with energy functional H, activity $z > 0$ and inverse temperature $\beta > 0$, denoted $P \in \mathscr{G}_{z,\beta}(H)$, if for every bounded measurable function f and for all bounded Borel sets $\Lambda \subset \mathbb{R}^d$, the following DLR equation holds

$$\int_{\mathscr{M}} f\, dP = \int_{\mathscr{M}} \int_{\mathscr{M}_\Lambda} f(\gamma_\Lambda \xi_{\Lambda^c})\, \Xi_\Lambda^{z,\beta}(\xi, d\gamma_\Lambda)\, P(d\xi), \tag{DLR}_\Lambda$$

where the Gibbsian probability kernel $\Xi_\Lambda^{z,\beta}$ is given by

$$\Xi_\Lambda^{z,\beta}(\xi, d\gamma_\Lambda) := \frac{e^{-\beta E_\phi\left(\gamma_\Lambda \,|\, \xi_\Lambda^c\right)}}{Z_\Lambda^{z,\beta}(\xi)}\, \pi_\Lambda^z(d\gamma_\Lambda), \quad \gamma_\Lambda \in \mathscr{M}_\Lambda,$$

where the normalisation factor is the *partition function*

$$Z_\Lambda^{z,\beta}(\xi) := \int_{\mathscr{M}_\Lambda} e^{-\beta E_\phi\left(\gamma_\Lambda \,|\, \xi_\Lambda^c\right)} \pi_\Lambda^z(d\gamma_\Lambda) \in (0,1].$$

Since $Z_\Lambda^{z,\beta}(\xi)$ is finite, the Gibbsian kernel is always *well-defined*.

The first question that arises in Gibbs point processes theory is whether there exists at least one solution to the DLR equations. This important and difficult problem has been studied for many different interactions and settings: from the the classical works of [90] and [77], to more recent works considering the case of geometrical interactions ([18, 21]), infinite-range pair potentials ([25]), and unbounded interactions in the context of marked point processes ([Z2]).

The existence of such a measure in the setting we present here is a known result (for example by D. Ruelle in [91]), which we state here for completeness:

Proposition 3.2. Let ϕ be a non-negative and symmetric pair potential. Let $\beta > 0$ and assume that

$$\forall x \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} \left(1 - e^{-\beta\phi(x,y)}\right) dy < +\infty.$$

Then, for any activity $z > 0$, there exists at least one Gibbs point process $P \in \mathcal{G}_{z,\beta}(H)$.

After the existence question, it is natural to explore that of the uniqueness (or lack thereof) of the Gibbs point process. This question has been of interest to the statistical mechanics community since the end of the 1960s. In particular, three techniques are commonly used to approach this problem: the *Dobrushin criterion* [29, 77], *cluster expansion* [64, 51], and *disagreement percolation* [97, 98, 49]. First introduced in the lattice setting and later extended to the point process setting, these techniques have different assumptions, and yield different parameter domains in which uniqueness holds.

In this chapter we first prove a simple and explicit uniqueness criterion derived from the standard Dobrushin technique, and then compare it to criteria coming from the two other techniques.

3.2 Uniqueness of the Gibbs point process

In Subsections 3.2.1 and 3.2.2, we derive a simple and explicit uniqueness region, by applying the discrete Dobrushin contraction criterion from [29] through a *discretisation* parameter a, and then considering the limit as a goes to 0.

The remainder of the chapter deals with the natural questions that arise from this result: in Subsection 3.2.3 we discuss the assumptions of Theorem 3.1, as well as possible generalisations that could be the subject of future works; in Subsection 3.2.4 we perform a numerical study to show that taking the limit for the mesh size $a \longrightarrow 0$ yields a larger uniqueness region than that of the Dobrushin criterion for any fixed $a > 0$. In Subsection 3.2.5 we compare this uniqueness region to the ones coming

from other approaches, namely *cluster expansion* and *disagreement percolation*.

3.2.1 Dobrushin uniqueness region

We present here the assumptions that are required in the statement of our main result. See Subsection 3.2.3 for some comments on these conditions.

Assumption 3.1. The pair potential ϕ satisfies the following conditions:

(A1) It is *non-negative* and admits a *hard-core* component close to the origin: there exists a measurable neighbourhood $U \subset \mathbb{R}^d$ of the origin (i.e. $B(0,\alpha) \subset U$, for some $\alpha > 0$), such that

$$\forall\, x, y \in \mathbb{R}^d :\ x - y \in U, \quad \phi(x,y) = +\infty.$$

(A2) Uniform regularity of the potential:

$$C(\beta) := \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left(1 - e^{-\beta\phi(x,y)}\right) dy < +\infty.$$

For the third assumption we first need to introduce some notations. Let $a > 0$, and divide the space $\mathbb{R}^d$ into cubes of side-length a, centred in the points of the lattice: for any $i \in a\mathbb{Z}^d$, these are

$$\Lambda_{a,i} := \left(i - \frac{a}{2}, i + \frac{a}{2}\right]^d,$$

then define the following "local supremum" of the Mayer function

$$\Psi_a(x,y) := \sup_{\bar{y}\in\Lambda_{a,i}} \left(1 - e^{-\beta\phi(x,\bar{y})}\right) \quad \text{if } y \in \Lambda_{a,i}. \tag{3.2}$$

The last assumption is given by the following

(A3) Regularity of the Mayer function:

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left(1 - e^{-\beta\phi(x,y)}\right) dy = \lim_{a\to 0} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \Psi_a(x,y)\, dy.$$

We are now ready to state the main result of this chapter:

Theorem 3.1. Let H be an energy functional as in (3.1), where the pair potential ϕ satisfies assumptions (A1), (A2), (A3). Furthermore, let $z > 0$ and $\beta > 0$, and assume

$$z < C(\beta)^{-1}. \tag{3.3}$$

There exists then a unique Gibbs point process $P \in \mathscr{G}_{z,\beta}(H)$.

3.2.2 Proof of Theorem 3.1

In this subsection we provide the proof to the above theorem. As already stated in the Introduction, the result is an application to the continuous setting of the classical Dobrushin technique for lattice models. To do this, we use a standard discretisation technique, via a parameter $a > 0$ that defines the mesh size a. The novelty of our result, and what leads to the explicit uniqueness region, is to consider the Dobrushin criterion in the limit as $a \longrightarrow 0$.

The Dobrushin contraction method in the lattice

The proof of Theorem 3.1 relies on the classical Dobrushin criterion [29]. In this subsection we describe the setting and the results as they apply to our model (for a general presentation see, for example, [45]).

Let $\mathcal{M}$ be a complete separable metric space, which we call the *spin space*. The *lattice configuration* space $\mathcal{M}^{a\mathbb{Z}^d}$ is equipped with the standard cylinder σ-algebra. Let $\Pi = (\Pi_\Gamma(\cdot|\omega))_{\Gamma\subset a\mathbb{Z}^d,\omega\in\mathcal{M}^{a\mathbb{Z}^d}}$ be a *lattice specification*, i.e. a consistent family of conditional probability measures indexed by a finite $\Gamma \subset a\mathbb{Z}^d$ and a lattice configuration $\omega \in \mathcal{M}^{a\mathbb{Z}^d}$. Furthermore, for any event A, $\Pi_\Gamma(A|\omega)$ only depends on the restriction ω_{Γ^c} of ω to $\mathcal{M}^{\Gamma^c}$.

Definition 3.3. A probability measure Q on $\mathcal{M}^{a\mathbb{Z}^d}$ is said to be a *Gibbs measure compatible with the specification* Π if, for any finite $\Gamma \subset a\mathbb{Z}^d$ and any bounded measurable function g, it satisfies

$$\int g(\omega)Q(d\omega) = \int g(\omega'_\Gamma\, \omega_{\Gamma^c})\Pi_\Gamma(d\omega'_\Gamma|\omega)Q(d\omega).$$

The original result, proved by Dobrushin in [29], reads:

Theorem 3.2. Let $\Pi_i := \Pi_{\{i\}}$. If

$$\sup_{i\in a\mathbb{Z}^d}\sum_{j\neq i}\sup_{\substack{\omega,\tilde\omega\in\mathcal{M}^{a\mathbb{Z}^d}\\ \omega_k=\tilde\omega_k\ \forall k\neq j}} d_{TV}\left(\Pi_i\left(\cdot|\omega\right),\Pi_i\left(\cdot|\tilde\omega\right)\right) < 1,$$

then there exists at most one Gibbs measure on $\mathcal{M}^{a\mathbb{Z}^d}$ compatible with the specification Π.

An easy generalisation is the following

Lemma 3.4. Let $A \subset \mathcal{M}^{a\mathbb{Z}^d}$ be such that $P(A) = 1$ for any Gibbs measure P in

$\mathcal{M}^{a\mathbb{Z}^d}$ compatible with the specification Π. Define, for any $i, j \in a\mathbb{Z}^d$,

$$k_{i,j}^A := \sup_{\substack{\omega,\tilde{\omega}\in A \\ \omega_k=\tilde{\omega}_k \ \forall k\neq j}} d_{TV}\left(\Pi_i(\cdot|\omega), \Pi_i(\cdot|\tilde{\omega})\right), \tag{3.4}$$

If

$$\sup_{i\in a\mathbb{Z}^d} \sum_{j\neq i} k_{i,j}^A < 1, \tag{3.5}$$

then there exists at most one Gibbs measure in $\mathcal{M}^{a\mathbb{Z}^d}$ compatible with the specification Π.

Remark. As seen in the above Lemma, the hard-core assumption (A1) allows us to restrict the set of possible boundary conditions in (3.4). This is also achieved by Dobrushin and Pecherski in [32], where they first extended the lattice result of Dobrushin to the continuous framework. However, it is less suited for finding an explicit uniqueness region, which is one of the goals of our work.

For uses of the Dobrushin–Pecherski criterion, see for example [77, 2].

Correspondence between continuous and lattice models

In order to apply Lemma 3.4, one must express the continuous model as a lattice model. The representation we make use of here does not lose any of the information from the continuous model, so that the uniqueness properties of the two models are indeed equivalent (see [32, 77]). In this it differs from the cell-gas model presented by A. Rebenko in [85], where the idea is that uniqueness for the lattice model implies uniqueness for the continuous one, but it is not a 1-1 correspondence.

Fix $a > 0$, and consider again the partition of $\mathbb{R}^d$ into cubes $\Lambda_{a,i} = i + (-a/2, a/2]^d$, $i \in a\mathbb{Z}^d$. Set the spin space to be the space of configurations supported on the closure of $\Lambda_{a,0}$, $\mathcal{M} := \mathscr{M}_{\bar{\Lambda}_{a,0}}$, endowed with its σ-algebra $\mathscr{F}_{\bar{\Lambda}_{a,0}}$. The lattice configurations are then subsets $\omega \subset \mathcal{M}^{a\mathbb{Z}^d}$. The correspondence between the two models is given by the following measurable embedding

$$\begin{aligned} T : \mathscr{M} &\longrightarrow \mathcal{M}^{a\mathbb{Z}^d} \\ \gamma &\longmapsto \omega = (\omega_j)_{j\in a\mathbb{Z}^d}, \end{aligned}$$

defined by setting, for all $j \in a\mathbb{Z}^d$,

$$\omega_j = \gamma_{\Lambda_{a,j}} - j := \{x - j : x \in \gamma_{\Lambda_{a,j}}\}.$$

Its inverse T^{-1} can then be naturally extended from $T(\mathscr{M})$ to the whole of $\mathcal{M}^{a\mathbb{Z}^d}$.

We next define the lattice specification by setting, for any finite $\Gamma \subset a\mathbb{Z}^d$,

$$\Pi_\Gamma(\cdot|\omega) := \Xi_\Lambda^{z,\beta}(\xi, \cdot),$$

where $\Lambda = \cup_{i\in\Gamma}\Lambda_{a,i}$ and $\xi = T^{-1}(\omega)$, and let $\mathscr{G}_{z,\beta}^{a\mathbb{Z}^d}(H)$ be the set of probability measures on $\mathcal{M}^{a\mathbb{Z}^d}$ compatible with the specification Π. Thanks to the hard-core assumption (A1), it can be seen ([77]) that, for $a > 0$ small enough, the map T induces a bijective map

$$\hat{T} : \mathscr{G}_{z,\beta}(H) \longrightarrow \mathscr{G}_{z,\beta}^{a\mathbb{Z}^d}(H).$$

We introduce the following

Definition 3.5. A configuration $\xi \in \mathscr{M}$ is said to be *admissible* if it satisfies the hard core condition, i.e.

$$x, y \in \xi \implies x - y \notin U.$$

Let $\mathscr{A} \subset \mathscr{M}$ denote the set of all admissible configurations.

It is easy to see that, for any $P \in \mathscr{G}_{z,\beta}(H)$, $P(\mathscr{A}) = 1$, so that we can restrict our study to only the admissible configurations. Indeed, thanks to Lemma 3.4 and the one-to-one correspondence between continuous and lattice models, in order to prove uniqueness of the original Gibbs point process, it is enough to show that, for some $a > 0$,

$$\sup_{i\in a\mathbb{Z}^d} \sum_{j\in a\mathbb{Z}^d} k_{i,j}^{(a)} < 1, \tag{3.6}$$

where we have set, for any $i, j \in a\mathbb{Z}^d$,

$$k_{i,j}^{(a)} := \sup_{\substack{\xi,\tilde{\xi}\in\mathscr{A} \\ \xi_{\Lambda_{a,j}^c}=\tilde{\xi}_{\Lambda_{a,j}^c}}} d_{TV}\left(\Xi_{\Lambda_{a,i}}^{z,\beta}(\xi, \cdot); \Xi_{\Lambda_{a,i}}^{z,\beta}(\tilde{\xi}, \cdot)\right).$$

Computation of the coefficients

In this subection we consider $a > 0$ small enough such that

$$\Lambda_{a,0} \subset B(0, \alpha) \subset U.$$

Thanks to Assumption (A1), this implies that every admissible configuration $\xi \in \mathscr{A}$ has at most one point in each cube $\Lambda_{a,k}$, $k \in a\mathbb{Z}^d$.

Fix $i \in a\mathbb{Z}^d$, and for the sake of simplicity, let Λ denote the cube $\Lambda_{a,i}$. For any $j \in a\mathbb{Z}^d \setminus \{i\}$, consider two configurations that coincide outside of the cube $\Lambda_{a,j}$, and

differ only inside:

$$\xi, \tilde{\xi} \in \mathscr{A} : \ \xi_{\Lambda_{a,j}^c} = \tilde{\xi}_{\Lambda_{a,j}^c}.$$

Without loss of generality, we can assume that $Z_\Lambda^{z,\beta}(\tilde{\xi}) \le Z_\Lambda^{z,\beta}(\xi)$. Using an equivalent formulation of the total variation distance, we write

$$\begin{aligned} & d_{TV}\left(\Xi_\Lambda^{z,\beta}(\xi,\cdot), \Xi_\Lambda^{z,\beta}(\tilde{\xi},\cdot)\right) \\ &= 1 - \int \min\left(\frac{e^{-\beta E_\phi(\gamma_\Lambda \mid \xi_\Lambda^c)}}{Z_\Lambda^{z,\beta}(\xi)}; \frac{e^{-E_\phi(\gamma_\Lambda \mid \tilde{\xi}_\Lambda^c)}}{Z_\Lambda^{z,\beta}(\tilde{\xi})}\right) \pi_\Lambda^z(d\gamma_\Lambda) \\ &= \int \left(\frac{e^{-\beta E_\phi(\gamma_\Lambda \mid \xi_\Lambda^c)}}{Z_\Lambda^{z,\beta}(\xi)} - \min\left(\frac{e^{-\beta E_\phi(\gamma_\Lambda \mid \tilde{\xi}_\Lambda^c)}}{Z_\Lambda^{z,\beta}(\xi)}; \frac{e^{-\beta E_\phi(\gamma_\Lambda \mid \tilde{\xi}_\Lambda^c)}}{Z_\Lambda^{z,\beta}(\tilde{\xi})}\right)\right) \pi_\Lambda^z(d\gamma_\Lambda). \end{aligned}$$

Since, thanks to the choice of α, at most one point is "admissible" in Λ, we have

$$d_{TV}\left(\Xi_\Lambda^{z,\beta}(\xi,\cdot), \Xi_\Lambda^{z,\beta}(\tilde{\xi},\cdot)\right) = z e^{-z|\Lambda|} \int_\Lambda \left[\frac{e^{-\beta E_\phi(x \mid \xi_\Lambda^c)}}{Z_\Lambda^{z,\beta}(\xi)} - \frac{e^{-\beta E_\phi(x \mid \tilde{\xi}_\Lambda^c)}}{Z_\Lambda^{z,\beta}(\tilde{\xi})}\right]^+ dx,$$

where $[\, y \,]^+ := \max(y, 0)$ denotes the *positive part* of $y \in \mathbb{R}$.

Since $e^{-z|\Lambda|} \le Z_\Lambda^{z,\beta}(\tilde{\xi}) \le Z_\Lambda^{z,\beta}(\xi) \le 1$, we have that

$$\begin{aligned} & e^{-z|\Lambda|}\left[\frac{e^{-\beta E_\phi(x \mid \xi_\Lambda^c)}}{Z_\Lambda^{z,\beta}(\xi)} - \frac{e^{-\beta E_\phi(x \mid \tilde{\xi}_\Lambda^c)}}{Z_\Lambda^{z,\beta}(\tilde{\xi})}\right]^+ \le e^{-z|\Lambda|}\left[\frac{e^{-\beta E_\phi(x \mid \xi_\Lambda^c)}}{Z_\Lambda^{z,\beta}(\xi)} - \frac{e^{-\beta E_\phi(x \mid \tilde{\xi}_\Lambda^c)}}{Z_\Lambda^{z,\beta}(\xi)}\right]^+ \\ & \le \left[e^{-\beta E_\phi(x \mid \xi_\Lambda^c)} - e^{-\beta E_\phi(x \mid \tilde{\xi}_\Lambda^c)}\right]^+, \end{aligned}$$

and we can now estimate

$$d_{TV}\left(\Xi_\Lambda^{z,\beta}(\xi,\cdot), \Xi_\Lambda^{z,\beta}(\tilde{\xi},\cdot)\right) \le z \int_\Lambda \left[e^{-\beta E_\phi(x \mid \xi_\Lambda^c)} - e^{-\beta E_\phi(x \mid \tilde{\xi}_\Lambda^c)}\right]^+ dx. \tag{3.7}$$

Noting that the right-hand side of (3.7) is largest when $\xi = \underline{o}$, and that, thanks to the hard-core assumption (A1), $\tilde{\xi}$ contains at most one point in the cube $\Lambda_{a,j}$, we obtain the following bound, that holds uniformly over all pairs of admissible boundary configurations:

$$\forall \xi, \tilde{\xi} \in \mathscr{A} \ : \ \xi_{\Lambda_{a,j}^c} = \tilde{\xi}_{\Lambda_{a,j}^c}, \ d_{TV}\left(\Xi_\Lambda^{z,\beta}(\xi,\cdot), \Xi_\Lambda^{z,\beta}(\tilde{\xi},\cdot)\right) \le z \sup_{y \in \Lambda_{a,j}} \int_\Lambda \left(1 - e^{-\beta\phi(x,y)}\right) dx,$$

In particular, we can take the supremum over the configurations $\xi, \tilde{\xi} \in \mathscr{A}$ such that

$\xi_{\Lambda_{a,j}^c} = \tilde{\xi}_{\Lambda_{a,j}^c}$:

$$k_{i,j}^{(a)} \leq z \sup_{y \in \Lambda_{a,j}} \int_{\Lambda_{a,i}} \left(1 - e^{-\beta\phi(x,y)}\right) dx.$$

Obtaining the bound

Summing the coefficients $k_{i,j}^{(a)}$ over all $j \in a\mathbb{Z}^d \setminus \{i\}$, leads to

$$\begin{aligned}\sum_{j \in a\mathbb{Z}^d \setminus \{i\}} k_{i,j}^{(a)} &\leq z \sum_{j \in a\mathbb{Z}^d \setminus \{i\}} \sup_{y \in \Lambda_{a,j}} \int_{\Lambda_{a,i}} \left(1 - e^{-\beta\phi(x,y)}\right) dx \\ &\leq z \int_{\Lambda_{a,i}} \sum_{j \in a\mathbb{Z}^d \setminus \{i\}} \sup_{y \in \Lambda_{a,j}} \left(1 - e^{-\beta\phi(x,y)}\right) dx.\end{aligned}$$

We want to obtain the expression of the local supremum $\Psi_a(x,y)$ of the Mayer function defined in (3.2). For this reason, we add an integration over $y \in \Lambda_{a,j}$ and rewrite the above as:

$$\begin{aligned}&z \int_{\Lambda_{a,i}} \sum_{j \in a\mathbb{Z}^d \setminus \{i\}} \frac{1}{|\Lambda_{a,j}|} \int_{\Lambda_{a,j}} \sup_{\bar{y} \in \Lambda_{a,j}} \left(1 - e^{-\beta\phi(x,\bar{y})}\right) dy\, dx \\ &= \frac{z}{|\Lambda_{a,i}|} \int_{\Lambda_{a,i}} \int_{\mathbb{R}^d \setminus \Lambda_{a,i}} \Psi_a(x,y) dy\, dx \\ &\leq z \sup_{x \in \Lambda_{a,i}} \int_{\mathbb{R}^d} \Psi_a(x,y) dy.\end{aligned}$$

Taking the supremum over all $i \in a\mathbb{Z}^d$ yields

$$\begin{aligned}&\sup_{i \in a\mathbb{Z}^d} \sum_{j \in a\mathbb{Z}^d} \sup_{\xi, \tilde{\xi} \in \mathscr{A}} d_{TV}\left(\Xi_{\Lambda_{a,i}}^{z,\beta}(\xi, \cdot), \Xi_{\Lambda_{a,i}}^{z,\beta}(\tilde{\xi}, \cdot)\right) \\ &\leq z \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \Psi_a(x,y) dy \xrightarrow{a \to 0} z \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(1 - e^{-\beta\phi(x,y)}\right) dy,\end{aligned} \tag{3.8}$$

where the last convergence follows from Assumption (A3). This means, in particular, that if $z \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(1 - e^{-\beta\phi(x,y)}\right) dy < 1$, there exists $a > 0$ such that the Dobrushin condition (3.6) is satisfied: $\sup_{i \in a\mathbb{Z}^d} \sum_{j \in a\mathbb{Z}^d \setminus \{i\}} k_{i,j}^{(a)} < 1$. This concludes the proof of Theorem 3.1.

3.2.3 Discussion about the assumptions and possible generalisations

The proof of Theorem 3.1 relies on the assumptions on the interaction made above, namely that it is coming from a non-negative pair potential ϕ which is hard-core

close to the origin (A1), with an integrability assumption (A2), and a technical regularity assumption (A3). We comment here on these requirements.

Firstly, we restrict our study to *non-negative* potentials ϕ in order to have a setting where all three uniqueness methods can be applied and compared. Moreover, extending this result to a more general pair potential, e.g. superstable and lower-regular, would cause some complications in the proof. For example, the observation made after (3.7) is no longer valid if the potential is allowed to take negative values.

Secondly, as we have already stated, the regularity assumption (A3) is purely technical, resulting from taking $a \longrightarrow 0$ in the proof. We remark that it is satisfied whenever the set of discontinuity of $\phi(x, .)$ is of measure 0, which is the case, for example, for the radial potentials with hard core presented in Subection 3.2.5.

Thirdly, the integrability assumption (A2) is quite standard, and seems unavoidable when using the Dobrushin criterion. Furthermore, a similar assumption is required when using the cluster expansion technique.

The most restricting requirement we make is therefore assumption (A1), which excludes interactions that do not have a hard-core component when two points are too close; notice how the hard core U is not necessarily a sphere $B(0, \alpha)$, but can be a more general neighbourhood of the origin, for example an ellipse where α is the length of the minor axis.

The hard-core component of the interaction makes sure that any boundary condition has at most one point if the cubes are small enough; this in turn allows us to derive the bound in (3.3). However, many interactions – like the widely known Strauss pairwise interaction from [96] – do not satisfy this assumption. In the case of a non-negative potential without hard core, as for the Strauss model with $\phi(x, y) = \mathbb{1}_{\{|x-y|\leq 1\}}$, we would still be able to apply the classical Dobrushin criterion and then consider the limit $a \longrightarrow 0$. However, since in this case there is no restriction on the number of points in each cube, the uniqueness region we would obtain is $z < b_d^{-1}$, where b_d is the volume of the unit ball. Notice how this criterion does not depend on β anymore.

One possible way to overcome this issue is to use the so-called Dobrushin–Pechersky criterion [32]. While the "general behaviour" of the uniqueness region obtained with this criterion is known (see [77]), the resulting conditions are too technical to obtain explicit values of the parameters. However, it seems possible to develop new techniques inspired by the proof of Theorem 3.1.

Heuristically, when taking $a \longrightarrow 0$, it is less and less likely for a boundary condition to have more than one point in a small cube of side-length a. This is precisely what we use the hard-core assumption (A1) for, so we formulate the following

Conjecture 3.3. Under assumptions (A2) and (A3), there is uniqueness of the Gibbs point process as soon as

$$z < C(\beta)^{-1}.$$

3.2.4 Optimality of Theorem 3.1 within the Dobrushin uniqueness regime

We believe the simplicity of the uniqueness region obtained in Theorem 3.1, as well as the ability to compare it to other criteria, is itself a justification for considering this method. It is of course natural to ask whether taking the limit for $a \longrightarrow 0$ actually yields a larger uniqueness region than what is obtained by just considering a fixed $a > 0$.

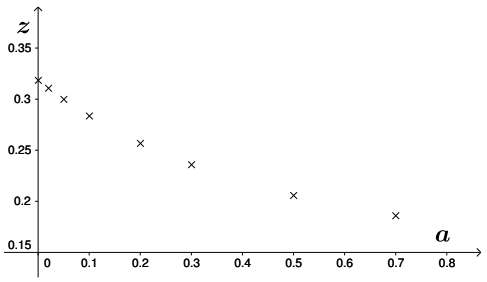

Figure 3.1. Dobrushin uniqueness bound $a \mapsto \bar{z}_\beta(a)$ for the Hard-sphere model, for several values of $a \in [0, 0.7]$. We find $\bar{z}_\beta(0) \simeq 0.318$.

This translates to comparing the uniqueness regions obtained by fixing some $a > 0$ and by taking the limit $a \longrightarrow 0$, i.e. respectively,

$$I_a := \{z > 0 : \sum_{\substack{j \in a\mathbb{Z}^d \\ j \neq i}} \sup_{\xi, \tilde{\xi}} d_{TV}\left(\Xi^{z,\beta}_{\Lambda_{a,i}}(\xi, \cdot), \Xi^{z,\beta}_{\Lambda_{a,i}}(\tilde{\xi}, \cdot)\right) < 1\} = \left(0, \bar{z}_\beta(a)\right), \ a > 0,$$

and

$$I_0 := \{z > 0 : z\, C(\beta) < 1\} = \left(0, \bar{z}_\beta(0)\right),$$

where $\bar{z}_\beta(a)$ is the supremum of the interval I_a, $a \geq 0$.

While we do not address this question in the general case, we have performed a numerical study in the simple case of the Hard-sphere model $\phi(x, y) = (+\infty)\mathbb{1}_{\{|x-y|<1\}}$ in dimension $d = 2$. Figure 3.1 displays the upper bound $\bar{z}_\beta(a)$ of the Dobrushin uniqueness interval that we have computed numerically for several values of a, showing that that taking the limit for $a \longrightarrow 0$ yields a larger Dobrushin uniqueness region than that of any fixed $a > 0$.

3.2.5 Comparison with the other uniqueness methods

In this subection we briefly describe the two methods of cluster expansion and disagreement percolation, and see how the uniqueness regions that they yield compare to our result via the Dobrushin criterion. In particular, we provide a visual comparison between the three uniqueness methods, displayed below in Figure 3.2 for two potentials with hard core, and in Figure 3.3 for the case of the Strauss potential if

Conjecture 3.3 holds.

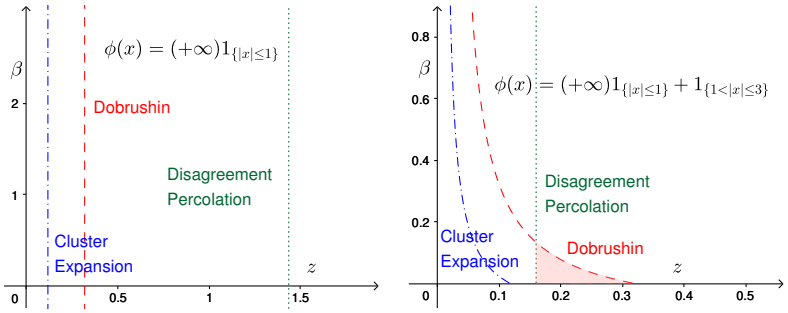

Figure 3.2. The uniqueness regions (shaded where our method yields a larger bound) for the three different methods consist in the regions below the respective curves. Depicted here for the pure hard-core potential $\phi(x, y) = (+\infty)\mathbb{1}_{\{|x-y|<1\}}$ on the left and for $\phi(x, y) = (+\infty)\mathbb{1}_{\{|x-y|<1\}} + \mathbb{1}_{\{1\leq|x-y|\leq 3\}}$ on the right.

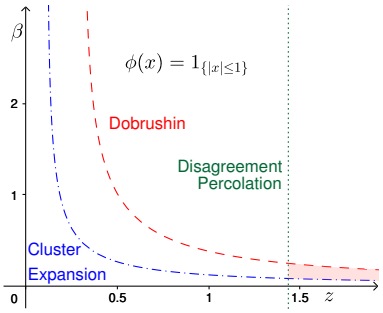

Figure 3.3. Uniqueness regions (shaded where our method yields a larger bound) for the Strauss potential $\phi(x, y) = \mathbb{1}_{\{|x-y|\leq 1\}}$, assuming Conjecture 3.3 is valid.

Cluster expansion

The method of cluster expansion was first developed for lattice systems in the 1980s (see e.g. [63]) and later extended to the continuous case. Indeed, different approaches exist to show convergence of the cluster terms (see e.g. [64, 80, 74]).

For continuous point processes, the technique (from Ruelle, see [90]) relies on a *series expansion of the correlation functions*. More precisely, it consists in showing that the correlation functions of a Gibbs point process can be expressed as an absolutely converging series of cluster terms, and uniqueness is then proved by considering a set of integral equations – the so-called *Kirkwood–Salsburg equations* – satisfied by the correlation functions, which can be reformulated as a fixed-point problem in an appropriately chosen Banach space, having therefore have a unique solution.

In [51], S. Jansen presents a cluster expansion criterion for Gibbs point processes with a repulsive interaction ϕ. However – as the most general form of this crite-

rion yields an implicit uniqueness region – in order to compare it with the domains obtained through the other methods, we only recall a specific uniqueness case, that allows however to obtain an explicit region.

Theorem 3.4 ([51]). Let H be an energy functional as in (3.1). Let $z > 0$ and $\beta > 0$, and suppose there exist a non-negative measurable function $\mathbf{a}$ and some $t > 0$ such that, for a.e. $x_0 \in \mathbb{R}^d$,

$$z e^t \int \left(1 - e^{-\beta\phi(x_0,y)}\right) e^{\mathbf{a}(y)} dy \leq \mathbf{a}(x_0).$$

There exists then a unique Gibbs point process $P \in \mathscr{G}_{z,\beta}(H)$.

Restricting $\mathbf{a}$ to be a constant $\mathbf{a} \equiv a \geq 0$, and remarking that $\max_{a\geq 0} ae^{-a} = 1/e$, the condition above holds for some $a \geq 0$ and $t > 0$ as soon as the classical Ruelle condition (e.g. Theorem 4.2.3 of [90]) holds

$$z < e^{-1} C(\beta)^{-1}. \tag{3.9}$$

By considering a non-constant function a in Theorem 3.4, one could hope to obtain a better bound than (3.9).

Moreover, using tree-graph estimates, Fernandez, Procacci, and Scoppola, were able to improve the classical cluster expansion bound. Their approach, based on tree-graph estimates, is first presented in [33], where they also present a comparison to the other methods. In particular, in [34], the authors study the specific case of the 2-dimensional hard-sphere model (here $C(\beta) = b_2$) and obtain from cluster expansion an improved bound which they then estimate numerically as

$$z < 0.5107\, b_2^{-1}.$$

While larger than bound (3.9), $z < (e\, C(\beta))^{-1}$, this uniqueness region is still smaller than the one we obtain, i.e. $z < b_2^{-1}$.

We remark that the restriction to non-negative potentials greatly simplifies the cluster expansion approach. Indeed, in this setting it is immediate to see that the correlation functions of any Gibbs point process are in the same Banach space where uniqueness holds, which is not in general true (proving this in the setting of [92] makes use of some superstability estimates).

Disagreement percolation

The method of disagreement percolation was introduced for lattice systems by van den Berg and Maes [97, 98]. The idea behind disagreement percolation is the construction of a coupling, sometimes called *disagreement coupling*, which compares

Gibbs specifications with two different boundary conditions outside of a given box, in such a way that the *disagreement points* between the two Gibbs specification are "connected" to the boundary of the box. If the probability of being connected to the boundary of an increasingly large box goes to zero, uniqueness holds.

Adapting the uniqueness result of [49] to our setting reads

Theorem 3.5 ([49]). Let H be an energy functional as in (3.1), where ϕ is a finite-range pair potential on $\mathbb{R}^d$, i.e. there exists $\mathfrak{r} > 0$ such that $\phi(x, y) = 0$ if $|x - y| > \mathfrak{r}$. Let $\beta > 0$ and

$$z < \frac{z_c(d)}{\mathfrak{r}^d},$$

where $z_c(d)$ is the *percolation threshold* of the Poisson–Boolean model in dimension d connecting points at distance at most one. There exists then at most one Gibbs point process $P \in \mathscr{G}_{z,\beta}(H)$.

Remark. We recall that the *Poisson–Boolean model* with fixed radius 1/2 is a Poisson point process of balls π^z where the configurations are collections of balls centred in the Poisson point in $\mathbb{R}^d$ and with radius 1/2. The percolation threshold $z_c(d)$ is the smallest intensity value at which there exists π^z-almost surely an infinite connected component appears (i.e. the model *percolates*).

Note that $z_c(d) \geq 1/b_d$ and, in the asymptotics as $d \longrightarrow \infty$, $z_c(d) \sim 1/b_d$, see [79]. We also remark that in dimension $d = 2$ a numerical simulation (see [83]) yields $z_c(2) \simeq 1.43629$.

In [49] the above bound is actually proved for interactions that do not necessarily come from a pair potential, without a hard core, and with a *finite random range* that may depend on the (unbounded) marks of each point of the configuration. However, it does not apply for infinite-range pair potentials, which are instead allowed in Theorem 3.1.

Finally, we remark that the disagreement percolation uniqueness region is independent of the parameter β and depends only on the range of the interaction. Typically, when β is large, the disagreement percolation uniqueness region is larger than the one obtained from our bound in Theorem 3.1, but when β is small, our uniqueness region is larger than the one coming from disagreement percolation (see the second potential of Figure 3.2).

Computer programmes A

A.1 Langevin diffusions

Code used to generate Figure 2.1.

```
import numpy as np
from math import sqrt
from pylab import plot, show, grid, axis, xlabel, ylabel,
    ↪ title
import six
import matplotlib.pyplot as plt

#function simulating a two dimensional diffusion
def langevin(x0,n,dt,delta,out):
  x0 = np.asarray(x0)
  r = np.random.normal(size=x0.shape + (n,), scale=delta*sqrt(
      ↪ dt))
  if out is None:
    out = np.empty(r.shape)
  L = np.empty(r.shape)
  np.cumsum(r, axis=-1, out=out)

  for i in range(1,n):
    L[0,i] = -2*L[0,i-1]**3*dt + out[0,i]
    L[1,i] = -2*L[1,i-1]**3*dt + out[1,i]
  return L

#computing the maximum displacement
def disp(x):
  D = np.sqrt(np.amax((x[0,:]-x[0,0])**2+(x[1,:]-x[1,0])**2))
  return D

def draw(x,d,c):
  # change default range so that new circles will work
```

```
  ax.set_xlim((-3, 7))
  ax.set_ylim((-5, 7))
  #plt.autoscale(False)
  beingsaved = plt.figure()
  plt.plot(x[0],x[1],c,linewidth=0.3)
  plt.plot(x[0,0],x[1,0],'x',color='k',markersize=5)
  #plt.plot(x[0,-1], x[1,-1], 'wo',markersize=3)
  circ=plt.Circle((x[0,0],x[1,0]),d,color=c,alpha=0.2)
  plt.gcf().gca().add_artist(circ)
  plt.savefig('Langevin.pdf', format='pdf',bbox_inches='tight'
      ↪ ,pad_inches=0.0)

# Parameter for the BM
delta1 = 2
delta2 = 2
# Total time.
T = 1
# Number of steps.
N = 100000
# Time step size
dt = T/N

#Initialisation
x1 = np.empty((2,N))
x2 = np.empty((2,N))
x1[:, 0] = 0.0
x2[:, 0] = 0.0

#starting points of diffusions
ic1 = 0.0
ic2 = 2.0

#creating the diffusions
x1 = langevin(x1[:,0], N, dt, delta1,out=x1)
d1 = disp(x1)
x2 = langevin(x2[:,0], N, dt, delta2,out=x2)
d2 = disp(x2)

ax = plt.gca()
ax.cla() #clear for plot
axis('equal')
```

```
ax.axis('off')
ax.patch.set_facecolor('#8FBC8F')
ax.patch.set_alpha(0.0)

#drawing the circles of maximum displacement radius
draw(x1+ic1,d1,'b')
draw(x2+ic2,d2,'r')

show()
```

A.2 Activity regime

Code used to generate Figures 2.5 and 2.6.

```
import numpy as np
import matplotlib.pyplot as plt

#parameters of the model
cphi = 1.0
cbeta = 1.0

zR = np.e**(-2*cphi-1)/cbeta

#constant c_z (see Equation 2.25)
def cz(num):
   #regime
   return (np.e**(2*cphi)*(1 + (2*np.pi)**(1/2)*np.e*np.log
      ↪ (1/(1-num/zR))))

#function f(z) (see page 68)
def f(num):
   return np.e**(2*cphi+num*cz(num)*cbeta)/cz(num)

x = np.linspace(0.000001,0.045,100)
y = f(x)

for i in range(100):
   print (x[i],y[i])

plt.xlabel('z')
plt.plot(x,y,'r')
plt.show()
```

List of frequently used symbols

Notation	Description	Page List
$\mathbb{R}^d$	d-dimensional Euclidean space	21
$\mathbb{Z}^d$	d-dimensional integer lattice	33
$\mathbb{N}^*$	Positive (non-zero) integers	22
dx, dx_Λ	Lebesgue measure on $\mathbb{R}^d$ or on $\Lambda \subset \mathbb{R}^d$	24
$\mathbf{R}$	Reference probability measure on the mark space $\mathscr{S}$	25
Λ	Finite volume: a bounded Borel set of $\mathbb{R}^d$	21
Λ_n	Cube $[-n, n)^d \subset \mathbb{R}^d$	31
$\lvert\Lambda\rvert$	d-dimensional Lebesgue measure of a set $\Lambda \subset \mathbb{R}^d$	21
b_d	Volume of the unit ball in $\mathbb{R}^d$	70
$\mathscr{B}(\mathbb{R}^d)$	Borel σ-algebra on $\mathbb{R}^d$	21
$\mathscr{B}_b(\mathbb{R}^d)$	Subset of bounded Borel sets on $\mathbb{R}^d$	21
$\mathscr{E}$	State space, either $\mathbb{R}^d \times \mathscr{S}$ or $\mathbb{R}^d$	21
$\mathscr{S}$	Space of marks; given by a normed space	21
$\mathscr{M}$	Set of point measures (configurations) on the state space $\mathscr{E}$	21
$\mathscr{M}_f$	Set of finite configurations on the state space $\mathscr{E}$	22
$\mathscr{M}^{\mathrm{temp}}$, $\mathscr{M}^{\mathbf{t}}$	Set of tempered configurations	23
γ, ξ, η	Point configurations on $\mathscr{E}$	21
$\underline{o}$	Empty configuration, supported on the empty set $\emptyset$	21
ω	Lattice configuration	80
π^z	Poisson point process on $\mathscr{E}$ with activity $z > 0$	24
H	Energy functional on $\mathscr{M}_f$	25
H_Λ	Conditional energy on Λ given the environment	26
$\mathscr{L}$	Space of tame and local functionals on $\mathscr{M}$	29
$\tau_{\mathscr{L}}$	Topology of local convergence on $\mathscr{P}(\mathscr{M})$ induced by the tame and local functionals	29
$\mathscr{G}_{z,\beta}(H), \mathscr{G}_{z,\beta}(\Phi)$	Set of Gibbs point processes with activity z associated to an energy functional H or a potential Φ	29
Ξ_Λ, $\Xi_\Lambda^{z,\beta}$	Gibbsian probability kernel	30

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